Chapter 2

Maxwell's theory of electromagnetism

2.1 The postulate

In 1864, James Clerk Maxwell proposed one of the most successful theories in the history of science. In a famous memoir to the Royal Society [125] he presented nine equations summarizing all known laws on electricity and magnetism. This was more than a mere cataloging of the laws of nature. By postulating the need for an additional term to make the set of equations self-consistent, Maxwell was able to put forth what is still considered a complete theory of macroscopic electromagnetism. The beauty of Maxwell's equations led Boltzmann to ask, "Was it a god who wrote these lines . . . ?" [185].

Since that time authors have struggled to find the best way to present Maxwell's theory. Although it is possible to study electromagnetics from an "empirical–inductive" viewpoint (roughly following the historical order of development beginning with static fields), it is only by postulating the complete theory that we can do justice to Maxwell's vision. His concept of the existence of an electromagnetic "field" (as introduced by Faraday) is fundamental to this theory, and has become one of the most significant principles of modern science.

We find controversy even over the best way to present Maxwell's equations. Maxwell worked at a time before vector notation was completely in place, and thus chose to use scalar variables and equations to represent the fields. Certainly the true beauty of Maxwell's equations emerges when they are written in vector form, and the use of tensors reduces the equations to their underlying physical simplicity. We shall use vector notation in this book because of its wide acceptance by engineers, but we still must decide whether it is more appropriate to present the vector equations in integral or point form.

On one side of this debate, the brilliant mathematician David Hilbert felt that the fundamental natural laws should be posited as axioms, each best described in terms of integral equations [154]. This idea has been championed by Truesdell and Toupin [199]. On the other side, we may quote from the great physicist Arnold Sommerfeld: "The general development of Maxwell's theory must proceed from its differential form; for special problems the integral form may, however, be more advantageous" ([185], p. 23). Special relativity flows naturally from the point forms, with fields easily converted between moving reference frames. For stationary media, it seems to us that the only difference between the two approaches arises in how we handle discontinuities in sources and materials. If we choose to use the point forms of Maxwell's equations, then we must also postulate the boundary conditions at surfaces of discontinuity. This is pointed out

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clearly by Tai [192], who also notes that if the integral forms are used, then their validity across regions of discontinuity should be stated as part of the postulate.

We have decided to use the point form in this text. In doing so we follow a long history begun by Hertz in 1890 [85] when he wrote down Maxwell's differential equations as a set of axioms, recognizing the equations as the launching point for the theory of electromagnetism. Also, by postulating Maxwell's equations in point form we can take full advantage of modern developments in the theory of partial differential equations; in particular, the idea of a "well-posed" theory determines what sort of information must be specified to make the postulate useful.

We must also decide which form of Maxwell's differential equations to use as the basis of our postulate. There are several competing forms, each differing on the manner in which materials are considered. The oldest and most widely used form was suggested by Minkowski in 1908 [130]. In the Minkowski form the differential equations contain no mention of the materials supporting the fields; all information about material media is relegated to the constitutive relationships. This places simplicity of the differential equations above intuitive understanding of the behavior of fields in materials. We choose the Maxwell–Minkowski form as the basis of our postulate, primarily for ease of manipulation. But we also recognize the value of other versions of Maxwell's equations. We shall present the basic ideas behind the Boffi form, which places some information about materials into the differential equations (although constitutive relationships are still required). Missing, however, is any information regarding the velocity of a moving medium. By using the polarization and magnetization vectors **P** and **M** rather than the fields **D** and **H**, it is sometimes easier to visualize the meaning of the field vectors and to understand (or predict) the nature of the constitutive relations.

The Chu and Amperian forms of Maxwell's equations have been promoted as useful alternatives to the Minkowski and Boffi forms. These include explicit information about the velocity of a moving material, and differ somewhat from the Boffi form in the physical interpretation of the electric and magnetic properties of matter. Although each of these models matter in terms of charged particles immersed in free space, magnetization in the Boffi and Amperian forms arises from electric current loops, while the Chu form employs magnetic dipoles. In all three forms polarization is modeled using electric dipoles. For a detailed discussion of the Chu and Amperian forms, the reader should consult the work of Kong [101], Tai [193], Penfield and Haus [145], or Fano, Chu and Adler [70].

Importantly, all of these various forms of Maxwell's equations produce the same values of the physical fields (at least external to the material where the fields are measurable).

We must include several other constituents, besides the field equations, to make the postulate complete. To form a complete field theory we need a source field, a mediating field, and a set of field differential equations. This allows us to mathematically describe the relationship between effect (the mediating field) and cause (the source field). In a well-posed postulate we must also include a set of constitutive relationships and a specification of some field relationship over a bounding surface and at an initial time. If the electromagnetic field is to have physical meaning, we must link it to some observable quantity such as force. Finally, to allow the solution of problems involving mathematical discontinuities we must specify certain boundary, or "jump," conditions.

2.1.1 The Maxwell–Minkowski equations

In Maxwell's macroscopic theory of electromagnetics, the source field consists of the vector field $\mathbf{J}(\mathbf{r}, t)$ (the current density) and the scalar field $\rho(\mathbf{r}, t)$ (the charge density).

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In Minkowski's form of Maxwell's equations, the mediating field is the *electromagnetic field* consisting of the set of four vector fields $\mathbf{E}(\mathbf{r},t)$, $\mathbf{D}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$, and $\mathbf{H}(\mathbf{r},t)$. The field equations are the four partial differential equations referred to as the *Maxwell–Minkowski equations*

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t), \tag{2.1}$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \mathbf{J}(\mathbf{r},t) + \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t), \tag{2.2}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t), \tag{2.3}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0, \tag{2.4}$$

along with the continuity equation

$$\nabla \cdot \mathbf{J}(\mathbf{r},t) = -\frac{\partial}{\partial t}\rho(\mathbf{r},t). \tag{2.5}$$

Here (2.1) is called *Faraday's law*, (2.2) is called *Ampere's law*, (2.3) is called *Gauss's law*, and (2.4) is called the *magnetic Gauss's law*. For brevity we shall often leave the dependence on $\mathbf{r}$ and t implicit, and refer to the Maxwell–Minkowski equations as simply the "Maxwell equations," or "Maxwell's equations."

Equations (2.1)–(2.5), the point forms of the field equations, describe the relationships between the fields and their sources at each point in space where the fields are continuously differentiable (i.e., the derivatives exist and are continuous). Such points are called *ordinary points*. We shall not attempt to define the fields at other points, but instead seek conditions relating the fields across surfaces containing these points. Normally this is necessary on surfaces across which either sources or material parameters are discontinuous.

The electromagnetic fields carry SI units as follows: $\mathbf{E}$ is measured in Volts per meter (V/m), $\mathbf{B}$ is measured in Teslas (T), $\mathbf{H}$ is measured in Amperes per meter (A/m), and $\mathbf{D}$ is measured in Coulombs per square meter (C/m^2). In older texts we find the units of $\mathbf{B}$ given as Webers per square meter (Wb/m^2) to reflect the role of $\mathbf{B}$ as a flux vector; in that case the Weber (Wb $=$ T$\cdot$m^2) is regarded as a unit of magnetic flux.

The interdependence of Maxwell's equations. It is often claimed that the divergence equations (2.3) and (2.4) may be derived from the curl equations (2.1) and (2.2). While this is true, it is *not* proper to say that only the two curl equations are required to describe Maxwell's theory. This is because an additional physical assumption, not present in the two curl equations, is required to complete the derivation. Either the divergence equations must be specified, or the values of certain constants that fix the initial conditions on the fields must be specified. It is customary to specify the divergence equations and include them with the curl equations to form the complete set we now call "Maxwell's equations."

To identify the interdependence we take the divergence of (2.1) to get

$$\nabla \cdot (\nabla \times \mathbf{E}) = \nabla \cdot \left(-\frac{\partial \mathbf{B}}{\partial t}\right),$$

hence

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0$$

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by (B.49). This requires that $\nabla \cdot \mathbf{B}$ be constant with time, say $\nabla \cdot \mathbf{B}(\mathbf{r}, t) = C_B(\mathbf{r})$. The constant C_B must be specified as part of the postulate of Maxwell's theory, and the choice we make is subject to experimental validation. We postulate that $C_B(\mathbf{r}) = 0$, which leads us to (2.4). Note that if we can identify a time prior to which $\mathbf{B}(\mathbf{r}, t) \equiv 0$, then $C_B(\mathbf{r})$ must vanish. For this reason, $C_B(\mathbf{r}) = 0$ and (2.4) are often called the "initial conditions" for Faraday's law [159]. Next we take the divergence of (2.2) to find that

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}).$$

Using (2.5) and (B.49), we obtain

$$\frac{\partial}{\partial t}(\rho - \nabla \cdot \mathbf{D}) = 0$$

and thus $\rho - \nabla \cdot \mathbf{D}$ must be some temporal constant $C_D(\mathbf{r})$. Again, we must postulate the value of C_D as part of the Maxwell theory. We choose $C_D(\mathbf{r}) = 0$ and thus obtain Gauss's law (2.3). If we can identify a time prior to which both $\mathbf{D}$ and ρ are everywhere equal to zero, then $C_D(\mathbf{r})$ must vanish. Hence $C_D(\mathbf{r}) = 0$ and (2.3) may be regarded as "initial conditions" for Ampere's law. Combining the two sets of initial conditions, we find that the curl equations imply the divergence equations as long as we can find a time prior to which all of the fields $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}$ and the sources $\mathbf{J}$ and ρ are equal to zero (since all the fields are related through the curl equations, and the charge and current are related through the continuity equation). Conversely, the empirical evidence supporting the two divergence equations implies that such a time should exist.

Throughout this book we shall refer to the two curl equations as the "fundamental" Maxwell equations, and to the two divergence equations as the "auxiliary" equations. The fundamental equations describe the relationships between the fields while, as we have seen, the auxiliary equations provide a sort of initial condition. This does not imply that the auxiliary equations are of lesser importance; indeed, they are required to establish uniqueness of the fields, to derive the wave equations for the fields, and to properly describe static fields.

Field vector terminology. Various terms are used for the field vectors, sometimes harkening back to the descriptions used by Maxwell himself, and often based on the physical nature of the fields. We are attracted to Sommerfeld's separation of the fields into *entities of intensity* $(\mathbf{E}, \mathbf{B})$ and *entities of quantity* $(\mathbf{D}, \mathbf{H})$. In this system $\mathbf{E}$ is called the *electric field strength*, $\mathbf{B}$ the *magnetic field strength*, $\mathbf{D}$ the *electric excitation*, and $\mathbf{H}$ the *magnetic excitation* [185]. Maxwell separated the fields into a set $(\mathbf{E}, \mathbf{H})$ of vectors that appear within line integrals to give work-related quantities, and a set $(\mathbf{B}, \mathbf{D})$ of vectors that appear within surface integrals to give flux-related quantities; we shall see this clearly when considering the integral forms of Maxwell's equations. By this system, authors such as Jones [97] and Ramo, Whinnery, and Van Duzer [153] call $\mathbf{E}$ the *electric intensity*, $\mathbf{H}$ the *magnetic intensity*, $\mathbf{B}$ the *magnetic flux density*, and $\mathbf{D}$ the *electric flux density*.

Maxwell himself designated names for each of the vector quantities. In his classic paper "A Dynamical Theory of the Electromagnetic Field," [178] Maxwell referred to the quantity we now designate $\mathbf{E}$ as the *electromotive force*, the quantity $\mathbf{D}$ as the *electric displacement* (with a time rate of change given by his now famous "displacement current"), the quantity $\mathbf{H}$ as the *magnetic force*, and the quantity $\mathbf{B}$ as the *magnetic*

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induction (although he described **B** as a density of lines of magnetic force). Maxwell also included a quantity designated *electromagnetic momentum* as an integral part of his theory. We now know this as the vector potential **A** which is not generally included as a part of the electromagnetics postulate.

Many authors follow the original terminology of Maxwell, with some slight modifications. For instance, Stratton [187] calls **E** the *electric field intensity*, **H** the *magnetic field intensity*, **D** the *electric displacement*, and **B** the *magnetic induction*. Jackson [91] calls **E** the *electric field*, **H** the *magnetic field*, **D** the *displacement*, and **B** the *magnetic induction*.

Other authors choose freely among combinations of these terms. For instance, Kong [101] calls **E** the *electric field strength*, **H** the *magnetic field strength*, **B** the *magnetic flux density*, and **D** the *electric displacement*. We do not wish to inject further confusion into the issue of nomenclature; still, we find it helpful to use as simple a naming system as possible. We shall refer to **E** as the *electric field*, **H** as the *magnetic field*, **D** as the *electric flux density* and **B** as the *magnetic flux density*. When we use the term *electromagnetic field* we imply the entire set of field vectors (**E**, **D**, **B**, **H**) used in Maxwell's theory.

Invariance of Maxwell's equations. Maxwell's differential equations are valid for any system in uniform relative motion with respect to the laboratory frame of reference in which we normally do our measurements. The field equations describe the relationships between the source and mediating fields *within that frame of reference*. This property was first proposed for moving material media by Minkowski in 1908 (using the term *covariance*) [130]. For this reason, Maxwell's equations expressed in the form (2.1)–(2.2) are referred to as the *Minkowski form*.

2.1.2 Connection to mechanics

Our postulate must include a connection between the abstract quantities of charge and field and a measurable physical quantity. A convenient means of linking electromagnetics to other classical theories is through mechanics. We postulate that charges experience mechanical forces given by the *Lorentz force equation*. If a small volume element dV contains a total charge $\rho\, dV$, then the force experienced by that charge when moving at velocity $\mathbf{v}$ in an electromagnetic field is

$$d\mathbf{F} = \rho\, dV\, \mathbf{E} + \rho\mathbf{v}\, dV \times \mathbf{B}. \tag{2.6}$$

As with any postulate, we verify this equation through experiment. Note that we write the Lorentz force in terms of charge $\rho\, dV$, rather than charge density ρ, since charge is an invariant quantity under a Lorentz transformation.

The important links between the electromagnetic fields and energy and momentum must also be postulated. We postulate that the quantity

$$\mathbf{S}_{em} = \mathbf{E} \times \mathbf{H} \tag{2.7}$$

represents the transport density of electromagnetic power, and that the quantity

$$\mathbf{g}_{em} = \mathbf{D} \times \mathbf{B} \tag{2.8}$$

represents the transport density of electromagnetic momentum.

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2.2 The well-posed nature of the postulate

It is important to investigate whether Maxwell's equations, along with the point form of the continuity equation, suffice as a useful theory of electromagnetics. Certainly we must agree that a theory is "useful" as long as it is defined as such by the scientists and engineers who employ it. In practice a theory is considered useful if it predicts accurately the behavior of nature under given circumstances, and even a theory that often fails may be useful if it is the best available. We choose here to take a more narrow view and investigate whether the theory is "well-posed."

A mathematical model for a physical problem is said to be *well-posed*, or *correctly set*, if three conditions hold:

1. the model has at least one solution (*existence*);
2. the model has at most one solution (*uniqueness*);
3. the solution is continuously dependent on the data supplied.

The importance of the first condition is obvious: if the electromagnetic model has no solution, it will be of little use to scientists and engineers. The importance of the second condition is equally obvious: if we apply two different solution methods to the same model and get two different answers, the model will not be very helpful in analysis or design work. The third point is more subtle; it is often extended in a practical sense to the following statement:

3′. Small changes in the data supplied produce equally small changes in the solution.

That is, the solution is not sensitive to errors in the data. To make sense of this we must decide which quantity is specified (the independent quantity) and which remains to be calculated (the dependent quantity). Commonly the source field (charge) is taken as the independent quantity, and the mediating (electromagnetic) field is computed from it; in such cases it can be shown that Maxwell's equations are well-posed. Taking the electromagnetic field to be the independent quantity, we can produce situations in which the computed quantity (charge or current) changes wildly with small changes in the specified fields. These situations (called *inverse problems*) are of great importance in remote sensing, where the field is measured and the properties of the object probed are thereby deduced.

At this point we shall concentrate on the "forward" problem of specifying the source field (charge) and computing the mediating field (the electromagnetic field). In this case we may question whether the first of the three conditions (existence) holds. We have twelve unknown quantities (the scalar components of the four vector fields), but only eight equations to describe them (from the scalar components of the two fundamental Maxwell equations and the two scalar auxiliary equations). With fewer equations than unknowns we cannot be sure that a solution exists, and we refer to Maxwell's equations as being *indefinite*. To overcome this problem we must specify more information in the form of constitutive relations among the field quantities **E**, **B**, **D**, **H**, and **J**. When these are properly formulated, the number of unknowns and the number of equations are equal and Maxwell's equations are in *definite form*. If we provide more equations than unknowns, the solution may be non-unique. When we model the electromagnetic properties of materials we must supply precisely the right amount of information in the constitutive relations, or our postulate will not be well-posed.

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Once Maxwell's equations are in definite form, standard methods for partial differential equations can be used to determine whether the electromagnetic model is well-posed. In a nutshell, the system (2.1)–(2.2) of hyperbolic differential equations is well-posed if and only if we specify **E** and **H** throughout a volume region V at some time instant and also specify, at all subsequent times,

1. the tangential component of **E** over all of the boundary surface S, or
2. the tangential component of **H** over all of S, or
3. the tangential component of **E** over part of S, and the tangential component of **H** over the remainder of S.

Proof of all three of the conditions of well-posedness is quite tedious, but a simplified uniqueness proof is often given in textbooks on electromagnetics. The procedure used by Stratton [187] is reproduced below. The interested reader should refer to Hansen [81] for a discussion of the existence of solutions to Maxwell's equations.

2.2.1 Uniqueness of solutions to Maxwell'sequations

Consider a simply connected region of space V bounded by a surface S, where both V and S contain only ordinary points. The fields within V are associated with a current distribution **J**, which may be internal to V (entirely or in part). By the initial conditions that imply the auxiliary Maxwell's equations, we know there is a time, say $t = 0$, prior to which the current is zero for all time, and thus by causality the fields throughout V are identically zero for all times $t < 0$. We next assume that the fields are specified throughout V at some time $t_0 > 0$, and seek conditions under which they are determined uniquely for all $t > t_0$.

Let the field set $(\mathbf{E}_1, \mathbf{D}_1, \mathbf{B}_1, \mathbf{H}_1)$ be a solution to Maxwell's equations (2.1)–(2.2) associated with the current **J** (along with an appropriate set of constitutive relations), and let $(\mathbf{E}_2, \mathbf{D}_2, \mathbf{B}_2, \mathbf{H}_2)$ be a second solution associated with **J**. To determine the conditions for uniqueness of the fields, we look for a situation that results in $\mathbf{E}_1 = \mathbf{E}_2$, $\mathbf{B}_1 = \mathbf{B}_2$, and so on. The electromagnetic fields must obey

$$\nabla \times \mathbf{E}_1 = -\frac{\partial \mathbf{B}_1}{\partial t},$$

$$\nabla \times \mathbf{H}_1 = \mathbf{J} + \frac{\partial \mathbf{D}_1}{\partial t},$$

$$\nabla \times \mathbf{E}_2 = -\frac{\partial \mathbf{B}_2}{\partial t},$$

$$\nabla \times \mathbf{H}_2 = \mathbf{J} + \frac{\partial \mathbf{D}_2}{\partial t}.$$

Subtracting, we have

$$\nabla \times (\mathbf{E}_1 - \mathbf{E}_2) = -\frac{\partial(\mathbf{B}_1 - \mathbf{B}_2)}{\partial t}, \tag{2.9}$$

$$\nabla \times (\mathbf{H}_1 - \mathbf{H}_2) = \frac{\partial(\mathbf{D}_1 - \mathbf{D}_2)}{\partial t}, \tag{2.10}$$

hence defining $\mathbf{E}_0 = \mathbf{E}_1 - \mathbf{E}_2$, $\mathbf{B}_0 = \mathbf{B}_1 - \mathbf{B}_2$, and so on, we have

$$\mathbf{E}_0 \cdot (\nabla \times \mathbf{H}_0) = \mathbf{E}_0 \cdot \frac{\partial \mathbf{D}_0}{\partial t}, \tag{2.11}$$

$$\mathbf{H}_0 \cdot (\nabla \times \mathbf{E}_0) = -\mathbf{H}_0 \cdot \frac{\partial \mathbf{B}_0}{\partial t}. \tag{2.12}$$

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Subtracting again, we have

$$\mathbf{E}_0 \cdot (\nabla \times \mathbf{H}_0) - \mathbf{H}_0 \cdot (\nabla \times \mathbf{E}_0) = \mathbf{H}_0 \cdot \frac{\partial \mathbf{B}_0}{\partial t} + \mathbf{E}_0 \cdot \frac{\partial \mathbf{D}_0}{\partial t},$$

hence

$$-\nabla \cdot (\mathbf{E}_0 \times \mathbf{H}_0) = \mathbf{E}_0 \cdot \frac{\partial \mathbf{D}_0}{\partial t} + \mathbf{H}_0 \cdot \frac{\partial \mathbf{B}_0}{\partial t}$$

by (B.44). Integrating both sides throughout V and using the divergence theorem on the left-hand side, we get

$$-\oint_S (\mathbf{E}_0 \times \mathbf{H}_0) \cdot \mathbf{dS} = \int_V \left(\mathbf{E}_0 \cdot \frac{\partial \mathbf{D}_0}{\partial t} + \mathbf{H}_0 \cdot \frac{\partial \mathbf{B}_0}{\partial t} \right) dV.$$

Breaking S into two arbitrary portions and using (B.6), we obtain

$$\int_{S_1} \mathbf{E}_0 \cdot (\hat{\mathbf{n}} \times \mathbf{H}_0)\, dS - \int_{S_2} \mathbf{H}_0 \cdot (\hat{\mathbf{n}} \times \mathbf{E}_0)\, dS = \int_V \left(\mathbf{E}_0 \cdot \frac{\partial \mathbf{D}_0}{\partial t} + \mathbf{H}_0 \cdot \frac{\partial \mathbf{B}_0}{\partial t} \right) dV.$$

Now if $\hat{\mathbf{n}} \times \mathbf{E}_0 = 0$ or $\hat{\mathbf{n}} \times \mathbf{H}_0 = 0$ over all of S, or some combination of these conditions holds over all of S, then

$$\int_V \left(\mathbf{E}_0 \cdot \frac{\partial \mathbf{D}_0}{\partial t} + \mathbf{H}_0 \cdot \frac{\partial \mathbf{B}_0}{\partial t} \right) dV = 0. \tag{2.13}$$

This expression implies a relationship between $\mathbf{E}_0$, $\mathbf{D}_0$, $\mathbf{B}_0$, and $\mathbf{H}_0$. Since V is arbitrary, we see that one possibility is simply to have $\mathbf{D}_0$ and $\mathbf{B}_0$ constant with time. However, since the fields are identically zero for $t < 0$, if they are constant for all time then those constant values must be zero. Another possibility is to have one of each pair $(\mathbf{E}_0, \mathbf{D}_0)$ and $(\mathbf{H}_0, \mathbf{B}_0)$ equal to zero. Then, by (2.9) and (2.10), $\mathbf{E}_0 = 0$ implies $\mathbf{B}_0 = 0$, and $\mathbf{D}_0 = 0$ implies $\mathbf{H}_0 = 0$. Thus $\mathbf{E}_1 = \mathbf{E}_2$, $\mathbf{B}_1 = \mathbf{B}_2$, and so on, and the solution is unique throughout V. However, we cannot in general rule out more complicated relationships. The number of possibilities depends on the additional constraints on the relationship between $\mathbf{E}_0$, $\mathbf{D}_0$, $\mathbf{B}_0$, and $\mathbf{H}_0$ that we must supply to describe the material supporting the field — i.e., the constitutive relationships. For a simple medium described by the time-constant permittivity ϵ and permeability μ, (13) becomes

$$\int_V \left(\mathbf{E}_0 \cdot \epsilon \frac{\partial \mathbf{E}_0}{\partial t} + \mathbf{H}_0 \cdot \mu \frac{\partial \mathbf{H}_0}{\partial t} \right) dV = 0,$$

or

$$\frac{1}{2} \frac{\partial}{\partial t} \int_V (\epsilon \mathbf{E}_0 \cdot \mathbf{E}_0 + \mu \mathbf{H}_0 \cdot \mathbf{H}_0)\, dV = 0.$$

Since the integrand is always positive or zero (and not constant with time, as mentioned above), the only possible conclusion is that $\mathbf{E}_0$ and $\mathbf{H}_0$ must both be zero, and thus the fields are unique.

When establishing more complicated constitutive relations, we must be careful to ensure that they lead to a unique solution, and that the condition for uniqueness is understood. In the case above, the assumption $\hat{\mathbf{n}} \times \mathbf{E}_0\big|_S = 0$ implies that the tangential components of $\mathbf{E}_1$ and $\mathbf{E}_2$ are identical over S — that is, we must give specific values of these quantities on S to ensure uniqueness. A similar statement holds for the condition $\hat{\mathbf{n}} \times \mathbf{H}_0\big|_S = 0$. Requiring that constitutive relations lead to a unique solution is known

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as *just setting*, and is one of several factors that must be considered, as discussed in the next section.

Uniqueness implies that the electromagnetic state of an isolated region of space may be determined without the knowledge of conditions outside the region. If we wish to solve Maxwell's equations for that region, we need know only the source density within the region and the values of the tangential fields over the bounding surface. The effects of a complicated external world are thus reduced to the specification of surface fields. This concept has numerous applications to problems in antennas, diffraction, and guided waves.

2.2.2 Constitutive relations

We now supply a set of constitutive relations to complete the conditions for well-posedness. We generally split these relations into two sets. The first describes the relationships between the electromagnetic field quantities, and the second describes mechanical interaction between the fields and resulting secondary sources. All of these relations depend on the properties of the medium supporting the electromagnetic field. Material phenomena are quite diverse, and it is remarkable that the Maxwell–Minkowski equations hold for all phenomena yet discovered. All material effects, from nonlinearity to chirality to temporal dispersion, are described by the constitutive relations.

The specification of constitutive relationships is required in many areas of physical science to describe the behavior of "ideal materials": mathematical models of actual materials encountered in nature. For instance, in continuum mechanics the constitutive equations describe the relationship between material motions and stress tensors [209]. Truesdell and Toupin [199] give an interesting set of "guiding principles" for the concerned scientist to use when constructing constitutive relations. These include consideration of *consistency* (with the basic conservation laws of nature), *coordinate invariance* (independence of coordinate system), *isotropy and aeolotropy* (dependence on, or independence of, orientation), *just setting* (constitutive parameters should lead to a unique solution), *dimensional invariance* (similarity), *material indifference* (non-dependence on the observer), and *equipresence* (inclusion of *all* relevant physical phenomena in *all* of the constitutive relations across disciplines).

The constitutive relations generally involve a set of constitutive parameters and a set of constitutive operators. The constitutive parameters may be as simple as constants of proportionality between the fields or they may be components in a dyadic relationship. The constitutive operators may be linear and integro-differential in nature, or may imply some nonlinear operation on the fields. If the constitutive parameters are spatially constant within a certain region, we term the medium *homogeneous* within that region. If the constitutive parameters vary spatially, the medium is *inhomogeneous*. If the constitutive parameters are constants with time, we term the medium *stationary*; if they are time-changing, the medium is *nonstationary*. If the constitutive operators involve time derivatives or integrals, the medium is said to be *temporally dispersive*; if space derivatives or integrals are involved, the medium is *spatially dispersive*. Examples of all these effects can be found in common materials. It is important to note that the constitutive parameters may depend on other physical properties of the material, such as temperature, mechanical stress, and isomeric state, just as the mechanical constitutive parameters of a material may depend on the electromagnetic properties (principle of equipresence).

Many effects produced by linear constitutive operators, such as those associated with

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temporal dispersion, have been studied primarily in the frequency domain. In this case temporal derivative and integral operations produce complex constitutive parameters. It is becoming equally important to characterize these effects directly in the time domain for use with direct time-domain field solving techniques such as the finite-difference time-domain (FDTD) method. We shall cover the very basic properties of dispersive media in this section. A detailed description of frequency-domain fields (and a discussion of complex constitutive parameters) is deferred until later in this book.

It is difficult to find a simple and consistent means for classifying materials by their electromagnetic effects. One way is to separate linear and nonlinear materials, then categorize linear materials by the way in which the fields are coupled through the constitutive relations:

1. *Isotropic* materials are those in which **D** is related to **E**, **B** is related to **H**, and the secondary source current **J** is related to **E**, with the field direction in each pair aligned.
2. In *anisotropic* materials the pairings are the same, but the fields in each pair are generally not aligned.
3. In *biisotropic* materials (such as chiral media) the fields **D** and **B** depend on both **E** and **H**, but with no realignment of **E** or **H**; for instance, **D** is given by the addition of a scalar times **E** plus a second scalar times **H**. Thus the contributions to **D** involve no changes to the directions of **E** and **H**.
4. *Bianisotropic* materials exhibit the most general behavior: **D** and **H** depend on both **E** and **B**, with an arbitrary realignment of either or both of these fields.

In 1888, Roentgen showed experimentally that a material isotropic in its own stationary reference frame exhibits bianisotropic properties when observed from a moving frame. Only recently have materials bianisotropic in their own rest frame been discovered. In 1894 Curie predicted that in a stationary material, based on symmetry, an electric field might produce magnetic effects and a magnetic field might produce electric effects. These effects, coined *magnetoelectric* by Landau and Lifshitz in 1957, were sought unsuccessfully by many experimentalists during the first half of the twentieth century. In 1959 the Soviet scientist I.E. Dzyaloshinskii predicted that, theoretically, the antiferromagnetic material chromium oxide (Cr_2O_3) should display magnetoelectric effects. The magnetoelectric effect was finally observed soon after by D.N. Astrov in a single crystal of Cr_2O_3 using a 10 kHz electric field. Since then the effect has been observed in many different materials. Recently, highly exotic materials with useful electromagnetic properties have been proposed and studied in depth, including chiroplasmas and chiroferrites [211]. As the technology of materials synthesis advances, a host of new and intriguing media will certainly be created.

The most general forms of the constitutive relations between the fields may be written in symbolic form as

$$\mathbf{D} = \mathbf{D}[\mathbf{E}, \mathbf{B}], \tag{2.14}$$

$$\mathbf{H} = \mathbf{H}[\mathbf{E}, \mathbf{B}]. \tag{2.15}$$

That is, **D** and **H** have some mathematically descriptive relationship to **E** and **B**. The specific forms of the relationships may be written in terms of dyadics as [102]

$$c\mathbf{D} = \bar{\mathbf{P}} \cdot \mathbf{E} + \bar{\mathbf{L}} \cdot (c\mathbf{B}), \tag{2.16}$$

$$\mathbf{H} = \bar{\mathbf{M}} \cdot \mathbf{E} + \bar{\mathbf{Q}} \cdot (c\mathbf{B}), \tag{2.17}$$

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where each of the quantities $\bar{\mathbf{P}}, \bar{\mathbf{L}}, \bar{\mathbf{M}}, \bar{\mathbf{Q}}$ may be dyadics in the usual sense, or dyadic operators containing space or time derivatives or integrals, or some nonlinear operations on the fields. We may write these expressions as a single matrix equation

$$\begin{bmatrix} c\mathbf{D} \\ \mathbf{H} \end{bmatrix} = [\bar{\mathbf{C}}] \begin{bmatrix} \mathbf{E} \\ c\mathbf{B} \end{bmatrix} \tag{2.18}$$

where the 6×6 matrix

$$[\bar{\mathbf{C}}] = \begin{bmatrix} \bar{\mathbf{P}} & \bar{\mathbf{L}} \\ \bar{\mathbf{M}} & \bar{\mathbf{Q}} \end{bmatrix}.$$

This most general relationship between fields is the property of a bianisotropic material.

We may wonder why $\mathbf{D}$ is not related to $(\mathbf{E}, \mathbf{B}, \mathbf{H})$, $\mathbf{E}$ to $(\mathbf{D}, \mathbf{B})$, etc. The reason is that since the field pairs $(\mathbf{E}, \mathbf{B})$ and $(\mathbf{D}, \mathbf{H})$ convert identically under a Lorentz transformation, a constitutive relation that maps fields as in (2.18) is form invariant, as are the Maxwell–Minkowski equations. That is, although the constitutive parameters may vary numerically between observers moving at different velocities, the form of the relationship given by (2.18) is maintained.

Many authors choose to relate $(\mathbf{D}, \mathbf{B})$ to $(\mathbf{E}, \mathbf{H})$, often because the expressions are simpler and can be more easily applied to specific problems. For instance, in a linear, isotropic material (as shown below) $\mathbf{D}$ is directly proportional to $\mathbf{E}$ and $\mathbf{B}$ is directly proportional to $\mathbf{H}$. To provide the appropriate expression for the constitutive relations, we need only remap (2.18). This gives

$$\mathbf{D} = \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E} + \bar{\boldsymbol{\xi}} \cdot \mathbf{H}, \tag{2.19}$$

$$\mathbf{B} = \bar{\boldsymbol{\zeta}} \cdot \mathbf{E} + \bar{\boldsymbol{\mu}} \cdot \mathbf{H}, \tag{2.20}$$

or

$$\begin{bmatrix} \mathbf{D} \\ \mathbf{B} \end{bmatrix} = \left[\bar{\mathbf{C}}_{EH}\right] \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}, \tag{2.21}$$

where the new constitutive parameters $\bar{\boldsymbol{\epsilon}}, \bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\zeta}}, \bar{\boldsymbol{\mu}}$ can be easily found from the original constitutive parameters $\bar{\mathbf{P}}, \bar{\mathbf{L}}, \bar{\mathbf{M}}, \bar{\mathbf{Q}}$. We do note, however, that in the form (2.19)–(2.20) the Lorentz invariance of the constitutive equations is not obvious.

In the following paragraphs we shall characterize some of the most common materials according to these classifications. With this approach effects such as temporal or spatial dispersion are not part of the classification process, but arise from the nature of the constitutive parameters. Hence we shall not dwell on the particulars of the constitutive parameters, but shall concentrate on the form of the constitutive relations.

Constitutive relations for fields in free space. In a vacuum the fields are related by the simple constitutive equations

$$\mathbf{D} = \epsilon_0 \mathbf{E}, \tag{2.22}$$

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B}. \tag{2.23}$$

The quantities μ_0 and ϵ_0 are, respectively, the *free-space permeability* and *permittivity constants*. It is convenient to use three numerical quantities to describe the electromagnetic properties of free space — μ_0, ϵ_0, and the speed of light c — and interrelate them through the equation

$$c = 1/(\mu_0 \epsilon_0)^{1/2}.$$

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Historically it has been the practice to define μ_0, measure c, and compute ϵ_0. In SI units

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m},$$
$$c = 2.998 \times 10^{8} \text{ m/s},$$
$$\epsilon_0 = 8.854 \times 10^{-12} \text{ F/m}.$$

With the two constitutive equations we have enough information to put Maxwell's equations into definite form. Traditionally (2.22) and (2.23) are substituted into (2.1)–(2.2) to give

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.24}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{2.25}$$

These are two vector equations in two vector unknowns (equivalently, six scalar equations in six scalar unknowns).

In terms of the general constitutive relation (2.18), we find that free space is isotropic with

$$\bar{\mathbf{P}} = \bar{\mathbf{Q}} = \frac{1}{\eta_0}\bar{\mathbf{I}}, \qquad \bar{\mathbf{L}} = \bar{\mathbf{M}} = 0,$$

where $\eta_0 = (\mu_0/\epsilon_0)^{1/2}$ is called the *intrinsic impedance of free space*. This emphasizes the fact that free space has, along with c, only a single empirical constant associated with it (i.e., ϵ_0 or η_0). Since no derivative or integral operators appear in the constitutive relations, free space is nondispersive.

Constitutive relations in a linear isotropic material. In a linear isotropic material there is proportionality between **D** and **E** and between **B** and **H**. The constants of proportionality are the *permittivity* ϵ and the *permeability* μ. If the material is nondispersive, the constitutive relations take the form

$$\mathbf{D} = \epsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H},$$

where ϵ and μ may depend on position for inhomogeneous materials. Often the permittivity and permeability are referenced to the permittivity and permeability of free space according to

$$\epsilon = \epsilon_r \epsilon_0, \qquad \mu = \mu_r \mu_0.$$

Here the dimensionless quantities ϵ_r and μ_r are called, respectively, the *relative permittivity* and *relative permeability.*

When dealing with the Maxwell–Boffi equations (§ 2.4) the difference between the material and free space values of **D** and **H** becomes important. Thus for linear isotropic materials we often write the constitutive relations as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E}, \tag{2.26}$$

$$\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \chi_m \mathbf{H}, \tag{2.27}$$

where the dimensionless quantities $\chi_e = \epsilon_r - 1$ and $\chi_m = \mu_r - 1$ are called, respectively, the *electric* and *magnetic susceptibilities* of the material. In terms of (2.18) we have

$$\bar{\mathbf{P}} = \frac{\epsilon_r}{\eta_0}\bar{\mathbf{I}}, \qquad \bar{\mathbf{Q}} = \frac{1}{\eta_0 \mu_r}\bar{\mathbf{I}}, \qquad \bar{\mathbf{L}} = \bar{\mathbf{M}} = 0.$$

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Generally a material will have either its electric or magnetic properties dominant. If $\mu_r = 1$ and $\epsilon_r \neq 1$ then the material is generally called a *perfect dielectric* or a *perfect insulator*, and is said to be an electric material. If $\epsilon_r = 1$ and $\mu_r \neq 1$, the material is said to be a magnetic material.

A linear isotropic material may also have conduction properties. In a *conducting material*, a constitutive relation is generally used to describe the mechanical interaction of field and charge by relating the electric field to a secondary electric current. For a nondispersive isotropic material, the current is aligned with, and proportional to, the electric field; there are no temporal operators in the constitutive relation, which is simply

$$\mathbf{J} = \sigma \mathbf{E}. \tag{2.28}$$

This is known as *Ohm's law*. Here σ is the *conductivity* of the material.

If $\mu_r \approx 1$ and σ is very small, the material is generally called a *good dielectric*. If σ is very large, the material is generally called a *good conductor*. The conditions by which we say the conductivity is "small" or "large" are usually established using the frequency response of the material. Materials that are good dielectrics over broad ranges of frequency include various glasses and plastics such as fused quartz, polyethylene, and teflon. Materials that are good conductors over broad ranges of frequency include common metals such as gold, silver, and copper.

For dispersive linear isotropic materials, the constitutive parameters become nonstationary (time dependent), and the constitutive relations involve time operators. (Note that the name *dispersive* describes the tendency for pulsed electromagnetic waves to spread out, or disperse, in materials of this type.) If we assume that the relationships given by (2.26), (2.27), and (2.28) retain their product form in the frequency domain, then by the convolution theorem we have in the time domain the constitutive relations

$$\mathbf{D}(\mathbf{r}, t) = \epsilon_0 \left(\mathbf{E}(\mathbf{r}, t) + \int_{-\infty}^{t} \chi_e(\mathbf{r}, t - t')\mathbf{E}(\mathbf{r}, t')\, dt' \right), \tag{2.29}$$

$$\mathbf{B}(\mathbf{r}, t) = \mu_0 \left(\mathbf{H}(\mathbf{r}, t) + \int_{-\infty}^{t} \chi_m(\mathbf{r}, t - t')\mathbf{H}(\mathbf{r}, t')\, dt' \right), \tag{2.30}$$

$$\mathbf{J}(\mathbf{r}, t) = \int_{-\infty}^{t} \sigma(\mathbf{r}, t - t')\mathbf{E}(\mathbf{r}, t')\, dt'. \tag{2.31}$$

These expressions were first introduced by Volterra in 1912 [199]. We see that for a linear dispersive material of this type the constitutive operators are time integrals, and that the behavior of $\mathbf{D}(t)$ depends not only on the value of $\mathbf{E}$ at time t, but on its values at all past times. Thus, in dispersive materials there is a "time lag" between the effect of the applied field and the polarization or magnetization that results. In the frequency domain, temporal dispersion is associated with complex values of the constitutive parameters, which, to describe a causal relationship, cannot be constant with frequency. The nonzero imaginary component is identified with the dissipation of electromagnetic energy as heat. Causality is implied by the upper limit being t in the convolution integrals, which indicates that $\mathbf{D}(t)$ cannot depend on future values of $\mathbf{E}(t)$. This assumption leads to a relationship between the real and imaginary parts of the frequency domain constitutive parameters as described through the Kronig–Kramers equations.

Constitutive relations for fields in perfect conductors. In a perfect electric conductor (PEC) or a perfect magnetic conductor (PMC) the fields are exactly specified as

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the null field:

$$\mathbf{E} = \mathbf{D} = \mathbf{B} = \mathbf{H} = 0.$$

By Ampere's and Faraday's laws we must also have $\mathbf{J} = \mathbf{J}_m = 0$; hence, by the continuity equation, $\rho = \rho_m = 0$.

In addition to the null field, we have the condition that the tangential electric field on the surface of a PEC must be zero. Similarly, the tangential magnetic field on the surface of a PMC must be zero. This implies (§ 2.8.3) that an electric surface current may exist on the surface of a PEC but not on the surface of a PMC, while a magnetic surface current may exist on the surface of a PMC but not on the surface of a PEC.

A PEC may be regarded as the limit of a conducting material as $\sigma \to \infty$. In many practical cases, good conductors such as gold and copper can be assumed to be perfect electric conductors, which greatly simplifies the application of boundary conditions. No physical material is known to behave as a PMC, but the concept is mathematically useful for applying symmetry conditions (in which a PMC is sometimes referred to as a "magnetic wall") and for use in developing equivalence theorems.

Constitutive relations in a linear anisotropic material. In a linear anisotropic material there are relationships between **B** and **H** and between **D** and **E**, but the field vectors are not aligned as in the isotropic case. We can thus write

$$\mathbf{D} = \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}, \qquad \mathbf{B} = \bar{\boldsymbol{\mu}} \cdot \mathbf{H}, \qquad \mathbf{J} = \bar{\boldsymbol{\sigma}} \cdot \mathbf{E},$$

where $\bar{\boldsymbol{\epsilon}}$ is called the *permittivity dyadic*, $\bar{\boldsymbol{\mu}}$ is the *permeability dyadic*, and $\bar{\boldsymbol{\sigma}}$ is the *conductivity dyadic*. In terms of the general constitutive relation (2.18) we have

$$\bar{\mathbf{P}} = c\bar{\boldsymbol{\epsilon}}, \qquad \bar{\mathbf{Q}} = \frac{\bar{\boldsymbol{\mu}}^{-1}}{c}, \qquad \bar{\mathbf{L}} = \bar{\mathbf{M}} = 0.$$

Many different types of materials demonstrate anisotropic behavior, including optical crystals, magnetized plasmas, and ferrites. Plasmas and ferrites are examples of *gyrotropic* media. With the proper choice of coordinate system, the frequency-domain permittivity or permeability can be written in matrix form as

$$[\tilde{\bar{\boldsymbol{\epsilon}}}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ -\epsilon_{12} & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}, \qquad [\tilde{\bar{\boldsymbol{\mu}}}] = \begin{bmatrix} \mu_{11} & \mu_{12} & 0 \\ -\mu_{12} & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{bmatrix}. \tag{2.32}$$

Each of the matrix entries may be complex. For the special case of a *lossless* gyrotropic material, the matrices become *hermitian*:

$$[\tilde{\bar{\boldsymbol{\epsilon}}}] = \begin{bmatrix} \epsilon & -j\delta & 0 \\ j\delta & \epsilon & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}, \qquad [\tilde{\bar{\boldsymbol{\mu}}}] = \begin{bmatrix} \mu & -j\kappa & 0 \\ j\kappa & \mu & 0 \\ 0 & 0 & \mu_3 \end{bmatrix}, \tag{2.33}$$

where ϵ, ϵ_3, δ, μ, μ_3, and κ are real numbers.

Crystals have received particular attention because of their birefringent properties. A birefringent crystal can be characterized by a symmetric permittivity dyadic that has real permittivity parameters in the frequency domain; equivalently, the constitutive relations do not involve constitutive operators. A coordinate system called the *principal system*, with axes called the *principal axes*, can always be found so that the permittivity dyadic in that system is diagonal:

$$[\tilde{\bar{\boldsymbol{\epsilon}}}] = \begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix}.$$

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The geometrical structure of a crystal determines the relationship between ϵ_x, ϵ_y, and ϵ_z. If $\epsilon_x = \epsilon_y < \epsilon_z$, then the crystal is *positive uniaxial* (e.g., quartz). If $\epsilon_x = \epsilon_y > \epsilon_z$, the crystal is *negative uniaxial* (e.g., calcite). If $\epsilon_x \neq \epsilon_y \neq \epsilon_z$, the crystal is *biaxial* (e.g., mica). In uniaxial crystals the z-axis is called the *optical axis*.

If the anisotropic material is dispersive, we can generalize the convolutional form of the isotropic dispersive media to obtain the constitutive relations

$$\mathbf{D}(\mathbf{r}, t) = \epsilon_0 \left(\mathbf{E}(\mathbf{r}, t) + \int_{-\infty}^{t} \bar{\chi}_e(\mathbf{r}, t - t') \cdot \mathbf{E}(\mathbf{r}, t')\, dt' \right), \tag{2.34}$$

$$\mathbf{B}(\mathbf{r}, t) = \mu_0 \left(\mathbf{H}(\mathbf{r}, t) + \int_{-\infty}^{t} \bar{\chi}_m(\mathbf{r}, t - t') \cdot \mathbf{H}(\mathbf{r}, t')\, dt' \right), \tag{2.35}$$

$$\mathbf{J}(\mathbf{r}, t) = \int_{-\infty}^{t} \bar{\sigma}(\mathbf{r}, t - t') \cdot \mathbf{E}(\mathbf{r}, t')\, dt'. \tag{2.36}$$

Constitutive relations for biisotropic materials. A biisotropic material is an isotropic magnetoelectric material. Here we have **D** related to **E** and **B**, and **H** related to **E** and **B**, but with no realignment of the fields as in anisotropic (or bianisotropic) materials. Perhaps the simplest example is the *Tellegen medium* devised by B.D.H. Tellegen in 1948 [196], having

$$\mathbf{D} = \epsilon\mathbf{E} + \xi\mathbf{H}, \tag{2.37}$$

$$\mathbf{B} = \xi\mathbf{E} + \mu\mathbf{H}. \tag{2.38}$$

Tellegen proposed that his hypothetical material be composed of small (but macroscopic) ferromagnetic particles suspended in a liquid. This is an example of a *synthetic* material, constructed from ordinary materials to have an exotic electromagnetic behavior. Other examples include artificial dielectrics made from metallic particles imbedded in lightweight foams [66], and *chiral materials* made from small metallic helices suspended in resins [112].

Chiral materials are also biisotropic, and have the constitutive relations

$$\mathbf{D} = \epsilon\mathbf{E} - \chi \frac{\partial \mathbf{H}}{\partial t}, \tag{2.39}$$

$$\mathbf{B} = \mu\mathbf{H} + \chi \frac{\partial \mathbf{E}}{\partial t}, \tag{2.40}$$

where the constitutive parameter χ is called the *chirality parameter*. Note the presence of temporal derivative operators. Alternatively,

$$\mathbf{D} = \epsilon(\mathbf{E} + \beta\nabla \times \mathbf{E}), \tag{2.41}$$

$$\mathbf{B} = \mu(\mathbf{H} + \beta\nabla \times \mathbf{H}), \tag{2.42}$$

by Faraday's and Ampere's laws. Chirality is a natural state of symmetry; many natural substances are chiral materials, including DNA and many sugars. The time derivatives in (2.39)–(2.40) produce rotation of the polarization of time harmonic electromagnetic waves propagating in chiral media.

Constitutive relations in nonlinear media. Nonlinear electromagnetic effects have been studied by scientists and engineers since the beginning of the era of electrical technology. Familiar examples include saturation and hysteresis in ferromagnetic materials

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and the behavior of p-n junctions in solid-state rectifiers. The invention of the laser extended interest in nonlinear effects to the realm of optics, where phenomena such as parametric amplification and oscillation, harmonic generation, and magneto-optic interactions have found applications in modern devices [174].

Provided that the external field applied to a nonlinear electric material is small compared to the internal molecular fields, the relationship between **E** and **D** can be expanded in a Taylor series of the electric field. For an anisotropic material exhibiting no hysteresis effects, the constitutive relation is [131]

$$D_i(\mathbf{r},t) = \epsilon_0 E_i(\mathbf{r},t) + \sum_{j=1}^{3} \chi_{ij}^{(1)} E_j(\mathbf{r},t) + \sum_{j,k=1}^{3} \chi_{ijk}^{(2)} E_j(\mathbf{r},t) E_k(\mathbf{r},t) + \\ + \sum_{j,k,l=1}^{3} \chi_{ijkl}^{(3)} E_j(\mathbf{r},t) E_k(\mathbf{r},t) E_l(\mathbf{r},t) + \cdots \tag{2.43}$$

where the index $i = 1, 2, 3$ refers to the three components of the fields **D** and **E**. The first sum in (2.43) is identical to the constitutive relation for linear anisotropic materials. Thus, $\chi_{ij}^{(1)}$ is identical to the susceptibility dyadic of a linear anisotropic medium considered earlier. The quantity $\chi_{ijk}^{(2)}$ is called the *second-order susceptibility*, and is a three-dimensional matrix (or third rank tensor) describing the nonlinear electric effects quadratic in **E**. Similarly $\chi_{ijkl}^{(3)}$ is called the *third-order susceptibility*, and is a four-dimensional matrix (or fourth rank tensor) describing the nonlinear electric effects cubic in **E**. Numerical values of $\chi_{ijk}^{(2)}$ and $\chi_{ijkl}^{(3)}$ are given in Shen [174] for a variety of crystals.

When the material shows hysteresis effects, **D** at any point **r** and time t is due not only to the value of **E** at that point and at that time, but to the values of **E** at all points and times. That is, the material displays both temporal and spatial dispersion.

2.3 Maxwell's equations in moving frames

The essence of special relativity is that the mathematical forms of Maxwell's equations are identical in all *inertial reference frames*: frames moving with uniform velocities relative to the *laboratory frame of reference* in which we perform our measurements. This *form invariance* of Maxwell's equations is a specific example of the general physical *principle of covariance*. In the laboratory frame we write the differential equations of Maxwell's theory as

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t},$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \mathbf{J}(\mathbf{r},t) + \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t},$$

$$\nabla \cdot \mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t),$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0,$$

$$\nabla \cdot \mathbf{J}(\mathbf{r},t) = -\frac{\partial \rho(\mathbf{r},t)}{\partial t}.$$

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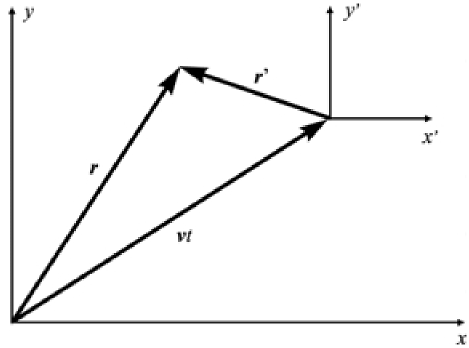

Figure 2.1: Primed coordinate system moving with velocity $\mathbf{v}$ relative to laboratory (unprimed) coordinate system.

Similarly, in an inertial frame having four-dimensional coordinates $(\mathbf{r}', t')$ we have

$$\nabla' \times \mathbf{E}'(\mathbf{r}', t') = -\frac{\partial \mathbf{B}'(\mathbf{r}', t')}{\partial t'},$$

$$\nabla' \times \mathbf{H}'(\mathbf{r}', t') = \mathbf{J}'(\mathbf{r}', t') + \frac{\partial \mathbf{D}'(\mathbf{r}', t')}{\partial t'},$$

$$\nabla' \cdot \mathbf{D}'(\mathbf{r}', t') = \rho'(\mathbf{r}', t'),$$

$$\nabla' \cdot \mathbf{B}'(\mathbf{r}', t') = 0,$$

$$\nabla' \cdot \mathbf{J}'(\mathbf{r}', t') = -\frac{\partial \rho'(\mathbf{r}', t')}{\partial t'}.$$

The primed fields measured in the moving system do *not* have the same numerical values as the unprimed fields measured in the laboratory. To convert between $\mathbf{E}$ and $\mathbf{E}'$, $\mathbf{B}$ and $\mathbf{B}'$, and so on, we must find a way to convert between the coordinates $(\mathbf{r}, t)$ and $(\mathbf{r}', t')$.

2.3.1 Field conversions under Galilean transformation

We shall assume that the primed coordinate system moves with constant velocity $\mathbf{v}$ relative to the laboratory frame (Figure 2.1). Prior to the early part of the twentieth century, converting between the primed and unprimed coordinate variables was intuitive and obvious: it was thought that time must be measured identically in each coordinate system, and that the relationship between the space variables can be determined simply by the displacement of the moving system at time $t = t'$. Under these assumptions, and under the further assumption that the two systems coincide at time $t = 0$, we can write

$$t' = t, \qquad x' = x - v_x t, \qquad y' = y - v_y t, \qquad z' = z - v_z t,$$

or simply

$$t' = t, \qquad \mathbf{r}' = \mathbf{r} - \mathbf{v}t.$$

This is called a *Galilean transformation.* We can use the chain rule to describe the manner in which differential operations transform, i.e., to relate derivatives with respect to the laboratory coordinates to derivatives with respect to the inertial coordinates. We have, for instance,

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial t}\frac{\partial}{\partial y'} + \frac{\partial z'}{\partial t}\frac{\partial}{\partial z'}$$

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$$
\begin{aligned}
&= \frac{\partial}{\partial t'} - v_x \frac{\partial}{\partial x'} - v_y \frac{\partial}{\partial y'} - v_z \frac{\partial}{\partial z'} \\
&= \frac{\partial}{\partial t'} - (\mathbf{v} \cdot \nabla'). \qquad (2.44)
\end{aligned}
$$

Similarly

$$
\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}, \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'},
$$

from which

$$
\nabla \times \mathbf{A}(\mathbf{r}, t) = \nabla' \times \mathbf{A}(\mathbf{r}, t), \qquad \nabla \cdot \mathbf{A}(\mathbf{r}, t) = \nabla' \cdot \mathbf{A}(\mathbf{r}, t), \qquad (2.45)
$$

for each vector field $\mathbf{A}$.

Newton was aware that the laws of mechanics are invariant with respect to Galilean transformations. Do Maxwell's equations also behave in this way? Let us use the Galilean transformation to determine which relationship between the primed and unprimed fields results in form invariance of Maxwell's equations. We first examine $\nabla' \times \mathbf{E}$, the spatial rate of change of the laboratory field with respect to the inertial frame spatial coordinates:

$$
\nabla' \times \mathbf{E} = \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial \mathbf{B}}{\partial t'} + (\mathbf{v} \cdot \nabla')\mathbf{B}
$$

by (2.45) and (2.44). Rewriting the last term by (B.45) we have

$$
(\mathbf{v} \cdot \nabla')\mathbf{B} = -\nabla' \times (\mathbf{v} \times \mathbf{B})
$$

since $\mathbf{v}$ is constant and $\nabla' \cdot \mathbf{B} = \nabla \cdot \mathbf{B} = 0$, hence

$$
\nabla' \times (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = -\frac{\partial \mathbf{B}}{\partial t'}. \qquad (2.46)
$$

Similarly

$$
\nabla' \times \mathbf{H} = \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t'} + \nabla' \times (\mathbf{v} \times \mathbf{D}) - \mathbf{v}(\nabla' \cdot \mathbf{D})
$$

where $\nabla' \cdot \mathbf{D} = \nabla \cdot \mathbf{D} = \rho$ so that

$$
\nabla' \times (\mathbf{H} - \mathbf{v} \times \mathbf{D}) = \frac{\partial \mathbf{D}}{\partial t'} - \rho \mathbf{v} + \mathbf{J}. \qquad (2.47)
$$

Also

$$
\nabla' \cdot \mathbf{J} = \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial \rho}{\partial t'} + (\mathbf{v} \cdot \nabla')\rho
$$

and we may use (B.42) to write

$$
(\mathbf{v} \cdot \nabla')\rho = \mathbf{v} \cdot (\nabla' \rho) = \nabla' \cdot (\rho \mathbf{v}),
$$

obtaining

$$
\nabla' \cdot (\mathbf{J} - \rho \mathbf{v}) = -\frac{\partial \rho}{\partial t'}. \qquad (2.48)
$$

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Equations (2.46), (2.47), and (2.48) show that the forms of Maxwell's equations in the inertial and laboratory frames are identical provided that

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{2.49}$$
$$\mathbf{D}' = \mathbf{D}, \tag{2.50}$$
$$\mathbf{H}' = \mathbf{H} - \mathbf{v} \times \mathbf{D}, \tag{2.51}$$
$$\mathbf{B}' = \mathbf{B}, \tag{2.52}$$
$$\mathbf{J}' = \mathbf{J} - \rho\mathbf{v}, \tag{2.53}$$
$$\rho' = \rho. \tag{2.54}$$

That is, (2.49)–(2.54) result in form invariance of Faraday's law, Ampere's law, and the continuity equation under a Galilean transformation. These equations express the fields measured by a moving observer in terms of those measured in the laboratory frame. To convert the opposite way, we need only use the principle of relativity. Neither observer can tell whether he or she is stationary — only that the other observer is moving relative to him or her. To obtain the fields in the laboratory frame we simply change the sign on $\mathbf{v}$ and swap primed with unprimed fields in (2.49)–(2.54):

$$\mathbf{E} = \mathbf{E}' - \mathbf{v} \times \mathbf{B}', \tag{2.55}$$
$$\mathbf{D} = \mathbf{D}', \tag{2.56}$$
$$\mathbf{H} = \mathbf{H}' + \mathbf{v} \times \mathbf{D}', \tag{2.57}$$
$$\mathbf{B} = \mathbf{B}', \tag{2.58}$$
$$\mathbf{J} = \mathbf{J}' + \rho'\mathbf{v}, \tag{2.59}$$
$$\rho = \rho'. \tag{2.60}$$

According to (2.53), a moving observer interprets charge stationary in the laboratory frame as an additional current moving opposite the direction of his or her motion. This seems reasonable. However, while $\mathbf{E}$ depends on both $\mathbf{E}'$ and $\mathbf{B}'$, the field $\mathbf{B}$ is unchanged under the transformation. Why should $\mathbf{B}$ have this special status? In fact, we may uncover an inconsistency among the transformations by considering free space where (2.22) and (2.23) hold: in this case (2.49) gives

$$\mathbf{D}'/\epsilon_0 = \mathbf{D}/\epsilon_0 + \mathbf{v} \times \mu_0\mathbf{H}$$

or

$$\mathbf{D}' = \mathbf{D} + \mathbf{v} \times \mathbf{H}/c^2$$

rather than (2.50). Similarly, from (2.51) we get

$$\mathbf{B}' = \mathbf{B} - \mathbf{v} \times \mathbf{E}/c^2$$

instead of (2.52). Using these, the set of transformations becomes

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{2.61}$$
$$\mathbf{D}' = \mathbf{D} + \mathbf{v} \times \mathbf{H}/c^2, \tag{2.62}$$
$$\mathbf{H}' = \mathbf{H} - \mathbf{v} \times \mathbf{D}, \tag{2.63}$$
$$\mathbf{B}' = \mathbf{B} - \mathbf{v} \times \mathbf{E}/c^2, \tag{2.64}$$
$$\mathbf{J}' = \mathbf{J} - \rho\mathbf{v}, \tag{2.65}$$
$$\rho' = \rho. \tag{2.66}$$

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These can also be written using dyadic notation as

$$\mathbf{E}' = \bar{\mathbf{I}} \cdot \mathbf{E} + \bar{\boldsymbol{\beta}} \cdot (c\mathbf{B}), \tag{2.67}$$

$$c\mathbf{B}' = -\bar{\boldsymbol{\beta}} \cdot \mathbf{E} + \bar{\mathbf{I}} \cdot (c\mathbf{B}), \tag{2.68}$$

and

$$c\mathbf{D}' = \bar{\mathbf{I}} \cdot (c\mathbf{D}) + \bar{\boldsymbol{\beta}} \cdot \mathbf{H}, \tag{2.69}$$

$$\mathbf{H}' = -\bar{\boldsymbol{\beta}} \cdot (c\mathbf{D}) + \bar{\mathbf{I}} \cdot \mathbf{H}, \tag{2.70}$$

where

$$[\bar{\beta}] = \begin{bmatrix} 0 & -\beta_z & \beta_y \\ \beta_z & 0 & -\beta_x \\ -\beta_y & \beta_x & 0 \end{bmatrix}$$

with $\boldsymbol{\beta} = \mathbf{v}/c$. This set of equations is self-consistent among Maxwell's equations. However, the equations are not consistent with the assumption of a Galilean transformation of the coordinates, and thus Maxwell's equations are not covariant under a Galilean transformation. Maxwell's equations are only covariant under a Lorentz transformation as described in the next section. Expressions (2.61)–(2.64) turn out to be accurate to order v/c, hence are the results of a *first-order Lorentz transformation*. Only when v is an appreciable fraction of c do the field conversions resulting from the first-order Lorentz transformation differ markedly from those resulting from a Galilean transformation; those resulting from the true Lorentz transformation require even higher velocities to differ markedly from the first-order expressions. Engineering accuracy is often accomplished using the Galilean transformation. This pragmatic observation leads to quite a bit of confusion when considering the large-scale forms of Maxwell's equations, as we shall soon see.

2.3.2 Field conversions under Lorentz transformation

To find the proper transformation under which Maxwell's equations are covariant, we must discard our notion that time progresses the same in the primed and the unprimed frames. The proper transformation of coordinates that guarantees covariance of Maxwell's equations is the *Lorentz transformation*

$$ct' = \gamma ct - \gamma \boldsymbol{\beta} \cdot \mathbf{r}, \tag{2.71}$$

$$\mathbf{r}' = \bar{\boldsymbol{\alpha}} \cdot \mathbf{r} - \gamma \boldsymbol{\beta} ct, \tag{2.72}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \bar{\boldsymbol{\alpha}} = \bar{\mathbf{I}} + (\gamma - 1)\frac{\boldsymbol{\beta}\boldsymbol{\beta}}{\beta^2}, \qquad \beta = |\boldsymbol{\beta}|.$$

This is obviously more complicated than the Galilean transformation; only as $\boldsymbol{\beta} \to 0$ are the Lorentz and Galilean transformations equivalent.

Not surprisingly, field conversions between inertial reference frames are more complicated with the Lorentz transformation than with the Galilean transformation. For simplicity we assume that the velocity of the moving frame has only an x-component: $\mathbf{v} = \hat{\mathbf{x}}v$. Later we can generalize this to any direction. Equations (2.71) and (2.72) become

$$x' = x + (\gamma - 1)x - \gamma vt, \tag{2.73}$$

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$$y' = y, \tag{2.74}$$

$$z' = z, \tag{2.75}$$

$$ct' = \gamma ct - \gamma \frac{v}{c} x, \tag{2.76}$$

and the chain rule gives

$$\frac{\partial}{\partial x} = \gamma \frac{\partial}{\partial x'} - \gamma \frac{v}{c^2} \frac{\partial}{\partial t'}, \tag{2.77}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'}, \tag{2.78}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'}, \tag{2.79}$$

$$\frac{\partial}{\partial t} = -\gamma v \frac{\partial}{\partial x'} + \gamma \frac{\partial}{\partial t'}. \tag{2.80}$$

We begin by examining Faraday's law in the laboratory frame. In component form we have

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}, \tag{2.81}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}, \tag{2.82}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t}. \tag{2.83}$$

These become

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \gamma v \frac{\partial B_x}{\partial x'} - \gamma \frac{\partial B_x}{\partial t'}, \tag{2.84}$$

$$\frac{\partial E_x}{\partial z'} - \gamma \frac{\partial E_z}{\partial x'} + \gamma \frac{v}{c^2} \frac{\partial E_z}{\partial t'} = \gamma v \frac{\partial B_y}{\partial x'} - \gamma \frac{\partial B_y}{\partial t'}, \tag{2.85}$$

$$\gamma \frac{\partial E_y}{\partial x'} - \gamma \frac{v}{c^2} \frac{\partial E_y}{\partial t'} - \frac{\partial E_x}{\partial y'} = \gamma v \frac{\partial B_z}{\partial x'} - \gamma \frac{\partial B_z}{\partial t'}, \tag{2.86}$$

after we use (2.77)–(2.80) to convert the derivatives in the laboratory frame to derivatives with respect to the moving frame coordinates. To simplify (2.84) we consider

$$\nabla \cdot \mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0.$$

Converting the laboratory frame coordinates to the moving frame coordinates, we have

$$\gamma \frac{\partial B_x}{\partial x'} - \gamma \frac{v}{c^2} \frac{\partial B_x}{\partial t'} + \frac{\partial B_y}{\partial y'} + \frac{\partial B_z}{\partial z'} = 0$$

or

$$-\gamma v \frac{\partial B_x}{\partial x'} = -\gamma \frac{v^2}{c^2} \frac{\partial B_x}{\partial t'} + v \frac{\partial B_y}{\partial y'} + v \frac{\partial B_z}{\partial z'}.$$

Substituting this into (2.84) and rearranging (2.85) and (2.86), we obtain

$$\frac{\partial}{\partial y'} \gamma (E_z + v B_y) - \frac{\partial}{\partial z'} \gamma (E_y - v B_z) = -\frac{\partial B_x}{\partial t'},$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial}{\partial x'} \gamma (E_z + v B_y) = -\frac{\partial}{\partial t'} \gamma \left(B_y + \frac{v}{c^2} E_z \right),$$

$$\frac{\partial}{\partial x'} \gamma (E_y - v B_z) - \frac{\partial E_x}{\partial y'} = -\frac{\partial}{\partial t'} \gamma \left(B_z - \frac{v}{c^2} E_y \right).$$

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Comparison with (2.81)–(2.83) shows that form invariance of Faraday's law under the Lorentz transformation requires

$$E'_x = E_x, \qquad E'_y = \gamma(E_y - vB_z), \qquad E'_z = \gamma(E_z + vB_y),$$

and

$$B'_x = B_x, \qquad B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \qquad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right).$$

To generalize $\mathbf{v}$ to any direction, we simply note that the components of the fields parallel to the velocity direction are identical in the moving and laboratory frames, while the components perpendicular to the velocity direction convert according to a simple cross product rule. After similar analyses with Ampere's and Gauss's laws (see Problem 2.2), we find that

$$\mathbf{E}'_\parallel = \mathbf{E}_\parallel, \qquad \mathbf{B}'_\parallel = \mathbf{B}_\parallel, \qquad \mathbf{D}'_\parallel = \mathbf{D}_\parallel, \qquad \mathbf{H}'_\parallel = \mathbf{H}_\parallel,$$

$$\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \boldsymbol{\beta} \times c\mathbf{B}_\perp), \tag{2.87}$$

$$c\mathbf{B}'_\perp = \gamma(c\mathbf{B}_\perp - \boldsymbol{\beta} \times \mathbf{E}_\perp), \tag{2.88}$$

$$c\mathbf{D}'_\perp = \gamma(c\mathbf{D}_\perp + \boldsymbol{\beta} \times \mathbf{H}_\perp), \tag{2.89}$$

$$\mathbf{H}'_\perp = \gamma(\mathbf{H}_\perp - \boldsymbol{\beta} \times c\mathbf{D}_\perp), \tag{2.90}$$

and

$$\mathbf{J}'_\parallel = \gamma(\mathbf{J}_\parallel - \rho\mathbf{v}), \tag{2.91}$$

$$\mathbf{J}'_\perp = \mathbf{J}_\perp, \tag{2.92}$$

$$c\rho' = \gamma(c\rho - \boldsymbol{\beta} \cdot \mathbf{J}), \tag{2.93}$$

where the symbols $\parallel$ and $\perp$ designate the components of the field parallel and perpendicular to $\mathbf{v}$, respectively.

These conversions are self-consistent, and the Lorentz transformation is the transformation under which Maxwell's equations are covariant. If $v^2 \ll c^2$, then $\gamma \approx 1$ and to first order (2.87)–(2.93) reduce to (2.61)–(2.66). If $v/c \ll 1$, then the first-order fields reduce to the Galilean fields (2.49)–(2.54).

To convert in the opposite direction, we can swap primed and unprimed fields and change the sign on $\mathbf{v}$:

$$\mathbf{E}_\perp = \gamma(\mathbf{E}'_\perp - \boldsymbol{\beta} \times c\mathbf{B}'_\perp), \tag{2.94}$$

$$c\mathbf{B}_\perp = \gamma(c\mathbf{B}'_\perp + \boldsymbol{\beta} \times \mathbf{E}'_\perp), \tag{2.95}$$

$$c\mathbf{D}_\perp = \gamma(c\mathbf{D}'_\perp - \boldsymbol{\beta} \times \mathbf{H}'_\perp), \tag{2.96}$$

$$\mathbf{H}_\perp = \gamma(\mathbf{H}'_\perp + \boldsymbol{\beta} \times c\mathbf{D}'_\perp), \tag{2.97}$$

and

$$\mathbf{J}_\parallel = \gamma(\mathbf{J}'_\parallel + \rho'\mathbf{v}), \tag{2.98}$$

$$\mathbf{J}_\perp = \mathbf{J}'_\perp, \tag{2.99}$$

$$c\rho = \gamma(c\rho' + \boldsymbol{\beta} \cdot \mathbf{J}'). \tag{2.100}$$

The conversion formulas can be written much more succinctly in dyadic notation:

$$\mathbf{E}' = \gamma\bar{\boldsymbol{\alpha}}^{-1} \cdot \mathbf{E} + \gamma\bar{\boldsymbol{\beta}} \cdot (c\mathbf{B}), \tag{2.101}$$

$$c\mathbf{B}' = -\gamma\bar{\boldsymbol{\beta}} \cdot \mathbf{E} + \gamma\bar{\boldsymbol{\alpha}}^{-1} \cdot (c\mathbf{B}), \tag{2.102}$$

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$$c\mathbf{D}' = \gamma\bar{\boldsymbol{\alpha}}^{-1}\cdot(c\mathbf{D}) + \gamma\bar{\boldsymbol{\beta}}\cdot\mathbf{H}, \tag{2.103}$$

$$\mathbf{H}' = -\gamma\bar{\boldsymbol{\beta}}\cdot(c\mathbf{D}) + \gamma\bar{\boldsymbol{\alpha}}^{-1}\cdot\mathbf{H}, \tag{2.104}$$

and

$$c\rho' = \gamma(c\rho - \boldsymbol{\beta}\cdot\mathbf{J}), \tag{2.105}$$

$$\mathbf{J}' = \bar{\boldsymbol{\alpha}}\cdot\mathbf{J} - \gamma\boldsymbol{\beta}c\rho, \tag{2.106}$$

where $\bar{\boldsymbol{\alpha}}^{-1}\cdot\bar{\boldsymbol{\alpha}} = \bar{\mathbf{I}}$, and thus $\bar{\boldsymbol{\alpha}}^{-1} = \bar{\boldsymbol{\alpha}} - \gamma\boldsymbol{\beta}\boldsymbol{\beta}$.

Maxwell's equations are covariant under a Lorentz transformation but not under a Galilean transformation; the laws of mechanics are invariant under a Galilean transformation but not under a Lorentz transformation. How then should we analyze interactions between electromagnetic fields and particles or materials? Einstein realized that the laws of mechanics needed revision to make them Lorentz covariant: in fact, under his theory of special relativity all physical laws should demonstrate Lorentz covariance. Interestingly, charge is then Lorentz invariant, whereas mass is not (recall that invariance refers to a quantity, whereas covariance refers to the form of a natural law). We shall not attempt to describe all the ramifications of special relativity, but instead refer the reader to any of the excellent and readable texts on the subject, including those by Bohm [14], Einstein [62], and Born [18], and to the nice historical account by Miller [130]. However, we shall examine the importance of Lorentz invariants in electromagnetic theory.

Lorentz invariants. Although the electromagnetic fields are not Lorentz invariant (e.g., the numerical value of $\mathbf{E}$ measured by one observer differs from that measured by another observer in uniform relative motion), several quantities do give identical values regardless of the velocity of motion. Most fundamental are the speed of light and the quantity of electric charge which, unlike mass, is the same in all frames of reference. Other important *Lorentz invariants* include $\mathbf{E}\cdot\mathbf{B}$, $\mathbf{H}\cdot\mathbf{D}$, and the quantities

$$\mathbf{B}\cdot\mathbf{B} - \mathbf{E}\cdot\mathbf{E}/c^2,$$

$$\mathbf{H}\cdot\mathbf{H} - c^2\mathbf{D}\cdot\mathbf{D},$$

$$\mathbf{B}\cdot\mathbf{H} - \mathbf{E}\cdot\mathbf{D},$$

$$c\mathbf{B}\cdot\mathbf{D} + \mathbf{E}\cdot\mathbf{H}/c.$$

(See Problem 2.3.) To see the importance of these quantities, consider the special case of fields in empty space. If $\mathbf{E}\cdot\mathbf{B} = 0$ in one reference frame, then it is zero in all reference frames. Then if $\mathbf{B}\cdot\mathbf{B} - \mathbf{E}\cdot\mathbf{E}/c^2 = 0$ in any reference frame, the ratio of E to B is always c^2 regardless of the reference frame in which the fields are measured. This is the characteristic of a plane wave in free space.

If $\mathbf{E}\cdot\mathbf{B} = 0$ and $c^2B^2 > E^2$, then we can find a reference frame using the conversion formulas (2.101)–(2.106) (see Problem 2.5) in which the electric field is zero but the magnetic field is nonzero. In this case we call the fields *purely magnetic* in any reference frame, even if both $\mathbf{E}$ and $\mathbf{B}$ are nonzero. Similarly, if $\mathbf{E}\cdot\mathbf{B} = 0$ and $c^2B^2 < E^2$ then we can find a reference frame in which the magnetic field is zero but the electric field is nonzero. We call fields of this type *purely electric.*

The Lorentz force is not Lorentz invariant. Consider a point charge at rest in the laboratory frame. While we measure only an electric field in the laboratory frame, an inertial observer measures both electric and magnetic fields. A test charge Q in the

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laboratory frame experiences the Lorentz force $\mathbf{F} = Q\mathbf{E}$; in an inertial frame the same charge experiences $\mathbf{F}' = Q\mathbf{E}' + Q\mathbf{v} \times \mathbf{B}'$ (see Problem 2.6). The conversion formulas show that $\mathbf{F}$ and $\mathbf{F}'$ are not identical.

We see that both $\mathbf{E}$ and $\mathbf{B}$ are integral components of the electromagnetic field: the separation of the field into electric and magnetic components depends on the motion of the reference frame in which measurements are made. This has obvious implications when considering static electric and magnetic fields.

Derivation of Maxwell's equations from Coulomb's law. Consider a point charge at rest in the laboratory frame. If the magnetic component of force on this charge arises naturally through motion of an inertial reference frame, and if this force can be expressed in terms of Coulomb's law in the laboratory frame, then perhaps the magnetic field can be derived directly from Coulomb's and the Lorentz transformation. Perhaps it is possible to derive all of Maxwell's theory with Coulomb's law and Lorentz invariance as the only postulates.

Several authors, notably Purcell [152] and Elliott [65], have used this approach. However, Jackson [91] has pointed out that many additional assumptions are required to deduce Maxwell's equations beginning with Coulomb's law. Feynman [73] is critical of the approach, pointing out that we must introduce a vector potential which adds to the scalar potential from electrostatics in order to produce an entity that transforms according to the laws of special relativity. In addition, the assumption of Lorentz invariance seems to involve circular reasoning since the Lorentz transformation was originally introduced to make Maxwell's equations covariant. But Lucas and Hodgson [117] point out that the Lorentz transformation can be deduced from other fundamental principles (such as causality and the isotropy of space), and that the postulate of a vector potential is reasonable. Schwartz [170] gives a detailed derivation of Maxwell's equations from Coulomb's law, outlining the necessary assumptions.

Transformation of constitutive relations. Minkowski's interest in the covariance of Maxwell's equations was aimed not merely at the relationship between fields in different moving frames of reference, but at an understanding of the electrodynamics of moving media. He wished to ascertain the effect of a moving material body on the electromagnetic fields in some region of space. By proposing the covariance of Maxwell's equations in materials as well as in free space, he extended Maxwell's theory to moving material bodies.

We have seen in (2.101)–(2.104) that $(\mathbf{E}, c\mathbf{B})$ and $(c\mathbf{D}, \mathbf{H})$ convert identically under a Lorentz transformation. Since the most general form of the constitutive relations relate $c\mathbf{D}$ and $\mathbf{H}$ to the field pair $(\mathbf{E}, c\mathbf{B})$ (see § 2.2.2) as

$$\begin{bmatrix} c\mathbf{D} \\ \mathbf{H} \end{bmatrix} = [\bar{\mathbf{C}}] \begin{bmatrix} \mathbf{E} \\ c\mathbf{B} \end{bmatrix},$$

this form of the constitutive relations must be Lorentz covariant. That is, in the reference frame of a moving material we have

$$\begin{bmatrix} c\mathbf{D}' \\ \mathbf{H}' \end{bmatrix} = [\bar{\mathbf{C}}'] \begin{bmatrix} \mathbf{E}' \\ c\mathbf{B}' \end{bmatrix},$$

and should be able to convert $[\bar{\mathbf{C}}']$ to $[\bar{\mathbf{C}}]$. We should be able to find the constitutive matrix describing the relationships among the fields observed in the laboratory frame.

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It is somewhat laborious to obtain the constitutive matrix $[\bar{\mathbf{C}}]$ for an arbitrary moving medium. Detailed expressions for isotropic, bianisotropic, gyrotropic, and uniaxial media are given by Kong [101]. The rather complicated expressions can be written in a more compact form if we consider the expressions for $\mathbf{B}$ and $\mathbf{D}$ in terms of the pair $(\mathbf{E},\mathbf{H})$. For a linear isotropic material such that $\mathbf{D}' = \epsilon'\mathbf{E}'$ and $\mathbf{B}' = \mu'\mathbf{H}'$ in the moving frame, the relationships in the laboratory frame are [101]

$$\mathbf{B} = \mu'\bar{\mathbf{A}} \cdot \mathbf{H} - \boldsymbol{\Omega} \times \mathbf{E}, \tag{2.107}$$

$$\mathbf{D} = \epsilon'\bar{\mathbf{A}} \cdot \mathbf{E} + \boldsymbol{\Omega} \times \mathbf{H}, \tag{2.108}$$

where

$$\bar{\mathbf{A}} = \frac{1-\beta^2}{1-n^2\beta^2}\left[\bar{\mathbf{I}} - \frac{n^2-1}{1-\beta^2}\boldsymbol{\beta}\boldsymbol{\beta}\right], \tag{2.109}$$

$$\boldsymbol{\Omega} = \frac{n^2-1}{1-n^2\beta^2}\frac{\boldsymbol{\beta}}{c}, \tag{2.110}$$

and where $n = c(\mu'\epsilon')^{1/2}$ is the optical index of the medium. A moving material that is isotropic in its own moving reference frame is bianisotropic in the laboratory frame. If, for instance, we tried to measure the relationship between the fields of a moving isotropic fluid, but used instruments that were stationary in our laboratory (e.g., attached to our measurement bench) we would find that $\mathbf{D}$ depends not only on $\mathbf{E}$ but also on $\mathbf{H}$, and that $\mathbf{D}$ aligns with neither $\mathbf{E}$ nor $\mathbf{H}$. That a moving material isotropic in its own frame of reference is bianisotropic in the laboratory frame was known long ago. Roentgen showed experimentally in 1888 that a dielectric moving through an electric field becomes magnetically polarized, while H.A. Wilson showed in 1905 that a dielectric moving through a magnetic field becomes electrically polarized [139].

If $v^2/c^2 \ll 1$, we can consider the form of the constitutive equations for a first-order Lorentz transformation. Ignoring terms to order v^2/c^2 in (2.109) and (2.110), we obtain $\bar{\mathbf{A}} = \bar{\mathbf{I}}$ and $\boldsymbol{\Omega} = \mathbf{v}(n^2-1)/c^2$. Then, by (2.107) and (2.108),

$$\mathbf{B} = \mu'\mathbf{H} - (n^2-1)\frac{\mathbf{v}\times\mathbf{E}}{c^2}, \tag{2.111}$$

$$\mathbf{D} = \epsilon'\mathbf{E} + (n^2-1)\frac{\mathbf{v}\times\mathbf{H}}{c^2}. \tag{2.112}$$

We can also derive these from the first-order field conversion equations (2.61)–(2.64). From (2.61) and (2.62) we have

$$\mathbf{D}' = \mathbf{D} + \mathbf{v}\times\mathbf{H}/c^2 = \epsilon'\mathbf{E}' = \epsilon'(\mathbf{E} + \mathbf{v}\times\mathbf{B}).$$

Eliminating $\mathbf{B}$ via (2.64), we have

$$\mathbf{D} + \mathbf{v}\times\mathbf{H}/c^2 = \epsilon'\mathbf{E} + \epsilon'\mathbf{v}\times(\mathbf{v}\times\mathbf{E}/c^2) + \epsilon'\mathbf{v}\times\mathbf{B}' = \epsilon'\mathbf{E} + \epsilon'\mathbf{v}\times\mathbf{B}'$$

where we have neglected terms of order v^2/c^2. Since $\mathbf{B}' = \mu'\mathbf{H}' = \mu'(\mathbf{H} - \mathbf{v}\times\mathbf{D})$, we have

$$\mathbf{D} + \mathbf{v}\times\mathbf{H}/c^2 = \epsilon'\mathbf{E} + \epsilon'\mu'\mathbf{v}\times\mathbf{H} - \epsilon'\mu'\mathbf{v}\times\mathbf{v}\times\mathbf{D}.$$

Using $n^2 = c^2\mu'\epsilon'$ and neglecting the last term since it is of order v^2/c^2, we obtain

$$\mathbf{D} = \epsilon'\mathbf{E} + (n^2-1)\frac{\mathbf{v}\times\mathbf{H}}{c^2},$$

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which is identical to the expression (2.112) obtained by approximating the exact result to first order. Similar steps produce (2.111). In a Galilean frame where $v/c \ll 1$, the expressions reduce to $\mathbf{D} = \epsilon'\mathbf{E}$ and $\mathbf{B} = \mu'\mathbf{H}$, and the isotropy of the fields is preserved.

For a conducting medium having

$$\mathbf{J}' = \sigma'\mathbf{E}'$$

in a moving reference frame, Cullwick [48] shows that in the laboratory frame

$$\mathbf{J} = \sigma'\gamma[\bar{\mathbf{I}} - \boldsymbol{\beta}\boldsymbol{\beta}] \cdot \mathbf{E} + \sigma'\gamma c\boldsymbol{\beta} \times \mathbf{B}.$$

For $v \ll c$ we can set $\gamma \approx 1$ and see that

$$\mathbf{J} = \sigma'(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

to first order.

Constitutive relations in deforming or rotating media. The transformations discussed in the previous paragraphs hold for media in uniform relative motion. When a material body undergoes deformation or rotation, the concepts of special relativity are not directly applicable. However, authors such as Pauli [144] and Sommerfeld [185] have maintained that Minkowski's theory is *approximately* valid for deforming or rotating media if $\mathbf{v}$ is taken to be the *instantaneous* velocity at each point within the body. The reasoning is that at any instant in time each point within the body has a velocity $\mathbf{v}$ that may be associated with some inertial reference frame (generally different for each point). Thus the constitutive relations for the material at that point, within some small time interval taken about the observation time, may be assumed to be those of a stationary material, and the relations measured by an observer within the laboratory frame may be computed using the inertial frame for that point. This *instantaneous rest-frame* theory is most accurate at small accelerations $d\mathbf{v}/dt$. Van Bladel [201] outlines its shortcomings. See also Anderson [3] and Mo [132] for detailed discussions of the electromagnetic properties of material media in accelerating frames of reference.

2.4 The Maxwell–Boffi equations

In any version of Maxwell's theory, the mediating field is the electromagnetic field described by four field vectors. In Minkowski's form of Maxwell's equations we use $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{H}$. As an alternative consider the electromagnetic field as represented by the vector fields $\mathbf{E}$, $\mathbf{B}$, $\mathbf{P}$, and $\mathbf{M}$, and described by

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.113}$$

$$\nabla \times (\mathbf{B}/\mu_0 - \mathbf{M}) = \mathbf{J} + \frac{\partial}{\partial t}(\epsilon_0\mathbf{E} + \mathbf{P}), \tag{2.114}$$

$$\nabla \cdot (\epsilon_0\mathbf{E} + \mathbf{P}) = \rho, \tag{2.115}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{2.116}$$

These *Maxwell–Boffi equations* are named after L. Boffi, who formalized them for moving media [13]. The quantity $\mathbf{P}$ is the *polarization vector*, and $\mathbf{M}$ is the *magnetization vector*.

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The use of **P** and **M** in place of **D** and **H** is sometimes called an application of the *principle of Ampere and Lorentz* [199].

Let us examine the ramification of using (2.113)–(2.116) as the basis for a postulate of electromagnetics. These equations are similar to the Maxwell–Minkowski equations used earlier; must we rebuild all the underpinning of a new postulate, or can we use our original arguments based on the Minkowski form? For instance, how do we invoke uniqueness if we no longer have the field **H**? What represents the flux of energy, formerly found using $\mathbf{E}\times\mathbf{H}$? And, importantly, are (2.113)–(2.114) form invariant under a Lorentz transformation?

It turns out that the set of vector fields $(\mathbf{E}, \mathbf{B}, \mathbf{P}, \mathbf{M})$ is merely a linear mapping of the set $(\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H})$. As pointed out by Tai [193], any linear mapping of the four field vectors from Minkowski's form onto any other set of four field vectors will preserve the covariance of Maxwell's equations. Boffi chose to keep **E** and **B** intact and to introduce only two new fields; he could have kept **H** and **D** instead, or used a mapping that introduced four completely new fields (as did Chu). Many authors retain **E** and **H**. This is somewhat more cumbersome since these vectors do not convert as a pair under a Lorentz transformation. A discussion of the idea of field vector "pairing" appears in § 2.6.

The usefulness of the Boffi form lies in the specific mapping chosen. Comparison of (2.113)–(2.116) to (2.1)–(2.4) quickly reveals that

$$\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E}, \tag{2.117}$$

$$\mathbf{M} = \mathbf{B}/\mu_0 - \mathbf{H}. \tag{2.118}$$

We see that **P** is the difference between **D** in a material and **D** in free space, while **M** is the difference between **H** in free space and **H** in a material. In free space, $\mathbf{P} = \mathbf{M} = 0$.

Equivalent polarization and magnetization sources. The Boffi formulation provides a new way to regard **E** and **B**. Maxwell grouped $(\mathbf{E}, \mathbf{H})$ as a pair of "force vectors" to be associated with line integrals (or curl operations in the point forms of his equations), and $(\mathbf{D}, \mathbf{B})$ as a pair of "flux vectors" associated with surface integrals (or divergence operations). That is, **E** is interpreted as belonging to the computation of "emf" as a line integral, while **B** is interpreted as a density of magnetic "flux" passing through a surface. Similarly, **H** yields the "mmf" about some closed path and **D** the electric flux through a surface. The introduction of **P** and **M** allows us to also regard **E** as a flux vector and **B** as a force vector — in essence, allowing the two fields **E** and **B** to take on the duties that required four fields in Minkowski's form. To see this, we rewrite the Maxwell–Boffi equations as

$$\begin{aligned}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \times \frac{\mathbf{B}}{\mu_0} &= \left(\mathbf{J} + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}\right) + \frac{\partial \epsilon_0 \mathbf{E}}{\partial t}, \\
\nabla \cdot (\epsilon_0 \mathbf{E}) &= (\rho - \nabla \cdot \mathbf{P}), \\
\nabla \cdot \mathbf{B} &= 0,
\end{aligned}$$

and compare them to the Maxwell–Minkowski equations for sources in free space:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

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$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J} + \frac{\partial \epsilon_0 \mathbf{E}}{\partial t},$$
$$\nabla \cdot (\epsilon_0 \mathbf{E}) = \rho,$$
$$\nabla \cdot \mathbf{B} = 0.$$

The forms are preserved if we identify $\partial \mathbf{P}/\partial t$ and $\nabla \times \mathbf{M}$ as new types of current density, and $\nabla \cdot \mathbf{P}$ as a new type of charge density. We define

$$\mathbf{J}_P = \frac{\partial \mathbf{P}}{\partial t} \tag{2.119}$$

as an *equivalent polarization current* density, and

$$\mathbf{J}_M = \nabla \times \mathbf{M}$$

as an *equivalent magnetization current* density (sometimes called the *equivalent Amperian currents of magnetized matter* [199]). We define

$$\rho_P = -\nabla \cdot \mathbf{P}$$

as an *equivalent polarization charge* density (sometimes called the *Poisson–Kelvin* equivalent charge distribution [199]). Then the Maxwell–Boffi equations become simply

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.120}$$
$$\nabla \times \frac{\mathbf{B}}{\mu_0} = (\mathbf{J} + \mathbf{J}_M + \mathbf{J}_P) + \frac{\partial \epsilon_0 \mathbf{E}}{\partial t}, \tag{2.121}$$
$$\nabla \cdot (\epsilon_0 \mathbf{E}) = (\rho + \rho_P), \tag{2.122}$$
$$\nabla \cdot \mathbf{B} = 0. \tag{2.123}$$

Here is the new view. A material can be viewed as composed of charged particles of matter immersed in free space. When these charges are properly considered as "equivalent" polarization and magnetization charges, all field effects (describable through flux and force vectors) can be handled by the two fields **E** and **B**. Whereas in Minkowski's form **D** diverges from ρ, in Boffi's form **E** diverges from a *total* charge density consisting of ρ and ρ_P. Whereas in the Minkowski form **H** curls around **J**, in the Boffi form **B** curls around the total current density consisting of **J**, $\mathbf{J}_M$, and $\mathbf{J}_P$.

This view was pioneered by Lorentz, who by 1892 considered matter as consisting of bulk molecules in a vacuum that would respond to an applied electromagnetic field [130]. The resulting motion of the charged particles of matter then became another source term for the "fundamental" fields **E** and **B**. Using this reasoning he was able to reduce the fundamental Maxwell equations to two equations in two unknowns, demonstrating a simplicity appealing to many (including Einstein). Of course, to apply this concept we must be able to describe how the charged particles respond to an applied field. Simple microscopic models of the constituents of matter are generally used: some combination of electric and magnetic dipoles, or of loops of electric and magnetic current.

The Boffi equations are mathematically appealing since they now specify both the curl and divergence of the two field quantities **E** and **B**. By the Helmholtz theorem we know that a field vector is uniquely specified when both its curl and divergence are given. But this assumes that the equivalent sources produced by **P** and **M** are true source fields in the same sense as **J**. We have precluded this by insisting in Chapter 1 that the source field must be independent of the mediating field it sources. If we view **P** and **M** as

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merely a mapping from the original vector fields of Minkowski's form, we still have four vector fields with which to contend. And with these must also be a mapping of the constitutive relationships, which now link the fields **E**, **B**, **P**, and **M**. Rather than argue the actual physical existence of the equivalent sources, we note that a real benefit of the new view is that under certain circumstances the equivalent source quantities can be determined through physical reasoning, hence we can create physical models of **P** and **M** and deduce their links to **E** and **B**. We may then find it easier to understand and deduce the constitutive relationships. However we do not in general consider **E** and **B** to be in any way more "fundamental" than **D** and **H**.

Covariance of the Boffi form. Because of the linear relationships (2.117) and (2.118), covariance of the Maxwell–Minkowski equations carries over to the Maxwell–Boffi equations. However, the conversion between fields in different moving reference frames will now involve **P** and **M**. Since Faraday's law is unchanged in the Boffi form, we still have

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \tag{2.124}$$
$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}, \tag{2.125}$$
$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \boldsymbol{\beta} \times c\mathbf{B}_{\perp}), \tag{2.126}$$
$$c\mathbf{B}'_{\perp} = \gamma(c\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E}_{\perp}). \tag{2.127}$$

To see how **P** and **M** convert, we note that in the laboratory frame $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ and $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$, while in the moving frame $\mathbf{D}' = \epsilon_0\mathbf{E}' + \mathbf{P}'$ and $\mathbf{H}' = \mathbf{B}'/\mu_0 - \mathbf{M}'$. Thus

$$\mathbf{P}'_{\parallel} = \mathbf{D}'_{\parallel} - \epsilon_0\mathbf{E}'_{\parallel} = \mathbf{D}_{\parallel} - \epsilon_0\mathbf{E}_{\parallel} = \mathbf{P}_{\parallel}$$

and

$$\mathbf{M}'_{\parallel} = \mathbf{B}'_{\parallel}/\mu_0 - \mathbf{H}'_{\parallel} = \mathbf{B}_{\parallel}/\mu_0 - \mathbf{H}_{\parallel} = \mathbf{M}_{\parallel}.$$

For the perpendicular components

$$\mathbf{D}'_{\perp} = \gamma(\mathbf{D}_{\perp} + \boldsymbol{\beta} \times \mathbf{H}_{\perp}/c) = \epsilon_0\mathbf{E}'_{\perp} + \mathbf{P}'_{\perp} = \epsilon_0\left[\gamma(\mathbf{E}_{\perp} + \boldsymbol{\beta} \times c\mathbf{B}_{\perp})\right] + \mathbf{P}'_{\perp};$$

substitution of $\mathbf{H}_{\perp} = \mathbf{B}_{\perp}/\mu_0 - \mathbf{M}_{\perp}$ then gives

$$\mathbf{P}'_{\perp} = \gamma(\mathbf{D}_{\perp} - \epsilon_0\mathbf{E}_{\perp}) - \gamma\epsilon_0\boldsymbol{\beta} \times c\mathbf{B}_{\perp} + \gamma\boldsymbol{\beta} \times \mathbf{B}_{\perp}/(c\mu_0) - \gamma\boldsymbol{\beta} \times \mathbf{M}_{\perp}/c$$

or

$$c\mathbf{P}'_{\perp} = \gamma(c\mathbf{P}_{\perp} - \boldsymbol{\beta} \times \mathbf{M}_{\perp}).$$

Similarly,

$$\mathbf{M}'_{\perp} = \gamma(\mathbf{M}_{\perp} + \boldsymbol{\beta} \times c\mathbf{P}_{\perp}).$$

Hence

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \quad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}, \quad \mathbf{P}'_{\parallel} = \mathbf{P}_{\parallel}, \quad \mathbf{M}'_{\parallel} = \mathbf{M}_{\parallel}, \quad \mathbf{J}'_{\perp} = \mathbf{J}_{\perp}, \tag{2.128}$$

and

$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \boldsymbol{\beta} \times c\mathbf{B}_{\perp}), \tag{2.129}$$
$$c\mathbf{B}'_{\perp} = \gamma(c\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E}_{\perp}), \tag{2.130}$$
$$c\mathbf{P}'_{\perp} = \gamma(c\mathbf{P}_{\perp} - \boldsymbol{\beta} \times \mathbf{M}_{\perp}), \tag{2.131}$$
$$\mathbf{M}'_{\perp} = \gamma(\mathbf{M}_{\perp} + \boldsymbol{\beta} \times c\mathbf{P}_{\perp}), \tag{2.132}$$
$$\mathbf{J}'_{\parallel} = \gamma(\mathbf{J}_{\parallel} - \rho\mathbf{v}). \tag{2.133}$$

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In the case of the first-order Lorentz transformation we can set $\gamma \approx 1$ to obtain

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{2.134}$$

$$\mathbf{B}' = \mathbf{B} - \frac{\mathbf{v} \times \mathbf{E}}{c^2}, \tag{2.135}$$

$$\mathbf{P}' = \mathbf{P} - \frac{\mathbf{v} \times \mathbf{M}}{c^2}, \tag{2.136}$$

$$\mathbf{M}' = \mathbf{M} + \mathbf{v} \times \mathbf{P}, \tag{2.137}$$

$$\mathbf{J}' = \mathbf{J} - \rho \mathbf{v}. \tag{2.138}$$

To convert from the moving frame to the laboratory frame we simply swap primed with unprimed fields and let $\mathbf{v} \to -\mathbf{v}$.

As a simple example, consider a linear isotropic medium having

$$\mathbf{D}' = \epsilon_0 \epsilon_r' \mathbf{E}', \qquad \mathbf{B}' = \mu_0 \mu_r' \mathbf{H}',$$

in a moving reference frame. From (117) we have

$$\mathbf{P}' = \epsilon_0 \epsilon_r' \mathbf{E}' - \epsilon_0 \mathbf{E}' = \epsilon_0 \chi_e' \mathbf{E}'$$

where $\chi_e' = \epsilon_r' - 1$ is the electric susceptibility of the moving material. Similarly (2.118) yields

$$\mathbf{M}' = \frac{\mathbf{B}'}{\mu_0} - \frac{\mathbf{B}'}{\mu_0 \mu_r'} = \frac{\mathbf{B}' \chi_m'}{\mu_0 \mu_r'}$$

where $\chi_m' = \mu_r' - 1$ is the magnetic susceptibility of the moving material. How are $\mathbf{P}$ and $\mathbf{M}$ related to $\mathbf{E}$ and $\mathbf{B}$ in the laboratory frame? For simplicity, we consider the first-order expressions. From (2.136) we have

$$\mathbf{P} = \mathbf{P}' + \frac{\mathbf{v} \times \mathbf{M}'}{c^2} = \epsilon_0 \chi_e' \mathbf{E}' + \frac{\mathbf{v} \times \mathbf{B}' \chi_m'}{\mu_0 \mu_r' c^2}.$$

Substituting for $\mathbf{E}'$ and $\mathbf{B}'$ from (2.134) and (2.135), and using $\mu_0 c^2 = 1/\epsilon_0$, we have

$$\mathbf{P} = \epsilon_0 \chi_e' (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \epsilon_0 \frac{\chi_m'}{\mu_r'} \mathbf{v} \times \left(\mathbf{B} - \frac{\mathbf{v} \times \mathbf{E}}{c^2} \right).$$

Neglecting the last term since it varies as v^2/c^2, we get

$$\mathbf{P} = \epsilon_0 \chi_e' \mathbf{E} + \epsilon_0 \left(\chi_e' + \frac{\chi_m'}{\mu_r'} \right) \mathbf{v} \times \mathbf{B}. \tag{2.139}$$

Similarly,

$$\mathbf{M} = \frac{\chi_m'}{\mu_0 \mu_r'} \mathbf{B} - \epsilon_0 \left(\chi_e' + \frac{\chi_m'}{\mu_r'} \right) \mathbf{v} \times \mathbf{E}. \tag{2.140}$$

2.5 Large-scale form of Maxwell's equations

We can write Maxwell's equations in a form that incorporates the spatial variation of the field in a certain region of space. To do this, we integrate the point form of Maxwell's

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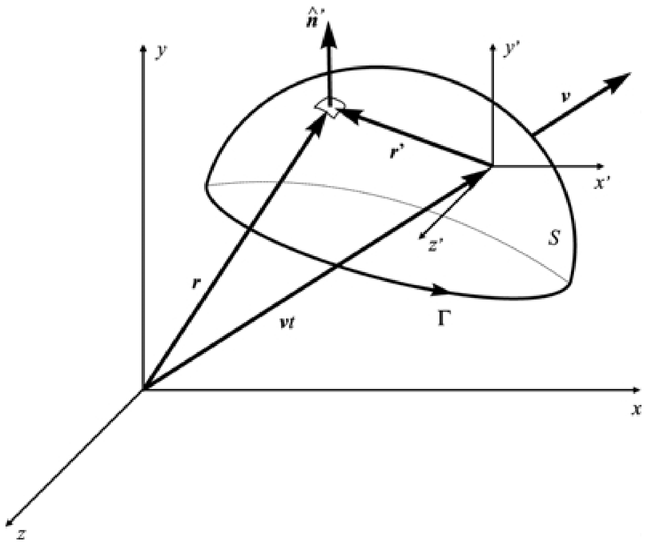

Figure 2.2: Open surface having velocity **v** relative to laboratory (unprimed) coordinate system. Surface is non-deforming.

equations over a region of space, then perform some succession of manipulations until we arrive at a form that provides us some benefit in our work with electromagnetic fields. The results are particularly useful for understanding the properties of electric and magnetic circuits, and for predicting the behavior of electrical machinery.

We shall consider two important situations: a mathematical surface that moves with constant velocity **v** and with constant shape, and a surface that moves and deforms arbitrarily.

2.5.1 Surface moving with constant velocity

Consider an open surface S moving with constant velocity **v** relative to the laboratory frame (Figure 2.2). Assume every point on the surface is an ordinary point. At any instant t we can express the relationship between the fields at points on S in either frame. In the laboratory frame we have

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J},$$

while in the moving frame

$$\nabla' \times \mathbf{E}' = -\frac{\partial \mathbf{B}'}{\partial t'}, \qquad \nabla' \times \mathbf{H}' = \frac{\partial \mathbf{D}'}{\partial t'} + \mathbf{J}'.$$

If we integrate over S and use Stokes's theorem, we get for the laboratory frame

$$\oint_\Gamma \mathbf{E} \cdot \mathbf{dl} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}, \tag{2.141}$$

$$\oint_\Gamma \mathbf{H} \cdot \mathbf{dl} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{dS} + \int_S \mathbf{J} \cdot \mathbf{dS}, \tag{2.142}$$

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and for the moving frame

$$\oint_{\Gamma'} \mathbf{E}' \cdot \mathbf{dl}' = -\int_{S'} \frac{\partial \mathbf{B}'}{\partial t'} \cdot \mathbf{dS}', \tag{2.143}$$

$$\oint_{\Gamma'} \mathbf{H}' \cdot \mathbf{dl}' = \int_{S'} \frac{\partial \mathbf{D}'}{\partial t'} \cdot \mathbf{dS}' + \int_{S'} \mathbf{J}' \cdot \mathbf{dS}'. \tag{2.144}$$

Here boundary contour Γ has sense determined by the right-hand rule. We use the notation Γ', S', etc., to indicate that all integrations for the moving frame are computed using space and time variables in that frame. Equation (2.141) is the *integral form of Faraday's law*, while (2.142) is the *integral form of Ampere's law*.

Faraday's law states that the net circulation of $\mathbf{E}$ about a contour Γ (sometimes called the *electromotive force* or *emf*) is determined by the flux of the time-rate of change of the flux vector $\mathbf{B}$ passing through the surface bounded by Γ. Ampere's law states that the circulation of $\mathbf{H}$ (sometimes called the *magnetomotive force* or *mmf*) is determined by the flux of the current $\mathbf{J}$ plus the flux of the time-rate of change of the flux vector $\mathbf{D}$. It is the term containing $\partial\mathbf{D}/\partial t$ that Maxwell recognized as necessary to make his equations consistent; since it has units of current, it is often referred to as the *displacement current* term.

Equations (2.141)–(2.142) are the large-scale or integral forms of Maxwell's equations. They are the integral-form equivalents of the point forms, and are form invariant under Lorentz transformation. If we express the fields in terms of the moving reference frame, we can write

$$\oint_{\Gamma'} \mathbf{E}' \cdot \mathbf{dl}' = -\frac{d}{dt}\int_{S'} \mathbf{B}' \cdot \mathbf{dS}', \tag{2.145}$$

$$\oint_{\Gamma'} \mathbf{H}' \cdot \mathbf{dl}' = \frac{d}{dt}\int_{S'} \mathbf{D}' \cdot \mathbf{dS}' + \int_{S'} \mathbf{J}' \cdot \mathbf{dS}'. \tag{2.146}$$

These hold for a stationary surface, since the surface would be stationary to an observer who moves with it. We are therefore justified in removing the partial derivative from the integral. Although the surfaces and contours considered here are purely mathematical, they often coincide with actual physical boundaries. The surface may surround a moving material medium, for instance, or the contour may conform to a wire moving in an electrical machine.

We can also convert the auxiliary equations to large-scale form. Consider a volume region V surrounded by a surface S that moves with velocity $\mathbf{v}$ relative to the laboratory frame (Figure 2.3). Integrating the point form of Gauss's law over V we have

$$\int_V \nabla \cdot \mathbf{D}\, dV = \int_V \rho\, dV.$$

Using the divergence theorem and recognizing that the integral of charge density is total charge, we obtain

$$\oint_S \mathbf{D} \cdot \mathbf{dS} = \int_V \rho\, dV = Q(t) \tag{2.147}$$

where $Q(t)$ is the total charge contained within V at time t. This large-scale form of Gauss's law states that the total flux of $\mathbf{D}$ passing through a closed surface is identical to the electric charge Q contained within. Similarly,

$$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0 \tag{2.148}$$

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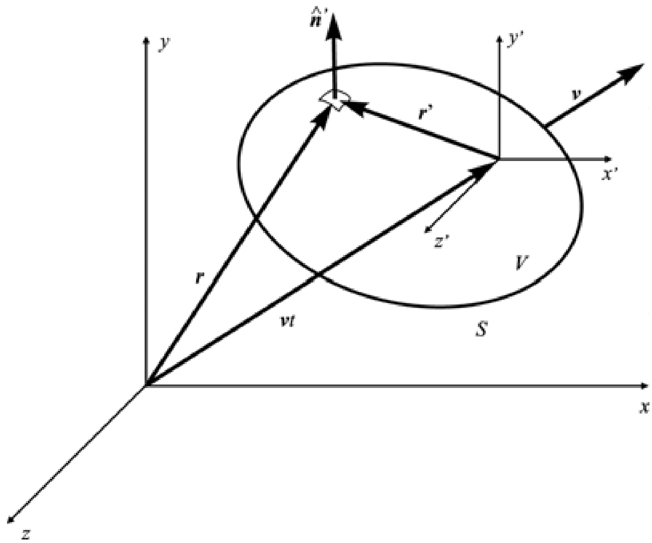

Figure 2.3: Non-deforming volume region having velocity **v** relative to laboratory (unprimed) coordinate system.

is the large-scale magnetic field Gauss's law. It states that the total flux of **B** passing through a closed surface is zero, since there are no magnetic charges contained within (i.e., magnetic charge does not exist).

Since charge is an invariant quantity, the large-scale forms of the auxiliary equations take the same form in a moving reference frame:

$$\oint_{S'} \mathbf{D}' \cdot \mathbf{dS}' = \int_{V'} \rho' \, dV' = Q(t) \tag{2.149}$$

and

$$\oint_{S'} \mathbf{B}' \cdot \mathbf{dS}' = 0. \tag{2.150}$$

The large-scale forms of the auxiliary equations may be derived from the large-scale forms of Faraday's and Ampere's laws. To obtain Gauss's law, we let the open surface in Ampere's law become a closed surface. Then $\oint \mathbf{H} \cdot \mathbf{dl}$ vanishes, and application of the large-scale form of the continuity equation (1.10) produces (2.147). The magnetic Gauss's law (2.148) is found from Faraday's law (2.141) by a similar transition from an open surface to a closed surface.

The values obtained from the expressions (2.141)–(2.142) will *not* match those obtained from (2.143)–(2.144), and we can use the Lorentz transformation field conversions to study how they differ. That is, we can write either side of the laboratory equations in terms of the moving reference frame fields, or vice versa. For most engineering applications where $v/c \ll 1$ this is not done via the Lorentz transformation field relations, but rather via the Galilean approximations to these relations (see Tai [194] for details on using the Lorentz transformation field relations). We consider the most common situation in the next section.

Kinematic form of the large-scale Maxwell equations. Confusion can result from the fact that the large-scale forms of Maxwell's equations can be written in a number of

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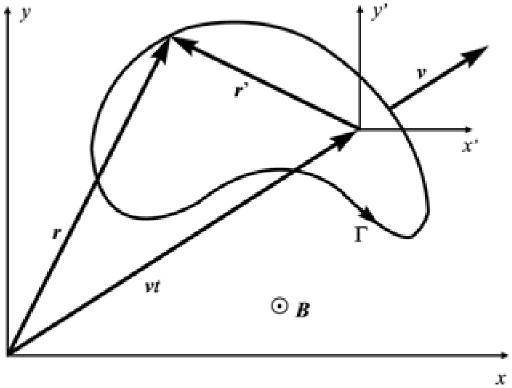

Figure 2.4: Non-deforming closed contour moving with velocity $\mathbf{v}$ through a magnetic field $\mathbf{B}$ given in the laboratory (unprimed) coordinate system.

ways. A popular formulation of Faraday's law, the *emf formulation*, revolves around the concept of electromotive force. Unfortunately, various authors offer different definitions of emf in a moving circuit.

Consider a non-deforming contour in space, moving with constant velocity $\mathbf{v}$ relative to the laboratory frame (Figure 2.4). In terms of the laboratory fields we have the large-scale form of Faraday's law (2.141). The flux term on the right-hand side of this equation can be written differently by employing the Helmholtz transport theorem (A.63). If a non-deforming surface S moves with uniform velocity $\mathbf{v}$ relative to the laboratory frame, and a vector field $\mathbf{A}(\mathbf{r}, t)$ is expressed in the stationary frame, then the time derivative of the flux of $\mathbf{A}$ through S is

$$\frac{d}{dt}\int_S \mathbf{A}\cdot\mathbf{dS} = \int_S \left[\frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}(\nabla\cdot\mathbf{A}) - \nabla\times(\mathbf{v}\times\mathbf{A})\right]\cdot\mathbf{dS}. \tag{2.151}$$

Using this with (2.141) we have

$$\oint_\Gamma \mathbf{E}\cdot\mathbf{dl} = -\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{dS} + \int_S \mathbf{v}(\nabla\cdot\mathbf{B})\cdot\mathbf{dS} - \int_S \nabla\times(\mathbf{v}\times\mathbf{B})\cdot\mathbf{dS}.$$

Remembering that $\nabla\cdot\mathbf{B} = 0$ and using Stokes's theorem on the last term, we obtain

$$\oint_\Gamma (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\mathbf{dl} = -\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{dS} = -\frac{d\Psi(t)}{dt} \tag{2.152}$$

where the *magnetic flux*

$$\int_S \mathbf{B}\cdot\mathbf{dS} = \Psi(t)$$

represents the flux of $\mathbf{B}$ through S. Following Sommerfeld [185], we may set

$$\mathbf{E}^* = \mathbf{E}+\mathbf{v}\times\mathbf{B}$$

to obtain the *kinematic form of Faraday's law*

$$\oint_\Gamma \mathbf{E}^*\cdot\mathbf{dl} = -\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{dS} = -\frac{d\Psi(t)}{dt}. \tag{2.153}$$

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(The asterisk should not be confused with the notation for complex conjugate.)

Much confusion arises from the similarity between (2.153) and (2.145). In fact, these expressions are different and give different results. This is because $\mathbf{B}'$ in (2.145) is measured *in the frame of the moving circuit*, while $\mathbf{B}$ in (2.153) is measured in the frame of the laboratory. Further confusion arises from various definitions of emf. Many authors (e.g., Hermann Weyl [213]) define emf to be the circulation of $\mathbf{E}^*$. In that case the emf is equal to the negative time rate of change of the flux of the *laboratory frame* magnetic field $\mathbf{B}$ through S. Since the Lorentz force experienced by a charge q moving *with the contour* is given by $q\mathbf{E}^* = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, this emf is the circulation of Lorentz force per unit charge along the contour. If the contour is aligned with a conducting circuit, then in some cases this emf can be given physical interpretation as the work required to move a charge around the entire circuit through the conductor against the Lorentz force. Unfortunately the usefulness of this definition of emf is lost if the time or space rate of change of the fields is so large that no true loop current can be established (hence Kirchoff's law cannot be employed). Such a problem must be treated as an electromagnetic "scattering" problem with consideration given to retardation effects. Detailed discussions of the physical interpretation of $\mathbf{E}^*$ in the definition of emf are given by Scanlon [165] and Cullwick [48].

Other authors choose to define emf as the circulation of the electric field *in the frame of the moving contour*. In this case the circulation of $\mathbf{E}'$ in (2.145) is the emf, and is related to the flux of the magnetic field *in the frame of the moving circuit*. As pointed out above, the result differs from that based on the Lorentz force. If we wish, we can also write this emf in terms of the fields expressed in the laboratory frame. To do this we must convert $\partial\mathbf{B}'/\partial t'$ to the laboratory fields using the rules for a Lorentz transformation. The result, given by Tai [194], is quite complicated and involves both the magnetic *and* electric laboratory-frame fields.

The moving-frame emf as computed from the Lorentz transformation is rarely used as a working definition of emf, mostly because circuits moving at relativistic velocities are seldom used by engineers. Unfortunately, more confusion arises for the case $v \ll c$, since for a Galilean frame the Lorentz-force and moving-frame emfs become identical. This is apparent if we use (2.52) to replace $\mathbf{B}'$ with the laboratory frame field $\mathbf{B}$, and (2.49) to replace $\mathbf{E}'$ with the combination of laboratory frame fields $\mathbf{E} + \mathbf{v} \times \mathbf{B}$. Then (2.145) becomes

$$\oint_\Gamma \mathbf{E}' \cdot \mathbf{dl} = \oint_\Gamma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{dl} = -\frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{dS},$$

which is identical to (2.153). For circuits moving with low velocity then, the circulation of $\mathbf{E}'$ can be interpreted as work per unit charge. As an added bit of confusion, the term

$$\oint_\Gamma (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{dl} = \int_S \nabla \times (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{dS}$$

is sometimes called *motional emf*, since it is the component of the circulation of $\mathbf{E}^*$ that is directly attributable to the motion of the circuit.

Although less commonly done, we can also rewrite Ampere's law (2.142) using (2.151). This gives

$$\oint_\Gamma \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS} + \frac{d}{dt}\int_S \mathbf{D} \cdot \mathbf{dS} - \int_S (\mathbf{v}\nabla \cdot \mathbf{D}) \cdot \mathbf{dS} + \int_S \nabla \times (\mathbf{v} \times \mathbf{D}) \cdot \mathbf{dS}.$$

Using $\nabla \cdot \mathbf{D} = \rho$ and using Stokes's theorem on the last term, we obtain

$$\oint_\Gamma (\mathbf{H} - \mathbf{v} \times \mathbf{D}) \cdot \mathbf{dl} = \frac{d}{dt}\int_S \mathbf{D} \cdot \mathbf{dS} + \int_S (\mathbf{J} - \rho\mathbf{v}) \cdot \mathbf{dS}.$$

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Finally, letting $\mathbf{H}^* = \mathbf{H} - \mathbf{v} \times \mathbf{D}$ and $\mathbf{J}^* = \mathbf{J} - \rho\mathbf{v}$ we can write the *kinematic form of Ampere's law*:

$$\oint_\Gamma \mathbf{H}^* \cdot \mathbf{dl} = \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{dS} + \int_S \mathbf{J}^* \cdot \mathbf{dS}. \tag{2.154}$$

In a Galilean frame where we use (2.49)–(2.54), we see that (2.154) is identical to

$$\oint_\Gamma \mathbf{H}' \cdot \mathbf{dl} = \frac{d}{dt} \int_S \mathbf{D}' \cdot \mathbf{dS} + \int_S \mathbf{J}' \cdot \mathbf{dS} \tag{2.155}$$

where the primed fields are measured in the frame of the moving contour. This equivalence does *not* hold when the Lorentz transformation is used to represent the primed fields.

Alternative form of the large-scale Maxwell equations. We can write Maxwell's equations in an alternative large-scale form involving only surface and volume integrals. This will be useful later for establishing the field jump conditions across a material or source discontinuity. Again we begin with Maxwell's equations in point form, but instead of integrating them over an open surface we integrate over a volume region V moving with velocity $\mathbf{v}$ (Figure 2.3). In the laboratory frame this gives

$$\int_V (\nabla \times \mathbf{E})\, dV = -\int_V \frac{\partial \mathbf{B}}{\partial t}\, dV,$$

$$\int_V (\nabla \times \mathbf{H})\, dV = \int_V \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right) dV.$$

An application of curl theorem (B.24) then gives

$$\oint_S (\hat{\mathbf{n}} \times \mathbf{E})\, dS = -\int_V \frac{\partial \mathbf{B}}{\partial t}\, dV, \tag{2.156}$$

$$\oint_S (\hat{\mathbf{n}} \times \mathbf{H})\, dS = \int_V \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right) dV. \tag{2.157}$$

Similar results are obtained for the fields in the moving frame:

$$\oint_{S'} (\hat{\mathbf{n}}' \times \mathbf{E}')\, dS' = -\int_{V'} \frac{\partial \mathbf{B}'}{\partial t'}\, dV',$$

$$\oint_{S'} (\hat{\mathbf{n}}' \times \mathbf{H}')\, dS' = \int_{V'} \left(\frac{\partial \mathbf{D}'}{\partial t'} + \mathbf{J}' \right) dV'.$$

These large-scale forms are an alternative to (2.141)–(2.144). They are also form-invariant under a Lorentz transformation.

An alternative to the kinematic formulation of (2.153) and (2.154) can be achieved by applying a kinematic identity for a moving volume region. If V is surrounded by a surface S that moves with velocity $\mathbf{v}$ relative to the laboratory frame, and if a vector field $\mathbf{A}$ is measured in the laboratory frame, then the vector form of the general transport theorem (A.68) states that

$$\frac{d}{dt} \int_V \mathbf{A}\, dV = \int_V \frac{\partial \mathbf{A}}{\partial t}\, dV + \oint_S \mathbf{A}(\mathbf{v} \cdot \hat{\mathbf{n}})\, dS. \tag{2.158}$$

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Applying this to (2.156) and (2.157) we have

$$\oint_S [\hat{\mathbf{n}} \times \mathbf{E} - (\mathbf{v} \cdot \hat{\mathbf{n}})\mathbf{B}]\, dS = -\frac{d}{dt}\int_V \mathbf{B}\, dV, \tag{2.159}$$

$$\oint_S [\hat{\mathbf{n}} \times \mathbf{H} + (\mathbf{v} \cdot \hat{\mathbf{n}})\mathbf{D}]\, dS = \int_V \mathbf{J}\, dV + \frac{d}{dt}\int_V \mathbf{D}\, dV. \tag{2.160}$$

We can also apply (2.158) to the large-scale form of the continuity equation (2.10) and obtain the expression for a volume region moving with velocity $\mathbf{v}$:

$$\oint_S (\mathbf{J} - \rho\mathbf{v}) \cdot \mathbf{dS} = -\frac{d}{dt}\int_V \rho\, dV.$$

2.5.2 Moving, deforming surfaces

Because (2.151) holds for arbitrarily moving surfaces, the kinematic versions (2.153) and (2.154) hold when $\mathbf{v}$ is interpreted as an instantaneous velocity. However, if the surface and contour lie within a material body that moves relative to the laboratory frame, the constitutive equations relating $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{J}$ in the laboratory frame differ from those relating the fields in the stationary frame of the body (if the body is not accelerating), and thus the concepts of § 2.3.2 must be employed. This is important when boundary conditions at a moving surface are needed. Particular care must be taken when the body accelerates, since the constitutive relations are then only approximate.

The representation (2.145)–(2.146) is also generally valid, provided we *define* the primed fields as those converted from laboratory fields using the Lorentz transformation with instantaneous velocity $\mathbf{v}$. Here we should use a different inertial frame for each point in the integration, and align the frame with the velocity vector $\mathbf{v}$ at the instant t. We certainly may do this since we can choose to integrate any function we wish. However, this representation may not find wide application.

We thus choose the following expressions, valid for arbitrarily moving surfaces containing only regular points, as our general forms of the large-scale Maxwell equations:

$$\oint_{\Gamma(t)} \mathbf{E}^* \cdot \mathbf{dl} = -\frac{d}{dt}\int_{S(t)} \mathbf{B} \cdot \mathbf{dS} = -\frac{d\Psi(t)}{dt},$$

$$\oint_{\Gamma(t)} \mathbf{H}^* \cdot \mathbf{dl} = \frac{d}{dt}\int_{S(t)} \mathbf{D} \cdot \mathbf{dS} + \int_{S(t)} \mathbf{J}^* \cdot \mathbf{dS},$$

where

$$\mathbf{E}^* = \mathbf{E} + \mathbf{v} \times \mathbf{B},$$
$$\mathbf{H}^* = \mathbf{H} - \mathbf{v} \times \mathbf{D},$$
$$\mathbf{J}^* = \mathbf{J} - \rho\mathbf{v},$$

and where all fields are taken to be measured in the laboratory frame with $\mathbf{v}$ the instantaneous velocity of points on the surface and contour relative to that frame. The constitutive parameters must be considered carefully if the contours and surfaces lie in a moving material medium.

Kinematic identity (2.158) is also valid for arbitrarily moving surfaces. Thus we have the following, valid for arbitrarily moving surfaces and volumes containing only regular

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points:

$$\oint_{S(t)} [\hat{\mathbf{n}} \times \mathbf{E} - (\mathbf{v} \cdot \hat{\mathbf{n}})\mathbf{B}]\, dS = -\frac{d}{dt} \int_{V(t)} \mathbf{B}\, dV,$$

$$\oint_{S(t)} [\hat{\mathbf{n}} \times \mathbf{H} + (\mathbf{v} \cdot \hat{\mathbf{n}})\mathbf{D}]\, dS = \int_{V(t)} \mathbf{J}\, dV + \frac{d}{dt} \int_{V(t)} \mathbf{D}\, dV.$$

We also find that the two Gauss's law expressions,

$$\oint_{S(t)} \mathbf{D} \cdot \mathbf{dS} = \int_{V(t)} \rho\, dV,$$

$$\oint_{S(t)} \mathbf{B} \cdot \mathbf{dS} = 0,$$

remain valid.

2.5.3 Large-scale form of the Boffi equations

The Maxwell–Boffi equations can be written in large-scale form using the same approach as with the Maxwell–Minkowski equations. Integrating (2.120) and (2.121) over an open surface S and applying Stokes's theorem, we have

$$\oint_{\Gamma} \mathbf{E} \cdot \mathbf{dl} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}, \tag{2.161}$$

$$\oint_{\Gamma} \mathbf{B} \cdot \mathbf{dl} = \mu_0 \int_S \left(\mathbf{J} + \mathbf{J}_M + \mathbf{J}_P + \frac{\partial \epsilon_0 \mathbf{E}}{\partial t} \right) \cdot \mathbf{dS}, \tag{2.162}$$

for fields in the laboratory frame, and

$$\oint_{\Gamma'} \mathbf{E}' \cdot \mathbf{dl}' = -\int_{S'} \frac{\partial \mathbf{B}'}{\partial t'} \cdot \mathbf{dS}',$$

$$\oint_{\Gamma'} \mathbf{B}' \cdot \mathbf{dl}' = \mu_0 \int_{S'} \left(\mathbf{J}' + \mathbf{J}'_M + \mathbf{J}'_P + \frac{\partial \epsilon_0 \mathbf{E}'}{\partial t'} \right) \cdot \mathbf{dS}',$$

for fields in a moving frame. We see that Faraday's law is unmodified by the introduction of polarization and magnetization, hence our prior discussion of emf for moving contours remains valid. However, Ampere's law must be interpreted somewhat differently. The flux vector $\mathbf{B}$ also acts as a force vector, and its circulation is proportional to the outflux of total current, consisting of $\mathbf{J}$ plus the equivalent magnetization and polarization currents plus the displacement current *in free space*, through the surface bounded by the circulation contour.

The large-scale forms of the auxiliary equations can be found by integrating (2.122) and (2.123) over a volume region and applying the divergence theorem. This gives

$$\oint_S \mathbf{E} \cdot \mathbf{dS} = \frac{1}{\epsilon_0} \int_V (\rho + \rho_P)\, dV,$$

$$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0,$$

for the laboratory frame fields, and

$$\oint_{S'} \mathbf{E}' \cdot \mathbf{dS}' = \frac{1}{\epsilon_0} \int_{V'} (\rho' + \rho'_P)\, dV',$$

$$\oint_{S'} \mathbf{B}' \cdot \mathbf{dS}' = 0,$$

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for the moving frame fields. Here we find the force vector **E** also acting as a flux vector, with the outflux of **E** over a closed surface proportional to the sum of the electric and polarization charges enclosed by the surface.

To provide the alternative representation, we integrate the point forms over V and use the curl theorem to obtain

$$\oint_S (\hat{\mathbf{n}} \times \mathbf{E})\, dS = -\int_V \frac{\partial \mathbf{B}}{\partial t}\, dV, \tag{2.163}$$

$$\oint_S (\hat{\mathbf{n}} \times \mathbf{B})\, dS = \mu_0 \int_V \left(\mathbf{J} + \mathbf{J}_M + \mathbf{J}_P + \frac{\partial \epsilon_0 \mathbf{E}}{\partial t}\right) dV, \tag{2.164}$$

for the laboratory frame fields, and

$$\oint_{S'} (\hat{\mathbf{n}}' \times \mathbf{E}')\, dS' = -\int_{V'} \frac{\partial \mathbf{B}'}{\partial t'}\, dV',$$

$$\oint_{S'} (\hat{\mathbf{n}}' \times \mathbf{B}')\, dS' = \mu_0 \int_{V'} \left(\mathbf{J}' + \mathbf{J}'_M + \mathbf{J}'_P + \frac{\partial \epsilon_0 \mathbf{E}'}{\partial t'}\right) dV',$$

for the moving frame fields.

The large-scale forms of the Boffi equations can also be put into kinematic form using either (2.151) or (2.158). Using (2.151) on (2.161) and (2.162) we have

$$\oint_{\Gamma(t)} \mathbf{E}^* \cdot \mathbf{dl} = -\frac{d}{dt} \int_{S(t)} \mathbf{B} \cdot \mathbf{dS}, \tag{2.165}$$

$$\oint_{\Gamma(t)} \mathbf{B}^\dagger \cdot \mathbf{dl} = \int_{S(t)} \mu_0 \mathbf{J}^\dagger \cdot \mathbf{dS} + \frac{1}{c^2} \frac{d}{dt} \int_{S(t)} \mathbf{E} \cdot \mathbf{dS}, \tag{2.166}$$

where

$$\mathbf{E}^* = \mathbf{E} + \mathbf{v} \times \mathbf{B},$$

$$\mathbf{B}^\dagger = \mathbf{B} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E},$$

$$\mathbf{J}^\dagger = \mathbf{J} + \mathbf{J}_M + \mathbf{J}_P - (\rho + \rho_P)\mathbf{v}.$$

Here $\mathbf{B}^\dagger$ is equivalent to the first-order Lorentz transformation representation of the field in the moving frame (2.64). (The dagger † should not be confused with the symbol for the hermitian operation.) Using (2.158) on (2.163) and (2.164) we have

$$\oint_{S(t)} [\hat{\mathbf{n}} \times \mathbf{E} - (\mathbf{v} \cdot \hat{\mathbf{n}})\mathbf{B}]\, dS = -\frac{d}{dt} \int_{V(t)} \mathbf{B}\, dV, \tag{2.167}$$

and

$$\oint_{S(t)} \left[\hat{\mathbf{n}} \times \mathbf{B} + \frac{1}{c^2}(\mathbf{v} \cdot \hat{\mathbf{n}})\mathbf{E}\right] dS = \mu_0 \int_{V(t)} (\mathbf{J} + \mathbf{J}_M + \mathbf{J}_P)\, dV + \frac{1}{c^2} \frac{d}{dt} \int_{V(t)} \mathbf{E}\, dV. \tag{2.168}$$

In each case the fields are measured in the laboratory frame, and **v** is measured with respect to the laboratory frame and may vary arbitrarily over the surface or contour.

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2.6 The nature of the four field quantities

Since the very inception of Maxwell's theory, its students have been distressed by the fact that while there are four electromagnetic fields $(\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H})$, there are only two fundamental equations (the curl equations) to describe their interrelationship. The relegation of additional required information to constitutive equations that vary widely between classes of materials seems to lessen the elegance of the theory. While some may find elegant the separation of equations into a set expressing the basic wave nature of electromagnetism and a set describing how the fields interact with materials, the history of the discipline is one of categorizing and pairing fields as "fundamental" and "supplemental" in hopes of reducing the model to two equations in two unknowns.

Lorentz led the way in this area. With his electrical theory of matter, all material effects could be interpreted in terms of atomic charge and current immersed in free space. We have seen how the Maxwell–Boffi equations seem to eliminate the need for **D** and **H**, and indeed for simple media where there is a linear relation between the remaining "fundamental" fields and the induced polarization and magnetization, it appears that only **E** and **B** are required. However, for more complicated materials that display nonlinear and bianisotropic effects we are only able to supplant **D** and **H** with two other fields **P** and **M**, along with (possibly complicated) constitutive relations relating them to **E** and **B**.

Even those authors who do not wish to eliminate two of the fields tend to categorize the fields into pairs based on physical arguments, implying that one or the other pair is in some way "more fundamental." Maxwell himself separated the fields into the pair $(\mathbf{E}, \mathbf{H})$ that appears within line integrals to give work and the pair $(\mathbf{B}, \mathbf{D})$ that appears within surface integrals to give flux. In what other ways might we pair the four vectors?

Most prevalent is the splitting of the fields into electric and magnetic pairs: $(\mathbf{E}, \mathbf{D})$ and $(\mathbf{B}, \mathbf{H})$. In Poynting's theorem $\mathbf{E} \cdot \mathbf{D}$ describes one component of stored energy (called "electric energy") and $\mathbf{B} \cdot \mathbf{H}$ describes another component (called "magnetic energy"). These pairs also occur in Maxwell's stress tensor. In statics, the fields decouple into electric and magnetic sets. But biisotropic and bianisotropic materials demonstrate how separation into electric and magnetic effects can become problematic.

In the study of electromagnetic waves, the ratio of E to H appears to be an important quantity, called the "intrinsic impedance." The pair $(\mathbf{E}, \mathbf{H})$ also determines the Poynting flux of power, and is required to establish the uniqueness of the electromagnetic field. In addition, constitutive relations for simple materials usually express $(\mathbf{D}, \mathbf{B})$ in terms of $(\mathbf{E}, \mathbf{H})$. Models for these materials are often conceived by viewing the fields $(\mathbf{E}, \mathbf{H})$ as interacting with the atomic structure in such a way as to produce secondary effects describable by $(\mathbf{D}, \mathbf{B})$. These considerations, along with Maxwell's categorization into a pair of work vectors and a pair of flux vectors, lead many authors to formulate electromagnetics with **E** and **H** as the "fundamental" quantities. But the pair $(\mathbf{B}, \mathbf{D})$ gives rise to electromagnetic momentum and is also perpendicular to the direction of wave propagation in an anisotropic material; in these senses, we might argue that these fields must be equally "fundamental."

Perhaps the best motivation for grouping fields comes from relativistic considerations. We have found that $(\mathbf{E}, \mathbf{B})$ transform together under a Lorentz transformation, as do $(\mathbf{D}, \mathbf{H})$. In each of these pairs we have one polar vector (**E** or **D**) and one axial vector (**B** or **H**). A polar vector retains its meaning under a change in handedness of the coordinate system, while an axial vector does not. The Lorentz force involves one polar vector (**E**)

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and one axial vector (**B**) that we also call "electric" and "magnetic." If we follow the lead of some authors and *choose* to define **E** and **B** through measurements of the Lorentz force, then we recognize that **B** must be axial since it is not measured directly, but as part of the cross product $\mathbf{v} \times \mathbf{B}$ that changes its meaning if we switch from a right-hand to a left-hand coordinate system. The other polar vector (**D**) and axial vector (**H**) arise through the "secondary" constitutive relations. Following this reasoning we might claim that **E** and **B** are "fundamental."

Sommerfeld also associates **E** with **B** and **D** with **H**. The vectors **E** and **B** are called *entities of intensity*, describing "how strong," while **D** and **H** are called *entities of quantity*, describing "how much." This is in direct analogy with stress (intensity) and strain (quantity) in materials. We might also say that the entities of intensity describe a "cause" while the entities of quantity describe an "effect." In this view **E** "induces" (causes) a polarization **P**, and the field $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ is the result. Similarly **B** creates **M**, and $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$ is the result. Interestingly, each of the terms describing energy and momentum in the electromagnetic field ($\mathbf{D} \cdot \mathbf{E}$, $\mathbf{B} \cdot \mathbf{H}$, $\mathbf{E} \times \mathbf{H}$, $\mathbf{D} \times \mathbf{B}$) involves the interaction of an entity of intensity with an entity of quantity.

Although there is a natural tendency to group things together based on conceptual similarity, there appears to be little reason to believe that any of the four field vectors are more "fundamental" than the rest. Perhaps we are fortunate that we can apply Maxwell's theory without worrying too much about such questions of underlying philosophy.

2.7 Maxwell's equations with magnetic sources

Researchers have yet to discover the "magnetic monopole": a magnetic source from which magnetic field would diverge. This has not stopped speculation on the form that Maxwell's equations might take if such a discovery were made. Arguments based on fundamental principles of physics (such as symmetry and conservation laws) indicate that in the presence of magnetic sources Maxwell's equations would assume the forms

$$\nabla \times \mathbf{E} = -\mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t}, \tag{2.169}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{2.170}$$

$$\nabla \cdot \mathbf{B} = \rho_m, \tag{2.171}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{2.172}$$

where $\mathbf{J}_m$ is a volume magnetic current density describing the flow of magnetic charge in exactly the same manner as **J** describes the flow of electric charge. The density of this magnetic charge is given by ρ_m and should, by analogy with electric charge density, obey a conservation law

$$\nabla \cdot \mathbf{J}_m + \frac{\partial \rho_m}{\partial t} = 0.$$

This is the magnetic source continuity equation.

It is interesting to inquire as to the units of $\mathbf{J}_m$ and ρ_m. From (2.169) we see that if **B** has units of Wb/m^2, then $\mathbf{J}_m$ has units of (Wb/s)/m^2. Similarly, (2.171) shows that ρ_m must have units of Wb/m^3. Hence magnetic charge is measured in Wb, magnetic current in Wb/s. This gives a nice symmetry with electric sources where charge is measured in

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C and current in C/s.[3] The physical symmetry is equally appealing: magnetic flux lines diverge from magnetic charge, and the total flux passing through a surface is given by the total magnetic charge contained within the surface. This is best seen by considering the large-scale forms of Maxwell's equations for stationary surfaces. We need only modify (2.145) to include the magnetic current term; this gives

$$\oint_\Gamma \mathbf{E} \cdot \mathbf{dl} = -\int_S \mathbf{J}_m \cdot \mathbf{dS} - \frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{dS}, \tag{2.173}$$

$$\oint_\Gamma \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS} + \frac{d}{dt}\int_S \mathbf{D} \cdot \mathbf{dS}. \tag{2.174}$$

If we modify (2.148) to include magnetic charge, we get the auxiliary equations

$$\oint_S \mathbf{D} \cdot \mathbf{dS} = \int_V \rho \, dV,$$

$$\oint_S \mathbf{B} \cdot \mathbf{dS} = \int_V \rho_m \, dV.$$

Any of the large-scale forms of Maxwell's equations can be similarly modified to include magnetic current and charge. For arbitrarily moving surfaces we have

$$\oint_{\Gamma(t)} \mathbf{E}^* \cdot \mathbf{dl} = -\frac{d}{dt}\int_{S(t)} \mathbf{B} \cdot \mathbf{dS} - \int_{S(t)} \mathbf{J}_m^* \cdot \mathbf{dS},$$

$$\oint_{\Gamma(t)} \mathbf{H}^* \cdot \mathbf{dl} = \frac{d}{dt}\int_{S(t)} \mathbf{D} \cdot \mathbf{dS} + \int_{S(t)} \mathbf{J}^* \cdot \mathbf{dS},$$

where

$$\mathbf{E}^* = \mathbf{E} + \mathbf{v} \times \mathbf{B},$$

$$\mathbf{H}^* = \mathbf{H} - \mathbf{v} \times \mathbf{D},$$

$$\mathbf{J}^* = \mathbf{J} - \rho\mathbf{v},$$

$$\mathbf{J}_m^* = \mathbf{J}_m - \rho_m\mathbf{v},$$

and all fields are taken to be measured in the laboratory frame with $\mathbf{v}$ the instantaneous velocity of points on the surface and contour relative to the laboratory frame. We also have the alternative forms

$$\oint_S (\hat{\mathbf{n}} \times \mathbf{E}) \, dS = \int_V \left(-\frac{\partial \mathbf{B}}{\partial t} - \mathbf{J}_m \right) dV, \tag{2.175}$$

$$\oint_S (\hat{\mathbf{n}} \times \mathbf{H}) \, dS = \int_V \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right) dV, \tag{2.176}$$

and

$$\oint_{S(t)} [\hat{\mathbf{n}} \times \mathbf{E} - (\mathbf{v} \cdot \hat{\mathbf{n}})\mathbf{B}] \, dS = -\int_{V(t)} \mathbf{J}_m \, dV - \frac{d}{dt}\int_{V(t)} \mathbf{B} \, dV, \tag{2.177}$$

$$\oint_{S(t)} [\hat{\mathbf{n}} \times \mathbf{H} + (\mathbf{v} \cdot \hat{\mathbf{n}})\mathbf{D}] \, dS = \int_{V(t)} \mathbf{J} \, dV + \frac{d}{dt}\int_{V(t)} \mathbf{D} \, dV, \tag{2.178}$$

[3] We note that if the modern unit of T is used to describe $\mathbf{B}$, then ρ_m is described using the more cumbersome units of T/m, while $\mathbf{J}_m$ is given in terms of T/s. Thus, magnetic charge is measured in Tm^2 and magnetic current in $(\text{Tm}^2)/\text{s}$.

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and the two Gauss's law expressions

$$\oint_{S(t)} \mathbf{D} \cdot \hat{\mathbf{n}}\, dS = \int_{V(t)} \rho\, dV,$$

$$\oint_{S(t)} \mathbf{B} \cdot \hat{\mathbf{n}}\, dS = \int_{V(t)} \rho_m\, dV.$$

Magnetic sources also allow us to develop equivalence theorems in which difficult problems involving boundaries are replaced by simpler problems involving magnetic sources. Although these sources may not physically exist, the mathematical solutions are completely valid.

2.8 Boundary (jump) conditions

If we restrict ourselves to regions of space without spatial (jump) discontinuities in either the sources or the constitutive relations, we can find meaningful solutions to the Maxwell differential equations. We also know that for given sources, if the fields are specified on a closed boundary and at an initial time the solutions are unique. The standard approach to treating regions that do contain spatial discontinuities is to isolate the discontinuities on surfaces. That is, we introduce surfaces that serve to separate space into regions in which the differential equations are solvable and the fields are well defined. To make the solutions in adjoining regions unique, we must specify the tangential fields on each side of the adjoining surface. If we can relate the fields across the boundary, we can propagate the solution from one region to the next; in this way, information about the source in one region is effectively passed on to the solution in an adjacent region. For uniqueness, only relations between the tangential components need be specified.

We shall determine the appropriate boundary conditions (BC's) via two distinct approaches. We first model a thin source layer and consider a discontinuous surface source layer as a limiting case of the continuous thin layer. With no true discontinuity, Maxwell's differential equations hold everywhere. We then consider a true spatial discontinuity between material surfaces (with possible surface sources lying along the discontinuity). We must then isolate the region containing the discontinuity and *postulate* a field relationship that is both physically meaningful and experimentally verifiable.

We shall also consider both stationary and moving boundary surfaces, and surfaces containing magnetic as well as electric sources.

2.8.1 Boundary conditions across a stationary, thin source layer

In § 1.3.3 we discussed how in the macroscopic sense a surface source is actually a volume distribution concentrated near a surface S. We write the charge and current in terms of the point $\mathbf{r}$ on the surface and the normal distance x from the surface at $\mathbf{r}$ as

$$\rho(\mathbf{r}, x, t) = \rho_s(\mathbf{r}, t) f(x, \Delta), \tag{2.179}$$

$$\mathbf{J}(\mathbf{r}, x, t) = \mathbf{J}_s(\mathbf{r}, t) f(x, \Delta), \tag{2.180}$$

where $f(x, \Delta)$ is the source density function obeying

$$\int_{-\infty}^{\infty} f(x, \Delta)\, dx = 1. \tag{2.181}$$

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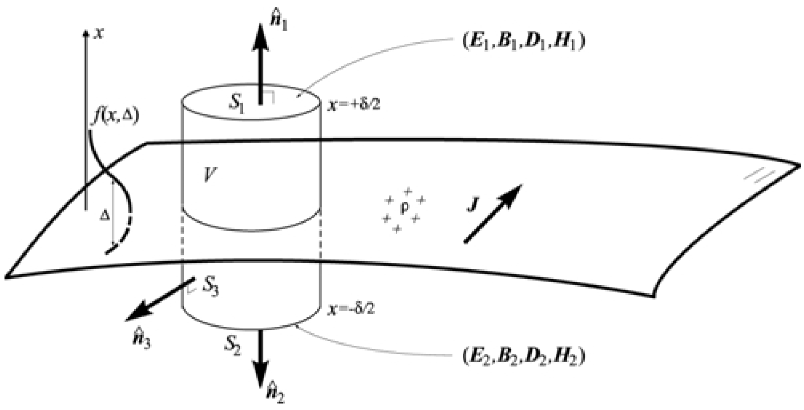

Figure 2.5: Derivation of the electromagnetic boundary conditions across a thin continuous source layer.

The parameter Δ describes the "width" of the source layer normal to the reference surface.

We use (2.156)–(2.157) to study field behavior across the source layer. Consider a volume region V that intersects the source layer as shown in Figure 2.5. Let the top and bottom surfaces be parallel to the reference surface, and label the fields on the top and bottom surfaces with subscripts 1 and 2, respectively. Since points on and within V are all regular, (2.157) yields

$$\int_{S_1} \hat{\mathbf{n}}_1 \times \mathbf{H}_1 \, dS + \int_{S_2} \hat{\mathbf{n}}_2 \times \mathbf{H}_2 \, dS + \int_{S_3} \hat{\mathbf{n}}_3 \times \mathbf{H} \, dS = \int_V \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) dV.$$

We now choose $\delta = k\Delta$ $(k > 1)$ so that most of the source lies within V. As $\Delta \to 0$ the thin source layer recedes to a surface layer, and the volume integral of displacement current and the integral of tangential $\mathbf{H}$ over S_3 both approach zero by continuity of the fields. By symmetry $S_1 = S_2$ and $\hat{\mathbf{n}}_1 = -\hat{\mathbf{n}}_2 = \hat{\mathbf{n}}_{12}$, where $\hat{\mathbf{n}}_{12}$ is the surface normal directed into region 1 from region 2. Thus

$$\int_{S_1} \hat{\mathbf{n}}_{12} \times (\mathbf{H}_1 - \mathbf{H}_2) \, dS = \int_V \mathbf{J} \, dV. \qquad (2.182)$$

Note that

$$\int_V \mathbf{J} \, dV = \int_{S_1} \int_{-\delta/2}^{\delta/2} \mathbf{J} \, dS \, dx = \int_{-\delta/2}^{\delta/2} f(x, \Delta) \, dx \int_{S_1} \mathbf{J}_s(\mathbf{r}, t) \, dS.$$

Since we assume that the majority of the source current lies within V, the integral can be evaluated using (2.181) to give

$$\int_{S_1} [\hat{\mathbf{n}}_{12} \times (\mathbf{H}_1 - \mathbf{H}_2) - \mathbf{J}_s] \, dS = 0,$$

hence

$$\hat{\mathbf{n}}_{12} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s.$$

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The tangential magnetic field across a thin source distribution is discontinuous by an amount equal to the surface current density.

Similar steps with Faraday's law give

$$\hat{\mathbf{n}}_{12} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0.$$

The tangential electric field is continuous across a thin source.

We can also derive conditions on the normal components of the fields, although these are not required for uniqueness. Gauss's law (2.147) applied to the volume V in Figure 2.5 gives

$$\int_{S_1} \mathbf{D}_1 \cdot \hat{\mathbf{n}}_1 \, dS + \int_{S_2} \mathbf{D}_2 \cdot \hat{\mathbf{n}}_2 \, dS + \int_{S_3} \mathbf{D} \cdot \hat{\mathbf{n}}_3 \, dS = \int_V \rho \, dV.$$

As $\Delta \to 0$, the thin source layer recedes to a surface layer. The integral of normal $\mathbf{D}$ over S_3 tends to zero by continuity of the fields. By symmetry $S_1 = S_2$ and $\hat{\mathbf{n}}_1 = -\hat{\mathbf{n}}_2 = \hat{\mathbf{n}}_{12}$. Thus

$$\int_{S_1} (\mathbf{D}_1 - \mathbf{D}_2) \cdot \hat{\mathbf{n}}_{12} \, dS = \int_V \rho \, dV. \tag{2.183}$$

The volume integral is

$$\int_V \rho \, dV = \int_{S_1} \int_{-\delta/2}^{\delta/2} \rho \, dS \, dx = \int_{-\delta/2}^{\delta/2} f(x, \Delta) \, dx \int_{S_1} \rho_s(\mathbf{r}, t) \, dS.$$

Since $\delta = k\Delta$ has been chosen so that most of the source charge lies within V, (2.181) gives

$$\int_{S_1} \left[(\mathbf{D}_1 - \mathbf{D}_2) \cdot \hat{\mathbf{n}}_{12} - \rho_s \right] dS = 0,$$

hence

$$(\mathbf{D}_1 - \mathbf{D}_2) \cdot \hat{\mathbf{n}}_{12} = \rho_s.$$

The normal component of $\mathbf{D}$ is discontinuous across a thin source distribution by an amount equal to the surface charge density. Similar steps with the magnetic Gauss's law yield

$$(\mathbf{B}_1 - \mathbf{B}_2) \cdot \hat{\mathbf{n}}_{12} = 0.$$

The normal component of $\mathbf{B}$ is continuous across a thin source layer.

We can follow similar steps when a thin magnetic source layer is present. When evaluating Faraday's law we must include magnetic surface current and when evaluating the magnetic Gauss's law we must include magnetic charge. However, since such sources are not physical we postpone their consideration until the next section, where appropriate boundary conditions are postulated rather than derived.

2.8.2 Boundary conditions across a stationary layer of field discontinuity

Provided that we model a surface source as a limiting case of a very thin but continuous volume source, we can derive boundary conditions across a surface layer. We might ask whether we can extend this idea to surfaces of materials where the constitutive parameters change from one region to another. Indeed, if we take Lorentz' viewpoint and visualize a material as a conglomerate of atomic charge, we should be able to apply this same idea. After all, a material should demonstrate a continuous transition (in the macroscopic

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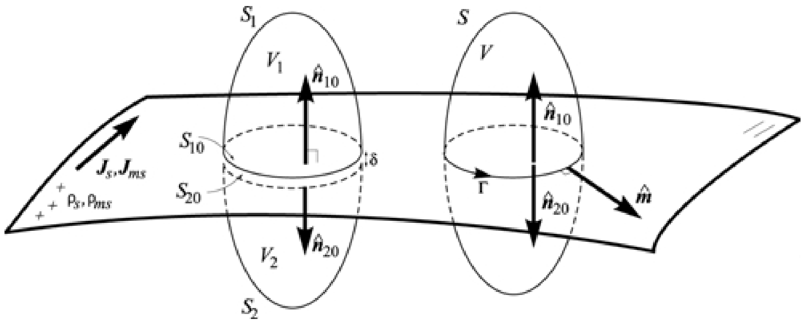

Figure 2.6: Derivation of the electromagnetic boundary conditions across a discontinuous source layer.

sense) across its boundary, and we can employ the Maxwell–Boffi equations to describe the relationship between the "equivalent" sources and the electromagnetic fields.

We should note, however, that the limiting concept is not without its critics. Stokes suggested as early as 1848 that jump conditions should never be derived from smooth solutions [199]. Let us therefore pursue the boundary conditions for a surface of true field discontinuity. This will also allow us to treat a material modeled as having a true discontinuity in its material parameters (which we can always take as a mathematical model of a more gradual transition) before we have studied in a deeper sense the physical properties of materials. This approach, taken by many textbooks, must be done carefully.

There is a logical difficulty with this approach, lying in the application of the large-scale forms of Maxwell's equations. Many authors postulate Maxwell's equations in point form, integrate to obtain the large-scale forms, then apply the large-scale forms to regions of discontinuity. Unfortunately, the large-scale forms thus obtained are only valid in the same regions where their point form antecedents were valid — discontinuities must be excluded. Schelkunoff [167] has criticized this approach, calling it a "swindle" rather than a proof, and has suggested that the proper way to handle true discontinuities is to postulate the large-scale forms of Maxwell's equations, *and* to include as part of the postulate the assumption that the large-scale forms are valid at points of field discontinuity. Does this mean we must reject our postulate of the point form Maxwell equations and reformulate everything in terms of the large-scale forms? Fortunately, no. Tai [192] has pointed out that it is still possible to postulate the point forms, as long as we also postulate appropriate boundary conditions that make the large-scale forms, as derived from the point forms, valid at surfaces of discontinuity. In essence, both approaches require an additional postulate for surfaces of discontinuity: the large scale forms require a postulate of applicability to discontinuous surfaces, and from there the boundary conditions can be derived; the point forms require a postulate of the boundary conditions that result in the large-scale forms being valid on surfaces of discontinuity. Let us examine how the latter approach works.

Consider a surface across which the constitutive relations are discontinuous, containing electric and magnetic surface currents and charges $\mathbf{J}_s$, ρ_s, $\mathbf{J}_{ms}$, and ρ_{ms} (Figure 2.6). We locate a volume region V_1 above the surface of discontinuity; this volume is bounded by a surface S_1 and another surface S_{10} which is parallel to, and a small distance $\delta/2$ above, the surface of discontinuity. A second volume region V_2 is similarly situated below the surface of discontinuity. Because these regions exclude the surface of discontinuity

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we can use (2.176) to get

$$\int_{S_1} \hat{\mathbf{n}} \times \mathbf{H}\, dS + \int_{S_{10}} \hat{\mathbf{n}} \times \mathbf{H}\, dS = \int_{V_1} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) dV,$$
$$\int_{S_2} \hat{\mathbf{n}} \times \mathbf{H}\, dS + \int_{S_{20}} \hat{\mathbf{n}} \times \mathbf{H}\, dS = \int_{V_2} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) dV.$$

Adding these we obtain

$$\int_{S_1+S_2} \hat{\mathbf{n}} \times \mathbf{H}\, dS - \int_{V_1+V_2} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) dV - \\ - \int_{S_{10}} \hat{\mathbf{n}}_{10} \times \mathbf{H}_1\, dS - \int_{S_{20}} \hat{\mathbf{n}}_{20} \times \mathbf{H}_2\, dS = 0, \tag{2.184}$$

where we have used subscripts to delineate the fields on each side of the discontinuity surface.

If δ is very small (but nonzero), then $\hat{\mathbf{n}}_{10} = -\hat{\mathbf{n}}_{20} = \hat{\mathbf{n}}_{12}$ and $S_{10} = S_{20}$. Letting $S_1 + S_2 = S$ and $V_1 + V_2 = V$, we can write (184) as

$$\int_S (\hat{\mathbf{n}} \times \mathbf{H})\, dS - \int_V \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) dV = \int_{S_{10}} \hat{\mathbf{n}}_{12} \times (\mathbf{H}_1 - \mathbf{H}_2)\, dS. \tag{2.185}$$

Now suppose we use the same volume region V, but let it intersect the surface of discontinuity (Figure 2.6), and suppose that the large-scale form of Ampere's law holds even if V contains points of field discontinuity. We must include the surface current in the computation. Since $\int_V \mathbf{J}\, dV$ becomes $\int_S \mathbf{J}_s\, dS$ on the surface, we have

$$\int_S (\hat{\mathbf{n}} \times \mathbf{H})\, dS - \int_V \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) dV = \int_{S_{10}} \mathbf{J}_s\, dS. \tag{2.186}$$

We wish to have this give the same value for the integrals over V and S as (2.185), which included in its derivation no points of discontinuity. This is true provided that

$$\hat{\mathbf{n}}_{12} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s. \tag{2.187}$$

Thus, under the condition (2.187) we may interpret the large-scale form of Ampere's law (as derived from the point form) as being valid for regions containing discontinuities. Note that this condition is not "derived," but must be regarded as a postulate that results in the large-scale form holding for surfaces of discontinuous field.

Similar reasoning can be used to determine the appropriate boundary condition on tangential $\mathbf{E}$ from Faraday's law. Corresponding to (2.185) we obtain

$$\int_S (\hat{\mathbf{n}} \times \mathbf{E})\, dS - \int_V \left(-\mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t}\right) dV = \int_{S_{10}} \hat{\mathbf{n}}_{12} \times (\mathbf{E}_1 - \mathbf{E}_2)\, dS. \tag{2.188}$$

Employing (2.175) over the region containing the field discontinuity surface we get

$$\int_S (\hat{\mathbf{n}} \times \mathbf{E})\, dS - \int_V \left(-\mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t}\right) dV = -\int_{S_{10}} \mathbf{J}_{ms}\, dS. \tag{2.189}$$

To have (2.188) and (2.189) produce identical results, we postulate

$$\hat{\mathbf{n}}_{12} \times (\mathbf{E}_1 - \mathbf{E}_2) = -\mathbf{J}_{ms} \tag{2.190}$$

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as the boundary condition appropriate to a surface of field discontinuity containing a magnetic surface current.

We can also postulate boundary conditions on the normal fields to make Gauss's laws valid for surfaces of discontinuous fields. Integrating (2.147) over the regions V_1 and V_2 and adding, we obtain

$$\int_{S_1+S_2} \mathbf{D}\cdot\hat{\mathbf{n}}\,dS - \int_{S_{10}} \mathbf{D}_1\cdot\hat{\mathbf{n}}_{10}\,dS - \int_{S_{20}} \mathbf{D}_2\cdot\hat{\mathbf{n}}_{20}\,dS = \int_{V_1+V_2} \rho\,dV.$$

As $\delta \to 0$ this becomes

$$\int_S \mathbf{D}\cdot\hat{\mathbf{n}}\,dS - \int_V \rho\,dV = \int_{S_{10}} (\mathbf{D}_1 - \mathbf{D}_2)\cdot\hat{\mathbf{n}}_{12}\,dS. \tag{2.191}$$

If we integrate Gauss's law over the entire region V, including the surface of discontinuity, we get

$$\oint_S \mathbf{D}\cdot\hat{\mathbf{n}}\,dS = \int_V \rho\,dV + \int_{S_{10}} \rho_s\,dS. \tag{2.192}$$

In order to get identical answers from (2.191) and (2.192), we must have

$$(\mathbf{D}_1 - \mathbf{D}_2)\cdot\hat{\mathbf{n}}_{12} = \rho_s$$

as the boundary condition appropriate to a surface of field discontinuity containing an electric surface charge. Similarly, we must postulate

$$(\mathbf{B}_1 - \mathbf{B}_2)\cdot\hat{\mathbf{n}}_{12} = \rho_{ms}$$

as the condition appropriate to a surface of field discontinuity containing a magnetic surface charge.

We can determine an appropriate boundary condition on current by using the large-scale form of the continuity equation. Applying (2.10) over each of the volume regions of Figure 2.6 and adding the results, we have

$$\int_{S_1+S_2} \mathbf{J}\cdot\hat{\mathbf{n}}\,dS - \int_{S_{10}} \mathbf{J}_1\cdot\hat{\mathbf{n}}_{10}\,dS - \int_{S_{20}} \mathbf{J}_2\cdot\hat{\mathbf{n}}_{20}\,dS = -\int_{V_1+V_2} \frac{\partial\rho}{\partial t}\,dV.$$

As $\delta \to 0$ we have

$$\int_S \mathbf{J}\cdot\hat{\mathbf{n}}\,dS - \int_{S_{10}} (\mathbf{J}_1 - \mathbf{J}_2)\cdot\hat{\mathbf{n}}_{12}\,dS = -\int_V \frac{\partial\rho}{\partial t}\,dV. \tag{2.193}$$

Applying the continuity equation over the entire region V and allowing it to intersect the discontinuity surface, we get

$$\int_S \mathbf{J}\cdot\hat{\mathbf{n}}\,dS + \int_\Gamma \mathbf{J}_s\cdot\hat{\mathbf{m}}\,dl = -\int_V \frac{\partial\rho}{\partial t}\,dV - \int_{S_{10}} \frac{\partial\rho_s}{\partial t}\,dS.$$

By the two-dimensional divergence theorem (B.20) we can write this as

$$\int_S \mathbf{J}\cdot\hat{\mathbf{n}}\,dS + \int_{S_{10}} \nabla_s\cdot\mathbf{J}_s\,dS = -\int_V \frac{\partial\rho}{\partial t}\,dV - \int_{S_{10}} \frac{\partial\rho_s}{\partial t}\,dS.$$

In order for this expression to produce the same values of the integrals over S and V as in (2.193) we require

$$\nabla_s\cdot\mathbf{J}_s = -\hat{\mathbf{n}}_{12}\cdot(\mathbf{J}_1 - \mathbf{J}_2) - \frac{\partial\rho_s}{\partial t},$$

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which we take as our postulate of the boundary condition on current across a surface containing discontinuities. A similar set of steps carried out using the continuity equation for magnetic sources yields

$$\nabla_s \cdot \mathbf{J}_{ms} = -\hat{\mathbf{n}}_{12} \cdot (\mathbf{J}_{m1} - \mathbf{J}_{m2}) - \frac{\partial \rho_{ms}}{\partial t}.$$

In summary, we have the following boundary conditions for fields across a surface containing discontinuities:

$$\hat{\mathbf{n}}_{12} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s, \tag{2.194}$$

$$\hat{\mathbf{n}}_{12} \times (\mathbf{E}_1 - \mathbf{E}_2) = -\mathbf{J}_{ms}, \tag{2.195}$$

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s, \tag{2.196}$$

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = \rho_{ms}, \tag{2.197}$$

and

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{J}_1 - \mathbf{J}_2) = -\nabla_s \cdot \mathbf{J}_s - \frac{\partial \rho_s}{\partial t}, \tag{2.198}$$

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{J}_{m1} - \mathbf{J}_{m2}) = -\nabla_s \cdot \mathbf{J}_{ms} - \frac{\partial \rho_{ms}}{\partial t}, \tag{2.199}$$

where $\hat{\mathbf{n}}_{12}$ points into region 1 from region 2.

2.8.3 Boundary conditions at the surface of a perfect conductor

We can easily specialize the results of the previous section to the case of perfect electric or magnetic conductors. In § 2.2.2 we saw that the constitutive relations for perfect conductors requires the null field within the material. In addition, a PEC requires zero tangential electric field, while a PMC requires zero tangential magnetic field. Using (2.194)–(2.199), we find that the boundary conditions for a perfect electric conductor are

$$\hat{\mathbf{n}} \times \mathbf{H} = \mathbf{J}_s, \tag{2.200}$$

$$\hat{\mathbf{n}} \times \mathbf{E} = 0, \tag{2.201}$$

$$\hat{\mathbf{n}} \cdot \mathbf{D} = \rho_s, \tag{2.202}$$

$$\hat{\mathbf{n}} \cdot \mathbf{B} = 0, \tag{2.203}$$

and

$$\hat{\mathbf{n}} \cdot \mathbf{J} = -\nabla_s \cdot \mathbf{J}_s - \frac{\partial \rho_s}{\partial t}, \qquad \hat{\mathbf{n}} \cdot \mathbf{J}_m = 0. \tag{2.204}$$

For a PMC the conditions are

$$\hat{\mathbf{n}} \times \mathbf{H} = 0, \tag{2.205}$$

$$\hat{\mathbf{n}} \times \mathbf{E} = -\mathbf{J}_{ms}, \tag{2.206}$$

$$\hat{\mathbf{n}} \cdot \mathbf{D} = 0, \tag{2.207}$$

$$\hat{\mathbf{n}} \cdot \mathbf{B} = \rho_{ms}, \tag{2.208}$$

and

$$\hat{\mathbf{n}} \cdot \mathbf{J}_m = -\nabla_s \cdot \mathbf{J}_{ms} - \frac{\partial \rho_{ms}}{\partial t}, \qquad \hat{\mathbf{n}} \cdot \mathbf{J} = 0. \tag{2.209}$$

We note that the normal vector $\hat{\mathbf{n}}$ points out of the conductor and into the adjacent region of nonzero fields.

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2.8.4 Boundary conditions across a stationary layer of field discontinuity using equivalent sources

So far we have avoided using the physical interpretation of the equivalent sources in the Maxwell–Boffi equations so that we might investigate the behavior of fields across true discontinuities. Now that we have the appropriate boundary conditions, it is interesting to interpret them in terms of the equivalent sources.

If we put $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$ into (2.194) and rearrange, we get

$$\hat{\mathbf{n}}_{12} \times (\mathbf{B}_1 - \mathbf{B}_2) = \mu_0(\mathbf{J}_s + \hat{\mathbf{n}}_{12} \times \mathbf{M}_1 - \hat{\mathbf{n}}_{12} \times \mathbf{M}_2). \tag{2.210}$$

The terms on the right involving $\hat{\mathbf{n}}_{12} \times \mathbf{M}$ have the units of surface current and are called *equivalent magnetization surface currents.* Defining

$$\mathbf{J}_{Ms} = -\hat{\mathbf{n}} \times \mathbf{M} \tag{2.211}$$

where $\hat{\mathbf{n}}$ is directed normally outward from the material region of interest, we can rewrite (2.210) as

$$\hat{\mathbf{n}}_{12} \times (\mathbf{B}_1 - \mathbf{B}_2) = \mu_0(\mathbf{J}_s + \mathbf{J}_{Ms1} + \mathbf{J}_{Ms2}). \tag{2.212}$$

We note that J_{Ms} replaces atomic charge moving along the surface of a material with an equivalent surface current in free space.

If we substitute $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ into (2.196) and rearrange, we get

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = \frac{1}{\epsilon_0}(\rho_s - \hat{\mathbf{n}}_{12} \cdot \mathbf{P}_1 + \hat{\mathbf{n}}_{12} \cdot \mathbf{P}_2). \tag{2.213}$$

The terms on the right involving $\hat{\mathbf{n}}_{12} \cdot \mathbf{P}$ have the units of surface charge and are called *equivalent polarization surface charges.* Defining

$$\rho_{Ps} = \hat{\mathbf{n}} \cdot \mathbf{P}, \tag{2.214}$$

we can rewrite (2.213) as

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = \frac{1}{\epsilon_0}(\rho_s + \rho_{Ps1} + \rho_{Ps2}). \tag{2.215}$$

We note that ρ_{Ps} replaces atomic charge adjacent to a surface of a material with an equivalent surface charge in free space.

In summary, the boundary conditions at a stationary surface of discontinuity written in terms of equivalent sources are

$$\hat{\mathbf{n}}_{12} \times (\mathbf{B}_1 - \mathbf{B}_2) = \mu_0(\mathbf{J}_s + \mathbf{J}_{Ms1} + \mathbf{J}_{Ms2}), \tag{2.216}$$

$$\hat{\mathbf{n}}_{12} \times (\mathbf{E}_1 - \mathbf{E}_2) = -\mathbf{J}_{ms}, \tag{2.217}$$

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = \frac{1}{\epsilon_0}(\rho_s + \rho_{Ps1} + \rho_{Ps2}), \tag{2.218}$$

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = \rho_{ms}. \tag{2.219}$$

2.8.5 Boundary conditions across a moving layer of field discontinuity

With a moving material body it is often necessary to apply boundary conditions describing the behavior of the fields across the surface of the body. If a surface of discontinuity moves with constant velocity v, the boundary conditions (2.194)–(2.199) hold as

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long as all fields are expressed *in the frame of the moving surface*. We can also derive boundary conditions for a deforming surface moving with arbitrary velocity by using equations (2.177)–(2.178). In this case all fields are expressed in the laboratory frame. Proceeding through the same set of steps that gave us (2.194)–(2.197), we find

$$\hat{\mathbf{n}}_{12} \times (\mathbf{H}_1 - \mathbf{H}_2) + (\hat{\mathbf{n}}_{12} \cdot \mathbf{v})(\mathbf{D}_1 - \mathbf{D}_2) = \mathbf{J}_s, \tag{2.220}$$

$$\hat{\mathbf{n}}_{12} \times (\mathbf{E}_1 - \mathbf{E}_2) - (\hat{\mathbf{n}}_{12} \cdot \mathbf{v})(\mathbf{B}_1 - \mathbf{B}_2) = -\mathbf{J}_{ms}, \tag{2.221}$$

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s, \tag{2.222}$$

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = \rho_{ms}. \tag{2.223}$$

Note that when $\hat{\mathbf{n}}_{12} \cdot \mathbf{v} = 0$ these boundary conditions reduce to those for a stationary surface. This occurs not only when $\mathbf{v} = 0$ but also when the velocity is parallel to the surface.

The reader must be wary when employing (2.220)–(2.223). Since the fields are measured in the laboratory frame, if the constitutive relations are substituted into the boundary conditions they must also be represented in the laboratory frame. It is probable that the material parameters would be known in the rest frame of the material, in which case a conversion to the laboratory frame would be necessary.

2.9 Fundamental theorems

In this section we shall consider some of the important theorems of electromagnetics that pertain directly to Maxwell's equations. They may be derived without reference to the solutions of Maxwell's equations, and are not connected with any specialization of the equations or any specific application or geometrical configuration. In this sense these theorems are fundamental to the study of electromagnetics.

2.9.1 Linearity

Recall that a mathematical operator L is *linear* if

$$L(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 L(f_1) + \alpha_2 L(f_2)$$

holds for any two functions $f_{1,2}$ in the domain of L and any two scalar constants $\alpha_{1,2}$. A standard observation regarding the equation

$$L(f) = s, \tag{2.224}$$

where L is a linear operator and s is a given forcing function, is that if f_1 and f_2 are solutions to

$$L(f_1) = s_1, \qquad L(f_2) = s_2, \tag{2.225}$$

respectively, and

$$s = s_1 + s_2, \tag{2.226}$$

then

$$f = f_1 + f_2 \tag{2.227}$$

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is a solution to (2.224). This is the *principle of superposition*; if convenient, we can decompose s in equation (2.224) as a sum (2.226) and solve the two resulting equations (2.225) independently. The solution to (2.224) is then (2.227), "by superposition." Of course, we are free to split the right side of (2.224) into more than two terms — the method extends directly to any finite number of terms.

Because the operators $\nabla\cdot$, $\nabla\times$, and $\partial/\partial t$ are all linear, Maxwell's equations can be treated by this method. If, for instance,

$$\nabla \times \mathbf{E}_1 = -\frac{\partial \mathbf{B}_1}{\partial t}, \qquad \nabla \times \mathbf{E}_2 = -\frac{\partial \mathbf{B}_2}{\partial t},$$

then

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

where $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ and $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$. The motivation for decomposing terms in a particular way is often based on physical considerations; we give one example here and defer others to later sections of the book. We saw earlier that Maxwell's equations can be written in terms of both electric and (fictitious) magnetic sources as in equations (2.169)–(2.172). Let $\mathbf{E} = \mathbf{E}_e + \mathbf{E}_m$ where $\mathbf{E}_e$ is produced by electric-type sources and $\mathbf{E}_m$ is produced by magnetic-type sources, and decompose the other fields similarly. Then

$$\nabla \times \mathbf{E}_e = -\frac{\partial \mathbf{B}_e}{\partial t}, \qquad \nabla \times \mathbf{H}_e = \mathbf{J} + \frac{\partial \mathbf{D}_e}{\partial t}, \qquad \nabla \cdot \mathbf{D}_e = \rho, \qquad \nabla \cdot \mathbf{B}_e = 0,$$

with a similar equation set for the magnetic sources. We may, if desired, solve these two equation sets independently for $\mathbf{E}_e$, $\mathbf{D}_e$, $\mathbf{B}_e$, $\mathbf{H}_e$ and $\mathbf{E}_m$, $\mathbf{D}_m$, $\mathbf{E}_m$, $\mathbf{H}_m$, and then use superposition to obtain the total fields $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, $\mathbf{H}$.

2.9.2 Duality

The intriguing symmetry of Maxwell's equations leads us to an observation that can reduce the effort required to compute solutions. Consider a closed surface S enclosing a region of space that includes an electric source current $\mathbf{J}$ and a magnetic source current $\mathbf{J}_m$. The fields ($\mathbf{E}_1$,$\mathbf{D}_1$,$\mathbf{B}_1$,$\mathbf{H}_1$) within the region (which may also contain arbitrary media) are described by

$$\nabla \times \mathbf{E}_1 = -\mathbf{J}_m - \frac{\partial \mathbf{B}_1}{\partial t}, \tag{2.228}$$

$$\nabla \times \mathbf{H}_1 = \mathbf{J} + \frac{\partial \mathbf{D}_1}{\partial t}, \tag{2.229}$$

$$\nabla \cdot \mathbf{D}_1 = \rho, \tag{2.230}$$

$$\nabla \cdot \mathbf{B}_1 = \rho_m. \tag{2.231}$$

Suppose we have been given a mathematical description of the sources $(\mathbf{J}, \mathbf{J}_m)$ and have solved for the field vectors $(\mathbf{E}_1, \mathbf{D}_1, \mathbf{B}_1, \mathbf{H}_1)$. Of course, we must also have been supplied with a set of boundary values and constitutive relations in order to make the solution unique. We note that if we replace the formula for $\mathbf{J}$ with the formula for $\mathbf{J}_m$ in (2.229) (and ρ with ρ_m in (2.230)) and also replace $\mathbf{J}_m$ with $-\mathbf{J}$ in (2.228) (and ρ_m with $-\rho$ in (2.231)) we get a new problem to solve, with a different solution. However, the symmetry of the equations allows us to specify the solution immediately. The new set of

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curl equations requires

$$\nabla \times \mathbf{E}_2 = \mathbf{J} - \frac{\partial \mathbf{B}_2}{\partial t}, \tag{2.232}$$

$$\nabla \times \mathbf{H}_2 = \mathbf{J}_m + \frac{\partial \mathbf{D}_2}{\partial t}. \tag{2.233}$$

As long as we can resolve the question of how the constitutive parameters must be altered to reflect these replacements, we can conclude by comparing (2.232) with (2.229) and (2.233) with (2.228) that the solution to these equations is merely

$$\mathbf{E}_2 = \mathbf{H}_1,$$
$$\mathbf{B}_2 = -\mathbf{D}_1,$$
$$\mathbf{D}_2 = \mathbf{B}_1,$$
$$\mathbf{H}_2 = -\mathbf{E}_1.$$

That is, if we have solved the original problem, we can use those solutions to find the new ones. This is an application of the general *principle of duality*.

Unfortunately, this approach is a little awkward since the units of the sources and fields in the two problems are different. We can make the procedure more convenient by multiplying Ampere's law by $\eta_0 = (\mu_0/\epsilon_0)^{1/2}$. Then we have

$$\nabla \times \mathbf{E} = -\mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t}, \tag{2.234}$$

$$\nabla \times (\eta_0 \mathbf{H}) = (\eta_0 \mathbf{J}) + \frac{\partial (\eta_0 \mathbf{D})}{\partial t}. \tag{2.235}$$

Thus if the original problem has solution $(\mathbf{E}_1, \eta_0\mathbf{D}_1, \mathbf{B}_1, \eta_0\mathbf{H}_1)$, then the dual problem with $\mathbf{J}$ replaced by $\mathbf{J}_m/\eta_0$ and $\mathbf{J}_m$ replaced by $-\eta_0\mathbf{J}$ has solution

$$\mathbf{E}_2 = \eta_0 \mathbf{H}_1, \tag{2.236}$$

$$\mathbf{B}_2 = -\eta_0 \mathbf{D}_1, \tag{2.237}$$

$$\eta_0 \mathbf{D}_2 = \mathbf{B}_1, \tag{2.238}$$

$$\eta_0 \mathbf{H}_2 = -\mathbf{E}_1. \tag{2.239}$$

The units on the quantities in the two problems are now identical.

Of course, the constitutive parameters for the dual problem must be altered from those of the original problem to reflect the change in field quantities. From (2.19) and (2.20) we know that the most general forms of the constitutive relations (those for linear, bianisotropic media) are

$$\mathbf{D}_1 = \bar{\boldsymbol{\xi}}_1 \cdot \mathbf{H}_1 + \bar{\epsilon}_1 \cdot \mathbf{E}_1, \tag{2.240}$$

$$\mathbf{B}_1 = \bar{\boldsymbol{\mu}}_1 \cdot \mathbf{H}_1 + \bar{\boldsymbol{\zeta}}_1 \cdot \mathbf{E}_1, \tag{2.241}$$

for the original problem, and

$$\mathbf{D}_2 = \bar{\boldsymbol{\xi}}_2 \cdot \mathbf{H}_2 + \bar{\epsilon}_2 \cdot \mathbf{E}_2, \tag{2.242}$$

$$\mathbf{B}_2 = \bar{\boldsymbol{\mu}}_2 \cdot \mathbf{H}_2 + \bar{\boldsymbol{\zeta}}_2 \cdot \mathbf{E}_2, \tag{2.243}$$

for the dual problem. Substitution of (2.236)–(2.239) into (2.240) and (2.241) gives

$$\mathbf{D}_2 = (-\bar{\boldsymbol{\zeta}}_1) \cdot \mathbf{H}_2 + \left(\frac{\bar{\boldsymbol{\mu}}_1}{\eta_0^2} \right) \cdot \mathbf{E}_2, \tag{2.244}$$

$$\mathbf{B}_2 = \left(\eta_0^2 \bar{\epsilon}_1 \right) \cdot \mathbf{H}_2 + (-\bar{\boldsymbol{\xi}}_1) \cdot \mathbf{E}_2. \tag{2.245}$$

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Comparing (2.244) with (2.242) and (2.245) with (2.243), we conclude that

$$\bar{\zeta}_2 = -\bar{\xi}_1, \qquad \bar{\xi}_2 = -\bar{\zeta}_1, \qquad \bar{\mu}_2 = \eta_0^2 \bar{\epsilon}_1, \qquad \bar{\epsilon}_2 = \frac{\bar{\mu}_1}{\eta_0^2}.$$

As an important special case, we see that for a linear, isotropic medium specified by a permittivity ϵ and permeability μ, the dual problem is obtained by replacing ϵ_r with μ_r and μ_r with ϵ_r. The solution to the dual problem is then given by

$$\mathbf{E}_2 = \eta_0 \mathbf{H}_1, \qquad \eta_0 \mathbf{H}_2 = -\mathbf{E}_1,$$

as before. We thus see that the medium in the dual problem must have electric properties numerically equal to the magnetic properties of the medium in the original problem, and magnetic properties numerically equal to the electric properties of the medium in the original problem. This is rather inconvenient for most applications. Alternatively, we may divide Ampere's law by $\eta = (\mu/\epsilon)^{1/2}$ instead of η_0. Then the dual problem has $\mathbf{J}$ replaced by $\mathbf{J}_m/\eta$, and $\mathbf{J}_m$ replaced by $-\eta\mathbf{J}$, and the solution to the dual problem is given by

$$\mathbf{E}_2 = \eta \mathbf{H}_1, \qquad \eta \mathbf{H}_2 = -\mathbf{E}_1.$$

In this case there is no need to swap ϵ_r and μ_r, since information about these parameters is incorporated into the replacement sources.

We must also remember that to obtain a unique solution we need to specify the boundary values of the fields. In a true dual problem, the boundary values of the fields used in the original problem are used on the swapped fields in the dual problem. A typical example of this is when the condition of zero tangential electric field on a perfect electric conductor is replaced by the condition of zero tangential magnetic field on the surface of a perfect magnetic conductor. However, duality can also be used to obtain the mathematical form of the field expressions, often in a homogeneous (source-free) situation, and boundary values can be applied later to specify the solution appropriate to the problem geometry. This approach is often used to compute waveguide modal fields and the electromagnetic fields scattered from objects. In these cases a TE/TM field decomposition is employed, and duality is used to find one part of the decomposition once the other is known.

Duality of electric and magnetic point source fields. By duality, we can sometimes use the known solution to one problem to solve a related problem by merely substituting different variables into the known mathematical expression. An example of this is the case in which we have solved for the fields produced by a certain distribution of electric sources and wish to determine the fields when the same distribution is used to describe magnetic sources.

Let us consider the case when the source distribution is that of a point current, or *Hertzian dipole*, immersed in free space. As we shall see in Chapter 5, the fields for a general source may be found by using the fields produced by these point sources. We begin by finding the fields produced by an electric dipole source at the origin aligned along the z-axis,

$$\mathbf{J} = \hat{\mathbf{z}} I_0 \delta(\mathbf{r}),$$

then use duality to find the fields produced by a magnetic current source $\mathbf{J}_m = \hat{\mathbf{z}} I_{m0} \delta(\mathbf{r})$.

The fields produced by the electric source must obey

$$\nabla \times \mathbf{E}_e = -\frac{\partial}{\partial t} \mu_0 \mathbf{H}_e, \qquad (2.246)$$

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$$\nabla \times \mathbf{H}_e = \hat{\mathbf{z}} I_0 \delta(\mathbf{r}) + \frac{\partial}{\partial t} \epsilon_0 \mathbf{E}_e, \tag{2.247}$$

$$\nabla \cdot \epsilon_0 \mathbf{E}_e = \rho, \tag{2.248}$$

$$\nabla \cdot \mathbf{H}_e = 0, \tag{2.249}$$

while those produced by the magnetic source must obey

$$\nabla \times \mathbf{E}_m = -\hat{\mathbf{z}} I_{m0} \delta(\mathbf{r}) - \frac{\partial}{\partial t} \mu_0 \mathbf{H}_m, \tag{2.250}$$

$$\nabla \times \mathbf{H}_m = \frac{\partial}{\partial t} \epsilon_0 \mathbf{E}_m, \tag{2.251}$$

$$\nabla \cdot \mathbf{E}_m = 0, \tag{2.252}$$

$$\nabla \cdot \mu_0 \mathbf{H}_m = \rho_m. \tag{2.253}$$

We see immediately that the second set of equations is the dual of the first, as long as we scale the sources appropriately. Multiplying (2.250) by $-I_0/I_{m0}$ and (2.251) by $I_0 \eta_0^2 / I_{m0}$, we have the curl equations

$$\nabla \times \left(-\frac{I_0}{I_{m0}} \mathbf{E}_m \right) = \hat{\mathbf{z}} I_0 \delta(\mathbf{r}) + \frac{\partial}{\partial t} \left(\mu_0 \frac{I_0}{I_{m0}} \mathbf{H}_m \right), \tag{2.254}$$

$$\nabla \times \left(\frac{I_0 \eta_0^2}{I_{m0}} \mathbf{H}_m \right) = -\frac{\partial}{\partial t} \left(-\epsilon_0 \frac{I_0 \eta_0^2}{I_{m0}} \mathbf{E}_m \right). \tag{2.255}$$

Comparing (2.255) with (2.246) and (2.254) with (2.247) we see that

$$\mathbf{E}_m = -\frac{I_{m0}}{I_0} \mathbf{H}_e, \qquad \mathbf{H}_m = \frac{I_{m0}}{I_0} \frac{\mathbf{E}_e}{\eta_0^2}.$$

We note that it is impossible to have a point current source without accompanying point charge sources terminating each end of the dipole current. The point charges are required to satisfy the continuity equation, and vary in time as the moving charge that establishes the current accumulates at the ends of the dipole. From (2.247) we see that the magnetic field curls around the combination of the electric field and electric current source, while from (2.246) the electric field curls around the magnetic field, and from (2.248) diverges from the charges located at the ends of the dipole. From (2.250) we see that the electric field must curl around the combination of the magnetic field and magnetic current source, while (2.251) and (2.253) show that the magnetic field curls around the electric field and diverges from the magnetic charge.

Duality in a source-free region. Consider a closed surface S enclosing a source-free region of space. For simplicity, assume that the medium within S is linear, isotropic, and homogeneous. The fields within S are described by Maxwell's equations

$$\nabla \times \mathbf{E}_1 = -\frac{\partial}{\partial t} \mu \mathbf{H}_1, \tag{2.256}$$

$$\nabla \times \eta \mathbf{H}_1 = \frac{\partial}{\partial t} \epsilon \eta \mathbf{E}_1, \tag{2.257}$$

$$\nabla \cdot \epsilon \mathbf{E}_1 = 0, \tag{2.258}$$

$$\nabla \cdot \mu \mathbf{H}_1 = 0. \tag{2.259}$$

Under these conditions the concept of duality takes on a different face. The symmetry of the equations is such that the mathematical form of the solution for $\mathbf{E}$ is the same as

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that for $\eta\mathbf{H}$. That is, the fields

$$\mathbf{E}_2 = \eta\mathbf{H}_1, \tag{2.260}$$

$$\mathbf{H}_2 = -\mathbf{E}_1/\eta, \tag{2.261}$$

are also a solution to Maxwell's equations, and thus the dual problem merely involves replacing $\mathbf{E}$ by $\eta\mathbf{H}$ and $\mathbf{H}$ by $-\mathbf{E}/\eta$. However, the final forms of $\mathbf{E}$ and $\mathbf{H}$ will *not* be identical after appropriate boundary values are imposed.

This form of duality is very important for the solution of fields within waveguides or the fields scattered by objects where the sources are located outside the region where the fields are evaluated.

2.9.3 Reciprocity

The reciprocity theorem, also called the *Lorentz reciprocity theorem*, describes a specific and often useful relationship between sources and the electromagnetic fields they produce. Under certain special circumstances we find that an interaction between independent source and mediating fields called "reaction" is a spatially symmetric quantity. The reciprocity theorem is used in the study of guided waves to establish the orthogonality of guided wave modes, in microwave network theory to obtain relationships between terminal characteristics, and in antenna theory to demonstrate the equivalence of transmission and reception patterns.

Consider a closed surface S enclosing a volume V. Assume that the fields within and on S are produced by two independent source fields. The source $(\mathbf{J}_a, \mathbf{J}_{ma})$ produces the field $(\mathbf{E}_a, \mathbf{D}_a, \mathbf{B}_a, \mathbf{H}_a)$ as described by Maxwell's equations

$$\nabla \times \mathbf{E}_a = -\mathbf{J}_{ma} - \frac{\partial \mathbf{B}_a}{\partial t}, \tag{2.262}$$

$$\nabla \times \mathbf{H}_a = \mathbf{J}_a + \frac{\partial \mathbf{D}_a}{\partial t}, \tag{2.263}$$

while the source field $(\mathbf{J}_b, \mathbf{J}_{mb})$ produces the field $(\mathbf{E}_b, \mathbf{D}_b, \mathbf{B}_b, \mathbf{H}_b)$ as described by

$$\nabla \times \mathbf{E}_b = -\mathbf{J}_{mb} - \frac{\partial \mathbf{B}_b}{\partial t}, \tag{2.264}$$

$$\nabla \times \mathbf{H}_b = \mathbf{J}_b + \frac{\partial \mathbf{D}_b}{\partial t}. \tag{2.265}$$

The sources may be distributed in any way relative to S: they may lie completely inside, completely outside, or partially inside and partially outside. Material media may lie within S, and their properties may depend on position.

Let us examine the quantity

$$R \equiv \nabla \cdot (\mathbf{E}_a \times \mathbf{H}_b - \mathbf{E}_b \times \mathbf{H}_a).$$

By (B.44) we have

$$R = \mathbf{H}_b \cdot \nabla \times \mathbf{E}_a - \mathbf{E}_a \cdot \nabla \times \mathbf{H}_b - \mathbf{H}_a \cdot \nabla \times \mathbf{E}_b + \mathbf{E}_b \cdot \nabla \times \mathbf{H}_a$$

so that by Maxwell's curl equations

$$R = \left[\mathbf{H}_a \cdot \frac{\partial \mathbf{B}_b}{\partial t} - \mathbf{H}_b \cdot \frac{\partial \mathbf{B}_a}{\partial t}\right] - \left[\mathbf{E}_a \cdot \frac{\partial \mathbf{D}_b}{\partial t} - \mathbf{E}_b \cdot \frac{\partial \mathbf{D}_a}{\partial t}\right] + \\ + \left[\mathbf{J}_a \cdot \mathbf{E}_b - \mathbf{J}_b \cdot \mathbf{E}_a - \mathbf{J}_{ma} \cdot \mathbf{H}_b + \mathbf{J}_{mb} \cdot \mathbf{H}_a\right].$$

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The useful relationships we seek occur when the first two bracketed quantities on the right-hand side of the above expression are zero. Whether this is true depends not only on the behavior of the fields, but on the properties of the medium at the point in question. Though we have assumed that the sources of the field sets are independent, it is apparent that they must share a similar time dependence in order for the terms within each of the bracketed quantities to cancel. Of special interest is the case where the two sources are both sinusoidal in time with identical frequencies, but with differing spatial distributions. We shall consider this case in detail in § 4.10.2 after we have discussed the properties of the time harmonic field. Importantly, we will find that only certain characteristics of the constitutive parameters allow cancellation of the bracketed terms; materials with these characteristics are called *reciprocal*, and the fields they support are said to display the property of *reciprocity*. To see what this property entails, we set the bracketed terms to zero and integrate over a volume V to obtain

$$\oint_S (\mathbf{E}_a \times \mathbf{H}_b - \mathbf{E}_b \times \mathbf{H}_a) \cdot \mathbf{dS} = \int_V (\mathbf{J}_a \cdot \mathbf{E}_b - \mathbf{J}_b \cdot \mathbf{E}_a - \mathbf{J}_{ma} \cdot \mathbf{H}_b + \mathbf{J}_{mb} \cdot \mathbf{H}_a)\, dV,$$

which is the time-domain version of the *Lorentz reciprocity theorem.*

Two special cases of this theorem are important to us. If all sources lie outside S, we have *Lorentz's lemma*

$$\oint_S (\mathbf{E}_a \times \mathbf{H}_b - \mathbf{E}_b \times \mathbf{H}_a) \cdot \mathbf{dS} = 0.$$

This remarkable expression shows that a relationship exists between the fields produced by completely independent sources, and is useful for establishing waveguide mode orthogonality for time harmonic fields. If sources reside within S but the surface integral is equal to zero, we have

$$\int_V (\mathbf{J}_a \cdot \mathbf{E}_b - \mathbf{J}_b \cdot \mathbf{E}_a - \mathbf{J}_{ma} \cdot \mathbf{H}_b + \mathbf{J}_{mb} \cdot \mathbf{H}_a)\, dV = 0.$$

This occurs when the surface is bounded by a special material (such as an impedance sheet or a perfect conductor), or when the surface recedes to infinity; the expression is useful for establishing the reciprocity conditions for networks and antennas. We shall interpret it for time harmonic fields in § 4.10.2.

2.9.4 Similitude

A common approach in physical science involves the introduction of normalized variables to provide for scaling of problems along with a chance to identify certain physically significant parameters. Similarity as a general principle can be traced back to the earliest attempts to describe physical effects with mathematical equations, with serious study undertaken by Galileo. Helmholtz introduced the first systematic investigation in 1873, and the concept was rigorized by Reynolds ten years later [216]. Similitude is now considered a fundamental guiding principle in the modeling of materials [199].

The process often begins with a consideration of the fundamental differential equations. In electromagnetics we may introduce a set of dimensionless field and source variables

$$\underline{\mathbf{E}}, \quad \underline{\mathbf{D}}, \quad \underline{\mathbf{B}}, \quad \underline{\mathbf{H}}, \quad \underline{\mathbf{J}}, \quad \underline{\rho}, \tag{2.266}$$

by setting

$$\mathbf{E} = \underline{\mathbf{E}} k_E, \quad \mathbf{B} = \underline{\mathbf{B}} k_B, \quad \mathbf{D} = \underline{\mathbf{D}} k_D,$$
$$\mathbf{H} = \underline{\mathbf{H}} k_H, \quad \mathbf{J} = \underline{\mathbf{J}} k_J, \quad \rho = \underline{\rho} k_\rho. \tag{2.267}$$

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Here we regard the quantities $k_E, k_B, \ldots$ as base units for the discussion, while the dimensionless quantities (2.266) serve to express the actual fields $\mathbf{E}, \mathbf{B}, \ldots$ in terms of these base units. Of course, the time and space variables can also be scaled: we can write

$$t = \underline{t} k_t, \qquad l = \underline{l} k_l, \tag{2.268}$$

if l is any length of interest. Again, the quantities $\underline{t}$ and $\underline{l}$ are dimensionless measure numbers used to express the actual quantities t and l relative to the chosen base amounts k_t and k_l. With (2.267) and (2.268), Maxwell's curl equations become

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{k_B}{k_E}\frac{k_l}{k_t}\frac{\partial \underline{\mathbf{B}}}{\partial \underline{t}}, \qquad \underline{\nabla} \times \underline{\mathbf{H}} = \frac{k_J k_l}{k_H}\underline{\mathbf{J}} + \frac{k_D}{k_H}\frac{k_l}{k_t}\frac{\partial \underline{\mathbf{D}}}{\partial \underline{t}} \tag{2.269}$$

while the continuity equation becomes

$$\underline{\nabla} \cdot \underline{\mathbf{J}} = -\frac{k_\rho}{k_J}\frac{k_l}{k_t}\frac{\partial \underline{\rho}}{\partial \underline{t}}, \tag{2.270}$$

where $\underline{\nabla}$ has been normalized by k_l. These are examples of field equations cast into dimensionless form — it is easily verified that the *similarity parameters*

$$\frac{k_B}{k_E}\frac{k_l}{k_t}, \quad \frac{k_J k_l}{k_H}, \quad \frac{k_D}{k_H}\frac{k_l}{k_t}, \quad \frac{k_\rho}{k_J}\frac{k_l}{k_t}, \tag{2.271}$$

are dimensionless. The idea behind electromagnetic similitude is that a given set of normalized values $\underline{\mathbf{E}}, \underline{\mathbf{B}}, \ldots$ can satisfy equations (2.269) and (2.270) for many different physical situations, provided that the numerical values of the coefficients (2.271) are all fixed across those situations. Indeed, the differential equations would be identical.

To make this discussion a bit more concrete, let us assume a conducting linear medium where

$$\mathbf{D} = \epsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H}, \qquad \mathbf{J} = \sigma \mathbf{E},$$

and use

$$\epsilon = \underline{\epsilon} k_\epsilon, \qquad \mu = \underline{\mu} k_\mu, \qquad \sigma = \underline{\sigma} k_\sigma,$$

to express the material parameters in terms of dimensionless values $\underline{\epsilon}$, $\underline{\mu}$, and $\underline{\sigma}$. Then

$$\underline{\mathbf{D}} = \frac{k_\epsilon k_E}{k_D}\underline{\epsilon}\underline{\mathbf{E}}, \qquad \underline{\mathbf{B}} = \frac{k_\mu k_H}{k_B}\underline{\mu}\underline{\mathbf{H}}, \qquad \underline{\mathbf{J}} = \frac{k_\sigma k_E}{k_J}\underline{\sigma}\underline{\mathbf{E}},$$

and equations (2.269) become

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\left(\frac{k_\mu k_l}{k_t}\frac{k_H}{k_E}\right)\underline{\mu}\frac{\partial \underline{\mathbf{H}}}{\partial \underline{t}},$$

$$\underline{\nabla} \times \underline{\mathbf{H}} = \left(k_\sigma k_l \frac{k_E}{k_H}\right)\underline{\sigma}\underline{\mathbf{E}} + \left(\frac{k_\epsilon k_l}{k_t}\frac{k_E}{k_H}\right)\underline{\epsilon}\frac{\partial \underline{\mathbf{E}}}{\partial \underline{t}}.$$

Defining

$$\alpha = \frac{k_\mu k_l}{k_t}\frac{k_H}{k_E}, \qquad \gamma = k_\sigma k_l \frac{k_E}{k_H}, \qquad \beta = \frac{k_\epsilon k_l}{k_t}\frac{k_E}{k_H},$$

we see that under the current assumptions similarity holds between two electromagnetics problems only if $\alpha\underline{\mu}$, $\gamma\underline{\sigma}$, and $\beta\underline{\epsilon}$ are numerically the same in both problems. A necessary condition for similitude, then, is that the products

$$(\alpha\underline{\mu})(\beta\underline{\epsilon}) = k_\mu k_\epsilon \left(\frac{k_l}{k_t}\right)^2 \underline{\mu}\underline{\epsilon}, \qquad (\alpha\underline{\mu})(\gamma\underline{\sigma}) = k_\mu k_\sigma \frac{k_l^2}{k_t}\underline{\mu}\underline{\sigma},$$

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(which do not involve k_E or k_H) stay constant between problems. We see, for example, that we may compensate for a halving of the length scale k_l by (a) a quadrupling of the permeability $\underline{\mu}$, or (b) a simultaneous halving of the time scale k_t and doubling of the conductivity $\underline{\sigma}$. A much less subtle special case is that for which $\underline{\sigma} = 0$, $k_\epsilon = \epsilon_0$, $k_\mu = \mu_0$, and $\underline{\epsilon} = \underline{\mu} = 1$; we then have free space and must simply maintain

$$k_l / k_t = \text{constant}$$

so that the time and length scales stay proportional. In the sinusoidal steady state, for instance, the frequency would be made to vary inversely with the length scale.

2.9.5 Conservation theorems

The misconception that Poynting's theorem can be "derived" from Maxwell's equations is widespread and ingrained. We must, in fact, *postulate* the idea that the electromagnetic field can be associated with an energy flux propagating at the speed of light. Since the form of the postulate is patterned after the well-understood laws of mechanics, we begin by developing the basic equations of momentum and energy balance in mechanical systems. Then we shall see whether it is sensible to ascribe these principles to the electromagnetic field.

Maxwell's theory allows us to describe, using Maxwell's equations, the behavior of the electromagnetic fields within a (possibly) finite region V of space. The presence of any sources or material objects outside V are made known through the specification of tangential fields over the boundary of V, as required for uniqueness. Thus, the influence of external effects can always be viewed as being transported across the boundary. This is true of mechanical as well as electromagnetic effects. A charged material body can be acted on by physical contact with another body, by gravitational forces, and by the Lorentz force, each effect resulting in momentum exchange across the boundary of the object. These effects must all be taken into consideration if we are to invoke momentum conservation, resulting in a very complicated situation. This suggests that we try to decompose the problem into simpler "systems" based on physical effects.

The system concept in the physical sciences. The idea of decomposing a complicated system into simpler, self-contained systems is quite common in the physical sciences. Penfield and Haus [145] invoke this concept by introducing an *electromagnetic system* where the effects of the Lorentz force equation are considered to accompany a *mechanical system* where effects of pressure, stress, and strain are considered, and a *thermodynamic system* where the effects of heat exchange are considered. These systems can all be interrelated in a variety of ways. For instance, as a material heats up it can expand, and the resulting mechanical forces can alter the electrical properties of the material. We will follow Penfield and Haus by considering separate electromagnetic and mechanical subsystems; other systems may be added analogously.

If we separate the various systems by physical effect, we will need to know how to "reassemble the information." Two conservation theorems are very helpful in this regard: conservation of energy, and conservation of momentum. Engineers often employ these theorems to make tacit use of the system idea. For instance, when studying electromagnetic waves propagating in a waveguide, it is common practice to compute wave attenuation by calculating the Poynting flux of power into the walls of the guide. The power lost from the wave is said to "heat up the waveguide walls," which indeed it does. This is an admission that the electromagnetic system is not "closed": it requires the

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inclusion of a thermodynamic system in order that energy be conserved. Of course, the detailed workings of the thermodynamic system are often ignored, indicating that any thermodynamic "feedback" mechanism is weak. In the waveguide example, for instance, the heating of the metallic walls does not alter their electromagnetic properties enough to couple back into an effect on the fields in the walls or in the guide. If such effects were important, they would have to be included in the conservation theorem via the boundary fields; it is therefore reasonable to associate with these fields a "flow" of energy or momentum into V. Thus, we wish to develop conservation laws that include not only the Lorentz force effects within V, but a flow of external effects into V through its boundary surface.

To understand how external influences may effect the electromagnetic subsystem, we look to the behavior of the mechanical subsystem as an analogue. In the electromagnetic system, effects are felt both internally to a region (because of the Lorentz force effect) and through the system boundary (by the dependence of the internal fields on the boundary fields). In the mechanical and thermodynamic systems, a region of mass is affected both internally (through transfer of heat and gravitational forces) and through interactions occurring across its surface (through transfers of energy and momentum, by pressure and stress). One beauty of electromagnetic theory is that we can find a mathematical symmetry between electromagnetic and mechanical effects which parallels the above conceptual symmetry. This makes applying conservation of energy and momentum to the total system (electromagnetic, thermodynamic, and mechanical) very convenient.

Conservation of momentum and energy in mechanical systems. We begin by reviewing the interactions of material bodies in a mechanical system. For simplicity we concentrate on fluids (analogous to charge in space); the extension of these concepts to solid bodies is straightforward.

Consider a fluid with mass density ρ_m. The momentum of a small subvolume of the fluid is given by $\rho_m \mathbf{v}\,dV$, where $\mathbf{v}$ is the velocity of the subvolume. So the momentum density is $\rho_m \mathbf{v}$. Newton's second law states that a force acting throughout the subvolume results in a change in its momentum given by

$$\frac{D}{Dt}(\rho_m \mathbf{v}\,dV) = \mathbf{f}\,dV, \tag{2.272}$$

where $\mathbf{f}$ is the volume force density and the D/Dt notation shows that we are interested in the rate of change of the momentum as observed by the moving fluid element (see § A.2). Here $\mathbf{f}$ could be the weight force, for instance. Addition of the results for all elements of the fluid body gives

$$\frac{D}{Dt}\int_V \rho_m \mathbf{v}\,dV = \int_V \mathbf{f}\,dV \tag{2.273}$$

as the change in momentum for the entire body. If on the other hand the force exerted on the body is through contact with its surface, the change in momentum is

$$\frac{D}{Dt}\int_V \rho_m \mathbf{v}\,dV = \oint_S \mathbf{t}\,dS \tag{2.274}$$

where $\mathbf{t}$ is the "surface traction."

We can write the time-rate of change of momentum in a more useful form by applying the Reynolds transport theorem (A.66):

$$\frac{D}{Dt}\int_V \rho_m \mathbf{v}\,dV = \int_V \frac{\partial}{\partial t}(\rho_m \mathbf{v})\,dV + \oint_S (\rho_m \mathbf{v})\mathbf{v}\cdot \mathbf{dS}. \tag{2.275}$$

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Superposing (2.273) and (2.274) and substituting into (2.275) we have

$$\int_V \frac{\partial}{\partial t}(\rho_m \mathbf{v})\, dV + \oint_S (\rho_m \mathbf{v})\mathbf{v} \cdot \mathbf{dS} = \int_V \mathbf{f}\, dV + \oint_S \mathbf{t}\, dS. \tag{2.276}$$

If we define the dyadic quantity

$$\bar{\mathbf{T}}_k = \rho_m \mathbf{v}\mathbf{v}$$

then (2.276) can be written as

$$\int_V \frac{\partial}{\partial t}(\rho_m \mathbf{v})\, dV + \oint_S \hat{\mathbf{n}} \cdot \bar{\mathbf{T}}_k\, dS = \int_V \mathbf{f}\, dV + \oint_S \mathbf{t}\, dS. \tag{2.277}$$

This *principle of linear momentum* [214] can be interpreted as a large-scale form of conservation of kinetic linear momentum. Here $\hat{\mathbf{n}} \cdot \bar{\mathbf{T}}_k$ represents the flow of kinetic momentum across S, and the sum of this momentum transfer and the change of momentum within V stands equal to the forces acting internal to V and upon S.

The surface traction may be related to the surface normal $\hat{\mathbf{n}}$ through a dyadic quantity $\bar{\mathbf{T}}_m$ called the mechanical *stress tensor*:

$$\mathbf{t} = \hat{\mathbf{n}} \cdot \bar{\mathbf{T}}_m.$$

With this we may write (2.277) as

$$\int_V \frac{\partial}{\partial t}(\rho_m \mathbf{v})\, dV + \oint_S \hat{\mathbf{n}} \cdot \bar{\mathbf{T}}_k\, dS = \int_V \mathbf{f}\, dV + \oint_S \hat{\mathbf{n}} \cdot \bar{\mathbf{T}}_m\, dS$$

and apply the dyadic form of the divergence theorem (B.19) to get

$$\int_V \frac{\partial}{\partial t}(\rho_m \mathbf{v})\, dV + \int_V \nabla \cdot (\rho_m \mathbf{v}\mathbf{v})\, dV = \int_V \mathbf{f}\, dV + \int_V \nabla \cdot \bar{\mathbf{T}}_m\, dV. \tag{2.278}$$

Combining the volume integrals and setting the integrand to zero we have

$$\frac{\partial}{\partial t}(\rho_m \mathbf{v}) + \nabla \cdot (\rho_m \mathbf{v}\mathbf{v}) = \mathbf{f} + \nabla \cdot \bar{\mathbf{T}}_m,$$

which is the point-form equivalent of (2.277). Note that the second term on the right-hand side is nonzero only for points residing on the surface of the body. Finally, letting $\mathbf{g}$ denote momentum density we obtain the simple expression

$$\nabla \cdot \bar{\mathbf{T}}_k + \frac{\partial \mathbf{g}_k}{\partial t} = \mathbf{f}_k, \tag{2.279}$$

where

$$\mathbf{g}_k = \rho_m \mathbf{v}$$

is the density of kinetic momentum and

$$\mathbf{f}_k = \mathbf{f} + \nabla \cdot \bar{\mathbf{T}}_m \tag{2.280}$$

is the total force density.

Equation (2.279) is somewhat analogous to the electric charge continuity equation (1.11). For each point of the body, the total outflux of kinetic momentum plus the time rate of change of kinetic momentum equals the total force. The resemblance to (1.11) is strong, except for the nonzero term on the right-hand side. The charge continuity

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equation represents a closed system: charge cannot spontaneously appear and add an extra term to the right-hand side of (1.11). On the other hand, the change in total momentum at a point can exceed that given by the momentum flowing out of the point if there is another "source" (e.g., gravity for an internal point, or pressure on a boundary point).

To obtain a momentum conservation expression that resembles the continuity equation, we must consider a "subsystem" with terms that exactly counterbalance the extra expressions on the right-hand side of (2.279). For a fluid acted on only by external pressure the sole effect enters through the traction term, and [145]

$$\nabla \cdot \bar{\mathbf{T}}_m = -\nabla p \tag{2.281}$$

where p is the pressure exerted on the fluid body. Now, using (B.63), we can write

$$-\nabla p = -\nabla \cdot \bar{\mathbf{T}}_p \tag{2.282}$$

where

$$\bar{\mathbf{T}}_p = p\bar{\mathbf{I}}$$

and $\bar{\mathbf{I}}$ is the unit dyad. Finally, using (2.282), (2.281), and (2.280) in (2.279), we obtain

$$\nabla \cdot (\bar{\mathbf{T}}_k + \bar{\mathbf{T}}_p) + \frac{\partial}{\partial t}\mathbf{g}_k = 0$$

and we have an expression for a closed system including all possible effects. Now, note that we can form the above expression as

$$\left(\nabla \cdot \bar{\mathbf{T}}_k + \frac{\partial}{\partial t}\mathbf{g}_k\right) + \left(\nabla \cdot \bar{\mathbf{T}}_p + \frac{\partial}{\partial t}\mathbf{g}_p\right) = 0 \tag{2.283}$$

where $\mathbf{g}_p = 0$ since there are no volume effects associated with pressure. This can be viewed as the sum of two closed subsystems

$$\nabla \cdot \bar{\mathbf{T}}_k + \frac{\partial}{\partial t}\mathbf{g}_k = 0, \tag{2.284}$$

$$\nabla \cdot \bar{\mathbf{T}}_p + \frac{\partial}{\partial t}\mathbf{g}_p = 0.$$

We now have the desired viewpoint. The conservation formula for the complete closed system can be viewed as a sum of formulas for open subsystems, each having the form of a conservation law for a closed system. In case we must include the effects of gravity, for instance, we need only determine $\bar{\mathbf{T}}_g$ and $\mathbf{g}_g$ such that

$$\nabla \cdot \bar{\mathbf{T}}_g + \frac{\partial}{\partial t}\mathbf{g}_g = 0$$

and add this new conservation equation to (2.283). If we can find a conservation expression of form similar to (2.284) for an "electromagnetic subsystem," we can include its effects along with the mechanical effects by merely adding together the conservation laws. We shall find just such an expression later in this section.

We stated in § 1.3 that there are four fundamental conservation principles. We have now discussed linear momentum; the principle of angular momentum follows similarly. Our next goal is to find an expression similar to (2.283) for conservation of energy. We may expect the conservation of energy expression to obey a similar law of superposition.

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We begin with the fundamental definition of work: for a particle moving with velocity $\mathbf{v}$ under the influence of a force $\mathbf{f}_k$ the work is given by $\mathbf{f}_k \cdot \mathbf{v}$. Dot multiplying (2.272) by $\mathbf{v}$ and replacing $\mathbf{f}$ by $\mathbf{f}_k$ (to represent both volume and surface forces), we get

$$\mathbf{v} \cdot \frac{D}{Dt}(\rho_m \mathbf{v})\, dV = \mathbf{v} \cdot \mathbf{f}_k\, dV$$

or equivalently

$$\frac{D}{Dt}\left(\frac{1}{2}\rho_m \mathbf{v} \cdot \mathbf{v}\right)\, dV = \mathbf{v} \cdot \mathbf{f}_k\, dV.$$

Integration over a volume and application of the Reynolds transport theorem (A.66) then gives

$$\int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m v^2\right)\, dV + \oint_S \hat{\mathbf{n}} \cdot \left(\mathbf{v}\frac{1}{2}\rho_m v^2\right)\, dS = \int_V \mathbf{f}_k \cdot \mathbf{v}\, dV.$$

Hence the sum of the time rate of change in energy internal to the body and the flow of kinetic energy across the boundary must equal the work done by internal and surface forces acting on the body. In point form,

$$\nabla \cdot \mathbf{S}_k + \frac{\partial}{\partial t} W_k = \mathbf{f}_k \cdot \mathbf{v} \tag{2.285}$$

where

$$\mathbf{S}_k = \mathbf{v}\frac{1}{2}\rho_m v^2$$

is the density of the flow of kinetic energy and

$$W_k = \frac{1}{2}\rho_m v^2$$

is the kinetic energy density. Again, the system is not closed (the right-hand side of (2.285) is not zero) because the balancing forces are not included. As was done with the momentum equation, the effect of the work done by the pressure forces can be described in a closed-system-type equation

$$\nabla \cdot \mathbf{S}_p + \frac{\partial}{\partial t} W_p = 0. \tag{2.286}$$

Combining (2.285) and (2.286) we have

$$\nabla \cdot (\mathbf{S}_k + \mathbf{S}_p) + \frac{\partial}{\partial t}(W_k + W_p) = 0,$$

the energy conservation equation for the closed system.

Conservation in the electromagnetic subsystem. We would now like to achieve closed-system conservation theorems for the electromagnetic subsystem so that we can add in the effects of electromagnetism. For the momentum equation, we can proceed exactly as we did with the mechanical system. We begin with

$$\mathbf{f}_{em} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}.$$

This force term should appear on one side of the point form of the momentum conservation equation. The term on the other side must involve the electromagnetic fields, since

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they are the mechanism for exerting force on the charge distribution. Substituting for $\mathbf{J}$ from (2.2) and for ρ from (2.3) we have

$$\mathbf{f}_{em} = \mathbf{E}(\nabla \cdot \mathbf{D}) - \mathbf{B} \times (\nabla \times \mathbf{H}) + \mathbf{B} \times \frac{\partial \mathbf{D}}{\partial t}.$$

Using

$$\mathbf{B} \times \frac{\partial \mathbf{D}}{\partial t} = -\frac{\partial}{\partial t}(\mathbf{D} \times \mathbf{B}) + \mathbf{D} \times \frac{\partial \mathbf{B}}{\partial t}$$

and substituting from Faraday's law for $\partial \mathbf{B}/\partial t$ we have

$$-\left[\mathbf{E}(\nabla \cdot \mathbf{D}) - \mathbf{D} \times (\nabla \times \mathbf{E}) + \mathbf{H}(\nabla \cdot \mathbf{B}) - \mathbf{B} \times (\nabla \times \mathbf{H})\right] + \frac{\partial}{\partial t}(\mathbf{D} \times \mathbf{B}) = -\mathbf{f}_{em}. \qquad (2.287)$$

Here we have also added the null term $\mathbf{H}(\nabla \cdot \mathbf{B})$.

The forms of (2.287) and (2.279) would be identical if the bracketed term could be written as the divergence of a dyadic function $\bar{\mathbf{T}}_{em}$. This is indeed possible for linear, homogeneous, bianisotropic media, provided that the constitutive matrix $[\bar{\mathbf{C}}_{EH}]$ in (2.21) is symmetric [101]. In that case

$$\bar{\mathbf{T}}_{em} = \frac{1}{2}(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H})\bar{\mathbf{I}} - \mathbf{DE} - \mathbf{BH}, \qquad (2.288)$$

which is called the *Maxwell stress tensor*. Let us demonstrate this equivalence for a linear, isotropic, homogeneous material. Putting $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu$ into (2.287) we obtain

$$\nabla \cdot \mathbf{T}_{em} = -\epsilon \mathbf{E}(\nabla \cdot \mathbf{E}) + \frac{1}{\mu}\mathbf{B} \times (\nabla \times \mathbf{B}) + \epsilon \mathbf{E} \times (\nabla \times \mathbf{E}) - \frac{1}{\mu}\mathbf{B}(\nabla \cdot \mathbf{B}). \qquad (2.289)$$

Now (B.46) gives

$$\nabla(\mathbf{A} \cdot \mathbf{A}) = 2\mathbf{A} \times (\nabla \times \mathbf{A}) + 2(\mathbf{A} \cdot \nabla)\mathbf{A}$$

so that

$$\mathbf{E}(\nabla \cdot \mathbf{E}) - \mathbf{E} \times (\nabla \times \mathbf{E}) = \mathbf{E}(\nabla \cdot \mathbf{E}) + (\mathbf{E} \cdot \nabla)\mathbf{E} - \frac{1}{2}\nabla(E^2).$$

Finally, (B.55) and (B.63) give

$$\mathbf{E}(\nabla \cdot \mathbf{E}) - \mathbf{E} \times (\nabla \times \mathbf{E}) = \nabla \cdot \left(\mathbf{EE} - \frac{1}{2}\bar{\mathbf{I}}\mathbf{E} \cdot \mathbf{E}\right).$$

Substituting this expression and a similar one for $\mathbf{B}$ into (2.289) we have

$$\nabla \cdot \bar{\mathbf{T}}_{em} = \nabla \cdot \left[\frac{1}{2}(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H})\,\bar{\mathbf{I}} - \mathbf{DE} - \mathbf{BH}\right],$$

which matches (2.288).

Replacing the term in brackets in (2.287) by $\nabla \cdot \bar{\mathbf{T}}_{em}$, we get

$$\nabla \cdot \bar{\mathbf{T}}_{em} + \frac{\partial \mathbf{g}_{em}}{\partial t} = -\mathbf{f}_{em} \qquad (2.290)$$

where

$$\mathbf{g}_{em} = \mathbf{D} \times \mathbf{B}.$$

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Equation (2.290) is the point form of the electromagnetic conservation of momentum theorem. It is mathematically identical in form to the mechanical theorem (2.279). Integration over a volume gives the large-scale form

$$\oint_S \bar{\mathbf{T}}_{em} \cdot \mathbf{dS} + \int_V \frac{\partial \mathbf{g}_{em}}{\partial t} \, dV = -\int_V \mathbf{f}_{em} \, dV. \tag{2.291}$$

If we interpret this as we interpreted the conservation theorems from mechanics, the first term on the left-hand side represents the flow of electromagnetic momentum across the boundary of V, while the second term represents the change in momentum within V. The sum of these two quantities is exactly compensated by the total Lorentz force acting on the charges within V. Thus we identify $\mathbf{g}_{em}$ as the transport density of electromagnetic momentum.

Because (2.290) is not zero on the right-hand side, it does not represent a closed system. If the Lorentz force is the only force acting on the charges within V, then the mechanical reaction to the Lorentz force should be described by Newton's third law. Thus we have the kinematic momentum conservation formula

$$\nabla \cdot \bar{\mathbf{T}}_k + \frac{\partial \mathbf{g}_k}{\partial t} = \mathbf{f}_k = -\mathbf{f}_{em}.$$

Subtracting this expression from (2.290) we obtain

$$\nabla \cdot (\bar{\mathbf{T}}_{em} - \bar{\mathbf{T}}_k) + \frac{\partial}{\partial t}(\mathbf{g}_{em} - \mathbf{g}_k) = 0, \tag{2.292}$$

which describes momentum conservation for the closed system.

It is also possible to derive a conservation theorem for electromagnetic energy that resembles the corresponding theorem for mechanical energy. Earlier we noted that $\mathbf{v} \cdot \mathbf{f}$ represents the volume density of work produced by moving an object at velocity $\mathbf{v}$ under the action of a force $\mathbf{f}$. For the electromagnetic subsystem the work is produced by charges moving against the Lorentz force. So the volume density of work delivered to the currents is

$$w_{em} = \mathbf{v} \cdot \mathbf{f}_{em} = \mathbf{v} \cdot (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) = (\rho\mathbf{v}) \cdot \mathbf{E} + \rho\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}). \tag{2.293}$$

Using (B.6) on the second term in (2.293) we get

$$w_{em} = (\rho\mathbf{v}) \cdot \mathbf{E} + \rho\mathbf{B} \cdot (\mathbf{v} \times \mathbf{v}).$$

The second term vanishes by definition of the cross product. This is the familiar property that the magnetic field does no work on moving charge. Hence

$$w_{em} = \mathbf{J} \cdot \mathbf{E}. \tag{2.294}$$

This important relation says that charge moving in an electric field experiences a force which results in energy transfer to (or from) the charge. We wish to write this energy transfer in terms of an energy flux vector, as we did with the mechanical subsystem.

As with our derivation of the conservation of electromagnetic momentum, we wish to relate the energy transfer to the electromagnetic fields. Substitution of $\mathbf{J}$ from (2.2) into (2.294) gives

$$w_{em} = (\nabla \times \mathbf{H}) \cdot \mathbf{E} - \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{E},$$

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hence

$$w_{em} = -\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{E}$$

by (B.44). Substituting for $\nabla \times \mathbf{E}$ from (2.1) we have

$$w_{em} = -\nabla \cdot (\mathbf{E} \times \mathbf{H}) - \left[\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right].$$

This is not quite of the form (2.285) since a single term representing the time rate of change of energy density is not present. However, for a linear isotropic medium in which ϵ and μ do not depend on time (i.e., a nondispersive medium) we have

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2} \epsilon \frac{\partial}{\partial t} (\mathbf{E} \cdot \mathbf{E}) = \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{D} \cdot \mathbf{E}), \tag{2.295}$$

$$\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} = \frac{1}{2} \mu \frac{\partial}{\partial t} (\mathbf{H} \cdot \mathbf{H}) = \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{H} \cdot \mathbf{B}). \tag{2.296}$$

Using this we obtain

$$\nabla \cdot \mathbf{S}_{em} + \frac{\partial}{\partial t} W_{em} = -\mathbf{f}_{em} \cdot \mathbf{v} = -\mathbf{J} \cdot \mathbf{E} \tag{2.297}$$

where

$$W_{em} = \frac{1}{2} (\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H})$$

and

$$\mathbf{S}_{em} = \mathbf{E} \times \mathbf{H}. \tag{2.298}$$

Equation (2.297) is the point form of the energy conservation theorem, also called *Poynting's theorem* after J.H. Poynting who first proposed it. The quantity $\mathbf{S}_{em}$ given in (2.298) is known as the *Poynting vector.* Integrating (2.297) over a volume and using the divergence theorem, we obtain the large-scale form

$$-\int_V \mathbf{J} \cdot \mathbf{E} \, dV = \int_V \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) \, dV + \oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{dS}. \tag{2.299}$$

This also holds for a nondispersive, linear, bianisotropic medium with a symmetric constitutive matrix [101, 185].

We see that the electromagnetic energy conservation theorem (2.297) is identical in form to the mechanical energy conservation theorem (2.285). Thus, if the system is composed of just the kinetic and electromagnetic subsystems, the mechanical force exactly balances the Lorentz force, and (2.297) and (2.285) add to give

$$\nabla \cdot (\mathbf{S}_{em} + \mathbf{S}_k) + \frac{\partial}{\partial t} (W_{em} + W_k) = 0, \tag{2.300}$$

showing that energy is conserved for the entire system.

As in the mechanical system, we identify W_{em} as the volume electromagnetic energy density in V, and $\mathbf{S}_{em}$ as the density of electromagnetic energy flowing across the boundary of V. This interpretation is somewhat controversial, as discussed below.

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Interpretation of the energy and momentum conservation theorems. There has been some controversy regarding Poynting's theorem (and, equally, the momentum conservation theorem). While there is no question that Poynting's theorem is mathematically correct, we may wonder whether we are justified in associating W_{em} with W_k and $\mathbf{S}_{em}$ with $\mathbf{S}_k$ merely because of the similarities in their mathematical expressions. Certainly there is some justification for associating W_k, the kinetic energy of particles, with W_{em}, since we shall show that for static fields the term $\frac{1}{2}(\mathbf{D}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{H})$ represents the energy required to assemble the charges and currents into a certain configuration. However, the term $\mathbf{S}_{em}$ is more problematic. In a mechanical system, $\mathbf{S}_k$ represents the flow of kinetic energy associated with moving particles — does that imply that $\mathbf{S}_{em}$ represents the flow of electromagnetic energy? That is the position generally taken, and it is widely supported by experimental evidence. However, the interpretation is not clear-cut.

If we associate $\mathbf{S}_{em}$ with the flow of electromagnetic energy at a point in space, then we must define what a flow of electromagnetic energy is. We naturally associate the flow of kinetic energy with moving particles; with what do we associate the flow of electromagnetic energy? Maxwell felt that electromagnetic energy must flow through space as a result of the mechanical stresses and strains associated with an unobserved substance called the "aether." A more modern interpretation is that the electromagnetic fields propagate as a wave through space at finite velocity; when those fields encounter a charged particle a force is exerted, work is done, and energy is "transferred" from the field to the particle. Hence the energy flow is associated with the "flow" of the electromagnetic wave.

Unfortunately, it is uncertain whether $\mathbf{E}\times\mathbf{H}$ is the appropriate quantity to associate with this flow, since only its divergence appears in Poynting's theorem. We could add any other term $\mathbf{S}'$ that satisfies $\nabla\cdot\mathbf{S}'=0$ to $\mathbf{S}_{em}$ in (2.297), and the conservation theorem would be unchanged. (Equivalently, we could add to (2.299) any term that integrates to zero over S.) There is no such ambiguity in the mechanical case because kinetic energy is rigorously defined. We are left, then, to postulate that $\mathbf{E}\times\mathbf{H}$ represents the density of energy flow associated with an electromagnetic wave (based on the symmetry with mechanics), and to look to experimental evidence as justification. In fact, experimental evidence does point to the correctness of this hypothesis, and the quantity $\mathbf{E}\times\mathbf{H}$ is widely and accurately used to compute the energy radiated by antennas, carried by waveguides, etc.

Confusion also arises regarding the interpretation of W_{em}. Since this term is so conveniently paired with the mechanical volume kinetic energy density in (2.300) it would seem that we should interpret it as an electromagnetic energy density. As such, we can think of this energy as "localized" in certain regions of space. This viewpoint has been criticized [187, 145, 69] since the large-scale form of energy conservation for a space region only requires that the total energy in the region be specified, and the integrand (energy density) giving this energy is not unique. It is also felt that energy should be associated with a "configuration" of objects (such as charged particles) and not with an arbitrary point in space. However, we retain the concept of localized energy because it is convenient and produces results consistent with experiment.

The validity of extending the static field interpretation of

$$\frac{1}{2}(\mathbf{D}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{H})$$

as the energy "stored" by a charge and a current arrangement to the time-varying case has also been questioned. If we do extend this view to the time-varying case, Poynting's theorem suggests that every point in space somehow has an energy density associated

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with it, and the flow of energy from that point (via $\mathbf{S}_{em}$) must be accompanied by a change in the stored energy at that point. This again gives a very useful and intuitively satisfying point of view. Since we can associate the flow of energy with the propagation of the electromagnetic fields, we can view the fields in any region of space as having the potential to do work on charged particles in that region. If there are charged particles in that region then work is done, accompanied by a transfer of energy to the particles and a reduction in the amplitudes of the fields.

We must also remember that the association of stored electromagnetic energy density W_{em} with the mechanical energy density W_k is only possible if the medium is nondispersive. If we cannot make the assumptions that justify (2.295) and (2.296), then Poynting's theorem must take the form

$$-\int_V \mathbf{J}\cdot\mathbf{E}\,dV = \int_V \left[\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right] dV + \oint_S (\mathbf{E}\times\mathbf{H})\cdot\mathbf{dS}. \tag{2.301}$$

For dispersive media, the volume term on the right-hand side describes not only the stored electromagnetic energy, but also the energy dissipated within the material produced by a time lag between the field applied to the medium and the resulting polarization or magnetization of the atoms. This is clearly seen in (2.29), which shows that $\mathbf{D}(t)$ depends on the value of $\mathbf{E}$ at time t and at all past times. The stored energy and dissipative terms are hard to separate, but we can see that there must always be a stored energy term by substituting $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ and $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$ into (2.301) to obtain

$$\begin{gathered}-\int_V [(\mathbf{J}+\mathbf{J}_P)\cdot\mathbf{E} + \mathbf{J}_H\cdot\mathbf{H}]\,dV = \\ \frac{1}{2}\frac{\partial}{\partial t}\int_V (\epsilon_0\mathbf{E}\cdot\mathbf{E} + \mu_0\mathbf{H}\cdot\mathbf{H})\,dV + \oint_S (\mathbf{E}\times\mathbf{H})\cdot\mathbf{dS}.\end{gathered} \tag{2.302}$$

Here J_P is the equivalent polarization current (2.119) and J_H is an analogous *magnetic polarization current* given by

$$\mathbf{J}_H = \mu_0\frac{\partial\mathbf{M}}{\partial t}.$$

In this form we easily identify the quantity

$$\frac{1}{2}(\epsilon_0\mathbf{E}\cdot\mathbf{E} + \mu_0\mathbf{H}\cdot\mathbf{H})$$

as the electromagnetic energy density for the fields $\mathbf{E}$ and $\mathbf{H}$ *in free space.* Any dissipation produced by polarization and magnetization lag is now handled by the interaction between the fields and equivalent current, just as $\mathbf{J}\cdot\mathbf{E}$ describes the interaction of the electric current (source and secondary) with the electric field. Unfortunately, the equivalent current interaction terms also include the additional stored energy that results from polarizing and magnetizing the material atoms, and again the effects are hard to separate.

Finally, let us consider the case of static fields. Setting the time derivative to zero in (2.299), we have

$$-\int_V \mathbf{J}\cdot\mathbf{E}\,dV = \oint_S (\mathbf{E}\times\mathbf{H})\cdot\mathbf{dS}.$$

This shows that energy flux is required to maintain steady current flow. For instance, we need both an electromagnetic and a thermodynamic subsystem to account for energy conservation in the case of steady current flow through a resistor. The Poynting flux

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describes the electromagnetic energy entering the resistor and the thermodynamic flux describes the heat dissipation. For the sum of the two subsystems conservation of energy requires

$$\nabla \cdot (\mathbf{S}_{em} + \mathbf{S}_{th}) = -\mathbf{J} \cdot \mathbf{E} + P_{th} = 0.$$

To compute the heat dissipation we can use

$$P_{th} = \mathbf{J} \cdot \mathbf{E} = -\nabla \cdot \mathbf{S}_{em}$$

and thus either use the boundary fields or the fields and current internal to the resistor to find the dissipated heat.

Boundary conditions on the Poynting vector. The large-scale form of Poynting's theorem may be used to determine the behavior of the Poynting vector on either side of a boundary surface. We proceed exactly as in § 2.8.2. Consider a surface S across which the electromagnetic sources and constitutive parameters are discontinuous (Figure 2.6). As before, let $\hat{\mathbf{n}}_{12}$ be the unit normal directed into region 1. We now simplify the notation and write $\mathbf{S}$ instead of $\mathbf{S}_{em}$. If we apply Poynting's theorem

$$\int_V \left(\mathbf{J} \cdot \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) dV + \oint_S \mathbf{S} \cdot \mathbf{n}\, dS = 0$$

to the two separate surfaces shown in Figure 2.6, we obtain

$$\int_V \left(\mathbf{J} \cdot \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) dV + \int_S \mathbf{S} \cdot \mathbf{n}\, dS = \int_{S_{10}} \hat{\mathbf{n}}_{12} \cdot (\mathbf{S}_1 - \mathbf{S}_2)\, dS. \tag{2.303}$$

If on the other hand we apply Poynting's theorem to the entire volume region including the surface of discontinuity and include the contribution produced by surface current, we get

$$\int_V \left(\mathbf{J} \cdot \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) dV + \int_S \mathbf{S} \cdot \mathbf{n}\, dS = -\int_{S_{10}} \mathbf{J}_s \cdot \mathbf{E}\, dS. \tag{2.304}$$

Since we are uncertain whether to use $\mathbf{E}_1$ or $\mathbf{E}_2$ in the surface term on the right-hand side, if we wish to have the integrals over V and S in (2.303) and (2.304) produce identical results we must postulate the two conditions

$$\hat{\mathbf{n}}_{12} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$$

and

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{S}_1 - \mathbf{S}_2) = -\mathbf{J}_s \cdot \mathbf{E}. \tag{2.305}$$

The first condition is merely the continuity of tangential electric field as originally postulated in § 2.8.2; it allows us to be nonspecific as to which value of $\mathbf{E}$ we use in the second condition, which is the desired boundary condition on $\mathbf{S}$.

It is interesting to note that (2.305) may also be derived directly from the two postulated boundary conditions on tangential $\mathbf{E}$ and $\mathbf{H}$. Here we write with the help of (B.6)

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{S}_1 - \mathbf{S}_2) = \hat{\mathbf{n}}_{12} \cdot (\mathbf{E}_1 \times \mathbf{H}_1 - \mathbf{E}_2 \times \mathbf{H}_2) = \mathbf{H}_1 \cdot (\hat{\mathbf{n}}_{12} \times \mathbf{E}_1) - \mathbf{H}_2 \cdot (\hat{\mathbf{n}}_{12} \times \mathbf{E}_2).$$

Since $\hat{\mathbf{n}}_{12} \times \mathbf{E}_1 = \hat{\mathbf{n}}_{12} \times \mathbf{E}_2 = \hat{\mathbf{n}}_{12} \times \mathbf{E}$, we have

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{S}_1 - \mathbf{S}_2) = (\mathbf{H}_1 - \mathbf{H}_2) \cdot (\hat{\mathbf{n}}_{12} \times \mathbf{E}) = [-\hat{\mathbf{n}}_{12} \times (\mathbf{H}_1 - \mathbf{H}_2)] \cdot \mathbf{E}.$$

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Finally, using $\hat{\mathbf{n}}_{12} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$ we arrive at (2.305).

The arguments above suggest an interesting way to look at the boundary conditions. Once we identify $\mathbf{S}$ with the flow of electromagnetic energy, we may consider the condition on normal $\mathbf{S}$ as a fundamental statement of the conservation of energy. This statement implies continuity of tangential $\mathbf{E}$ in order to have an unambiguous interpretation for the meaning of the term $\mathbf{J}_s \cdot \mathbf{E}$. Then, with continuity of tangential $\mathbf{E}$ established, we can derive the condition on tangential $\mathbf{H}$ directly.

An alternative formulation of the conservation theorems. As we saw in the paragraphs above, our derivation of the conservation theorems lacks strong motivation. We manipulated Maxwell's equations until we found expressions that resembled those for mechanical momentum and energy, but in the process found that the validity of the expressions is somewhat limiting. For instance, we needed to assume a linear, homogeneous, bianisotropic medium in order to identify the Maxwell stress tensor (2.288) and the energy densities in Poynting's theorem (2.299). In the end, we were reduced to postulating the meaning of the individual terms in the conservation theorems in order for the whole to have meaning.

An alternative approach is popular in physics. It involves postulating a single Lagrangian density function for the electromagnetic field, and then applying the stationary property of the action integral. The results are precisely the same conservation expressions for linear momentum and energy as obtained from manipulating Maxwell's equations (plus the equation for conservation of angular momentum), obtained with fewer restrictions regarding the constitutive relations. This process also separates the stored energy, Maxwell stress tensor, momentum density, and Poynting vector as natural components of a tensor equation, allowing a better motivated interpretation of the meaning of these components. Since this approach is also a powerful tool in mechanics, its application is more strongly motivated than merely manipulating Maxwell's equations. Of course, some knowledge of the structure of the electromagnetic field is required to provide an appropriate postulate of the Lagrangian density. Interested readers should consult Kong [101], Jackson [91], Doughty [57], or Tolstoy [198].

2.10 The wave nature of the electromagnetic field

Throughout this chapter our goal has been a fundamental understanding of Maxwell's theory of electromagnetics. We have concentrated on developing and understanding the equations relating the field quantities, but have done little to understand the nature of the field itself. We would now like to investigate, in a very general way, the behavior of the field. We shall not attempt to solve a vast array of esoteric problems, but shall instead concentrate on a few illuminating examples.

The electromagnetic field can take on a wide variety of characteristics. Static fields differ qualitatively from those which undergo rapid time variations. Time-varying fields exhibit wave behavior and carry energy away from their sources. In the case of slow time variation this wave nature may often be neglected in favor of the nearby coupling of sources we know as the inductance effect, hence circuit theory may suffice to describe the field-source interaction. In the case of extremely rapid oscillations, particle concepts may be needed to describe the field.

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The dynamic coupling between the various field vectors in Maxwell's equations provides a means of characterizing the field. Static fields are characterized by decoupling of the electric and magnetic fields. Quasistatic fields exhibit some coupling, but the wave characteristic of the field is ignored. Tightly coupled fields are dominated by the wave effect, but may still show a static-like spatial distribution near the source. Any such "near-zone" effects are generally ignored for fields at light-wave frequencies, and the particle nature of light must often be considered.

2.10.1 Electromagnetic waves

An early result of Maxwell's theory was the prediction and later verification by Heinrich Hertz of the existence of electromagnetic waves. We now know that nearly any time-varying source produces waves, and that these waves have certain important properties. An electromagnetic wave is a propagating electromagnetic field that travels with finite velocity as a disturbance through a medium. The field itself is the disturbance, rather than merely representing a physical displacement or other effect on the medium. This fact is fundamental for understanding how electromagnetic waves can travel through a true vacuum. Many specific characteristics of the wave, such as velocity and polarization, depend on the properties of the medium through which it propagates. The evolution of the disturbance also depends on these properties: we say that a material exhibits "dispersion" if the disturbance undergoes a change in its temporal behavior as the wave progresses. As waves travel they carry energy and momentum away from their source. This energy may be later returned to the source or delivered to some distant location. Waves are also capable of transferring energy to, or withdrawing energy from, the medium through which they propagate. When energy is carried outward from the source never to return, we refer to the process as "electromagnetic radiation." The effects of radiated fields can be far-reaching; indeed, radio astronomers observe waves that originated at the very edges of the universe.

Light is an electromagnetic phenomenon, and many of the familiar characteristics of light that we recognize from our everyday experience may be applied to all electromagnetic waves. For instance, radio waves bend (or "refract") in the ionosphere much as light waves bend while passing through a prism. Microwaves reflect from conducting surfaces in the same way that light waves reflect from a mirror; detecting these reflections forms the basis of radar. Electromagnetic waves may also be "confined" by reflecting boundaries to form waves standing in one or more directions. With this concept we can use waveguides or transmission lines to guide electromagnetic energy from spot to spot, or to concentrate it in the cavity of a microwave oven.

The manifestations of electromagnetic waves are so diverse that no one book can possibly describe the entire range of phenomena or application. In this section we shall merely introduce the reader to some of the most fundamental concepts of electromagnetic wave behavior. In the process we shall also introduce the three most often studied types of traveling electromagnetic waves: plane waves, spherical waves, and cylindrical waves. In later sections we shall study some of the complicated interactions of these waves with objects and boundaries, in the form of guided waves and scattering problems.

Mathematically, electromagnetic waves arise as a subset of solutions to Maxwell's equations. These solutions obey the electromagnetic "wave equation," which may be derived from Maxwell's equations under certain circumstances. Not all electromagnetic fields satisfy the wave equation. Obviously, time-invariant fields cannot represent evolving wave disturbances, and must obey the static field equations. Time-varying fields in cer-

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tain metals may obey the diffusion equation rather than the wave equation, and must thereby exhibit different behavior. In the study of quasistatic fields we often ignore the displacement current term in Maxwell's equations, producing solutions that are most important near the sources of the fields and having little associated radiation. When the displacement term is significant we produce solutions with the properties of waves.

2.10.2 Wave equation for bianisotropic materials

In deriving electromagnetic wave equations we transform the first-order coupled partial differential equations we know as Maxwell's equations into uncoupled second-order equations. That is, we perform a set of operations (and make appropriate assumptions) to reduce the set of four differential equations in the four unknown fields **E**, **D**, **B**, and **H**, into a set of differential equations each involving a single unknown (usually **E** or **H**). It is possible to derive wave equations for **E** and **H** even for the most general cases of inhomogeneous, bianisotropic media, as long as the constitutive parameters $\bar{\boldsymbol{\mu}}$ and $\bar{\boldsymbol{\xi}}$ are constant with time. Substituting the constitutive relations (2.19)–(2.20) into the Maxwell–Minkowski curl equations (2.169)–(2.170) we get

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\bar{\boldsymbol{\zeta}} \cdot \mathbf{E} + \bar{\boldsymbol{\mu}} \cdot \mathbf{H}) - \mathbf{J}_m, \tag{2.306}$$

$$\nabla \times \mathbf{H} = \frac{\partial}{\partial t}(\bar{\epsilon} \cdot \mathbf{E} + \bar{\boldsymbol{\xi}} \cdot \mathbf{H}) + \mathbf{J}. \tag{2.307}$$

Separate equations for **E** and **H** are facilitated by introducing a new dyadic operator $\bar{\nabla}$, which when dotted with a vector field **V** gives the curl:

$$\bar{\nabla} \cdot \mathbf{V} = \nabla \times \mathbf{V}. \tag{2.308}$$

It is easy to verify that in rectangular coordinates $\bar{\nabla}$ is

$$[\bar{\nabla}] = \begin{bmatrix} 0 & -\partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & -\partial/\partial x \\ -\partial/\partial y & \partial/\partial x & 0 \end{bmatrix}.$$

With this notation, Maxwell's curl equations (2.306)–(2.307) become simply

$$\left(\bar{\nabla} + \frac{\partial}{\partial t}\bar{\zeta}\right) \cdot \mathbf{E} = -\frac{\partial}{\partial t}\bar{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{J}_m, \tag{2.309}$$

$$\left(\bar{\nabla} - \frac{\partial}{\partial t}\bar{\boldsymbol{\xi}}\right) \cdot \mathbf{H} = \frac{\partial}{\partial t}\bar{\epsilon} \cdot \mathbf{E} + \mathbf{J}. \tag{2.310}$$

Obtaining separate equations for **E** and **H** is straightforward. Defining the inverse dyadic $\bar{\boldsymbol{\mu}}^{-1}$ through

$$\bar{\boldsymbol{\mu}} \cdot \bar{\boldsymbol{\mu}}^{-1} = \bar{\boldsymbol{\mu}}^{-1} \cdot \bar{\boldsymbol{\mu}} = \bar{\mathbf{I}},$$

we can write (2.309) as

$$\frac{\partial}{\partial t}\mathbf{H} = -\bar{\boldsymbol{\mu}}^{-1} \cdot \left(\bar{\nabla} + \frac{\partial}{\partial t}\bar{\zeta}\right) \cdot \mathbf{E} - \bar{\boldsymbol{\mu}}^{-1} \cdot \mathbf{J}_m \tag{2.311}$$

where we have assumed that $\bar{\boldsymbol{\mu}}$ is independent of time. Assuming that $\bar{\boldsymbol{\xi}}$ is also independent of time, we can differentiate (2.310) with respect to time to obtain

$$\left(\bar{\nabla} - \frac{\partial}{\partial t}\bar{\boldsymbol{\xi}}\right) \cdot \frac{\partial \mathbf{H}}{\partial t} = \frac{\partial^2}{\partial t^2}(\bar{\epsilon} \cdot \mathbf{E}) + \frac{\partial \mathbf{J}}{\partial t}.$$

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Substituting $\partial\mathbf{H}/\partial t$ from (2.311) and rearranging, we get

$$\left[\left(\bar{\nabla} - \frac{\partial}{\partial t}\bar{\boldsymbol{\xi}}\right)\cdot\bar{\boldsymbol{\mu}}^{-1}\cdot\left(\bar{\nabla} + \frac{\partial}{\partial t}\bar{\zeta}\right) + \frac{\partial^2}{\partial t^2}\bar{\epsilon}\right]\cdot\mathbf{E} = -\left(\bar{\nabla} - \frac{\partial}{\partial t}\bar{\boldsymbol{\xi}}\right)\cdot\bar{\boldsymbol{\mu}}^{-1}\cdot\mathbf{J}_m - \frac{\partial\mathbf{J}}{\partial t}. \tag{2.312}$$

This is the general wave equation for $\mathbf{E}$. Using an analogous set of steps, and assuming $\bar{\boldsymbol{\epsilon}}$ and $\bar{\boldsymbol{\zeta}}$ are independent of time, we can find

$$\left[\left(\bar{\nabla} + \frac{\partial}{\partial t}\bar{\zeta}\right)\cdot\bar{\boldsymbol{\epsilon}}^{-1}\cdot\left(\bar{\nabla} - \frac{\partial}{\partial t}\bar{\boldsymbol{\xi}}\right) + \frac{\partial^2}{\partial t^2}\bar{\boldsymbol{\mu}}\right]\cdot\mathbf{H} = \left(\bar{\nabla} + \frac{\partial}{\partial t}\bar{\zeta}\right)\cdot\bar{\boldsymbol{\epsilon}}^{-1}\cdot\mathbf{J} - \frac{\partial\mathbf{J}_m}{\partial t}. \tag{2.313}$$

This is the wave equation for $\mathbf{H}$. The case in which the constitutive parameters are time-dependent will be handled using frequency domain techniques in later chapters.

Wave equations for anisotropic, isotropic, and homogeneous media are easily obtained from (2.312) and (2.313) as special cases. For example, the wave equations for a homogeneous, isotropic medium can be found by setting $\bar{\boldsymbol{\zeta}} = \bar{\boldsymbol{\xi}} = 0$, $\bar{\boldsymbol{\mu}} = \mu\bar{\mathbf{I}}$, and $\bar{\boldsymbol{\epsilon}} = \epsilon\bar{\mathbf{I}}$:

$$\frac{1}{\mu}\bar{\nabla}\cdot(\bar{\nabla}\cdot\mathbf{E}) + \epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} = -\frac{1}{\mu}\bar{\nabla}\cdot\mathbf{J}_m - \frac{\partial\mathbf{J}}{\partial t},$$

$$\frac{1}{\epsilon}\bar{\nabla}\cdot(\bar{\nabla}\cdot\mathbf{H}) + \mu\frac{\partial^2\mathbf{H}}{\partial t^2} = \frac{1}{\epsilon}\bar{\nabla}\cdot\mathbf{J} - \frac{\partial\mathbf{J}_m}{\partial t}.$$

Returning to standard curl notation we find that these become

$$\nabla\times(\nabla\times\mathbf{E}) + \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} = -\nabla\times\mathbf{J}_m - \mu\frac{\partial\mathbf{J}}{\partial t}, \tag{2.314}$$

$$\nabla\times(\nabla\times\mathbf{H}) + \mu\epsilon\frac{\partial^2\mathbf{H}}{\partial t^2} = \nabla\times\mathbf{J} - \epsilon\frac{\partial\mathbf{J}_m}{\partial t}. \tag{2.315}$$

In each of the wave equations it appears that operations on the electromagnetic fields have been separated from operations on the source terms. However, we have not yet invoked any coupling between the fields and sources associated with secondary interactions. That is, we need to separate the impressed sources, which are independent of the fields they source, with secondary sources resulting from interactions between the sourced fields and the medium in which the fields exist. The simple case of an isotropic conducting medium will be discussed below.

Wave equation using equivalent sources. An alternative approach for studying wave behavior in general media is to use the Maxwell–Boffi form of the field equations

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{2.316}$$

$$\nabla\times\frac{\mathbf{B}}{\mu_0} = (\mathbf{J} + \mathbf{J}_M + \mathbf{J}_P) + \frac{\partial\epsilon_0\mathbf{E}}{\partial t}, \tag{2.317}$$

$$\nabla\cdot(\epsilon_0\mathbf{E}) = (\rho + \rho_P), \tag{2.318}$$

$$\nabla\cdot\mathbf{B} = 0. \tag{2.319}$$

Taking the curl of (2.316) we have

$$\nabla\times(\nabla\times\mathbf{E}) = -\frac{\partial}{\partial t}\nabla\times\mathbf{B}.$$

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Substituting for $\nabla \times \mathbf{B}$ from (2.317) we then obtain

$$\nabla \times (\nabla \times \mathbf{E}) + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu_0 \frac{\partial}{\partial t}(\mathbf{J} + \mathbf{J}_M + \mathbf{J}_P), \tag{2.320}$$

which is the wave equation for $\mathbf{E}$. Taking the curl of (2.317) and substituting from (2.316) we obtain the wave equation

$$\nabla \times (\nabla \times \mathbf{B}) + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = \mu_0 \nabla \times (\mathbf{J} + \mathbf{J}_M + \mathbf{J}_P) \tag{2.321}$$

for $\mathbf{B}$. Solution of the wave equations is often facilitated by writing the curl-curl operation in terms of the vector Laplacian. Using (B.47), and substituting for the divergence from (2.318) and (2.319), we can write the wave equations as

$$\nabla^2 \mathbf{E} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\epsilon_0} \nabla(\rho + \rho_P) + \mu_0 \frac{\partial}{\partial t}(\mathbf{J} + \mathbf{J}_M + \mathbf{J}_P), \tag{2.322}$$

$$\nabla^2 \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = -\mu_0 \nabla \times (\mathbf{J} + \mathbf{J}_M + \mathbf{J}_P). \tag{2.323}$$

The simplicity of these equations relative to (2.312) and (2.313) is misleading. We have not considered the constitutive equations relating the polarization $\mathbf{P}$ and magnetization $\mathbf{M}$ to the fields, nor have we considered interactions leading to secondary sources.

2.10.3 Wave equation in a conducting medium

As an example of the type of wave equation that arises when secondary sources are included, consider a homogeneous isotropic conducting medium described by permittivity ϵ, permeability μ, and conductivity σ. In a conducting medium we must separate the source field into a causative impressed term $\mathbf{J}^i$ that is independent of the fields it sources, and a secondary term $\mathbf{J}^s$ that is an effect of the sourced fields. In an isotropic conducting medium the effect is described by Ohm's law $\mathbf{J}^s = \sigma \mathbf{E}$. Writing the total current as $\mathbf{J} = \mathbf{J}^i + \mathbf{J}^s$, and assuming that $\mathbf{J}_m = 0$, we write the wave equation (2.314) as

$$\nabla \times (\nabla \times \mathbf{E}) + \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu \frac{\partial(\mathbf{J}^i + \sigma \mathbf{E})}{\partial t}. \tag{2.324}$$

Using (B.47) and substituting $\nabla \cdot \mathbf{E} = \rho/\epsilon$, we can write the wave equation for $\mathbf{E}$ as

$$\nabla^2 \mathbf{E} - \mu\sigma \frac{\partial \mathbf{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu \frac{\partial \mathbf{J}^i}{\partial t} + \frac{1}{\epsilon} \nabla \rho. \tag{2.325}$$

Substituting $\mathbf{J} = \mathbf{J}^i + \sigma \mathbf{E}$ into (2.315) and using (B.47), we obtain

$$\nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} + \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = \nabla \times \mathbf{J}^i + \sigma \nabla \times \mathbf{E}.$$

Since $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ and $\nabla \cdot \mathbf{H} = \nabla \cdot \mathbf{B}/\mu = 0$, we have

$$\nabla^2 \mathbf{H} - \mu\sigma \frac{\partial \mathbf{H}}{\partial t} - \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = -\nabla \times \mathbf{J}^i. \tag{2.326}$$

This is the wave equation for $\mathbf{H}$.

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2.10.4 Scalar wave equation for a conducting medium

In many applications, particularly those involving planar boundary surfaces, it is convenient to decompose the vector wave equation into cartesian components. Using $\nabla^2 \mathbf{V} = \hat{\mathbf{x}}\nabla^2 V_x + \hat{\mathbf{y}}\nabla^2 V_y + \hat{\mathbf{z}}\nabla^2 V_z$ in (2.325) and in (2.326), we find that the rectangular components of **E** and **H** must obey the scalar wave equation

$$\nabla^2 \psi(\mathbf{r}, t) - \mu\sigma \frac{\partial \psi(\mathbf{r}, t)}{\partial t} - \mu\epsilon \frac{\partial^2 \psi(\mathbf{r}, t)}{\partial t^2} = s(\mathbf{r}, t). \tag{2.327}$$

For the electric field wave equation we have

$$\psi = E_\alpha, \qquad s = \mu \frac{\partial J^i_\alpha}{\partial t} + \frac{1}{\epsilon} \hat{\boldsymbol{\alpha}} \cdot \nabla \rho,$$

where $\alpha = x, y, z$. For the magnetic field wave equations we have

$$\psi = H_\alpha, \qquad s = \hat{\boldsymbol{\alpha}} \cdot (-\nabla \times \mathbf{J}^i).$$

2.10.5 Fields determined by Maxwell's equations vs. fields determined by the wave equation

Although we derive the wave equations directly from Maxwell's equations, we may wonder whether the solutions to second-order differential equations such as (2.314)–(2.315) are necessarily the same as the solutions to the first-order Maxwell equations. Hansen and Yaghjian [81] show that if all information about the fields is supplied by the sources $\mathbf{J}(\mathbf{r}, t)$ and $\rho(\mathbf{r}, t)$, rather than by specification of field values on boundaries, the solutions to Maxwell's equations and the wave equations are equivalent as long as the second derivatives of the quantities

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) - \rho(\mathbf{r}, t)/\epsilon, \qquad \nabla \cdot \mathbf{H}(\mathbf{r}, t),$$

are continuous functions of $\mathbf{r}$ and t. If boundary values are supplied in an attempt to guarantee uniqueness, then solutions to the wave equation and to Maxwell's equations may differ. This is particularly important when comparing numerical solutions obtained directly from Maxwell's equations (using the FDTD method, say) to solutions obtained from the wave equation. "Spurious" solutions having no physical significance are a continual plague for engineers who employ numerical techniques. The interested reader should see Jiang [94].

We note that these conclusions do not hold for static fields. The conditions for equivalence of the first-order and second-order static field equations are considered in § 3.2.4.

2.10.6 Transient uniform plane waves in a conducting medium

We can learn a great deal about the wave nature of the electromagnetic field by solving the wave equation (2.325) under simple circumstances. In Chapter 5 we shall solve for the field produced by an arbitrary distribution of impressed sources, but here we seek a simple solution to the homogeneous form of the equation. This allows us to study the phenomenology of wave propagation without worrying about the consequences of specific source functions. We shall also assume a high degree of symmetry so that we are not bogged down in details about the vector directions of the field components.

We seek a solution of the wave equation in which the fields are invariant over a chosen planar surface. The resulting fields are said to comprise a *uniform plane wave*. Although

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we can envision a uniform plane wave as being created by a uniform surface source of doubly-infinite extent, plane waves are also useful as models for spherical waves over localized regions of the wavefront.

We choose the plane of field invariance to be the xy-plane and later generalize the resulting solution to any planar surface by a simple rotation of the coordinate axes. Since the fields vary with z only we choose to write the wave equation (2.325) in rectangular coordinates, giving for a source-free region of space[4]

$$\hat{\mathbf{x}}\frac{\partial^2 E_x(z,t)}{\partial z^2} + \hat{\mathbf{y}}\frac{\partial^2 E_y(z,t)}{\partial z^2} + \hat{\mathbf{z}}\frac{\partial^2 E_z(z,t)}{\partial z^2} - \mu\sigma\frac{\partial \mathbf{E}(z,t)}{\partial t} - \mu\epsilon\frac{\partial^2 \mathbf{E}(z,t)}{\partial t^2} = 0. \tag{2.328}$$

If we return to Maxwell's equations, we soon find that not all components of $\mathbf{E}$ are present in the plane-wave solution. Faraday's law states that

$$\nabla \times \mathbf{E}(z,t) = -\hat{\mathbf{x}}\frac{\partial E_y(z,t)}{\partial z} + \hat{\mathbf{y}}\frac{\partial E_x(z,t)}{\partial z} = \hat{\mathbf{z}} \times \frac{\partial \mathbf{E}(z,t)}{\partial z} = -\mu\frac{\partial \mathbf{H}(z,t)}{\partial t}. \tag{2.329}$$

We see that $\partial H_z/\partial t = 0$, hence H_z must be constant with respect to time. Because a nonzero constant field component would not exhibit wave-like behavior, we can only have $H_z = 0$ in our wave solution. Similarly, Ampere's law in a homogeneous conducting region free from impressed sources states that

$$\nabla \times \mathbf{H}(z,t) = \mathbf{J} + \frac{\partial \mathbf{D}(z,t)}{\partial t} = \sigma\mathbf{E}(z,t) + \epsilon\frac{\partial \mathbf{E}(z,t)}{\partial t}$$

or

$$-\hat{\mathbf{x}}\frac{\partial H_y(z,t)}{\partial z} + \hat{\mathbf{y}}\frac{\partial H_x(z,t)}{\partial z} = \hat{\mathbf{z}} \times \frac{\partial \mathbf{H}(z,t)}{\partial z} = \sigma\mathbf{E}(z,t) + \epsilon\frac{\partial \mathbf{E}(z,t)}{\partial t}. \tag{2.330}$$

This implies that

$$\sigma E_z(z,t) + \epsilon\frac{\partial E_z(z,t)}{\partial t} = 0,$$

which is a differential equation for E_z with solution

$$E_z(z,t) = E_0(z)\, e^{-\frac{\sigma}{\epsilon}t}.$$

Since we are interested only in wave-type solutions, we choose $E_z = 0$.

Hence $E_z = H_z = 0$, and thus both $\mathbf{E}$ and $\mathbf{H}$ are perpendicular to the z-direction. Using (2.329) and (2.330), we also see that

$$\begin{aligned}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{H}) &= \mathbf{E}\cdot\frac{\partial \mathbf{H}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{E}}{\partial t}\\ &= -\frac{1}{\mu}\mathbf{E}\cdot\left(\hat{\mathbf{z}} \times \frac{\partial \mathbf{E}}{\partial z}\right) - \mathbf{H}\cdot\left(\frac{\sigma}{\epsilon}\mathbf{E}\right) + \frac{1}{\epsilon}\mathbf{H}\cdot\left(\hat{\mathbf{z}} \times \frac{\partial \mathbf{H}}{\partial z}\right)\end{aligned}$$

or

$$\left(\frac{\partial}{\partial t} + \frac{\sigma}{\epsilon}\right)(\mathbf{E}\cdot\mathbf{H}) = \frac{1}{\mu}\hat{\mathbf{z}}\cdot\left(\mathbf{E} \times \frac{\partial \mathbf{E}}{\partial z}\right) - \frac{1}{\epsilon}\hat{\mathbf{z}}\cdot\left(\mathbf{H} \times \frac{\partial \mathbf{H}}{\partial z}\right).$$

We seek solutions of the type $\mathbf{E}(z,t) = \hat{\mathbf{p}}E(z,t)$ and $\mathbf{H}(z,t) = \hat{\mathbf{q}}H(z,t)$, where $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ are constant unit vectors. Under this condition we have $\mathbf{E} \times \partial\mathbf{E}/\partial z = 0$ and $\mathbf{H} \times \partial\mathbf{H}/\partial z = 0$, giving

$$\left(\frac{\partial}{\partial t} + \frac{\sigma}{\epsilon}\right)(\mathbf{E}\cdot\mathbf{H}) = 0.$$

[4]The term "source free" applied to a conducting region implies that the region is devoid of impressed sources *and*, because of the relaxation effect, has no free charge. See the discussion in Jones [97].

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Thus we also have $\mathbf{E} \cdot \mathbf{H} = 0$, and find that $\mathbf{E}$ must be perpendicular to $\mathbf{H}$. So $\mathbf{E}$, $\mathbf{H}$, and $\hat{\mathbf{z}}$ comprise a mutually orthogonal triplet of vectors. A wave having this property is said to be *TEM to the z-direction* or simply TEM_z. Here "TEM" stands for *transverse electromagnetic*, indicating the orthogonal relationship between the field vectors and the z-direction. Note that

$$\hat{\mathbf{p}} \times \hat{\mathbf{q}} = \pm\hat{\mathbf{z}}.$$

The constant direction described by $\hat{\mathbf{p}}$ is called the *polarization* of the plane wave.

We are now ready to solve the source-free wave equation (2.328). If we dot both sides of the homogeneous expression by $\hat{\mathbf{p}}$ we obtain

$$\hat{\mathbf{p}} \cdot \hat{\mathbf{x}} \frac{\partial^2 E_x}{\partial z^2} + \hat{\mathbf{p}} \cdot \hat{\mathbf{y}} \frac{\partial^2 E_y}{\partial z^2} - \mu\sigma \frac{\partial(\hat{\mathbf{p}} \cdot \mathbf{E})}{\partial t} - \mu\epsilon \frac{\partial^2(\hat{\mathbf{p}} \cdot \mathbf{E})}{\partial t^2} = 0.$$

Noting that

$$\hat{\mathbf{p}} \cdot \hat{\mathbf{x}} \frac{\partial^2 E_x}{\partial z^2} + \hat{\mathbf{p}} \cdot \hat{\mathbf{y}} \frac{\partial^2 E_y}{\partial z^2} = \frac{\partial^2}{\partial z^2}(\hat{\mathbf{p}} \cdot \hat{\mathbf{x}} E_x + \hat{\mathbf{p}} \cdot \hat{\mathbf{y}} E_y) = \frac{\partial^2}{\partial z^2}(\hat{\mathbf{p}} \cdot \mathbf{E}),$$

we have the wave equation

$$\frac{\partial^2 E(z,t)}{\partial z^2} - \mu\sigma \frac{\partial E(z,t)}{\partial t} - \mu\epsilon \frac{\partial^2 E(z,t)}{\partial t^2} = 0. \tag{2.331}$$

Similarly, dotting both sides of (2.326) with $\hat{\mathbf{q}}$ and setting $\mathbf{J}^i = 0$ we obtain

$$\frac{\partial^2 H(z,t)}{\partial z^2} - \mu\sigma \frac{\partial H(z,t)}{\partial t} - \mu\epsilon \frac{\partial^2 H(z,t)}{\partial t^2} = 0. \tag{2.332}$$

In a source-free homogeneous conducting region $\mathbf{E}$ and $\mathbf{H}$ satisfy identical wave equations.

Solutions are considered in § A.1. There we solve for the total field for all z, t given the value of the field and its derivative over the $z = 0$ plane. This solution can be directly applied to find the total field of a plane wave reflected by a perfect conductor. Let us begin by considering the lossless case where $\sigma = 0$, and assuming the region $z < 0$ contains a perfect electric conductor. The conditions on the field in the $z = 0$ plane are determined by the required boundary condition on a perfect conductor: the tangential electric field must vanish. From (2.330) we see that since $\mathbf{E} \perp \hat{\mathbf{z}}$, requiring

$$\left.\frac{\partial H(z,t)}{\partial z}\right|_{z=0} = 0 \tag{2.333}$$

gives $\mathbf{E}(0,t) = 0$ and thus satisfies the boundary condition. Writing

$$H(0,t) = H_0 f(t), \qquad \left.\frac{\partial H(z,t)}{\partial z}\right|_{z=0} = H_0 g(t) = 0, \tag{2.334}$$

and setting $\Omega = 0$ in (A.41) we obtain the solution to (2.332):

$$H(z,t) = \frac{H_0}{2} f\left(t - \frac{z}{v}\right) + \frac{H_0}{2} f\left(t + \frac{z}{v}\right), \tag{2.335}$$

where $v = 1/(\mu\epsilon)^{1/2}$. Since we designate the vector direction of $\mathbf{H}$ as $\hat{\mathbf{q}}$, the vector field is

$$\mathbf{H}(z,t) = \hat{\mathbf{q}} \frac{H_0}{2} f\left(t - \frac{z}{v}\right) + \hat{\mathbf{q}} \frac{H_0}{2} f\left(t + \frac{z}{v}\right). \tag{2.336}$$

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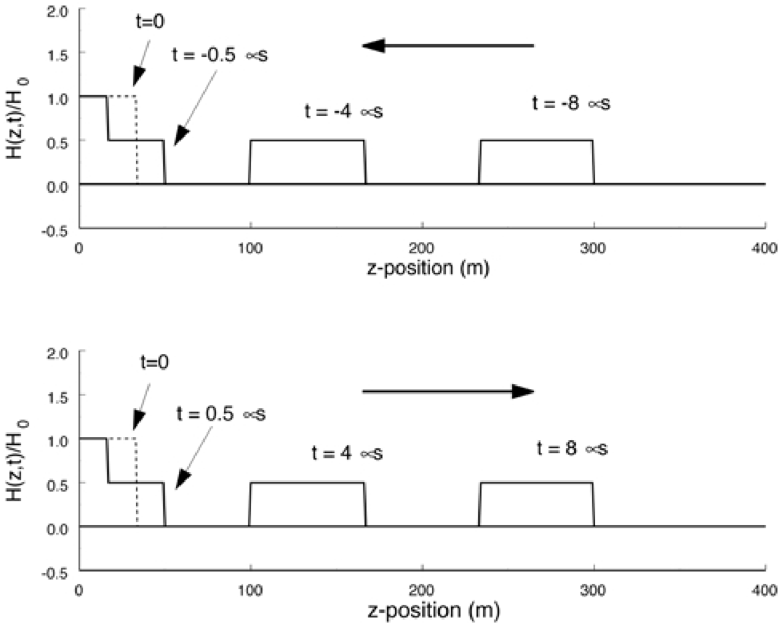

Figure 2.7: Propagation of a transient plane wave in a lossless medium.

From (2.329) we also have the solution for $\mathbf{E}(z,t)$:

$$\mathbf{E}(z,t) = \hat{\mathbf{p}}\frac{v\mu H_0}{2} f\left(t - \frac{z}{v}\right) - \hat{\mathbf{p}}\frac{v\mu H_0}{2} f\left(t + \frac{z}{v}\right), \tag{2.337}$$

where

$$\hat{\mathbf{p}} \times \hat{\mathbf{q}} = \hat{\mathbf{z}}.$$

The boundary conditions $E(0,t) = 0$ and $H(0,t) = H_0 f(t)$ are easily verified by substitution.

This solution displays the quintessential behavior of electromagnetic waves. We may interpret the term $f(t + z/v)$ as a wave field disturbance, propagating at velocity v in the $-z$-direction, incident from $z > 0$ upon the conductor. The term $f(t - z/v)$ represents a wave field disturbance propagating in the $+z$-direction with velocity v, reflected from the conductor. By "propagating" we mean that if we increment time, the disturbance will occupy a spatial position determined by incrementing z by vt. For free space where $v = 1/(\mu_0\epsilon_0)^{1/2}$, the velocity of propagation is the speed of light c.

A specific example should serve to clarify our interpretation of the wave solution. Taking $\mu = \mu_0$ and $\epsilon = 81\epsilon_0$, representing typical constitutive values for fresh water, we can plot (2.335) as a function of position for fixed values of time. The result is shown in Figure 2.7, where we have chosen

$$f(t) = \mathrm{rect}(t/\tau) \tag{2.338}$$

with $\tau = 1$ μs. We see that the disturbance is spatially distributed as a rectangular pulse of extent $L = 2v\tau = 66.6$ m, where $v = 3.33 \times 10^7$ m/s is the wave velocity,

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and where 2τ is the temporal duration of the pulse. At $t = -8\ \mu$s the leading edge of the pulse is at $z = 233$ m, while at $-4\ \mu$s the pulse has traveled a distance $z = vt = (3.33 \times 10^7) \times (4 \times 10^{-6}) = 133$ m in the $-z$-direction, and the leading edge is thus at 100 m. At $t = -1\ \mu$s the leading edge strikes the conductor and begins to induce a current in the conductor surface. This current sets up the reflected wave, which begins to travel in the opposite ($+z$) direction. At $t = -0.5\ \mu$s a portion of the wave has begun to travel in the $+z$-direction while the trailing portion of the disturbance continues to travel in the $-z$-direction. At $t = 1\ \mu$s the wave has been completely reflected from the surface, and thus consists only of the component traveling in the $+z$-direction. Note that if we plot the total field in the $z = 0$ plane, the sum of the forward and backward traveling waves produces the pulse waveform (2.338) as expected.

Using the expressions for $\mathbf{E}$ and $\mathbf{H}$ we can determine many interesting characteristics of the wave. We see that the terms $f(t \pm z/v)$ represent the components of the waves traveling in the $\mp z$-directions, respectively. If we were to isolate these waves from each other (by, for instance, measuring them as functions of time at a position where they do not overlap) we would find from (2.336) and (2.337) that the ratio of E to H for a wave traveling in either direction is

$$\left|\frac{E(z,t)}{H(z,t)}\right| = v\mu = (\mu/\epsilon)^{1/2},$$

independent of the time and position of the measurement. This ratio, denoted by η and carrying units of ohms, is called the *intrinsic impedance* of the medium through which the wave propagates. Thus, if we let $E_0 = \eta H_0$ we can write

$$\mathbf{E}(z,t) = \hat{\mathbf{p}}\frac{E_0}{2} f\left(t - \frac{z}{v}\right) - \hat{\mathbf{p}}\frac{E_0}{2} f\left(t + \frac{z}{v}\right). \tag{2.339}$$

We can easily determine the current induced in the conductor by applying the boundary condition (2.200):

$$\mathbf{J}_s = \hat{\mathbf{n}} \times \mathbf{H}|_{z=0} = \hat{\mathbf{z}} \times [H_0 \hat{\mathbf{q}} f(t)] = -\hat{\mathbf{p}} H_0 f(t). \tag{2.340}$$

We can also determine the pressure exerted on the conductor due to the Lorentz force interaction between the fields and the induced current. The total force on the conductor can be computed by integrating the Maxwell stress tensor (2.288) over the xy-plane[5]:

$$\mathbf{F}_{em} = -\int_S \bar{\mathbf{T}}_{em} \cdot \mathbf{dS}.$$

The surface traction is

$$\mathbf{t} = \bar{\mathbf{T}}_{em} \cdot \hat{\mathbf{n}} = \left[\frac{1}{2}(\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H})\bar{\mathbf{I}} - \mathbf{DE} - \mathbf{BH}\right] \cdot \hat{\mathbf{z}}.$$

Since $\mathbf{E}$ and $\mathbf{H}$ are both normal to $\hat{\mathbf{z}}$, the last two terms in this expression are zero. Also, the boundary condition on $\mathbf{E}$ implies that it vanishes in the xy-plane. Thus

$$\mathbf{t} = \frac{1}{2}(\mathbf{B}\cdot\mathbf{H})\hat{\mathbf{z}} = \hat{\mathbf{z}}\frac{\mu}{2}H^2(t).$$

[5]We may neglect the momentum term in (2.291), which is small compared to the stress tensor term. See Problem 2.20.

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With $H_0 = E_0/\eta$ we have

$$\mathbf{t} = \hat{\mathbf{z}} \frac{E_0^2}{2\eta^2} \mu f^2(t). \tag{2.341}$$

As a numerical example, consider a high-altitude nuclear electromagnetic pulse (HEMP) generated by the explosion of a large nuclear weapon in the upper atmosphere. Such an explosion could generate a transient electromagnetic wave of short (sub-microsecond) duration with an electric field amplitude of 50,000 V/m in air [200]. Using (2.341), we find that the wave would exert a peak pressure of $P = |\mathbf{t}| = .011$ Pa $= 1.6 \times 10^{-6}$ lb/in^2 if reflected from a perfect conductor at normal incidence. Obviously, even for this extreme field level the pressure produced by a transient electromagnetic wave is quite small. However, from (2.340) we find that the current induced in the conductor would have a peak value of 133 A/m. Even a small portion of this current could destroy a sensitive electronic circuit if it were to leak through an opening in the conductor. This is an important concern for engineers designing circuitry to be used in high-field environments, and demonstrates why the concepts of current and voltage can often supersede the concept of force in terms of importance.

Finally, let us see how the terms in the Poynting power balance theorem relate. Consider a cubic region V bounded by the planes $z = z_1$ and $z = z_2$, $z_2 > z_1$. We choose the field waveform $f(t)$ and locate the planes so that we can isolate either the forward or backward traveling wave. Since there is no current in V, Poynting's theorem (2.299) becomes

$$\frac{1}{2}\frac{\partial}{\partial t}\int_V (\epsilon \mathbf{E}\cdot\mathbf{E} + \mu \mathbf{H}\cdot\mathbf{H})\, dV = -\oint_S (\mathbf{E}\times\mathbf{H})\cdot \mathbf{dS}.$$

Consider the wave traveling in the $-z$-direction. Substitution from (2.336) and (2.337) gives the time-rate of change of stored energy as

$$\begin{aligned} S_{\text{cube}}(t) &= \frac{1}{2}\frac{\partial}{\partial t}\int_V \left[\epsilon E^2(z,t) + \mu H^2(z,t)\right] dV \\ &= \frac{1}{2}\frac{\partial}{\partial t}\int_x\int_y dx\,dy \int_{z_1}^{z_2} \left[\epsilon \frac{(v\mu)^2 H_0^2}{4} f^2\left(t + \frac{z}{v}\right) + \mu \frac{H_0^2}{4} f^2\left(t+\frac{z}{v}\right)\right] dz \\ &= \frac{1}{2}\frac{\partial}{\partial t}\mu \frac{H_0^2}{2}\int_x\int_y dx\,dy \int_{z_1}^{z_2} f^2\left(t+\frac{z}{v}\right) dz. \end{aligned}$$

Integration over x and y gives the area A of the cube face. Putting $u = t + z/v$ we see that

$$S = A\mu \frac{H_0^2}{4}\frac{\partial}{\partial t}\int_{t+z_1/v}^{t+z_2/v} f^2(u) v\, du.$$

Leibnitz' rule for differentiation (A.30) then gives

$$S_{\text{cube}}(t) = A\frac{\mu v H_0^2}{4}\left[f^2\left(t+\frac{z_2}{v}\right) - f^2\left(t+\frac{z_1}{v}\right)\right]. \tag{2.342}$$

Again substituting for $E(t + z/v)$ and $H(t + z/v)$ we can write

$$\begin{aligned} S_{\text{cube}}(t) &= -\oint_S (\mathbf{E}\times\mathbf{H})\cdot\mathbf{dS} \\ &= -\int_x\int_y \frac{v\mu H_0^2}{4} f^2\left(t+\frac{z_1}{v}\right)(-\hat{\mathbf{p}}\times\hat{\mathbf{q}})\cdot(-\hat{\mathbf{z}})\, dx\,dy \, - \\ &\quad - \int_x\int_y \frac{v\mu H_0^2}{4} f^2\left(t+\frac{z_2}{v}\right)(-\hat{\mathbf{p}}\times\hat{\mathbf{q}})\cdot(\hat{\mathbf{z}})\, dx\,dy. \end{aligned}$$

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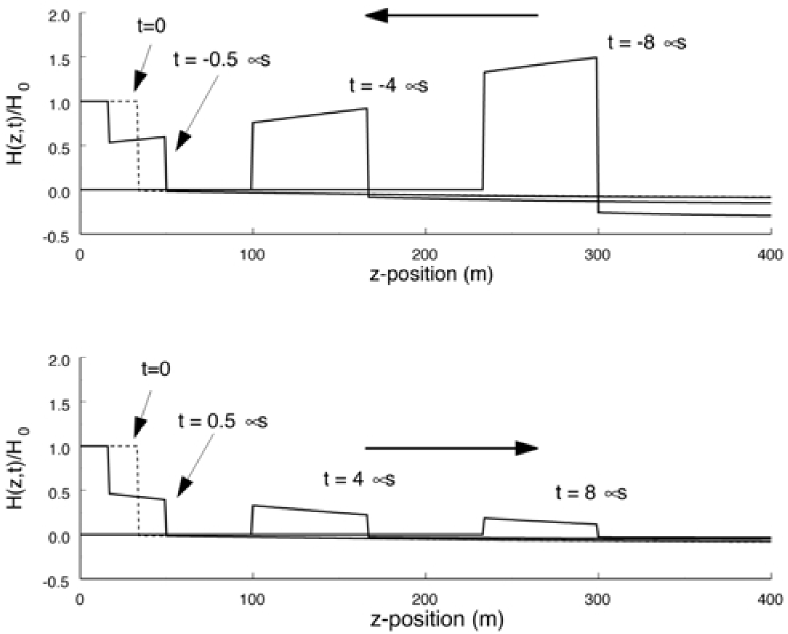

Figure 2.8: Propagation of a transient plane wave in a dissipative medium.

The second term represents the energy change in V produced by the backward traveling wave entering the cube by passing through the plane at $z = z_2$, while the first term represents the energy change in V produced by the wave exiting the cube by passing through the plane $z = z_1$. Contributions from the sides, top, and bottom are zero since $\mathbf{E} \times \mathbf{H}$ is perpendicular to $\hat{\mathbf{n}}$ over those surfaces. Since $\hat{\mathbf{p}} \times \hat{\mathbf{q}} = \hat{\mathbf{z}}$, we get

$$S_{\text{cube}}(t) = A\frac{\mu v H_0^2}{4}\left[f^2\left(t + \frac{z_2}{v}\right) - f^2\left(t + \frac{z_1}{v}\right)\right],$$

which matches (2.342) and thus verifies Poynting's theorem. We may interpret this result as follows. The propagating electromagnetic disturbance carries energy through space. The energy within any region is associated with the field in that region, and can change with time as the propagating wave carries a flux of energy across the boundary of the region. The energy continues to propagate even if the source is changed or is extinguished altogether. That is, the behavior of the leading edge of the disturbance is determined by causality — it is affected by obstacles it encounters, but not by changes in the source that occur after the leading edge has been established.

When propagating through a dissipative region a plane wave takes on a somewhat different character. Again applying the conditions (2.333) and (2.334), we obtain from (2.991) the solution to the wave equation (2.332):

$$H(z,t) = \frac{H_0}{2}e^{-\frac{\Omega}{v}z}f\left(t - \frac{z}{v}\right) + \frac{H_0}{2}e^{\frac{\Omega}{v}z}f\left(t + \frac{z}{v}\right) -$$

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$$-\frac{z\Omega^2 H_0}{2v}e^{-\Omega t}\int_{t-\frac{z}{v}}^{t+\frac{z}{v}} f(u)e^{\Omega u}\frac{J_1\left(\frac{\Omega}{v}\sqrt{z^2-(t-u)^2v^2}\right)}{\frac{\Omega}{v}\sqrt{z^2-(t-u)^2v^2}}\,du \tag{2.343}$$

where $\Omega = \sigma/2\epsilon$. The first two terms resemble those for the lossless case, modified by an exponential damping factor. This accounts for the loss in amplitude that must accompany the transfer of energy from the propagating wave to joule loss (heat) within the conducting medium. The remaining term appears only when the medium is lossy, and results in an extension of the disturbance through the medium because of the currents induced by the passing wavefront. This "wake" follows the leading edge of the disturbance as is shown clearly in Figure 2.8. Here we have repeated the calculation of Figure 2.7, but with $\sigma = 2\times10^{-4}$, approximating the conductivity of fresh water. As the wave travels to the left it attenuates and leaves a trailing remnant behind. Upon reaching the conductor it reflects much as in the lossless case, resulting in a time dependence at $z = 0$ given by the finite-duration rectangular pulse (2.338). In order for the pulse to be of finite duration, the wake left by the reflected pulse must exactly cancel the wake associated with the incident pulse that continues to arrive after the reflection. As the reflected pulse sweeps forward, the wake is obliterated everywhere behind.

If we were to verify the Poynting theorem for a dissipative medium (which we shall not attempt because of the complexity of the computation), we would need to include the $\mathbf{E}\cdot\mathbf{J}$ term. Here $\mathbf{J}$ is the induced conduction current and the integral of $\mathbf{E}\cdot\mathbf{J}$ accounts for the joule loss within a region V balanced by the difference in Poynting energy flux carried into and out of V.

Once we have the fields for a wave propagating along the z-direction, it is a simple matter to generalize these results to any propagation direction. Assume that $\hat{\mathbf{u}}$ is normal to the surface of a plane over which the fields are invariant. Then $u = \hat{\mathbf{u}}\cdot\mathbf{r}$ describes the distance from the origin along the direction $\hat{\mathbf{u}}$. We need only replace z by $\hat{\mathbf{u}}\cdot\mathbf{r}$ in any of the expressions obtained above to determine the fields of a plane wave propagating in the u-direction. We must also replace the orthogonality condition $\hat{\mathbf{p}}\times\hat{\mathbf{q}} = \hat{\mathbf{z}}$ with

$$\hat{\mathbf{p}}\times\hat{\mathbf{q}} = \hat{\mathbf{u}}.$$

For instance, the fields associated with a wave propagating through a lossless medium in the positive u-direction are, from (2.336)–(2.337),

$$\mathbf{H}(\mathbf{r},t) = \hat{\mathbf{q}}\frac{H_0}{2}f\left(t-\frac{\hat{\mathbf{u}}\cdot\mathbf{r}}{v}\right), \qquad \mathbf{E}(\mathbf{r},t) = \hat{\mathbf{p}}\frac{v\mu H_0}{2}f\left(t-\frac{\hat{\mathbf{u}}\cdot\mathbf{r}}{v}\right).$$

2.10.7 Propagation of cylindrical waves in a lossless medium

Much as we envisioned a uniform plane wave arising from a uniform planar source, we can imagine a uniform cylindrical wave arising from a uniform line source. Although this line source must be infinite in extent, uniform cylindrical waves (unlike plane waves) display the physical behavior of diverging from their source while carrying energy outwards to infinity.

A *uniform cylindrical wave* has fields that are invariant over a cylindrical surface: $\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\rho,t)$, $\mathbf{H}(\mathbf{r},t) = \mathbf{H}(\rho,t)$. For simplicity, we shall assume that waves propagate in a homogeneous, isotropic, linear, and lossless medium described by permittivity ϵ and permeability μ. From Maxwell's equations we find that requiring the fields to be independent of ϕ and z puts restrictions on the remaining vector components. Faraday's

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law states

$$\nabla \times \mathbf{E}(\rho, t) = -\hat{\boldsymbol{\phi}}\frac{\partial E_z(\rho, t)}{\partial \rho} + \hat{\mathbf{z}}\frac{1}{\rho}\frac{\partial}{\partial \rho}[\rho E_\phi(\rho, t)] = -\mu\frac{\partial \mathbf{H}(\rho, t)}{\partial t}. \tag{2.344}$$

Equating components we see that $\partial H_\rho/\partial t = 0$, and because our interest lies in wave solutions we take $H_\rho = 0$. Ampere's law in a homogeneous lossless region free from impressed sources states in a similar manner

$$\nabla \times \mathbf{H}(\rho, t) = -\hat{\boldsymbol{\phi}}\frac{\partial H_z(\rho, t)}{\partial \rho} + \hat{\mathbf{z}}\frac{1}{\rho}\frac{\partial}{\partial \rho}[\rho H_\phi(\rho, t)] = \epsilon\frac{\partial \mathbf{E}(\rho, t)}{\partial t}. \tag{2.345}$$

Equating components we find that $E_\rho = 0$. Since $E_\rho = H_\rho = 0$, both $\mathbf{E}$ and $\mathbf{H}$ are perpendicular to the ρ-direction. Note that if there is only a z-component of $\mathbf{E}$ then there is only a ϕ-component of $\mathbf{H}$. This case, termed *electric polarization*, results in

$$\frac{\partial E_z(\rho, t)}{\partial \rho} = \mu\frac{\partial H_\phi(\rho, t)}{\partial t}.$$

Similarly, if there is only a z-component of $\mathbf{H}$ then there is only a ϕ-component of $\mathbf{E}$. This case, termed *magnetic polarization*, results in

$$-\frac{\partial H_z(\rho, t)}{\partial \rho} = \epsilon\frac{\partial E_\phi(\rho, t)}{\partial t}.$$

Since $\mathbf{E} = \hat{\boldsymbol{\phi}}E_\phi + \hat{\mathbf{z}}E_z$ and $\mathbf{H} = \hat{\boldsymbol{\phi}}H_\phi + \hat{\mathbf{z}}H_z$, we can always decompose a cylindrical electromagnetic wave into cases of electric and magnetic polarization. In each case the resulting field is TEM$_\rho$ since the vectors $\mathbf{E}, \mathbf{H}, \hat{\boldsymbol{\rho}}$ are mutually orthogonal.

Wave equations for E_z in the electric polarization case and for H_z in the magnetic polarization case can be found in the usual manner. Taking the curl of (2.344) and substituting from (2.345) we find

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{E}) &= -\hat{\mathbf{z}}\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial E_z}{\partial \rho}\right) - \hat{\boldsymbol{\phi}}\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}[\rho E_\phi]\right) \\ &= -\frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\frac{1}{v^2}\left[\hat{\mathbf{z}}\frac{\partial^2 E_z}{\partial t^2} + \hat{\boldsymbol{\phi}}\frac{\partial^2 E_\phi}{\partial t^2}\right]\end{aligned}$$

where $v = 1/(\mu\epsilon)^{1/2}$. Noting that $E_\phi = 0$ for the electric polarization case we obtain the wave equation for E_z. A similar set of steps beginning with the curl of (2.345) gives an identical equation for H_z. Thus

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\begin{bmatrix} E_z \\ H_z \end{bmatrix}\right) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\begin{bmatrix} E_z \\ H_z \end{bmatrix} = 0. \tag{2.346}$$

We can obtain a solution for (2.346) in much the same way as we do for the wave equations in § A.1. We begin by substituting for $E_z(\rho, t)$ in terms of its temporal Fourier representation

$$E_z(\rho, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{E}_z(\rho, \omega)e^{j\omega t}\, d\omega$$

to obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\tilde{E}_z(\rho, \omega)\right) + \frac{\omega^2}{v^2}\tilde{E}_z(\rho, \omega)\right]e^{j\omega t}\, d\omega = 0.$$

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The Fourier integral theorem implies that the integrand is zero. Then, expanding out the ρ derivatives, we find that $\tilde{E}_z(\rho, \omega)$ obeys the ordinary differential equation

$$\frac{d^2\tilde{E}_z}{d\rho^2} + \frac{1}{\rho}\frac{d\tilde{E}_z}{d\rho} + k^2\tilde{E}_z = 0$$

where $k = \omega/v$. This is merely Bessel's differential equation (A.124). It is a second-order equation with two independent solutions chosen from the list

$$J_0(k\rho), \quad Y_0(k\rho), \quad H_0^{(1)}(k\rho), \quad H_0^{(2)}(k\rho).$$

We find that $J_0(k\rho)$ and $Y_0(k\rho)$ are useful for describing standing waves between boundaries while $H_0^{(1)}(k\rho)$ and $H_0^{(2)}(k\rho)$ are useful for describing waves propagating in the ρ-direction. Of these, $H_0^{(1)}(k\rho)$ represents waves traveling inward while $H_0^{(2)}(k\rho)$ represents waves traveling outward. Concentrating on the outward traveling wave we find that

$$\tilde{E}_z(\rho, \omega) = \tilde{A}(\omega)\left[-j\frac{\pi}{2}H_0^{(2)}(k\rho)\right] = \tilde{A}(\omega)\tilde{g}(\rho, \omega).$$

Here $A(t) \leftrightarrow \tilde{A}(\omega)$ is the disturbance waveform, assumed to be a real, causal function. To make $E_z(\rho, t)$ real we require that the inverse transform of $\tilde{g}(\rho, \omega)$ be real. This requires the inclusion of the $-j\pi/2$ factor in $\tilde{g}(\rho, \omega)$. Inverting we have

$$E_z(\rho, t) = A(t) * g(\rho, t) \tag{2.347}$$

where $g(\rho, t) \leftrightarrow (-j\pi/2)H_0^{(2)}(k\rho)$.

The inverse transform needed to obtain $g(\rho, t)$ may be found in Campbell [26]:

$$g(\rho, t) = \mathcal{F}^{-1}\left\{-j\frac{\pi}{2}H_0^{(2)}\left(\omega\frac{\rho}{v}\right)\right\} = \frac{U\left(t - \frac{\rho}{v}\right)}{\sqrt{t^2 - \frac{\rho^2}{v^2}}},$$

where $U(t)$ is the unit step function defined in (A.5). Substituting this into (2.347) and writing the convolution in integral form we have

$$E_z(\rho, t) = \int_{-\infty}^{\infty} A(t - t')\frac{U(t' - \rho/v)}{\sqrt{t'^2 - \rho^2/v^2}}\,dt'.$$

The change of variable $x = t' - \rho/v$ then gives

$$E_z(\rho, t) = \int_0^{\infty} \frac{A(t - x - \rho/v)}{\sqrt{x^2 + 2x\rho/v}}\,dx. \tag{2.348}$$

Those interested in the details of the inverse transform should see Chew [33].

As an example, consider a lossless medium with $\mu_r = 1$, $\epsilon_r = 81$, and a waveform

$$A(t) = E_0[U(t) - U(t - \tau)]$$

where $\tau = 2$ μs. This situation is the same as that in the plane wave example above, except that the pulse waveform begins at $t = 0$. Substituting for $A(t)$ into (2.348) and using the integral

$$\int \frac{dx}{\sqrt{x}\sqrt{x + a}} = 2\ln\left[\sqrt{x} + \sqrt{x + a}\,\right]$$

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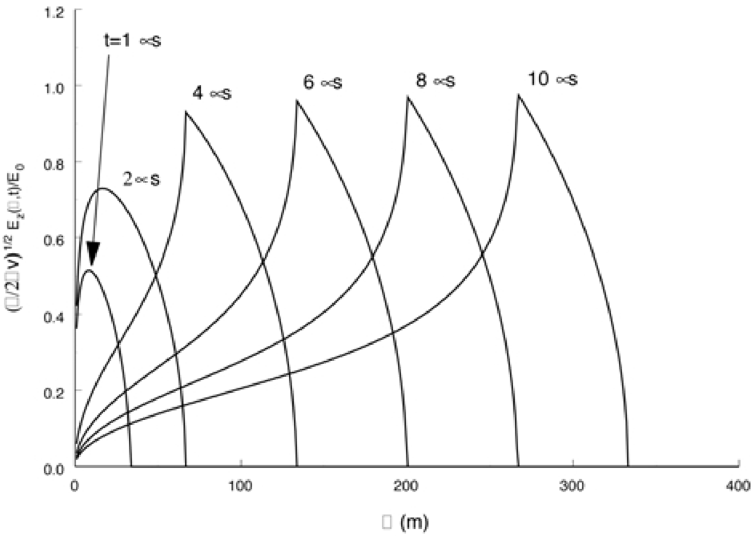

Figure 2.9: Propagation of a transient cylindrical wave in a lossless medium.

we can write the electric field in closed form as

$$E_z(\rho, t) = 2E_0 \ln \left[\frac{\sqrt{x_2} + \sqrt{x_2 + 2\rho/v}}{\sqrt{x_1} + \sqrt{x_1 + 2\rho/v}} \right], \tag{2.349}$$

where $x_2 = \max[0, t - \rho/v]$ and $x_1 = \max[0, t - \rho/v - \tau]$. The field is plotted in Figure 2.9 for various values of time. Note that the leading edge of the disturbance propagates outward at a velocity v and a wake trails behind the disturbance. This wake is similar to that for a plane wave in a dissipative medium, but it exists in this case even though the medium is lossless. We can think of the wave as being created by a line source of infinite extent, pulsed by the disturbance waveform. Although current changes simultaneously everywhere along the line, it takes the disturbance longer to propagate to an observation point in the $z = 0$ plane from source points $z \neq 0$ than from the source point at $z = 0$. Thus, the field at an arbitrary observation point ρ arrives from different source points at different times. If we look at Figure 2.9 we note that there is always a nonzero field near $\rho = 0$ (or any value of $\rho < vt$) regardless of the time, since at any given t the disturbance is arriving from some point along the line source.

We also see in Figure 2.9 that as ρ becomes large the peak value of the propagating disturbance approaches a certain value. This value occurs at $t_m = \rho/v + \tau$ or, equivalently, $\rho_m = v(t - \tau)$. If we substitute this value into (2.349) we find that

$$E_z(\rho, t_m) = 2E_0 \ln \left[\sqrt{\frac{\tau}{2\rho/v}} + \sqrt{1 + \frac{\tau}{2\rho/v}} \right].$$

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For large values of ρ/v,

$$E_z(\rho, t_m) \approx 2E_0 \ln\left[1 + \sqrt{\frac{\tau}{2\rho/v}}\right].$$

Using $\ln(1 + x) \approx x$ when $x \ll 1$, we find that

$$E_z(\rho, t_m) \approx E_0\sqrt{\frac{2\tau v}{\rho}}.$$

Thus, as $\rho \to \infty$ we have $\mathbf{E}\times\mathbf{H} \sim 1/\rho$ and the flux of energy passing through a cylindrical surface of area $\rho\, d\phi\, dz$ is independent of ρ. This result is similar to that seen for spherical waves where $\mathbf{E} \times \mathbf{H} \sim 1/r^2$.

2.10.8 Propagation of spherical waves in a lossless medium

In the previous section we found solutions that describe uniform cylindrical waves dependent only on the radial variable ρ. It turns out that similar solutions are not possible in spherical coordinates; fields that only depend on r cannot satisfy Maxwell's equations since, as shown in § 2.10.9, a source having the appropriate symmetry for the production of uniform spherical waves in fact produces no field at all external to the region it occupies. As we shall see in Chapter 5, the fields produced by localized sources are in general quite complex. However, certain solutions that are only slightly nonuniform may be found, and these allow us to investigate the most important properties of spherical waves. We shall find that spherical waves diverge from a localized point source and expand outward with finite velocity, carrying energy away from the source.

Consider a homogeneous, lossless, source-free region of space characterized by permittivity ϵ and permeability μ. We seek solutions to the wave equation that are TEM_r in spherical coordinates ($H_r = E_r = 0$), and independent of the azimuthal angle ϕ. Thus we may write

$$\mathbf{E}(\mathbf{r}, t) = \hat{\boldsymbol{\theta}} E_\theta(r, \theta, t) + \hat{\boldsymbol{\phi}} E_\phi(r, \theta, t),$$
$$\mathbf{H}(\mathbf{r}, t) = \hat{\boldsymbol{\theta}} H_\theta(r, \theta, t) + \hat{\boldsymbol{\phi}} H_\phi(r, \theta, t).$$

Maxwell's equations show that not all of these vector components are required. Faraday's law states that

$$\nabla \times \mathbf{E}(r, \theta, t) = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta\, E_\phi(r, \theta, t)] - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial r}[r E_\phi(r, \theta, t)] + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial r}[r E_\theta(r, \theta, t)]$$
$$= -\mu\frac{\partial \mathbf{H}(r, \theta, t)}{\partial t}. \tag{2.350}$$

Since we require $H_r = 0$ we must have

$$\frac{\partial}{\partial\theta}[\sin\theta\, E_\phi(r, \theta, t)] = 0.$$

This implies that either $E_\phi \sim 1/\sin\theta$ or $E_\phi = 0$. We shall choose $E_\phi = 0$ and investigate whether the resulting fields satisfy the remaining Maxwell equations.

In a source-free region of space we have $\nabla \cdot \mathbf{D} = \epsilon\nabla \cdot \mathbf{E} = 0$. Since we now have only a θ-component of the electric field, this requires

$$\frac{1}{r}\frac{\partial}{\partial\theta}E_\theta(r, \theta, t) + \frac{\cot\theta}{r}E_\theta(r, \theta, t) = 0.$$

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From this we see that when $E_\phi = 0$ the component E_θ must obey

$$E_\theta(r, \theta, t) = \frac{f_E(r, t)}{\sin\theta}.$$

By (2.350) there is only a ϕ-component of magnetic field, and it must obey $H_\phi(r, \theta, t) = f_H(r, t)/\sin\theta$ where

$$-\mu \frac{\partial}{\partial t} f_H(r, t) = \frac{1}{r}\frac{\partial}{\partial r}[r f_E(r, t)]. \tag{2.351}$$

Thus the spherical wave has the property $\mathbf{E} \perp \mathbf{H} \perp \mathbf{r}$, and is TEM to the r-direction.

We can obtain a wave equation for E_θ by taking the curl of (2.350) and substituting from Ampere's law:

$$\nabla \times (\nabla \times \mathbf{E}) = -\hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial^2}{\partial r^2}[r E_\theta] = \nabla \times \left[-\mu \frac{\partial}{\partial t}\mathbf{H}\right] = -\mu \frac{\partial}{\partial t}\left[\sigma \mathbf{E} + \epsilon \frac{\partial}{\partial t}\mathbf{E}\right].$$

This gives

$$\frac{\partial^2}{\partial r^2}[r f_E(r, t)] - \mu\sigma \frac{\partial}{\partial t}[r f_E(r, t)] - \mu\epsilon \frac{\partial^2}{\partial t^2}[r f_E(r, t)] = 0, \tag{2.352}$$

which is the desired wave equation for $\mathbf{E}$. Proceeding similarly we find that H_ϕ obeys

$$\frac{\partial^2}{\partial r^2}[r f_H(r, t)] - \mu\sigma \frac{\partial}{\partial t}[r f_H(r, t)] - \mu\epsilon \frac{\partial^2}{\partial t^2}[r f_H(r, t)] = 0. \tag{2.353}$$

We see that the wave equation for $r f_E$ is identical to that for the plane wave field E_z (2.331). Thus, we can use the solution obtained in § A.1, as we did with the plane wave, with a few subtle differences. First, we cannot have $r < 0$. Second, we do not anticipate a solution representing a wave traveling in the $-r$-direction — i.e., a wave converging toward the origin. (In other situations we might need such a solution in order to form a standing wave between two spherical boundary surfaces, but here we are only interested in the basic propagating behavior of spherical waves.) Thus, we choose as our solution the term (A.45) and find for a lossless medium where $\Omega = 0$

$$E_\theta(r, \theta, t) = \frac{1}{r \sin\theta} A\left(t - \frac{r}{v}\right). \tag{2.354}$$

From (2.351) we see that

$$H_\phi = \frac{1}{\mu v}\frac{1}{r \sin\theta} A\left(t - \frac{r}{v}\right). \tag{2.355}$$

Since $\mu v = (\mu/\epsilon)^{1/2} = \eta$, we can also write this as

$$\mathbf{H} = \frac{\hat{\mathbf{r}} \times \mathbf{E}}{\eta}.$$

We note that our solution is not appropriate for unbounded space since the fields have a singularity at $\theta = 0$. Thus we must exclude the z-axis. This can be accomplished by using PEC cones of angles θ_1 and θ_2, $\theta_2 > \theta_1$. Because the electric field $\mathbf{E} = \hat{\boldsymbol{\theta}} E_\theta$ is normal to these cones, the boundary condition that tangential $\mathbf{E}$ vanishes is satisfied.

It is informative to see how the terms in the Poynting power balance theorem relate for a spherical wave. Consider the region between the spherical surfaces $r = r_1$ and $r = r_2$, $r_2 > r_1$. Since there is no current within the volume region, Poynting's theorem (2.299) becomes

$$\frac{1}{2}\frac{\partial}{\partial t}\int_V (\epsilon \mathbf{E} \cdot \mathbf{E} + \mu \mathbf{H} \cdot \mathbf{H})\, dV = -\oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{dS}. \tag{2.356}$$

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From (2.354) and (2.355), the time-rate of change of stored energy is

$$\begin{aligned}
P_{\text{sphere}}(t) &= \frac{1}{2}\frac{\partial}{\partial t}\int_V [\epsilon E^2(r,\theta,t) + \mu H^2(r,\theta,t)]\,dV \\
&= \frac{1}{2}\frac{\partial}{\partial t}\int_0^{2\pi} d\phi \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sin\theta}\int_{r_1}^{r_2}\left[\epsilon\frac{1}{r^2}A^2\left(t-\frac{r}{v}\right) + \mu\frac{1}{r^2}\frac{1}{(v\mu)^2}A^2\left(t-\frac{r}{v}\right)\right] r^2\,dr \\
&= 2\pi\epsilon F\frac{\partial}{\partial t}\int_{r_1}^{r_2} A^2\left(t-\frac{r}{v}\right)\,dr
\end{aligned}$$

where

$$F = \ln\left[\frac{\tan(\theta_2/2)}{\tan(\theta_1/2)}\right].$$

Putting $u = t - r/v$ we see that

$$P_{\text{sphere}}(t) = -2\pi\epsilon F\frac{\partial}{\partial t}\int_{t-r_1/v}^{t-r_2/v} A^2(u)v\,du.$$

An application of Leibnitz' rule for differentiation (A.30) gives

$$P_{\text{sphere}}(t) = -\frac{2\pi}{\eta}F\left[A^2\left(t-\frac{r_2}{v}\right) - A^2\left(t-\frac{r_1}{v}\right)\right]. \tag{2.357}$$

Next we find the Poynting flux term:

$$\begin{aligned}
P_{\text{sphere}}(t) &= -\oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S} \\
&= -\int_0^{2\pi} d\phi\int_{\theta_1}^{\theta_2}\left[\frac{1}{r_1}A\left(t-\frac{r_1}{v}\right)\hat{\boldsymbol{\theta}}\right]\times\left[\frac{1}{r_1}\frac{1}{\mu v}A\left(t-\frac{r_1}{v}\right)\hat{\boldsymbol{\phi}}\right]\cdot(-\hat{\mathbf{r}})r_1^2\frac{d\theta}{\sin\theta} - \\
&\quad -\int_0^{2\pi} d\phi\int_{\theta_1}^{\theta_2}\left[\frac{1}{r_2}A\left(t-\frac{r_2}{v}\right)\hat{\boldsymbol{\theta}}\right]\times\left[\frac{1}{r_2}\frac{1}{\mu v}A\left(t-\frac{r_2}{v}\right)\hat{\boldsymbol{\phi}}\right]\cdot\hat{\mathbf{r}}r_2^2\frac{d\theta}{\sin\theta}.
\end{aligned}$$

The first term represents the power carried by the traveling wave into the volume region by passing through the spherical surface at $r = r_1$, while the second term represents the power carried by the wave out of the region by passing through the surface $r = r_2$. Integration gives

$$P_{\text{sphere}}(t) = -\frac{2\pi}{\eta}F\left[A^2\left(t-\frac{r_2}{v}\right) - A^2\left(t-\frac{r_1}{v}\right)\right], \tag{2.358}$$

which matches (2.357), thus verifying Poynting's theorem.

It is also interesting to compute the total energy passing through a surface of radius r_0. From (2.358) we see that the flux of energy (power density) passing outward through the surface $r = r_0$ is

$$P_{\text{sphere}}(t) = \frac{2\pi}{\eta}FA^2\left(t-\frac{r_0}{v}\right).$$

The total energy associated with this flux can be computed by integrating over all time: we have

$$E = \frac{2\pi}{\eta}F\int_{-\infty}^{\infty}A^2\left(t-\frac{r_0}{v}\right)dt = \frac{2\pi}{\eta}F\int_{-\infty}^{\infty}A^2(u)\,du$$

after making the substitution $u = t - r_0/v$. The total energy passing through a spherical surface is independent of the radius of the sphere. This is an important property of spherical waves. The $1/r$ dependence of the electric and magnetic fields produces a power density that decays with distance in precisely the right proportion to compensate for the r^2-type increase in the surface area through which the power flux passes.

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2.10.9 Nonradiating sources

Not all time-dependent sources produce electromagnetic waves. In fact, certain localized source distributions produce no fields external to the region containing the sources. Such distributions are said to be *nonradiating*, and the fields they produce (within their source regions) lack wave characteristics.

Let us consider a specific example involving two concentric spheres. The inner sphere, carrying a uniformly distributed total charge $-Q$, is rigid and has a fixed radius a; the outer sphere, carrying uniform charge $+Q$, is a flexible balloon that can be stretched to any radius $b = b(t)$. The two surfaces are initially stationary, some external force being required to hold them in place. Now suppose we apply a time-varying force that results in $b(t)$ changing from $b(t_1) = b_1$ to $b(t_2) = b_2 > b_1$. This creates a radially directed time-varying current $\hat{\mathbf{r}}J_r(\mathbf{r}, t)$. By symmetry J_r depends only on r and produces a field $\mathbf{E}$ that depends only on r and is directed radially. An application of Gauss's law over a sphere of radius $r_0 > b_2$, which contains zero total charge, gives

$$4\pi r_0^2 E_r(r_0, t) = 0,$$

hence $\mathbf{E}(\mathbf{r}, t) = 0$ for $r > r_0$ and all time t. So $\mathbf{E} = 0$ external to the current distribution and no outward traveling wave is produced. Gauss's law also shows that $\mathbf{E} = 0$ inside the rigid sphere, while between the spheres

$$\mathbf{E}(\mathbf{r}, t) = -\hat{\mathbf{r}}\frac{Q}{4\pi\epsilon_0 r^2}.$$

Now work is certainly required to stretch the balloon and overcome the Lorentz force between the two charged surfaces. But an application of Poynting's theorem over a surface enclosing both spheres shows that no energy is carried away by an electromagnetic wave. Where does the expended energy go? The presence of only two nonzero terms in Poynting's theorem clearly indicates that the power term $\int_V \mathbf{E} \cdot \mathbf{J}\, dV$ corresponding to the external work must be balanced exactly by a change in stored energy. As the radius of the balloon increases, so does the region of nonzero field as well as the stored energy.

In free space any current source expressible in the form

$$\mathbf{J}(\mathbf{r}, t) = \nabla\left(\frac{\partial\psi(\mathbf{r}, t)}{\partial t}\right) \tag{2.359}$$

and localized to a volume region V, such as the current in the example above, is nonradiating. Indeed, Ampere's law states that

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \nabla\left(\frac{\partial\psi(\mathbf{r}, t)}{\partial t}\right) \tag{2.360}$$

for $\mathbf{r} \in V$; taking the curl we have

$$\nabla \times (\nabla \times \mathbf{H}) = \epsilon_0 \frac{\partial \nabla \times \mathbf{E}}{\partial t} + \nabla \times \nabla\left(\frac{\partial\psi(\mathbf{r}, t)}{\partial t}\right).$$

But the second term on the right is zero, so

$$\nabla \times (\nabla \times \mathbf{H}) = \epsilon_0 \frac{\partial \nabla \times \mathbf{E}}{\partial t}$$

and this equation holds for all $\mathbf{r}$. By Faraday's law we can rewrite it as

$$\left((\nabla \times \nabla\times) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{H}(\mathbf{r}, t) = 0.$$

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So $\mathbf{H}$ obeys the homogeneous wave equation everywhere, and $\mathbf{H} = 0$ follows from causality. The laws of Ampere and Faraday may also be combined with (2.359) to show that

$$\left((\nabla \times \nabla\times) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left[\mathbf{E}(\mathbf{r}, t) + \frac{1}{\epsilon_0}\nabla\psi(\mathbf{r}, t)\right] = 0$$

for all $\mathbf{r}$. By causality

$$\mathbf{E}(\mathbf{r}, t) = -\frac{1}{\epsilon_0}\nabla\psi(\mathbf{r}, t) \tag{2.361}$$

everywhere. But since $\psi(\mathbf{r}, t) = 0$ external to V, we must also have $\mathbf{E} = 0$ there. Note that $\mathbf{E} = -\nabla\psi/\epsilon_0$ is consistent with Ampere's law (2.360) provided that $\mathbf{H} = 0$ everywhere.

We see that sources having spherical symmetry such that

$$\mathbf{J}(\mathbf{r}, t) = \hat{\mathbf{r}}J_r(r, t) = \nabla\left(\frac{\partial\psi(r, t)}{\partial t}\right) = \hat{\mathbf{r}}\frac{\partial^2\psi(r, t)}{\partial r\,\partial t}$$

obey (2.359) and are therefore nonradiating. Hence the fields associated with any outward traveling spherical wave must possess some angular variation. This holds, for example, for the fields far removed from a time-varying source of finite extent.

As pointed out by Lindell [113], nonradiating sources are not merely hypothetical. The outflowing currents produced by a highly symmetric nuclear explosion in outer space or in a homogeneous atmosphere would produce no electromagnetic field outside the source region. The large electromagnetic-pulse effects discussed in § 2.10.6 are due to inhomogeneities in the earth's atmosphere. We also note that the fields produced by a radiating source $\mathbf{J}^r(\mathbf{r}, t)$ do not change external to the source if we superpose a nonradiating component $\mathbf{J}^{nr}(\mathbf{r}, t)$ to create a new source $\mathbf{J} = \mathbf{J}^{nr} + \mathbf{J}^r$. We say that the two sources $\mathbf{J}$ and $\mathbf{J}^r$ are *equivalent* for the region V external to the sources. This presents difficulties in remote sensing where investigators are often interested in reconstructing an unknown source by probing the fields external to (and usually far away from) the source region. Unique reconstruction is possible only if the fields within the source region are also measured.

For the time harmonic case, Devaney and Wolf [54] provide the most general possible form for a nonradiating source. See § 4.11.9 for details.

2.11 Problems

2.1 Consider the constitutive equations (2.16)–(2.17) relating $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{H}$ in a bianisotropic medium. Using the definition for $\mathbf{P}$ and $\mathbf{M}$, show that the constitutive equations relating $\mathbf{E}$, $\mathbf{B}$, $\mathbf{P}$, and $\mathbf{M}$ are

$$\mathbf{P} = \left(\frac{1}{c}\bar{\mathbf{P}} - \epsilon_0\bar{\mathbf{I}}\right)\cdot\mathbf{E} + \bar{\mathbf{L}}\cdot\mathbf{B},$$

$$\mathbf{M} = -\bar{\mathbf{M}}\cdot\mathbf{E} - \left(c\bar{\mathbf{Q}} - \frac{1}{\mu_0}\bar{\mathbf{I}}\right)\cdot\mathbf{B}.$$

Also find the constitutive equations relating $\mathbf{E}$, $\mathbf{H}$, $\mathbf{P}$, and $\mathbf{M}$.

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2.2 Consider Ampere's law and Gauss's law written in terms of rectangular components in the laboratory frame of reference. Assume that an inertial frame moves with velocity $\mathbf{v} = \hat{\mathbf{x}}v$ with respect to the laboratory frame. Using the Lorentz transformation given by (2.73)–(2.76), show that

$$\begin{aligned} c\mathbf{D}'_\perp &= \gamma(c\mathbf{D}_\perp + \boldsymbol{\beta} \times \mathbf{H}_\perp), \\ \mathbf{H}'_\perp &= \gamma(\mathbf{H}_\perp - \boldsymbol{\beta} \times c\mathbf{D}_\perp), \\ \mathbf{J}'_\parallel &= \gamma(\mathbf{J}_\parallel - \rho\mathbf{v}), \\ \mathbf{J}'_\perp &= \mathbf{J}_\perp, \\ c\rho' &= \gamma(c\rho - \boldsymbol{\beta} \cdot \mathbf{J}), \end{aligned}$$

where "$\perp$" means perpendicular to the direction of the velocity and "$\parallel$" means parallel to the direction of the velocity.

2.3 Show that the following quantities are invariant under Lorentz transformation:

(a) $\mathbf{E} \cdot \mathbf{B}$,
(b) $\mathbf{H} \cdot \mathbf{D}$,
(c) $\mathbf{B} \cdot \mathbf{B} - \mathbf{E} \cdot \mathbf{E}/c^2$,
(d) $\mathbf{H} \cdot \mathbf{H} - c^2\mathbf{D} \cdot \mathbf{D}$,
(e) $\mathbf{B} \cdot \mathbf{H} - \mathbf{E} \cdot \mathbf{D}$,
(f) $c\mathbf{B} \cdot \mathbf{D} + \mathbf{E} \cdot \mathbf{H}/c$.

2.4 Show that if $c^2B^2 > E^2$ holds in one reference frame, then it holds in all other reference frames. Repeat for the inequality $c^2B^2 < E^2$.

2.5 Show that if $\mathbf{E} \cdot \mathbf{B} = 0$ and $c^2B^2 > E^2$ holds in one reference frame, then a reference frame may be found such that $\mathbf{E} = 0$. Show that if $\mathbf{E} \cdot \mathbf{B} = 0$ and $c^2B^2 < E^2$ holds in one reference frame, then a reference frame may be found such that $\mathbf{B} = 0$.

2.6 A test charge Q at rest in the laboratory frame experiences a force $\mathbf{F} = Q\mathbf{E}$ as measured by an observer in the laboratory frame. An observer in an inertial frame measures a force on the charge given by $\mathbf{F}' = Q\mathbf{E}' + Q\mathbf{v} \times \mathbf{B}'$. Show that $\mathbf{F} \neq \mathbf{F}'$ and find the formula for converting between $\mathbf{F}$ and $\mathbf{F}'$.

2.7 Consider a material moving with velocity $\mathbf{v}$ with respect to the laboratory frame of reference. When the fields are measured in the moving frame, the material is found to be isotropic with $\mathbf{D}' = \epsilon'\mathbf{E}'$ and $\mathbf{B}' = \mu'\mathbf{H}'$. Show that the fields measured in the laboratory frame are given by (2.107) and (2.108), indicating that the material is bianisotropic when measured in the laboratory frame.

2.8 Show that by assuming $v^2/c^2 \ll 1$ in (2.61)–(2.64) we may obtain (2.111).

2.9 Derive the following expressions that allow us to convert the value of the magnetization measured in the laboratory frame of reference to the value measured in a moving frame:

$$\mathbf{M}'_\perp = \gamma(\mathbf{M}_\perp + \boldsymbol{\beta} \times c\mathbf{P}_\perp), \qquad \mathbf{M}'_\parallel = \mathbf{M}_\parallel.$$

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2.10 Beginning with the expressions (2.61)–(2.64) for the field conversions under a first-order Lorentz transformation, show that

$$\mathbf{P}' = \mathbf{P} - \frac{\mathbf{v} \times \mathbf{M}}{c^2}, \qquad \mathbf{M}' = \mathbf{M} + \mathbf{v} \times \mathbf{P}.$$

2.11 Consider a simple isotropic material moving through space with velocity $\mathbf{v}$ relative to the laboratory frame. The relative permittivity and permeability of the material measured in the moving frame are ϵ_r' and μ_r', respectively. Show that the magnetization as measured in the laboratory frame is related to the laboratory frame electric field and magnetic flux density as

$$\mathbf{M} = \frac{\chi_m'}{\mu_0 \mu_r'} \mathbf{B} - \epsilon_0 \left(\chi_e' + \frac{\chi_m'}{\mu_r'} \right) \mathbf{v} \times \mathbf{E}$$

when a first-order Lorentz transformation is used. Here $\chi_e' = \epsilon_r' - 1$ and $\chi_m' = \mu_r' - 1$.

2.12 Consider a simple isotropic material moving through space with velocity $\mathbf{v}$ relative to the laboratory frame. The relative permittivity and permeability of the material measured in the moving frame are ϵ_r' and μ_r', respectively. Derive the formulas for the magnetization and polarization in the laboratory frame in terms of $\mathbf{E}$ and $\mathbf{B}$ measured in the laboratory frame by using the Lorentz transformations (2.128) and (2.129)–(2.132). Show that these expressions reduce to (2.139) and (2.140) under the assumption of a first-order Lorentz transformation ($v^2/c^2 \ll 1$).

2.13 Derive the kinematic form of the large-scale Maxwell–Boffi equations (2.165) and (2.166). Derive the alternative form of the large-scale Maxwell–Boffi equations (2.167) and (2.168).

2.14 Modify the kinematic form of the Maxwell–Boffi equations (2.165)–(2.166) to account for the presence of magnetic sources. Repeat for the alternative forms (2.167)–(2.168).

2.15 Consider a thin magnetic source distribution concentrated near a surface S. The magnetic charge and current densities are given by

$$\rho_m(\mathbf{r}, x, t) = \rho_{ms}(\mathbf{r}, t) f(x, \Delta), \qquad \mathbf{J}_m(\mathbf{r}, x, t) = \mathbf{J}_{ms}(\mathbf{r}, t) f(x, \Delta),$$

where $f(x, \Delta)$ satisfies

$$\int_{-\infty}^{\infty} f(x, \Delta)\, dx = 1.$$

Let $\Delta \to 0$ and derive the boundary conditions on $(\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H})$ across S.

2.16 Beginning with the kinematic forms of Maxwell's equations (2.177)–(2.178), derive the boundary conditions for a moving surface

$$\hat{\mathbf{n}}_{12} \times (\mathbf{H}_1 - \mathbf{H}_2) + (\hat{\mathbf{n}}_{12} \cdot \mathbf{v})(\mathbf{D}_1 - \mathbf{D}_2) = \mathbf{J}_s,$$
$$\hat{\mathbf{n}}_{12} \times (\mathbf{E}_1 - \mathbf{E}_2) - (\hat{\mathbf{n}}_{12} \cdot \mathbf{v})(\mathbf{B}_1 - \mathbf{B}_2) = -\mathbf{J}_{ms}.$$

2.17 Beginning with Maxwell's equations and the constitutive relationships for a bianisotropic medium (2.19)–(2.20), derive the wave equation for $\mathbf{H}$ (2.313). Specialize the result for the case of an anisotropic medium.

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2.18 Consider an isotropic but inhomogeneous material, so that

$$\mathbf{D}(\mathbf{r}, t) = \epsilon(\mathbf{r})\mathbf{E}(\mathbf{r}, t), \qquad \mathbf{B}(\mathbf{r}, t) = \mu(\mathbf{r})\mathbf{H}(\mathbf{r}, t).$$

Show that the wave equations for the fields within this material may be written as

$$\nabla^2\mathbf{E} - \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} + \nabla\left[\mathbf{E}\cdot\left(\frac{\nabla\epsilon}{\epsilon}\right)\right] - (\nabla\times\mathbf{E})\times\left(\frac{\nabla\mu}{\mu}\right) = \mu\frac{\partial\mathbf{J}}{\partial t} + \nabla\left(\frac{\rho}{\epsilon}\right),$$
$$\nabla^2\mathbf{H} - \mu\epsilon\frac{\partial^2\mathbf{H}}{\partial t^2} + \nabla\left[\mathbf{H}\cdot\left(\frac{\nabla\mu}{\mu}\right)\right] - (\nabla\times\mathbf{H})\times\left(\frac{\nabla\epsilon}{\epsilon}\right) = -\nabla\times\mathbf{J} - \mathbf{J}\times\left(\frac{\nabla\epsilon}{\epsilon}\right).$$

2.19 Consider a homogeneous, isotropic material in which $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$. Using the definitions of the equivalent sources, show that the wave equations (2.322)–(2.323) are equivalent to (2.314)–(2.315).

2.20 When we calculate the force on a conductor produced by an incident plane wave, we often neglect the momentum term

$$\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}).$$

Compute this term for the plane wave field (2.336) in free space at the surface of the conductor and compare to the term obtained from the Maxwell stress tensor (2.341). What is the relative difference in amplitude?

2.21 When a material is only slightly conducting, and thus Ω is very small, we often neglect the third term in the plane wave solution (2.343). Reproduce the plot of Figure 2.8 with this term omitted and compare. Discuss how the omitted term affects the shape of the propagating waveform.

2.22 A total charge Q is evenly distributed over a spherical surface. The surface expands outward at constant velocity so that the radius of the surface is $b = vt$ at time t. (a) Use Gauss's law to find $\mathbf{E}$ everywhere as a function of time. (b) Show that $\mathbf{E}$ may be found from a potential function

$$\psi(\mathbf{r}, t) = \frac{Q}{4\pi r}(r - vt)U(r - vt)$$

according to (2.361). Here $U(t)$ is the unit step function. (c) Write down the form of $\mathbf{J}$ for the expanding sphere and show that since it may be found from (2.359) it is a nonradiating source.

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