Appendix E

Properties of special functions

E.1 Bessel functions

Notation

z = complex number; ν, x = real numbers; n = integer
$J_\nu(z)$ = ordinary Bessel function of the first kind
$N_\nu(z)$ = ordinary Bessel function of the second kind
$I_\nu(z)$ = modified Bessel function of the first kind
$K_\nu(z)$ = modified Bessel function of the second kind
$H_\nu^{(1)}$ = Hankel function of the first kind
$H_\nu^{(2)}$ = Hankel function of the second kind
$j_n(z)$ = ordinary spherical Bessel function of the first kind
$n_n(z)$ = ordinary spherical Bessel function of the second kind
$h_n^{(1)}(z)$ = spherical Hankel function of the first kind
$h_n^{(2)}(z)$ = spherical Hankel function of the second kind
$f'(z) = df(z)/dz$ = derivative with respect to argument

Differential equations

$$\frac{d^2 Z_\nu(z)}{dz^2} + \frac{1}{z}\frac{dZ_\nu(z)}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) Z_\nu(z) = 0 \tag{E.1}$$

$$Z_\nu(z) = \begin{cases} J_\nu(z) \\ N_\nu(z) \\ H_\nu^{(1)}(z) \\ H_\nu^{(2)}(z) \end{cases} \tag{E.2}$$

$$N_\nu(z) = \frac{\cos(\nu\pi) J_\nu(z) - J_{-\nu}(z)}{\sin(\nu\pi)}, \quad \nu \neq n, \quad |\arg(z)| < \pi \tag{E.3}$$

$$H_\nu^{(1)}(z) = J_\nu(z) + jN_\nu(z) \tag{E.4}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - jN_\nu(z) \tag{E.5}$$

$$\frac{d^2\bar{Z}_\nu(x)}{dz^2} + \frac{1}{z}\frac{d\bar{Z}_\nu(z)}{dz} - \left(1 + \frac{\nu^2}{z^2}\right)\bar{Z}_\nu = 0 \tag{E.6}$$

$$\bar{Z}_\nu(z) = \begin{cases} I_\nu(z) \\ K_\nu(z) \end{cases} \tag{E.7}$$

$$L(z) = \begin{cases} I_\nu(z) \\ e^{j\nu\pi}K_\nu(z) \end{cases} \tag{E.8}$$

$$I_\nu(z) = e^{-j\nu\pi/2}J_\nu(ze^{j\pi/2}), \quad -\pi < \arg(z) \le \frac{\pi}{2} \tag{E.9}$$

$$I_\nu(z) = e^{j3\nu\pi/2}J_\nu(ze^{-j3\pi/2}), \quad \frac{\pi}{2} < \arg(z) \le \pi \tag{E.10}$$

$$K_\nu(z) = \frac{j\pi}{2}e^{j\nu\pi/2}H_\nu^{(1)}(ze^{j\pi/2}), \quad -\pi < \arg(z) \le \frac{\pi}{2} \tag{E.11}$$

$$K_\nu(z) = -\frac{j\pi}{2}e^{-j\nu\pi/2}H_\nu^{(2)}(ze^{-j\pi/2}), \quad -\frac{\pi}{2} < \arg(z) \le \pi \tag{E.12}$$

$$I_n(x) = j^{-n}J_n(jx) \tag{E.13}$$

$$K_n(x) = \frac{\pi}{2}j^{n+1}H_n^{(1)}(jx) \tag{E.14}$$

$$\frac{d^2z_n(z)}{dz^2} + \frac{2}{z}\frac{dz_n(z)}{dz} + \left[1 - \frac{n(n+1)}{z^2}\right]z_n(z) = 0, \quad n = 0, \pm 1, \pm 2, \ldots \tag{E.15}$$

$$z_n(z) = \begin{cases} j_n(z) \\ n_n(z) \\ h_n^{(1)}(z) \\ h_n^{(2)}(z) \end{cases} \tag{E.16}$$

$$j_n(z) = \sqrt{\frac{\pi}{2z}}J_{n+\frac{1}{2}}(z) \tag{E.17}$$

$$n_n(z) = \sqrt{\frac{\pi}{2z}}N_{n+\frac{1}{2}}(z) \tag{E.18}$$

$$h_n^{(1)}(z) = \sqrt{\frac{\pi}{2z}}H_{n+\frac{1}{2}}^{(1)}(z) = j_n(z) + jn_n(z) \tag{E.19}$$

$$h_n^{(2)}(z) = \sqrt{\frac{\pi}{2z}}H_{n+\frac{1}{2}}^{(2)}(z) = j_n(z) - jn_n(z) \tag{E.20}$$

$$n_n(z) = (-1)^{n+1}j_{-(n+1)}(z) \tag{E.21}$$

Orthogonality relationships

$$\int_0^a J_\nu\left(\frac{p_{\nu m}}{a}\rho\right)J_\nu\left(\frac{p_{\nu n}}{a}\rho\right)\rho\,d\rho = \delta_{mn}\frac{a^2}{2}J_{\nu+1}^2(p_{\nu n}) = \delta_{mn}\frac{a^2}{2}\left[J_\nu'(p_{\nu n})\right]^2, \quad \nu > -1 \tag{E.22}$$

$$\int_0^a J_\nu\left(\frac{p'_{\nu m}}{a}\rho\right) J_\nu\left(\frac{p'_{\nu n}}{a}\rho\right)\rho\, d\rho = \delta_{mn}\frac{a^2}{2}\left(1-\frac{\nu^2}{p'^2_{\nu m}}\right) J_\nu^2(p'_{\nu m}), \quad \nu > -1 \tag{E.23}$$

$$\int_0^\infty J_\nu(\alpha x) J_\nu(\beta x) x\, dx = \frac{1}{\alpha}\delta(\alpha-\beta) \tag{E.24}$$

$$\int_0^a j_l\left(\frac{\alpha_{lm}}{a}r\right) j_l\left(\frac{\alpha_{ln}}{a}r\right) r^2\, dr = \delta_{mn}\frac{a^3}{2} j_{n+1}^2(\alpha_{ln}a) \tag{E.25}$$

$$\int_{-\infty}^\infty j_m(x) j_n(x)\, dx = \delta_{mn}\frac{\pi}{2n+1}, \quad m, n \geq 0 \tag{E.26}$$

$$J_m(p_{mn}) = 0 \tag{E.27}$$

$$J'_m(p'_{mn}) = 0 \tag{E.28}$$

$$j_m(\alpha_{mn}) = 0 \tag{E.29}$$

$$j'_m(\alpha'_{mn}) = 0 \tag{E.30}$$

Specific examples

$$j_0(z) = \frac{\sin z}{z} \tag{E.31}$$

$$n_0(z) = -\frac{\cos z}{z} \tag{E.32}$$

$$h_0^{(1)}(z) = -\frac{j}{z}e^{jz} \tag{E.33}$$

$$h_0^{(2)}(z) = \frac{j}{z}e^{-jz} \tag{E.34}$$

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z} \tag{E.35}$$

$$n_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z} \tag{E.36}$$

$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right)\sin z - \frac{3}{z^2}\cos z \tag{E.37}$$

$$n_2(z) = \left(-\frac{3}{z^3} + \frac{1}{z}\right)\cos z - \frac{3}{z^2}\sin z \tag{E.38}$$

Functional relationships

$$J_n(-z) = (-1)^n J_n(z) \tag{E.39}$$

$$I_n(-z) = (-1)^n I_n(z) \tag{E.40}$$

$$j_n(-z) = (-1)^n j_n(z) \tag{E.41}$$

$$n_n(-z) = (-1)^{n+1} n_n(z) \tag{E.42}$$

$$J_{-n}(z) = (-1)^n J_n(z) \tag{E.43}$$

$$N_{-n}(z) = (-1)^n N_n(z) \tag{E.44}$$

$$I_{-n}(z) = I_n(z) \tag{E.45}$$

$$K_{-n}(z) = K_n(z) \tag{E.46}$$

$$j_{-n}(z) = (-1)^n n_{n-1}(z), \quad n > 0 \tag{E.47}$$

Power series

$$J_n(z) = \sum_{k=0}^{\infty} (-1)^k \frac{(z/2)^{n+2k}}{k!(n+k)!} \tag{E.48}$$

$$I_n(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{n+2k}}{k!(n+k)!} \tag{E.49}$$

Small argument approximations $|z| \ll 1.$

$$J_n(z) \approx \frac{1}{n!} \left(\frac{z}{2}\right)^n \tag{E.50}$$

$$J_\nu(z) \approx \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu \tag{E.51}$$

$$N_0(z) \approx \frac{2}{\pi}(\ln z + 0.5772157 - \ln 2) \tag{E.52}$$

$$N_n(z) \approx -\frac{(n-1)!}{\pi} \left(\frac{2}{z}\right)^n, \quad n > 0 \tag{E.53}$$

$$N_\nu(z) \approx -\frac{\Gamma(\nu)}{\pi} \left(\frac{2}{z}\right)^\nu, \quad \nu > 0 \tag{E.54}$$

$$I_n(z) \approx \frac{1}{n!} \left(\frac{z}{2}\right)^n \tag{E.55}$$

$$I_\nu(z) \approx \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu \tag{E.56}$$

$$j_n(z) \approx \frac{2^n n!}{(2n+1)!} z^n \tag{E.57}$$

$$n_n(z) \approx -\frac{(2n)!}{2^n n!} z^{-(n+1)} \tag{E.58}$$

Large argument approximations $|z| \gg 1.$

$$J_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4} - \frac{\nu\pi}{2}\right), \quad |\arg(z)| < \pi \tag{E.59}$$

$$N_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4} - \frac{\nu\pi}{2}\right), \quad |\arg(z)| < \pi \tag{E.60}$$

$$H_\nu^{(1)}(z) \approx \sqrt{\frac{2}{\pi z}} e^{j\left(z - \frac{\pi}{4} - \frac{\nu\pi}{2}\right)}, \quad -\pi < \arg(z) < 2\pi \tag{E.61}$$

$$H_\nu^{(2)}(z) \approx \sqrt{\frac{2}{\pi z}} e^{-j\left(z - \frac{\pi}{4} - \frac{\nu\pi}{2}\right)}, \quad -2\pi < \arg(z) < \pi \tag{E.62}$$

$$I_\nu(z) \approx \sqrt{\frac{1}{2\pi z}} e^z, \quad |\arg(z)| < \frac{\pi}{2} \tag{E.63}$$

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z}, \quad |\arg(z)| < \frac{3\pi}{2} \tag{E.64}$$

$$j_n(z) \approx \frac{1}{z} \sin\left(z - \frac{n\pi}{2}\right), \quad |\arg(z)| < \pi \tag{E.65}$$

$$n_n(z) \approx -\frac{1}{z} \cos\left(z - \frac{n\pi}{2}\right), \quad |\arg(z)| < \pi \tag{E.66}$$

$$h_n^{(1)}(z) \approx (-j)^{n+1} \frac{e^{jz}}{z}, \quad -\pi < \arg(z) < 2\pi \tag{E.67}$$

$$h_n^{(2)}(z) \approx j^{n+1} \frac{e^{-jz}}{z}, \quad -2\pi < \arg(z) < \pi \tag{E.68}$$

Recursion relationships

$$zZ_{\nu-1}(z) + zZ_{\nu+1}(z) = 2\nu Z_\nu(z) \tag{E.69}$$

$$Z_{\nu-1}(z) - Z_{\nu+1}(z) = 2Z'_\nu(z) \tag{E.70}$$

$$zZ'_\nu(z) + \nu Z_\nu(z) = zZ_{\nu-1}(z) \tag{E.71}$$

$$zZ'_\nu(z) - \nu Z_\nu(z) = -zZ_{\nu+1}(z) \tag{E.72}$$

$$zL_{\nu-1}(z) - zL_{\nu+1}(z) = 2\nu L_\nu(z) \tag{E.73}$$

$$L_{\nu-1}(z) + L_{\nu+1}(z) = 2L'_\nu(z) \tag{E.74}$$

$$zL'_\nu(z) + \nu L_\nu(z) = zL_{\nu-1}(z) \tag{E.75}$$

$$zL'_\nu(z) - \nu L_\nu(z) = zL_{\nu+1}(z) \tag{E.76}$$

$$zz_{n-1}(z) + zz_{n+1}(z) = (2n+1)z_n(z) \tag{E.77}$$

$$nz_{n-1}(z) - (n+1)z_{n+1}(z) = (2n+1)z'_n(z) \tag{E.78}$$

$$zz'_n(z) + (n+1)z_n(z) = zz_{n-1}(z) \tag{E.79}$$

$$-zz'_n(z) + nz_n(z) = zz_{n+1}(z) \tag{E.80}$$

Integral representations

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-jn\theta + jz\sin\theta}\, d\theta \tag{E.81}$$

$$J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - z\sin\theta)\, d\theta \tag{E.82}$$

$$J_n(z) = \frac{1}{2\pi} j^{-n} \int_{-\pi}^{\pi} e^{jz\cos\theta} \cos(n\theta)\, d\theta \tag{E.83}$$

$$I_n(z) = \frac{1}{\pi} \int_0^{\pi} e^{z\cos\theta} \cos(n\theta)\, d\theta \tag{E.84}$$

$$K_n(z) = \int_0^{\infty} e^{-z\cosh(t)} \cosh(nt)\, dt, \quad |\arg(z)| < \frac{\pi}{2} \tag{E.85}$$

$$j_n(z) = \frac{z^n}{2^{n+1} n!} \int_0^{\pi} \cos(z\cos\theta) \sin^{2n+1}\theta\, d\theta \tag{E.86}$$

$$j_n(z) = \frac{(-j)^n}{2} \int_0^{\pi} e^{jz\cos\theta} P_n(\cos\theta) \sin\theta\, d\theta \tag{E.87}$$

Wronskians and cross products

$$J_\nu(z)N_{\nu+1}(z) - J_{\nu+1}(z)N_\nu(z) = -\frac{2}{\pi z} \tag{E.88}$$

$$H_\nu^{(2)}(z)H_{\nu+1}^{(1)}(z) - H_\nu^{(1)}(z)H_{\nu+1}^{(2)}(z) = \frac{4}{j\pi z} \tag{E.89}$$

$$I_\nu(z)K_{\nu+1}(z) + I_{\nu+1}(z)K_\nu(z) = \frac{1}{z} \tag{E.90}$$

$$I_\nu(z)K_\nu'(z) - I_\nu'(z)K_\nu(z) = -\frac{1}{z} \tag{E.91}$$

$$J_\nu(z)H_\nu^{(1)\prime}(z) - J_\nu'(z)H_\nu^{(1)}(z) = \frac{2j}{\pi z} \tag{E.92}$$

$$J_\nu(z)H_\nu^{(2)\prime}(z) - J_\nu'(z)H_\nu^{(2)}(z) = -\frac{2j}{\pi z} \tag{E.93}$$

$$H_\nu^{(1)}(z)H_\nu^{(2)\prime}(z) - H_\nu^{(1)\prime}(z)H_\nu^{(2)}(z) = -\frac{4j}{\pi z} \tag{E.94}$$

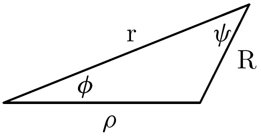
$$j_n(z)n_{n-1}(z) - j_{n-1}(z)n_n(z) = \frac{1}{z^2} \tag{E.95}$$

$$j_{n+1}(z)n_{n-1}(z) - j_{n-1}(z)n_{n+1}(z) = \frac{2n+1}{z^3} \tag{E.96}$$

$$j_n(z)n_n'(z) - j_n'(z)n_n(z) = \frac{1}{z^2} \tag{E.97}$$

$$h_n^{(1)}(z)h_n^{(2)\prime}(z) - h_n^{(1)\prime}(z)h_n^{(2)}(z) = -\frac{2j}{z^2} \tag{E.98}$$

Summation formulas



R, r, ρ, ϕ, ψ as shown.
$R = \sqrt{r^2 + \rho^2 - 2r\rho\cos\phi}$.

$$e^{j\nu\psi}Z_\nu(zR) = \sum_{k=-\infty}^{\infty} J_k(z\rho)Z_{\nu+k}(zr)e^{jk\phi}, \quad \rho < r, \quad 0 < \psi < \frac{\pi}{2} \tag{E.99}$$

$$e^{jn\psi}J_n(zR) = \sum_{k=-\infty}^{\infty} J_k(z\rho)J_{n+k}(zr)e^{jk\phi} \tag{E.100}$$

$$e^{jz\rho\cos\phi} = \sum_{k=0}^{\infty} j^k(2k+1)j_k(z\rho)P_k(\cos\phi) \tag{E.101}$$

For $\rho < r$ and $0 < \psi < \pi/2$,

$$\frac{e^{jzR}}{R} = \frac{j\pi}{2\sqrt{r\rho}}\sum_{k=0}^{\infty}(2k+1)J_{k+\frac{1}{2}}(z\rho)H_{k+\frac{1}{2}}^{(1)}(zr)P_k(\cos\phi) \tag{E.102}$$

$$\frac{e^{-jzR}}{R} = -\frac{j\pi}{2\sqrt{r\rho}}\sum_{k=0}^{\infty}(2k+1)J_{k+\frac{1}{2}}(z\rho)H_{k+\frac{1}{2}}^{(2)}(zr)P_k(\cos\phi) \tag{E.103}$$

Integrals

$$\int x^{\nu+1} J_\nu(x)\,dx = x^{\nu+1} J_{\nu+1}(x) + C \tag{E.104}$$

$$\int Z_\nu(ax)Z_\nu(bx)x\,dx = x\frac{[bZ_\nu(ax)Z_{\nu-1}(bx) - aZ_{\nu-1}(ax)Z_\nu(bx)]}{a^2-b^2} + C, \quad a \neq b \tag{E.105}$$

$$\int xZ_\nu^2(ax)\,dx = \frac{x^2}{2}\left[Z_\nu^2(ax) - Z_{\nu-1}(ax)Z_{\nu+1}(ax)\right] + C \tag{E.106}$$

$$\int_0^\infty J_\nu(ax)\,dx = \frac{1}{a}, \quad \nu > -1, \quad a > 0 \tag{E.107}$$

Fourier–Bessel expansion of a function

$$f(\rho) = \sum_{m=1}^\infty a_m J_\nu\left(p_{\nu m}\frac{\rho}{a}\right), \quad 0 \le \rho \le a, \quad \nu > -1 \tag{E.108}$$

$$a_m = \frac{2}{a^2 J_{\nu+1}^2(p_{\nu m})}\int_0^a f(\rho)J_\nu\left(p_{\nu m}\frac{\rho}{a}\right)\rho\,d\rho \tag{E.109}$$

$$f(\rho) = \sum_{m=1}^\infty b_m J_\nu\left(p'_{\nu m}\frac{\rho}{a}\right), \quad 0 \le \rho \le a, \quad \nu > -1 \tag{E.110}$$

$$b_m = \frac{2}{a^2\left(1 - \frac{\nu^2}{p'^2_{\nu m}}J_\nu^2(p'_{\nu m})\right)}\int_0^a f(\rho)J_\nu\left(\frac{p'_{\nu m}}{a}\rho\right)\rho\,d\rho \tag{E.111}$$

Series of Bessel functions

$$e^{jz\cos\phi} = \sum_{k=-\infty}^\infty j^k J_k(z)e^{jk\phi} \tag{E.112}$$

$$e^{jz\cos\phi} = J_0(z) + 2\sum_{k=1}^\infty j^k J_k(z)\cos\phi \tag{E.113}$$

$$\sin z = 2\sum_{k=0}^\infty (-1)^k J_{2k+1}(z) \tag{E.114}$$

$$\cos z = J_0(z) + 2\sum_{k=1}^\infty (-1)^k J_{2k}(z) \tag{E.115}$$

E.2 Legendre functions

Notation

x, y, θ = real numbers; l, m, n = integers;
$P_n^m(\cos\theta)$ = associated Legendre function of the first kind

$Q_n^m(\cos\theta)$ = associated Legendre function of the second kind
$P_n(\cos\theta) = P_n^0(\cos\theta)$ = Legendre polynomial
$Q_n(\cos\theta) = Q_n^0(\cos\theta)$ = Legendre function of the second kind

Differential equation $x = \cos\theta$.

$$(1-x^2)\frac{d^2R_n^m(x)}{dx^2} - 2x\frac{dR_n^m(x)}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right]R_n^m(x) = 0, \quad -1 \le x \le 1 \quad \text{(E.116)}$$

$$R_n^m(x) = \begin{cases} P_n^m(x) \\ Q_n^m(x) \end{cases} \quad \text{(E.117)}$$

Orthogonality relationships

$$\int_{-1}^{1} P_l^m(x)P_n^m(x)\,dx = \delta_{ln}\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} \quad \text{(E.118)}$$

$$\int_0^{\pi} P_l^m(\cos\theta)P_n^m(\cos\theta)\sin\theta\,d\theta = \delta_{ln}\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} \quad \text{(E.119)}$$

$$\int_{-1}^{1} \frac{P_n^m(x)P_n^k(x)}{1-x^2}\,dx = \delta_{mk}\frac{1}{m}\frac{(n+m)!}{(n-m)!} \quad \text{(E.120)}$$

$$\int_0^{\pi} \frac{P_n^m(\cos\theta)P_n^k(\cos\theta)}{\sin\theta}\,d\theta = \delta_{mk}\frac{1}{m}\frac{(n+m)!}{(n-m)!} \quad \text{(E.121)}$$

$$\int_{-1}^{1} P_l(x)P_n(x)\,dx = \delta_{ln}\frac{2}{2n+1} \quad \text{(E.122)}$$

$$\int_0^{\pi} P_l(\cos\theta)P_n(\cos\theta)\sin\theta\,d\theta = \delta_{ln}\frac{2}{2n+1} \quad \text{(E.123)}$$

Specific examples

$$P_0(x) = 1 \quad \text{(E.124)}$$

$$P_1(x) = x = \cos(\theta) \quad \text{(E.125)}$$

$$P_2(x) = \frac{1}{2}(3x^2-1) = \frac{1}{4}(3\cos 2\theta + 1) \quad \text{(E.126)}$$

$$P_3(x) = \frac{1}{2}(5x^3-3x) = \frac{1}{8}(5\cos 3\theta + 3\cos\theta) \quad \text{(E.127)}$$

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3) = \frac{1}{64}(35\cos 4\theta + 20\cos 2\theta + 9) \quad \text{(E.128)}$$

$$P_5(x) = \frac{1}{8}(63x^5-70x^3+15x) = \frac{1}{128}(63\cos 5\theta + 35\cos 3\theta + 30\cos\theta) \quad \text{(E.129)}$$

$$Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \ln\left(\cot\frac{\theta}{2}\right) \quad \text{(E.130)}$$

$$Q_1(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1 = \cos\theta\ln\left(\cot\frac{\theta}{2}\right) - 1 \quad \text{(E.131)}$$

$$Q_2(x) = \frac{1}{4}(3x^2 - 1)\ln\left(\frac{1+x}{1-x}\right) - \frac{3}{2}x \tag{E.132}$$

$$Q_3(x) = \frac{1}{4}(5x^3 - 3x)\ln\left(\frac{1+x}{1-x}\right) - \frac{5}{2}x^2 + \frac{2}{3} \tag{E.133}$$

$$Q_4(x) = \frac{1}{16}(35x^4 - 30x^2 + 3)\ln\left(\frac{1+x}{1-x}\right) - \frac{35}{8}x^3 + \frac{55}{24}x \tag{E.134}$$

$$P_1^1(x) = -(1-x^2)^{1/2} = -\sin\theta \tag{E.135}$$

$$P_2^1(x) = -3x(1-x^2)^{1/2} = -3\cos\theta\sin\theta \tag{E.136}$$

$$P_2^2(x) = 3(1-x^2) = 3\sin^2\theta \tag{E.137}$$

$$P_3^1(x) = -\frac{3}{2}(5x^2-1)(1-x^2)^{1/2} = -\frac{3}{2}(5\cos^2\theta - 1)\sin\theta \tag{E.138}$$

$$P_3^2(x) = 15x(1-x^2) = 15\cos\theta\sin^2\theta \tag{E.139}$$

$$P_3^3(x) = -15(1-x^2)^{3/2} = -15\sin^3\theta \tag{E.140}$$

$$P_4^1(x) = -\frac{5}{2}(7x^3-3x)(1-x^2)^{1/2} = -\frac{5}{2}(7\cos^3\theta - 3\cos\theta)\sin\theta \tag{E.141}$$

$$P_4^2(x) = \frac{15}{2}(7x^2-1)(1-x^2) = \frac{15}{2}(7\cos^2\theta - 1)\sin^2\theta \tag{E.142}$$

$$P_4^3(x) = -105x(1-x^2)^{3/2} = -105\cos\theta\sin^3\theta \tag{E.143}$$

$$P_4^4(x) = 105(1-x^2)^2 = 105\sin^4\theta \tag{E.144}$$

Functional relationships

$$P_n^m(x) = \begin{cases} 0, & m > n, \\ (-1)^m \frac{(1-x^2)^{m/2}}{2^n n!}\frac{d^{n+m}(x^2-1)^n}{dx^{n+m}}, & m \le n. \end{cases} \tag{E.145}$$

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n} \tag{E.146}$$

$$R_n^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m R_n(x)}{dx^m} \tag{E.147}$$

$$P_n^{-m}(x) = (-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(x) \tag{E.148}$$

$$P_n(-x) = (-1)^n P_n(x) \tag{E.149}$$

$$Q_n(-x) = (-1)^{n+1} Q_n(x) \tag{E.150}$$

$$P_n^m(-x) = (-1)^{n+m} P_n^m(x) \tag{E.151}$$

$$Q_n^m(-x) = (-1)^{n+m+1} Q_n^m(x) \tag{E.152}$$

$$P_n^m(1) = \begin{cases} 1, & m = 0, \\ 0, & m > 0. \end{cases} \tag{E.153}$$

$$|P_n(x)| \le P_n(1) = 1 \tag{E.154}$$

$$P_n(0) = \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}+1\right)} \cos\frac{n\pi}{2} \tag{E.155}$$

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x) \tag{E.156}$$

Power series

$$P_n(x) = \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{(n-k)!(k!)^2 2^{k+1}} \left[(1-x)^k + (-1)^n (1+x)^k\right] \tag{E.157}$$

Recursion relationships

$$(n+1-m) R_{n+1}^m(x) + (n+m) R_{n-1}^m(x) = (2n+1) x R_n^m(x) \tag{E.158}$$

$$(1-x^2) R_n^{m\prime}(x) = (n+1) x R_n^m(x) - (n-m+1) R_{n+1}^m(x) \tag{E.159}$$

$$(2n+1) x R_n(x) = (n+1) R_{n+1}(x) + n R_{n-1}(x) \tag{E.160}$$

$$(x^2-1) R_n'(x) = (n+1)[R_{n+1}(x) - x R_n(x)] \tag{E.161}$$

$$R_{n+1}'(x) - R_{n-1}'(x) = (2n+1) R_n(x) \tag{E.162}$$

Integral representations

$$P_n(\cos\theta) = \frac{\sqrt{2}}{\pi} \int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right) u}{\sqrt{\cos\theta - \cos u}}\, du \tag{E.163}$$

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left[x + (x^2-1)^{1/2} \cos\theta\right]^n \, d\theta \tag{E.164}$$

Addition formula

$$P_n(\cos\gamma) = P_n(\cos\theta) P_n(\cos\theta') + \\ + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\phi - \phi'), \tag{E.165}$$

$$\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi') \tag{E.166}$$

Summations

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} = \sum_{n=0}^{\infty} \frac{r_<^n}{r_>^{n+1}} P_n(\cos\gamma) \tag{E.167}$$

$$\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi') \tag{E.168}$$

$$r_< = \min\left\{|\mathbf{r}|, |\mathbf{r}'|\right\}, \quad r_> = \max\left\{|\mathbf{r}|, |\mathbf{r}'|\right\} \tag{E.169}$$

Integrals

$$\int P_n(x)\,dx = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1} + C \tag{E.170}$$

$$\int_{-1}^{1} x^m P_n(x)\,dx = 0, \quad m < n \tag{E.171}$$

$$\int_{-1}^{1} x^n P_n(x)\,dx = \frac{2^{n+1}(n!)^2}{(2n+1)!} \tag{E.172}$$

$$\int_{-1}^{1} x^{2k} P_{2n}(x)\,dx = \frac{2^{2n+1}(2k)!(k+n)!}{(2k+2n+1)!(k-n)!} \tag{E.173}$$

$$\int_{-1}^{1} \frac{P_n(x)}{\sqrt{1-x}}\,dx = \frac{2\sqrt{2}}{2n+1} \tag{E.174}$$

$$\int_{-1}^{1} \frac{P_{2n}(x)}{\sqrt{1-x^2}}\,dx = \left[\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!}\right]^2 \tag{E.175}$$

$$\int_{0}^{1} P_{2n+1}(x)\,dx = (-1)^n \frac{(2n)!}{2n+2}\frac{1}{(2^n n!)^2} \tag{E.176}$$

Fourier–Legendre series expansion of a function

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \quad -1 \le x \le 1 \tag{E.177}$$

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x) P_n(x)\,dx \tag{E.178}$$

E.3 Spherical harmonics

Notation

θ, ϕ = real numbers; m, n = integers
$Y_{nm}(\theta, \phi)$ = spherical harmonic function

Differential equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2} + \frac{1}{a^2}\lambda Y(\theta,\phi) = 0 \tag{E.179}$$

$$\lambda = a^2 n(n+1) \tag{E.180}$$

$$Y_{nm}(\theta,\phi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\,P_n^m(\cos\theta)e^{jm\theta} \tag{E.181}$$

Orthogonality relationships

$$\int_{-\pi}^{\pi}\int_{0}^{\pi} Y^*_{n'm'}(\theta,\phi)Y_{nm}(\theta,\phi)\sin\theta\, d\theta\, d\phi = \delta_{n'n}\delta_{m'm} \tag{E.182}$$

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} Y^*_{nm}(\theta',\phi')Y_{nm}(\theta,\phi) = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta') \tag{E.183}$$

Specific examples

$$Y_{00}(\theta,\phi) = \sqrt{\frac{1}{4\pi}} \tag{E.184}$$

$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta \tag{E.185}$$

$$Y_{11}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{j\phi} \tag{E.186}$$

$$Y_{20}(\theta,\phi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \tag{E.187}$$

$$Y_{21}(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{j\phi} \tag{E.188}$$

$$Y_{22}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2j\phi} \tag{E.189}$$

$$Y_{30}(\theta,\phi) = \sqrt{\frac{7}{4\pi}}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) \tag{E.190}$$

$$Y_{31}(\theta,\phi) = -\sqrt{\frac{21}{64\pi}}\sin\theta\left(5\cos^2\theta - 1\right)e^{j\phi} \tag{E.191}$$

$$Y_{32}(\theta,\phi) = \sqrt{\frac{105}{32\pi}}\sin^2\theta\cos\theta e^{2j\phi} \tag{E.192}$$

$$Y_{33}(\theta,\phi) = -\sqrt{\frac{35}{64\pi}}\sin^3\theta e^{3j\phi} \tag{E.193}$$

Functional relationships

$$Y_{n0}(\theta,\phi) = \sqrt{\frac{2n+1}{4\pi}}P_n(\cos\theta) \tag{E.194}$$

$$Y_{n,-m}(\theta,\phi) = (-1)^m Y^*_{nm}(\theta,\phi) \tag{E.195}$$

Addition formulas

$$P_n(\cos\gamma) = \frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y_{nm}(\theta,\phi)Y^*_{nm}(\theta',\phi') \tag{E.196}$$

$$\begin{aligned} P_n(\cos\gamma) &= P_n(\cos\theta)P_n(\cos\theta') + \\ &+ \sum_{m=-n}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos\left[m(\phi-\phi')\right] \end{aligned} \tag{E.197}$$

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi') \tag{E.198}$$

Series

$$\sum_{m=-n}^{n} |Y_{nm}(\theta,\phi)|^2 = \frac{2n+1}{4\pi} \tag{E.199}$$

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} \\ &= 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{2n+1} \frac{r_<^n}{r_>^{n+1}} Y_{nm}^*(\theta',\phi') Y_{nm}(\theta,\phi), \end{aligned} \tag{E.200}$$

$$r_< = \min\left\{|\mathbf{r}|, |\mathbf{r}'|\right\}, \quad r_> = \max\left\{|\mathbf{r}|, |\mathbf{r}'|\right\} \tag{E.201}$$

Series expansion of a function

$$f(\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} Y_{nm}(\theta,\phi) \tag{E.202}$$

$$a_{nm} = \int_{-\pi}^{\pi} \int_{0}^{\pi} f(\theta,\phi) Y_{nm}^*(\theta,\phi) \sin\theta \, d\theta \, d\phi \tag{E.203}$$