

Appendix B

Useful identities

Algebraic identities for vectors and dyadics

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \quad (\text{B.1})$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \quad (\text{B.2})$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \quad (\text{B.3})$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \quad (\text{B.4})$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \quad (\text{B.5})$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \quad (\text{B.6})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{B} \times (\mathbf{A} \times \mathbf{C}) + \mathbf{C} \times (\mathbf{B} \times \mathbf{A}) \quad (\text{B.7})$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{A} \cdot [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] = (\mathbf{B} \cdot \mathbf{D})(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D}) \quad (\text{B.8})$$

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = \mathbf{C}[\mathbf{A} \cdot (\mathbf{B} \times \mathbf{D})] - \mathbf{D}[\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})] \quad (\text{B.9})$$

$$\mathbf{A} \times [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] = (\mathbf{B} \cdot \mathbf{D})(\mathbf{A} \times \mathbf{C}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \times \mathbf{D}) \quad (\text{B.10})$$

$$\mathbf{A} \cdot (\bar{\mathbf{c}} \cdot \mathbf{B}) = (\mathbf{A} \cdot \bar{\mathbf{c}}) \cdot \mathbf{B} \quad (\text{B.11})$$

$$\mathbf{A} \times (\bar{\mathbf{c}} \times \mathbf{B}) = (\mathbf{A} \times \bar{\mathbf{c}}) \times \mathbf{B} \quad (\text{B.12})$$

$$\mathbf{C} \cdot (\bar{\mathbf{a}} \cdot \bar{\mathbf{b}}) = (\mathbf{C} \cdot \bar{\mathbf{a}}) \cdot \bar{\mathbf{b}} \quad (\text{B.13})$$

$$(\bar{\mathbf{a}} \cdot \bar{\mathbf{b}}) \cdot \mathbf{C} = \bar{\mathbf{a}} \cdot (\bar{\mathbf{b}} \cdot \mathbf{C}) \quad (\text{B.14})$$

$$\mathbf{A} \cdot (\mathbf{B} \times \bar{\mathbf{c}}) = -\mathbf{B} \cdot (\mathbf{A} \times \bar{\mathbf{c}}) = (\mathbf{A} \times \mathbf{B}) \cdot \bar{\mathbf{c}} \quad (\text{B.15})$$

$$\mathbf{A} \times (\mathbf{B} \times \bar{\mathbf{c}}) = \mathbf{B} \cdot (\mathbf{A} \times \bar{\mathbf{c}}) - \bar{\mathbf{c}}(\mathbf{A} \cdot \mathbf{B}) \quad (\text{B.16})$$

$$\mathbf{A} \cdot \bar{\mathbf{I}} = \bar{\mathbf{I}} \cdot \mathbf{A} = \mathbf{A} \quad (\text{B.17})$$

Integral theorems

Note: S bounds V , Γ bounds S , $\hat{\mathbf{n}}$ is normal to S at \mathbf{r} , $\hat{\mathbf{l}}$ and $\hat{\mathbf{m}}$ are tangential to S at \mathbf{r} , $\hat{\mathbf{l}}$ is tangential to the contour Γ , $\hat{\mathbf{m}} \times \hat{\mathbf{l}} = \hat{\mathbf{n}}$, $d\mathbf{l} = \hat{\mathbf{l}} dl$, and $d\mathbf{S} = \hat{\mathbf{n}} dS$.

Divergence theorem

$$\int_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot d\mathbf{S} \quad (\text{B.18})$$

$$\int_V \nabla \cdot \bar{\mathbf{a}} dV = \oint_S \hat{\mathbf{n}} \cdot \bar{\mathbf{a}} dS \quad (\text{B.19})$$

$$\int_S \nabla_s \cdot \mathbf{A} dS = \oint_\Gamma \hat{\mathbf{m}} \cdot \mathbf{A} dl \quad (\text{B.20})$$

Gradient theorem

$$\int_V \nabla a dV = \oint_S a \mathbf{dS} \quad (\text{B.21})$$

$$\int_V \nabla \mathbf{A} dV = \oint_S \hat{\mathbf{n}} \mathbf{A} dS \quad (\text{B.22})$$

$$\int_V \nabla_s a dS = \oint_\Gamma \hat{\mathbf{m}} a dl \quad (\text{B.23})$$

Curl theorem

$$\int_V (\nabla \times \mathbf{A}) dV = - \oint_S \mathbf{A} \times \mathbf{dS} \quad (\text{B.24})$$

$$\int_V (\nabla \times \bar{\mathbf{a}}) dV = \oint_S \hat{\mathbf{n}} \times \bar{\mathbf{a}} dS \quad (\text{B.25})$$

$$\int_S \nabla_s \times \mathbf{A} dS = \oint_\Gamma \hat{\mathbf{m}} \times \mathbf{A} dl \quad (\text{B.26})$$

Stokes's theorem

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{dS} = \oint_\Gamma \mathbf{A} \cdot \mathbf{dl} \quad (\text{B.27})$$

$$\int_S \hat{\mathbf{n}} \cdot (\nabla \times \bar{\mathbf{a}}) dS = \oint_\Gamma \mathbf{dl} \cdot \bar{\mathbf{a}} \quad (\text{B.28})$$

Green's first identity for scalar fields

$$\int_V (\nabla a \cdot \nabla b + a \nabla^2 b) dV = \oint_S a \frac{\partial b}{\partial n} dS \quad (\text{B.29})$$

Green's second identity for scalar fields (Green's theorem)

$$\int_V (a \nabla^2 b - b \nabla^2 a) dV = \oint_S \left(a \frac{\partial b}{\partial n} - b \frac{\partial a}{\partial n} \right) dS \quad (\text{B.30})$$

Green's first identity for vector fields

$$\begin{aligned} \int_V \{(\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{B}) - \mathbf{A} \cdot [\nabla \times (\nabla \times \mathbf{B})]\} dV = \\ \int_V \nabla \cdot [\mathbf{A} \times (\nabla \times \mathbf{B})] dV = \oint_S [\mathbf{A} \times (\nabla \times \mathbf{B})] \cdot \mathbf{dS} \end{aligned} \quad (\text{B.31})$$

Green's second identity for vector fields

$$\begin{aligned} \int_V \{[\mathbf{B} \cdot [\nabla \times (\nabla \times \mathbf{A})] - \mathbf{A} \cdot [\nabla \times (\nabla \times \mathbf{B})]\} dV = \\ \oint_S [\mathbf{A} \times (\nabla \times \mathbf{B}) - \mathbf{B} \times (\nabla \times \mathbf{A})] \cdot \mathbf{dS} \end{aligned} \quad (\text{B.32})$$

Helmholtz theorem

$$\begin{aligned} \mathbf{A}(\mathbf{r}) = & -\nabla \left[\int_V \frac{\nabla' \cdot \mathbf{A}(\mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|} dV' - \oint_S \frac{\mathbf{A}(\mathbf{r}') \cdot \hat{\mathbf{n}}'}{4\pi|\mathbf{r} - \mathbf{r}'|} dS' \right] + \\ & + \nabla \times \left[\int_V \frac{\nabla' \times \mathbf{A}(\mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|} dV' + \oint_S \frac{\mathbf{A}(\mathbf{r}') \times \hat{\mathbf{n}}'}{4\pi|\mathbf{r} - \mathbf{r}'|} dS' \right] \end{aligned} \quad (\text{B.33})$$

Miscellaneous identities

$$\oint_S \mathbf{dS} = 0 \quad (\text{B.34})$$

$$\int_S \hat{\mathbf{n}} \times (\nabla a) dS = \oint_\Gamma a \mathbf{dl} \quad (\text{B.35})$$

$$\int_S (\nabla a \times \nabla b) \cdot \mathbf{dS} = \int_\Gamma a \nabla b \cdot \mathbf{dl} = - \int_\Gamma b \nabla a \cdot \mathbf{dl} \quad (\text{B.36})$$

$$\oint \mathbf{dl} \mathbf{A} = \int_S \hat{\mathbf{n}} \times (\nabla \mathbf{A}) dS \quad (\text{B.37})$$

Derivative identities

$$\nabla(a + b) = \nabla a + \nabla b \quad (\text{B.38})$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \quad (\text{B.39})$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \quad (\text{B.40})$$

$$\nabla(ab) = a\nabla b + b\nabla a \quad (\text{B.41})$$

$$\nabla \cdot (a\mathbf{B}) = a\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla a \quad (\text{B.42})$$

$$\nabla \times (a\mathbf{B}) = a\nabla \times \mathbf{B} - \mathbf{B} \times \nabla a \quad (\text{B.43})$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \quad (\text{B.44})$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \quad (\text{B.45})$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \quad (\text{B.46})$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad (\text{B.47})$$

$$\nabla \cdot (\nabla a) = \nabla^2 a \quad (\text{B.48})$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad (\text{B.49})$$

$$\nabla \times (\nabla a) = 0 \quad (\text{B.50})$$

$$\nabla \times (a\nabla b) = \nabla a \times \nabla b \quad (\text{B.51})$$

$$\nabla^2(ab) = a\nabla^2 b + 2(\nabla a) \cdot (\nabla b) + b\nabla^2 a \quad (\text{B.52})$$

$$\nabla^2(a\mathbf{B}) = a\nabla^2 \mathbf{B} + \mathbf{B}\nabla^2 a + 2(\nabla a \cdot \nabla)\mathbf{B} \quad (\text{B.53})$$

$$\nabla^2 \bar{\mathbf{a}} = \nabla(\nabla \cdot \bar{\mathbf{a}}) - \nabla \times (\nabla \times \bar{\mathbf{a}}) \quad (\text{B.54})$$

$$\nabla \cdot (\mathbf{AB}) = (\nabla \cdot \mathbf{A})\mathbf{B} + \mathbf{A} \cdot (\nabla \mathbf{B}) = (\nabla \cdot \mathbf{A})\mathbf{B} + (\mathbf{A} \cdot \nabla)\mathbf{B} \quad (\text{B.55})$$

$$\nabla \times (\mathbf{AB}) = (\nabla \times \mathbf{A})\mathbf{B} - \mathbf{A} \times (\nabla \mathbf{B}) \quad (\text{B.56})$$

$$\nabla \cdot (\nabla \times \bar{\mathbf{a}}) = 0 \quad (\text{B.57})$$

$$\nabla \times (\nabla \mathbf{A}) = 0 \quad (\text{B.58})$$

$$\nabla(\mathbf{A} \times \mathbf{B}) = (\nabla \mathbf{A}) \times \mathbf{B} - (\nabla \mathbf{B}) \times \mathbf{A} \quad (\text{B.59})$$

$$\nabla(a\mathbf{B}) = (\nabla a)\mathbf{B} + a(\nabla \mathbf{B}) \quad (\text{B.60})$$

$$\nabla \cdot (a\bar{\mathbf{b}}) = (\nabla a) \cdot \bar{\mathbf{b}} + a(\nabla \cdot \bar{\mathbf{b}}) \quad (\text{B.61})$$

$$\nabla \times (a\bar{\mathbf{b}}) = (\nabla a) \times \bar{\mathbf{b}} + a(\nabla \times \bar{\mathbf{b}}) \quad (\text{B.62})$$

$$\nabla \cdot (a\bar{\mathbf{I}}) = \nabla a \quad (\text{B.63})$$

$$\nabla \times (a\bar{\mathbf{I}}) = \nabla a \times \bar{\mathbf{I}} \quad (\text{B.64})$$

Identities involving the displacement vector

Note: $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, $R = |\mathbf{R}|$, $\hat{\mathbf{R}} = \mathbf{R}/R$, $f'(x) = df(x)/dx$.

$$\nabla f(R) = -\nabla' f(R) = \hat{\mathbf{R}} f'(R) \quad (\text{B.65})$$

$$\nabla R = \hat{\mathbf{R}} \quad (\text{B.66})$$

$$\nabla \left(\frac{1}{R} \right) = -\frac{\hat{\mathbf{R}}}{R^2} \quad (\text{B.67})$$

$$\nabla \left(\frac{e^{-jkR}}{R} \right) = -\hat{\mathbf{R}} \left(\frac{1}{R} + jk \right) \frac{e^{-jkR}}{R} \quad (\text{B.68})$$

$$\nabla \cdot [f(R)\hat{\mathbf{R}}] = -\nabla' \cdot [f(R)\hat{\mathbf{R}}] = 2\frac{f(R)}{R} + f'(R) \quad (\text{B.69})$$

$$\nabla \cdot \mathbf{R} = 3 \quad (\text{B.70})$$

$$\nabla \cdot \hat{\mathbf{R}} = \frac{2}{R} \quad (\text{B.71})$$

$$\nabla \cdot \left(\hat{\mathbf{R}} \frac{e^{-jkR}}{R} \right) = \left(\frac{1}{R} - jk \right) \frac{e^{-jkR}}{R} \quad (\text{B.72})$$

$$\nabla \times [f(R)\hat{\mathbf{R}}] = 0 \quad (\text{B.73})$$

$$\nabla^2 \left(\frac{1}{R} \right) = -4\pi \delta(\mathbf{R}) \quad (\text{B.74})$$

$$(\nabla^2 + k^2) \frac{e^{-jkR}}{R} = -4\pi \delta(\mathbf{R}) \quad (\text{B.75})$$

Identities involving the plane-wave function

Note: \mathbf{E} is a constant vector, $k = |\mathbf{k}|$.

$$\nabla (e^{-j\mathbf{k}\cdot\mathbf{r}}) = -j\mathbf{k}e^{-j\mathbf{k}\cdot\mathbf{r}} \quad (\text{B.76})$$

$$\nabla \cdot (\mathbf{E}e^{-j\mathbf{k}\cdot\mathbf{r}}) = -j\mathbf{k} \cdot \mathbf{E}e^{-j\mathbf{k}\cdot\mathbf{r}} \quad (\text{B.77})$$

$$\nabla \times (\mathbf{E}e^{-j\mathbf{k}\cdot\mathbf{r}}) = -j\mathbf{k} \times \mathbf{E}e^{-j\mathbf{k}\cdot\mathbf{r}} \quad (\text{B.78})$$

$$\nabla^2 (\mathbf{E}e^{-j\mathbf{k}\cdot\mathbf{r}}) = -k^2\mathbf{E}e^{-j\mathbf{k}\cdot\mathbf{r}} \quad (\text{B.79})$$

Identities involving the transverse/longitudinal decomposition

Note: $\hat{\mathbf{u}}$ is a constant unit vector, $A_u \equiv \hat{\mathbf{u}} \cdot \mathbf{A}$, $\partial/\partial u \equiv \hat{\mathbf{u}} \cdot \nabla$, $\mathbf{A}_t \equiv \mathbf{A} - \hat{\mathbf{u}}A_u$, $\nabla_t \equiv \nabla - \hat{\mathbf{u}}\partial/\partial u$.

$$\mathbf{A} = \mathbf{A}_t + \hat{\mathbf{u}}A_u \quad (\text{B.80})$$

$$\nabla = \nabla_t + \hat{\mathbf{u}}\frac{\partial}{\partial u} \quad (\text{B.81})$$

$$\hat{\mathbf{u}} \cdot \mathbf{A}_t = 0 \quad (\text{B.82})$$

$$(\hat{\mathbf{u}} \cdot \nabla_t)\phi = 0 \quad (\text{B.83})$$

$$\nabla_t\phi = \nabla\phi - \hat{\mathbf{u}}\frac{\partial\phi}{\partial u} \quad (\text{B.84})$$

$$\hat{\mathbf{u}} \cdot (\nabla\phi) = (\hat{\mathbf{u}} \cdot \nabla)\phi = \frac{\partial\phi}{\partial u} \quad (\text{B.85})$$

$$\hat{\mathbf{u}} \cdot (\nabla_t\phi) = 0 \quad (\text{B.86})$$

$$\nabla_t \cdot (\hat{\mathbf{u}}\phi) = 0 \quad (\text{B.87})$$

$$\nabla_t \times (\hat{\mathbf{u}}\phi) = -\hat{\mathbf{u}} \times \nabla_t\phi \quad (\text{B.88})$$

$$\nabla_t \times (\hat{\mathbf{u}} \times \mathbf{A}) = \hat{\mathbf{u}}\nabla_t \cdot \mathbf{A}_t \quad (\text{B.89})$$

$$\hat{\mathbf{u}} \times (\nabla_t \times \mathbf{A}) = \nabla_t A_u \quad (\text{B.90})$$

$$\hat{\mathbf{u}} \times (\nabla_t \times \mathbf{A}_t) = 0 \quad (\text{B.91})$$

$$\hat{\mathbf{u}} \cdot (\hat{\mathbf{u}} \times \mathbf{A}) = 0 \quad (\text{B.92})$$

$$\hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \mathbf{A}) = -\mathbf{A}_t \quad (\text{B.93})$$

$$\nabla\phi = \nabla_t\phi + \hat{\mathbf{u}}\frac{\partial\phi}{\partial u} \quad (\text{B.94})$$

$$\nabla \cdot \mathbf{A} = \nabla_t \cdot \mathbf{A}_t + \frac{\partial A_u}{\partial u} \quad (\text{B.95})$$

$$\nabla \times \mathbf{A} = \nabla_t \times \mathbf{A}_t + \hat{\mathbf{u}} \times \left[\frac{\partial \mathbf{A}_t}{\partial u} - \nabla_t A_u \right] \quad (\text{B.96})$$

$$\nabla^2\phi = \nabla_t^2\phi + \frac{\partial^2\phi}{\partial u^2} \quad (\text{B.97})$$

$$\nabla \times \nabla \times \mathbf{A} = \left[\nabla_t \times \nabla_t \times \mathbf{A}_t - \frac{\partial^2 \mathbf{A}_t}{\partial u^2} + \nabla_t \frac{\partial A_u}{\partial u} \right] + \hat{\mathbf{u}} \left[\frac{\partial}{\partial u} (\nabla_t \cdot \mathbf{A}_t) - \nabla_t^2 A_u \right] \quad (\text{B.98})$$

$$\nabla^2 \mathbf{A} = \left[\nabla_t (\nabla_t \cdot \mathbf{A}_t) + \frac{\partial^2 \mathbf{A}_t}{\partial u^2} - \nabla_t \times \nabla_t \times \mathbf{A}_t \right] + \hat{\mathbf{u}} \nabla^2 A_u \quad (\text{B.99})$$