Appendix A

Mathematical appendix

A.1 The Fourier transform

The Fourier transform permits us to decompose a complicated field structure into elemental components. This can simplify the computation of fields and provide physical insight into their spatiotemporal behavior. In this section we review the properties of the transform and demonstrate its usefulness in solving field equations.

One-dimensional case

Let f be a function of a single variable x. The Fourier transform of $f(x)$ is the function $F(k)$ defined by the integral

$$\mathcal{F}\{f(x)\} = F(k) = \int_{-\infty}^{\infty} f(x)e^{-jkx}\,dx. \tag{A.1}$$

Note that x and the corresponding transform variable k must have reciprocal units: if x is time in seconds, then k is a *temporal frequency* in radians per second; if x is a length in meters, then k is a *spatial frequency* in radians per meter. We sometimes refer to $F(k)$ as the *frequency spectrum* of $f(x)$.

Not every function has a Fourier transform. The existence of (A.1) can be guaranteed by a set of sufficient conditions such as the following:

1. f is absolutely integrable: $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$;
2. f has no infinite discontinuities;
3. f has at most finitely many discontinuities and finitely many extrema in any finite interval (a, b).

While such rigor is certainly of mathematical value, it may be of less ultimate use to the engineer than the following heuristic observation offered by Bracewell [22]: *a good mathematical model of a physical process should be Fourier transformable.* That is, if the Fourier transform of a mathematical model does not exist, the model cannot precisely describe a physical process.

The usefulness of the transform hinges on our ability to recover f through the inverse transform:

$$\mathcal{F}^{-1}\{F(k)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\,e^{jkx}\,dk. \tag{A.2}$$

When this is possible we write

$$f(x) \leftrightarrow F(k)$$

and say that $f(x)$ and $F(k)$ form a Fourier transform pair. The *Fourier integral theorem* states that

$$\mathcal{F}\,\mathcal{F}^{-1}\{f(x)\} = \mathcal{F}^{-1}\,\mathcal{F}\{f(x)\} = f(x),$$

except at points of discontinuity of f. At a jump discontinuity the inversion formula returns the average value of the one-sided limits $f(x^+)$ and $f(x^-)$ of $f(x)$. At points of continuity the forward and inverse transforms are unique.

Transform theorems and properties. We now review some basic facts pertaining to the Fourier transform. Let $f(x) \leftrightarrow F(k) = R(k) + jX(k)$, and $g(x) \leftrightarrow G(k)$.

1. *Linearity.* $\alpha f(x) + \beta g(x) \leftrightarrow \alpha F(k) + \beta G(k)$ if α and β are arbitrary constants. This follows directly from the linearity of the transform integral, and makes the transform useful for solving linear differential equations (e.g., Maxwell's equations).
2. *Symmetry.* The property $F(x) \leftrightarrow 2\pi f(-k)$ is helpful when interpreting transform tables in which transforms are listed only in the forward direction.
3. *Conjugate function.* We have $f^*(x) \leftrightarrow F^*(-k)$.
4. *Real function.* If f is real, then $F(-k) = F^*(k)$. Also,

$$R(k) = \int_{-\infty}^{\infty} f(x)\cos kx\,dx, \quad X(k) = -\int_{-\infty}^{\infty} f(x)\sin kx\,dx,$$

and

$$f(x) = \frac{1}{\pi}\,\mathrm{Re}\int_0^{\infty} F(k)e^{jkx}\,dk.$$

A real function is completely determined by its positive frequency spectrum. It is obviously advantageous to know this when planning to collect spectral data.

5. *Real function with reflection symmetry.* If f is real and even, then $X(k) \equiv 0$ and

$$R(k) = 2\int_0^{\infty} f(x)\cos kx\,dx, \quad f(x) = \frac{1}{\pi}\int_0^{\infty} R(k)\cos kx\,dk.$$

If f is real and odd, then $R(k) \equiv 0$ and

$$X(k) = -2\int_0^{\infty} f(x)\sin kx\,dx, \quad f(x) = -\frac{1}{\pi}\int_0^{\infty} X(k)\sin kx\,dk.$$

(Recall that f is even if $f(-x) = f(x)$ for all x. Similarly f is odd if $f(-x) = -f(x)$ for all x.)

6. *Causal function.* Recall that f is causal if $f(x) = 0$ for $x < 0$.
(a) If f is real and causal, then

$$X(k) = -\frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} R(k')\cos k'x\sin kx\,dk'\,dx,$$
$$R(k) = -\frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} X(k')\sin k'x\cos kx\,dk'\,dx.$$

(b) If f is real and causal, and $f(0)$ is finite, then $R(k)$ and $X(k)$ are related by the *Hilbert transforms*

$$X(k) = -\frac{1}{\pi}\,\text{P.V.}\int_{-\infty}^{\infty}\frac{R(k)}{k-k'}\,dk', \quad R(k) = \frac{1}{\pi}\,\text{P.V.}\int_{-\infty}^{\infty}\frac{X(k)}{k-k'}\,dk'.$$

(c) If f is causal and has finite energy, it is not possible to have $F(k) = 0$ for $k_1 < k < k_2$. That is, the transform of a causal function cannot vanish over an interval.

A causal function is completely determined by the real or imaginary part of its spectrum. As with item 4, this is helpful when performing calculations or measurements in the frequency domain. If the function is not band-limited however, truncation of integrals will give erroneous results.

7. *Time-limited vs. band-limited functions.* Assume $t_2 > t_1$. If $f(t) = 0$ for both $t < t_1$ and $t > t_2$, then it is not possible to have $F(k) = 0$ for both $k < k_1$ and $k > k_2$ where $k_2 > k_1$. That is, a time-limited signal cannot be band-limited. Similarly, a band-limited signal cannot be time-limited.
8. *Null function. If the forward or inverse transform of a function is identically zero, then the function is identically zero.* This important consequence of the Fourier integral theorem is useful when solving homogeneous partial differential equations in the frequency domain.
9. *Space or time shift.* For any fixed x_0,

$$f(x - x_0) \;\leftrightarrow\; F(k)e^{-jkx_0}. \tag{A.3}$$

A temporal or spatial shift affects only the phase of the transform, not the magnitude.

10. *Frequency shift.* For any fixed k_0,

$$f(x)e^{jk_0x} \;\leftrightarrow\; F(k - k_0).$$

Note that if $f \leftrightarrow F$ where f is real, then frequency-shifting F causes f to become complex — again, this is important if F has been obtained experimentally or through computation in the frequency domain.

11. *Similarity.* We have

$$f(\alpha x) \;\leftrightarrow\; \frac{1}{|\alpha|}F\left(\frac{k}{\alpha}\right),$$

where α is any real constant. "Reciprocal spreading" is exhibited by the Fourier transform pair; dilation in space or time results in compression in frequency, and vice versa.

12. *Convolution.* We have

$$\int_{-\infty}^{\infty} f_1(x')\,f_2(x - x')\,dx' \;\leftrightarrow\; F_1(k)F_2(k)$$

and

$$f_1(x)\,f_2(x) \;\leftrightarrow\; \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(k')F_2(k - k')\,dk'.$$

The first of these is particularly useful when a problem has been solved in the frequency domain and the solution is found to be a product of two or more functions of k.

13. *Parseval's identity.* We have

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(k)|^2\,dk.$$

Computations of energy in the time and frequency domains always give the same result.

14. *Differentiation.* We have

$$\frac{d^n f(x)}{dx^n} \leftrightarrow (jk)^n F(k) \quad \text{and} \quad (-jx)^n f(x) \leftrightarrow \frac{d^n F(k)}{dk^n}.$$

The Fourier transform can convert a differential equation in the x domain into an algebraic equation in the k domain, and vice versa.

15. *Integration.* We have

$$\int_{-\infty}^{x} f(u)\,du \leftrightarrow \pi F(k)\delta(k) + \frac{F(k)}{jk}$$

where $\delta(k)$ is the Dirac delta or unit impulse.

Generalized Fourier transforms and distributions. It is worth noting that many useful functions are not Fourier transformable in the sense given above. An example is the signum function

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases}$$

Although this function lacks a Fourier transform in the usual sense, for practical purposes it may still be safely associated with what is known as a *generalized Fourier transform.* A treatment of this notion would be out of place here; however, the reader should certainly be prepared to encounter an entry such as

$$\operatorname{sgn}(x) \leftrightarrow 2/jk$$

in a standard Fourier transform table. Other functions can be regarded as possessing transforms when *generalized functions* are permitted into the discussion. An important example of a generalized function is the Dirac delta $\delta(x)$, which has enormous value in describing distributions that are very thin, such as the charge layers often found on conductor surfaces. We shall not delve into the intricacies of distribution theory. However, we can hardly avoid dealing with generalized functions; to see this we need look no further than the simple function $\cos k_0 x$ with its transform pair

$$\cos k_0 x \leftrightarrow \pi[\delta(k + k_0) + \delta(k - k_0)].$$

The reader of this book must therefore know the standard facts about $\delta(x)$: that it acquires meaning only as part of an integrand, and that it satisfies the *sifting property*

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x)\,dx = f(x_0)$$

for any continuous function f. With $f(x) = 1$ we obtain the familiar relation

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

With $f(x) = e^{-jkx}$ we obtain

$$\int_{-\infty}^{\infty} \delta(x) e^{-jkx}\, dx = 1,$$

thus

$$\delta(x) \leftrightarrow 1.$$

It follows that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{jkx}\, dk = \delta(x). \tag{A.4}$$

Useful transform pairs. Some of the more common Fourier transforms that arise in the study of electromagnetics are given in Appendix C. These often involve the simple functions defined here:

1. *Unit step function*

$$U(x) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases} \tag{A.5}$$

2. *Signum function*

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases} \tag{A.6}$$

3. *Rectangular pulse function*

$$\operatorname{rect}(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases} \tag{A.7}$$

4. *Triangular pulse function*

$$\Lambda(x) = \begin{cases} 1 - |x|, & |x| < 1, \\ 0, & |x| > 1. \end{cases} \tag{A.8}$$

5. *Sinc function*

$$\operatorname{sinc}(x) = \frac{\sin x}{x}. \tag{A.9}$$

Transforms of multi-variable functions

Fourier transformations can be performed over multiple variables by successive applications of (A.1). For example, the two-dimensional Fourier transform over x_1 and x_2 of the function $f(x_1, x_2, x_3, \ldots, x_N)$ is the quantity $F(k_{x_1}, k_{x_2}, x_3, \ldots, x_N)$ given by

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x_1, x_2, x_3, \ldots, x_N)\, e^{-jk_{x_1}x_1}\, dx_1 \right] e^{-jk_{x_2}x_2}\, dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3, \ldots, x_N)\, e^{-jk_{x_1}x_1} e^{-jk_{x_2}x_2}\, dx_1\, dx_2.$$

The two-dimensional inverse transform is computed by multiple application of (A.2), recovering $f(x_1, x_2, x_3, \ldots, x_N)$ through the operation

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_{x_1}, k_{x_2}, x_3, \ldots, x_N)\, e^{jk_{x_1}x_1} e^{jk_{x_2}x_2}\, dk_{x_1}\, dk_{x_2}.$$

Higher-dimensional transforms and inversions are done analogously.

Transforms of separable functions. If we are able to write

$$f(x_1, x_2, x_3, \ldots, x_N) = f_1(x_1, x_3, \ldots, x_N) f_2(x_2, x_3, \ldots, x_N),$$

then successive transforms on the variables x_1 and x_2 result in

$$f(x_1, x_2, x_3, \ldots, x_N) \;\leftrightarrow\; F_1(k_{x_1}, x_3, \ldots, x_N) F_2(k_{x_2}, x_3, \ldots, x_N).$$

In this case a multi-variable transform can be obtained with the help of a table of one-dimensional transforms. If, for instance,

$$f(x, y, z) = \delta(x - x')\delta(y - y')\delta(z - z'),$$

then we obtain

$$F(k_x, k_y, k_z) = e^{-jk_x x'} e^{-jk_y y'} e^{-jk_z z'}$$

by three applications of (A.1).

A more compact notation for multi-dimensional functions and transforms makes use of the vector notation $\mathbf{k} = \hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y + \hat{\mathbf{z}}k_z$ and $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$ where $\mathbf{r}$ is the position vector. In the example above, for instance, we could have written

$$\delta(x - x')\delta(y - y')\delta(z - z') = \delta(\mathbf{r} - \mathbf{r}'),$$

and

$$F(\mathbf{k}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\mathbf{r} - \mathbf{r}') e^{-j\mathbf{k}\cdot\mathbf{r}}\, dx\, dy\, dz = e^{-j\mathbf{k}\cdot\mathbf{r}'}.$$

Fourier–Bessel transform. If x_1 and x_2 have the same dimensions, it may be convenient to recast the two-dimensional Fourier transform in polar coordinates. Let $x_1 = \rho\cos\phi$, $k_{x_1} = p\cos\theta$, $x_2 = \rho\sin\phi$, and $k_{x_2} = p\sin\theta$, where p and ρ are defined on $(0, \infty)$ and ϕ and θ are defined on $(-\pi, \pi)$. Then

$$F(p, \theta, x_3, \ldots, x_N) = \int_{-\pi}^{\pi}\int_{0}^{\infty} f(\rho, \phi, x_3, \ldots, x_N)\, e^{-jp\rho\cos(\phi-\theta)}\, \rho\, d\rho\, d\phi. \tag{A.10}$$

If f is independent of ϕ (due to rotational symmetry about an axis transverse to x_1 and x_2), then the ϕ integral can be computed using the identity

$$J_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-jx\cos(\phi-\theta)}\, d\phi.$$

Thus (A.10) becomes

$$F(p, x_3, \ldots, x_N) = 2\pi\int_{0}^{\infty} f(\rho, x_3, \ldots, x_N) J_0(\rho p)\, \rho\, d\rho, \tag{A.11}$$

showing that F is independent of the angular variable θ. Expression (A.11) is termed the *Fourier–Bessel transform* of f. The reader can easily verify that f can be recovered from F through

$$f(\rho, x_3, \ldots, x_N) = \int_0^\infty F(p, x_3, \ldots, x_N) J_0(\rho p)\, p\, dp,$$

the inverse Fourier–Bessel transform.

A review of complex contour integration

Some powerful techniques for the evaluation of integrals rest on complex variable theory. In particular, the computation of the Fourier inversion integral is often aided by these techniques. We therefore provide a brief review of this material. For a fuller discussion the reader may refer to one of many widely available textbooks on complex analysis.

We shall denote by $f(z)$ a complex valued function of a complex variable z. That is,

$$f(z) = u(x, y) + jv(x, y),$$

where the real and imaginary parts $u(x, y)$ and $v(x, y)$ of f are each functions of the real and imaginary parts x and y of z:

$$z = x + jy = \mathrm{Re}(z) + j\,\mathrm{Im}(z).$$

Here $j = \sqrt{-1}$, as is mostly standard in the electrical engineering literature.

Limits, differentiation, and analyticity. Let $w = f(z)$, and let $z_0 = x_0 + jy_0$ and $w_0 = u_0 + jv_0$ be points in the complex z and w planes, respectively. We say that w_0 is the limit of $f(z)$ as z approaches z_0, and write

$$\lim_{z \to z_0} f(z) = w_0,$$

if and only if both $u(x, y) \to u_0$ and $v(x, y) \to v_0$ as $x \to x_0$ and $y \to y_0$ independently. The derivative of $f(z)$ at a point $z = z_0$ is defined by the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

if it exists. Existence requires that the derivative be independent of direction of approach; that is, $f'(z_0)$ cannot depend on the manner in which $z \to z_0$ in the complex plane. (This turns out to be a much stronger condition than simply requiring that the functions u and v be differentiable with respect to the variables x and y.) We say that $f(z)$ is *analytic* at z_0 if it is differentiable at z_0 and at all points in some neighborhood of z_0.

If $f(z)$ is not analytic at z_0 but every neighborhood of z_0 contains a point at which $f(z)$ is analytic, then z_0 is called a *singular point of* $f(z)$.

Laurent expansions and residues. Although Taylor series can be used to expand complex functions around points of analyticity, we must often expand functions around points z_0 at or near which the functions fail to be analytic. For this we use the *Laurent*

expansion, a generalization of the Taylor expansion involving both positive and negative powers of $z - z_0$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The numbers a_n are the coefficients of the Laurent expansion of $f(z)$ at point $z = z_0$. The first series on the right is the *principal part* of the Laurent expansion, and the second series is the *regular part*. The regular part is an ordinary power series, hence it converges in some disk $|z - z_0| < R$ where $R \geq 0$. Putting $\zeta = 1/(z - z_0)$, the principal part becomes $\sum_{n=1}^{\infty} a_{-n} \zeta^n$; this power series converges for $|\zeta| < \rho$ where $\rho \geq 0$, hence the principal part converges for $|z - z_0| > 1/\rho \triangleq r$. When $r < R$, the Laurent expansion converges in the annulus $r < |z - z_0| < R$; when $r > R$, it diverges everywhere in the complex plane.

The function $f(z)$ has an *isolated singularity* at point z_0 if $f(z)$ is not analytic at z_0 but is analytic in the "punctured disk" $0 < |z - z_0| < R$ for some $R > 0$. Isolated singularities are classified by reference to the Laurent expansion. Three types can arise:

1. *Removable singularity.* The point z_0 is a removable singularity of $f(z)$ if the principal part of the Laurent expansion of $f(z)$ about z_0 is identically zero (i.e., if $a_n = 0$ for $n = -1, -2, -3, \ldots$).
2. *Pole of order k.* The point z_0 is a pole of order k if the principal part of the Laurent expansion about z_0 contains only finitely many terms that form a polynomial of degree k in $(z - z_0)^{-1}$. A pole of order 1 is called a *simple pole*.
3. *Essential singularity.* The point z_0 is an essential singularity of $f(z)$ if the principal part of the Laurent expansion of $f(z)$ about z_0 contains infinitely many terms (i.e., if $a_{-n} \neq 0$ for infinitely many n).

The coefficient a_{-1} in the Laurent expansion of $f(z)$ about an isolated singular point z_0 is the *residue of* $f(z)$ at z_0. It can be shown that

$$a_{-1} = \frac{1}{2\pi j} \oint_{\Gamma} f(z)\, dz \tag{A.12}$$

where Γ is any simple closed curve oriented counterclockwise and containing in its interior z_0 and no other singularity of $f(z)$. Particularly useful to us is the formula for evaluation of residues at pole singularities. If $f(z)$ has a pole of order k at $z = z_0$, then the residue of $f(z)$ at z_0 is given by

$$a_{-1} = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} [(z - z_0)^k f(z)]. \tag{A.13}$$

Cauchy–Goursat and residue theorems. It can be shown that if $f(z)$ is analytic at all points on and within a simple closed contour C, then

$$\oint_C f(z)\, dz = 0.$$

This central result is known as the *Cauchy–Goursat theorem*. We shall not offer a proof, but shall proceed instead to derive a useful consequence known as the *residue theorem*.

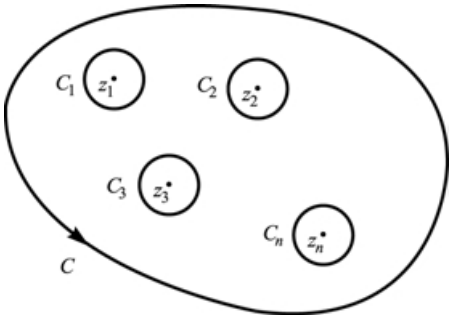

Figure A.1: Derivation of the residue theorem.

Figure A.1 depicts a simple closed curve C enclosing n isolated singularities of a function $f(z)$. We assume that $f(z)$ is analytic on and elsewhere within C. Around each singular point z_k we have drawn a circle C_k so small that it encloses no singular point other than z_k; taken together, the C_k $(k = 1, \ldots, n)$ and C form the boundary of a region in which $f(z)$ is everywhere analytic. By the Cauchy–Goursat theorem

$$\int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0.$$

Hence

$$\frac{1}{2\pi j} \int_C f(z)\,dz = \sum_{k=1}^{n} \frac{1}{2\pi j} \int_{C_k} f(z)\,dz,$$

where now the integrations are all performed in a counterclockwise sense. By (A.12)

$$\int_C f(z)\,dz = 2\pi j \sum_{k=1}^{n} r_k \tag{A.14}$$

where $r_1, \ldots, r_n$ are the residues of $f(z)$ at the singularities within C.

Contour deformation. Suppose f is analytic in a region D and Γ is a simple closed curve in D. If Γ can be continuously deformed to another simple closed curve Γ' without passing out of D, then

$$\int_{\Gamma'} f(z)\,dz = \int_{\Gamma} f(z)\,dz. \tag{A.15}$$

To see this, consider Figure A.2 where we have introduced another set of curves $\pm\gamma$; these new curves are assumed parallel and infinitesimally close to each other. Let C be the composite curve consisting of Γ, $+\gamma$, $-\Gamma'$, and $-\gamma$, in that order. Since f is analytic on and within C, we have

$$\int_C f(z)\,dz = \int_{\Gamma} f(z)\,dz + \int_{+\gamma} f(z)\,dz + \int_{-\Gamma'} f(z)\,dz + \int_{-\gamma} f(z)\,dz = 0.$$

But $\int_{-\Gamma'} f(z)\,dz = -\int_{\Gamma'} f(z)\,dz$ and $\int_{-\gamma} f(z)\,dz = -\int_{+\gamma} f(z)\,dz$, hence (A.15) follows. The contour deformation principle often permits us to replace an integration contour by one that is more convenient.

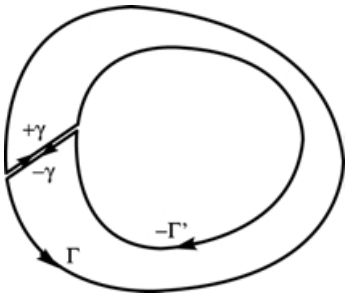

Figure A.2: Derivation of the contour deformation principle.

Principal value integrals. We must occasionally carry out integrations of the form

$$I = \int_{-\infty}^{\infty} f(x)\,dx$$

where $f(x)$ has a finite number of singularities x_k $(k = 1, \ldots, n)$ along the real axis. Such singularities in the integrand force us to interpret I as an improper integral. With just one singularity present at point x_1, for instance, we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\varepsilon\to 0}\int_{-\infty}^{x_1-\varepsilon} f(x)\,dx + \lim_{\eta\to 0}\int_{x_1+\eta}^{\infty} f(x)\,dx$$

provided that both limits exist. When both limits do not exist, we may still be able to obtain a well-defined result by computing

$$\lim_{\varepsilon\to 0}\left(\int_{-\infty}^{x_1-\varepsilon} f(x)\,dx + \int_{x_1+\varepsilon}^{\infty} f(x)\,dx\right)$$

(i.e., by taking $\eta = \varepsilon$ so that the limits are "symmetric"). This quantity is called the *Cauchy principal value* of I and is denoted

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx.$$

More generally, we have

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{x_1-\varepsilon} f(x)\,dx + \int_{x_1+\varepsilon}^{x_2-\varepsilon} f(x)\,dx + \right.$$
$$\left. + \cdots + \int_{x_{n-1}+\varepsilon}^{x_n-\varepsilon} f(x)\,dx + \int_{x_n+\varepsilon}^{\infty} f(x)\,dx\right)$$

for n singularities $x_1 < \cdots < x_n$.

In a large class of problems $f(z)$ (i.e., $f(x)$ with x replaced by the complex variable z) is analytic everywhere except for the presence of finitely many simple poles. Some of these may lie on the real axis (at points $x_1 < \cdots < x_n$, say), and some may not. Consider now the integration contour C shown in Figure A.3. We choose R so large and ε so small that C encloses all the poles of f that lie in the upper half of the complex

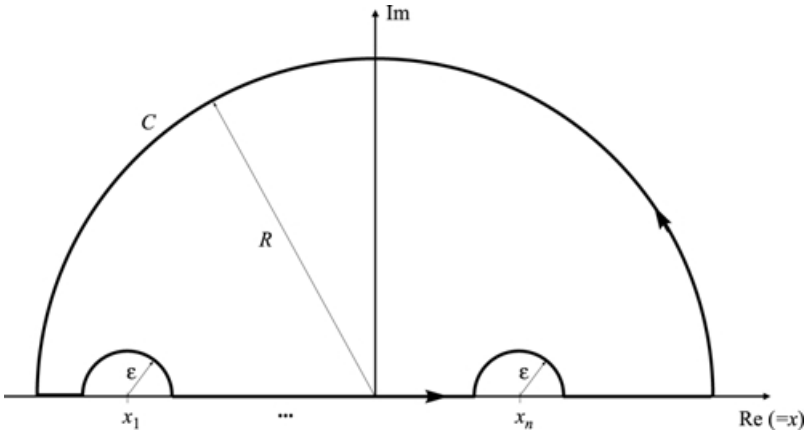

Figure A.3: Complex plane technique for evaluating a principal value integral.

plane. In many problems of interest the integral of f around the large semicircle tends to zero as $R \to \infty$ and the integrals around the small semicircles are well-behaved as $\varepsilon \to 0$. It may then be shown that

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\, dx = \pi j \sum_{k=1}^{n} r_k + 2\pi j \sum_{\text{UHP}} r_k$$

where r_k is the residue at the kth simple pole. The first sum on the right accounts for the contributions of those poles that lie on the real axis; note that it is associated with a factor πj instead of $2\pi j$, since these terms arose from integrals over semicircles rather than over full circles. The second sum, of course, is extended only over those poles that reside in the upper half-plane.

Fourier transform solution of the 1-D wave equation

Successive applications of the Fourier transform can reduce a partial differential equation to an ordinary differential equation, and finally to an algebraic equation. After the algebraic equation is solved by standard techniques, Fourier inversion can yield a solution to the original partial differential equation. We illustrate this by solving the one-dimensional inhomogeneous wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi(x, y, z, t) = S(x, y, z, t), \tag{A.16}$$

where the field ψ is the desired unknown and S is the known source term. For uniqueness of solution we must specify ψ and $\partial\psi/\partial z$ over some $z =$ constant plane. Assume that

$$\psi(x, y, z, t)\Big|_{z=0} = f(x, y, t), \tag{A.17}$$

$$\frac{\partial}{\partial z} \psi(x, y, z, t)\Big|_{z=0} = g(x, y, t). \tag{A.18}$$

We begin by positing inverse temporal Fourier transform relationships for ψ and S:

$$\psi(x, y, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\psi}(x, y, z, \omega) e^{j\omega t}\, d\omega,$$

$$S(x, y, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{S}(x, y, z, \omega) e^{j\omega t}\, d\omega.$$

Substituting into (A.16), passing the derivatives through the integral, calculating the derivatives, and combining the inverse transforms, we obtain

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\left(\frac{\partial^2}{\partial z^2} + k^2 \right) \tilde{\psi}(x, y, z, \omega) - \tilde{S}(x, y, z, \omega) \right] e^{j\omega t}\, d\omega = 0$$

where $k = \omega/c$. By the Fourier integral theorem

$$\left(\frac{\partial^2}{\partial z^2} + k^2 \right) \tilde{\psi}(x, y, z, \omega) - \tilde{S}(x, y, z, \omega) = 0. \tag{A.19}$$

We have thus converted a partial differential equation into an ordinary differential equation. A spatial transform on z will now convert the ordinary differential equation into an algebraic equation. We write

$$\tilde{\psi}(x, y, z, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\psi}^z(x, y, k_z, \omega) e^{jk_z z}\, dk_z,$$
$$\tilde{S}(x, y, z, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{S}^z(x, y, k_z, \omega) e^{jk_z z}\, dk_z,$$

in (A.19), pass the derivatives through the integral sign, compute the derivatives, and set the integrand to zero to get

$$(k^2 - k_z^2)\tilde{\psi}^z(x, y, k_z, \omega) - \tilde{S}^z(x, y, k_z, \omega) = 0;$$

hence

$$\tilde{\psi}^z(x, y, k_z, \omega) = -\frac{\tilde{S}^z(x, y, k_z, \omega)}{(k_z - k)(k_z + k)}. \tag{A.20}$$

The price we pay for such an easy solution is that we must now perform a two-dimensional Fourier inversion to obtain $\psi(x, y, z, t)$ from $\tilde{\psi}^z(x, y, k_z, \omega)$. It turns out to be easiest to perform the spatial inverse transform first, so let us examine

$$\tilde{\psi}(x, y, z, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\psi}^z(x, y, k_z, \omega) e^{jk_z z}\, dk_z.$$

By (A.20) we have

$$\tilde{\psi}(x, y, z, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\tilde{S}^z(x, y, k_z, \omega)] \left[\frac{-1}{(k_z - k)(k_z + k)} \right] e^{jk_z z}\, dk_z,$$

where the integrand involves a product of two functions. With

$$\tilde{g}^z(k_z, \omega) = \frac{-1}{(k_z - k)(k_z + k)},$$

the convolution theorem gives

$$\tilde{\psi}(x, y, z, \omega) = \int_{-\infty}^{\infty} \tilde{S}(x, y, \zeta, \omega) \tilde{g}(z - \zeta, \omega)\, d\zeta \tag{A.21}$$

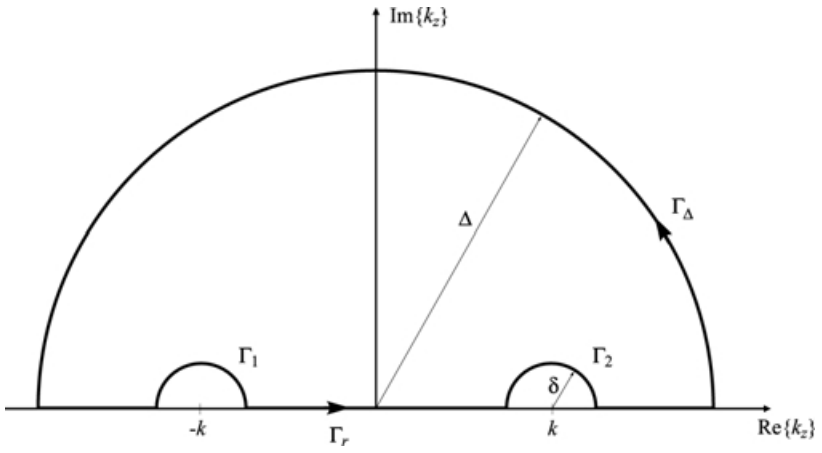

Figure A.4: Contour used to compute inverse transform in solution of the 1-D wave equation.

where

$$\tilde{g}(z,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{g}^z(k_z,\omega)e^{jk_z z}\,dk_z = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-1}{(k_z-k)(k_z+k)}e^{jk_z z}\,dk_z.$$

To compute this integral we use complex plane techniques. The domain of integration extends along the real k_z-axis in the complex k_z-plane; because of the poles at $k_z = \pm k$, we must treat the integral as a principal value integral. Denoting

$$I(k_z) = \frac{-e^{jk_z z}}{2\pi(k_z-k)(k_z+k)},$$

we have

$$\begin{aligned}\int_{-\infty}^{\infty} I(k_z)\,dk_z &= \lim \int_{\Gamma_r} I(k_z)\,dk_z \\ &= \lim \int_{-\Delta}^{-k-\delta} I(k_z)\,dk_z + \lim \int_{-k+\delta}^{k-\delta} I(k_z)\,dk_z + \lim \int_{k+\delta}^{\Delta} I(k_z)\,dk_z\end{aligned}$$

where the limits take $\delta \to 0$ and $\Delta \to \infty$. Our k_z-plane contour takes detours around the poles using semicircles of radius δ, and is closed using a semicircle of radius Δ (Figure A.4). Note that if $z > 0$, we must close the contour in the upper half-plane.

By Cauchy's integral theorem

$$\int_{\Gamma_r} I(k_z)\,dk_z + \int_{\Gamma_1} I(k_z)\,dk_z + \int_{\Gamma_2} I(k_z)\,dk_z + \int_{\Gamma_\Delta} I(k_z)\,dk_z = 0.$$

Thus

$$\int_{-\infty}^{\infty} I(k_z)\,dk_z = -\lim_{\delta\to 0}\int_{\Gamma_1} I(k_z)\,dk_z - \lim_{\delta\to 0}\int_{\Gamma_2} I(k_z)\,dk_z - \lim_{\Delta\to\infty}\int_{\Gamma_\Delta} I(k_z)\,dk_z.$$

The contribution from the semicircle of radius Δ can be computed by writing k_z in polar coordinates as $k_z = \Delta e^{j\theta}$:

$$\lim_{\Delta\to\infty}\int_{\Gamma_\Delta} I(k_z)\,dk_z = \frac{1}{2\pi}\lim_{\Delta\to\infty}\int_0^{\pi} \frac{-e^{jz\Delta e^{j\theta}}}{(\Delta e^{j\theta}-k)(\Delta e^{j\theta}+k)} j\Delta e^{j\theta}\,d\theta.$$

Using Euler's identity we can write

$$\lim_{\Delta\to\infty}\int_{\Gamma_\Delta} I(k_z)\,dk_z = \frac{1}{2\pi}\lim_{\Delta\to\infty}\int_0^\pi \frac{-e^{-\Delta z\sin\theta}e^{j\Delta z\cos\theta}}{\Delta^2 e^{2j\theta}}\,j\Delta e^{j\theta}\,d\theta.$$

Thus, as long as $z > 0$ the integrand will decay exponentially as $\Delta \to \infty$, and

$$\lim_{\Delta\to\infty}\int_{\Gamma_\Delta} I(k_z)\,dk_z \to 0.$$

Similarly, $\int_{\Gamma_\Delta} I(k_z)\,dk_z \to 0$ when $z < 0$ if we close the semicircle in the lower half-plane. Thus,

$$\int_{-\infty}^{\infty} I(k_z)\,dk_z = -\lim_{\delta\to 0}\int_{\Gamma_1} I(k_z)\,dk_z - \lim_{\delta\to 0}\int_{\Gamma_2} I(k_z)\,dk_z. \tag{A.22}$$

The integrals around the poles can also be computed by writing k_z in polar coordinates. Writing $k_z = -k + \delta e^{j\theta}$ we find

$$\begin{aligned}\lim_{\delta\to 0}\int_{\Gamma_1} I(k_z)\,dk_z &= \frac{1}{2\pi}\lim_{\delta\to 0}\int_\pi^0 \frac{-e^{jz(-k+\delta e^{j\theta})}j\delta e^{j\theta}}{(-k+\delta e^{j\theta}-k)(-k+\delta e^{j\theta}+k)}\,d\theta \\ &= \frac{1}{2\pi}\int_0^\pi \frac{e^{-jkz}}{-2k}\,j\,d\theta = -\frac{j}{4k}e^{-jkz}.\end{aligned}$$

Similarly, using $k_z = k + \delta e^{j\theta}$, we obtain

$$\lim_{\delta\to 0}\int_{\Gamma_2} I(k_z)\,dk_z = \frac{j}{4k}e^{jkz}.$$

Substituting these into (A.22) we have

$$\tilde{g}(z,\omega) = \frac{j}{4k}e^{-jkz} - \frac{j}{4k}e^{jkz} = \frac{1}{2k}\sin kz, \tag{A.23}$$

valid for $z > 0$. For $z < 0$, we close in the lower half-plane instead and get

$$\tilde{g}(z,\omega) = -\frac{1}{2k}\sin kz. \tag{A.24}$$

Substituting (A.23) and (A.24) into (A.21) we obtain

$$\tilde{\psi}(x,y,z,\omega) = \int_{-\infty}^{z}\tilde{S}(x,y,\zeta,\omega)\frac{\sin k(z-\zeta)}{2k}\,d\zeta - \frac{1}{2k}\int_z^\infty \tilde{S}(x,y,\zeta,\omega)\frac{\sin k(z-\zeta)}{2k}\,d\zeta$$

where we have been careful to separate the two cases considered above. To make things a bit easier when we apply the boundary conditions, let us rewrite the above expression. Splitting the domain of integration we write

$$\begin{aligned}\tilde{\psi}(x,y,z,\omega) = &\int_{-\infty}^{0}\tilde{S}(x,y,\zeta,\omega)\frac{\sin k(z-\zeta)}{2k}\,d\zeta + \int_0^z \tilde{S}(x,y,\zeta,\omega)\frac{\sin k(z-\zeta)}{k}\,d\zeta - \\ &- \int_0^\infty \tilde{S}(x,y,\zeta,\omega)\frac{\sin k(z-\zeta)}{2k}\,d\zeta.\end{aligned}$$

Expansion of the trigonometric functions then gives

$$\tilde{\psi}(x,y,z,\omega) = \int_0^z \tilde{S}(x,y,\zeta,\omega)\frac{\sin k(z-\zeta)}{k}\,d\zeta + \\ + \frac{\sin kz}{2k}\int_{-\infty}^{0} \tilde{S}(x,y,\zeta,\omega)\cos k\zeta\,d\zeta - \frac{\cos kz}{2k}\int_{-\infty}^{0} \tilde{S}(x,y,\zeta,\omega)\sin k\zeta\,d\zeta - \\ - \frac{\sin kz}{2k}\int_{0}^{\infty} \tilde{S}(x,y,\zeta,\omega)\cos k\zeta\,d\zeta + \frac{\cos kz}{2k}\int_{0}^{\infty} \tilde{S}(x,y,\zeta,\omega)\sin k\zeta\,d\zeta.$$

The last four integrals are independent of z, so we can represent them with functions constant in z. Finally, rewriting the trigonometric functions as exponentials we have

$$\tilde{\psi}(x,y,z,\omega) = \int_0^z \tilde{S}(x,y,\zeta,\omega)\frac{\sin k(z-\zeta)}{k}\,d\zeta + \tilde{A}(x,y,\omega)e^{-jkz} + \tilde{B}(x,y,\omega)e^{jkz}. \tag{A.25}$$

This formula for $\tilde{\psi}$ was found as a solution to the inhomogeneous ordinary differential equation (A.19). Hence, to obtain the complete solution we should add any possible solutions of the homogeneous differential equation. Since these are exponentials, (A.25) in fact represents the complete solution, where $\tilde{A}$ and $\tilde{B}$ are considered unknown and can be found using the boundary conditions.

If we are interested in the frequency-domain solution to the wave equation, then we are done. However, since our boundary conditions (A.17) and (A.18) pertain to the time domain, we must temporally inverse transform before we can apply them. Writing the sine function in (A.25) in terms of exponentials, we can express the time-domain solution as

$$\tilde{\psi}(x,y,z,t) = \int_0^z \mathcal{F}^{-1}\left\{\frac{c}{2}\frac{\tilde{S}(x,y,\zeta,\omega)}{j\omega}e^{j\frac{\omega}{c}(z-\zeta)} - \frac{c}{2}\frac{\tilde{S}(x,y,\zeta,\omega)}{j\omega}e^{-j\frac{\omega}{c}(z-\zeta)}\right\}d\zeta + \\ + \mathcal{F}^{-1}\left\{\tilde{A}(x,y,\omega)e^{-j\frac{\omega}{c}z}\right\} + \mathcal{F}^{-1}\left\{\tilde{B}(x,y,\omega)e^{j\frac{\omega}{c}z}\right\}. \tag{A.26}$$

A combination of the Fourier integration and time-shifting theorems gives the general identity

$$\mathcal{F}^{-1}\left\{\frac{\tilde{S}(x,y,\zeta,\omega)}{j\omega}e^{-j\omega t_0}\right\} = \int_{-\infty}^{t-t_0} S(x,y,\zeta,\tau)\,d\tau, \tag{A.27}$$

where we have assumed that $\tilde{S}(x,y,\zeta,0) = 0$. Using this in (A.26) along with the time-shifting theorem we obtain

$$\psi(x,y,z,t) = \frac{c}{2}\int_0^z \left\{\int_{-\infty}^{t-\frac{\zeta-z}{c}} S(x,y,\zeta,\tau)\,d\tau - \int_{-\infty}^{t-\frac{z-\zeta}{c}} S(x,y,\zeta,\tau)\,d\tau\right\}d\zeta + \\ + a\left(x,y,t-\frac{z}{c}\right) + b\left(x,y,t+\frac{z}{c}\right),$$

or

$$\psi(x,y,z,t) = \frac{c}{2}\int_0^z \int_{t-\frac{z-\zeta}{c}}^{t+\frac{z-\zeta}{c}} S(x,y,\zeta,\tau)\,d\tau\,d\zeta + a\left(x,y,t-\frac{z}{c}\right) + b\left(x,y,t+\frac{z}{c}\right) \tag{A.28}$$

where

$$a(x,y,t) = \mathcal{F}^{-1}[\tilde{A}(x,y,\omega)], \qquad b(x,y,t) = \mathcal{F}^{-1}[\tilde{B}(x,y,\omega)].$$

To calculate $a(x, y, t)$ and $b(x, y, t)$, we must use the boundary conditions (A.17) and (A.18). To apply (A.17), we put $z = 0$ into (A.28) to give

$$a(x, y, t) + b(x, y, t) = f(x, y, t). \tag{A.29}$$

Using (A.18) is a bit more complicated since we must compute $\partial\psi/\partial z$, and z is a parameter in the limits of the integral describing ψ. To compute the derivative we apply Leibnitz' rule for differentiation:

$$\frac{d}{d\alpha}\int_{\phi(\alpha)}^{\theta(\alpha)} f(x,\alpha)\,dx = \left(\frac{d\theta}{d\alpha}\right) f\left(\theta(\alpha),\alpha\right) - \left(\frac{d\phi}{d\alpha}\right) f\left(\phi(\alpha),\alpha\right) + \int_{\phi(\alpha)}^{\theta(\alpha)} \frac{\partial f}{\partial\alpha}\,dx. \tag{A.30}$$

Using this on the integral term in (A.28) we have

$$\frac{\partial}{\partial z}\left[\frac{c}{2}\int_0^z \left(\int_{t-\frac{z-\zeta}{c}}^{t+\frac{z-\zeta}{c}} S(x, y, \zeta, \tau)\,d\tau\right) d\zeta\right] = \frac{c}{2}\int_0^z \frac{\partial}{\partial z}\left(\int_{t-\frac{z-\zeta}{c}}^{t+\frac{z-\zeta}{c}} S(x, y, \zeta, \tau)\,d\tau\right) d\zeta,$$

which is zero at $z = 0$. Thus

$$\left.\frac{\partial\psi}{\partial z}\right|_{z=0} = g(x, y, t) = -\frac{1}{c}a'(x, y, t) + \frac{1}{c}b'(x, y, t)$$

where $a' = \partial a/\partial t$ and $b' = \partial b/\partial t$. Integration gives

$$-a(x, y, t) + b(x, y, t) = c\int_{-\infty}^{t} g(x, y, \tau)\,d\tau. \tag{A.31}$$

Equations (A.29) and (A.31) represent two algebraic equations in the two unknown functions a and b. The solutions are

$$2a(x, y, t) = f(x, y, t) - c\int_{-\infty}^{t} g(x, y, \tau)\,d\tau,$$
$$2b(x, y, t) = f(x, y, t) + c\int_{-\infty}^{t} g(x, y, \tau)\,d\tau.$$

Finally, substitution of these into (A.28) gives us the solution to the inhomogeneous wave equation

$$\psi(x, y, z, t) = \frac{c}{2}\int_0^z \int_{t-\frac{z-\zeta}{c}}^{t+\frac{z-\zeta}{c}} S(x, y, \zeta, \tau)\,d\tau\,d\zeta + \frac{1}{2}\left[f\left(x, y, t - \frac{z}{c}\right) + f\left(x, y, t + \frac{z}{c}\right)\right] + \\ + \frac{c}{2}\int_{t-\frac{z}{c}}^{t+\frac{z}{c}} g(x, y, \tau)\,d\tau. \tag{A.32}$$

This is known as the *D'Alembert solution.* The terms $f(x, y, t \mp z/c)$ contribute to ψ as waves propagating away from the plane $z = 0$ in the $\pm z$-directions, respectively. The integral over the forcing term S is seen to accumulate values of S over a time interval determined by $z - \zeta$.

The boundary conditions could have been applied while still in the temporal frequency domain (but not the spatial frequency domain, since the spatial position z is lost). But to do this, we would need the boundary conditions to be in the temporal frequency domain. This is easily accomplished by transforming them to give

$$\left.\tilde{\psi}(x, y, z, \omega)\right|_{z=0} = \tilde{f}(x, y, \omega),$$
$$\left.\frac{\partial}{\partial z}\tilde{\psi}(x, y, z, \omega)\right|_{z=0} = \tilde{g}(x, y, \omega).$$

Applying these to (A.25) (and again using Leibnitz' rule) we have

$$\begin{aligned}\tilde{A}(x,y,\omega)+\tilde{B}(x,y,\omega)&=\tilde{f}(x,y,\omega),\\ -jk\tilde{A}(x,y,\omega)+jk\tilde{B}(x,y,\omega)&=\tilde{g}(x,y,\omega),\end{aligned}$$

hence

$$\begin{aligned}2\tilde{A}(x,y,\omega)&=\tilde{f}(x,y,\omega)-c\frac{\tilde{g}(x,y,\omega)}{j\omega},\\ 2\tilde{B}(x,y,\omega)&=\tilde{f}(x,y,\omega)+c\frac{\tilde{g}(x,y,\omega)}{j\omega}.\end{aligned}$$

Finally, substituting these back into (A.25) and expanding the sine function we obtain the frequency-domain solution that obeys the given boundary conditions:

$$\begin{aligned}\tilde{\psi}(x,y,z,\omega)=&\frac{c}{2}\int_0^z\left[\frac{\tilde{S}(x,y,\zeta,\omega)e^{j\frac{\omega}{c}(z-\zeta)}}{j\omega}-\frac{\tilde{S}(x,y,\zeta,\omega)e^{-j\frac{\omega}{c}(z-\zeta)}}{j\omega}\right]d\zeta+\\ &+\frac{1}{2}\left[\tilde{f}(x,y,\omega)e^{j\frac{\omega}{c}z}+\tilde{f}(x,y,\omega)e^{-j\frac{\omega}{c}z}\right]+\\ &+\frac{c}{2}\left[\frac{\tilde{g}(x,y,\omega)e^{j\frac{\omega}{c}z}}{j\omega}-\frac{\tilde{g}(x,y,\omega)e^{-j\frac{\omega}{c}z}}{j\omega}\right].\end{aligned}$$

This is easily inverted using (A.27) to give (A.32).

Fourier transform solution of the 1-D homogeneous wave equation for dissipative media

Wave propagation in dissipative media can be studied using the one-dimensional wave equation

$$\left(\frac{\partial^2}{\partial z^2}-\frac{2\Omega}{v^2}\frac{\partial}{\partial t}-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\psi(x,y,z,t)=S(x,y,z,t). \tag{A.33}$$

This equation is nearly identical to the wave equation for lossless media studied in the previous section, except for the addition of the $\partial\psi/\partial t$ term. This extra term will lead to important physical consequences regarding the behavior of the wave solutions.

We shall solve (A.33) using the Fourier transform approach of the previous section, but to keep the solution simple we shall only consider the homogeneous problem. We begin by writing ψ in terms of its inverse temporal Fourier transform:

$$\psi(x,y,z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\psi}(x,y,z,\omega)e^{j\omega t}\,d\omega.$$

Substituting this into the homogeneous version of (A.33) and taking the time derivatives, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[(j\omega)^2+2\Omega(j\omega)-v^2\frac{\partial^2}{\partial z^2}\right]\tilde{\psi}(x,y,z,\omega)e^{j\omega t}\,d\omega=0.$$

The Fourier integral theorem leads to

$$\frac{\partial^2\tilde{\psi}(x,y,z,\omega)}{\partial z^2}-\kappa^2\tilde{\psi}(x,y,z,\omega)=0 \tag{A.34}$$

where

$$\kappa = \frac{1}{v}\sqrt{p^2 + 2\Omega p}$$

with $p = j\omega$.

We can solve the homogeneous ordinary differential equation (A.34) by inspection:

$$\tilde{\psi}(x, y, z, \omega) = \tilde{A}(x, y, \omega)e^{-\kappa z} + \tilde{B}(x, y, \omega)e^{\kappa z}. \tag{A.35}$$

Here $\tilde{A}$ and $\tilde{B}$ are frequency-domain coefficients to be determined. We can either specify these coefficients directly, or solve for them by applying specific boundary conditions. We examine each possibility below.

Solution to wave equation by direct application of boundary conditions. The solution to the wave equation (A.33) will be unique if we specify functions $f(x, y, t)$ and $g(x, y, t)$ such that

$$\begin{aligned} \psi(x, y, z, t)\Big|_{z=0} &= f(x, y, t), \\ \frac{\partial}{\partial z}\psi(x, y, z, t)\Big|_{z=0} &= g(x, y, t). \end{aligned} \tag{A.36}$$

Assuming the Fourier transform pairs $f(x, y, t) \leftrightarrow \tilde{f}(x, y, \omega)$ and $g(x, y, t) \leftrightarrow \tilde{g}(x, y, \omega)$, we can apply the boundary conditions (A.36) in the frequency domain:

$$\begin{aligned} \tilde{\psi}(x, y, z, \omega)\Big|_{z=0} &= \tilde{f}(x, y, \omega), \\ \frac{\partial}{\partial z}\tilde{\psi}(x, y, z, \omega)\Big|_{z=0} &= \tilde{g}(x, y, \omega). \end{aligned}$$

From these we find

$$\tilde{A} + \tilde{B} = \tilde{f}, \qquad -\kappa\tilde{A} + \kappa\tilde{B} = \tilde{g}, v$$

or

$$\tilde{A} = \frac{1}{2}\left[\tilde{f} - \frac{\tilde{g}}{\kappa}\right], \qquad \tilde{B} = \frac{1}{2}\left[\tilde{f} + \frac{\tilde{g}}{\kappa}\right].$$

Substitution into (A.35) gives

$$\begin{aligned} \tilde{\psi}(x, y, z, \omega) &= \tilde{f}(x, y, \omega)\cosh\kappa z + \tilde{g}(x, y, \omega)\frac{\sinh\kappa z}{\kappa} \\ &= \tilde{f}(x, y, \omega)\frac{\partial}{\partial z}\tilde{Q}(x, y, z, \omega) + \tilde{g}(x, y, \omega)\tilde{Q}(x, y, z, \omega) \\ &= \tilde{\psi}_1(x, y, z, \omega) + \tilde{\psi}_2(x, y, z, \omega) \end{aligned}$$

where $\tilde{Q} = \sinh\kappa z/\kappa$. Assuming that $Q(x, y, z, t) \leftrightarrow \tilde{Q}(x, y, z, \omega)$, we can employ the convolution theorem to immediately write down $\psi(x, y, z, t)$:

$$\begin{aligned} \psi(x, y, z, t) &= f(x, y, t) * \frac{\partial}{\partial z}Q(x, y, z, t) + g(x, y, z, t) * Q(x, y, z, t) \\ &= \psi_1(x, y, z, t) + \psi_2(x, y, z, t). \end{aligned} \tag{A.37}$$

To find ψ we must first compute the inverse transform of $\tilde{Q}$. Here we resort to a tabulated result [26]:

$$\frac{\sinh\left[a\sqrt{p+\lambda}\sqrt{p+\mu}\right]}{\sqrt{p+\lambda}\sqrt{p+\mu}} \leftrightarrow \frac{1}{2}e^{-\frac{1}{2}(\mu+\lambda)t}J_0\left(\frac{1}{2}(\lambda-\mu)\sqrt{a^2-t^2}\right), \quad -a < t < a.$$

Here a is a positive, finite real quantity, and λ and μ are finite complex quantities. Outside the range $|t| < a$ the time-domain function is zero.

Letting $a = z/v$, $\mu = 0$, and $\lambda = 2\Omega$ in the above expression, we find

$$Q(x, y, z, t) = \frac{v}{2} e^{-\Omega t} J_0\left(\frac{\Omega}{v}\sqrt{z^2 - v^2 t^2}\right)[U(t + z/v) - U(t - z/v)] \tag{A.38}$$

where $U(x)$ is the unit step function (A.5). From (A.37) we see that

$$\psi_2(x, y, z, t) = \int_{-\infty}^{\infty} g(x, y, t - \tau) Q(x, y, z, \tau)\, d\tau = \int_{-z/v}^{z/v} g(x, y, t - \tau) Q(x, y, z, \tau)\, d\tau.$$

Using the change of variables $u = t - \tau$ and substituting (A.38), we then have

$$\psi_2(x, y, z, t) = \frac{v}{2} e^{-\Omega t} \int_{t - \frac{z}{v}}^{t + \frac{z}{v}} g(x, y, u) e^{\Omega u} J_0\left(\frac{\Omega}{v}\sqrt{z^2 - (t - u)^2 v^2}\right) du. \tag{A.39}$$

To find ψ_1 we must compute $\partial Q/\partial z$. Using the product rule we have

$$\begin{aligned}\frac{\partial Q(x, y, z, t)}{\partial z} &= \frac{v}{2} e^{-\Omega t} J_0\left(\frac{\Omega}{v}\sqrt{z^2 - v^2 t^2}\right) \frac{\partial}{\partial z}[U(t + z/v) - U(t - z/v)] + \\ &+ \frac{v}{2} e^{-\Omega t}[U(t + z/v) - U(t - z/v)] \frac{\partial}{\partial z} J_0\left(\frac{\Omega}{v}\sqrt{z^2 - v^2 t^2}\right).\end{aligned}$$

Next, using $dU(x)/dx = \delta(x)$ and remembering that $J_0'(x) = -J_1(x)$ and $J_0(0) = 1$, we can write

$$\begin{aligned}\frac{\partial Q(x, y, z, t)}{\partial z} &= \frac{1}{2} e^{-\Omega t}[\delta(t + z/v) + \delta(t - z/v)] - \\ &- \frac{z\Omega^2}{2v} e^{-\Omega t} \frac{J_1\left(\frac{\Omega}{v}\sqrt{z^2 - v^2 t^2}\right)}{\frac{\Omega}{v}\sqrt{z^2 - v^2 t^2}}[U(t + z/v) - U(t - z/v)].\end{aligned}$$

Convolving this expression with $f(x, y, t)$ we obtain

$$\begin{aligned}\psi_1(x, y, z, t) &= \frac{1}{2} e^{-\frac{\Omega}{v} z} f\left(x, y, t - \frac{z}{v}\right) + \frac{1}{2} e^{\frac{\Omega}{v} z} f\left(x, y, t + \frac{z}{v}\right) - \\ &- \frac{z\Omega^2}{2v} e^{-\Omega t} \int_{t - \frac{z}{v}}^{t + \frac{z}{v}} f(x, y, u) e^{\Omega u} \frac{J_1\left(\frac{\Omega}{v}\sqrt{z^2 - (t - u)^2 v^2}\right)}{\frac{\Omega}{v}\sqrt{z^2 - (t - u)^2 v^2}}\, du.\end{aligned} \tag{A.40}$$

Finally, adding (A.40) and (A.39), we obtain

$$\begin{aligned}\psi(x, y, z, t) &= \frac{1}{2} e^{-\frac{\Omega}{v} z} f\left(x, y, t - \frac{z}{v}\right) + \frac{1}{2} e^{\frac{\Omega}{v} z} f\left(x, y, t + \frac{z}{v}\right) - \\ &- \frac{z\Omega^2}{2v} e^{-\Omega t} \int_{t - \frac{z}{v}}^{t + \frac{z}{v}} f(x, y, u) e^{\Omega u} \frac{J_1\left(\frac{\Omega}{v}\sqrt{z^2 - (t - u)^2 v^2}\right)}{\frac{\Omega}{v}\sqrt{z^2 - (t - u)^2 v^2}}\, du + \\ &+ \frac{v}{2} e^{-\Omega t} \int_{t - \frac{z}{v}}^{t + \frac{z}{v}} g(x, y, u) e^{\Omega u} J_0\left(\frac{\Omega}{v}\sqrt{z^2 - (t - u)^2 v^2}\right) du.\end{aligned} \tag{A.41}$$

Note that when $\Omega = 0$ this reduces to

$$\psi(x, y, z, t) = \frac{1}{2} f\left(x, y, t - \frac{z}{v}\right) + \frac{1}{2} f\left(x, y, t + \frac{z}{v}\right) + \frac{v}{2} \int_{t - \frac{z}{v}}^{t + \frac{z}{v}} g(x, y, u)\, du,$$

which matches (A.32) for the homogeneous case where $S = 0$.

Solution to wave equation by specification of wave amplitudes. An alternative to direct specification of boundary conditions is specification of the amplitude functions $\tilde{A}(x, y, \omega)$ and $\tilde{B}(x, y, \omega)$ or their inverse transforms $A(x, y, t)$ and $B(x, y, t)$. If we specify the time-domain functions we can write $\psi(x, y, z, t)$ as the inverse transform of (A.35). For example, a wave traveling in the $+z$-direction behaves as

$$\psi(x, y, z, t) = A(x, y, t) * F^{+}(x, y, z, t) \tag{A.42}$$

where

$$F^{+}(x, y, z, t) \leftrightarrow e^{-\kappa z} = e^{-\frac{z}{v}\sqrt{p^2+2\Omega p}}.$$

We can find F^{+} using the following Fourier transform pair [26]):

$$e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}} \leftrightarrow e^{-\frac{\rho}{v}x}\delta(t - x/v) + \frac{\sigma x}{v}e^{-\rho t}\frac{I_1\left(\sigma\sqrt{t^2-(x/v)^2}\right)}{\sqrt{t^2-(x/v)^2}}, \quad \frac{x}{v} < t. \tag{A.43}$$

Here x is real and positive and $I_1(x)$ is the modified Bessel function of the first kind and order 1. Outside the range $x/v < t$ the time-domain function is zero. Letting $\rho = \Omega$ and $\sigma = \Omega$ we find

$$F^{+}(x, y, z, t) = \frac{\Omega^2 z}{v}e^{-\Omega t}\frac{I_1(\Omega\sqrt{t^2-(z/v)^2})}{\Omega\sqrt{t^2-(z/v)^2}}U(t - z/v) + e^{-\frac{\Omega}{v}z}\delta(t - z/v). \tag{A.44}$$

Note that F^{+} is a real functions of time, as expected.

Substituting (A.44) into (A.42) and writing the convolution in integral form we have

$$\psi(x, y, z, t) = \int_{z/v}^{\infty} A(x, y, t-\tau)\left[\frac{\Omega^2 z}{v}e^{-\Omega\tau}\frac{I_1(\Omega\sqrt{\tau^2-(z/v)^2})}{\Omega\sqrt{\tau^2-(z/v)^2}}\right]d\tau + \\ + e^{-\frac{\Omega}{v}z}A\left(x, y, t - \frac{z}{v}\right), \quad z > 0. \tag{A.45}$$

The 3-D Green's function for waves in dissipative media

To understand the fields produced by bounded sources within a dissipative medium we may wish to investigate solutions to the wave equation in three dimensions. The Green's function approach requires the solution to

$$\left(\nabla^2 - \frac{2\Omega}{v^2}\frac{\partial}{\partial t} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r}|\mathbf{r}'; t) = -\delta(t)\delta(\mathbf{r} - \mathbf{r}') \\ = -\delta(t)\delta(x - x')\delta(y - y')\delta(z - z').$$

That is, we are interested in the impulse response of a point source located at $\mathbf{r} = \mathbf{r}'$. We begin by substituting the inverse temporal Fourier transform relations

$$G(\mathbf{r}|\mathbf{r}'; t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}(\mathbf{r}|\mathbf{r}'; \omega)e^{j\omega t}\,d\omega,$$
$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{j\omega t}\,d\omega,$$

obtaining

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\left(\nabla^2 - j\omega\frac{2\Omega}{v^2} - \frac{1}{v^2}(j\omega)^2\right)\tilde{G}(\mathbf{r}|\mathbf{r}'; \omega) + \delta(\mathbf{r} - \mathbf{r}')\right]e^{j\omega t}d\omega = 0.$$

By the Fourier integral theorem we have

$$(\nabla^2 + k^2)\tilde{G}(\mathbf{r}|\mathbf{r}';\omega) = -\delta(\mathbf{r}-\mathbf{r}'). \tag{A.46}$$

This is known as the *Helmholtz equation*. Here

$$k = \frac{1}{v}\sqrt{\omega^2 - j2\omega\Omega} \tag{A.47}$$

is called the *wavenumber*.

To solve the Helmholtz equation we write $\tilde{G}$ in terms of a 3-dimensional inverse Fourier transform. Substitution of

$$\tilde{G}(\mathbf{r}|\mathbf{r}';\omega) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \tilde{G}^r(\mathbf{k}|\mathbf{r}';\omega)e^{j\mathbf{k}\cdot\mathbf{r}}\,d^3k,$$
$$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{j\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,d^3k,$$

into (A.46) gives

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\left[\nabla^2\left(\tilde{G}^r(\mathbf{k}|\mathbf{r}';\omega)e^{j\mathbf{k}\cdot\mathbf{r}}\right) + k^2\tilde{G}^r(\mathbf{k}|\mathbf{r}';\omega)e^{j\mathbf{k}\cdot\mathbf{r}} + e^{j\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\right]d^3k = 0.$$

Here

$$\mathbf{k} = \hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y + \hat{\mathbf{z}}k_z$$

with $|\mathbf{k}|^2 = k_x^2 + k_y^2 + k_z^2 = K^2$. Carrying out the derivatives and invoking the Fourier integral theorem we have

$$(K^2 - k^2)\tilde{G}^r(\mathbf{k}|\mathbf{r}';\omega) = e^{-j\mathbf{k}\cdot\mathbf{r}'}.$$

Solving for $\tilde{G}$ and substituting it into the inverse transform relation we have

$$\tilde{G}(\mathbf{r}|\mathbf{r}';\omega) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{e^{j\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{(K-k)(K+k)}\,d^3k. \tag{A.48}$$

To compute the inverse transform integral in (A.48) we write the 3-D transform variable in spherical coordinates:

$$\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') = KR\cos\theta, \qquad d^3k = K^2\sin\theta\,dK\,d\theta\,d\phi,$$

where $R = |\mathbf{r}-\mathbf{r}'|$ and θ is the angle between $\mathbf{k}$ and $\mathbf{r}-\mathbf{r}'$. Hence (A.48) becomes

$$\begin{aligned}\tilde{G}(\mathbf{r}|\mathbf{r}';\omega) &= \frac{1}{(2\pi)^3}\int_0^{\infty}\frac{K^2\,dK}{(K-k)(K+k)}\int_0^{2\pi}d\phi\int_0^{\pi}e^{jKR\cos\theta}\sin\theta\,d\theta \\ &= \frac{2}{(2\pi)^2R}\int_0^{\infty}\frac{K\sin(KR)}{(K-k)(K+k)}\,dK,\end{aligned}$$

or, equivalently,

$$\begin{aligned}\tilde{G}(\mathbf{r}|\mathbf{r}';\omega) = &\frac{1}{2jR(2\pi)^2}\int_{-\infty}^{\infty}\frac{e^{jKR}}{(K-k)(K+k)}K\,dK - \\ &- \frac{1}{2jR(2\pi)^2}\int_{-\infty}^{\infty}\frac{e^{-jkR}}{(K-k)(K+k)}K\,dK.\end{aligned}$$

We can compute the integrals over K using the complex plane technique. We consider K to be a complex variable, and note that for dissipative media we have $k = k_r + jk_i$, where $k_r > 0$ and $k_i < 0$. Thus the integrand has poles at $K = \pm k$. For the integral involving e^{+jKR} we close the contour in the upper half-plane using a semicircle of radius Δ and use Cauchy's residue theorem. Then at all points on the semicircle the integrand decays exponentially as $\Delta \to \infty$, and there is no contribution to the integral from this part of the contour. The real-line integral is thus equal to $2\pi j$ times the residue at $K = -k$:

$$\int_{-\infty}^{\infty} \frac{e^{jKR}}{(K-k)(K+k)} K\, dK = 2\pi j \frac{e^{-jkR}}{-2k}(-k).$$

For the term involving e^{-jKR} we close in the lower half-plane and again the contribution from the infinite semicircle vanishes. In this case our contour is clockwise and so the real line integral is $-2\pi j$ times the residue at $K = k$:

$$\int_{-\infty}^{\infty} \frac{e^{-jKR}}{(K-k)(K+k)} K\, dK = -2\pi j \frac{e^{-jkR}}{2k} k.$$

Thus

$$\tilde{G}(\mathbf{r}|\mathbf{r}';\omega) = \frac{e^{-jkR}}{4\pi R}. \tag{A.49}$$

Note that if $\Omega = 0$ then this reduces to

$$\tilde{G}(\mathbf{r}|\mathbf{r}';\omega) = \frac{e^{-j\omega R/v}}{4\pi R}. \tag{A.50}$$

Our last step is to find the temporal Green's function. Let $p = j\omega$. Then we can write

$$\tilde{G}(\mathbf{r}|\mathbf{r}';\omega) = \frac{e^{\kappa R}}{4\pi R}$$

where

$$\kappa = -jk = \frac{1}{v}\sqrt{p^2 + 2\Omega p}.$$

We may find the inverse transform using (A.43). Letting $x = R$, $\rho = \Omega$, and $\sigma = \Omega$ we find

$$G(\mathbf{r}|\mathbf{r}';t) = e^{-\frac{\Omega}{v}R}\frac{\delta(t-R/v)}{4\pi R} + \frac{\Omega^2}{4\pi v}e^{-\Omega t}\frac{I_1\left(\Omega\sqrt{t^2-(R/v)^2}\right)}{\Omega\sqrt{t^2-(R/v)^2}}U\left(t-\frac{R}{v}\right).$$

We note that in the case of no dissipation where $\Omega = 0$ this reduces to

$$G(\mathbf{r}|\mathbf{r}';t) = \frac{\delta(t-R/v)}{4\pi R}$$

which is the inverse transform of (A.50).

Fourier transform representation of the static Green's function

In the study of static fields, we shall be interested in the solution to the partial differential equation

$$\nabla^2 G(\mathbf{r}|\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') = -\delta(x-x')\delta(y-y')\delta(z-z'). \tag{A.51}$$

Here $G(\mathbf{r}|\mathbf{r}')$, called the "static Green's function," represents the potential at point $\mathbf{r}$ produced by a unit point source at point $\mathbf{r}'$.

In Chapter 3 we find that $G(\mathbf{r}|\mathbf{r}') = 1/4\pi|\mathbf{r}-\mathbf{r}'|$. In a variety of problems it is also useful to have G written in terms of an inverse Fourier transform over the variables x and y. Letting G^r form a three-dimensional Fourier transform pair with G, we can write

$$G(\mathbf{r}|\mathbf{r}') = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} G^r(k_x,k_y,k_z|\mathbf{r}')e^{jk_x x}e^{jk_y y}e^{jk_z z}\,dk_x\,dk_y\,dk_z.$$

Substitution into (A.51) along with the inverse transformation representation for the delta function (A.4) gives

$$\begin{aligned}&\frac{1}{(2\pi)^3}\nabla^2\int_{-\infty}^{\infty} G^r(k_x,k_y,k_z|\mathbf{r}')e^{jk_x x}e^{jk_y y}e^{jk_z z}\,dk_x\,dk_y\,dk_z\\ &\quad = -\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{jk_x(x-x')}e^{jk_y(y-y')}e^{jk_z(z-z')}\,dk_x\,dk_y\,dk_z.\end{aligned}$$

We then combine the integrands and move the Laplacian operator through the integral to obtain

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\left[\nabla^2\left(G^r(\mathbf{k}|\mathbf{r}')e^{j\mathbf{k}\cdot\mathbf{r}}\right)+e^{j\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\right]d^3k = 0,$$

where $\mathbf{k} = \hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y + \hat{\mathbf{z}}k_z$. Carrying out the derivatives,

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\left[\left(-k_x^2-k_y^2-k_z^2\right)G^r(\mathbf{k}|\mathbf{r}')+e^{-j\mathbf{k}\cdot\mathbf{r}'}\right]e^{j\mathbf{k}\cdot\mathbf{r}}\,d^3k = 0.$$

Letting $k_x^2+k_y^2 = k_\rho^2$ and invoking the Fourier integral theorem we get the algebraic equation

$$\left(-k_\rho^2-k_z^2\right)G^r(\mathbf{k}|\mathbf{r}')+e^{-j\mathbf{k}\cdot\mathbf{r}'} = 0,$$

which we can easily solve for G^r:

$$G^r(\mathbf{k}|\mathbf{r}') = \frac{e^{-j\mathbf{k}\cdot\mathbf{r}'}}{k_\rho^2+k_z^2}. \tag{A.52}$$

Equation (A.52) gives us a 3-D transform representation for the Green's function. Since we desire the 2-D representation, we shall have to perform the inverse transform over k_z. Writing

$$G^{xy}(k_x,k_y,z|\mathbf{r}') = \frac{1}{2\pi}\int_{-\infty}^{\infty} G^r(k_x,k_y,k_z|\mathbf{r}')e^{jk_z z}\,dk_z$$

we have

$$G^{xy}(k_x,k_y,z|\mathbf{r}') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-jk_x x'}e^{-jk_y y'}e^{jk_z(z-z')}}{k_\rho^2+k_z^2}\,dk_z. \tag{A.53}$$

To compute this integral, we let k_z be a complex variable and consider a closed contour in the complex plane, consisting of a semicircle and the real axis. As previously discussed, we compute the principal value integral as the semicircle radius $\Delta\to\infty$, and find that the contribution along the semicircle reduces to zero. Hence we can use Cauchy's residue theorem (A.14) to obtain the real-line integral:

$$G^{xy}(k_x,k_y,z|\mathbf{r}') = 2\pi j\,\text{res}\left\{\frac{1}{2\pi}\frac{e^{-jk_x x'}e^{-jk_y y'}e^{jk_z(z-z')}}{k_\rho^2+k_z^2}\right\}.$$

Here $\text{res}\{f(k_z)\}$ denotes the residues of the function $f(k_z)$. The integrand in (A.53) has poles of order 1 at $k_z = \pm jk_\rho$, $k_\rho \geq 0$. If $z - z' > 0$ we close in the upper half-plane and enclose only the pole at $k_z = jk_\rho$. Computing the residue using (A.13), we obtain

$$G^{xy}(k_x, k_y, z|\mathbf{r}') = j\frac{e^{-jk_x x'}e^{-jk_y y'}e^{-k_\rho(z-z')}}{2jk_\rho}, \qquad z > z'.$$

Since $z > z'$ this function decays for increasing z, as expected physically. For $z - z' < 0$ we close in the lower half-plane, enclosing the pole at $k_z = -jk_\rho$ and incurring an additional negative sign since our contour is now clockwise. Evaluating the residue we have

$$G^{xy}(k_x, k_y, z|\mathbf{r}') = -j\frac{e^{-jk_x x'}e^{-jk_y y'}e^{k_\rho(z-z')}}{-2jk_\rho}, \qquad z < z'.$$

We can combine both cases $z > z'$ and $z < z'$ by using the absolute value function:

$$G^{xy}(k_x, k_y, z|\mathbf{r}') = \frac{e^{-jk_x x'}e^{-jk_y y'}e^{-k_\rho|z-z'|}}{2k_\rho}. \tag{A.54}$$

Finally, we substitute (A.54) into the inverse transform formula. This gives the Green's function representation

$$G(\mathbf{r}|\mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{e^{-k_\rho|z-z'|}}{2k_\rho}e^{j\mathbf{k}_\rho\cdot(\mathbf{r}-\mathbf{r}')}\,d^2k_\rho, \tag{A.55}$$

where $\mathbf{k}_\rho = \hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y$, $k_\rho = |\mathbf{k}_\rho|$, and $d^2k_\rho = dk_x\,dk_y$.

On occasion we may wish to represent the solution of the homogeneous (Laplace) equation

$$\nabla^2\psi(\mathbf{r}) = 0$$

in terms of a 2-D Fourier transform. In this case we represent ψ as a 2-D inverse transform and substitute to obtain

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\nabla^2\left(\psi^{xy}(k_x, k_y, z)e^{jk_x x}e^{jk_y y}\right)\,dk_x\,dk_y = 0.$$

Carrying out the derivatives and invoking the Fourier integral theorem we find that

$$\left(\frac{\partial^2}{\partial z^2} - k_\rho^2\right)\psi^{xy}(k_x, k_y, z) = 0.$$

Hence

$$\psi^{xy}(k_x, k_y, z) = Ae^{k_\rho z} + Be^{-k_\rho z}$$

where A and B are constants with respect to z. Inverse transformation gives

$$\psi(\mathbf{r}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left[A(\mathbf{k}_\rho)e^{k_\rho z} + B(\mathbf{k}_\rho)e^{-k_\rho z}\right]e^{j\mathbf{k}_\rho\cdot\mathbf{r}}\,d^2k_\rho. \tag{A.56}$$

A.2 Vector transport theorems

We are often interested in the time rate of change of some field integrated over a moving volume or surface. Such a derivative may be used to describe the transport of a physical quantity (e.g., charge, momentum, energy) through space. Many of the relevant theorems are derived in this section. The results find application in the development of the large-scale forms of Maxwell equations, the continuity equation, and the Poynting theorem.

Partial, total, and material derivatives

The key to understanding transport theorems lies in the difference between the various means of time-differentiating a field. Consider a scalar field $T(\mathbf{r}, t)$ (which could represent one component of a vector or dyadic field). If we fix our position within the field and examine how the field varies with time, we describe the *partial derivative* of T. However, this may not be the most useful means of measuring the time rate of change of a field. For instance, in mechanics we might be interested in the rate at which water cools as it sinks to the bottom of a container. In this case, T could represent temperature. We could create a "depth profile" at any given time (i.e., measure $T(\mathbf{r}, t_0)$ for some fixed t_0) by taking simultaneous data from a series of temperature probes at varying depths. We could also create a temporal profile at any given depth (i.e., measure $T(\mathbf{r}_0, t)$ for some fixed $\mathbf{r}_0$) by taking continuous data from a probe fixed at that depth. But neither of these would describe how an individual sinking water particle "experiences" a change in temperature over time.

Instead, we could use a probe that descends along with a particular water packet (i.e., volume element), measuring the time rate of temperature change of that element. This rate of change is called the *convective* or *material derivative*, since it corresponds to a situation in which a physical material quantity is followed as the derivative is calculated. We anticipate that this quantity will depend on (1) the time rate of change of T at each fixed point that the particle passes, and (2) the spatial rate of change of T as well as the rapidity with which the packet of interest is swept through that space gradient. The faster the packet descends, or the faster the temperature cools with depth, the larger the material derivative should be.

To compute the material derivative we describe the position of a water packet by the vector

$$\mathbf{r}(t) = \hat{\mathbf{x}}x(t) + \hat{\mathbf{y}}y(t) + \hat{\mathbf{z}}z(t).$$

Because no two packets can occupy the same place at the same time, the specification of $\mathbf{r}(0) = \mathbf{r}_0$ uniquely describes (or "tags") a particular packet. The time rate of change of $\mathbf{r}$ with $\mathbf{r}_0$ held constant (the material derivative of the position vector) is thus the velocity field $\mathbf{u}(\mathbf{r}, t)$ of the fluid:

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\mathbf{r}_0} = \frac{D\mathbf{r}}{Dt} = \mathbf{u}. \tag{A.57}$$

Here we use the "big D" notation to denote the material derivative, thereby avoiding confusion with the partial and total derivatives described below.

To describe the time rate of change of the temperature of a particular water packet, we only need to hold $\mathbf{r}_0$ constant while we examine the change. If we write the temperature

as

$$T(\mathbf{r},t) = T(\mathbf{r}(\mathbf{r}_0,t),t) = T[x(\mathbf{r}_0,t),\, y(\mathbf{r}_0,t),\, z(\mathbf{r}_0,t),\, t],$$

then we can use the chain rule to find the time rate of change of T with $\mathbf{r}_0$ held constant:

$$\begin{aligned}\frac{DT}{Dt} &= \left(\frac{dT}{dt}\right)_{\mathbf{r}_0} \\ &= \left(\frac{\partial T}{\partial x}\right)\left(\frac{dx}{dt}\right)_{\mathbf{r}_0} + \left(\frac{\partial T}{\partial y}\right)\left(\frac{dy}{dt}\right)_{\mathbf{r}_0} + \left(\frac{\partial T}{\partial z}\right)\left(\frac{dz}{dt}\right)_{\mathbf{r}_0} + \frac{\partial T}{\partial t}.\end{aligned}$$

We recognize the partial derivatives of the coordinates as the components of the material velocity (A.57), and thus can write

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_x\frac{\partial T}{\partial x} + u_y\frac{\partial T}{\partial y} + u_z\frac{\partial T}{\partial z} = \frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T.$$

As expected, the material derivative depends on both the local time rate of change and the spatial rate of change of temperature.

Suppose next that our probe is motorized and can travel about in the sinking water. If the probe sinks faster than the surrounding water, the time rate of change (measured by the probe) should exceed the material derivative. Let the probe position and velocity be

$$\mathbf{r}(t) = \hat{\mathbf{x}}x(t) + \hat{\mathbf{y}}y(t) + \hat{\mathbf{z}}z(t), \qquad \mathbf{v}(\mathbf{r},t) = \hat{\mathbf{x}}\frac{dx(t)}{dt} + \hat{\mathbf{y}}\frac{dy(t)}{dt} + \hat{\mathbf{z}}\frac{dz(t)}{dt}.$$

We can use the chain rule to determine the time rate of change of the temperature observed by the probe, but in this case we do *not* constrain the velocity components to represent the moving fluid. Thus, we merely obtain

$$\begin{aligned}\frac{dT}{dt} &= \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} + \frac{\partial T}{\partial t} \\ &= \frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T.\end{aligned}$$

This is called the *total derivative* of the temperature field.

In summary, the time rate of change of a scalar field T seen by an observer moving with arbitrary velocity $\mathbf{v}$ is given by the total derivative

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T. \tag{A.58}$$

If the velocity of the observer happens to match the velocity $\mathbf{u}$ of a moving substance, the time rate of change is the material derivative

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T. \tag{A.59}$$

We can obtain the material derivative of a vector field $\mathbf{F}$ by component-wise application of (A.59):

$$\begin{aligned}\frac{D\mathbf{F}}{Dt} &= \frac{D}{Dt}\left[\hat{\mathbf{x}}F_x + \hat{\mathbf{y}}F_y + \hat{\mathbf{z}}F_z\right] \\ &= \hat{\mathbf{x}}\frac{\partial F_x}{\partial t} + \hat{\mathbf{y}}\frac{\partial F_y}{\partial t} + \hat{\mathbf{z}}\frac{\partial F_z}{\partial t} + \hat{\mathbf{x}}\left[\mathbf{u}\cdot(\nabla F_x)\right] + \hat{\mathbf{y}}\left[\mathbf{u}\cdot(\nabla F_y)\right] + \hat{\mathbf{z}}\left[\mathbf{u}\cdot(\nabla F_z)\right].\end{aligned}$$

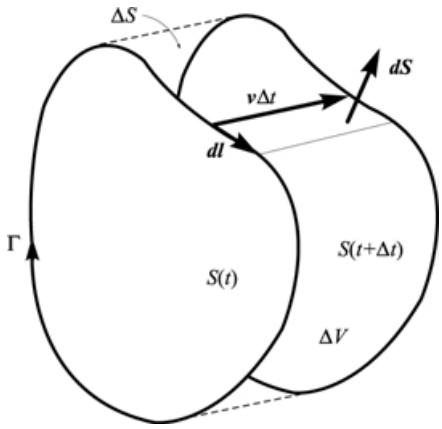

Figure A.5: Derivation of the Helmholtz transport theorem.

Using the notation

$$\mathbf{u} \cdot \nabla = u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}$$

we can write

$$\frac{D\mathbf{F}}{Dt} = \frac{\partial \mathbf{F}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{F}. \tag{A.60}$$

This is the material derivative of a vector field $\mathbf{F}$ when $\mathbf{u}$ describes the motion of a physical material. Similarly, the total derivative of a vector field is

$$\frac{d\mathbf{F}}{dt} = \frac{\partial \mathbf{F}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{F}$$

where $\mathbf{v}$ is arbitrary.

The Helmholtz and Reynolds transport theorems

We choose the intuitive approach taken by Tai [190] and Whitaker [214]. Consider an open surface $S(t)$ moving through space and possibly deforming as it moves. The velocity of the points comprising the surface is given by the vector field $\mathbf{v}(\mathbf{r}, t)$. We are interested in computing the time derivative of the flux of a vector field $\mathbf{F}(\mathbf{r}, t)$ through $S(t)$:

$$\begin{aligned} \psi(t) &= \frac{d}{dt} \int_{S(t)} \mathbf{F}(\mathbf{r}, t) \cdot \mathbf{dS} \\ &= \lim_{\Delta t \to 0} \frac{\int_{S(t+\Delta t)} \mathbf{F}(\mathbf{r}, t + \Delta t) \cdot \mathbf{dS} - \int_{S(t)} \mathbf{F}(\mathbf{r}, t) \cdot \mathbf{dS}}{\Delta t}. \end{aligned} \tag{A.61}$$

Here $S(t+\Delta t) = S_2$ is found by extending each point on $S(t) = S_1$ through a displacement $\mathbf{v}\Delta t$, as shown in Figure A.5. Substituting the Taylor expansion

$$\mathbf{F}(\mathbf{r}, t + \Delta t) = \mathbf{F}(\mathbf{r}, t) + \frac{\partial \mathbf{F}(\mathbf{r}, t)}{\partial t} \Delta t + \cdots$$

into (A.61), we find that only the first two terms give non-zero contributions to the integral and

$$\psi(t) = \int_{S(t)} \frac{\partial \mathbf{F}(\mathbf{r},t)}{\partial t} \cdot \mathbf{dS} + \lim_{\Delta t \to 0} \frac{\int_{S_2} \mathbf{F}(\mathbf{r},t) \cdot \mathbf{dS} - \int_{S_1} \mathbf{F}(\mathbf{r},t) \cdot \mathbf{dS}}{\Delta t}. \tag{A.62}$$

The second term on the right can be evaluated with the help of Figure A.5. As the surface moves through a displacement $\mathbf{v}\Delta t$ it sweeps out a volume region ΔV that is bounded on the back by S_1, on the front by S_2, and on the side by a surface $S_3 = \Delta S$. We can thus compute the two surface integrals in (A.62) as the difference between contributions from the surface enclosing ΔV and the side surface ΔS (remembering that the normal to S_1 in (A.62) points *into* ΔV). Thus

$$\begin{aligned}\psi(t) &= \int_{S(t)} \frac{\partial \mathbf{F}(\mathbf{r},t)}{\partial t} \cdot \mathbf{dS} + \lim_{\Delta t \to 0} \frac{\oint_{S_1+S_2+\Delta S} \mathbf{F}(\mathbf{r},t) \cdot \mathbf{dS} - \int_{\Delta S} \mathbf{F}(\mathbf{r},t) \cdot \mathbf{dS}_3}{\Delta t} \\ &= \int_{S(t)} \frac{\partial \mathbf{F}(\mathbf{r},t)}{\partial t} \cdot \mathbf{dS} + \lim_{\Delta t \to 0} \frac{\int_{\Delta V} \nabla \cdot \mathbf{F}(\mathbf{r},t)\, dV_3 - \int_{\Delta S} \mathbf{F}(\mathbf{r},t) \cdot \mathbf{dS}_3}{\Delta t}\end{aligned}$$

by the divergence theorem. To compute the integrals over ΔS and ΔV we note from Figure A.5 that the incremental surface and volume elements are just

$$\mathbf{dS}_3 = \mathbf{dl} \times (\mathbf{v}\Delta t), \qquad dV_3 = (\mathbf{v}\Delta t) \cdot \mathbf{dS}.$$

Then, since $\mathbf{F} \cdot [\mathbf{dl} \times (\mathbf{v}\Delta t)] = \Delta t (\mathbf{v} \times \mathbf{F}) \cdot \mathbf{dl}$, we have

$$\psi(t) = \int_{S(t)} \frac{\partial \mathbf{F}(\mathbf{r},t)}{\partial t} \cdot \mathbf{dS} + \lim_{\Delta t \to 0} \frac{\Delta t \int_{S(t)} [\mathbf{v}\nabla \cdot \mathbf{F}(\mathbf{r},t)] \cdot \mathbf{dS}}{\Delta t} - \lim_{\Delta t \to 0} \frac{\Delta t \oint_{\Gamma} [\mathbf{v} \times \mathbf{F}(\mathbf{r},t)] \cdot \mathbf{dl}}{\Delta t}.$$

Taking the limit and using Stokes's theorem on the last integral we have finally

$$\frac{d}{dt} \int_{S(t)} \mathbf{F} \cdot \mathbf{dS} = \int_{S(t)} \left[\frac{\partial \mathbf{F}}{\partial t} + \mathbf{v}\nabla \cdot \mathbf{F} - \nabla \times (\mathbf{v} \times \mathbf{F}) \right] \cdot \mathbf{dS}, \tag{A.63}$$

which is the *Helmholtz transport theorem* [190, 43].

In case the surface corresponds to a moving physical material, we may wish to write the Helmholtz transport theorem in terms of the material derivative. We can set $\mathbf{v} = \mathbf{u}$ and use

$$\nabla \times (\mathbf{u} \times \mathbf{F}) = \mathbf{u}(\nabla \cdot \mathbf{F}) - \mathbf{F}(\nabla \cdot \mathbf{u}) + (\mathbf{F} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{F}$$

and (A.60) to obtain

$$\frac{d}{dt} \int_{S(t)} \mathbf{F} \cdot \mathbf{dS} = \int_{S(t)} \left[\frac{D\mathbf{F}}{Dt} + \mathbf{F}(\nabla \cdot \mathbf{u}) - (\mathbf{F} \cdot \nabla)\mathbf{u} \right] \cdot \mathbf{dS}.$$

If $S(t)$ in (A.63) is closed, enclosing a volume region $V(t)$, then

$$\oint_{S(t)} [\nabla \times (\mathbf{v} \times \mathbf{F})] \cdot \mathbf{dS} = \int_{V(t)} \nabla \cdot [\nabla \times (\mathbf{v} \times \mathbf{F})]\, dV = 0$$

by the divergence theorem and (B.49). In this case the Helmholtz transport theorem becomes

$$\frac{d}{dt} \oint_{S(t)} \mathbf{F} \cdot \mathbf{dS} = \oint_{S(t)} \left[\frac{\partial \mathbf{F}}{\partial t} + \mathbf{v}\nabla \cdot \mathbf{F} \right] \cdot \mathbf{dS}. \tag{A.64}$$

We now come to an essential tool that we employ throughout the book. Using the divergence theorem we can rewrite (A.64) as

$$\frac{d}{dt}\int_{V(t)} \nabla\cdot\mathbf{F}\,dV = \int_{V(t)} \nabla\cdot\frac{\partial\mathbf{F}}{\partial t}\,dV + \oint_{S(t)} (\nabla\cdot\mathbf{F})\mathbf{v}\cdot\mathbf{dS}.$$

Replacing $\nabla\cdot\mathbf{F}$ by the scalar field ρ we have

$$\frac{d}{dt}\int_{V(t)} \rho\,dV = \int_{V(t)} \frac{\partial\rho}{\partial t}\,dV + \oint_{S(t)} \rho\mathbf{v}\cdot\mathbf{dS}. \tag{A.65}$$

In this *general form* of the transport theorem $\mathbf{v}$ is an arbitrary velocity. In most applications $\mathbf{v} = \mathbf{u}$ describes the motion of a material substance; then

$$\frac{D}{Dt}\int_{V(t)} \rho\,dV = \int_{V(t)} \frac{\partial\rho}{\partial t}\,dV + \oint_{S(t)} \rho\mathbf{u}\cdot\mathbf{dS}, \tag{A.66}$$

which is the *Reynolds transport theorem* [214]. The D/Dt notation implies that $V(t)$ retains exactly the same material elements as it moves and deforms to follow the material substance.

We may rewrite the Reynolds transport theorem in various forms. By the divergence theorem we have

$$\frac{d}{dt}\int_{V(t)} \rho\,dV = \int_{V(t)} \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right] dV.$$

Setting $\mathbf{v} = \mathbf{u}$, using (B.42), and using (A.59) for the material derivative of ρ, we obtain

$$\frac{D}{Dt}\int_{V(t)} \rho\,dV = \int_{V(t)} \left[\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u}\right] dV. \tag{A.67}$$

We may also generate a vector form of the general transport theorem by taking ρ in (A.65) to be a component of a vector. Assembling all of the components we have

$$\frac{d}{dt}\int_{V(t)} \mathbf{A}\,dV = \int_{V(t)} \frac{\partial\mathbf{A}}{\partial t}\,dV + \oint_{S(t)} \mathbf{A}(\mathbf{v}\cdot\hat{\mathbf{n}})\,dS. \tag{A.68}$$

A.3 Dyadic analysis

Dyadic analysis was introduced in the late nineteenth century by Gibbs to generalize vector analysis to problems in which the components of vectors are related in a linear manner. It has now been widely supplanted by tensor theory, but maintains a foothold in engineering where the transformation properties of tensors are not paramount (except, of course, in considerations such as those involving special relativity). Terms such as "tensor permittivity" and "dyadic permittivity" are often used interchangeably.

Component form representation. We wish to write one vector field $\mathbf{A}(\mathbf{r}, t)$ as a linear function of another vector field $\mathbf{B}(\mathbf{r}, t)$:

$$\mathbf{A} = f(\mathbf{B}).$$

By this we mean that each component of $\mathbf{A}$ is a linear combination of the components of $\mathbf{B}$:

$$\begin{aligned}
A_1(\mathbf{r},t) &= a_{11'}\,B_{1'}(\mathbf{r},t) + a_{12'}\,B_{2'}(\mathbf{r},t) + a_{13'}\,B_{3'}(\mathbf{r},t),\\
A_2(\mathbf{r},t) &= a_{21'}\,B_{1'}(\mathbf{r},t) + a_{22'}\,B_{2'}(\mathbf{r},t) + a_{23'}\,B_{3'}(\mathbf{r},t),\\
A_3(\mathbf{r},t) &= a_{31'}\,B_{1'}(\mathbf{r},t) + a_{32'}\,B_{2'}(\mathbf{r},t) + a_{33'}\,B_{3'}(\mathbf{r},t).
\end{aligned}$$

Here the $a_{ij'}$ may depend on space and time (or frequency). The prime on the second index indicates that $\mathbf{A}$ and $\mathbf{B}$ may be expressed in distinct coordinate frames $(\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3)$ and $(\hat{\mathbf{i}}_{1'}, \hat{\mathbf{i}}_{2'}, \hat{\mathbf{i}}_{3'})$, respectively. We have

$$\begin{aligned}
A_1 &= \left(a_{11'}\hat{\mathbf{i}}_{1'} + a_{12'}\hat{\mathbf{i}}_{2'} + a_{13'}\hat{\mathbf{i}}_{3'}\right)\cdot\left(\hat{\mathbf{i}}_{1'}B_{1'} + \hat{\mathbf{i}}_{2'}B_{2'} + \hat{\mathbf{i}}_{3'}B_{3'}\right),\\
A_2 &= \left(a_{21'}\hat{\mathbf{i}}_{1'} + a_{22'}\hat{\mathbf{i}}_{2'} + a_{23'}\hat{\mathbf{i}}_{3'}\right)\cdot\left(\hat{\mathbf{i}}_{1'}B_{1'} + \hat{\mathbf{i}}_{2'}B_{2'} + \hat{\mathbf{i}}_{3'}B_{3'}\right),\\
A_3 &= \left(a_{31'}\hat{\mathbf{i}}_{1'} + a_{32'}\hat{\mathbf{i}}_{2'} + a_{33'}\hat{\mathbf{i}}_{3'}\right)\cdot\left(\hat{\mathbf{i}}_{1'}B_{1'} + \hat{\mathbf{i}}_{2'}B_{2'} + \hat{\mathbf{i}}_{3'}B_{3'}\right),
\end{aligned}$$

and since $\mathbf{B} = \hat{\mathbf{i}}_{1'}B_{1'} + \hat{\mathbf{i}}_{2'}B_{2'} + \hat{\mathbf{i}}_{3'}B_{3'}$ we can write

$$\mathbf{A} = \hat{\mathbf{i}}_1(\mathbf{a}'_1\cdot\mathbf{B}) + \hat{\mathbf{i}}_2(\mathbf{a}'_2\cdot\mathbf{B}) + \hat{\mathbf{i}}_3(\mathbf{a}'_3\cdot\mathbf{B})$$

where

$$\begin{aligned}
\mathbf{a}'_1 &= a_{11'}\hat{\mathbf{i}}_{1'} + a_{12'}\hat{\mathbf{i}}_{2'} + a_{13'}\hat{\mathbf{i}}_{3'},\\
\mathbf{a}'_2 &= a_{21'}\hat{\mathbf{i}}_{1'} + a_{22'}\hat{\mathbf{i}}_{2'} + a_{23'}\hat{\mathbf{i}}_{3'},\\
\mathbf{a}'_3 &= a_{31'}\hat{\mathbf{i}}_{1'} + a_{32'}\hat{\mathbf{i}}_{2'} + a_{33'}\hat{\mathbf{i}}_{3'}.
\end{aligned}$$

In shorthand notation

$$\mathbf{A} = \bar{\mathbf{a}}\cdot\mathbf{B} \tag{A.69}$$

where

$$\bar{\mathbf{a}} = \hat{\mathbf{i}}_1\mathbf{a}'_1 + \hat{\mathbf{i}}_2\mathbf{a}'_2 + \hat{\mathbf{i}}_3\mathbf{a}'_3. \tag{A.70}$$

Written out, the quantity $\bar{\mathbf{a}}$ looks like

$$\begin{aligned}
\bar{\mathbf{a}} = {} & a_{11'}(\hat{\mathbf{i}}_1\hat{\mathbf{i}}_{1'}) + a_{12'}(\hat{\mathbf{i}}_1\hat{\mathbf{i}}_{2'}) + a_{13'}(\hat{\mathbf{i}}_1\hat{\mathbf{i}}_{3'}) +\\
& + a_{21'}(\hat{\mathbf{i}}_2\hat{\mathbf{i}}_{1'}) + a_{22'}(\hat{\mathbf{i}}_2\hat{\mathbf{i}}_{2'}) + a_{23'}(\hat{\mathbf{i}}_2\hat{\mathbf{i}}_{3'}) +\\
& + a_{31'}(\hat{\mathbf{i}}_3\hat{\mathbf{i}}_{1'}) + a_{32'}(\hat{\mathbf{i}}_3\hat{\mathbf{i}}_{2'}) + a_{33'}(\hat{\mathbf{i}}_3\hat{\mathbf{i}}_{3'}).
\end{aligned}$$

Terms such as $\hat{\mathbf{i}}_1\hat{\mathbf{i}}_{1'}$ are called *dyads*, while sums of dyads such as $\bar{\mathbf{a}}$ are called *dyadics*. The components $a_{ij'}$ of $\bar{\mathbf{a}}$ may be conveniently placed into an array:

$$[\bar{\mathbf{a}}] = \begin{bmatrix} a_{11'} & a_{12'} & a_{13'}\\ a_{21'} & a_{22'} & a_{23'}\\ a_{31'} & a_{32'} & a_{33'}\end{bmatrix}.$$

Writing

$$[\mathbf{A}] = \begin{bmatrix} A_1\\ A_2\\ A_3\end{bmatrix}, \qquad [\mathbf{B}] = \begin{bmatrix} B_{1'}\\ B_{2'}\\ B_{3'}\end{bmatrix},$$

we see that $\mathbf{A} = \bar{\mathbf{a}} \cdot \mathbf{B}$ can be written as

$$[\mathbf{A}] = [\bar{\mathbf{a}}]\,[\mathbf{B}] = \begin{bmatrix} a_{11'} & a_{12'} & a_{13'} \\ a_{21'} & a_{22'} & a_{23'} \\ a_{31'} & a_{32'} & a_{33'} \end{bmatrix} \begin{bmatrix} B_{1'} \\ B_{2'} \\ B_{3'} \end{bmatrix}.$$

Note carefully that in (A.69) $\bar{\mathbf{a}}$ operates on $\mathbf{B}$ from the left. A reorganization of the components of $\bar{\mathbf{a}}$ allows us to write

$$\bar{\mathbf{a}} = \mathbf{a}_1 \hat{\mathbf{i}}_{1'} + \mathbf{a}_2 \hat{\mathbf{i}}_{2'} + \mathbf{a}_3 \hat{\mathbf{i}}_{3'} \tag{A.71}$$

where

$$\begin{aligned} \mathbf{a}_1 &= a_{11'}\,\hat{\mathbf{i}}_1 + a_{21'}\,\hat{\mathbf{i}}_2 + a_{31'}\,\hat{\mathbf{i}}_3, \\ \mathbf{a}_2 &= a_{12'}\,\hat{\mathbf{i}}_1 + a_{22'}\,\hat{\mathbf{i}}_2 + a_{32'}\,\hat{\mathbf{i}}_3, \\ \mathbf{a}_3 &= a_{13'}\,\hat{\mathbf{i}}_1 + a_{23'}\,\hat{\mathbf{i}}_2 + a_{33'}\,\hat{\mathbf{i}}_3. \end{aligned}$$

We may now consider using $\bar{\mathbf{a}}$ to operate on a vector $\mathbf{C} = \hat{\mathbf{i}}_1 C_1 + \hat{\mathbf{i}}_2 C_2 + \hat{\mathbf{i}}_3 C_3$ from the right:

$$\mathbf{C} \cdot \bar{\mathbf{a}} = (\mathbf{C} \cdot \mathbf{a}_1)\hat{\mathbf{i}}_{1'} + (\mathbf{C} \cdot \mathbf{a}_2)\hat{\mathbf{i}}_{2'} + (\mathbf{C} \cdot \mathbf{a}_3)\hat{\mathbf{i}}_{3'}.$$

In matrix form $\mathbf{C} \cdot \bar{\mathbf{a}}$ is

$$[\bar{\mathbf{a}}]^T\,[\mathbf{C}] = \begin{bmatrix} a_{11'} & a_{21'} & a_{31'} \\ a_{12'} & a_{22'} & a_{32'} \\ a_{13'} & a_{23'} & a_{33'} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

where the superscript "T" denotes the matrix transpose operation. That is,

$$\mathbf{C} \cdot \bar{\mathbf{a}} = \bar{\mathbf{a}}^T \cdot \mathbf{C}$$

where $\bar{\mathbf{a}}^T$ is the transpose of $\bar{\mathbf{a}}$.

If the primed and unprimed frames coincide, then

$$\begin{aligned} \bar{\mathbf{a}} = {} & a_{11}(\hat{\mathbf{i}}_1\hat{\mathbf{i}}_1) + a_{12}(\hat{\mathbf{i}}_1\hat{\mathbf{i}}_2) + a_{13}(\hat{\mathbf{i}}_1\hat{\mathbf{i}}_3) + \\ & + a_{21}(\hat{\mathbf{i}}_2\hat{\mathbf{i}}_1) + a_{22}(\hat{\mathbf{i}}_2\hat{\mathbf{i}}_2) + a_{23}(\hat{\mathbf{i}}_2\hat{\mathbf{i}}_3) + \\ & + a_{31}(\hat{\mathbf{i}}_3\hat{\mathbf{i}}_1) + a_{32}(\hat{\mathbf{i}}_3\hat{\mathbf{i}}_2) + a_{33}(\hat{\mathbf{i}}_3\hat{\mathbf{i}}_3). \end{aligned}$$

In this case we may compare the results of $\bar{\mathbf{a}} \cdot \mathbf{B}$ and $\mathbf{B} \cdot \bar{\mathbf{a}}$ for a given vector $\mathbf{B} = \hat{\mathbf{i}}_1 B_1 + \hat{\mathbf{i}}_2 B_2 + \hat{\mathbf{i}}_3 B_3$. We leave it to the reader to verify that in general

$$\mathbf{B} \cdot \bar{\mathbf{a}} \neq \bar{\mathbf{a}} \cdot \mathbf{B}.$$

Vector form representation. We can express dyadics in coordinate-free fashion if we expand the concept of a dyad to permit entities such as $\mathbf{AB}$. Here $\mathbf{A}$ and $\mathbf{B}$ are called the *antecedent* and *consequent*, respectively. The operation rules

$$(\mathbf{AB}) \cdot \mathbf{C} = \mathbf{A}(\mathbf{B} \cdot \mathbf{C}), \qquad \mathbf{C} \cdot (\mathbf{AB}) = (\mathbf{C} \cdot \mathbf{A})\mathbf{B},$$

define the anterior and posterior products of $\mathbf{AB}$ with a vector $\mathbf{C}$, and give results consistent with our prior component notation. Sums of dyads such as $\mathbf{AB} + \mathbf{CD}$ are called *dyadic polynomials*, or dyadics. The simple dyadic

$$\mathbf{AB} = (A_1\hat{\mathbf{i}}_1 + A_2\hat{\mathbf{i}}_2 + A_3\hat{\mathbf{i}}_3)(B_{1'}\hat{\mathbf{i}}_{1'} + B_{2'}\hat{\mathbf{i}}_{2'} + B_{3'}\hat{\mathbf{i}}_{3'})$$

can be represented in component form using

$$\mathbf{AB} = \hat{\mathbf{i}}_1 \mathbf{a}'_1 + \hat{\mathbf{i}}_2 \mathbf{a}'_2 + \hat{\mathbf{i}}_3 \mathbf{a}'_3$$

where

$$\begin{aligned}
\mathbf{a}'_1 &= A_1 B_{1'} \hat{\mathbf{i}}_{1'} + A_1 B_{2'} \hat{\mathbf{i}}_{2'} + A_1 B_{3'} \hat{\mathbf{i}}_{3'}, \\
\mathbf{a}'_2 &= A_2 B_{1'} \hat{\mathbf{i}}_{1'} + A_2 B_{2'} \hat{\mathbf{i}}_{2'} + A_2 B_{3'} \hat{\mathbf{i}}_{3'}, \\
\mathbf{a}'_3 &= A_3 B_{1'} \hat{\mathbf{i}}_{1'} + A_3 B_{2'} \hat{\mathbf{i}}_{2'} + A_3 B_{3'} \hat{\mathbf{i}}_{3'},
\end{aligned}$$

or using

$$\mathbf{AB} = \mathbf{a}_1 \hat{\mathbf{i}}_{1'} + \mathbf{a}_2 \hat{\mathbf{i}}_{2'} + \mathbf{a}_3 \hat{\mathbf{i}}_{3'}$$

where

$$\begin{aligned}
\mathbf{a}_1 &= \hat{\mathbf{i}}_1 A_1 B_{1'} + \hat{\mathbf{i}}_2 A_2 B_{1'} + \hat{\mathbf{i}}_3 A_3 B_{1'}, \\
\mathbf{a}_2 &= \hat{\mathbf{i}}_1 A_1 B_{2'} + \hat{\mathbf{i}}_2 A_2 B_{2'} + \hat{\mathbf{i}}_3 A_3 B_{2'}, \\
\mathbf{a}_3 &= \hat{\mathbf{i}}_1 A_1 B_{3'} + \hat{\mathbf{i}}_2 A_2 B_{3'} + \hat{\mathbf{i}}_3 A_3 B_{3'}.
\end{aligned}$$

Note that if we write $\bar{\mathbf{a}} = \mathbf{AB}$ then $a_{ij} = A_i B_{j'}$.

A simple dyad **AB** by itself cannot represent a general dyadic $\bar{\mathbf{a}}$; only six independent quantities are available in **AB** (the three components of **A** and the three components of **B**), while an arbitrary dyadic has nine independent components. However, it can be shown that any dyadic can be written as a sum of three dyads:

$$\bar{\mathbf{a}} = \mathbf{AB} + \mathbf{CD} + \mathbf{EF}.$$

This is called a *vector representation* of $\bar{\mathbf{a}}$. If **V** is a vector, the distributive laws

$$\begin{aligned}
\bar{\mathbf{a}} \cdot \mathbf{V} &= (\mathbf{AB} + \mathbf{CD} + \mathbf{EF}) \cdot \mathbf{V} = \mathbf{A}(\mathbf{B} \cdot \mathbf{V}) + \mathbf{C}(\mathbf{D} \cdot \mathbf{V}) + \mathbf{E}(\mathbf{F} \cdot \mathbf{V}), \\
\mathbf{V} \cdot \bar{\mathbf{a}} &= \mathbf{V} \cdot (\mathbf{AB} + \mathbf{CD} + \mathbf{EF}) = (\mathbf{V} \cdot \mathbf{A})\mathbf{B} + (\mathbf{V} \cdot \mathbf{C})\mathbf{D} + (\mathbf{V} \cdot \mathbf{E})\mathbf{F},
\end{aligned}$$

apply.

Dyadic algebra and calculus. The cross product of a vector with a dyadic produces another dyadic. If $\bar{\mathbf{a}} = \mathbf{AB} + \mathbf{CD} + \mathbf{EF}$ then by definition

$$\begin{aligned}
\bar{\mathbf{a}} \times \mathbf{V} &= \mathbf{A}(\mathbf{B} \times \mathbf{V}) + \mathbf{C}(\mathbf{D} \times \mathbf{V}) + \mathbf{E}(\mathbf{F} \times \mathbf{V}), \\
\mathbf{V} \times \bar{\mathbf{a}} &= (\mathbf{V} \times \mathbf{A})\mathbf{B} + (\mathbf{V} \times \mathbf{C})\mathbf{D} + (\mathbf{V} \times \mathbf{E})\mathbf{F}.
\end{aligned}$$

The corresponding component forms are

$$\begin{aligned}
\bar{\mathbf{a}} \times \mathbf{V} &= \hat{\mathbf{i}}_1(\mathbf{a}'_1 \times \mathbf{V}) + \hat{\mathbf{i}}_2(\mathbf{a}'_2 \times \mathbf{V}) + \hat{\mathbf{i}}_3(\mathbf{a}'_3 \times \mathbf{V}), \\
\mathbf{V} \times \bar{\mathbf{a}} &= (\mathbf{V} \times \mathbf{a}_1)\hat{\mathbf{i}}_{1'} + (\mathbf{V} \times \mathbf{a}_2)\hat{\mathbf{i}}_{2'} + (\mathbf{V} \times \mathbf{a}_3)\hat{\mathbf{i}}_{3'},
\end{aligned}$$

where we have used (A.70) and (A.71), respectively. Interactions between dyads or dyadics may also be defined. The dot product of two dyads **AB** and **CD** is a dyad given by

$$(\mathbf{AB}) \cdot (\mathbf{CD}) = \mathbf{A}(\mathbf{B} \cdot \mathbf{C})\mathbf{D} = (\mathbf{B} \cdot \mathbf{C})(\mathbf{AD}).$$

The dot product of two dyadics can be found by applying the distributive property.

If α is a scalar, then the product $\alpha\bar{\mathbf{a}}$ is a dyadic with components equal to α times the components of $\bar{\mathbf{a}}$. Dyadic addition may be accomplished by adding individual dyadic components as long as the dyadics are expressed in the same coordinate system. Subtraction is accomplished by adding the negative of a dyadic, which is defined through scalar multiplication by -1.

Some useful dyadic identities appear in Appendix B. Many more can be found in Van Bladel [202].

The various vector derivatives may also be extended to dyadics. Computations are easiest in rectangular coordinates, since $\hat{\mathbf{i}}_1 = \hat{\mathbf{x}}$, $\hat{\mathbf{i}}_2 = \hat{\mathbf{y}}$, and $\hat{\mathbf{i}}_3 = \hat{\mathbf{z}}$ are constant with position. The dyadic

$$\bar{\mathbf{a}} = \mathbf{a}_x\hat{\mathbf{x}} + \mathbf{a}_y\hat{\mathbf{y}} + \mathbf{a}_z\hat{\mathbf{z}}$$

has divergence

$$\nabla \cdot \bar{\mathbf{a}} = (\nabla \cdot \mathbf{a}_x)\hat{\mathbf{x}} + (\nabla \cdot \mathbf{a}_y)\hat{\mathbf{y}} + (\nabla \cdot \mathbf{a}_z)\hat{\mathbf{z}},$$

and curl

$$\nabla \times \bar{\mathbf{a}} = (\nabla \times \mathbf{a}_x)\hat{\mathbf{x}} + (\nabla \times \mathbf{a}_y)\hat{\mathbf{y}} + (\nabla \times \mathbf{a}_z)\hat{\mathbf{z}}.$$

Note that the divergence of a dyadic is a vector while the curl of a dyadic is a dyadic. The gradient of a vector $\mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$ is

$$\nabla\mathbf{a} = (\nabla a_x)\hat{\mathbf{x}} + (\nabla a_y)\hat{\mathbf{y}} + (\nabla a_z)\hat{\mathbf{z}},$$

a dyadic quantity.

The dyadic derivatives may be expressed in coordinate-free notation by using the vector representation. The dyadic $\mathbf{AB}$ has divergence

$$\nabla \cdot (\mathbf{AB}) = (\nabla \cdot \mathbf{A})\mathbf{B} + \mathbf{A} \cdot (\nabla\mathbf{B})$$

and curl

$$\nabla \times (\mathbf{AB}) = (\nabla \times \mathbf{A})\mathbf{B} - \mathbf{A} \times (\nabla\mathbf{B}).$$

The Laplacian of a dyadic is a dyadic given by

$$\nabla^2\bar{\mathbf{a}} = \nabla(\nabla \cdot \bar{\mathbf{a}}) - \nabla \times (\nabla \times \bar{\mathbf{a}}).$$

The divergence theorem for dyadics is

$$\int_V \nabla \cdot \bar{\mathbf{a}}\, dV = \oint_S \hat{\mathbf{n}} \cdot \bar{\mathbf{a}}\, dS.$$

Some of the other common differential and integral identities for dyadics can be found in Van Bladel [202] and Tai [192].

Special dyadics. We say that $\bar{\mathbf{a}}$ is *symmetric* if

$$\mathbf{B} \cdot \bar{\mathbf{a}} = \bar{\mathbf{a}} \cdot \mathbf{B}$$

for any vector $\mathbf{B}$. This requires $\bar{\mathbf{a}}^T = \bar{\mathbf{a}}$, i.e., $a_{ij'} = a_{ji'}$. We say that $\bar{\mathbf{a}}$ is *antisymmetric* if

$$\mathbf{B} \cdot \bar{\mathbf{a}} = -\bar{\mathbf{a}} \cdot \mathbf{B}$$

for any $\mathbf{B}$. In this case $\bar{\mathbf{a}}^T = -\bar{\mathbf{a}}$. That is, $a_{ij'} = -a_{ji'}$ and $a_{ii'} = 0$. A symmetric dyadic has only six independent components while an antisymmetric dyadic has only three. The

reader can verify that any dyadic can be decomposed into symmetric and antisymmetric parts as

$$\bar{\mathbf{a}} = \frac{1}{2}\left(\bar{\mathbf{a}} + \bar{\mathbf{a}}^T\right) + \frac{1}{2}\left(\bar{\mathbf{a}} - \bar{\mathbf{a}}^T\right).$$

A simple example of a symmetric dyadic is the *unit dyadic* $\bar{\mathbf{I}}$ defined by

$$\bar{\mathbf{I}} = \hat{\mathbf{i}}_1\hat{\mathbf{i}}_1 + \hat{\mathbf{i}}_2\hat{\mathbf{i}}_2 + \hat{\mathbf{i}}_3\hat{\mathbf{i}}_3.$$

This quantity often arises in the manipulation of dyadic equations, and satisfies

$$\mathbf{A}\cdot\bar{\mathbf{I}} = \bar{\mathbf{I}}\cdot\mathbf{A} = \mathbf{A}$$

for any vector $\mathbf{A}$. In matrix form $\bar{\mathbf{I}}$ is the identity matrix:

$$[\bar{\mathbf{I}}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The components of a dyadic may be complex. We say that $\bar{\mathbf{a}}$ is *hermitian* if

$$\mathbf{B}\cdot\bar{\mathbf{a}} = \bar{\mathbf{a}}^*\cdot\mathbf{B} \tag{A.72}$$

holds for any $\mathbf{B}$. This requires that $\bar{\mathbf{a}}^* = \bar{\mathbf{a}}^T$. Taking the transpose we can write

$$\bar{\mathbf{a}} = (\bar{\mathbf{a}}^*)^T = \bar{\mathbf{a}}^\dagger$$

where "†" stands for the conjugate-transpose operation. We say that $\bar{\mathbf{a}}$ is *anti-hermitian* if

$$\mathbf{B}\cdot\bar{\mathbf{a}} = -\bar{\mathbf{a}}^*\cdot\mathbf{B} \tag{A.73}$$

for arbitrary $\mathbf{B}$. In this case $\bar{\mathbf{a}}^* = -\bar{\mathbf{a}}^T$. Any complex dyadic can be decomposed into hermitian and anti-hermitian parts:

$$\bar{\mathbf{a}} = \frac{1}{2}\left(\bar{\mathbf{a}}^H + \bar{\mathbf{a}}^A\right) \tag{A.74}$$

where

$$\bar{\mathbf{a}}^H = \bar{\mathbf{a}} + \bar{\mathbf{a}}^\dagger, \qquad \bar{\mathbf{a}}^A = \bar{\mathbf{a}} - \bar{\mathbf{a}}^\dagger. \tag{A.75}$$

A dyadic identity important in the study of material parameters is

$$\mathbf{B}\cdot\bar{\mathbf{a}}^*\cdot\mathbf{B}^* = \mathbf{B}^*\cdot\bar{\mathbf{a}}^\dagger\cdot\mathbf{B}. \tag{A.76}$$

We show this by decomposing $\bar{\mathbf{a}}$ according to (A.74), giving

$$\mathbf{B}\cdot\bar{\mathbf{a}}^*\cdot\mathbf{B}^* = \frac{1}{2}\left(\left[\mathbf{B}^*\cdot\bar{\mathbf{a}}^H\right]^* + \left[\mathbf{B}^*\cdot\bar{\mathbf{a}}^A\right]^*\right)\cdot\mathbf{B}^*$$

where we have used $(\mathbf{B}\cdot\bar{\mathbf{a}})^* = (\mathbf{B}^*\cdot\bar{\mathbf{a}}^*)$. Applying (A.72) and (A.73) we obtain

$$\begin{aligned}
\mathbf{B}\cdot\bar{\mathbf{a}}^*\cdot\mathbf{B}^* &= \frac{1}{2}\left(\left[\bar{\mathbf{a}}^{H*}\cdot\mathbf{B}^*\right]^* - \left[\bar{\mathbf{a}}^{A*}\cdot\mathbf{B}^*\right]^*\right)\cdot\mathbf{B}^* \\
&= \mathbf{B}^*\cdot\frac{1}{2}\left(\left[\bar{\mathbf{a}}^H\cdot\mathbf{B}\right] - \left[\bar{\mathbf{a}}^A\cdot\mathbf{B}\right]\right) \\
&= \mathbf{B}^*\cdot\left(\frac{1}{2}\left[\bar{\mathbf{a}}^H - \bar{\mathbf{a}}^A\right]\cdot\mathbf{B}\right).
\end{aligned}$$

Since the term in brackets is $\bar{\mathbf{a}}^H - \bar{\mathbf{a}}^A = 2\bar{\mathbf{a}}^\dagger$ by (A.75), the identity is proved.

A.4 Boundary value problems

Many physical phenomena may be described mathematically as the solutions to *boundary value problems*. The desired physical quantity (usually called a "field") in a certain region of space is found by solving one or more partial differential equations subject to certain conditions over the boundary surface. The boundary conditions may specify the values of the field, some manipulated version of the field (such as the normal derivative), or a relationship between fields in adjoining regions. If the field varies with time as well as space, initial or final values of the field must also be specified. Particularly important is whether a boundary value problem is *well-posed* and therefore has a unique solution which depends continuously on the data supplied. This depends on the forms of the differential equation and boundary conditions. The well-posedness of Maxwell's equations is discussed in § 2.2.

The importance of boundary value problems has led to an array of techniques, both analytical and numerical, for solving them. Many problems (such as boundary value problems involving Laplace's equation) may be solved in several different ways. Uniqueness permits an engineer to focus attention on which technique will yield the most efficient solution. In this section we concentrate on the separation of variables technique, which is widely applied in the solution of Maxwell's equations. We first discuss eigenvalue problems and then give an overview of separation of variables. Finally we consider a number of example problems in each of the three common coordinate systems.

Sturm–Liouville problems and eigenvalues

The partial differential equations of electromagnetics can often be reduced to ordinary differential equations. In some cases symmetry permits us to reduce the number of dimensions by inspection; in other cases, we may employ an integral transform (e.g., the Fourier transform) or separation of variables. The resulting ordinary differential equations may be viewed as particular cases of the *Sturm–Liouville differential equation*

$$\frac{d}{dx}\left[p(x)\frac{d\psi(x)}{dx}\right] + q(x)\psi(x) + \lambda\sigma(x)\psi(x) = 0, \qquad x \in [a, b]. \tag{A.77}$$

In linear operator notation

$$\mathcal{L}[\psi(x)] = -\lambda\sigma(x)\psi(x), \tag{A.78}$$

where $\mathcal{L}$ is the linear Sturm–Liouville operator

$$\mathcal{L} = \left(\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)\right).$$

Obviously $\psi(x) = 0$ satisfies (A.78). However, for certain values of λ dependent on p, q, σ, and the boundary conditions we impose, (A.78) has non-trivial solutions. Each λ that satisfies (A.78) is an *eigenvalue* of $\mathcal{L}$, and any non-trivial solution associated with that eigenvalue is an *eigenfunction*. Taken together, the eigenvalues of an operator form its *eigenvalue spectrum*.

We shall restrict ourselves to the case in which $\mathcal{L}$ is *self-adjoint*. Assume p, q, and σ are real and continuous on $[a, b]$. It is straightforward to show that for any two functions $u(x)$ and $v(x)$ Lagrange's identity

$$u\,\mathcal{L}[v] - v\,\mathcal{L}[u] = \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right] \tag{A.79}$$

holds. Integration gives Green's formula

$$\int_a^b (u\,\mathcal{L}[v] - v\,\mathcal{L}[u])\,dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b.$$

The operator $\mathcal{L}$ is self-adjoint if its associated boundary conditions are such that

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b = 0. \tag{A.80}$$

Possible sets of conditions include the *homogeneous* boundary conditions

$$\alpha_1\psi(a) + \beta_1\psi'(a) = 0, \quad \alpha_2\psi(b) + \beta_2\psi'(b) = 0, \tag{A.81}$$

and the *periodic* boundary conditions

$$\psi(a) = \psi(b), \quad p(a)\psi'(a) = p(b)\psi'(b). \tag{A.82}$$

By imposing one of these sets on (A.78) we obtain a *Sturm–Liouville problem.*

The self-adjoint Sturm–Liouville operator has some nice properties. Each eigenvalue is real, and the eigenvalues form a denumerable set with no cluster point. Moreover, eigenfunctions corresponding to distinct eigenvalues are orthogonal, and the eigenfunctions form a complete set. Hence we can expand any sufficiently smooth function in terms of the eigenfunctions of a problem. We discuss this further below.

A *regular* Sturm–Liouville problem involves a self-adjoint operator $\mathcal{L}$ with $p(x) > 0$ and $\sigma(x) > 0$ everywhere, and the homogeneous boundary conditions (A.81). If p or σ vanishes at an endpoint of $[a, b]$, or an endpoint is at infinity, the problem is *singular.* The harmonic differential equation can form the basis of regular problems, while problems involving Bessel's and Legendre's equations are singular. Regular Sturm–Liouville problems have additional properties. There are infinitely many eigenvalues. There is a smallest eigenvalue but no largest eigenvalue, and the eigenvalues can be ordered as $\lambda_0 < \lambda_1 < \cdots < \lambda_n \cdots$. Associated with each λ_n is a unique (to an arbitrary multiplicative constant) eigenfunction ψ_n that has exactly n zeros in (a, b).

If a problem is singular because $p = 0$ at an endpoint, we can also satisfy (A.80) by demanding that ψ be bounded at that endpoint (a *singularity condition*) and that any regular Sturm–Liouville boundary condition hold at the other endpoint. This is the case for Bessel's and Legendre's equations discussed below.

Orthogonality of the eigenfunctions. Let $\mathcal{L}$ be self-adjoint, and let ψ_m and ψ_n be eigenfunctions associated with λ_m and λ_n, respectively. Then by (A.80) we have

$$\int_a^b (\psi_m(x)\,\mathcal{L}[\psi_n(x)] - \psi_n(x)\,\mathcal{L}[\psi_m(x)])\,dx = 0.$$

But $\mathcal{L}[\psi_n(x)] = -\lambda_n\sigma(x)\psi_n(x)$ and $\mathcal{L}[\psi_m(x)] = -\lambda_m\sigma(x)\psi_m(x)$. Hence

$$(\lambda_m - \lambda_n)\int_a^b \psi_m(x)\psi_n(x)\sigma(x)\,dx = 0,$$

and $\lambda_m \neq \lambda_n$ implies that

$$\int_a^b \psi_m(x)\psi_n(x)\sigma(x)\,dx = 0. \tag{A.83}$$

We say that ψ_m and ψ_n are orthogonal with respect to the weight function $\sigma(x)$.

Eigenfunction expansion of an arbitrary function. If $\mathcal{L}$ is self-adjoint, then its eigenfunctions form a *complete set*. This means that any piecewise smooth function may be represented as a weighted series of eigenfunctions. Specifically, if f and f' are piecewise continuous on $[a, b]$, then f may be represented as the *generalized Fourier series*

$$f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x). \tag{A.84}$$

Convergence of the series is uniform and gives, at any point of (a, b), the average value $[f(x^+) + f(x^-)]/2$ of the one-sided limits $f(x^+)$ and $f(x^-)$ of $f(x)$. The c_n can be found using orthogonality condition (A.83): multiply (A.84) by $\psi_m \sigma$ and integrate to obtain

$$\int_a^b f(x)\psi_m(x)\sigma(x)\,dx = \sum_{n=0}^{\infty} c_n \int_a^b \psi_n(x)\psi_m(x)\sigma(x)\,dx,$$

hence

$$c_n = \frac{\int_a^b f(x)\psi_n(x)\sigma(x)\,dx}{\int_a^b \psi_n^2(x)\sigma(x)\,dx}. \tag{A.85}$$

These coefficients ensure that the series *converges in mean* to f; i.e., the mean-square error

$$\int_a^b \left| f(x) - \sum_{n=0}^{\infty} c_n \psi_n(x) \right|^2 \sigma(x)\,dx$$

is minimized. Truncation to finitely-many terms generally results in oscillations (*Gibb's phenomena*) near points of discontinuity of f. The c_n are easier to compute if the ψ_n are orthonormal with

$$\int_a^b \psi_n^2(x)\sigma(x)\,dx = 1$$

for each n.

Uniqueness of the eigenfunctions. If both ψ_1 and ψ_2 are associated with the same eigenvalue λ, then

$$\mathcal{L}[\psi_1(x)] + \lambda\sigma(x)\psi_1(x) = 0, \quad \mathcal{L}[\psi_2(x)] + \lambda\sigma(x)\psi_2(x) = 0,$$

hence

$$\psi_1(x)\,\mathcal{L}[\psi_2(x)] - \psi_2(x)\,\mathcal{L}[\psi_1(x)] = 0.$$

By (A.79) we have

$$\frac{d}{dx}\left[p(x)\left(\psi_1(x)\frac{d\psi_2(x)}{dx} - \psi_2(x)\frac{d\psi_1(x)}{dx} \right)\right] = 0$$

or

$$p(x)\left(\psi_1(x)\frac{d\psi_2(x)}{dx} - \psi_2(x)\frac{d\psi_1(x)}{dx} \right) = C$$

where C is constant. Either of (A.81) implies $C = 0$, hence

$$\frac{d}{dx}\left(\frac{\psi_2(x)}{\psi_1(x)} \right) = 0$$

so that $\psi_1(x) = K\psi_2(x)$ for some constant K. So under homogeneous boundary conditions, every eigenvalue is associated with a unique eigenfunction.

This is false for the periodic boundary conditions (A.82). Eigenfunction expansion then becomes difficult, as we can no longer assume eigenfunction orthogonality. However, the Gram–Schmidt algorithm may be used to construct orthogonal eigenfunctions. We refer the interested reader to Haberman [79].

The harmonic differential equation. The ordinary differential equation

$$\frac{d^2\psi(x)}{dx^2} = -k^2\psi(x) \tag{A.86}$$

is Sturm–Liouville with $p \equiv 1$, $q \equiv 0$, $\sigma \equiv 1$, and $\lambda = k^2$. Suppose we take $[a, b] = [0, L]$ and adopt the homogeneous boundary conditions

$$\psi(0) = 0 \quad \text{and} \quad \psi(L) = 0. \tag{A.87}$$

Since $p(x) > 0$ and $\sigma(x) > 0$ on $[0, L]$, equations (A.86) and (A.87) form a regular Sturm–Liouville problem. Thus we should have an infinite number of discrete eigenvalues. A power series technique yields the two independent solutions

$$\psi_a(x) = A_a \sin kx, \quad \psi_b(x) = A_b \cos kx,$$

to (A.86); hence by linearity the most general solution is

$$\psi(x) = A_a \sin kx + A_b \cos kx. \tag{A.88}$$

The condition at $x = 0$ gives $A_a \sin 0 + A_b \cos 0 = 0$, hence $A_b = 0$. The other condition then requires

$$A_a \sin kL = 0. \tag{A.89}$$

Since $A_a = 0$ would give $\psi \equiv 0$, we satisfy (A.89) by choosing $k = k_n = n\pi/L$ for $n = 1, 2, \ldots$. Because $\lambda = k^2$, the eigenvalues are

$$\lambda_n = (n\pi/L)^2$$

with corresponding eigenfunctions

$$\psi_n(x) = \sin k_n x.$$

Note that $\lambda = 0$ is not an eigenvalue; eigenfunctions are nontrivial by definition, and $\sin(0\pi x/L) \equiv 0$. Likewise, the differential equation associated with $\lambda = 0$ can be solved easily, but only its trivial solution can fit homogeneous boundary conditions: with $k = 0$, (A.86) becomes $d^2\psi(x)/dx^2 = 0$, giving $\psi(x) = ax + b$; this can satisfy (A.87) only with $a = b = 0$.

These "eigensolutions" obey the properties outlined earlier. In particular the ψ_n are orthogonal,

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2}\delta_{mn},$$

and the eigenfunction expansion of a piecewise continuous function f is given by

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

where, with $\sigma(x) = 1$ in (A.85), we have

$$c_n = \frac{\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx} = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Hence we recover the standard Fourier sine series for $f(x)$.

With little extra effort we can examine the eigenfunctions resulting from enforcement of the periodic boundary conditions

$$\psi(0) = \psi(L) \quad \text{and} \quad \psi'(0) = \psi'(L).$$

The general solution (A.88) still holds, so we have the choices $\psi(x) = \sin kx$ and $\psi(x) = \cos kx$. Evidently both

$$\psi(x) = \sin\left(\frac{2n\pi x}{L}\right) \quad \text{and} \quad \psi(x) = \cos\left(\frac{2n\pi x}{L}\right)$$

satisfy the boundary conditions for $n = 1, 2, \ldots$. Thus each eigenvalue $(2n\pi/L)^2$ is associated with two eigenfunctions.

Bessel's differential equation. Bessel's equation

$$\frac{d}{dx}\left(x \frac{d\psi(x)}{dx}\right) + \left(k^2 x - \frac{\nu^2}{x}\right) \psi(x) = 0 \tag{A.90}$$

occurs when problems are solved in circular-cylindrical coordinates. Comparison with (A.77) shows that $\lambda = k^2$, $p(x) = x$, $q(x) = -\nu^2/x$, and $\sigma(x) = x$. We take $[a, b] = [0, L]$ along with the boundary conditions

$$\psi(L) = 0 \quad \text{and} \quad |\psi(0)| < \infty. \tag{A.91}$$

Although the resulting Sturm–Liouville problem is singular, the specified conditions (A.91) maintain satisfaction of (A.80). The eigenfunctions are orthogonal because (A.80) is satisfied by having $\psi(L) = 0$ and $p(x)\, d\psi(x)/dx \to 0$ as $x \to 0$.

As a second-order ordinary differential equation, (A.90) has two solutions denoted by

$$J_\nu(kx) \quad \text{and} \quad N_\nu(kx),$$

and termed *Bessel functions*. Their properties are summarized in Appendix E.1. The function $J_\nu(x)$, the Bessel function of the first kind and order ν, is well-behaved in $[0, L]$. The function $N_\nu(x)$, the Bessel function of the second kind and order ν, is unbounded at $x = 0$; hence it is excluded as an eigenfunction of the Sturm–Liouville problem.

The condition at $x = L$ shows that the eigenvalues are defined by

$$J_\nu(kL) = 0.$$

We denote the mth root of $J_\nu(x) = 0$ by $p_{\nu m}$. Then

$$k_{\nu m} = \sqrt{\lambda_{\nu m}} = p_{\nu m}/L.$$

The infinitely many eigenvalues are ordered as $\lambda_{\nu 1} < \lambda_{\nu 2} < \ldots$. Associated with eigenvalue $\lambda_{\nu m}$ is a single eigenfunction $J_\nu(\sqrt{\lambda_{\nu m}}x)$. The orthogonality relation is

$$\int_0^L J_\nu\left(\frac{p_{\nu m}}{L}x\right) J_\nu\left(\frac{p_{\nu n}}{L}x\right) x\, dx = 0, \quad m \neq n.$$

Since the eigenfunctions are also complete, we can expand any piecewise continuous function f in a *Fourier–Bessel series*

$$f(x) = \sum_{m=1}^{\infty} c_m J_\nu \left(p_{\nu m} \frac{x}{L} \right), \quad 0 \le x \le L, \ \nu > -1.$$

By (A.85) and (E.22) we have

$$c_m = \frac{2}{L^2 J_{\nu+1}^2(p_{\nu m})} \int_0^L f(x) J_\nu \left(p_{\nu m} \frac{x}{L} \right) x \, dx.$$

The associated Legendre equation. Legendre's equation occurs when problems are solved in spherical coordinates. It is often written in one of two forms. Letting θ be the polar angle of spherical coordinates ($0 \le \theta \le \pi$), the equation is

$$\frac{d}{d\theta} \left(\sin\theta \frac{d\psi(\theta)}{d\theta} \right) + \left(\lambda \sin\theta - \frac{m^2}{\sin\theta} \right) \psi(\theta) = 0.$$

This is Sturm–Liouville with $p(\theta) = \sin\theta$, $\sigma(\theta) = \sin\theta$, and $q(\theta) = -m^2/\sin\theta$. The boundary conditions

$$|\psi(0)| < \infty \quad \text{and} \quad |\psi(\pi)| < \infty$$

define a singular problem: the conditions are not homogeneous, $p(\theta) = 0$ at both endpoints, and $q(\theta) < 0$. Despite this, the Legendre problem does share properties of a regular Sturm–Liouville problem — including eigenfunction orthogonality and completeness.

Using $x = \cos\theta$, we can put Legendre's equation into its other common form

$$\frac{d}{dx} \left([1 - x^2] \frac{d\psi(x)}{dx} \right) + \left(\lambda - \frac{m^2}{1 - x^2} \right) \psi(x) = 0, \tag{A.92}$$

where $-1 \le x \le 1$. It is found that ψ is bounded at $x = \pm 1$ only if

$$\lambda = n(n+1)$$

where $n \ge m$ is an integer. These λ are the eigenvalues of the Sturm–Liouville problem, and the corresponding $\psi_n(x)$ are the eigenfunctions.

As a second-order partial differential equation, (A.92) has two solutions known as *associated Legendre functions*. The solution bounded at both $x = \pm 1$ is the associated Legendre function of the first kind, denoted $P_n^m(x)$. The second solution, unbounded at $x = \pm 1$, is the associated Legendre function of the second kind $Q_n^m(x)$. Appendix E.2 tabulates some properties of these functions.

For fixed m, each λ_{mn} is associated with a single eigenfunction $P_n^m(x)$. Since $P_n^m(x)$ is bounded at $x = \pm 1$, and since $p(\pm 1) = 0$, the eigenfunctions obey Lagrange's identity (A.79), hence are orthogonal on $[-1, 1]$ with respect to the weight function $\sigma(x) = 1$. Evaluation of the orthogonality integral leads to

$$\int_{-1}^{1} P_l^m(x) P_n^m(x) \, dx = \delta_{ln} \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \tag{A.93}$$

or equivalently

$$\int_0^\pi P_l^m(\cos\theta) P_n^m(\cos\theta) \sin\theta \, d\theta = \delta_{ln} \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}.$$

For $m = 0$, $P_n^m(x)$ is a polynomial of degree n. Each such *Legendre polynomial*, denoted $P_n(x)$, is given by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2 - 1)^n}{dx^n}.$$

It turns out that

$$P_n^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m P_n(x)}{dx^m},$$

giving $P_n^m(x) = 0$ for $m > n$.

Because the Legendre polynomials form a complete set in the interval $[-1, 1]$, we may expand any sufficiently smooth function in a *Fourier–Legendre series*

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x).$$

Convergence in mean is guaranteed if

$$c_n = \frac{2n + 1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx,$$

found using (A.85) along with (A.93).

In practice, the associated Legendre functions appear along with exponential functions in the solutions to spherical boundary value problems. The combined functions are known as *spherical harmonics*, and form solutions to two-dimensional Sturm–Liouville problems. We consider these next.

Higher-dimensional SL problems: Helmholtz's equation. Replacing d/dx by ∇, we generalize the Sturm–Liouville equation to higher dimensions:

$$\nabla \cdot [p(\mathbf{r}) \nabla \psi(\mathbf{r})] + q(\mathbf{r}) \psi(\mathbf{r}) + \lambda \sigma(\mathbf{r}) \psi(\mathbf{r}) = 0,$$

where q, p, σ, ψ are real functions. Of particular interest is the case $q(\mathbf{r}) = 0$, $p(\mathbf{r}) = \sigma(\mathbf{r}) = 1$, giving the Helmholtz equation

$$\nabla^2 \psi(\mathbf{r}) + \lambda \psi(\mathbf{r}) = 0. \tag{A.94}$$

In most boundary value problems, ψ or its normal derivative is specified on the surface of a bounded region. We obtain a three-dimensional analogue to the regular Sturm–Liouville problem by assuming the homogeneous boundary conditions

$$\alpha \psi(\mathbf{r}) + \beta \hat{\mathbf{n}} \cdot \nabla \psi(\mathbf{r}) = 0 \tag{A.95}$$

on the closed surface, where $\hat{\mathbf{n}}$ is the outward unit normal.

The problem consisting of (A.94) and (A.95) has properties analogous to those of the regular one-dimensional Sturm–Liouville problem. All eigenvalues are real. There are infinitely many eigenvalues. There is a smallest eigenvalue but no largest eigenvalue. However, associated with an eigenvalue there may be many eigenfunctions $\psi_\lambda(\mathbf{r})$. The eigenfunctions are orthogonal with

$$\int_V \psi_{\lambda_1}(\mathbf{r}) \psi_{\lambda_2}(\mathbf{r})\, dV = 0, \qquad \lambda_1 \neq \lambda_2.$$

They are also complete and can be used to represent any piecewise smooth function $f(\mathbf{r})$ according to

$$f(\mathbf{r}) = \sum_{\lambda} a_\lambda \psi_\lambda(\mathbf{r}),$$

which converges in mean when

$$a_{\lambda_m} = \frac{\int_V f(\mathbf{r})\psi_{\lambda_m}(\mathbf{r})\,dV}{\int_V \psi^2_{\lambda_m}(\mathbf{r})\,dV}.$$

These properties are shared by the two-dimensional eigenvalue problem involving an open surface S with boundary contour Γ.

Spherical harmonics. We now inspect solutions to the two-dimensional eigenvalue problem

$$\nabla^2 Y(\theta,\phi) + \frac{\lambda}{a^2} Y(\theta,\phi) = 0$$

over the surface of a sphere of radius a. Since the sphere has no boundary contour, we demand that $Y(\theta,\phi)$ be bounded in θ and periodic in ϕ. In the next section we shall apply separation of variables and show that

$$Y_{nm}(\theta,\phi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\, P_n^m(\cos\theta) e^{jm\phi}$$

where $\lambda = n(n+1)$. Note that Q_n^m does not appear as it is not bounded at $\theta = 0, \pi$. The functions Y_{nm} are called *spherical harmonics* (sometimes *zonal* or *tesseral* harmonics, depending on the values of n and m). As expressed above they are in orthonormal form, because the orthogonality relationships for the exponential and associated Legendre functions yield

$$\int_{-\pi}^{\pi}\int_{0}^{\pi} Y^*_{n'm'}(\theta,\phi) Y_{nm}(\theta,\phi)\sin\theta\,d\theta\,d\phi = \delta_{n'n}\delta_{m'm}. \tag{A.96}$$

As solutions to the Sturm–Liouville problem, these functions form a complete set on the surface of a sphere. Hence they can be used to represent any piecewise smooth function $f(\theta,\phi)$ as

$$f(\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{nm} Y_{nm}(\theta,\phi),$$

where

$$a_{nm} = \int_{-\pi}^{\pi}\int_{0}^{\pi} f(\theta,\phi) Y^*_{nm}(\theta,\phi)\sin\theta\,d\theta\,d\phi$$

by (A.96). The summation index m ranges from $-n$ to n because $P_n^m = 0$ for $m > n$. For negative index we can use

$$Y_{n,-m}(\theta,\phi) = (-1)^m Y^*_{nm}(\theta,\phi).$$

Some properties of the spherical harmonics are tabulated in Appendix E.3.

Separation of variables

We now consider a technique that finds widespread application in solving boundary value problems, applying as it does to many important partial differential equations such as Laplace's equation, the diffusion equation, and the scalar and vector wave equations. These equations are related to the scalar Helmholtz equation

$$\nabla^2\psi(\mathbf{r}) + k^2\psi(\mathbf{r}) = 0 \tag{A.97}$$

where k is a complex constant. If k is real and we supply the appropriate boundary conditions, we have the higher-dimensional Sturm–Liouville problem with $\lambda = k^2$. We shall not pursue the extension of Sturm–Liouville theory to complex values of k.

Laplace's equation is Helmholtz's equation with $k = 0$. With $\lambda = k^2 = 0$ it might appear that Laplace's equation does not involve eigenvalues; however, separation of variables does lead us to lower-dimensional eigenvalue problems to which our previous methods apply. Solutions to the scalar or vector wave equations usually begin with Fourier transformation on the time variable, or with an initial separation of the time variable to reach a Helmholtz form.

The separation of variables idea is simple. We seek a solution to (A.97) in the form of a product of functions each of a single variable. If ψ depends on all three spatial dimensions, then we seek a solution of the type

$$\psi(u, v, w) = U(u)V(v)W(w),$$

where u, v, and w are the coordinate variables used to describe the problem. If ψ depends on only two coordinates, we may seek a product solution involving two functions each dependent on a single coordinate; alternatively, may use the three-variable solution and choose constants so that the result shows no variation with one coordinate. The Helmholtz equation is considered *separable* if it can be reduced to a set of independent ordinary differential equations, each involving a single coordinate variable. The ordinary differential equations, generally of second order, can be solved by conventional techniques resulting in solutions of the form

$$\begin{aligned} U(u) &= A_u U_A(u, k_u, k_v, k_w) + B_u U_B(u, k_u, k_v, k_w), \\ V(v) &= A_v V_A(v, k_u, k_v, k_w) + B_v V_B(v, k_u, k_v, k_w), \\ W(w) &= A_w W_A(w, k_u, k_v, k_w) + B_w W_B(w, k_u, k_v, k_w). \end{aligned}$$

The constants k_u, k_v, k_w are called *separation constants* and are found, along with the amplitude constants A, B, by applying boundary conditions appropriate for a given problem. At least one separation constant depends on (or equals) k, so only two are independent. In many cases k_u, k_v, and k_w become the discrete eigenvalues of the respective differential equations, and correspond to eigenfunctions $U(u, k_u, k_v, k_w)$, $V(v, k_u, k_v, k_w)$, and $W(w, k_u, k_v, k_w)$. In other cases the separation constants form a continuous spectrum of values, often when a Fourier transform solution is employed.

The Helmholtz equation can be separated in eleven different orthogonal coordinate systems [134]. Undoubtedly the most important of these are the rectangular, circular-cylindrical, and spherical systems, and we shall consider each in detail. We do note, however, that separability in a certain coordinate system does not imply that all problems expressed in that coordinate system can be easily handled using the resulting solutions. Only when the geometry and boundary conditions are simple do the solutions lend themselves to easy application; often other solution techniques are more appropriate.

Although rigorous conditions can be set forth to guarantee solvability by separation of variables [119], we prefer the following, more heuristic list:

1. Use a coordinate system that allows the given partial differential equation to separate into ordinary differential equations.
2. The problem's boundaries must be such that those boundaries not at infinity coincide with a single level surface of the coordinate system.
3. Use superposition to reduce the problem to one involving a single nonhomogeneous boundary condition. Then:
   (a) Solve the resulting Sturm–Liouville problem in one or two dimensions, with homogeneous boundary conditions on all boundaries. Then use a discrete eigenvalue expansion (Fourier series) and eigenfunction orthogonality to satisfy the remaining nonhomogeneous condition.
   (b) If a Sturm–Liouville problem cannot be formulated with the homogeneous boundary conditions (because, for instance, one boundary is at infinity), use a Fourier integral (continuous expansion) to satisfy the remaining nonhomogeneous condition.

If a Sturm–Liouville problem cannot be formulated, discovering the form of the integral transform to use can be difficult. In these cases other approaches, such as conformal mapping, may prove easier.

Solutions in rectangular coordinates. In rectangular coordinates the Helmholtz equation is

$$\frac{\partial^2\psi(x,y,z)}{\partial x^2}+\frac{\partial^2\psi(x,y,z)}{\partial y^2}+\frac{\partial^2\psi(x,y,z)}{\partial z^2}+k^2\psi(x,y,z)=0. \tag{A.98}$$

We seek a solution of the form $\psi(x,y,z)=X(x)Y(y)Z(z)$; substitution into (A.98) followed by division through by $X(x)Y(y)Z(z)$ gives

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}+\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}+\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}=-k^2. \tag{A.99}$$

At this point we require the *separation argument*. The left-hand side of (A.99) is a sum of three functions, each involving a single independent variable, whereas the right-hand side is constant. But the only functions of independent variables that always sum to a constant are themselves constants. Thus we may equate each term on the left to a different constant:

$$\begin{aligned}\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}&=-k_x^2,\\ \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}&=-k_y^2,\\ \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}&=-k_z^2,\end{aligned} \tag{A.100}$$

provided that

$$k_x^2+k_y^2+k_z^2=k^2.$$

The negative signs in (A.100) have been introduced for convenience.

Let us discuss the general solutions of equations (A.100). If $k_x = 0$, the two independent solutions for $X(x)$ are

$$X(x) = a_x x \quad \text{and} \quad X(x) = b_x$$

where a_x and b_x are constants. If $k_x \neq 0$, solutions may be chosen from the list of functions

$$e^{-jk_x x}, \quad e^{jk_x x}, \quad \sin k_x x, \quad \cos k_x x,$$

any two of which are independent. Because

$$\sin x = (e^{jx} - e^{-jx})/2j \quad \text{and} \quad \cos x = (e^{jx} + e^{-jx})/2, \tag{A.101}$$

the six possible solutions for $k_x \neq 0$ are

$$X(x) = \begin{cases} A_x e^{jk_x x} + B_x e^{-jk_x x}, \\ A_x \sin k_x x + B_x \cos k_x x, \\ A_x \sin k_x x + B_x e^{-jk_x x}, \\ A_x e^{jk_x x} + B_x \sin k_x x, \\ A_x e^{jk_x x} + B_x \cos k_x x, \\ A_x e^{-jk_x x} + B_x \cos k_x x. \end{cases} \tag{A.102}$$

We may base our choice on convenience (e.g., the boundary conditions may be amenable to one particular form) or on the desired behavior of the solution (e.g., standing waves vs. traveling waves). If k is complex, then so may be k_x, k_y, or k_z; observe that with imaginary arguments the complex exponentials are actually real exponentials, and the trigonometric functions are actually hyperbolic functions.

The solutions for $Y(y)$ and $Z(z)$ are identical to those for $X(x)$. We can write, for instance,

$$X(x) = \begin{cases} A_x e^{jk_x x} + B_x e^{-jk_x x}, & k_x \neq 0, \\ a_x x + b_x, & k_x = 0, \end{cases} \tag{A.103}$$

$$Y(y) = \begin{cases} A_y e^{jk_y y} + B_y e^{-jk_y y}, & k_y \neq 0, \\ a_y y + b_y, & k_y = 0, \end{cases} \tag{A.104}$$

$$Z(z) = \begin{cases} A_z e^{jk_z z} + B_z e^{-jk_z z}, & k_z \neq 0, \\ a_z z + b_z, & k_z = 0. \end{cases} \tag{A.105}$$

Examples. Let us begin by solving the simple equation

$$\nabla^2 V(x) = 0.$$

Since V depends only on x we can use (A.103)–(A.105) with $k_y = k_z = 0$ and $a_y = a_z = 0$. Moreover $k_x = 0$ because $k_x^2 + k_y^2 + k_z^2 = k^2 = 0$ for Laplace's equation. The general solution is therefore

$$V(x) = a_x x + b_x.$$

Boundary conditions must be specified to determine a_x and b_x; for instance, the conditions $V(0) = 0$ and $V(L) = V_0$ yield $V(x) = V_0 x/L$.

Next let us solve

$$\nabla^2 \psi(x, y) = 0.$$

We produce a lack of z-dependence in ψ by letting $k_z = 0$ and choosing $a_z = 0$. Moreover, $k_x^2 = -k_y^2$ since Laplace's equation requires $k = 0$. This leads to three possibilities. If $k_x = k_y = 0$, we have the product solution

$$\psi(x, y) = (a_x x + b_x)(a_y y + b_y). \tag{A.106}$$

If k_y is real and nonzero, then

$$\psi(x, y) = (A_x e^{-k_y x} + B_x e^{k_y x})(A_y e^{jk_y y} + B_y e^{-jk_y y}). \tag{A.107}$$

Using the relations

$$\sinh u = (e^u - e^{-u})/2 \quad \text{and} \quad \cosh u = (e^u + e^{-u})/2 \tag{A.108}$$

along with (A.101), we can rewrite (A.107) as

$$\psi(x, y) = (A_x \sinh k_y x + B_x \cosh k_y x)(A_y \sin k_y y + B_y \cos k_y y). \tag{A.109}$$

(We can reuse the constant names A_x, B_x, A_y, B_y, since the constants are unknown at this point.) If k_x is real and nonzero we have

$$\psi(x, y) = (A_x \sin k_x x + B_x \cos k_x x)(A_y \sinh k_x y + B_y \cosh k_x y). \tag{A.110}$$

Consider the problem consisting of Laplace's equation

$$\nabla^2 V(x, y) = 0 \tag{A.111}$$

holding in the region $0 < x < L_1$, $0 < y < L_2$, $-\infty < z < \infty$, together with the boundary conditions

$$V(0, y) = V_1, \quad V(L_1, y) = V_2, \quad V(x, 0) = V_3, \quad V(x, L_2) = V_4.$$

The solution $V(x, y)$ represents the potential within a conducting tube with each wall held at a different potential. Superposition applies: since Laplace's equation is linear we can write the solution as the sum of solutions to four different sub-problems. Each sub-problem has homogeneous boundary conditions on one independent variable and inhomogeneous conditions on the other, giving a Sturm–Liouville problem in one of the variables. For instance, let us examine the solutions found above in relation to the sub-problem consisting of Laplace's equation (A.111) in the region $0 < x < L_1$, $0 < y < L_2$, $-\infty < z < \infty$, subject to the conditions

$$V(0, y) = V(L_1, y) = V(x, 0) = 0, \quad V(x, L_2) = V_4 \neq 0.$$

First we try (A.106). The boundary condition at $x = 0$ gives

$$V(0, y) = (a_x(0) + b_x)(a_y y + b_y) = 0,$$

which holds for all $y \in (0, L_2)$ only if $b_x = 0$. The condition at $x = L_1$,

$$V(L_1, y) = a_x L_1(a_y y + b_y) = 0,$$

then requires $a_x = 0$. But $a_x = b_x = 0$ gives $V(x, y) = 0$, and the condition at $y = L_2$ cannot be satisfied; clearly (A.106) was inappropriate. Next we examine (A.109). The condition at $x = 0$ gives

$$V(0, y) = (A_x \sinh 0 + B_x \cosh 0)(A_y \sin k_y y + B_y \cos k_y y) = 0,$$

hence $B_x = 0$. The condition at $x = L_1$ implies

$$V(L_1, y) = [A_x \sinh(k_y L_1)](A_y \sin k_y y + B_y \cos k_y y) = 0.$$

This can hold if either $A_x = 0$ or $k_y = 0$, but the case $k_y = 0$ $(= k_x)$ was already considered. Thus $A_x = 0$ and the trivial solution reappears. Our last candidate is (A.110). The condition at $x = 0$ requires

$$V(0, y) = (A_x \sin 0 + B_x \cos 0)(A_y \sinh k_x y + B_y \cosh k_x y) = 0,$$

which implies $B_x = 0$. Next we have

$$V(L_1, y) = [A_x \sin(k_x L_1)](A_y \sinh k_y y + B_y \cosh k_y y) = 0.$$

We avoid $A_x = 0$ by setting $\sin(k_x L_1) = 0$ so that $k_{x_n} = n\pi/L_1$ for $n = 1, 2, \ldots$. (Here $n = 0$ is omitted because it would produce a trivial solution.) These are eigenvalues corresponding to the eigenfunctions $X_n(x) = \sin(k_{x_n} x)$, and were found in § A.4 for the harmonic equation. At this point we have a family of solutions

$$V_n(x, y) = \sin(k_{x_n} x)[A_{y_n} \sinh(k_{x_n} y) + B_{y_n} \cosh(k_{x_n} y)], \quad n = 1, 2, \ldots.$$

The subscript n on the left identifies V_n as the eigensolution associated with eigenvalue k_{x_n}. It remains to satisfy boundary conditions at $y = 0, L_2$. At $y = 0$ we have

$$V_n(x, 0) = \sin(k_{x_n} x)[A_{y_n} \sinh 0 + B_{y_n} \cosh 0] = 0,$$

hence $B_{y_n} = 0$ and

$$V_n(x, y) = A_{y_n} \sin(k_{x_n} x) \sinh(k_{x_n} y), \quad n = 1, 2, \ldots. \tag{A.112}$$

It is clear that no single eigensolution (A.112) can satisfy the one remaining boundary condition. However, we are guaranteed that a series of solutions can represent the constant potential on $y = L_2$; recall that as a solution to a regular Sturm–Liouville problem, the trigonometric functions are complete (hence they could represent any well-behaved function on the interval $0 \le x \le L_1$). In fact, the resulting series is a Fourier sine series for the constant potential at $y = L_2$. So let

$$V(x, y) = \sum_{n=1}^{\infty} V_n(x, y) = \sum_{n=1}^{\infty} A_{y_n} \sin(k_{x_n} x) \sinh(k_{x_n} y).$$

The remaining boundary condition requires

$$V(x, L_2) = \sum_{n=1}^{\infty} A_{y_n} \sin(k_{x_n} x) \sinh(k_{x_n} L_2) = V_4.$$

The constants A_{y_n} can be found using orthogonality; multiplying through by $\sin(k_{x_m} x)$ and integrating, we have

$$\sum_{n=1}^{\infty} A_{y_n} \sinh(k_{x_n} L_2) \int_0^{L_1} \sin\left(\frac{m\pi x}{L_1}\right) \sin\left(\frac{n\pi x}{L_1}\right) dx = V_4 \int_0^{L_1} \sin\left(\frac{m\pi x}{L_1}\right) dx.$$

The integral on the left equals $\delta_{mn} L_1/2$ where δ_{mn} is the Kronecker delta given by

$$\delta_{mn} = \begin{cases} 1, & m = n, \\ 0, & n \neq m. \end{cases}$$

After evaluating the integral on the right we obtain

$$\sum_{n=1}^{\infty} A_{y_n} \delta_{mn} \sinh(k_{x_n} L_2) = \frac{2V_4(1-\cos m\pi)}{m\pi},$$

hence

$$A_{y_m} = \frac{2V_4(1-\cos m\pi)}{m\pi \sinh(k_{x_m} L_2)}.$$

The final solution for this sub-problem is therefore

$$V(x, y) = \sum_{n=1}^{\infty} \frac{2V_4(1-\cos n\pi)}{n\pi \sinh\left(\frac{n\pi L_2}{L_1}\right)} \sin\left(\frac{n\pi x}{L_1}\right) \sinh\left(\frac{n\pi y}{L_1}\right).$$

The remaining three sub-problems are left for the reader.

Let us again consider (A.111), this time for

$$0 \le x \le L_1, \quad 0 \le y < \infty, \quad -\infty < z < \infty,$$

and subject to

$$V(0, y) = V(L_1, y) = 0, \quad V(x, 0) = V_0.$$

Let us try the solution form that worked in the previous example:

$$V(x, y) = [A_x \sin(k_x x) + B_x \cos(k_x x)][A_y \sinh(k_x y) + B_y \cosh(k_x y)].$$

The boundary conditions at $x = 0, L_1$ are the same as before so we have

$$V_n(x, y) = \sin(k_{x_n} x)[A_{y_n} \sinh(k_{x_n} y) + B_{y_n} \cosh(k_{x_n} y)], \quad n = 1, 2, \ldots.$$

To find A_{y_n} and B_{y_n} we note that V cannot grow without bound as $y \to \infty$. Individually the hyperbolic functions grow exponentially. However, using (A.108) we see that $B_{y_n} = -A_{y_n}$ gives

$$V_n(x, y) = A_{y_n} \sin(k_{x_n} x) e^{-k_{x_n} y}$$

where A_{y_n} is a new unknown constant. (Of course, we could have chosen this exponential dependence at the beginning.) Lastly, we can impose the boundary condition at $y = 0$ on the infinite series of eigenfunctions

$$V(x, y) = \sum_{n=1}^{\infty} A_{y_n} \sin(k_{x_n} x) e^{-k_{x_n} y}$$

to find A_{y_n}. The result is

$$V(x, y) = \sum_{n=1}^{\infty} \frac{2V_0}{\pi n}(1-\cos n\pi) \sin(k_{x_n} x) e^{-k_{x_n} y}.$$

As in the previous example, the solution is a discrete superposition of eigenfunctions.

The problem consisting of (A.111) holding for

$$0 \le x \le L_1, \quad 0 \le y < \infty, \quad -\infty < z < \infty,$$

along with

$$V(0, y) = 0, \quad V(L_1, y) = V_0 e^{-ay}, \quad V(x, 0) = 0,$$

requires a continuous superposition of eigenfunctions to satisfy the boundary conditions. Let us try

$$V(x, y) = [A_x \sinh k_y x + B_x \cosh k_y x][A_y \sin k_y y + B_y \cos k_y y].$$

The conditions at $x = 0$ and $y = 0$ require that $B_x = B_y = 0$. Thus

$$V_{k_y}(x, y) = A \sinh k_y x \sin k_y y.$$

A single function of this form cannot satisfy the remaining condition at $x = L_1$. So we form a continuous superposition

$$V(x, y) = \int_0^\infty A(k_y) \sinh k_y x \sin k_y y \, dk_y. \tag{A.113}$$

By the condition at $x = L_1$

$$\int_0^\infty A(k_y) \sinh(k_y L_1) \sin k_y y \, dk_y = V_0 e^{-ay}. \tag{A.114}$$

We can find the amplitude function $A(k_y)$ by using the orthogonality property

$$\delta(y - y') = \frac{2}{\pi} \int_0^\infty \sin xy \sin xy' \, dx. \tag{A.115}$$

Multiplying both sides of (A.114) by $\sin k'_y y$ and integrating, we have

$$\int_0^\infty A(k_y) \sinh(k_y L_1) \left[\int_0^\infty \sin k_y y \sin k'_y y \, dy \right] dk_y = \int_0^\infty V_0 e^{-ay} \sin k'_y y \, dy.$$

We can evaluate the term in brackets using (A.115) to obtain

$$\int_0^\infty A(k_y) \sinh(k_y L_1) \frac{\pi}{2} \delta(k_y - k'_y) \, dk_y = \int_0^\infty V_0 e^{-ay} \sin k'_y y \, dy,$$

hence

$$\frac{\pi}{2} A(k'_y) \sinh(k'_y L_1) = V_0 \int_0^\infty e^{-ay} \sin k'_y y \, dy.$$

We then evaluate the integral on the right, solve for $A(k_y)$, and substitute into (A.113) to obtain

$$V(x, y) = \frac{2V_0}{\pi} \int_0^\infty \frac{k_y}{a^2 + k_y^2} \frac{\sinh(k_y x)}{\sinh(k_y L_1)} \sin k_y y \, dk_y.$$

Note that our application of the orthogonality property is merely a calculation of the inverse Fourier sine transform. Thus we could have found the amplitude coefficient by reference to a table of transforms.

We can use the Fourier transform solution even when the domain is infinite in more than one dimension. Suppose we solve (A.111) in the region

$$0 \le x < \infty, \quad 0 \le y < \infty, \quad -\infty < z < \infty,$$

subject to

$$V(0, y) = V_0 e^{-ay}, \quad V(x, 0) = 0.$$

Because of the condition at $y = 0$ let us use

$$V(x, y) = (A_x e^{-k_y x} + B_x e^{k_y x})(A_y \sin k_y y + B_y \cos k_y y).$$

The solution form

$$V_{k_y}(x, y) = B(k_y) e^{-k_y x} \sin k_y y$$

satisfies the finiteness condition and the homogeneous condition at $y = 0$. The remaining condition can be satisfied by a continuous superposition of solutions:

$$V(x, y) = \int_0^\infty B(k_y) e^{-k_y x} \sin k_y y \, dk_y.$$

We must have

$$V_0 e^{-ay} = \int_0^\infty B(k_y) \sin k_y y \, dk_y.$$

Use of the orthogonality relationship (A.115) yields the amplitude spectrum $B(k_y)$, and we find that

$$V(x, y) = \frac{2}{\pi} \int_0^\infty e^{-k_y x} \frac{k_y}{a^2 + k_y^2} \sin k_y y \, dk_y. \tag{A.116}$$

As a final example in rectangular coordinates let us consider a problem in which ψ depends on all three variables:

$$\nabla^2 \psi(x, y, z) + k^2 \psi(x, y, z) = 0$$

for

$$0 \le x \le L_1, \quad 0 \le y \le L_2, \quad 0 \le z \le L_3,$$

subject to

$$\begin{aligned} \psi(0, y, z) &= \psi(L_1, y, z) = 0, \\ \psi(x, 0, z) &= \psi(x, L_2, z) = 0, \\ \psi(x, y, 0) &= \psi(x, y, L_3) = 0. \end{aligned}$$

Here $k \neq 0$ is a constant. This is a three-dimensional eigenvalue problem as described in § A.4, where $\lambda = k^2$ are the eigenvalues and the closed surface is a rectangular box. Physically, the wave function ψ represents the so-called *eigenvalue* or *normal mode* solutions for the "TM modes" of a rectangular cavity. Since $k_x^2 + k_y^2 + k_z^2 = k^2$, we might have one or two separation constants equal to zero, but not all three. We find, however, that the only solution with a zero separation constant that can fit the boundary conditions is the trivial solution. In light of the boundary conditions and because we expect standing waves in the box, we take

$$\begin{aligned} \psi(x, y, z) = {} & [A_x \sin(k_x x) + B_x \cos(k_x x)] \cdot \\ & \cdot \; [A_y \sin(k_y y) + B_y \cos(k_y y)] \cdot \\ & \cdot \; [A_z \sin(k_z z) + B_z \cos(k_z z)]. \end{aligned}$$

The conditions $\psi(0, y, z) = \psi(x, 0, z) = \psi(x, y, 0) = 0$ give $B_x = B_y = B_z = 0$. The conditions at $x = L_1$, $y = L_2$, and $z = L_3$ require the separation constants to assume the discrete values $k_x = k_{x_m} = m\pi/L_1$, $k_y = k_{y_n} = n\pi/L_2$, and $k_z = k_{z_p} = p\pi/L_3$, where $k_{x_m}^2 + k_{y_n}^2 + k_{z_p}^2 = k_{mnp}^2$ and $m, n, p = 1, 2, \ldots$. Associated with each of these

eigenvalues is an eigenfunction of a one-dimensional Sturm–Liouville problem. For the three-dimensional problem, an eigenfunction

$$\psi_{mnp}(x, y, z) = A_{mnp} \sin(k_{x_m} x) \sin(k_{y_n} y) \sin(k_{z_p} z)$$

is associated with each three-dimensional eigenvalue k_{mnp}. Each choice of m, n, p produces a discrete cavity resonance frequency at which the boundary conditions can be satisfied. Depending on the values of $L_{1,2,3}$, we may have more than one eigenfunction associated with an eigenvalue. For example, if $L_1 = L_2 = L_3 = L$ then $k_{121} = k_{211} = k_{112} = \sqrt{6}\pi/L$. However, the eigenfunctions associated with this single eigenvalue are all different:

$$\begin{aligned}
\psi_{121} &= \sin(k_{x_1} x) \sin(k_{y_2} y) \sin(k_{z_1} z), \\
\psi_{211} &= \sin(k_{x_2} x) \sin(k_{y_1} y) \sin(k_{z_1} z), \\
\psi_{112} &= \sin(k_{x_1} x) \sin(k_{y_1} y) \sin(k_{z_2} z).
\end{aligned}$$

When more than one cavity mode corresponds to a given resonant frequency, we call the modes *degenerate*. By completeness, we can represent any well-behaved function as

$$f(x, y, z) = \sum_{m,n,p} A_{mnp} \sin(k_{x_m} x) \sin(k_{y_n} y) \sin(k_{z_p} z).$$

The A_{mnp} are found using orthogonality. When such expansions are used to solve problems involving objects (such as excitation probes) inside the cavity, they are termed *normal mode expansions* of the cavity field.

Solutions in cylindrical coordinates. In cylindrical coordinates the Helmholtz equation is

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi(\rho, \phi, z)}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi(\rho, \phi, z)}{\partial \phi^2} + \frac{\partial^2 \psi(\rho, \phi, z)}{\partial z^2} + k^2 \psi(\rho, \phi, z) = 0. \quad \text{(A.117)}$$

With $\psi(\rho, \phi, z) = P(\rho)\Phi(\phi)Z(z)$ we obtain

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial (P\Phi Z)}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 (P\Phi Z)}{\partial \phi^2} + \frac{\partial^2 (P\Phi Z)}{\partial z^2} + k^2 (P\Phi Z) = 0;$$

carrying out the ρ derivatives and dividing through by $P\Phi Z$ we have

$$-\frac{1}{Z}\frac{d^2 Z}{dz^2} = k^2 + \frac{1}{\rho^2 \Phi}\frac{d^2 \Phi}{d\phi^2} + \frac{1}{\rho P}\frac{dP}{d\rho} + \frac{1}{P}\frac{d^2 P}{d\rho^2}.$$

The left side depends on z while the right side depends on ρ and ϕ, hence both must equal the same constant k_z^2:

$$-\frac{1}{Z}\frac{d^2 Z}{dz^2} = k_z^2, \quad \text{(A.118)}$$

$$k^2 + \frac{1}{\rho^2 \Phi}\frac{d^2 \Phi}{d\phi^2} + \frac{1}{\rho P}\frac{dP}{d\rho} + \frac{1}{P}\frac{d^2 P}{d\rho^2} = k_z^2. \quad \text{(A.119)}$$

We have separated the z-dependence from the dependence on the other variables. For the harmonic equation (A.118),

$$Z(z) = \begin{cases} A_z \sin k_z z + B_z \cos k_z z, & k_z \neq 0, \\ a_z z + b_z, & k_z = 0. \end{cases} \quad \text{(A.120)}$$

Of course we could use exponentials or a combination of exponentials and trigonometric functions instead. Rearranging (A.119) and multiplying through by ρ^2, we obtain

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \left(k^2 - k_z^2\right)\rho^2 + \frac{\rho}{P}\frac{dP}{d\rho} + \frac{\rho^2}{P}\frac{d^2P}{d\rho^2}.$$

The left and right sides depend only on ϕ and ρ, respectively; both must equal some constant k_ϕ^2:

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = k_\phi^2, \tag{A.121}$$

$$\left(k^2 - k_z^2\right)\rho^2 + \frac{\rho}{P}\frac{dP}{d\rho} + \frac{\rho^2}{P}\frac{d^2P}{d\rho^2} = k_\phi^2. \tag{A.122}$$

The variables ρ and ϕ are thus separated, and harmonic equation (A.121) has solutions

$$\Phi(\phi) = \begin{cases} A_\phi \sin k_\phi\phi + B_\phi \cos k_\phi\phi, & k_\phi \neq 0, \\ a_\phi\phi + b_\phi, & k_\phi = 0. \end{cases} \tag{A.123}$$

Equation (A.122) is a bit more involved. In rearranged form it is

$$\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} + \left(k_c^2 - \frac{k_\phi^2}{\rho^2}\right)P = 0 \tag{A.124}$$

where

$$k_c^2 = k^2 - k_z^2.$$

The solution depends on whether any of k_z, k_ϕ, or k_c are zero. If $k_c = k_\phi = 0$, then

$$\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} = 0$$

so that

$$P(\rho) = a_\rho \ln\rho + b_\rho.$$

If $k_c = 0$ but $k_\phi \neq 0$, we have

$$\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} - \frac{k_\phi^2}{\rho^2}P = 0$$

so that

$$P(\rho) = a_\rho\rho^{-k_\phi} + b_\rho\rho^{k_\phi}. \tag{A.125}$$

This includes the case $k = k_z = 0$ (Laplace's equation). If $k_c \neq 0$ then (A.124) is Bessel's differential equation. For noninteger k_ϕ the two independent solutions are denoted $J_{k_\phi}(z)$ and $J_{-k_\phi}(z)$, where $J_\nu(z)$ is the ordinary Bessel function of the first kind of order ν. For k_ϕ an integer n, $J_n(z)$ and $J_{-n}(z)$ are not independent and a second independent solution denoted $N_n(z)$ must be introduced. This is the ordinary Bessel function of the second kind, order n. As it is also independent when the order is noninteger, $J_\nu(z)$ and $N_\nu(z)$ are often chosen as solutions whether ν is integer or not. Linear combinations of these independent solutions may be used to produce new independent solutions. The functions

$H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are the *Hankel functions* of the first and second kind of order ν, and are related to the Bessel functions by

$$H_\nu^{(1)}(z) = J_\nu(z) + jN_\nu(z),$$
$$H_\nu^{(2)}(z) = J_\nu(z) - jN_\nu(z).$$

The argument z can be complex (as can ν, but this shall not concern us). When z is imaginary we introduce two new functions $I_\nu(z)$ and $K_\nu(z)$, defined for integer order by

$$I_n(z) = j^{-n} J_n(jz),$$
$$K_n(z) = \frac{\pi}{2} j^{n+1} H_n^{(1)}(jz).$$

Expressions for noninteger order are given in Appendix E.1.

Bessel functions cannot be expressed in terms of simple, standard functions. However, a series solution to (A.124) produces many useful relationships between Bessel functions of differing order and argument. The *recursion relations* for Bessel functions serve to connect functions of various orders and their derivatives. See Appendix E.1.

Of the six possible solutions to (A.124),

$$R(\rho) = \begin{cases} A_\rho J_\nu(k_c\rho) + B_\rho N_\nu(k_c\rho), \\ A_\rho J_\nu(k_c\rho) + B_\rho H_\nu^{(1)}(k_c\rho), \\ A_\rho J_\nu(k_c\rho) + B_\rho H_\nu^{(2)}(k_c\rho), \\ A_\rho N_\nu(k_c\rho) + B_\rho H_\nu^{(1)}(k_c\rho), \\ A_\rho N_\nu(k_c\rho) + B_\rho H_\nu^{(2)}(k_c\rho), \\ A_\rho H_\nu^{(1)}(k_c\rho) + B_\rho H_\nu^{(2)}(k_c\rho), \end{cases}$$

which do we choose? Again, we are motivated by convenience and the physical nature of the problem. If the argument is real or imaginary, we often consider large or small argument behavior. For x real and large,

$$J_\nu(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \nu\frac{\pi}{2}\right),$$
$$N_\nu(x) \to \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \nu\frac{\pi}{2}\right),$$
$$H_\nu^{(1)}(x) \to \sqrt{\frac{2}{\pi x}} e^{j\left(x - \frac{\pi}{4} - \nu\frac{\pi}{2}\right)},$$
$$H_\nu^{(2)}(x) \to \sqrt{\frac{2}{\pi x}} e^{-j\left(x - \frac{\pi}{4} - \nu\frac{\pi}{2}\right)},$$
$$I_\nu(x) \to \sqrt{\frac{1}{2\pi x}} e^{x},$$
$$K_\nu(x) \to \sqrt{\frac{\pi}{2x}} e^{-x},$$

while for x real and small,

$$J_0(x) \to 1,$$
$$N_0(x) \to \frac{2}{\pi}\left(\ln x + 0.5772157 - \ln 2\right),$$

$$J_\nu(x) \to \frac{1}{\nu!}\left(\frac{x}{2}\right)^\nu,$$
$$N_\nu(x) \to -\frac{(\nu-1)!}{\pi}\left(\frac{2}{x}\right)^\nu.$$

Because $J_\nu(x)$ and $N_\nu(x)$ oscillate for large argument, they can represent standing waves along the radial direction. However, $N_\nu(x)$ is unbounded for small x and is inappropriate for regions containing the origin. The Hankel functions become complex exponentials for large argument, hence represent traveling waves. Finally, $K_\nu(x)$ is unbounded for small x and cannot be used for regions containing the origin, while $I_\nu(x)$ increases exponentially for large x and cannot be used for unbounded regions.

Examples. Consider the boundary value problem for Laplace's equation

$$\nabla^2 V(\rho,\phi) = 0 \tag{A.126}$$

in the region

$$0 \le \rho \le \infty, \quad 0 \le \phi \le \phi_0, \quad -\infty < z < \infty,$$

where the boundary conditions are

$$V(\rho,0) = 0, \quad V(\rho,\phi_0) = V_0.$$

Since there is no z-dependence we let $k_z = 0$ in (A.120) and choose $a_z = 0$. Then $k_c^2 = k^2 - k_z^2 = 0$ since $k = 0$. There are two possible solutions, depending on whether k_ϕ is zero. First let us try $k_\phi \neq 0$. Using (A.123) and (A.125) we have

$$V(\rho,\phi) = [A_\phi \sin(k_\phi\phi) + B_\phi \cos(k_\phi\phi)][a_\rho \rho^{-k_\phi} + b_\rho \rho^{k_\phi}]. \tag{A.127}$$

Assuming $k_\phi > 0$ we must have $b_\rho = 0$ to keep the solution finite. The condition $V(\rho,0) = 0$ requires $B_\phi = 0$. Thus

$$V(\rho,\phi) = A_\phi \sin(k_\phi\phi)\rho^{-k_\phi}.$$

Our final boundary condition requires

$$V(\rho,\phi_0) = V_0 = A_\phi \sin(k_\phi\phi_0)\rho^{-k_\phi}.$$

Because this cannot hold for all ρ, we must resort to $k_\phi = 0$ and

$$V(\rho,\phi) = (a_\phi\phi + b_\phi)(a_\rho \ln\rho + b_\rho). \tag{A.128}$$

Proper behavior as $\rho \to \infty$ dictates that $a_\rho = 0$. $V(\rho,0) = 0$ requires $b_\phi = 0$. Thus $V(\rho,\phi) = V(\phi) = b_\phi\phi$. The constant b_ϕ is found from the remaining boundary condition: $V(\phi_0) = V_0 = b_\phi\phi_0$ so that $b_\phi = V_0/\phi_0$. The final solution is

$$V(\phi) = V_0\phi/\phi_0.$$

It is worthwhile to specialize this to $\phi_0 = \pi/2$ and compare with the solution to the same problem found earlier using rectangular coordinates. With $a = 0$ in (A.116) we have

$$V(x,y) = \frac{2}{\pi}\int_0^\infty e^{-k_y x}\frac{\sin k_y y}{k_y}\,dk_y.$$

Despite its much more complicated form, this must be the same solution by uniqueness. Next let us solve (A.126) subject to the "split cylinder" conditions

$$V(a,\phi) = \begin{cases} V_0, & 0 < \phi < \pi, \\ 0, & -\pi < \phi < 0. \end{cases}$$

Because there is no z-dependence we choose $k_z = a_z = 0$ and have $k_c^2 = k^2 - k_z^2 = 0$. Since $k_\phi = 0$ would violate the boundary conditions at $\rho = a$, we use

$$V(\rho,\phi) = (a_\rho \rho^{-k_\phi} + b_\rho \rho^{k_\phi})(A_\phi \sin k_\phi\phi + B_\phi \cos k_\phi\phi).$$

The potential must be single-valued in ϕ: $V(\rho,\phi+2n\pi) = V(\rho,\phi)$. This is only possible if k_ϕ is an integer, say $k_\phi = m$. Then

$$V_m(\rho,\phi) = \begin{cases} (A_m \sin m\phi + B_m \cos m\phi)\rho^m, & \rho < a, \\ (C_m \sin m\phi + D_m \cos m\phi)\rho^{-m}, & \rho > a. \end{cases}$$

On physical grounds we have discarded ρ^{-m} for $\rho < a$ and ρ^m for $\rho > a$. To satisfy the boundary conditions at $\rho = a$ we must use an infinite series of the complete set of eigensolutions. For $\rho < a$ the boundary condition requires

$$B_0 + \sum_{m=1}^{\infty} (A_m \sin m\phi + B_m \cos m\phi)a^m = \begin{cases} V_0, & 0 < \phi < \pi, \\ 0, & -\pi < \phi < 0. \end{cases}$$

Application of the orthogonality relations

$$\int_{-\pi}^{\pi} \cos m\phi \cos n\phi \, d\phi = \frac{2\pi}{\epsilon_n}\delta_{mn}, \quad m,n = 0,1,2,\ldots, \tag{A.129}$$

$$\int_{-\pi}^{\pi} \sin m\phi \sin n\phi \, d\phi = \pi\delta_{mn}, \quad m,n = 1,2,\ldots, \tag{A.130}$$

$$\int_{-\pi}^{\pi} \cos m\phi \sin n\phi \, d\phi = 0, \quad m,n = 0,1,2,\ldots, \tag{A.131}$$

where

$$\epsilon_n = \begin{cases} 1, & n = 0, \\ 2, & n > 0, \end{cases} \tag{A.132}$$

is Neumann's number, produces appropriate values for the constants A_m and B_m. The full solution is

$$V(\rho,\phi) = \begin{cases} \dfrac{V_0}{2} + \dfrac{V_0}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{n}\left(\frac{\rho}{a}\right)^n \sin n\phi, & \rho < a, \\ \dfrac{V_0}{2} + \dfrac{V_0}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{n}\left(\frac{a}{\rho}\right)^n \sin n\phi, & \rho > a. \end{cases}$$

The boundary value problem

$$\begin{aligned} \nabla^2 V(\rho,\phi,z) &= 0, \quad 0 \le \rho \le a, \ -\pi \le \phi \le \pi, \ 0 \le z \le h, \\ V(\rho,\phi,0) &= 0, \quad 0 \le \rho \le a, \ -\pi \le \phi \le \pi, \\ V(a,\phi,z) &= 0, \quad -\pi \le \phi \le \pi, \ 0 \le z \le h, \\ V(\rho,\phi,h) &= V_0, \quad 0 \le \rho \le a, \ -\pi \le \phi \le \pi, \end{aligned}$$

describes the potential within a grounded "canister" with top at potential V_0. Symmetry precludes ϕ-dependence, hence $k_\phi = a_\phi = 0$. Since $k = 0$ (Laplace's equation) we also have $k_c^2 = k^2 - k_z^2 = -k_z^2$. Thus we have either k_z real and $k_c = jk_z$, or k_c real and $k_z = jk_c$. With k_z real we have

$$V(\rho, z) = [A_z \sin k_z z + B_z \cos k_z z][A_\rho K_0(k_z \rho) + B_\rho I_0(k_z \rho)]; \tag{A.133}$$

with k_c real we have

$$V(\rho, z) = [A_z \sinh k_c z + B_z \cosh k_c z][A_\rho J_0(k_c \rho) + B_\rho N_0(k_c \rho)]. \tag{A.134}$$

The functions K_0 and I_0 are inappropriate for use in this problem, and we proceed to (A.134). Since N_0 is unbounded for small argument, we need $B_\rho = 0$. The condition $V(\rho, 0) = 0$ gives $B_z = 0$, thus

$$V(\rho, z) = A_z \sinh(k_c z) J_0(k_c \rho).$$

The oscillatory nature of J_0 means that we can satisfy the condition at $\rho = a$:

$$V(a, z) = A_z \sinh(k_c z) J_0(k_c a) = 0 \quad \text{for} \quad 0 \le z < h$$

if $J_0(k_c a) = 0$. Letting p_{0m} denote the mth root of $J_0(x) = 0$ for $m = 1, 2, \ldots$, we have $k_{c_m} = p_{0m}/a$. Because we cannot satisfy the boundary condition at $z = h$ with a single eigensolution, we use the superposition

$$V(\rho, z) = \sum_{m=1}^{\infty} A_m \sinh\left(\frac{p_{0m} z}{a}\right) J_0\left(\frac{p_{0m}\rho}{a}\right).$$

We require

$$V(\rho, h) = \sum_{m=1}^{\infty} A_m \sinh\left(\frac{p_{0m} h}{a}\right) J_0\left(\frac{p_{0m}\rho}{a}\right) = V_0, \tag{A.135}$$

where the A_m can be evaluated by orthogonality of the functions $J_0(p_{0m}\rho/a)$. If $p_{\nu m}$ is the mth root of $J_\nu(x) = 0$, then

$$\int_0^a J_\nu\left(\frac{p_{\nu m}\rho}{a}\right) J_\nu\left(\frac{p_{\nu n}\rho}{a}\right) \rho \, d\rho = \delta_{mn} \frac{a^2}{2} J_\nu'^2(p_{\nu n}) = \delta_{mn} \frac{a^2}{2} J_{\nu+1}^2(p_{\nu n}) \tag{A.136}$$

where $J_\nu'(x) = dJ_\nu(x)/dx$. Multiplying (A.135) by $\rho J_0(p_{0n}\rho/a)$ and integrating, we have

$$A_n \sinh\left(\frac{p_{0n} h}{a}\right) \frac{a^2}{2} J_0'^2(p_{0n} a) = \int_0^a V_0 J_0\left(\frac{p_{0n}\rho}{a}\right) \rho \, d\rho.$$

Use of (E.105),

$$\int x^{n+1} J_n(x) \, dx = x^{n+1} J_{n+1}(x) + C,$$

allows us to evaluate

$$\int_0^a J_0\left(\frac{p_{0n}\rho}{a}\right) \rho \, d\rho = \frac{a^2}{p_{0n}} J_1(p_{0n}).$$

With this we finish calculating A_m and have

$$V(\rho, z) = 2V_0 \sum_{m=1}^{\infty} \frac{\sinh(\frac{p_{0m}}{a} z) J_0(\frac{p_{0m}}{a}\rho)}{p_{0m} \sinh(\frac{p_{0m}}{a} h) J_1(p_{0m})}$$

as the desired solution.

Finally, let us assume that $k > 0$ and solve

$$\nabla^2\psi(\rho,\phi,z) + k^2\psi(\rho,\phi,z) = 0$$

where $0 \le \rho \le a$, $-\pi \le \phi \le \pi$, and $-\infty < z < \infty$, subject to the condition

$$\hat{\mathbf{n}} \cdot \nabla\psi(\rho,\phi,z)\Big|_{\rho=a} = \frac{\partial\psi(\rho,\phi,z)}{\partial\rho}\Big|_{\rho=a} = 0$$

for $-\pi \le \phi \le \pi$ and $-\infty < z < \infty$. The solution to this problem leads to the transverse-electric (TE_z) fields in a lossless circular waveguide, where ψ represents the z-component of the magnetic field. Although there is symmetry with respect to ϕ, we seek ϕ-dependent solutions; the resulting complete eigenmode solution will permit us to expand any well-behaved function within the waveguide in terms of a normal mode (eigenfunction) series. In this problem none of the constants k, k_z, or k_ϕ equal zero, except as a special case. However, the field must be single-valued in ϕ and thus k_ϕ must be an integer m. We consider our possible choices for $P(\rho)$, $Z(z)$, and $\Phi(\phi)$. Since $k_c^2 = k^2 - k_z^2$ and $k^2 > 0$ is arbitrary, we must consider various possibilities for the signs of k_c^2 and k_z^2. We can rule out $k_c^2 < 0$ based on consideration of the behavior of the functions I_m and K_m. We also need not consider $k_c < 0$, since this gives the same solution as $k_c > 0$. We are then left with two possible cases. Writing $k_z^2 = k^2 - k_c^2$, we see that either $k > k_c$ and $k_z^2 > 0$, or $k < k_c$ and $k_z^2 < 0$. For $k_z^2 > 0$ we write

$$\psi(\rho,\phi,z) = [A_z e^{-jk_z z} + B_z e^{jk_z z}][A_\phi \sin m\phi + B_\phi \cos m\phi]J_m(k_c\rho).$$

Here the terms involving $e^{\mp jk_z z}$ represent waves propagating in the $\pm z$ directions. The boundary condition at $\rho = a$ requires

$$J_m'(k_c a) = 0$$

where $J_m'(x) = dJ_m(x)/dx$. Denoting the nth zero of $J_m'(x)$ by p_{mn}' we have $k_c = k_{cm} = p_{mn}'/a$. This gives the eigensolutions

$$\psi_m = [A_{zm}e^{-jk_z z} + B_{zm}e^{jk_z z}][A_{\phi m}\sin m\phi + B_{\phi m}\cos m\phi]k_c J_m\left(\frac{p_{mn}'\rho}{a}\right).$$

The undetermined constants A_{zm}, B_{zm}, $A_{\rho m}$, $B_{\rho m}$ could be evaluated when the individual eigensolutions are used to represent a function in terms of a modal expansion. For the case $k_z^2 < 0$ we again choose complex exponentials in z; however, $k_z = -j\alpha$ gives $e^{\mp jk_z z} = e^{\mp\alpha z}$ and attenuation along z. The reader can verify that the eigensolutions are

$$\psi_m = [A_{zm}e^{-\alpha z} + B_{zm}e^{\alpha z}][A_{\phi m}\sin m\phi + B_{\phi m}\cos m\phi]k_c J_m\left(\frac{p_{mn}'\rho}{a}\right)$$

where now $k_c^2 = k^2 + \alpha^2$.

We have used Bessel function completeness in the examples above. This property is a consequence of the Sturm–Liouville problem first studied in § A.4. We often use Fourier–Bessel series to express functions over finite intervals. Over infinite intervals we use the Fourier–Bessel transform.

The Fourier–Bessel series can be generalized to Bessel functions of noninteger order, and to the derivatives of Bessel functions. Let $f(\rho)$ be well-behaved over the interval $[0, a]$. Then the series

$$f(\rho) = \sum_{m=1}^{\infty} C_m J_\nu\left(p_{\nu m}\frac{\rho}{a}\right), \quad 0 \le \rho \le a,\ \nu > -1$$

converges, and the constants are

$$C_m = \frac{2}{a^2 J_{\nu+1}^2(p_{\nu m})} \int_0^a f(\rho) J_\nu \left(p_{\nu m} \frac{\rho}{a} \right) \rho \, d\rho$$

by (A.136). Here $p_{\nu m}$ is the mth root of $J_\nu(x)$. An alternative form of the series uses $p'_{\nu m}$, the roots of $J'_\nu(x)$, and is given by

$$f(\rho) = \sum_{m=1}^{\infty} D_m J_\nu \left(p'_{\nu m} \frac{\rho}{a} \right), \quad 0 \le \rho \le a, \ \nu > -1.$$

In this case the expansion coefficients are found using the orthogonality relationship

$$\int_0^a J_\nu \left(\frac{p'_{\nu m}}{a} \rho \right) J_\nu \left(\frac{p'_{\nu n}}{a} \rho \right) \rho \, d\rho = \delta_{mn} \frac{a^2}{2} \left(1 - \frac{\nu^2}{p'^2_{\nu m}} \right) J_\nu^2(p'_{\nu m}),$$

and are

$$D_m = \frac{2}{a^2 \left(1 - \frac{\nu^2}{p'^2_{\nu m}} J_\nu^2(p'_{\nu m}) \right)} \int_0^a f(\rho) J_\nu \left(\frac{p'_{\nu m}}{a} \rho \right) \rho \, d\rho.$$

Solutions in spherical coordinates. If into Helmholtz's equation

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi(r,\theta,\phi)}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi(r,\theta,\phi)}{\partial \theta} \right) + \\ + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi(r,\theta,\phi)}{\partial \phi^2} + k^2 \psi(r,\theta,\phi) = 0$$

we put $\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$ and multiply through by $r^2 \sin^2\theta / \psi(r,\theta,\phi)$, we obtain

$$\frac{\sin^2\theta}{R(r)} \frac{d}{dr} \left(r^2 \frac{dR(r)}{dr} \right) + \frac{\sin\theta}{\Theta(\theta)} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta(\theta)}{d\theta} \right) + k^2 r^2 \sin^2\theta = -\frac{1}{\Phi(\phi)} \frac{d^2\Phi(\phi)}{d\phi^2}.$$

Since the right side depends only on ϕ while the left side depends only on r and θ, both sides must equal some constant μ^2:

$$\frac{\sin^2\theta}{R(r)} \frac{d}{dr} \left(r^2 \frac{dR(r)}{dr} \right) + \frac{\sin\theta}{\Theta(\theta)} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta(\theta)}{d\theta} \right) + k^2 r^2 \sin^2\theta = \mu^2, \tag{A.137}$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} + \mu^2 \Phi(\phi) = 0. \tag{A.138}$$

We have thus separated off the ϕ-dependence. Harmonic ordinary differential equation (A.138) has solutions

$$\Phi(\phi) = \begin{cases} A_\phi \sin\mu\phi + B_\phi \cos\mu\phi, & \mu \neq 0, \\ a_\phi \phi + b_\phi, & \mu = 0. \end{cases}$$

(We could have used complex exponentials to describe $\Phi(\phi)$, or some combination of exponentials and trigonometric functions, but it is conventional to use only trigonometric functions.) Rearranging (A.137) and dividing through by $\sin^2\theta$ we have

$$\frac{1}{R(r)} \frac{d}{dr} \left(r^2 \frac{dR(r)}{dr} \right) + k^2 r^2 = -\frac{1}{\sin\theta\Theta(\theta)} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta(\theta)}{d\theta} \right) + \frac{\mu^2}{\sin^2\theta}.$$

We introduce a new constant k_θ^2 to separate r from θ:

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)+k^2r^2=k_\theta^2, \tag{A.139}$$

$$-\frac{1}{\sin\theta\Theta(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)+\frac{\mu^2}{\sin^2\theta}=k_\theta^2. \tag{A.140}$$

Equation (A.140),

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)+\left(k_\theta^2-\frac{\mu^2}{\sin^2\theta}\right)\Theta(\theta)=0,$$

can be put into a standard form by letting

$$\eta=\cos\theta \tag{A.141}$$

and $k_\theta^2=\nu(\nu+1)$ where ν is a parameter:

$$(1-\eta^2)\frac{d^2\Theta(\eta)}{d\eta^2}-2\eta\frac{d\Theta(\eta)}{d\eta}+\left[\nu(\nu+1)-\frac{\mu^2}{1-\eta^2}\right]\Theta(\eta)=0,\quad -1\le\eta\le 1.$$

This is the *associated Legendre equation*. It has two independent solutions called *associated Legendre functions of the first* and *second kinds*, denoted $P_\nu^\mu(\eta)$ and $Q_\nu^\mu(\eta)$, respectively. In these functions, all three quantities μ,ν,η may be arbitrary complex constants as long as $\nu+\mu\neq -1,-2,\ldots$. But (A.141) shows that η is real in our discussion; μ will generally be real also, and will be an integer whenever $\Phi(\phi)$ is single-valued. The choice of ν is somewhat more complicated. The function $P_\nu^\mu(\eta)$ diverges at $\eta=\pm1$ unless ν is an integer, while $Q_\nu^\mu(\eta)$ diverges at $\eta=\pm1$ regardless of whether ν is an integer. In § A.4 we required that $P_\nu^\mu(\eta)$ be bounded on $[-1,1]$ to have a Sturm–Liouville problem with suitable orthogonality properties. By (A.141) we must exclude $Q_\nu^\mu(\eta)$ for problems containing the z-axis, and restrict ν to be an integer n in $P_\nu^\mu(\eta)$ for such problems. In case the z-axis is excluded, we choose $\nu=n$ whenever possible, because the finite sums $P_n^m(\eta)$ and $Q_n^m(\eta)$ are much easier to manipulate than $P_\nu^\mu(\eta)$ and $Q_\nu^\mu(\eta)$. In many problems we must count on completeness of the Legendre polynomials $P_n(\eta)=P_n^0(\eta)$ or spherical harmonics $Y_{mn}(\theta,\phi)$ in order to satisfy the boundary conditions. In this book we shall consider only those boundary value problems that can be solved using integer values of ν and μ, hence choose

$$\Theta(\theta)=A_\theta P_n^m(\cos\theta)+B_\theta Q_n^m(\cos\theta). \tag{A.142}$$

Single-valuedness in $\Phi(\phi)$ is a consequence of having $\mu=m$, and $\phi=$ constant boundary surfaces are thereby disallowed.

The associated Legendre functions have many important properties. For instance,

$$P_n^m(\eta)=\begin{cases}0, & m>n,\\ (-1)^m\dfrac{(1-\eta^2)^{m/2}}{2^n n!}\dfrac{d^{n+m}(\eta^2-1)^n}{d\eta^{n+m}}, & m\le n.\end{cases} \tag{A.143}$$

The case $m=0$ receives particular attention because it corresponds to azimuthal invariance (ϕ-independence). We define $P_n^0(\eta)=P_n(\eta)$ where $P_n(\eta)$ is the Legendre polynomial

of order n. From (A.143), we see that[4]

$$P_n(\eta) = \frac{1}{2^n n!} \frac{d^n(\eta^2 - 1)^n}{d\eta^n}$$

is a polynomial of degree n, and that

$$P_n^m(\eta) = (-1)^m (1 - \eta^2)^{m/2} \frac{d^m}{d\eta^m} P_n(\eta).$$

Both the associated Legendre functions and the Legendre polynomials obey orthogonality relations and many recursion formulas.

In problems where the z-axis is included, the product $\Theta(\theta)\Phi(\phi)$ is sometimes defined as the spherical harmonic

$$Y_{nm}(\theta, \phi) = \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta) e^{jm\theta}.$$

These functions, which are complete over the surface of a sphere, were treated earlier in this section.

Remembering that $k_r^2 = \nu(\nu + 1)$, the r-dependent equation (A.139) becomes

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR(r)}{dr} \right) + \left(k^2 + \frac{n(n+1)}{r^2} \right) R(r) = 0. \tag{A.144}$$

When $k = 0$ we have

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} - \frac{n(n+1)}{r^2} R(r) = 0$$

so that

$$R(r) = A_r r^n + B_r r^{-(n+1)}.$$

When $k \neq 0$, the substitution $\bar{R}(r) = \sqrt{kr} R(r)$ puts (A.144) into the form

$$r^2 \frac{d^2\bar{R}(r)}{dr^2} + r \frac{d\bar{R}(r)}{dr} + \left[k^2 r^2 - \left(n + \frac{1}{2} \right)^2 \right] \bar{R}(r) = 0,$$

which we recognize as Bessel's equation of half-integer order. Thus

$$R(r) = \frac{\bar{R}(r)}{\sqrt{kr}} = \frac{Z_{n+\frac{1}{2}}(kr)}{\sqrt{kr}}.$$

For convenience we define the *spherical Bessel functions*

$$\begin{aligned}
j_n(z) &= \sqrt{\frac{\pi}{2z}} J_{n+\frac{1}{2}}(z), \\
n_n(z) &= \sqrt{\frac{\pi}{2z}} N_{n+\frac{1}{2}}(z) = (-1)^{n+1} \sqrt{\frac{\pi}{2z}} J_{-(n+\frac{1}{2})}(z), \\
h_n^{(1)}(z) &= \sqrt{\frac{\pi}{2z}} H_{n+\frac{1}{2}}^{(1)}(z) = j_n(z) + j n_n(z), \\
h_n^{(2)}(z) &= \sqrt{\frac{\pi}{2z}} H_{n+\frac{1}{2}}^{(2)}(z) = j_n(z) - j n_n(z).
\end{aligned}$$

[4]Care must be taken when consulting tables of Legendre functions and their properties. In particular, one must be on the lookout for possible disparities regarding the factor $(-1)^m$ (cf., [76, 1, 109, 8] vs. [5, 187]). Similar care is needed with $Q_n^m(x)$.

These can be written as finite sums involving trigonometric functions and inverse powers of z. We have, for instance,

$$\begin{aligned} j_0(z) &= \frac{\sin z}{z}, \\ n_0(z) &= -\frac{\cos z}{z}, \\ j_1(z) &= \frac{\sin z}{z^2} - \frac{\cos z}{z}, \\ n_1(z) &= -\frac{\cos z}{z^2} - \frac{\sin z}{z}. \end{aligned}$$

We can now write $R(r)$ as a linear combination of any two of the spherical Bessel functions $j_n, n_n, h_n^{(1)}, h_n^{(2)}$:

$$R(r) = \begin{cases} A_r j_n(kr) + B_r n_n(kr), \\ A_r j_n(kr) + B_r h_n^{(1)}(kr), \\ A_r j_n(kr) + B_r h_n^{(2)}(kr), \\ A_r n_n(kr) + B_r h_n^{(1)}(kr), \\ A_r n_n(kr) + B_r h_n^{(2)}(kr), \\ A_r h_n^{(1)}(kr) + B_r h_n^{(2)}(kr). \end{cases} \tag{A.145}$$

Imaginary arguments produce *modified spherical Bessel functions*; the interested reader is referred to Gradsteyn [76] or Abramowitz [1].

Examples. The problem

$$\begin{aligned} \nabla^2 V(r, \theta, \phi) = 0, &\quad \theta_0 \le \theta \le \pi/2,\ 0 \le r < \infty,\ -\pi \le \phi \le \pi, \\ V(r, \theta_0, \phi) = V_0, &\quad -\pi \le \phi \le \pi,\ 0 \le r < \infty, \\ V(r, \pi/2, \phi) = 0, &\quad -\pi \le \phi \le \pi,\ 0 \le r < \infty, \end{aligned}$$

gives the potential field between a cone and the $z = 0$ plane. Azimuthal symmetry prompts us to choose $\mu = a_\phi = 0$. Since $k = 0$ we have

$$R(r) = A_r r^n + B_r r^{-(n+1)}. \tag{A.146}$$

Noting that positive and negative powers of r are unbounded for large and small r, respectively, we take $n = B_r = 0$. Hence the solution depends only on θ:

$$V(r, \theta, \phi) = V(\theta) = A_\theta P_0^0(\cos\theta) + B_\theta Q_0^0(\cos\theta).$$

We must retain Q_0^0 since the solution region does not contain the z-axis. Using

$$P_0^0(\cos\theta) = 1 \quad \text{and} \quad Q_0^0(\cos\theta) = \ln\cot(\theta/2)$$

(cf., Appendix E.2), we have

$$V(\theta) = A_\theta + B_\theta \ln\cot(\theta/2).$$

A straightforward application of the boundary conditions gives $A_\theta = 0$ and $B_\theta = V_0/\ln\cot(\theta_0/2)$, hence

$$V(\theta) = V_0 \frac{\ln\cot(\theta/2)}{\ln\cot(\theta_0/2)}.$$

Next we solve the boundary value problem

$$\begin{aligned}\nabla^2 V(r,\theta,\phi) &= 0,\\ V(a,\theta,\phi) &= -V_0, \quad \pi/2 \le \theta < \pi,\ -\pi \le \phi \le \pi,\\ V(a,\theta,\phi) &= +V_0, \quad 0 < \theta \le \pi/2,\ -\pi \le \phi \le \pi,\end{aligned}$$

for both $r > a$ and $r < a$. This yields the potential field of a conducting sphere split into top and bottom hemispheres and held at a potential difference of $2V_0$. Azimuthal symmetry gives $\mu = 0$. The two possible solutions for $\Theta(\theta)$ are

$$\Theta(\theta) = \begin{cases} A_\theta + B_\theta \ln\cot(\theta/2), & n = 0,\\ A_\theta P_n(\cos\theta), & n \neq 0,\end{cases}$$

where we have discarded $Q_0^0(\cos\theta)$ because the region of interest contains the z-axis. The $n = 0$ solution cannot match the boundary conditions; neither can a single term of the type $A_\theta P_n(\cos\theta)$, but a series of these latter terms can. We use

$$V(r,\theta) = \sum_{n=0}^{\infty} V_n(r,\theta) = \sum_{n=0}^{\infty} [A_r r^n + B_r r^{-(n+1)}] P_n(\cos\theta). \tag{A.147}$$

The terms $r^{-(n+1)}$ and r^n are not allowed, respectively, for $r < a$ and $r > a$. For $r < a$ then,

$$V(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta).$$

Letting $V_0(\theta)$ be the potential on the surface of the split sphere, we impose the boundary condition:

$$V(a,\theta) = V_0(\theta) = \sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta), \quad 0 \le \theta \le \pi.$$

This is a Fourier–Legendre expansion of $V_0(\theta)$. The A_n are evaluated by orthogonality. Multiplying by $P_m(\cos\theta)\sin\theta$ and integrating from $\theta = 0$ to π, we obtain

$$\sum_{n=0}^{\infty} A_n a^n \int_0^\pi P_n(\cos\theta) P_m(\cos\theta)\sin\theta\, d\theta = \int_0^\pi V_0(\theta) P_m(\cos\theta)\sin\theta\, d\theta.$$

Using orthogonality relationship (A.93) and the given $V_0(\theta)$ we have

$$A_m a^m \frac{2}{2m+1} = V_0 \int_0^{\pi/2} P_m(\cos\theta)\sin\theta\, d\theta - V_0 \int_{\pi/2}^{\pi} P_m(\cos\theta)\sin\theta\, d\theta.$$

The substitution $\eta = \cos\theta$ gives

$$\begin{aligned} A_m a^m \frac{2}{2m+1} &= V_0 \int_0^1 P_m(\eta)\, d\eta - V_0 \int_{-1}^0 P_m(\eta)\, d\eta\\ &= V_0 \int_0^1 P_m(\eta)\, d\eta - V_0 \int_0^1 P_m(-\eta)\, d\eta;\end{aligned}$$

then $P_m(-\eta) = (-1)^m P_m(\eta)$ gives

$$A_m = a^{-m} \frac{2m+1}{2} V_0 [1 - (-1)^m] \int_0^1 P_m(\eta)\, d\eta.$$

Because $A_m = 0$ for m even, we can put $m = 2n + 1$ $(n = 0, 1, 2, \ldots)$ and have

$$A_{2n+1} = \frac{(4n+3)V_0}{a^{2n+1}} \int_0^1 P_{2n+1}(\eta)\, d\eta = \frac{V_0(-1)^n}{a^{2n+1}} \frac{4n+3}{2n+2} \frac{(2n!)}{(2^n n!)^2}$$

by (E.176). Hence

$$V(r,\theta) = \sum_{n=0}^{\infty} V_0(-1)^n \frac{4n+3}{2n+2} \frac{(2n!)}{(2^n n!)^2} \left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\cos\theta)$$

for $r < a$. The case $r > a$ is left to the reader.

Finally, consider

$$\nabla^2\psi(x,y,z) + k^2\psi(x,y,z) = 0, \quad 0 \le r \le a,\ 0 \le \theta \le \pi,\ -\pi \le \phi \le \pi,$$
$$\psi(a,\theta,\phi) = 0, \quad 0 \le \theta \le \pi,\ -\pi \le \phi \le \pi,$$

where $k \neq 0$ is constant. This is a three-dimensional eigenvalue problem. Wave function ψ represents the solutions for the electromagnetic field within a spherical cavity for modes TE to r. Despite the prevailing symmetry, we choose solutions that vary with both θ and ϕ. We are motivated by a desire to solve problems involving cavity excitation, and eigenmode completeness will enable us to represent any piecewise continuous function within the cavity. We employ spherical harmonics because the boundary surface is a sphere. These exclude $Q_m^n(\cos\theta)$, which is appropriate since our problem contains the z-axis. Since $k \neq 0$ we must choose a radial dependence from (A.145). Small-argument behavior rules out n_n, $h_n^{(1)}$, and $h_n^{(2)}$, leaving us with

$$\psi(r,\theta,\phi) = A_{mn} j_n(kr) Y_{nm}(\theta,\phi)$$

or, equivalently,

$$\psi(r,\theta,\phi) = A_{mn} j_n(kr) P_n^m(\cos\theta) e^{jm\phi}.$$

The eigenvalues $\lambda = k^2$ are found by applying the condition at $r = a$:

$$\psi(a,\theta,\phi) = A_{mn} j_n(ka) Y_{nm}(\theta,\phi) = 0,$$

requiring $j_n(ka) = 0$. Denoting the qth root of $j_n(x) = 0$ by α_{nq}, we have $k_{nq} = \alpha_{nq}/a$ and corresponding eigenfunctions

$$\psi_{mnq}(r,\theta,\phi) = A_{mnq} j_n(k_{nq} r) Y_{nm}(\theta,\phi).$$

The eigenvalues are proportional to the resonant frequencies of the cavity and the eigenfunctions can be used to find the modal field distributions. Since the eigenvalues are independent of m, we may have several eigenfunctions ψ_{mnq} associated with each k_{mnq}. The only limitation is that we must keep $m \le n$ to have $P_m^n(\cos\theta)$ nonzero. This is another instance of mode degeneracy. There are $2n$ degenerate modes associated with each resonant frequency (one for each of $e^{\pm jn\phi}$). By completeness we can expand any piecewise continuous function within or on the sphere as a series

$$f(r,\theta,\phi) = \sum_{m,n,q} A_{mnq} j_n(k_{nq} r) Y_{nm}(\theta,\phi).$$