Appendix D

Coordinate systems

Rectangular coordinate system

Coordinate variables

$$u = x, \quad -\infty < x < \infty \tag{D.1}$$
$$v = y, \quad -\infty < y < \infty \tag{D.2}$$
$$w = z, \quad -\infty < z < \infty \tag{D.3}$$

Vector algebra

$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z \tag{D.4}$$

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \tag{D.5}$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{D.6}$$

Dyadic representation

$$\begin{aligned} \bar{\mathbf{a}} = {} & \hat{\mathbf{x}}a_{xx}\hat{\mathbf{x}} + \hat{\mathbf{x}}a_{xy}\hat{\mathbf{y}} + \hat{\mathbf{x}}a_{xz}\hat{\mathbf{z}} + \\ & + \hat{\mathbf{y}}a_{yx}\hat{\mathbf{x}} + \hat{\mathbf{y}}a_{yy}\hat{\mathbf{y}} + \hat{\mathbf{y}}a_{yz}\hat{\mathbf{z}} + \\ & + \hat{\mathbf{z}}a_{zx}\hat{\mathbf{x}} + \hat{\mathbf{z}}a_{zy}\hat{\mathbf{y}} + \hat{\mathbf{z}}a_{zz}\hat{\mathbf{z}} \end{aligned} \tag{D.7}$$

$$\bar{\mathbf{a}} = \hat{\mathbf{x}}\mathbf{a}'_x + \hat{\mathbf{y}}\mathbf{a}'_y + \hat{\mathbf{z}}\mathbf{a}'_z = \mathbf{a}_x\hat{\mathbf{x}} + \mathbf{a}_y\hat{\mathbf{y}} + \mathbf{a}_z\hat{\mathbf{z}} \tag{D.8}$$

$$\mathbf{a}'_x = a_{xx}\hat{\mathbf{x}} + a_{xy}\hat{\mathbf{y}} + a_{xz}\hat{\mathbf{z}} \tag{D.9}$$
$$\mathbf{a}'_y = a_{yx}\hat{\mathbf{x}} + a_{yy}\hat{\mathbf{y}} + a_{yz}\hat{\mathbf{z}} \tag{D.10}$$
$$\mathbf{a}'_z = a_{zx}\hat{\mathbf{x}} + a_{zy}\hat{\mathbf{y}} + a_{zz}\hat{\mathbf{z}} \tag{D.11}$$

$$\mathbf{a}_x = a_{xx}\hat{\mathbf{x}} + a_{yx}\hat{\mathbf{y}} + a_{zx}\hat{\mathbf{z}} \tag{D.12}$$
$$\mathbf{a}_y = a_{xy}\hat{\mathbf{x}} + a_{yy}\hat{\mathbf{y}} + a_{zy}\hat{\mathbf{z}} \tag{D.13}$$
$$\mathbf{a}_z = a_{xz}\hat{\mathbf{x}} + a_{yz}\hat{\mathbf{y}} + a_{zz}\hat{\mathbf{z}} \tag{D.14}$$

Differential operations

$$\mathbf{dl} = \hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz \tag{D.15}$$

$$dV = dx\,dy\,dz \tag{D.16}$$

$$dS_x = dy\,dz \tag{D.17}$$

$$dS_y = dx\,dz \tag{D.18}$$

$$dS_z = dx\,dy \tag{D.19}$$

$$\nabla f = \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z} \tag{D.20}$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{D.21}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{D.22}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{D.23}$$

$$\nabla^2 \mathbf{F} = \hat{\mathbf{x}}\nabla^2 F_x + \hat{\mathbf{y}}\nabla^2 F_y + \hat{\mathbf{z}}\nabla^2 F_z \tag{D.24}$$

Separation of the Helmholtz equation

$$\frac{\partial^2 \psi(x,y,z)}{\partial x^2} + \frac{\partial^2 \psi(x,y,z)}{\partial y^2} + \frac{\partial^2 \psi(x,y,z)}{\partial z^2} + k^2 \psi(x,y,z) = 0 \tag{D.25}$$

$$\psi(x,y,z) = X(x)Y(y)Z(z) \tag{D.26}$$

$$k_x^2 + k_y^2 + k_z^2 = k^2 \tag{D.27}$$

$$\frac{d^2 X(x)}{dx^2} + k_x^2 X(x) = 0 \tag{D.28}$$

$$\frac{d^2 Y(y)}{dy^2} + k_y^2 Y(y) = 0 \tag{D.29}$$

$$\frac{d^2 Z(z)}{dz^2} + k_z^2 Z(z) = 0 \tag{D.30}$$

$$X(x) = \begin{cases} A_x F_1(k_x x) + B_x F_2(k_x x), & k_x \neq 0, \\ a_x x + b_x, & k_x = 0. \end{cases} \tag{D.31}$$

$$Y(y) = \begin{cases} A_y F_1(k_y y) + B_y F_2(k_y y), & k_y \neq 0, \\ a_y y + b_y, & k_y = 0. \end{cases} \tag{D.32}$$

$$Z(z) = \begin{cases} A_z F_1(k_z z) + B_z F_2(k_z z), & k_z \neq 0, \\ a_z z + b_z, & k_z = 0. \end{cases} \tag{D.33}$$

$$F_1(\xi), F_2(\xi) = \begin{cases} e^{j\xi} \\ e^{-j\xi} \\ \sin(\xi) \\ \cos(\xi) \end{cases} \tag{D.34}$$

Cylindrical coordinate system

Coordinate variables

$$u = \rho, \quad 0 \leq \rho < \infty \tag{D.35}$$
$$v = \phi, \quad -\pi \leq \phi \leq \pi \tag{D.36}$$
$$w = z, \quad -\infty < z < \infty \tag{D.37}$$

$$x = \rho \cos \phi \tag{D.38}$$
$$y = \rho \sin \phi \tag{D.39}$$
$$z = z \tag{D.40}$$

$$\rho = \sqrt{x^2 + y^2} \tag{D.41}$$
$$\phi = \tan^{-1} \frac{y}{x} \tag{D.42}$$
$$z = z \tag{D.43}$$

Vector algebra

$$\hat{\boldsymbol{\rho}} = \hat{\mathbf{x}} \cos \phi + \hat{\mathbf{y}} \sin \phi \tag{D.44}$$
$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin \phi + \hat{\mathbf{y}} \cos \phi \tag{D.45}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}} \tag{D.46}$$

$$\mathbf{A} = \hat{\boldsymbol{\rho}} A_\rho + \hat{\boldsymbol{\phi}} A_\phi + \hat{\mathbf{z}} A_z \tag{D.47}$$

$$\mathbf{A} \cdot \mathbf{B} = A_\rho B_\rho + A_\phi B_\phi + A_z B_z \tag{D.48}$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\boldsymbol{\rho}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_\rho & A_\phi & A_z \\ B_\rho & B_\phi & B_z \end{vmatrix} \tag{D.49}$$

Dyadic representation

$$\begin{aligned}\bar{\mathbf{a}} = {} & \hat{\boldsymbol{\rho}} a_{\rho\rho} \hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\rho}} a_{\rho\phi} \hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\rho}} a_{\rho z} \hat{\mathbf{z}} + \\ & + \hat{\boldsymbol{\phi}} a_{\phi\rho} \hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\phi}} a_{\phi\phi} \hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}} a_{\phi z} \hat{\mathbf{z}} + \\ & + \hat{\mathbf{z}} a_{z\rho} \hat{\boldsymbol{\rho}} + \hat{\mathbf{z}} a_{z\phi} \hat{\boldsymbol{\phi}} + \hat{\mathbf{z}} a_{zz} \hat{\mathbf{z}} \end{aligned} \tag{D.50}$$

$$\bar{\mathbf{a}} = \hat{\boldsymbol{\rho}} \mathbf{a}'_\rho + \hat{\boldsymbol{\phi}} \mathbf{a}'_\phi + \hat{\mathbf{z}} \mathbf{a}'_z = \mathbf{a}_\rho \hat{\boldsymbol{\rho}} + \mathbf{a}_\phi \hat{\boldsymbol{\phi}} + \mathbf{a}_z \hat{\mathbf{z}} \tag{D.51}$$

$$\mathbf{a}'_\rho = a_{\rho\rho} \hat{\boldsymbol{\rho}} + a_{\rho\phi} \hat{\boldsymbol{\phi}} + a_{\rho z} \hat{\mathbf{z}} \tag{D.52}$$

$$\mathbf{a}'_\phi = a_{\phi\rho} \hat{\boldsymbol{\rho}} + a_{\phi\phi} \hat{\boldsymbol{\phi}} + a_{\phi z} \hat{\mathbf{z}} \tag{D.53}$$

$$\mathbf{a}'_z = a_{z\rho} \hat{\boldsymbol{\rho}} + a_{z\phi} \hat{\boldsymbol{\phi}} + a_{zz} \hat{\mathbf{z}} \tag{D.54}$$

$$\mathbf{a}_\rho = a_{\rho\rho} \hat{\boldsymbol{\rho}} + a_{\phi\rho} \hat{\boldsymbol{\phi}} + a_{z\rho} \hat{\mathbf{z}} \tag{D.55}$$

$$\mathbf{a}_\phi = a_{\rho\phi} \hat{\boldsymbol{\rho}} + a_{\phi\phi} \hat{\boldsymbol{\phi}} + a_{z\phi} \hat{\mathbf{z}} \tag{D.56}$$

$$\mathbf{a}_z = a_{\rho z} \hat{\boldsymbol{\rho}} + a_{\phi z} \hat{\boldsymbol{\phi}} + a_{zz} \hat{\mathbf{z}} \tag{D.57}$$

Differential operations

$$\mathbf{dl} = \hat{\boldsymbol{\rho}}\, d\rho + \hat{\boldsymbol{\phi}} \rho\, d\phi + \hat{\mathbf{z}}\, dz \tag{D.58}$$

$$dV = \rho\, d\rho\, d\phi\, dz \tag{D.59}$$

$$dS_\rho = \rho\, d\phi\, dz, \tag{D.60}$$

$$dS_\phi = d\rho\, dz, \tag{D.61}$$

$$dS_z = \rho\, d\rho\, d\phi \tag{D.62}$$

$$\nabla f = \hat{\boldsymbol{\rho}} \frac{\partial f}{\partial \rho} + \hat{\boldsymbol{\phi}} \frac{1}{\rho} \frac{\partial f}{\partial \phi} + \hat{\mathbf{z}} \frac{\partial f}{\partial z} \tag{D.63}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho F_\rho \right) + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \tag{D.64}$$

$$\nabla \times \mathbf{F} = \frac{1}{\rho} \begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} \tag{D.65}$$

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \tag{D.66}$$

$$\nabla^2 \mathbf{F} = \hat{\boldsymbol{\rho}} \left(\nabla^2 F_\rho - \frac{2}{\rho^2} \frac{\partial F_\phi}{\partial \phi} - \frac{F_\rho}{\rho^2} \right) + \hat{\boldsymbol{\phi}} \left(\nabla^2 F_\phi + \frac{2}{\rho^2} \frac{\partial F_\rho}{\partial \phi} - \frac{F_\phi}{\rho^2} \right) + \hat{\mathbf{z}} \nabla^2 F_z \tag{D.67}$$

Separation of the Helmholtz equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi(\rho,\phi,z)}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi(\rho,\phi,z)}{\partial\phi^2}+\frac{\partial^2\psi(\rho,\phi,z)}{\partial z^2}+k^2\psi(\rho,\phi,z)=0 \tag{D.68}$$

$$\psi(\rho,\phi,z)=P(\rho)\Phi(\phi)Z(z) \tag{D.69}$$

$$k_c^2=k^2-k_z^2 \tag{D.70}$$

$$\frac{d^2P(\rho)}{d\rho^2}+\frac{1}{\rho}\frac{dP(\rho)}{d\rho}+\left(k_c^2-\frac{k_\phi^2}{\rho^2}\right)P(\rho)=0 \tag{D.71}$$

$$\frac{\partial^2\Phi(\phi)}{\partial\phi^2}+k_\phi^2\Phi(\phi)=0 \tag{D.72}$$

$$\frac{d^2Z(z)}{dz^2}+k_z^2Z(z)=0 \tag{D.73}$$

$$Z(z)=\begin{cases}A_zF_1(k_zz)+B_zF_2(k_zz), & k_z\neq 0,\\ a_zz+b_z, & k_z=0.\end{cases} \tag{D.74}$$

$$\Phi(\phi)=\begin{cases}A_\phi F_1(k_\phi\phi)+B_\phi F_2(k_\phi\phi), & k_\phi\neq 0,\\ a_\phi\phi+b_\phi, & k_\phi=0.\end{cases} \tag{D.75}$$

$$P(\rho)=\begin{cases}a_\rho\ln\rho+b_\rho, & k_c=k_\phi=0,\\ a_\rho\rho^{-k_\phi}+b_\rho\rho^{k_\phi}, & k_c=0 \text{ and } k_\phi\neq 0,\\ A_\rho G_1(k_c\rho)+B_\rho G_2(k_c\rho), & \text{otherwise.}\end{cases} \tag{D.76}$$

$$F_1(\xi),F_2(\xi)=\begin{cases}e^{j\xi}\\ e^{-j\xi}\\ \sin(\xi)\\ \cos(\xi)\end{cases} \tag{D.77}$$

$$G_1(\xi),G_2(\xi)=\begin{cases}J_{k_\phi}(\xi)\\ N_{k_\phi}(\xi)\\ H^{(1)}_{k_\phi}(\xi)\\ H^{(2)}_{k_\phi}(\xi)\end{cases} \tag{D.78}$$

Spherical coordinate system

Coordinate variables

$$u=r,\quad 0\le r<\infty \tag{D.79}$$

$$v=\theta,\quad 0\le\theta\le\pi \tag{D.80}$$

$$w=\phi,\quad -\pi\le\phi\le\pi \tag{D.81}$$

$$x = r\sin\theta\cos\phi \tag{D.82}$$
$$y = r\sin\theta\sin\phi \tag{D.83}$$
$$z = r\cos\theta \tag{D.84}$$

$$r = \sqrt{x^2 + y^2 + z^2} \tag{D.85}$$
$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} \tag{D.86}$$
$$\phi = \tan^{-1}\frac{y}{x} \tag{D.87}$$

Vector algebra

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta \tag{D.88}$$
$$\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta \tag{D.89}$$
$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi \tag{D.90}$$

$$\mathbf{A} = \hat{\mathbf{r}}A_r + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi \tag{D.91}$$

$$\mathbf{A}\cdot\mathbf{B} = A_rB_r + A_\theta B_\theta + A_\phi B_\phi \tag{D.92}$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_r & A_\theta & A_\phi \\ B_r & B_\theta & B_\phi \end{vmatrix} \tag{D.93}$$

Dyadic representation

$$\begin{aligned}\bar{\mathbf{a}} = {} & \hat{\mathbf{r}}a_{rr}\hat{\mathbf{r}} + \hat{\mathbf{r}}a_{r\theta}\hat{\boldsymbol{\theta}} + \hat{\mathbf{r}}a_{r\phi}\hat{\boldsymbol{\phi}} + \\ & + \hat{\boldsymbol{\theta}}a_{\theta r}\hat{\mathbf{r}} + \hat{\boldsymbol{\theta}}a_{\theta\theta}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\theta}}a_{\theta\phi}\hat{\boldsymbol{\phi}} + \\ & + \hat{\boldsymbol{\phi}}a_{\phi r}\hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}a_{\phi\theta}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}a_{\phi\phi}\hat{\boldsymbol{\phi}}\end{aligned} \tag{D.94}$$

$$\bar{\mathbf{a}} = \hat{\mathbf{r}}\mathbf{a}'_r + \hat{\boldsymbol{\theta}}\mathbf{a}'_\theta + \hat{\boldsymbol{\phi}}\mathbf{a}'_\phi = \mathbf{a}_r\hat{\mathbf{r}} + \mathbf{a}_\theta\hat{\boldsymbol{\theta}} + \mathbf{a}_\phi\hat{\boldsymbol{\phi}} \tag{D.95}$$

$$\mathbf{a}'_r = a_{rr}\hat{\mathbf{r}} + a_{r\theta}\hat{\boldsymbol{\theta}} + a_{r\phi}\hat{\boldsymbol{\phi}} \tag{D.96}$$
$$\mathbf{a}'_\theta = a_{\theta r}\hat{\mathbf{r}} + a_{\theta\theta}\hat{\boldsymbol{\theta}} + a_{\theta\phi}\hat{\boldsymbol{\phi}} \tag{D.97}$$
$$\mathbf{a}'_\phi = a_{\phi r}\hat{\mathbf{r}} + a_{\phi\theta}\hat{\boldsymbol{\theta}} + a_{\phi\phi}\hat{\boldsymbol{\phi}} \tag{D.98}$$

$$\mathbf{a}_r = a_{rr}\hat{\mathbf{r}} + a_{\theta r}\hat{\boldsymbol{\theta}} + a_{\phi r}\hat{\boldsymbol{\phi}} \tag{D.99}$$
$$\mathbf{a}_\theta = a_{r\theta}\hat{\mathbf{r}} + a_{\theta\theta}\hat{\boldsymbol{\theta}} + a_{\phi\theta}\hat{\boldsymbol{\phi}} \tag{D.100}$$
$$\mathbf{a}_\phi = a_{r\phi}\hat{\mathbf{r}} + a_{\theta\phi}\hat{\boldsymbol{\theta}} + a_{\phi\phi}\hat{\boldsymbol{\phi}} \tag{D.101}$$

Differential operations

$$\mathbf{dl} = \hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\theta}} r\,d\theta + \hat{\boldsymbol{\phi}} r\sin\theta\,d\phi \tag{D.102}$$

$$dV = r^2 \sin\theta\,dr\,d\theta\,d\phi \tag{D.103}$$

$$dS_r = r^2\sin\theta\,d\theta\,d\phi \tag{D.104}$$

$$dS_\theta = r\sin\theta\,dr\,d\phi \tag{D.105}$$

$$dS_\phi = r\,dr\,d\theta \tag{D.106}$$

$$\nabla f = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} \tag{D.107}$$

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta F_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi} \tag{D.108}$$

$$\nabla\times\mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix} \tag{D.109}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \tag{D.110}$$

$$\begin{aligned}\nabla^2\mathbf{F} = {} & \hat{\mathbf{r}}\left[\nabla^2 F_r - \frac{2}{r^2}\left(F_r + \frac{\cos\theta}{\sin\theta}F_\theta + \frac{1}{\sin\theta}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_\theta}{\partial\theta}\right)\right] + \\ & + \hat{\boldsymbol{\theta}}\left[\nabla^2 F_\theta - \frac{1}{r^2}\left(\frac{1}{\sin^2\theta}F_\theta - 2\frac{\partial F_r}{\partial\theta} + 2\frac{\cos\theta}{\sin^2\theta}\frac{\partial F_\phi}{\partial\phi}\right)\right] + \\ & + \hat{\boldsymbol{\phi}}\left[\nabla^2 F_\phi - \frac{1}{r^2}\left(\frac{1}{\sin^2\theta}F_\phi - 2\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - 2\frac{\cos\theta}{\sin^2\theta}\frac{\partial F_\theta}{\partial\phi}\right)\right]\end{aligned} \tag{D.111}$$

Separation of the Helmholtz equation

$$\begin{aligned}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi(r,\theta,\phi)}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi(r,\theta,\phi)}{\partial\theta}\right) + \\ + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi(r,\theta,\phi)}{\partial\phi^2} + k^2\psi(r,\theta,\phi) = 0\end{aligned} \tag{D.112}$$

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \tag{D.113}$$

$$\eta = \cos\theta \tag{D.114}$$

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) + k^2r^2 = n(n+1) \tag{D.115}$$

$$(1-\eta^2)\frac{d^2\Theta(\eta)}{d\eta^2} - 2\eta\frac{d\Theta(\eta)}{d\eta} + \left[n(n+1) - \frac{\mu^2}{1-\eta^2}\right]\Theta(\eta) = 0, \quad -1 \le \eta \le 1 \tag{D.116}$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} + \mu^2\Phi(\phi) = 0 \tag{D.117}$$

$$\Phi(\phi) = \begin{cases} A_\phi \sin(\mu\phi) + B_\phi \cos(\mu\phi), & \mu \neq 0, \\ a_\phi \phi + b_\phi, & \mu = 0. \end{cases} \tag{D.118}$$

$$\Theta(\theta) = A_\theta P_n^\mu(\cos\theta) + B_\theta Q_n^\mu(\cos\theta) \tag{D.119}$$

$$R(r) = \begin{cases} R(r) = A_r r^n + B_r r^{-(n+1)}, & k = 0, \\ A_r F_1(kr) + B_r F_2(kr), & \text{otherwise.} \end{cases} \tag{D.120}$$

$$F_1(\xi), F_2(\xi) = \begin{cases} j_n(\xi) \\ n_n(\xi) \\ h_n^{(1)}(\xi) \\ h_n^{(2)}(\xi) \end{cases} \tag{D.121}$$