Chapter 3

The static electromagnetic field

3.1 Static fields and steady currents

Perhaps the most carefully studied area of electromagnetics is that in which the fields are time-invariant. This area, known generally as *statics*, offers (1) the most direct opportunities for solution of the governing equations, and (2) the clearest physical pictures of the electromagnetic field. We therefore devote the present chapter to a treatment of static fields. We begin to seek and examine specific solutions to the field equations; however, our selection of examples is shaped by a search for insight into the behavior of the field itself, rather than a desire to catalog the solutions of numerous statics problems.

We note at the outset that a static field is physically sensible only as a limiting case of a time-varying field as the latter approaches a time-invariant equilibrium, and then only in local regions. The static field equations we shall study thus represent an idealized model of the physical fields.

If we examine the Maxwell–Minkowski equations (2.1)–(2.4) and set the time derivatives to zero, we obtain the *static field Maxwell equations*

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0, \tag{3.1}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r}), \tag{3.2}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r}), \tag{3.3}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0. \tag{3.4}$$

We note that if the fields are to be everywhere time-invariant, then the sources $\mathbf{J}$ and ρ must also be everywhere time-invariant. Under this condition the dynamic coupling between the fields described by Maxwell's equations disappears; any connection between $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{H}$ imposed by the time-varying nature of the field is gone. For static fields we also require that any dynamic coupling between fields in the constitutive relations vanish. In this *static field limit* we cannot derive the divergence equations from the curl equations, since we can no longer use the initial condition argument that the fields were identically zero prior to some time.

The static field equations are useful for approximating many physical situations in which the fields rapidly settle to a local, macroscopically-static state. This may occur so rapidly and so completely that, in a practical sense, the static equations describe the fields within our ability to measure and to compute. Such is the case when a capacitor is rapidly charged using a battery in series with a resistor; for example, a 1 pF capacitor charging through a 1 Ω resistor reaches 99.99% of its total charge static limit within 10 ps.

3.1.1 Decoupling of the electric and magnetic fields

For the remainder of this chapter we shall assume that there is no coupling between **E** and **H** or between **D** and **B** in the constitutive relations. Then the static equations decouple into two independent sets of equations in terms of two independent sets of fields. The static electric field set (**E**,**D**) is described by the equations

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0, \tag{3.5}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r}). \tag{3.6}$$

Integrating these over a stationary contour and surface, respectively, we have the large-scale forms

$$\oint_\Gamma \mathbf{E} \cdot \mathbf{dl} = 0, \tag{3.7}$$

$$\oint_S \mathbf{D} \cdot \mathbf{dS} = \int_V \rho \, dV. \tag{3.8}$$

The static magnetic field set (**B**,**H**) is described by

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r}), \tag{3.9}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0, \tag{3.10}$$

or, in large-scale form,

$$\oint_\Gamma \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS}, \tag{3.11}$$

$$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0. \tag{3.12}$$

We can also specialize the Maxwell–Boffi equations to static form. Assuming that the fields, sources, and equivalent sources are time-invariant, the electrostatic field **E**(**r**) is described by the point-form equations

$$\nabla \times \mathbf{E} = 0, \tag{3.13}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \left(\rho - \nabla \cdot \mathbf{P}\right), \tag{3.14}$$

or the equivalent large-scale equations

$$\oint_\Gamma \mathbf{E} \cdot \mathbf{dl} = 0, \tag{3.15}$$

$$\oint_S \mathbf{E} \cdot \mathbf{dS} = \frac{1}{\epsilon_0} \int_V \left(\rho - \nabla \cdot \mathbf{P}\right) dV. \tag{3.16}$$

Similarly, the magnetostatic field **B** is described by

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \nabla \times \mathbf{M}\right), \tag{3.17}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3.18}$$

or

$$\oint_\Gamma \mathbf{B} \cdot \mathbf{dl} = \mu_0 \int_S \left(\mathbf{J} + \nabla \times \mathbf{M}\right) \cdot \mathbf{dS}, \tag{3.19}$$

$$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0. \tag{3.20}$$

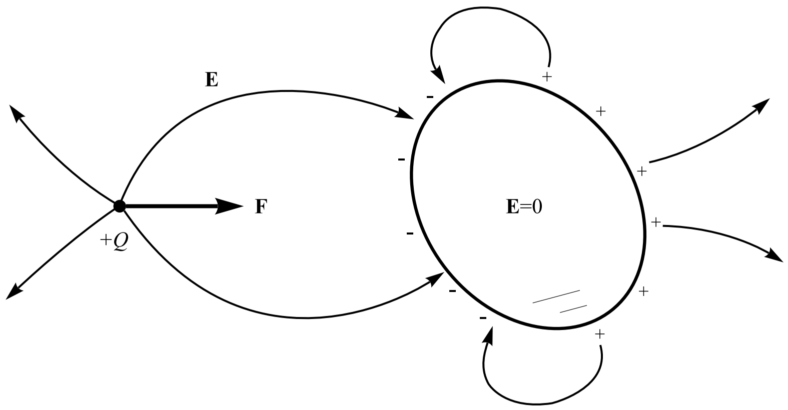

Figure 3.1: Positive point charge in the vicinity of an insulated, uncharged conductor.

It is important to note that any separation of the electromagnetic field into independent static electric and magnetic portions is illusory. As we mentioned in § 2.3.2, the electric and magnetic components of the EM field depend on the motion of the observer. An observer stationary with respect to a single charge measures only a static electric field, while an observer in uniform motion with respect to the charge measures both electric and magnetic fields.

3.1.2 Static field equilibrium and conductors

Suppose we could arrange a group of electric charges into a static configuration in free space. The charges would produce an electric field, resulting in a force on the distribution via the Lorentz force law, and hence would begin to move. Regardless of how we arrange the charges they cannot maintain their original static configuration without the help of some mechanical force to counterbalance the electrical force. This is a statement of Earnshaw's theorem, discussed in detail in § 3.4.2.

The situation is similar for charges within and on electric conductors. A conductor is a material having many charges free to move under external influences, both electric and non-electric. In a metallic conductor, electrons move against a background lattice of positive charges. An *uncharged conductor* is neutral: the amount of negative charge carried by the electrons is equal to the positive charge in the background lattice. The distribution of charges in an uncharged conductor is such that the macroscopic electric field is zero inside and outside the conductor. When the conductor is exposed to an additional electric field, the electrons move under the influence of the Lorentz force, creating a *conduction current*. Rather than accelerating indefinitely, conduction electrons experience collisions with the lattice, thereby giving up their kinetic energy. Macroscopically, the charge motion can be described in terms of a time-average velocity, hence a macroscopic current density can be assigned to the density of moving charge. The relationship between the applied, or "impressed," field and the resulting current density is given by *Ohm's law*; in a linear, isotropic, nondispersive material this is

$$\mathbf{J}(\mathbf{r}, t) = \sigma(\mathbf{r})\mathbf{E}(\mathbf{r}, t). \tag{3.21}$$

The conductivity σ describes the impediment to charge motion through the lattice: the

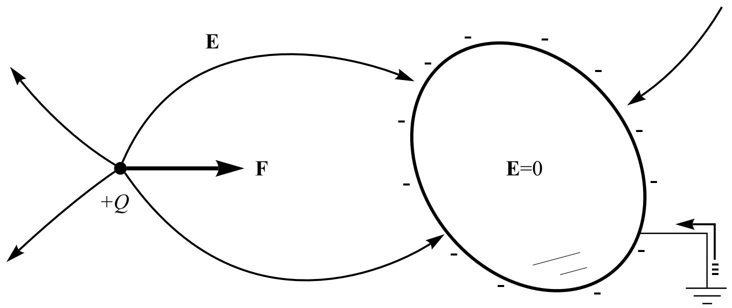

Figure 3.2: Positive point charge near a grounded conductor.

higher the conductivity, the farther an electron may move on average before undergoing a collision.

Let us examine how a state of equilibrium is established in a conductor. We shall consider several important situations. First, suppose we bring a positively charged particle into the vicinity of a neutral, insulated conductor (we say that a conductor is "insulated" if no means exists for depositing excess charge onto the conductor). The Lorentz force on the free electrons in the conductor results in their motion toward the particle (Figure 3.1). A reaction force $\mathbf{F}$ attracts the particle to the conductor. If the particle and the conductor are both held rigidly in space by an external mechanical force, the electrons within the conductor continue to move toward the surface. In a metal, when these electrons reach the surface and try to continue further they experience a rapid reversal in the direction of the Lorentz force, drawing them back toward the surface. A sufficiently large force (described by the *work function* of the metal) will be able to draw these charges from the surface, but anything less will permit the establishment of a stable equilibrium at the surface. If σ is large then equilibrium is established quickly, and a nonuniform static charge distribution appears on the conductor surface. The electric field within the conductor must settle to zero at equilibrium, since a nonzero field would be associated with a current $\mathbf{J} = \sigma\mathbf{E}$. In addition, the component of the field tangential to the surface must be zero or the charge would be forced to move along the surface. *At equilibrium, the field within and tangential to a conductor must be zero.* Note also that equilibrium cannot be established without external forces to hold the conductor and particle in place.

Next, suppose we bring a positively charged particle into the vicinity of a *grounded* (rather than insulated) conductor as in Figure 3.2. Use of the term "grounded" means that the conductor is attached via a filamentary conductor to a remote reservoir of charge known as *ground*; in practical applications the earth acts as this charge reservoir. Charges are drawn from or returned to the reservoir, without requiring any work, in response to the Lorentz force on the charge within the conducting body. As the particle approaches, negative charge is drawn to the body and then along the surface until a static equilibrium is re-established. Unlike the insulated body, the grounded conductor in equilibrium has excess negative charge, the amount of which depends on the proximity of the particle. Again, both particle and conductor must be held in place by external mechanical forces, and the total field produced by both the static charge on the conductor and the particle must be zero at points interior to the conductor.

Finally, consider the process whereby excess charge placed inside a conducting body redistributes as equilibrium is established. We assume an isotropic, homogeneous conducting body with permittivity ϵ and conductivity σ. An initially static charge with density $\rho_0(\mathbf{r})$ is introduced at time $t = 0$. The charge density must obey the continuity equation

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) = -\frac{\partial \rho(\mathbf{r}, t)}{\partial t};$$

since $\mathbf{J} = \sigma \mathbf{E}$, we have

$$\sigma \nabla \cdot \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \rho(\mathbf{r}, t)}{\partial t}.$$

By Gauss's law, $\nabla \cdot \mathbf{E}$ can be eliminated:

$$\frac{\sigma}{\epsilon} \rho(\mathbf{r}, t) = -\frac{\partial \rho(\mathbf{r}, t)}{\partial t}.$$

Solving this differential equation for the unknown $\rho(\mathbf{r}, t)$ we have

$$\rho(\mathbf{r}, t) = \rho_0(\mathbf{r}) e^{-\sigma t/\epsilon}. \tag{3.22}$$

The charge density within a homogeneous, isotropic conducting body decreases exponentially with time, regardless of the original charge distribution and shape of the body. Of course, the total charge must be constant, and thus charge within the body travels to the surface where it distributes itself in such a way that the field internal to the body approaches zero at equilibrium. The rate at which the volume charge dissipates is determined by the *relaxation time* ϵ/σ; for copper (a good conductor) this is an astonishingly small 10^{-19} s. Even distilled water, a relatively poor conductor, has $\epsilon/\sigma = 10^{-6}$ s. Thus we see how rapidly static equilibrium can be approached.

3.1.3 Steady current

Since time-invariant fields must arise from time-invariant sources, we have from the continuity equation

$$\nabla \cdot \mathbf{J}(\mathbf{r}) = 0. \tag{3.23}$$

In large-scale form this is

$$\oint_S \mathbf{J} \cdot \mathbf{dS} = 0. \tag{3.24}$$

A current with the property (3.23) is said to be a *steady current*. By (3.24), a steady current must be completely lineal (and infinite in extent) or must form closed loops. However, if a current forms loops then the individual moving charges must undergo acceleration (from the change in direction of velocity). Since a single accelerating particle radiates energy in the form of an electromagnetic wave, we might expect a large steady loop current to produce a great deal of radiation. In fact, if we superpose the fields produced by the many particles comprising a steady current, we find that a steady current produces no radiation [91]. Remarkably, to obtain this result we must consider the exact relativistic fields, and thus our finding is precise within the limits of our macroscopic assumptions.

If we try to create a steady current in free space, the flowing charges will tend to disperse because of the Lorentz force from the field set up by the charges, and the resulting current will not form closed loops. A beam of electrons or ions will produce both an electric field (because of the nonzero net charge of the beam) and a magnetic field (because of the current). At nonrelativistic particle speeds, the electric field produces an outward force on the charges that is much greater than the inward (or *pinch*) force produced by the magnetic field. Application of an additional, external force will allow the creation of a *collimated beam* of charge, as occurs in an electron tube where a series of permanent magnets can be used to create a beam of steady current.

More typically, steady currents are created using wire conductors to guide the moving charge. When an external force, such as the electric field created by a battery, is applied to an uncharged conductor, the free electrons will begin to move through the positive lattice, forming a current. Each electron moves only a short distance before colliding with the positive lattice, and if the wire is bent into a loop the resulting macroscopic current will be steady in the sense that the temporally and spatially averaged microscopic current will obey $\nabla \cdot \mathbf{J} = 0$. We note from the examples above that any charges attempting to leave the surface of the wire are drawn back by the electrostatic force produced by the resulting imbalance in electrical charge. For conductors, the "drift" velocity associated with the moving electrons is proportional to the applied field:

$$\mathbf{u}_d = -\mu_e \mathbf{E}$$

where μ_e is the *electron mobility*. The mobility of copper ($3.2 \times 10^{-3}\text{m}^2/\text{V} \cdot \text{s}$) is such that an applied field of 1 V/m results in a drift velocity of only a third of a centimeter per second.

Integral properties of a steady current. Steady currents obey several useful integral properties. To develop these properties we need an integral identity. Let $f(\mathbf{r})$ and $g(\mathbf{r})$ be scalar functions, continuous and with continuous derivatives in a volume region V. Let $\mathbf{J}$ represent a steady current field of finite extent, completely contained within V. We begin by using (B.42) to expand

$$\nabla \cdot (fg\mathbf{J}) = fg(\nabla \cdot \mathbf{J}) + \mathbf{J} \cdot \nabla(fg).$$

Noting that $\nabla \cdot \mathbf{J} = 0$ and using (B.41), we get

$$\nabla \cdot (fg\mathbf{J}) = (f\mathbf{J}) \cdot \nabla g + (g\mathbf{J}) \cdot \nabla f.$$

Now let us integrate over V and employ the divergence theorem:

$$\oint_S (fg)\mathbf{J} \cdot \mathbf{dS} = \int_V [(f\mathbf{J}) \cdot \nabla g + (g\mathbf{J}) \cdot \nabla f]\, dV.$$

Since $\mathbf{J}$ is contained entirely within S, we must have $\hat{\mathbf{n}} \cdot \mathbf{J} = 0$ everywhere on S. Hence

$$\int_V [(f\mathbf{J}) \cdot \nabla g + (g\mathbf{J}) \cdot \nabla f]\, dV = 0. \tag{3.25}$$

We can obtain a useful relation by letting $f = 1$ and $g = x_i$ in (3.25), where $(x, y, z) = (x_1, x_2, x_3)$. This gives

$$\int_V J_i(\mathbf{r})\, dV = 0, \tag{3.26}$$

where $J_1 = J_x$ and so on. Hence the volume integral of any rectangular component of $\mathbf{J}$ is zero. Similarly, letting $f = g = x_i$ we find that

$$\int_V x_i J_i(\mathbf{r})\, dV = 0. \tag{3.27}$$

With $f = x_i$ and $g = x_j$ we obtain

$$\int_V \left[x_i \mathbf{J}_j(\mathbf{r}) + x_j \mathbf{J}_i(\mathbf{r})\right] dV = 0. \tag{3.28}$$

3.2 Electrostatics

3.2.1 The electrostatic potential and work

The equation

$$\oint_\Gamma \mathbf{E} \cdot \mathbf{dl} = 0 \tag{3.29}$$

satisfied by the electrostatic field $\mathbf{E}(\mathbf{r})$ is particularly interesting. A field with zero circulation is said to be *conservative*. To see why, let us examine the work required to move a particle of charge Q around a closed path in the presence of $\mathbf{E}(\mathbf{r})$. Since work is the line integral of force and $\mathbf{B} = 0$, the work expended by the external system moving the charge against the Lorentz force is

$$W = -\oint_\Gamma (Q\mathbf{E} + Q\mathbf{v} \times \mathbf{B}) \cdot \mathbf{dl} = -Q \oint_\Gamma \mathbf{E} \cdot \mathbf{dl} = 0.$$

This property is analogous to the conservation property for a classical gravitational field: any potential energy gained by raising a point mass is lost when the mass is lowered.

Direct experimental verification of the electrostatic conservative property is difficult, aside from the fact that the motion of Q may alter $\mathbf{E}$ by interacting with the sources of $\mathbf{E}$. By moving Q with nonuniform velocity (i.e., with acceleration at the beginning of the loop, direction changes in transit, and deceleration at the end) we observe a radiative loss of energy, and this energy cannot be regained by the mechanical system providing the motion. To avoid this problem we may assume that the charge is moved so slowly, or in such small increments, that it does not radiate. We shall use this concept later to determine the "assembly energy" in a charge distribution.

The electrostatic potential. By the point form of (3.29),

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0,$$

we can introduce a scalar field $\Phi = \Phi(\mathbf{r})$ such that

$$\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}). \tag{3.30}$$

The function Φ carries units of volts and is known as the *electrostatic potential*. Let us consider the work expended by an external agent in moving a charge between points P_1 at $\mathbf{r}_1$ and P_2 at $\mathbf{r}_2$:

$$W_{21} = -Q \int_{P_1}^{P_2} -\nabla\Phi(\mathbf{r}) \cdot \mathbf{dl} = Q \int_{P_1}^{P_2} d\Phi(\mathbf{r}) = Q\left[\Phi(\mathbf{r}_2) - \Phi(\mathbf{r}_1)\right].$$

The work W_{21} is clearly independent of the path taken between P_1 and P_2; the quantity

$$V_{21} = \frac{W_{21}}{Q} = \Phi(\mathbf{r}_2) - \Phi(\mathbf{r}_1) = -\int_{P_1}^{P_2} \mathbf{E} \cdot \mathbf{dl}, \tag{3.31}$$

called the *potential difference*, has an obvious physical meaning as work per unit charge required to move a particle against an electric field between two points.

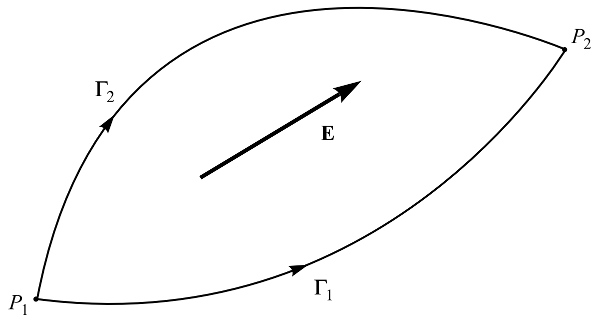

Figure 3.3: Demonstration of path independence of the electric field line integral.

Of course, the large-scale form (3.29) also implies the path-independence of work in the electrostatic field. Indeed, we may pass an arbitrary closed contour Γ through P_1 and P_2 and then split it into two pieces Γ_1 and Γ_2 as shown in Figure 3.3. Since

$$-Q \oint_{\Gamma_1 - \Gamma_2} \mathbf{E} \cdot \mathbf{dl} = -Q \int_{\Gamma_1} \mathbf{E} \cdot \mathbf{dl} + Q \int_{\Gamma_2} \mathbf{E} \cdot \mathbf{dl} = 0,$$

we have

$$-Q \int_{\Gamma_1} \mathbf{E} \cdot \mathbf{dl} = -Q \int_{\Gamma_2} \mathbf{E} \cdot \mathbf{dl}$$

as desired.

We sometimes refer to $\Phi(\mathbf{r})$ as the *absolute electrostatic potential*. Choosing a suitable reference point P_0 at location $\mathbf{r}_0$ and writing the potential difference as

$$V_{21} = [\Phi(\mathbf{r}_2) - \Phi(\mathbf{r}_0)] - [\Phi(\mathbf{r}_1) - \Phi(\mathbf{r}_0)],$$

we can justify calling $\Phi(\mathbf{r})$ the *absolute potential referred to* P_0. Note that P_0 might describe a locus of points, rather than a single point, since many points can be at the same potential. Although we can choose any reference point without changing the resulting value of $\mathbf{E}$ found from (3.30), for simplicity we often choose $\mathbf{r}_0$ such that $\Phi(\mathbf{r}_0) = 0$.

Several properties of the electrostatic potential make it convenient for describing static electric fields. We know that, at equilibrium, the electrostatic field within a conducting body must vanish. By (3.30) the potential at all points within the body must therefore have the same constant value. It follows that the surface of a conductor is an *equipotential surface*: a surface for which $\Phi(\mathbf{r})$ is constant.

As an infinite reservoir of charge that can be tapped through a filamentary conductor, the entity we call "ground" must also be an equipotential object. If we connect a conductor to ground, we have seen that charge may flow freely onto the conductor. Since no work is expended, "grounding" a conductor obviously places the conductor at the same absolute potential as ground. For this reason, ground is often assigned the role as the potential reference with an absolute potential of zero volts. Later we shall see that for sources of finite extent ground must be located at infinity.

3.2.2 Boundary conditions

Boundary conditions for the electrostatic field. The boundary conditions found for the dynamic electric field remain valid in the electrostatic case. Thus

$$\hat{\mathbf{n}}_{12} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \tag{3.32}$$

and

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s. \tag{3.33}$$

Here $\hat{\mathbf{n}}_{12}$ points into region 1 from region 2. Because the static curl and divergence equations are independent, so are the boundary conditions (3.32) and (3.33).

For a linear and isotropic dielectric where $\mathbf{D} = \epsilon\mathbf{E}$, equation (3.33) becomes

$$\hat{\mathbf{n}}_{12} \cdot (\epsilon_1\mathbf{E}_1 - \epsilon_2\mathbf{E}_2) = \rho_s. \tag{3.34}$$

Alternatively, using $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ we can write (3.33) as

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = \frac{1}{\epsilon_0}(\rho_s + \rho_{Ps1} + \rho_{Ps2}) \tag{3.35}$$

where

$$\rho_{Ps} = \hat{\mathbf{n}} \cdot \mathbf{P}$$

is the polarization surface charge with $\hat{\mathbf{n}}$ pointing outward from the material body.

We can also write the boundary conditions in terms of the electrostatic potential. With $\mathbf{E} = -\nabla\Phi$, equation (3.32) becomes

$$\Phi_1(\mathbf{r}) = \Phi_2(\mathbf{r}) \tag{3.36}$$

for all points $\mathbf{r}$ on the surface. Actually Φ_1 and Φ_2 may differ by a constant; because this constant is eliminated when the gradient is taken to find $\mathbf{E}$, it is generally ignored. We can write (3.35) as

$$\epsilon_0\left(\frac{\partial\Phi_1}{\partial n} - \frac{\partial\Phi_2}{\partial n}\right) = -\rho_s - \rho_{Ps1} - \rho_{Ps2}$$

where the normal derivative is taken in the $\hat{\mathbf{n}}_{12}$ direction. For a linear, isotropic dielectric (3.33) becomes

$$\epsilon_1\frac{\partial\Phi_1}{\partial n} - \epsilon_2\frac{\partial\Phi_2}{\partial n} = -\rho_s. \tag{3.37}$$

Again, we note that (3.36) and (3.37) are independent.

Boundary conditions for steady electric current. The boundary condition on the normal component of current found in § 2.8.2 remains valid in the steady current case. Assume that the boundary exists between two linear, isotropic conducting regions having constitutive parameters (ϵ_1,σ_1) and (ϵ_2,σ_2), respectively. By (2.198) we have

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{J}_1 - \mathbf{J}_2) = -\nabla_s \cdot \mathbf{J}_s \tag{3.38}$$

where $\hat{\mathbf{n}}_{12}$ points into region 1 from region 2. A surface current will not appear on the boundary between two regions having finite conductivity, although a surface charge may accumulate there during the transient period when the currents are established [31]. If charge is influenced to move from the surface, it will move into the adjacent regions,

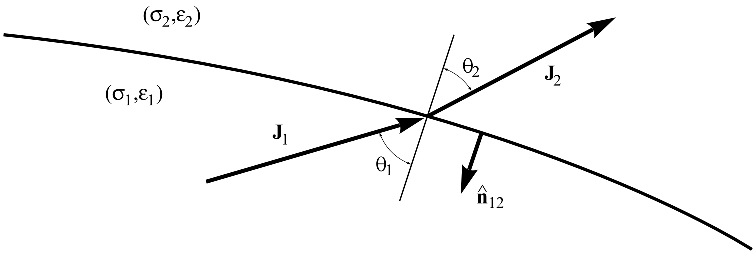

Figure 3.4: Refraction of steady current at a material interface.

rather than along the surface, and a new charge will replace it, supplied by the current. Thus, for finite conducting regions (3.38) becomes

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{J}_1 - \mathbf{J}_2) = 0. \tag{3.39}$$

A boundary condition on the tangential component of current can also be found. Substituting $\mathbf{E} = \mathbf{J}/\sigma$ into (3.32) we have

$$\hat{\mathbf{n}}_{12} \times \left(\frac{\mathbf{J}_1}{\sigma_1} - \frac{\mathbf{J}_2}{\sigma_2} \right) = 0.$$

We can also write this as

$$\frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2} \tag{3.40}$$

where

$$\mathbf{J}_{1t} = \hat{\mathbf{n}}_{12} \times \mathbf{J}_1, \qquad \mathbf{J}_{2t} = \hat{\mathbf{n}}_{12} \times \mathbf{J}_2.$$

We may combine the boundary conditions for the normal components of current and electric field to better understand the behavior of current at a material boundary. Substituting $\mathbf{E} = \mathbf{J}/\sigma$ into (3.34) we have

$$\frac{\epsilon_1}{\sigma_1} J_{1n} - \frac{\epsilon_2}{\sigma_2} J_{2n} = \rho_s \tag{3.41}$$

where $J_{1n} = \hat{\mathbf{n}}_{12} \cdot \mathbf{J}_1$ and $J_{2n} = \hat{\mathbf{n}}_{12} \cdot \mathbf{J}_2$. Combining (3.41) with (3.39), we have

$$\rho_s = J_{1n} \left(\frac{\epsilon_1}{\sigma_1} - \frac{\epsilon_2}{\sigma_2} \right) = E_{1n} \left(\epsilon_1 - \frac{\sigma_1}{\sigma_2} \epsilon_2 \right) = J_{2n} \left(\frac{\epsilon_1}{\sigma_1} - \frac{\epsilon_2}{\sigma_2} \right) = E_{2n} \left(\epsilon_1 \frac{\sigma_2}{\sigma_1} - \epsilon_2 \right)$$

where

$$E_{1n} = \hat{\mathbf{n}}_{12} \cdot \mathbf{E}_1, \qquad E_{2n} = \hat{\mathbf{n}}_{12} \cdot \mathbf{E}_2.$$

Unless $\epsilon_1 \sigma_2 - \sigma_1 \epsilon_2 = 0$, a surface charge will exist on the interface between dissimilar current-carrying conductors.

We may also combine the vector components of current on each side of the boundary to determine the effects of the boundary on current direction (Figure 3.4). Let $\theta_{1,2}$ denote the angle between $\mathbf{J}_{1,2}$ and $\hat{\mathbf{n}}_{12}$ so that

$$J_{1n} = J_1 \cos\theta_1, \qquad J_{1t} = J_1 \sin\theta_1$$
$$J_{2n} = J_2 \cos\theta_2, \qquad J_{2t} = J_2 \sin\theta_2.$$

Then $J_1 \cos\theta_1 = J_2 \cos\theta_2$ by (3.39), while $\sigma_2 J_1 \sin\theta_1 = \sigma_1 J_2 \sin\theta_2$ by (3.40). Hence

$$\sigma_2 \tan\theta_1 = \sigma_1 \tan\theta_2. \tag{3.42}$$

It is interesting to consider the case of current incident from a conducting material onto an insulating material. If region 2 is an insulator, then $J_{2n} = J_{2t} = 0$; by (3.39) we have $J_{1n} = 0$. But (3.40) does not require $J_{1t} = 0$; with $\sigma_2 = 0$ the right-hand side of (3.40) is indeterminate and thus J_{1t} may be nonzero. In other words, when current moving through a conductor approaches an insulating surface, it bends and flows tangential to the surface. This concept is useful in explaining how wires guide current.

Interestingly, (3.42) shows that when $\sigma_2 \ll \sigma_1$ we have $\theta_2 \to 0$; current passing from a conducting region into a slightly-conducting region does so normally.

3.2.3 Uniqueness of the electrostatic field

In § 2.2.1 we found that the electromagnetic field is unique within a region V when the tangential component of $\mathbf{E}$ is specified over the surrounding surface. Unfortunately, this condition is not appropriate in the electrostatic case. We should remember that an additional requirement for uniqueness of solution to Maxwell's equations is that the field be specified throughout V at some time t_0. For a static field this would completely determine $\mathbf{E}$ without need for the surface field!

Let us determine conditions for uniqueness beginning with the static field equations. Consider a region V surrounded by a surface S. Static charge may be located entirely or partially within V, or entirely outside V, and produces a field within V. The region may also contain any arrangement of conductors or other materials. Suppose $(\mathbf{D}_1, \mathbf{E}_1)$ and $(\mathbf{D}_2, \mathbf{E}_2)$ represent solutions to the static field equations within V with source $\rho(\mathbf{r})$. We wish to find conditions that guarantee both $\mathbf{E}_1 = \mathbf{E}_2$ and $\mathbf{D}_1 = \mathbf{D}_2$.

Since $\nabla \cdot \mathbf{D}_1 = \rho$ and $\nabla \cdot \mathbf{D}_2 = \rho$, the difference field $\mathbf{D}_0 = \mathbf{D}_2 - \mathbf{D}_1$ obeys the homogeneous equation

$$\nabla \cdot \mathbf{D}_0 = 0. \tag{3.43}$$

Consider the quantity

$$\nabla \cdot (\mathbf{D}_0 \Phi_0) = \Phi_0(\nabla \cdot \mathbf{D}_0) + \mathbf{D}_0 \cdot (\nabla \Phi_0)$$

where $\mathbf{E}_0 = \mathbf{E}_2 - \mathbf{E}_1 = -\nabla\Phi_0 = -\nabla(\Phi_2 - \Phi_1)$. We integrate over V and use the divergence theorem and (3.43) to obtain

$$\oint_S \Phi_0 \, [\mathbf{D}_0 \cdot \hat{\mathbf{n}}] \, dS = \int_V \mathbf{D}_0 \cdot (\nabla \Phi_0) \, dV = -\int_V \mathbf{D}_0 \cdot \mathbf{E}_0 \, dV. \tag{3.44}$$

Now suppose that $\Phi_0 = 0$ everywhere on S, or that $\hat{\mathbf{n}} \cdot \mathbf{D}_0 = 0$ everywhere on S, or that $\Phi_0 = 0$ over part of S and $\hat{\mathbf{n}} \cdot \mathbf{D}_0 = 0$ elsewhere on S. Then

$$\int_V \mathbf{D}_0 \cdot \mathbf{E}_0 \, dV = 0. \tag{3.45}$$

Since V is arbitrary, either $\mathbf{D}_0 = 0$ or $\mathbf{E}_0 = 0$. Assuming $\mathbf{E}$ and $\mathbf{D}$ are linked by the constitutive relations, we have $\mathbf{E}_1 = \mathbf{E}_2$ and $\mathbf{D}_1 = \mathbf{D}_2$.

Hence the fields within V are unique provided that either Φ, the normal component of $\mathbf{D}$, or some combination of the two, is specified over S. We often use a multiply-connected surface to exclude conductors. By (3.33) we see that specification of the normal component of $\mathbf{D}$ on a conductor is equivalent to specification of the surface charge density. Thus we must specify the potential or surface charge density over all conducting surfaces.

One other condition results in zero on the left-hand side of (3.44). If S recedes to infinity and Φ_0 and $\mathbf{D}_0$ decrease sufficiently fast, then (3.45) still holds and uniqueness is guaranteed. If $\mathbf{D}, \mathbf{E} \sim 1/r^2$ as $r \to \infty$, then $\Phi \sim 1/r$ and the surface integral in (3.44) tends to zero since the area of an expanding sphere increases only as r^2. We shall find later in this section that for sources of finite extent the fields do indeed vary inversely with distance squared from the source, hence we may allow S to expand and encompass all space.

For the case in which conducting bodies are immersed in an infinite homogeneous medium and the static fields must be determined throughout all space, a multiply-connected surface is used with one part receding to infinity and the remaining parts surrounding the conductors. Here uniqueness is guaranteed by specifying the potentials or charges on the surfaces of the conducting bodies.

3.2.4 Poisson's and Laplace's equations

For computational purposes it is often convenient to deal with the differential versions

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0, \tag{3.46}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r}), \tag{3.47}$$

of the electrostatic field equations. We must supplement these with constitutive relations between $\mathbf{E}$ and $\mathbf{D}$; at this point we focus our attention on linear, isotropic materials for which

$$\mathbf{D}(\mathbf{r}) = \epsilon(\mathbf{r})\mathbf{E}(\mathbf{r}).$$

Using this in (3.47) along with $\mathbf{E} = -\nabla\Phi$ (justified by (3.46)), we can write

$$\nabla \cdot [\epsilon(\mathbf{r})\nabla\Phi(\mathbf{r})] = -\rho(\mathbf{r}). \tag{3.48}$$

This is *Poisson's equation.* The corresponding homogeneous equation

$$\nabla \cdot [\epsilon(\mathbf{r})\nabla\Phi(\mathbf{r})] = 0, \tag{3.49}$$

holding at points $\mathbf{r}$ where $\rho(\mathbf{r}) = 0$, is *Laplace's equation.* Equations (3.48) and (3.49) are valid for inhomogeneous media. By (B.42) we can write

$$\nabla\Phi(\mathbf{r}) \cdot \nabla\epsilon(\mathbf{r}) + \epsilon(\mathbf{r})\nabla \cdot [\nabla\Phi(\mathbf{r})] = -\rho(\mathbf{r}).$$

For a homogeneous medium, $\nabla\epsilon = 0$; since $\nabla \cdot (\nabla\Phi) \equiv \nabla^2\Phi$, we have

$$\nabla^2\Phi(\mathbf{r}) = -\rho(\mathbf{r})/\epsilon \tag{3.50}$$

in such a medium. Correspondingly,

$$\nabla^2\Phi(\mathbf{r}) = 0$$

at points where $\rho(\mathbf{r}) = 0$.

Poisson's and Laplace's equations can be solved by separation of variables, Fourier transformation, conformal mapping, and numerical techniques such as the finite difference and moment methods. In Appendix A we consider the separation of variables solution to Laplace's equation in three major coordinate systems for a variety of problems. For an introduction to numerical techniques the reader is referred to the books by Sadiku [162], Harrington [82], and Peterson et al. [146]. Solution to Poisson's equation is often undertaken using the method of Green's functions, which we shall address later in this section. We shall also consider the solution to Laplace's equation for bodies immersed in an applied, or "impressed," field.

Uniqueness of solution to Poisson's equation. Before attempting any solutions, we must ask two very important questions. How do we know that solving the second-order differential equation produces the same values for $\mathbf{E} = -\nabla\Phi$ as solving the first-order equations directly for $\mathbf{E}$? And, if these solutions are the same, what are the conditions for uniqueness of solution to Poisson's and Laplace's equations? To answer the first question, a sufficient condition is to have Φ twice differentiable. We shall not attempt to prove this, but shall instead show that the condition for uniqueness of the second-order equations is the same as that for the first-order equations.

Consider a region of space V surrounded by a surface S. Static charge may be located entirely or partially within V, or entirely outside V, and produces a field within V. This region may also contain any arrangement of conductors or other materials. Now, assume that Φ_1 and Φ_2 represent solutions to the static field equations within V with source $\rho(\mathbf{r})$. We wish to find conditions under which $\Phi_1 = \Phi_2$.

Since we have

$$\nabla \cdot [\epsilon(\mathbf{r})\nabla\Phi_1(\mathbf{r})] = -\rho(\mathbf{r}), \qquad \nabla \cdot [\epsilon(\mathbf{r})\nabla\Phi_2(\mathbf{r})] = -\rho(\mathbf{r}),$$

the difference field $\Phi_0 = \Phi_2 - \Phi_1$ obeys

$$\nabla \cdot [\epsilon(\mathbf{r})\nabla\Phi_0(\mathbf{r})] = 0. \tag{3.51}$$

That is, Φ_0 obeys Laplace's equation. Now consider the quantity

$$\nabla \cdot (\epsilon\Phi_0\nabla\Phi_0) = \epsilon|\nabla\Phi_0|^2 + \Phi_0\nabla \cdot (\epsilon\nabla\Phi_0).$$

Integration over V and use of the divergence theorem and (3.51) gives

$$\oint_S \Phi_0(\mathbf{r})\,[\epsilon(\mathbf{r})\nabla\Phi_0(\mathbf{r})] \cdot \mathbf{dS} = \int_V \epsilon(\mathbf{r})|\nabla\Phi_0(\mathbf{r})|^2\,dV.$$

As with the first order equations, we see that specifying either $\Phi(\mathbf{r})$ or $\epsilon(\mathbf{r})\nabla\Phi(\mathbf{r}) \cdot \hat{\mathbf{n}}$ over S results in $\Phi_0(\mathbf{r}) = 0$ throughout V, hence $\Phi_1 = \Phi_2$. As before, specifying $\epsilon(\mathbf{r})\nabla\Phi(\mathbf{r}) \cdot \hat{\mathbf{n}}$ for a conducting surface is equivalent to specifying the surface charge on S.

Integral solution to Poisson's equation: the static Green's function. The method of Green's functions is one of the most useful techniques for solving Poisson's equation. We seek a solution for a single point source, then use Green's second identity to write the solution for an arbitrary charge distribution in terms of a superposition integral.

We seek the solution to Poisson's equation for a region of space V as shown in Figure 3.5. The region is assumed homogeneous with permittivity ϵ, and its surface is multiply-connected, consisting of a bounding surface S_B and any number of closed surfaces internal to V. We denote by S the composite surface consisting of S_B and the N internal surfaces S_n, $n = 1, \ldots, N$. The internal surfaces are used to exclude material bodies, such as the

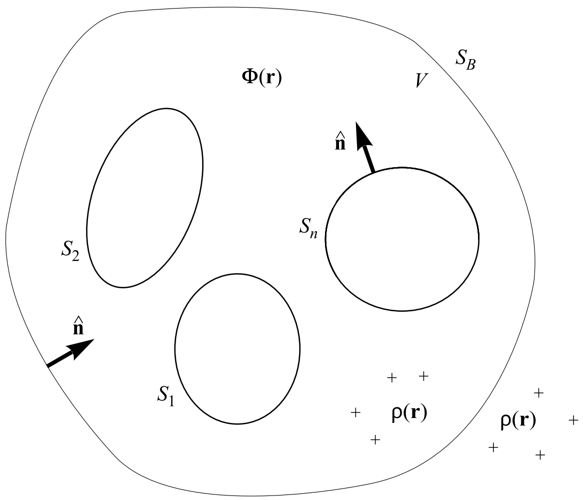

Figure 3.5: Computation of potential from known sources and values on bounding surfaces.

plates of a capacitor, which may be charged and on which the potential is assumed to be known. To solve for $\Phi(\mathbf{r})$ within V we must know the potential produced by a point source. This potential, called the *Green's function*, is denoted $G(\mathbf{r}|\mathbf{r}')$; it has two arguments because it satisfies Poisson's equation at $\mathbf{r}$ when the source is located at $\mathbf{r}'$:

$$\nabla^2 G(\mathbf{r}|\mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'). \tag{3.52}$$

Later we shall demonstrate that in all cases of interest to us the Green's function is symmetric in its arguments:

$$G(\mathbf{r}'|\mathbf{r}) = G(\mathbf{r}|\mathbf{r}'). \tag{3.53}$$

This property of G is known as *reciprocity.*

Our development rests on the mathematical result (B.30) known as *Green's second identity.* We can derive this by subtracting the identities

$$\nabla\cdot(\phi\nabla\psi) = \phi\nabla\cdot(\nabla\psi) + (\nabla\phi)\cdot(\nabla\psi),$$
$$\nabla\cdot(\psi\nabla\phi) = \psi\nabla\cdot(\nabla\phi) + (\nabla\psi)\cdot(\nabla\phi),$$

to obtain

$$\nabla\cdot(\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi.$$

Integrating this over a volume region V with respect to the dummy variable $\mathbf{r}'$ and using the divergence theorem, we obtain

$$\int_V [\phi(\mathbf{r}')\nabla'^2\psi(\mathbf{r}') - \psi(\mathbf{r}')\nabla'^2\phi(\mathbf{r}')]\,dV' = -\oint_S [\phi(\mathbf{r}')\nabla'\psi(\mathbf{r}') - \psi(\mathbf{r}')\nabla'\phi(\mathbf{r}')]\cdot\mathbf{dS}'.$$

The negative sign on the right-hand side occurs because $\hat{\mathbf{n}}$ is an *inward* normal to V. Finally, since $\partial\psi(\mathbf{r}')/\partial n' = \hat{\mathbf{n}}'\cdot\nabla'\psi(\mathbf{r}')$, we have

$$\int_V [\phi(\mathbf{r}')\nabla'^2\psi(\mathbf{r}') - \psi(\mathbf{r}')\nabla'^2\phi(\mathbf{r}')]\,dV' = -\oint_S \left[\phi(\mathbf{r}')\frac{\partial\psi(\mathbf{r}')}{\partial n'} - \psi(\mathbf{r}')\frac{\partial\phi(\mathbf{r}')}{\partial n'}\right] dS'$$

as desired.

To solve for Φ in V we shall make some seemingly unmotivated substitutions into this identity. First note that by (3.52) and (3.53) we can write

$$\nabla'^2 G(\mathbf{r}|\mathbf{r}') = -\delta(\mathbf{r}' - \mathbf{r}). \tag{3.54}$$

We now set $\phi(\mathbf{r}') = \Phi(\mathbf{r}')$ and $\psi(\mathbf{r}') = G(\mathbf{r}|\mathbf{r}')$ to obtain

$$\int_V [\Phi(\mathbf{r}')\nabla'^2 G(\mathbf{r}|\mathbf{r}') - G(\mathbf{r}|\mathbf{r}')\nabla'^2\Phi(\mathbf{r}')]\, dV' = \\ -\oint_S \left[\Phi(\mathbf{r}')\frac{\partial G(\mathbf{r}|\mathbf{r}')}{\partial n'} - G(\mathbf{r}|\mathbf{r}')\frac{\partial \Phi(\mathbf{r}')}{\partial n'}\right] dS', \tag{3.55}$$

hence

$$\int_V \left[\Phi(\mathbf{r}')\delta(\mathbf{r}' - \mathbf{r}) - G(\mathbf{r}|\mathbf{r}')\frac{\rho(\mathbf{r}')}{\epsilon}\right] dV' = \oint_S \left[\Phi(\mathbf{r}')\frac{\partial G(\mathbf{r}|\mathbf{r}')}{\partial n'} - G(\mathbf{r}|\mathbf{r}')\frac{\partial \Phi(\mathbf{r}')}{\partial n'}\right] dS'.$$

By the sifting property of the Dirac delta

$$\Phi(\mathbf{r}) = \int_V G(\mathbf{r}|\mathbf{r}')\frac{\rho(\mathbf{r}')}{\epsilon}\, dV' + \oint_{S_B} \left[\Phi(\mathbf{r}')\frac{\partial G(\mathbf{r}|\mathbf{r}')}{\partial n'} - G(\mathbf{r}|\mathbf{r}')\frac{\partial \Phi(\mathbf{r}')}{\partial n'}\right] dS' + \\ + \sum_{n=1}^{N} \oint_{S_n} \left[\Phi(\mathbf{r}')\frac{\partial G(\mathbf{r}|\mathbf{r}')}{\partial n'} - G(\mathbf{r}|\mathbf{r}')\frac{\partial \Phi(\mathbf{r}')}{\partial n'}\right] dS'. \tag{3.56}$$

With this we may compute the potential anywhere within V in terms of the charge density within V and the values of the potential and its normal derivative over S. We must simply determine $G(\mathbf{r}|\mathbf{r}')$ first.

Let us take a moment to specialize (3.56) to the case of unbounded space. Provided that the sources are of finite extent, as $S_B \to \infty$ we shall find that

$$\Phi(\mathbf{r}) = \int_V G(\mathbf{r}|\mathbf{r}')\frac{\rho(\mathbf{r}')}{\epsilon}\, dV' + \sum_{n=1}^{N} \oint_{S_n} \left[\Phi(\mathbf{r}')\frac{\partial G(\mathbf{r}|\mathbf{r}')}{\partial n'} - G(\mathbf{r}|\mathbf{r}')\frac{\partial \Phi(\mathbf{r}')}{\partial n'}\right] dS'.$$

A useful derivative identity. Many differential operations on the displacement vector $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ occur in the study of electromagnetics. The identities

$$\nabla R = -\nabla' R = \hat{\mathbf{R}}, \qquad \nabla\left(\frac{1}{R}\right) = -\nabla'\left(\frac{1}{R}\right) = -\frac{\hat{\mathbf{R}}}{R^2}, \tag{3.57}$$

for example, follow from direct differentiation of the rectangular coordinate representation

$$\mathbf{R} = \hat{\mathbf{x}}(x - x') + \hat{\mathbf{y}}(y - y') + \hat{\mathbf{z}}(z - z').$$

The identity

$$\nabla^2\left(\frac{1}{R}\right) = -4\pi\delta(\mathbf{r} - \mathbf{r}'), \tag{3.58}$$

crucial to potential theory, is more difficult to establish. We shall prove the equivalent version

$$\nabla'^2\left(\frac{1}{R}\right) = -4\pi\delta(\mathbf{r}' - \mathbf{r})$$

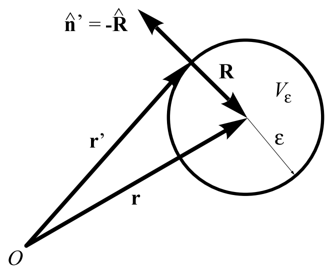

Figure 3.6: Geometry for establishing the singular property of $\nabla^2(1/R)$.

by showing that

$$\int_V f(\mathbf{r}')\nabla'^2\left(\frac{1}{R}\right)dV' = \begin{cases} -4\pi f(\mathbf{r}), & \mathbf{r}\in V, \\ 0, & \mathbf{r}\notin V, \end{cases} \tag{3.59}$$

holds for any continuous function $f(\mathbf{r})$. By direct differentiation we have

$$\nabla'^2\left(\frac{1}{R}\right) = 0 \text{ for } \mathbf{r}' \neq \mathbf{r},$$

hence the second part of (3.59) is established. This also shows that if $\mathbf{r}\in V$ then the domain of integration in (3.59) can be restricted to a sphere of arbitrarily small radius ε centered at $\mathbf{r}$ (Figure 3.6). The result we seek is found in the limit as $\varepsilon\to 0$. Thus we are interested in computing

$$\int_V f(\mathbf{r}')\nabla'^2\left(\frac{1}{R}\right)dV' = \lim_{\varepsilon\to 0}\int_{V_\varepsilon} f(\mathbf{r}')\nabla'^2\left(\frac{1}{R}\right)dV'.$$

Since f is continuous at $\mathbf{r}'=\mathbf{r}$, we have by the mean value theorem

$$\int_V f(\mathbf{r}')\nabla'^2\left(\frac{1}{R}\right)dV' = f(\mathbf{r})\lim_{\varepsilon\to 0}\int_{V_\varepsilon} \nabla'^2\left(\frac{1}{R}\right)dV'.$$

The integral over V_ε can be computed using $\nabla'^2(1/R) = \nabla'\cdot\nabla'(1/R)$ and the divergence theorem:

$$\int_{V_\varepsilon} \nabla'^2\left(\frac{1}{R}\right)dV' = \int_{S_\varepsilon} \hat{\mathbf{n}}'\cdot\nabla'\left(\frac{1}{R}\right)dS',$$

where S_ε bounds V_ε. Noting that $\hat{\mathbf{n}}' = -\hat{\mathbf{R}}$, using (57), and writing the integral in spherical coordinates $(\varepsilon,\theta,\phi)$ centered at the point $\mathbf{r}$, we have

$$\int_V f(\mathbf{r}')\nabla'^2\left(\frac{1}{R}\right)dV' = f(\mathbf{r})\lim_{\varepsilon\to 0}\int_0^{2\pi}\int_0^{\pi} -\hat{\mathbf{R}}\cdot\left(\frac{\hat{\mathbf{R}}}{\varepsilon^2}\right)\varepsilon^2\sin\theta\,d\theta\,d\phi = -4\pi f(\mathbf{r}).$$

Hence the first part of (3.59) is also established.

The Green's function for unbounded space. In view of (3.58), one solution to (3.52) is

$$G(\mathbf{r}|\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{3.60}$$

This simple Green's function is generally used to find the potential produced by charge in unbounded space. Here $N = 0$ (no internal surfaces) and $S_B \to \infty$. Thus

$$\Phi(\mathbf{r}) = \int_V G(\mathbf{r}|\mathbf{r}')\frac{\rho(\mathbf{r}')}{\epsilon}\,dV' + \lim_{S_B\to\infty}\oint_{S_B}\left[\Phi(\mathbf{r}')\frac{\partial G(\mathbf{r}|\mathbf{r}')}{\partial n'} - G(\mathbf{r}|\mathbf{r}')\frac{\partial\Phi(\mathbf{r}')}{\partial n'}\right]dS'.$$

We have seen that the Green's function varies inversely with distance from the source, and thus expect that, as a superposition of point-source potentials, $\Phi(\mathbf{r})$ will also vary inversely with distance from a source of finite extent as that distance becomes large with respect to the size of the source. The normal derivatives then vary inversely with distance squared. Thus, each term in the surface integrand will vary inversely with distance cubed, while the surface area itself varies with distance squared. The result is that the surface integral vanishes as the surface recedes to infinity, giving

$$\Phi(\mathbf{r}) = \int_V G(\mathbf{r}|\mathbf{r}')\frac{\rho(\mathbf{r}')}{\epsilon}\,dV'.$$

By (3.60) we then have

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV' \tag{3.61}$$

where the integration is performed over all of space. Since

$$\lim_{\mathbf{r}\to\infty}\Phi(\mathbf{r}) = 0,$$

points at infinity are a convenient reference for the absolute potential.

Later we shall need to know the amount of work required to move a charge Q from infinity to a point P located at $\mathbf{r}$. If a potential field is produced by charge located in unbounded space, moving an additional charge into position requires the work

$$W_{21} = -Q\int_\infty^P \mathbf{E}\cdot\mathbf{dl} = Q[\Phi(\mathbf{r}) - \Phi(\infty)] = Q\Phi(\mathbf{r}). \tag{3.62}$$

Coulomb's law. We can obtain $\mathbf{E}$ from (61) by direct differentiation. We have

$$\mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\epsilon}\nabla\int_V\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV' = -\frac{1}{4\pi\epsilon}\int_V\rho(\mathbf{r}')\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dV',$$

hence

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon}\int_V\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,dV' \tag{3.63}$$

by (3.57). So Coulomb's law follows from the two fundamental postulates of electrostatics (3.5) and (3.6).

Green's function for unbounded space: two dimensions. We define the two-dimensional Green's function as the potential at a point $\mathbf{r} = \boldsymbol{\rho} + \hat{\mathbf{z}}z$ produced by a z-directed line source of constant density located at $\mathbf{r}' = \boldsymbol{\rho}'$. Perhaps the simplest way to compute this is to first find $\mathbf{E}$ produced by a line source on the z-axis. By (3.63) we have

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon}\int_\Gamma\rho_l(z')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,dl'.$$

Then, since $\mathbf{r} = \hat{\mathbf{z}}z + \hat{\boldsymbol{\rho}}\rho$, $\mathbf{r}' = \hat{\mathbf{z}}z'$, and $dl' = dz'$, we have

$$\mathbf{E}(\boldsymbol{\rho}) = \frac{\rho_l}{4\pi\epsilon}\int_{-\infty}^{\infty}\frac{\hat{\boldsymbol{\rho}}\rho + \hat{\mathbf{z}}(z-z')}{\left[\rho^2 + (z-z')^2\right]^{3/2}}\,dz'.$$

Carrying out the integration we find that $\mathbf{E}$ has only a ρ-component which varies only with ρ:

$$\mathbf{E}(\boldsymbol{\rho}) = \hat{\boldsymbol{\rho}}\frac{\rho_l}{2\pi\epsilon\rho}. \tag{3.64}$$

The absolute potential referred to a radius ρ_0 can be found by computing the line integral of $\mathbf{E}$ from ρ to ρ_0:

$$\Phi(\boldsymbol{\rho}) = -\frac{\rho_l}{2\pi\epsilon}\int_{\rho_0}^{\rho}\frac{d\rho'}{\rho'} = \frac{\rho_l}{2\pi\epsilon}\ln\left(\frac{\rho_0}{\rho}\right).$$

We may choose any reference point ρ_0 except $\rho_0 = 0$ or $\rho_0 = \infty$. This choice is equivalent to the addition of an arbitrary constant, hence we can also write

$$\Phi(\boldsymbol{\rho}) = \frac{\rho_l}{2\pi\epsilon}\ln\left(\frac{1}{\rho}\right) + C. \tag{3.65}$$

The potential for a general two-dimensional charge distribution in unbounded space is by superposition

$$\Phi(\boldsymbol{\rho}) = \int_{S_T}\frac{\rho_T(\boldsymbol{\rho}')}{\epsilon}G(\boldsymbol{\rho}|\boldsymbol{\rho}')\,dS', \tag{3.66}$$

where the Green's function is the potential of a unit line source located at $\boldsymbol{\rho}'$:

$$G(\boldsymbol{\rho}|\boldsymbol{\rho}') = \frac{1}{2\pi}\ln\left(\frac{\rho_0}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|}\right). \tag{3.67}$$

Here S_T denotes the transverse (xy) plane, and ρ_T denotes the two-dimensional charge distribution $(\mathrm{C/m^2})$ within that plane.

We note that the potential field (3.66) of a two-dimensional source decreases logarithmically with distance. Only the potential produced by a source of finite extent decreases inversely with distance.

Dirichlet and Neumann Green's functions. The unbounded space Green's function may be inconvenient for expressing the potential in a region having internal surfaces. In fact, (3.56) shows that to use this function we would be forced to specify both Φ and its normal derivative over all surfaces. This, of course, would exceed the actual requirements for uniqueness.

Many functions can satisfy (3.52). For instance,

$$G(\mathbf{r}|\mathbf{r}') = \frac{A}{|\mathbf{r}-\mathbf{r}'|} + \frac{B}{|\mathbf{r}-\mathbf{r}_i|} \tag{3.68}$$

satisfies (3.52) if $\mathbf{r}_i \notin V$. Evaluation of (3.55) with the Green's function (3.68) reproduces the general formulation (3.56) since the Laplacian of the second term in (3.68) is identically zero in V. In fact, we can add any function to the free-space Green's function, provided that the additional term obeys Laplace's equation within V:

$$G(\mathbf{r}|\mathbf{r}') = \frac{A}{|\mathbf{r}-\mathbf{r}'|} + F(\mathbf{r}|\mathbf{r}'), \qquad \nabla'^2 F(\mathbf{r}|\mathbf{r}') = 0. \tag{3.69}$$

A good choice for $G(\mathbf{r}|\mathbf{r}')$ will minimize the effort required to evaluate $\Phi(\mathbf{r})$. Examining (3.56) we notice two possibilities. If we demand that

$$G(\mathbf{r}|\mathbf{r}') = 0 \text{ for all } \mathbf{r}' \in S \tag{3.70}$$

then the surface integral terms in (3.56) involving $\partial\Phi/\partial n'$ will vanish. The Green's function satisfying (3.70) is known as the *Dirichlet Green's function*. Let us designate it by G_D and use reciprocity to write (3.70) as

$$G_D(\mathbf{r}|\mathbf{r}') = 0 \text{ for all } \mathbf{r} \in S.$$

The resulting specialization of (3.56),

$$\Phi(\mathbf{r}) = \int_V G_D(\mathbf{r}|\mathbf{r}')\frac{\rho(\mathbf{r}')}{\epsilon}\,dV' + \oint_{S_B} \Phi(\mathbf{r}')\frac{\partial G_D(\mathbf{r}|\mathbf{r}')}{\partial n'}\,dS' + \\ + \sum_{n=1}^{N} \oint_{S_n} \Phi(\mathbf{r}')\frac{\partial G_D(\mathbf{r}|\mathbf{r}')}{\partial n'}\,dS', \tag{3.71}$$

requires the specification of Φ (but not its normal derivative) over the boundary surfaces. In case S_B and S_n surround and are adjacent to perfect conductors, the Dirichlet boundary condition has an important physical meaning. The corresponding Green's function is the potential at point $\mathbf{r}$ produced by a point source at $\mathbf{r}'$ in the presence of the conductors when the conductors are grounded — i.e., held at zero potential. Then we must specify the actual constant potentials on the conductors to determine Φ everywhere within V using (3.71). The additional term $F(\mathbf{r}|\mathbf{r}')$ in (3.69) accounts for the potential produced by surface charges on the grounded conductors.

By analogy with (3.70) it is tempting to try to define another electrostatic Green's function according to

$$\frac{\partial G(\mathbf{r}|\mathbf{r}')}{\partial n'} = 0 \text{ for all } \mathbf{r}' \in S. \tag{3.72}$$

But this choice is not permissible if V is a finite-sized region. Let us integrate (3.54) over V and employ the divergence theorem and the sifting property to get

$$\oint_S \frac{\partial G(\mathbf{r}|\mathbf{r}')}{\partial n'}\,dS' = -1; \tag{3.73}$$

in conjunction with this, equation (3.72) would imply the false statement $0 = -1$. Suppose instead that we introduce a Green's function according to

$$\frac{\partial G(\mathbf{r}|\mathbf{r}')}{\partial n'} = -\frac{1}{A} \text{ for all } \mathbf{r}' \in S. \tag{3.74}$$

where A is the total area of S. This choice avoids a contradiction in (3.73); it does not nullify any terms in (3.56), but does reduce the surface integral terms involving Φ to constants. Taken together, these terms all comprise a single additive constant on the right-hand side; although the corresponding potential $\Phi(\mathbf{r})$ is thereby determined only to within this additive constant, the value of $\mathbf{E}(\mathbf{r}) = -\nabla\Phi(\mathbf{r})$ will be unaffected. By reciprocity we can rewrite (3.74) as

$$\frac{\partial G_N(\mathbf{r}|\mathbf{r}')}{\partial n} = -\frac{1}{A} \text{ for all } \mathbf{r} \in S. \tag{3.75}$$

The Green's function G_N so defined is known as the *Neumann Green's function*. Observe that if V is not finite-sized then $A \to \infty$ and according to (3.74) the choice (3.72) becomes allowable.

Finding the Green's function that obeys one of the boundary conditions for a given geometry is often a difficult task. Nevertheless, certain canonical geometries make the Green's function approach straightforward and simple. Such is the case in image theory, when a charge is located near a simple conducting body such as a ground screen or a sphere. In these cases the function $F(\mathbf{r}|\mathbf{r}')$ consists of a single correction term as in (3.68). We shall consider these simple cases in examples to follow.

Reciprocity of the static Green's function. It remains to show that

$$G(\mathbf{r}|\mathbf{r}') = G(\mathbf{r}'|\mathbf{r})$$

for any of the Green's functions introduced above. The unbounded-space Green's function is reciprocal by inspection; $|\mathbf{r} - \mathbf{r}'|$ is unaffected by interchanging $\mathbf{r}$ and $\mathbf{r}'$. However, we can give a more general treatment covering this case as well as the Dirichlet and Neumann cases. We begin with

$$\nabla^2 G(\mathbf{r}|\mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}').$$

In Green's second identity let

$$\phi(\mathbf{r}) = G(\mathbf{r}|\mathbf{r}_a), \qquad \psi(\mathbf{r}) = G(\mathbf{r}|\mathbf{r}_b),$$

where $\mathbf{r}_a$ and $\mathbf{r}_b$ are arbitrary points, and integrate over the unprimed coordinates. We have

$$\int_V [G(\mathbf{r}|\mathbf{r}_a)\nabla^2 G(\mathbf{r}|\mathbf{r}_b) - G(\mathbf{r}|\mathbf{r}_b)\nabla^2 G(\mathbf{r}|\mathbf{r}_a)]\, dV =$$
$$- \oint_S \left[G(\mathbf{r}|\mathbf{r}_a)\frac{\partial G(\mathbf{r}|\mathbf{r}_b)}{\partial n} - G(\mathbf{r}|\mathbf{r}_b)\frac{\partial G(\mathbf{r}|\mathbf{r}_a)}{\partial n} \right] dS.$$

If G is the unbounded-space Green's function, the surface integral must vanish since $S_B \to \infty$. It must also vanish under Dirichlet or Neumann boundary conditions. Since

$$\nabla^2 G(\mathbf{r}|\mathbf{r}_a) = -\delta(\mathbf{r} - \mathbf{r}_a), \qquad \nabla^2 G(\mathbf{r}|\mathbf{r}_b) = -\delta(\mathbf{r} - \mathbf{r}_b),$$

we have

$$\int_V [G(\mathbf{r}|\mathbf{r}_a)\delta(\mathbf{r} - \mathbf{r}_b) - G(\mathbf{r}|\mathbf{r}_b)\delta(\mathbf{r} - \mathbf{r}_a)]\, dV = 0,$$

hence

$$G(\mathbf{r}_b|\mathbf{r}_a) = G(\mathbf{r}_a|\mathbf{r}_b)$$

by the sifting property. By the arbitrariness of $\mathbf{r}_a$ and $\mathbf{r}_b$, reciprocity is established.

Electrostatic shielding. The Dirichlet Green's function can be used to explain *electrostatic shielding*. We consider a closed, grounded, conducting shell with charge outside but not inside (Figure 3.7). By (3.71) the potential at points inside the shell is

$$\Phi(\mathbf{r}) = \oint_{S_B} \Phi(\mathbf{r}')\frac{\partial G_D(\mathbf{r}|\mathbf{r}')}{\partial n'}\, dS',$$

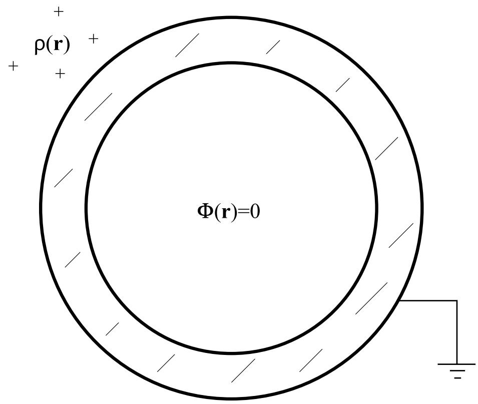

Figure 3.7: Electrostatic shielding by a conducting shell.

where S_B is tangential to the inner surface of the shell and we have used $\rho = 0$ within the shell. Because $\Phi(\mathbf{r}') = 0$ for all $\mathbf{r}'$ on S_B, we have

$$\Phi(\mathbf{r}) = 0$$

everywhere in the region enclosed by the shell. This result is independent of the charge outside the shell, and the interior region is "shielded" from the effects of that charge.

Conversely, consider a grounded conducting shell with charge contained inside. If we surround the outside of the shell by a surface S_1 and let S_B recede to infinity, then (3.71) becomes

$$\Phi(\mathbf{r}) = \lim_{S_B \to \infty} \oint_{S_B} \Phi(\mathbf{r}') \frac{\partial G_D(\mathbf{r}|\mathbf{r}')}{\partial n'} \, dS' + \oint_{S_1} \Phi(\mathbf{r}') \frac{\partial G_D(\mathbf{r}|\mathbf{r}')}{\partial n'} \, dS'.$$

Again there is no charge in V (since the charge lies completely inside the shell). The contribution from S_B vanishes. Since S_1 lies adjacent to the outer surface of the shell, $\Phi(\mathbf{r}') \equiv 0$ on S_1. Thus $\Phi(\mathbf{r}) = 0$ for all points outside the conducting shell.

Example solution to Poisson's equation: planar layered media. For simple geometries Poisson's equation may be solved as part of a *boundary value problem* (§ A.4). Occasionally such a solution has an appealing interpretation as the superposition of potentials produced by the physical charge and its "images." We shall consider here the case of planar media and subsequently use the results to predict the potential produced by charge near a conducting sphere.

Consider a layered dielectric medium where various regions of space are separated by planes at constant values of z. Material region i occupies volume region V_i and has permittivity ϵ_i; it may or may not contain source charge. The solution to Poisson's equation is given by (3.56). The contribution

$$\Phi^p(\mathbf{r}) = \int_V G(\mathbf{r}|\mathbf{r}') \frac{\rho(\mathbf{r}')}{\epsilon} \, dV'$$

produced by sources within V is known as the *primary potential.* The term

$$\Phi^s(\mathbf{r}) = \oint_S \left[\Phi(\mathbf{r}') \frac{\partial G(\mathbf{r}|\mathbf{r}')}{\partial n'} - G(\mathbf{r}|\mathbf{r}') \frac{\partial \Phi(\mathbf{r}')}{\partial n'} \right] dS',$$

on the other hand, involves an integral over the surface fields and is known as the *secondary potential*. This term is linked to effects outside V. Since the "sources" of Φ^s (i.e., the surface fields) lie on the boundary of V, Φ^s satisfies Laplace's equation within V. We may therefore use other, more convenient, representations of Φ^s provided they satisfy Laplace's equation. However, as solutions to a homogeneous equation they are of indefinite form until linked to appropriate boundary values.

Since the geometry is invariant in the x and y directions, we represent each potential function in terms of a 2-D Fourier transform over these variables. We leave the z dependence intact so that we may apply boundary conditions directly in the spatial domain. The transform representations of the Green's functions for the primary and secondary potentials are derived in Appendix A. From (A.55) we see that the primary potential within region V_i can be written as

$$\Phi_i^p(\mathbf{r}) = \int_{V_i} G^p(\mathbf{r}|\mathbf{r}')\frac{\rho(\mathbf{r}')}{\epsilon_i}\,dV' \tag{3.76}$$

where

$$G^p(\mathbf{r}|\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{e^{-k_\rho|z-z'|}}{2k_\rho}e^{j\mathbf{k}_\rho\cdot(\mathbf{r}-\mathbf{r}')}\,d^2k_\rho \tag{3.77}$$

is the primary Green's function with $\mathbf{k}_\rho = \hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y$, $k_\rho = |\mathbf{k}_\rho|$, and $d^2k_\rho = dk_x\,dk_y$. We also find in (A.56) that a solution of Laplace's equation can be written as

$$\Phi^s(\mathbf{r}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left[A(\mathbf{k}_\rho)e^{k_\rho z} + B(\mathbf{k}_\rho)e^{-k_\rho z}\right]e^{j\mathbf{k}_\rho\cdot\mathbf{r}}\,d^2k_\rho \tag{3.78}$$

where $A(\mathbf{k}_\rho)$ and $B(\mathbf{k}_\rho)$ must be found by the application of appropriate boundary conditions.

As a simple example, consider a charge distribution $\rho(\mathbf{r})$ in free space above a grounded conducting plane located at $z = 0$. We wish to find the potential in the region $z > 0$ using the Fourier transform representation of the potentials. The total potential is a sum of primary and secondary terms:

$$\Phi(x, y, z) = \int_V\left[\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{e^{-k_\rho|z-z'|}}{2k_\rho}e^{j\mathbf{k}_\rho\cdot(\mathbf{r}-\mathbf{r}')}\,d^2k_\rho\right]\frac{\rho(\mathbf{r}')}{\epsilon_0}\,dV' + \\ + \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left[B(\mathbf{k}_\rho)e^{-k_\rho z}\right]e^{j\mathbf{k}_\rho\cdot\mathbf{r}}\,d^2k_\rho,$$

where the integral is over the region $z > 0$. Here we have set $A(\mathbf{k}_\rho) = 0$ because $e^{k_\rho z}$ grows with increasing z. Since the plane is grounded we must have $\Phi(x, y, 0) = 0$. Because $z < z'$ when we apply this condition, we have $|z - z'| = z' - z$ and thus

$$\Phi(x, y, 0) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left[\int_V\frac{\rho(\mathbf{r}')}{\epsilon_0}\frac{e^{-k_\rho z'}}{2k_\rho}e^{-j\mathbf{k}_\rho\cdot\mathbf{r}'}\,dV' + B(\mathbf{k}_\rho)\right]e^{j\mathbf{k}_\rho\cdot\mathbf{r}}\,d^2k_\rho = 0.$$

Invoking the Fourier integral theorem we find

$$B(\mathbf{k}_\rho) = -\int_V\frac{\rho(\mathbf{r}')}{\epsilon_0}\frac{e^{-k_\rho z'}}{2k_\rho}e^{-j\mathbf{k}_\rho\cdot\mathbf{r}'}\,dV',$$

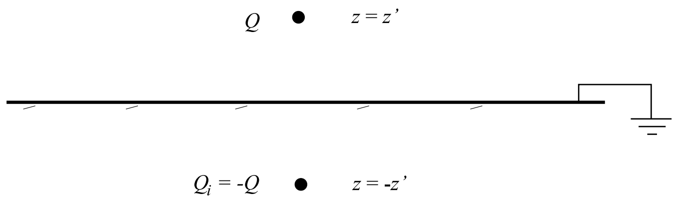

Figure 3.8: Construction of electrostatic Green's function for a ground plane.

hence the total potential is

$$\begin{aligned}\Phi(x, y, z) &= \int_V \left[\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{e^{-k_\rho|z-z'|} - e^{-k_\rho(z+z')}}{2k_\rho} e^{j\mathbf{k}_\rho\cdot(\mathbf{r}-\mathbf{r}')}\, d^2k_\rho \right] \frac{\rho(\mathbf{r}')}{\epsilon_0}\, dV' \\ &= \int_V G(\mathbf{r}|\mathbf{r}') \frac{\rho(\mathbf{r}')}{\epsilon_0}\, dV'\end{aligned}$$

where $G(\mathbf{r}|\mathbf{r}')$ is the Green's function for the region above a grounded planar conductor. We can interpret this Green's function as a sum of the primary Green's function (3.77) and a secondary Green's function

$$G^s(\mathbf{r}|\mathbf{r}') = -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{e^{-k_\rho(z+z')}}{2k_\rho} e^{j\mathbf{k}_\rho\cdot(\mathbf{r}-\mathbf{r}')}\, d^2k_\rho. \tag{3.79}$$

For $z > 0$ the term $z + z'$ can be replaced by $|z + z'|$. Then, comparing (3.79) with (3.77), we see that

$$G^s(\mathbf{r}\,|\,x', y', z') = -G^p(\mathbf{r}\,|\,x', y', -z') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'_i|} \tag{3.80}$$

where $\mathbf{r}'_i = \hat{\mathbf{x}}x' + \hat{\mathbf{y}}y' - \hat{\mathbf{z}}z'$. Because the Green's function is the potential of a point charge, we may interpret the secondary Green's function as produced by a negative unit charge placed in a position $-z'$ immediately beneath the positive unit charge that produces G^p (Figure 3.8). This secondary charge is the "image" of the primary charge. That two such charges would produce a null potential on the ground plane is easily verified.

As a more involved example, consider a charge distribution $\rho(\mathbf{r})$ above a planar interface separating two homogeneous dielectric media. Region 1 occupies $z > 0$ and has permittivity ϵ_1, while region 2 occupies $z < 0$ and has permittivity ϵ_2. In region 1 we can write the total potential as a sum of primary and secondary components, discarding the term that grows with z:

$$\begin{aligned}\Phi_1(x, y, z) &= \int_V \left[\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{e^{-k_\rho|z-z'|}}{2k_\rho} e^{j\mathbf{k}_\rho\cdot(\mathbf{r}-\mathbf{r}')}\, d^2k_\rho \right] \frac{\rho(\mathbf{r}')}{\epsilon_1}\, dV' + \\ &\quad + \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \left[B(\mathbf{k}_\rho) e^{-k_\rho z} \right] e^{j\mathbf{k}_\rho\cdot\mathbf{r}}\, d^2k_\rho.\end{aligned} \tag{3.81}$$

With no source in region 2, the potential there must obey Laplace's equation and therefore consists of only a secondary component:

$$\Phi_2(\mathbf{r}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \left[A(\mathbf{k}_\rho) e^{k_\rho z} \right] e^{j\mathbf{k}_\rho\cdot\mathbf{r}}\, d^2k_\rho. \tag{3.82}$$

To determine A and B we impose (3.36) and (3.37). By (3.36) we have

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left[\int_V \frac{\rho(\mathbf{r}')}{\epsilon_1}\frac{e^{-k_\rho z'}}{2k_\rho}e^{-j\mathbf{k}_\rho\cdot\mathbf{r}'}\,dV' + B(\mathbf{k}_\rho) - A(\mathbf{k}_\rho)\right]e^{j\mathbf{k}_\rho\cdot\mathbf{r}}\,d^2k_\rho = 0,$$

hence

$$\int_V \frac{\rho(\mathbf{r}')}{\epsilon_1}\frac{e^{-k_\rho z'}}{2k_\rho}e^{-j\mathbf{k}_\rho\cdot\mathbf{r}'}\,dV' + B(\mathbf{k}_\rho) - A(\mathbf{k}_\rho) = 0$$

by the Fourier integral theorem. Applying (3.37) at $z = 0$ with $\hat{\mathbf{n}}_{12} = \hat{\mathbf{z}}$, and noting that there is no excess surface charge, we find

$$\int_V \rho(\mathbf{r}')\frac{e^{-k_\rho z'}}{2k_\rho}e^{-j\mathbf{k}_\rho\cdot\mathbf{r}'}\,dV' - \epsilon_1 B(\mathbf{k}_\rho) - \epsilon_2 A(\mathbf{k}_\rho) = 0.$$

The solutions

$$A(\mathbf{k}_\rho) = \frac{2\epsilon_1}{\epsilon_1+\epsilon_2}\int_V \frac{\rho(\mathbf{r}')}{\epsilon_1}\frac{e^{-k_\rho z'}}{2k_\rho}e^{-j\mathbf{k}_\rho\cdot\mathbf{r}'}\,dV',$$
$$B(\mathbf{k}_\rho) = \frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\int_V \frac{\rho(\mathbf{r}')}{\epsilon_1}\frac{e^{-k_\rho z'}}{2k_\rho}e^{-j\mathbf{k}_\rho\cdot\mathbf{r}'}\,dV',$$

are then substituted into (3.81) and (3.82) to give

$$\begin{aligned}\Phi_1(\mathbf{r}) &= \int_V\left[\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{e^{-k_\rho|z-z'|}+\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}e^{-k_\rho(z+z')}}{2k_\rho}e^{j\mathbf{k}_\rho\cdot(\mathbf{r}-\mathbf{r}')}\,d^2k_\rho\right]\frac{\rho(\mathbf{r}')}{\epsilon_1}\,dV' \\ &= \int_V G_1(\mathbf{r}|\mathbf{r}')\frac{\rho(\mathbf{r}')}{\epsilon_1}\,dV', \\ \Phi_2(\mathbf{r}) &= \int_V\left[\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\frac{e^{-k_\rho(z'-z)}}{2k_\rho}e^{j\mathbf{k}_\rho\cdot(\mathbf{r}-\mathbf{r}')}\,d^2k_\rho\right]\frac{\rho(\mathbf{r}')}{\epsilon_2}\,dV' \\ &= \int_V G_2(\mathbf{r}|\mathbf{r}')\frac{\rho(\mathbf{r}')}{\epsilon_2}\,dV'.\end{aligned}$$

Since $z' > z$ for all points in region 2, we can replace $z' - z$ by $|z - z'|$ in the formula for Φ_2.

As with the previous example, let us compare the result to the form of the primary Green's function (3.77). We see that

$$G_1(\mathbf{r}|\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\frac{1}{4\pi|\mathbf{r}-\mathbf{r}_1'|},$$
$$G_2(\mathbf{r}|\mathbf{r}') = \frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\frac{1}{4\pi|\mathbf{r}-\mathbf{r}_2'|},$$

where $\mathbf{r}_1' = \hat{\mathbf{x}}x' + \hat{\mathbf{y}}y' - \hat{\mathbf{z}}z'$ and $\mathbf{r}_2' = \hat{\mathbf{x}}x' + \hat{\mathbf{y}}y' + \hat{\mathbf{z}}z'$. So we can also write

$$\Phi_1(\mathbf{r}) = \frac{1}{4\pi}\int_V\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|} + \frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\frac{1}{|\mathbf{r}-\mathbf{r}_1'|}\right]\frac{\rho(\mathbf{r}')}{\epsilon_1}\,dV',$$
$$\Phi_2(\mathbf{r}) = \frac{1}{4\pi}\int_V\left[\frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\frac{1}{|\mathbf{r}-\mathbf{r}_2'|}\right]\frac{\rho(\mathbf{r}')}{\epsilon_2}\,dV'.$$

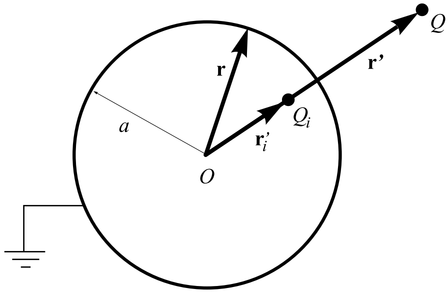

Figure 3.9: Green's function for a grounded conducting sphere.

Note that $\Phi_2 \to \Phi_1$ as $\epsilon_2 \to \epsilon_1$.

There is an image interpretation for the secondary Green's functions. The secondary Green's function for region 1 appears as a potential produced by an image of the primary charge located at $-z'$ in an infinite medium of permittivity ϵ_1, and with an amplitude of $(\epsilon_1 - \epsilon_2)/(\epsilon_1 + \epsilon_2)$ times the primary charge. The Green's function in region 2 is produced by an image charge located at z' (i.e., at the location of the primary charge) in an infinite medium of permittivity ϵ_2 with an amplitude of $2\epsilon_2/(\epsilon_1 + \epsilon_2)$ times the primary charge.

Example solution to Poisson's equation: conducting sphere. As an example involving a nonplanar geometry, consider the potential produced by a source near a grounded conducting sphere in free space (Figure 3.9). Based on our experience with planar layered media, we hypothesize that the secondary potential will be produced by an image charge; hence we try the simple Green's function

$$G^s(\mathbf{r}|\mathbf{r}') = \frac{A(\mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}_i'|}$$

where the amplitude A and location $\mathbf{r}_i'$ of the image are to be determined. We further assume, based on our experience with planar problems, that the image charge will reside inside the sphere along a line joining the origin to the primary charge. Since $\mathbf{r} = a\hat{\mathbf{r}}$ for all points on the sphere, the total Green's function must obey the Dirichlet condition

$$G(\mathbf{r}|\mathbf{r}')|_{r=a} = \left.\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}\right|_{r=a} + \left.\frac{A(\mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}_i'|}\right|_{r=a} = \frac{1}{4\pi|a\hat{\mathbf{r}} - r'\hat{\mathbf{r}}'|} + \frac{A(\mathbf{r}')}{4\pi|a\hat{\mathbf{r}} - r_i'\hat{\mathbf{r}}'|} = 0$$

in order to have the potential, given by (3.56), vanish on the sphere surface. Factoring a from the first denominator and r_i' from the second we obtain

$$\frac{1}{4\pi a|\hat{\mathbf{r}} - \frac{r'}{a}\hat{\mathbf{r}}'|} + \frac{A(\mathbf{r}')}{4\pi r_i'|\frac{a}{r_i'}\hat{\mathbf{r}} - \hat{\mathbf{r}}'|} = 0.$$

Now $|k\hat{\mathbf{r}} - k'\hat{\mathbf{r}}'| = k^2 + k'^2 - 2kk'\cos\gamma$ where γ is the angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$ and k, k' are constants; this means that $|k\hat{\mathbf{r}} - \hat{\mathbf{r}}'| = |\hat{\mathbf{r}} - k\hat{\mathbf{r}}'|$. Hence as long as we choose

$$\frac{r'}{a} = \frac{a}{r_i'}, \qquad \frac{A}{r_i'} = -\frac{1}{a},$$

the total Green's function vanishes everywhere on the surface of the sphere. The image charge is therefore located within the sphere at $\mathbf{r}_i' = a^2\mathbf{r}'/r'^2$ and has amplitude $A =$

$-a/r'$. (Note that both the location and amplitude of the image depend on the location of the primary charge.) With this Green's function and (3.71), the potential of an arbitrary source placed near a grounded conducting sphere is

$$\Phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{\epsilon} \frac{1}{4\pi} \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{a/r'}{|\mathbf{r} - \frac{a^2}{r'^2}\mathbf{r}'|} \right] dV'.$$

The Green's function may be used to compute the surface charge density induced on the sphere by a unit point charge: it is merely necessary to find the normal component of electric field from the gradient of $\Phi(\mathbf{r})$. We leave this as an exercise for the reader, who may then integrate the surface charge and thereby show that the total charge induced on the sphere is equal to the image charge. So the total charge induced on a grounded sphere by a point charge q at a point $r = r'$ is $Q = -qa/r'$.

It is possible to find the total charge induced on the sphere without finding the image charge first. This is an application of Green's reciprocation theorem (§ 3.4.4). According to (3.211), if we can find the potential V_P at a point $\mathbf{r}$ produced by the sphere when it is isolated and carrying a total charge Q_0, then the total charge Q induced on the grounded sphere in the vicinity of a point charge q placed at $\mathbf{r}$ is given by

$$Q = -qV_P/V_1$$

where V_1 is the potential of the isolated sphere. We can apply this formula by noting that an isolated sphere carrying charge Q_0 produces a field $\mathbf{E}(\mathbf{r}) = \hat{\mathbf{r}}Q_0/4\pi\epsilon r^2$. Integration from a radius r to infinity gives the potential referred to infinity: $\Phi(\mathbf{r}) = Q_0/4\pi\epsilon r$. So the potential of the isolated sphere is $V_1 = Q_0/4\pi\epsilon a$, while the potential at radius r' is $V_P = Q_0/4\pi\epsilon r'$. Substitution gives $Q = -qa/r'$ as before.

3.2.5 Force and energy

Maxwell's stress tensor. The electrostatic version of Maxwell's stress tensor can be obtained from (2.288) by setting $\mathbf{B} = \mathbf{H} = 0$:

$$\bar{\mathbf{T}}_e = \frac{1}{2}(\mathbf{D} \cdot \mathbf{E})\bar{\mathbf{I}} - \mathbf{DE}. \tag{3.83}$$

The total electric force on the charges in a region V bounded by the surface S is given by the relation

$$\mathbf{F}_e = -\oint_S \bar{\mathbf{T}}_e \cdot \mathbf{dS} = \int_V \mathbf{f}_e \, dV$$

where $\mathbf{f}_e = \rho\mathbf{E}$ is the electric force volume density.

In particular, suppose that S is adjacent to a solid conducting body embedded in a dielectric having permittivity $\epsilon(\mathbf{r})$. Since all the charge is at the surface of the conductor, the force within V acts directly on the surface. Thus, $-\bar{\mathbf{T}}_e \cdot \hat{\mathbf{n}}$ is the surface force density (*traction*) $\mathbf{t}$. Using $\mathbf{D} = \epsilon\mathbf{E}$, and remembering that the fields are normal to the conductor, we find that

$$\bar{\mathbf{T}}_e \cdot \hat{\mathbf{n}} = \frac{1}{2}\epsilon E_n^2 \hat{\mathbf{n}} - \epsilon \mathbf{EE} \cdot \hat{\mathbf{n}} = -\frac{1}{2}\epsilon E_n^2 \hat{\mathbf{n}} = -\frac{1}{2}\rho_s \mathbf{E}.$$

The surface force density is perpendicular to the surface.

As a simple but interesting example, consider the force acting on a rigid conducting sphere of radius a carrying total charge Q in a homogeneous medium. At equilibrium the charge is distributed uniformly with surface density $\rho_s = Q/4\pi a^2$, producing a field $\mathbf{E} = \hat{\mathbf{r}}Q/4\pi\epsilon r^2$ external to the sphere. Hence a force density

$$\mathbf{t} = \frac{1}{2}\hat{\mathbf{r}}\frac{Q^2}{\epsilon(4\pi a^2)^2}$$

acts at each point on the surface. This would cause the sphere to expand outward if the structural integrity of the material were to fail. Integration over the entire sphere yields

$$\mathbf{F} = \frac{1}{2}\frac{Q^2}{\epsilon(4\pi a^2)^2}\int_S \hat{\mathbf{r}}\,dS = 0.$$

However, integration of $\mathbf{t}$ over the upper hemisphere yields

$$\mathbf{F} = \frac{1}{2}\frac{Q^2}{\epsilon(4\pi a^2)^2}\int_0^{2\pi}\int_0^{\pi/2}\hat{\mathbf{r}}a^2\sin\theta\,d\theta\,d\phi.$$

Substitution of $\hat{\mathbf{r}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ leads immediately to $F_x = F_y = 0$, but the z-component is

$$F_z = \frac{1}{2}\frac{Q^2}{\epsilon(4\pi a^2)^2}\int_0^{2\pi}\int_0^{\pi/2} a^2\cos\theta\sin\theta\,d\theta\,d\phi = \frac{Q^2}{32\epsilon\pi a^2}.$$

This result can also be obtained by integrating $-\bar{\mathbf{T}}_e\cdot\hat{\mathbf{n}}$ over the entire xy-plane with $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$. Since $-\bar{\mathbf{T}}_e\cdot(-\hat{\mathbf{z}}) = \hat{\mathbf{z}}\frac{\epsilon}{2}\mathbf{E}\cdot\mathbf{E}$ we have

$$\mathbf{F} = \hat{\mathbf{z}}\frac{1}{2}\frac{Q^2}{(4\pi\epsilon)^2}\int_0^{2\pi}\int_a^{\infty}\frac{r\,dr\,d\phi}{r^4} = \hat{\mathbf{z}}\frac{Q^2}{32\epsilon\pi a^2}.$$

As a more challenging example, consider two identical line charges parallel to the z-axis and located at $x = \pm d/2$, $y = 0$ in free space. We can find the force on one line charge due to the other by integrating Maxwell's stress tensor over the yz-plane. From (3.64) we find that the total electric field on the yz-plane is

$$\mathbf{E}(y,z) = \frac{y}{y^2 + (d/2)^2}\frac{\rho_l}{\pi\epsilon_0}\hat{\mathbf{y}}$$

where ρ_l is the line charge density. The force density for either line charge is $-\bar{\mathbf{T}}_e\cdot\hat{\mathbf{n}}$, where we use $\hat{\mathbf{n}} = \pm\hat{\mathbf{x}}$ to obtain the force on the charge at $x = \mp d/2$. The force density for the charge at $x = -d/2$ is

$$\bar{\mathbf{T}}_e\cdot\hat{\mathbf{n}} = \frac{1}{2}(\mathbf{D}\cdot\mathbf{E})\bar{\mathbf{I}}\cdot\hat{\mathbf{x}} - \mathbf{DE}\cdot\hat{\mathbf{x}} = \frac{\epsilon_0}{2}\left[\frac{y}{y^2 + (d/2)^2}\frac{\rho_l}{\pi\epsilon_0}\right]^2\hat{\mathbf{x}}$$

and the total force is

$$\mathbf{F}_- = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\rho_l^2}{2\pi^2\epsilon_0}\frac{y^2}{\left[y^2 + (d/2)^2\right]^2}\hat{\mathbf{x}}\,dy\,dz.$$

On a per unit length basis the force is

$$\frac{\mathbf{F}_-}{l} = -\hat{\mathbf{x}}\frac{\rho_l^2}{2\pi^2\epsilon_0}\int_{-\infty}^{\infty}\frac{y^2}{[y^2 + (d/2)^2]^2}\,dy = -\hat{\mathbf{x}}\frac{\rho_l^2}{2\pi d\epsilon_0}.$$

Note that the force is repulsive as expected.

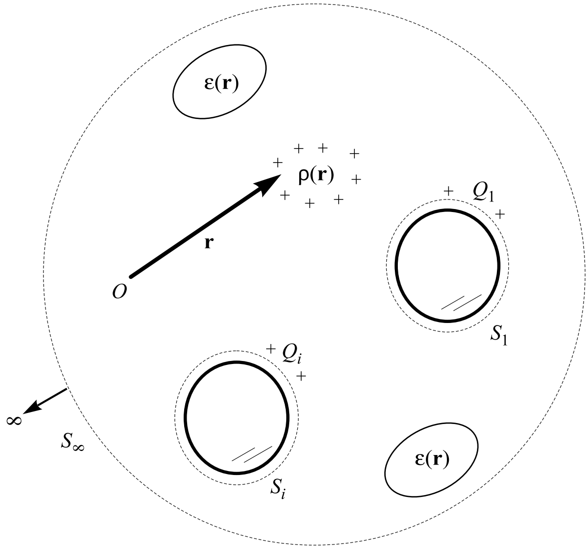

Figure 3.10: Computation of electrostatic stored energy via the assembly energy of a charge distribution.

Electrostatic stored energy. In § 2.9.5 we considered the energy relations for the electromagnetic field. Those relations remain valid in the static case. Since our interpretation of the dynamic relations was guided in part by our knowledge of the energy stored in a static field, we must, for completeness, carry out a study of that effect here.

The energy of a static configuration is taken to be the work required to assemble the configuration from a chosen starting point. For a configuration of static charges, the *stored electric energy* is the energy required to assemble the configuration, starting with all charges removed to infinite distance (the assumed zero potential reference). If the assembled charges are not held in place by an external mechanical force they will move, thereby converting stored electric energy into other forms of energy (e.g., kinetic energy and radiation).

By (3.62), the work required to move a point charge q from a reservoir at infinity to a point P at $\mathbf{r}$ in a potential field Φ is

$$W = q\Phi(\mathbf{r}).$$

If instead we have a continuous charge density ρ present, and wish to increase this to $\rho + \delta\rho$ by bringing in a small quantity of charge $\delta\rho$, a total work

$$\delta W = \int_{V_\infty} \delta\rho(\mathbf{r})\Phi(\mathbf{r})\, dV \tag{3.84}$$

is required, and the potential field is increased to $\Phi + \delta\Phi$. Here V_∞ denotes all of space. (We could restrict the integral to the region containing the charge, but we shall find it helpful to extend the domain of integration to all of space.)

Now consider the situation shown in Figure 3.10. Here we have charge in the form of both volume densities and surface densities on conducting bodies. Also present may be linear material bodies. We can think of assembling the charge in two distinctly different ways. We could, for instance, bring small portions of charge (or point charges) together to form the distribution ρ. Or, we could slowly build up ρ by adding infinitesimal, but spatially identical, distributions. That is, we can create the distribution ρ from a zero initial state by repeatedly adding a charge distribution

$$\delta\rho(\mathbf{r}) = \rho(\mathbf{r})/N,$$

where N is a large number. Whenever we add $\delta\rho$ we must perform the work given by (3.84), but we also increase the potential proportionately (remembering that all materials are assumed linear). At each step, more work is required. The total work is

$$W = \sum_{n=1}^{N} \int_{V_\infty} \delta\rho(\mathbf{r})[(n-1)\delta\Phi(\mathbf{r})]\, dV = \left[\sum_{n=1}^{N}(n-1)\right] \int_{V_\infty} \frac{\rho(\mathbf{r})}{N} \frac{\Phi(\mathbf{r})}{N}\, dV. \tag{3.85}$$

We must use an infinite number of steps so that no energy is lost to radiation at any step (since the charge we add each time is infinitesimally small). Using

$$\sum_{n=1}^{N}(n-1) = N(N-1)/2,$$

(3.85) becomes

$$W = \frac{1}{2} \int_{V_\infty} \rho(\mathbf{r})\Phi(\mathbf{r})\, dV \tag{3.86}$$

as $N \to \infty$. Finally, since some assembled charge will be in the form of a volume density and some in the form of the surface density on conductors, we can generalize (3.86) to

$$W = \frac{1}{2} \int_{V'} \rho(\mathbf{r})\Phi(\mathbf{r})\, dV + \frac{1}{2} \sum_{i=1}^{I} Q_i V_i. \tag{3.87}$$

Here V' is the region outside the conductors, Q_i is the total charge on the ith conductor ($i = 1, \ldots, I$), and V_i is the absolute potential (referred to infinity) of the ith conductor.

An intriguing property of electrostatic energy is that the charges on the conductors will arrange themselves, while seeking static equilibrium, into a minimum-energy configuration (Thomson's theorem).

In keeping with our field-centered view of electromagnetics, we now wish to write the energy (3.86) entirely in terms of the field vectors $\mathbf{E}$ and $\mathbf{D}$. Since $\rho = \nabla \cdot \mathbf{D}$ we have

$$W = \frac{1}{2} \int_{V_\infty} [\nabla \cdot \mathbf{D}(\mathbf{r})]\Phi(\mathbf{r})\, dV.$$

Then, by (B.42),

$$W = \frac{1}{2} \int_{V_\infty} \nabla \cdot [\Phi(\mathbf{r})\mathbf{D}(\mathbf{r})]\, dV - \frac{1}{2} \int_{V_\infty} \mathbf{D}(\mathbf{r}) \cdot [\nabla\Phi(\mathbf{r})]\, dV.$$

Use of the divergence theorem and (3.30) leads to

$$W = \frac{1}{2} \oint_{S_\infty} \Phi(\mathbf{r})\mathbf{D}(\mathbf{r}) \cdot \mathbf{dS} + \frac{1}{2} \int_{V_\infty} \mathbf{D}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})\, dV$$

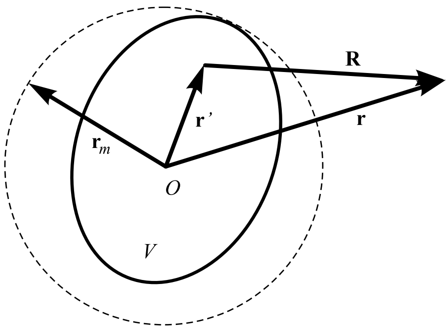

Figure 3.11: Multipole expansion.

where S_∞ is the bounding surface that recedes toward infinity to encompass all of space. Because $\Phi \sim 1/r$ and $D \sim 1/r^2$ as $r \to \infty$, the integral over S_∞ tends to zero and

$$W = \frac{1}{2} \int_{V_\infty} \mathbf{D}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) \, dV. \tag{3.88}$$

Hence we may compute the assembly energy in terms of the fields supported by the charge ρ.

It is significant that the assembly energy W is identical to the term within the time derivative in Poynting's theorem (2.299). Hence our earlier interpretation, that this term represents the time-rate of change of energy "stored" in the electric field, has a firm basis. Of course, the assembly energy is a static concept, and our generalization to dynamic fields is purely intuitive. We also face similar questions regarding the meaning of energy density, and whether energy can be "localized" in space. The discussions in § 2.9.5 still apply.

3.2.6 Multipole expansion

Consider an arbitrary but spatially localized charge distribution of total charge Q in an unbounded homogeneous medium (Figure 3.11). We have already obtained the potential (3.61) of the source; as we move the observation point away, Φ should decrease in a manner roughly proportional to $1/r$. The actual variation depends on the nature of the charge distribution and can be complicated. Often this dependence is dominated by a specific inverse power of distance for observation points far from the source, and we can investigate it by expanding the potential in powers of $1/r$. Although such *multipole expansions* of the potential are rarely used to perform actual computations, they can provide insight into both the behavior of static fields and the physical meaning of the polarization vector $\mathbf{P}$.

Let us place our origin of coordinates somewhere within the charge distribution, as shown in Figure 3.11, and expand the Green's function spatial dependence in a three-dimensional Taylor series about the origin:

$$\frac{1}{R} = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathbf{r}' \cdot \nabla')^n \frac{1}{R}\bigg|_{\mathbf{r}'=0} = \frac{1}{r} + (\mathbf{r}' \cdot \nabla') \frac{1}{R}\bigg|_{\mathbf{r}'=0} + \frac{1}{2} (\mathbf{r}' \cdot \nabla')^2 \frac{1}{R}\bigg|_{\mathbf{r}'=0} + \cdots, \tag{3.89}$$

where $R = |\mathbf{r} - \mathbf{r}'|$. Convergence occurs if $|\mathbf{r}| > |\mathbf{r}'|$. In the notation $(\mathbf{r}' \cdot \nabla')^n$ we interpret a power on a derivative operator as the order of the derivative. Substituting (3.89) into

(3.61) and writing the derivatives in Cartesian coordinates we obtain

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon}\int_V \rho(\mathbf{r}')\left[\frac{1}{R}\bigg|_{\mathbf{r}'=0} + (\mathbf{r}'\cdot\nabla')\frac{1}{R}\bigg|_{\mathbf{r}'=0} + \frac{1}{2}(\mathbf{r}'\cdot\nabla')^2\frac{1}{R}\bigg|_{\mathbf{r}'=0} + \cdots\right] dV'. \qquad (3.90)$$

For the second term we can use (3.57) to write

$$(\mathbf{r}'\cdot\nabla')\frac{1}{R}\bigg|_{\mathbf{r}'=0} = \mathbf{r}'\cdot\left(\nabla'\frac{1}{R}\right)\bigg|_{\mathbf{r}'=0} = \mathbf{r}'\cdot\left(\frac{\hat{\mathbf{R}}}{R^2}\right)\bigg|_{\mathbf{r}'=0} = \mathbf{r}'\cdot\frac{\hat{\mathbf{r}}}{r^2}. \qquad (3.91)$$

The third term is complicated. Let us denote (x, y, z) by (x_1, x_2, x_3) and perform an expansion in rectangular coordinates:

$$(\mathbf{r}'\cdot\nabla')^2\frac{1}{R}\bigg|_{\mathbf{r}'=0} = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i' x_j' \frac{\partial^2}{\partial x_i'\partial x_j'}\frac{1}{R}\bigg|_{\mathbf{r}'=0}.$$

It turns out [172] that this can be written as

$$(\mathbf{r}'\cdot\nabla')^2\frac{1}{R}\bigg|_{\mathbf{r}'=0} = \frac{1}{r^3}\hat{\mathbf{r}}\cdot(3\mathbf{r}'\mathbf{r}' - r'^2\bar{\mathbf{I}})\cdot\hat{\mathbf{r}}.$$

Substitution into (3.90) gives

$$\Phi(r) = \frac{Q}{4\pi\epsilon r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon r^2} + \frac{1}{2}\frac{\hat{\mathbf{r}}\cdot\bar{\mathbf{Q}}\cdot\hat{\mathbf{r}}}{4\pi\epsilon r^3} + \cdots, \qquad (3.92)$$

which is the multipole expansion for $\Phi(r)$. It converges for all $r > r_m$ where r_m is the radius of the smallest sphere completely containing the charge centered at $\mathbf{r}' = 0$ (Figure 3.11). In (3.92) the terms Q, $\mathbf{p}$, $\bar{\mathbf{Q}}$, and so on are called the *multipole moments* of $\rho(\mathbf{r})$. The first moment is merely the total charge

$$Q = \int_V \rho(\mathbf{r}')\, dV'.$$

The second moment is the *electric dipole moment vector*

$$\mathbf{p} = \int_V \mathbf{r}'\rho(\mathbf{r}')\, dV'.$$

The third moment is the *electric quadrupole moment dyadic*

$$\bar{\mathbf{Q}} = \int_V (3\mathbf{r}'\mathbf{r}' - r'^2\bar{\mathbf{I}})\rho(\mathbf{r}')\, dV'.$$

The expansion (3.92) allows us to identify the dominant power of r for $r \gg r_m$. The first nonzero term in (3.92) dominates the potential at points far from the source. Interestingly, the first nonvanishing moment is independent of the location of the origin of $\mathbf{r}'$, while all subsequent higher moments depend on the location of the origin [91]. We can see this most easily through a few simple examples.

For a single point charge q located at $\mathbf{r}_0$ we can write $\rho(\mathbf{r}) = q\delta(\mathbf{r} - \mathbf{r}_0)$. The first moment of ρ is

$$Q = \int_V q\delta(\mathbf{r}' - \mathbf{r}_0)\, dV' = q.$$

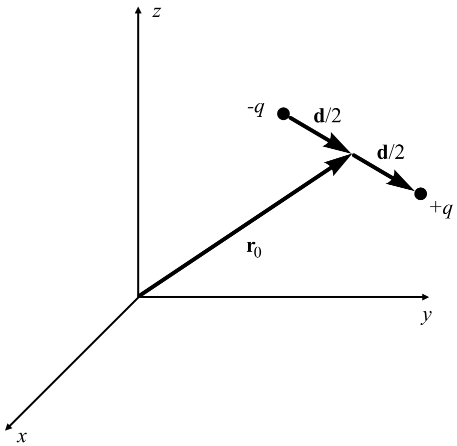

Figure 3.12: A dipole distribution.

Note that this is independent of $\mathbf{r}_0$. The second moment

$$\mathbf{p} = \int_V \mathbf{r}' q \delta(\mathbf{r}' - \mathbf{r}_0)\, dV' = q\mathbf{r}_0$$

depends on $\mathbf{r}_0$, as does the third moment

$$\bar{\mathbf{Q}} = \int_V (3\mathbf{r}'\mathbf{r}' - r'^2\bar{\mathbf{I}}) q \delta(\mathbf{r}' - \mathbf{r}_0)\, dV' = q(3\mathbf{r}_0\mathbf{r}_0 - r_0^2\bar{\mathbf{I}}).$$

If $\mathbf{r}_0 = 0$ then only the first moment is nonzero; that this must be the case is obvious from (3.61).

For the dipole of Figure 3.12 we can write

$$\rho(\mathbf{r}) = -q\delta(\mathbf{r} - \mathbf{r}_0 + \mathbf{d}/2) + q\delta(\mathbf{r} - \mathbf{r}_0 - \mathbf{d}/2).$$

In this case

$$Q = -q + q = 0, \qquad \mathbf{p} = q\mathbf{d}, \qquad \bar{\mathbf{Q}} = q[3(\mathbf{r}_0\mathbf{d} + \mathbf{d}\mathbf{r}_0) - 2(\mathbf{r}_0 \cdot \mathbf{d})\bar{\mathbf{I}}].$$

Only the first nonzero moment, in this case $\mathbf{p}$, is independent of $\mathbf{r}_0$. For $\mathbf{r}_0 = 0$ the only nonzero multipole moment would be the dipole moment $\mathbf{p}$. If the dipole is aligned along the z-axis with $\mathbf{d} = d\hat{\mathbf{z}}$ and $\mathbf{r}_0 = 0$, then the exact potential is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon}\frac{p\cos\theta}{r^2}.$$

By (3.30) we have

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon}\frac{p}{r^3}(\hat{\mathbf{r}}2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta), \tag{3.93}$$

which is the classic result for the electric field of a dipole.

Finally, consider the quadrupole shown in Figure 3.13. The charge density is

$$\rho(\mathbf{r}) = -q\delta(\mathbf{r} - \mathbf{r}_0) + q\delta(\mathbf{r} - \mathbf{r}_0 - \mathbf{d}_1) + q\delta(\mathbf{r} - \mathbf{r}_0 - \mathbf{d}_2) - q\delta(\mathbf{r} - \mathbf{r}_0 - \mathbf{d}_1 - \mathbf{d}_2).$$

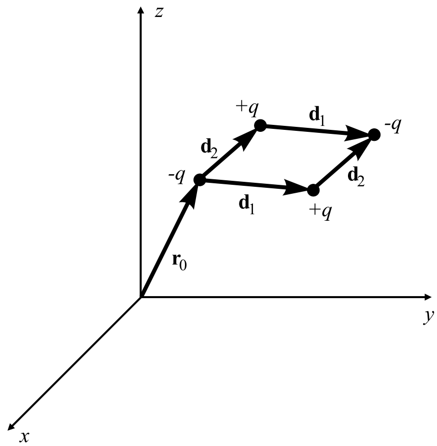

Figure 3.13: A quadrupole distribution.

Carrying through the details, we find that the first two moments of ρ vanish, while the third is given by

$$\bar{\mathbf{Q}} = q[-3(\mathbf{d}_1\mathbf{d}_2 + \mathbf{d}_2\mathbf{d}_1) + 2(\mathbf{d}_1 \cdot \mathbf{d}_2)\bar{\mathbf{I}}].$$

As expected, it is independent of $\mathbf{r}_0$.

It is tedious to carry (3.92) beyond the quadrupole term using the Taylor expansion. Another approach is to expand $1/R$ in spherical harmonics. Referring to Appendix E.3 we find that

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{2n+1} \frac{r'^n}{r^{n+1}} Y_{nm}^*(\theta', \phi') Y_{nm}(\theta, \phi)$$

(see Jackson [91] or Arfken [5] for a detailed derivation). This expansion converges for $|\mathbf{r}| > |\mathbf{r}_m|$. Substitution into (3.61) gives

$$\Phi(\mathbf{r}) = \frac{1}{\epsilon} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \left[\frac{1}{2n+1} \sum_{m=-n}^{n} q_{nm} Y_{nm}(\theta, \phi) \right] \tag{3.94}$$

where

$$q_{nm} = \int_V \rho(\mathbf{r}') r'^n Y_{nm}^*(\theta', \phi') \, dV'.$$

We can now identify any inverse power of r in the multipole expansion, but at the price of dealing with a double summation. For a charge distribution with axial symmetry (no ϕ-variation), only the coefficient q_{n0} is nonzero. The relation

$$Y_{n0}(\theta, \phi) = \sqrt{\frac{2n+1}{4\pi}} P_n(\cos\theta)$$

allows us to simplify (3.94) and obtain

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} q_n P_n(\cos\theta) \tag{3.95}$$

where

$$q_n = 2\pi \int_{r'} \int_{\theta'} \rho(r', \theta') r'^n P_n(\cos\theta') r'^2 \sin\theta' \, d\theta' \, dr'.$$

As a simple example consider a spherical distribution of charge given by

$$\rho(\mathbf{r}) = \frac{3Q}{\pi a^3} \cos\theta, \qquad r \le a.$$

This can be viewed as two adjacent hemispheres carrying total charges $\pm Q$. Since $\cos\theta = P_1(\cos\theta)$, we compute

$$\begin{aligned} q_n &= 2\pi \int_0^a \int_0^\pi \frac{3Q}{\pi a^3} P_1(\cos\theta') r'^n P_n(\cos\theta') r'^2 \sin\theta' \, d\theta' \, dr' \\ &= 2\pi \frac{3Q}{\pi a^3} \frac{a^{n+3}}{n+3} \int_0^\pi P_1(\cos\theta) P_n(\cos\theta') \sin\theta' \, d\theta'. \end{aligned}$$

Using the orthogonality relation (E.123) we find

$$q_n = 2\pi \frac{3Q}{\pi a^3} \frac{a^{n+3}}{n+3} \delta_{1n} \frac{2}{2n+1}.$$

Hence the only nonzero coefficient is $q_1 = Qa$ and

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon} \frac{1}{r^2} Qa P_1(\cos\theta) = \frac{Qa}{4\pi\epsilon r^2} \cos\theta.$$

This is the potential of a dipole having moment $\mathbf{p} = \hat{\mathbf{z}} Qa$. Thus we could replace the sphere with point charges $\mp Q$ at $z = \mp a/2$ without changing the field for $r > a$.

Physical interpretation of the polarization vector in a dielectric. We have used the Maxwell–Minkowski equations to determine the electrostatic potential of a charge distribution in the presence of a dielectric medium. Alternatively, we can use the Maxwell–Boffi equations

$$\nabla \times \mathbf{E} = 0, \tag{3.96}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} (\rho - \nabla \cdot \mathbf{P}). \tag{3.97}$$

Equation (3.96) allows us to define a scalar potential through (3.30). Substitution into (3.97) gives

$$\nabla^2 \Phi(\mathbf{r}) = -\frac{1}{\epsilon_0} [\rho(\mathbf{r}) + \rho_P(\mathbf{r})] \tag{3.98}$$

where $\rho_P = -\nabla \cdot \mathbf{P}$. This has the form of Poisson's equation (3.50), but with charge density term $\rho(\mathbf{r}) + \rho_P(\mathbf{r})$. Hence the solution is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}') - \nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV'.$$

To this we must add any potential produced by surface sources such as ρ_s. If there is a discontinuity in the dielectric region, there is also a surface polarization source $\rho_{Ps} = \hat{\mathbf{n}} \cdot \mathbf{P}$

according to (3.35). Separating the volume into regions with bounding surfaces S_i across which the permittivity is discontinuous, we may write

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV' + \frac{1}{4\pi\epsilon_0}\int_S \frac{\rho_s(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dS' + \\ + \sum_i \left[\frac{1}{4\pi\epsilon_0}\int_{V_i} \frac{-\nabla'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV' + \frac{1}{4\pi\epsilon_0}\oint_{S_i} \frac{\hat{\mathbf{n}}'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dS'\right], \tag{3.99}$$

where $\hat{\mathbf{n}}$ points outward from region i. Using the divergence theorem on the fourth term and employing (B.42), we obtain

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV' + \frac{1}{4\pi\epsilon_0}\int_S \frac{\rho_s(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dS' + \\ + \sum_i \left[\frac{1}{4\pi\epsilon_0}\int_{V_i} \mathbf{P}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dV'\right].$$

Since $\nabla'(1/R) = \hat{\mathbf{R}}/R^2$, the third term is a sum of integrals of the form

$$\frac{1}{4\pi\epsilon}\int_{V_i} \mathbf{P}(\mathbf{r}')\cdot\frac{\hat{\mathbf{R}}}{R^2}\,dV.$$

Comparing this to the second term of (3.92), we see that this integral represents a volume superposition of dipole terms where $\mathbf{P}$ is a volume density of dipole moments.

Thus, a dielectric with permittivity ϵ is equivalent to a volume distribution of dipoles in free space. No higher-order moments are required, and no zero-order moments are needed since any net charge is included in ρ. Note that we have arrived at this conclusion based only on Maxwell's equations and the assumption of a linear, isotropic relationship between $\mathbf{D}$ and $\mathbf{E}$. Assuming our macroscopic theory is correct, we are tempted to make assumptions about the behavior of matter on a microscopic level (e.g., atoms exposed to fields are polarized and their electron clouds are displaced from their positively charged nuclei), but this area of science is better studied from the viewpoints of particle physics and quantum mechanics.

Potential of an azimuthally-symmetric charged spherical surface. In several of our example problems we shall be interested in evaluating the potential of a charged spherical surface. When the charge is azimuthally-symmetric, the potential is particularly simple.

We will need the value of the integral

$$F(\mathbf{r}) = \frac{1}{4\pi}\int_S \frac{f(\theta')}{|\mathbf{r}-\mathbf{r}'|}\,dS' \tag{3.100}$$

where $\mathbf{r} = r\hat{\mathbf{r}}$ describes an arbitrary observation point and $\mathbf{r}' = a\hat{\mathbf{r}}'$ identifies the source point on the surface of the sphere of radius a. The integral is most easily done using the expansion (E.200) for $|\mathbf{r}-\mathbf{r}'|^{-1}$ in spherical harmonics. We have

$$F(\mathbf{r}) = a^2\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{Y_{nm}(\theta,\phi)}{2n+1}\frac{r_<^n}{r_>^{n+1}}\int_{-\pi}^{\pi}\int_0^{\pi} f(\theta')Y_{nm}^*(\theta',\phi')\sin\theta'\,d\theta'\,d\phi'$$

where $r_< = \min\{r, a\}$ and $r_> = \max\{r, a\}$. Using orthogonality of the exponentials we find that only the $m = 0$ terms contribute:

$$F(\mathbf{r}) = 2\pi a^2 \sum_{n=0}^{\infty} \frac{Y_{n0}(\theta, \phi)}{2n+1} \frac{r_<^n}{r_>^{n+1}} \int_0^{\pi} f(\theta') Y_{n0}^*(\theta', \phi') \sin\theta' \, d\theta'.$$

Finally, since

$$Y_{n0} = \sqrt{\frac{2n+1}{4\pi}} P_n(\cos\theta)$$

we have

$$F(\mathbf{r}) = \frac{1}{2} a^2 \sum_{n=0}^{\infty} P_n(\cos\theta) \frac{r_<^n}{r_>^{n+1}} \int_0^{\pi} f(\theta') P_n(\cos\theta') \sin\theta' \, d\theta'. \tag{3.101}$$

As an example, suppose $f(\theta) = \cos\theta = P_1(\cos\theta)$. Then

$$F(\mathbf{r}) = \frac{1}{2} a^2 \sum_{n=0}^{\infty} P_n(\cos\theta) \frac{r_<^n}{r_>^{n+1}} \int_0^{\pi} P_1(\cos\theta') P_n(\cos\theta') \sin\theta' \, d\theta'.$$

The orthogonality of the Legendre polynomials can be used to show that

$$\int_0^{\pi} P_1(\cos\theta') P_n(\cos\theta') \sin\theta' \, d\theta' = \frac{2}{3} \delta_{1n},$$

hence

$$F(\mathbf{r}) = \frac{a^2}{3} \cos\theta \frac{r_<}{r_>^2}. \tag{3.102}$$

3.2.7 Field produced by a permanently polarized body

Certain materials, called *electrets*, exhibit polarization in the absence of an external electric field. A permanently polarized material produces an electric field both internal and external to the material, hence there must be a charge distribution to support the fields. We can interpret this charge as being caused by the permanent separation of atomic charge within the material, but if we are only interested in the macroscopic field then we need not worry about the microscopic implications of such materials. Instead, we can use the Maxwell–Boffi equations and find the potential produced by the material by using (3.99). Thus, the field of an electret with known polarization $\mathbf{P}$ occupying volume region V in free space is dipolar in nature and is given by

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{-\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV' + \frac{1}{4\pi\epsilon_0} \oint_S \frac{\hat{\mathbf{n}}' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dS'$$

where $\hat{\mathbf{n}}$ points out of the volume region V.

As an example, consider a material sphere of radius a, permanently polarized along its axis with uniform polarization $\mathbf{P}(\mathbf{r}) = \hat{\mathbf{z}} P_0$. We have the equivalent source densities

$$\rho_p = -\nabla \cdot \mathbf{P} = 0, \qquad \rho_{Ps} = \hat{\mathbf{n}} \cdot \mathbf{P} = \hat{\mathbf{r}} \cdot \hat{\mathbf{z}} P_0 = P_0 \cos\theta.$$

Then

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_S \frac{\rho_{Ps}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dS' = \frac{1}{4\pi\epsilon_0} \oint_S \frac{P_0 \cos\theta'}{|\mathbf{r} - \mathbf{r}'|} \, dS'.$$

The integral takes the form (3.100), hence by (3.102) the solution is

$$\Phi(\mathbf{r}) = P_0 \frac{a^2}{3\epsilon_0} \cos\theta \frac{r_<}{r_>^2}. \tag{3.103}$$

If we are interested only in the potential for $r > a$, we can use the multipole expansion (3.95) to obtain

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} q_n P_n(\cos\theta), \qquad r > a$$

where

$$q_n = 2\pi \int_0^{\pi} \rho_{Ps}(\theta') a^n P_n(\cos\theta') a^2 \sin\theta' \, d\theta'.$$

Substituting for ρ_{Ps} and remembering that $\cos\theta = P_1(\cos\theta)$, we have

$$q_n = 2\pi a^{n+2} P_0 \int_0^{\pi} P_1(\cos\theta') P_n(\cos\theta') \sin\theta' \, d\theta'.$$

Using the orthogonality relation (E.123) we find

$$q_n = 2\pi a^{n+2} P_0 \delta_{1n} \frac{2}{2n+1}.$$

Therefore the only nonzero coefficient is

$$q_1 = \frac{4\pi a^3 P_0}{3}$$

and

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \frac{4\pi a^3 P_0}{3} P_1(\cos\theta) = \frac{P_0 a^3}{3\epsilon_0 r^2} \cos\theta, \qquad r > a.$$

This is a dipole field, and matches (3.103) as expected.

3.2.8 Potential of a dipole layer

Surface charge layers sometimes occur in bipolar form, such as in the membrane surrounding an animal cell. These can be modeled as a dipole layer consisting of parallel surface charges of opposite sign.

Consider a surface S located in free space. Parallel to this surface, and a distance $\Delta/2$ below, is located a surface charge layer of density $\rho_s(\mathbf{r}) = P_s(\mathbf{r})$. Also parallel to S, but a distance $\Delta/2$ above, is a surface charge layer of density $\rho_s(\mathbf{r}) = -P_s(\mathbf{r})$. We define the *surface dipole moment density* D_s as

$$D_s(\mathbf{r}) = \Delta \, P_s(\mathbf{r}). \tag{3.104}$$

Letting the position vector $\mathbf{r}'_0$ point to the surface S we can write the potential (3.61) produced by the two charge layers as

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{S^+} P_s(\mathbf{r}') \frac{1}{|\mathbf{r} - \mathbf{r}'_0 - \hat{\mathbf{n}}' \frac{\Delta}{2}|} \, dS' - \frac{1}{4\pi\epsilon_0} \int_{S^-} P_s(\mathbf{r}') \frac{1}{|\mathbf{r} - \mathbf{r}'_0 + \hat{\mathbf{n}}' \frac{\Delta}{2}|} \, dS'.$$

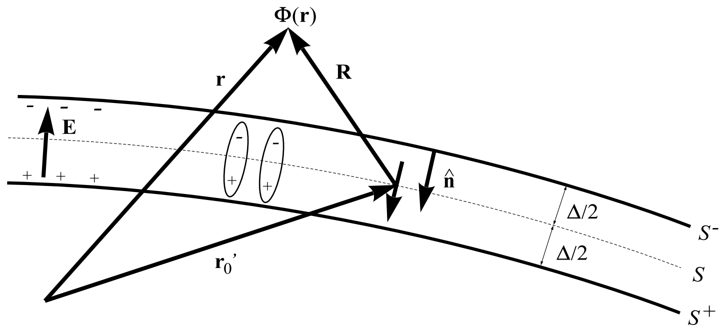

Figure 3.14: A dipole layer.

We are interested in the case in which the two charge layers collapse onto the surface S, and wish to compute the potential produced by a given dipole moment density. When $\Delta \to 0$ we have $\mathbf{r}_0' \to \mathbf{r}'$ and may write

$$\Phi(\mathbf{r}) = \lim_{\Delta\to 0} \frac{1}{4\pi\epsilon_0} \int_S \frac{D_s(\mathbf{r}')}{\Delta} \left[\frac{1}{|\mathbf{R} - \hat{\mathbf{n}}'\frac{\Delta}{2}|} - \frac{1}{|\mathbf{R} + \hat{\mathbf{n}}'\frac{\Delta}{2}|} \right] dS',$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. By the binomial theorem, the limit of the term in brackets can be written as

$$\lim_{\Delta\to 0} \left(\left[R^2 + \left(\frac{\Delta}{2}\right)^2 - 2\mathbf{R}\cdot\hat{\mathbf{n}}'\frac{\Delta}{2} \right]^{-\frac{1}{2}} - \left[R^2 + \left(\frac{\Delta}{2}\right)^2 + 2\mathbf{R}\cdot\hat{\mathbf{n}}'\frac{\Delta}{2} \right]^{-\frac{1}{2}} \right)$$
$$= \lim_{\Delta\to 0} \left(R^{-1} \left[1 + \frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{n}}'}{R}\frac{\Delta}{2} \right] - R^{-1} \left[1 - \frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{n}}'}{R}\frac{\Delta}{2} \right] \right) = \Delta\hat{\mathbf{n}}' \cdot \frac{\mathbf{R}}{R^3}.$$

Thus

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_S \mathbf{D}_s(\mathbf{r}') \cdot \frac{\mathbf{R}}{R^3}\, dS' \tag{3.105}$$

where $\mathbf{D}_s = \hat{\mathbf{n}} D_s$ is the surface vector dipole moment density. The potential of a dipole layer decreases more rapidly ($\sim 1/r^2$) than that of a unipolar charge layer. We saw similar behavior in the dipole term of the multipole expansion (3.92) for a general charge distribution.

We can use (3.105) to study the behavior of the potential across a dipole layer. As we approach the layer from above, the greatest contribution to Φ comes from the charge region immediately beneath the observation point. Assuming that the surface dipole moment density is continuous beneath the point, we can compute the difference in the fields across the layer at point $\mathbf{r}$ by replacing the arbitrary surface layer by a disk of constant surface dipole moment density $\mathbf{D}_0 = \mathbf{D}_s(\mathbf{r})$. For simplicity we center the disk at $z = 0$ in the xy-plane as shown in Figure 3.15 and compute the potential difference ΔV across the layer; i.e., $\Delta V = \Phi(h) - \Phi(-h)$ on the disk axis as $h \to 0$. Using (3.105) along with $\mathbf{r}' = \pm h\hat{\mathbf{z}} - \rho'\hat{\boldsymbol{\rho}}'$, we obtain

$$\Delta V = \lim_{h\to 0} \left[\frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^a [\hat{\mathbf{z}} D_0] \cdot \frac{\hat{\mathbf{z}} h - \hat{\boldsymbol{\rho}}'\rho'}{(h^2 + \rho'^2)^{3/2}} \rho'\, d\rho'\, d\phi' - \right.$$
$$\left. - \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^a [\hat{\mathbf{z}} D_0] \cdot \frac{-\hat{\mathbf{z}} h - \hat{\boldsymbol{\rho}}'\rho'}{(h^2 + \rho'^2)^{3/2}} \rho'\, d\rho'\, d\phi' \right]$$

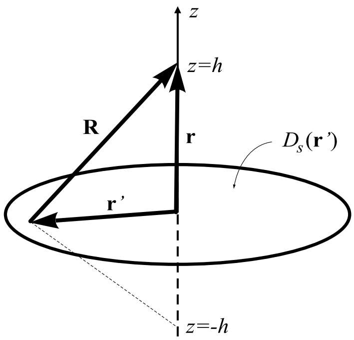

Figure 3.15: Auxiliary disk for studying the potential distribution across a dipole layer.

where a is the disk radius. Integration yields

$$\Delta V = \frac{D_0}{2\epsilon_0} \lim_{h\to 0} \left[\frac{-2}{\sqrt{1+\left(\frac{a}{h}\right)^2}} + 2 \right] = \frac{D_0}{\epsilon_0},$$

independent of a. Generalizing this to an arbitrary surface dipole moment density, we find that the boundary condition on the potential is given by

$$\Phi_2(\mathbf{r}) - \Phi_1(\mathbf{r}) = \frac{D_s(\mathbf{r})}{\epsilon_0} \tag{3.106}$$

where "1" denotes the positive side of the dipole moments and "2" the negative side. Physically, the potential difference in (3.106) is produced by the line integral of **E** "internal" to the dipole layer. Since there is no field internal to a unipolar surface layer, V is continuous across a surface containing charge ρ_s but having $D_s = 0$.

3.2.9 Behavior of electric charge density near a conducting edge

Sharp corners are often encountered in the application of electrostatics to practical geometries. The behavior of the charge distribution near these corners must be understood in order to develop numerical techniques for solving more complicated problems. We can use a simple model of a corner if we restrict our interest to the region near the edge. Consider the intersection of two planes as shown in Figure 3.16. The region near the intersection represents the corner we wish to study. We assume that the planes are held at zero potential and that the charge on the surface is induced by a two-dimensional charge distribution $\rho(\mathbf{r})$, or by a potential difference between the edge and another conductor far removed from the edge.

We can find the potential in the region near the edge by solving Laplace's equation in cylindrical coordinates. This problem is studied in Appendix A where the separation of variables solution is found to be either (A.127) or (A.128). Using (A.128) and enforcing $\Phi = 0$ at both $\phi = 0$ and $\phi = \beta$, we obtain the null solution. Hence the solution must take the form (A.127):

$$\Phi(\rho, \phi) = [A_\phi \sin(k_\phi \phi) + B_\phi \cos(k_\phi \phi)][a_\rho \rho^{-k_\phi} + b_\rho \rho^{k_\phi}]. \tag{3.107}$$

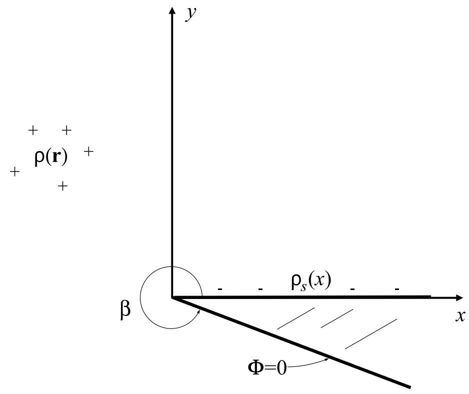

Figure 3.16: A conducting edge.

Since the origin is included we cannot have negative powers of ρ and must put $a_\rho = 0$. The boundary condition $\Phi(\rho, 0) = 0$ requires $B_\phi = 0$. The condition $\Phi(\rho, \beta) = 0$ then requires $\sin(k_\phi \beta) = 0$, which holds only if $k_\phi = n\pi/\beta$, $n = 1, 2, \ldots$. The general solution for the potential near the edge is therefore

$$\Phi(\rho, \phi) = \sum_{n=1}^{N} A_n \sin\left(\frac{n\pi}{\beta}\phi\right) \rho^{n\pi/\beta} \tag{3.108}$$

where the constants A_n depend on the excitation source or system of conductors. (Note that if the corner is held at potential $V_0 \neq 0$, we must merely add V_0 to the solution.) The charge on the conducting surfaces can be computed from the boundary condition on normal **D**. Using (3.30) we have

$$E_\phi = -\frac{1}{\rho}\frac{\partial}{\partial \phi} \sum_{n=1}^{N} A_n \sin\left(\frac{n\pi}{\beta}\phi\right) \rho^{n\pi/\beta} = -\sum_{n=1}^{N} A_n \frac{n\pi}{\beta} \cos\left(\frac{n\pi}{\beta}\phi\right) \rho^{(n\pi/\beta)-1},$$

hence

$$\rho_s(x) = -\epsilon \sum_{n=1}^{N} A_n \frac{n\pi}{\beta} x^{(n\pi/\beta)-1}$$

on the surface at $\phi = 0$. Near the edge, at small values of x, the variation of ρ_s is dominated by the lowest power of x. (Here we ignore those special excitation arrangements that produce $A_1 = 0$.) Thus

$$\rho_s(x) \sim x^{(\pi/\beta)-1}.$$

The behavior of the charge clearly depends on the wedge angle β. For a sharp edge (half plane) we put $\beta = 2\pi$ and find that the field varies as $x^{-1/2}$. This *square-root edge singularity* is very common on thin plates, fins, etc., and means that charge tends to accumulate near the edge of a flat conducting surface. For a right-angle corner where $\beta = 3\pi/2$, there is the somewhat weaker singularity $x^{-1/3}$. When $\beta = \pi$, the two surfaces fold out into an infinite plane and the charge, not surprisingly, is invariant with x to lowest order near the folding line. When $\beta < \pi$ the corner becomes interior and we find that the charge density varies with a positive power of distance from the edge. For very sharp interior angles the power is large, meaning that little charge accumulates on the inner surfaces near an interior corner.

3.2.10 Solution to Laplace's equation for bodies immersed in an impressed field

An important class of problems is based on the idea of placing a body into an existing electric field, assuming that the field arises from sources so remote that the introduction of the body does not alter the original field. The pre-existing field is often referred to as the *applied* or *impressed field*, and the solution external to the body is usually formulated as the sum of the applied field and a *secondary* or *scattered field* that satisfies Laplace's equation. This total field differs from the applied field, and must satisfy the appropriate boundary condition on the body. If the body is a conductor then the total potential must be constant everywhere on the boundary surface. If the body is a solid homogeneous dielectric then the total potential field must be continuous across the boundary.

As an example, consider a dielectric sphere of permittivity ϵ and radius a, centered at the origin and immersed in a constant electric field $\mathbf{E}_0(\mathbf{r}) = E_0\hat{\mathbf{z}}$. By (3.30) the applied potential field is $\Phi_0(\mathbf{r}) = -E_0 z = -E_0 r\cos\theta$ (to within a constant). Outside the sphere $(r > a)$ we write the total potential field as

$$\Phi_2(\mathbf{r}) = \Phi_0(\mathbf{r}) + \Phi^s(\mathbf{r})$$

where $\Phi^s(\mathbf{r})$ is the secondary or scattered potential. Since Φ^s must satisfy Laplace's equation, we can write it as a separation of variables solution (§ A.4). By azimuthal symmetry the potential has an r-dependence as in (A.146), and a θ-dependence as in (A.142) with $B_\theta = 0$ and $m = 0$. Thus Φ^s has a representation identical to (A.147), except that we cannot use terms that are unbounded as $r \to \infty$. We therefore use

$$\Phi^s(r,\theta) = \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta). \tag{3.109}$$

The potential inside the sphere also obeys Laplace's equation, so we can use the same form (A.147) while discarding terms unbounded at the origin. Thus

$$\Phi_1(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) \tag{3.110}$$

for $r < a$. To find the constants A_n and B_n we apply (3.36) and (3.37) to the total field. Application of (3.36) at $r = a$ gives

$$-E_0 a\cos\theta + \sum_{n=0}^{\infty} B_n a^{-(n+1)} P_n(\cos\theta) = \sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta).$$

Multiplying through by $P_m(\cos\theta)\sin\theta$, integrating from $\theta = 0$ to $\theta = \pi$, and using the orthogonality relationship (E.123), we obtain

$$-E_0 a + a^{-2} B_1 = A_1 a, \tag{3.111}$$

$$B_n a^{-(n+1)} = A_n a^n, \quad n \neq 1, \tag{3.112}$$

where we have used $P_1(\cos\theta) = \cos\theta$. Next, since $\rho_s = 0$, equation (3.37) requires that

$$\epsilon_1 \frac{\partial \Phi_1(\mathbf{r})}{\partial r} = \epsilon_2 \frac{\partial \Phi_2(\mathbf{r})}{\partial r}$$

at $r = a$. This gives

$$-\epsilon_0 E_0 \cos\theta + \epsilon_0 \sum_{n=0}^{\infty} [-(n+1)B_n] a^{-n-2} P_n(\cos\theta) = \epsilon \sum_{n=0}^{\infty} [n A_n] a^{n-1} P_n(\cos\theta).$$

By orthogonality of the Legendre functions we have

$$-\epsilon_0 E_0 - 2\epsilon_0 B_1 a^{-3} = \epsilon A_1, \tag{3.113}$$

$$-\epsilon_0 (n+1) B_n a^{-n-2} = \epsilon n A_n a^{n-1}, \quad n \neq 1. \tag{3.114}$$

Equations (3.112) and (3.114) cannot hold simultaneously unless $A_n = B_n = 0$ for $n \neq 1$. Solving (3.111) and (3.113) we have

$$A_1 = -E_0 \frac{3\epsilon_0}{\epsilon + 2\epsilon_0}, \qquad B_1 = E_0 a^3 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}.$$

Hence

$$\Phi_1(\mathbf{r}) = -E_0 \frac{3\epsilon_0}{\epsilon + 2\epsilon_0} r \cos\theta = -E_0 z \frac{3\epsilon_0}{\epsilon + 2\epsilon_0}, \tag{3.115}$$

$$\Phi_2(\mathbf{r}) = -E_0 r \cos\theta + E_0 \frac{a^3}{r^2} \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \cos\theta. \tag{3.116}$$

Interestingly, the electric field

$$\mathbf{E}_1(\mathbf{r}) = -\nabla\Phi_1(\mathbf{r}) = \hat{\mathbf{z}} E_0 \frac{3\epsilon_0}{\epsilon + 2\epsilon_0}$$

inside the sphere is constant with position and is aligned with the applied external field. However, it is weaker than the applied field since $\epsilon > \epsilon_0$. To explain this, we compute the polarization charge within and on the sphere. Using $\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P}$ we have

$$\mathbf{P}_1 = \hat{\mathbf{z}}(\epsilon - \epsilon_0) E_0 \frac{3\epsilon_0}{\epsilon + 2\epsilon_0}. \tag{3.117}$$

The volume polarization charge density $-\nabla \cdot \mathbf{P}$ is zero, while the polarization surface charge density is

$$\rho_{Ps} = \hat{\mathbf{r}} \cdot \mathbf{P} = (\epsilon - \epsilon_0) E_0 \frac{3\epsilon_0}{\epsilon + 2\epsilon_0} \cos\theta.$$

Hence the secondary electric field can be attributed to an induced surface polarization charge, and is in a direction opposing the applied field. According to the Maxwell–Boffi viewpoint we should be able to replace the sphere by the surface polarization charge immersed in free space, and use the formula (3.61) to reproduce (3.115) and (3.116). This is left as an exercise for the reader.

3.3 Magnetostatics

The large-scale forms of the magnetostatic field equations are

$$\oint_\Gamma \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS}, \tag{3.118}$$

$$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0, \tag{3.119}$$

while the point forms are

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r}), \tag{3.120}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0. \tag{3.121}$$

Note the interesting dichotomy between the electrostatic field equations and the magnetostatic field equations. Whereas the electrostatic field exhibits zero curl and a divergence proportional to the source (charge), the magnetostatic field has zero divergence and a curl proportional to the source (current). Because the vector relationship between the magnetostatic field and its source is of a more complicated nature than the scalar relationship between the electrostatic field and its source, more effort is required to develop a strong understanding of magnetic phenomena. Also, it must always be remembered that although the equations describing the electrostatic and magnetostatic field sets decouple, the phenomena themselves remain linked. Since current is moving charge, electrical phenomena are associated with the establishment of the current that supports a magnetostatic field. We know, for example, that in order to have current in a wire an electric field must be present to drive electrons through the wire.

The magnetic scalar potential. Under certain conditions the equations of magnetostatics have the same form as those of electrostatics. If $\mathbf{J} = 0$ in a region V, the magnetostatic equations are

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0, \tag{3.122}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0; \tag{3.123}$$

compare with (3.5)–(3.6) when $\rho = 0$. Using (3.122) we can define a *magnetic scalar potential* Φ_m:

$$\mathbf{H} = -\nabla\Phi_m. \tag{3.124}$$

The negative sign is chosen for consistency with (3.30). We can then define a magnetic potential difference between two points as

$$V_{m21} = -\int_{P_1}^{P_2} \mathbf{H} \cdot \mathbf{dl} = -\int_{P_1}^{P_2} -\nabla\Phi_m(\mathbf{r}) \cdot \mathbf{dl} = \int_{P_1}^{P_2} d\Phi_m(\mathbf{r}) = \Phi_m(\mathbf{r}_2) - \Phi_m(\mathbf{r}_1).$$

Unlike the electrostatic potential difference, V_{m21} is not unique. Consider Figure 3.17, which shows a plane passing through the cross-section of a wire carrying total current I. Although there is no current within the region V (external to the wire), equation (3.118) still gives

$$\int_{\Gamma_2} \mathbf{H} \cdot \mathbf{dl} - \int_{\Gamma_3} \mathbf{H} \cdot \mathbf{dl} = I.$$

Thus

$$\int_{\Gamma_2} \mathbf{H} \cdot \mathbf{dl} = \int_{\Gamma_3} \mathbf{H} \cdot \mathbf{dl} + I,$$

and the integral $\int_\Gamma \mathbf{H} \cdot \mathbf{dl}$ is not path-independent. However,

$$\int_{\Gamma_1} \mathbf{H} \cdot \mathbf{dl} = \int_{\Gamma_2} \mathbf{H} \cdot \mathbf{dl}$$

since no current passes through the surface bounded by $\Gamma_1 - \Gamma_2$. So we can artificially impose uniqueness by demanding that no path cross a cut such as that indicated by the line L in the figure.

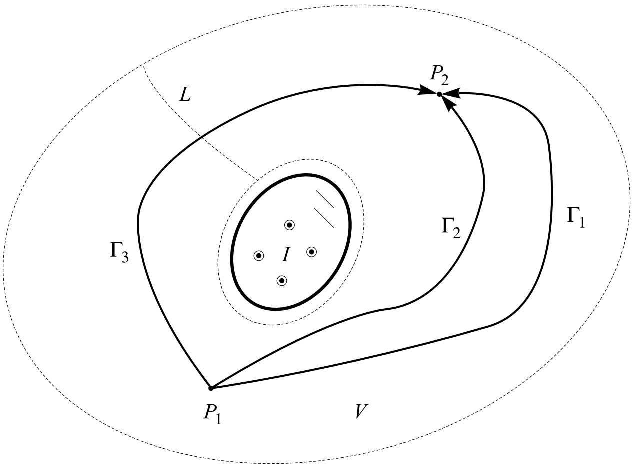

Figure 3.17: Magnetic potential.

Because V_{m21} is not unique, the field $\mathbf{H}$ is nonconservative. In point form this is shown by the fact that $\nabla \times \mathbf{H}$ is not identically zero. We are not too concerned about energy-related implications of the nonconservative nature of $\mathbf{H}$; the electric point charge has no magnetic analogue that might fail to conserve potential energy if moved around in a magnetic field.

Assuming a linear, isotropic region where $\mathbf{B}(\mathbf{r}) = \mu(\mathbf{r})\mathbf{H}(\mathbf{r})$, we can substitute (3.124) into (3.123) and expand to obtain

$$\nabla\mu(\mathbf{r}) \cdot \nabla\Phi_m(\mathbf{r}) + \mu(\mathbf{r})\nabla^2\Phi_m(\mathbf{r}) = 0.$$

For a homogeneous medium this reduces to Laplace's equation

$$\nabla^2\Phi_m = 0.$$

We can also obtain an analogue to Poisson's equation of electrostatics if we use

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = -\mu_0\nabla\Phi_m + \mu_0\mathbf{M}$$

in (3.123); we have

$$\nabla^2\Phi_m = -\rho_M \tag{3.125}$$

where

$$\rho_M = -\nabla \cdot \mathbf{M}$$

is called the *equivalent magnetization charge density*. This form can be used to describe fields of permanent magnets in the absence of $\mathbf{J}$. Comparison with (3.98) shows that ρ_M is analogous to the polarization charge ρ_P.

Since Φ_m obeys Poisson's equation, the details regarding uniqueness and the construction of solutions follow from those of the electrostatic case. If we include the possibility of a surface density of magnetization charge, then the integral solution for Φ_m in unbounded space is

$$\Phi_m(\mathbf{r}) = \frac{1}{4\pi}\int_V \frac{\rho_M(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, dV' + \frac{1}{4\pi}\int_S \frac{\rho_{Ms}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, dS'. \tag{3.126}$$

Here ρ_{Ms}, the surface density of magnetization charge, is identified as $\hat{\mathbf{n}} \cdot \mathbf{M}$ in the boundary condition (3.152).

3.3.1 The magnetic vector potential

Although the magnetic scalar potential is useful for describing fields of permanent magnets and for solving certain boundary value problems, it does not include the effects of source current. A second type of potential function, called the *magnetic vector potential*, can be used with complete generality to describe the magnetostatic field. Because $\nabla \cdot \mathbf{B} = 0$, we can write by (B.49)

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) \tag{3.127}$$

where $\mathbf{A}$ is the vector potential. Now $\mathbf{A}$ is not determined by (3.127) alone, since the gradient of any scalar field can be added to $\mathbf{A}$ without changing the value of $\nabla \times \mathbf{A}$. Such "gauge transformations" are discussed in Chapter 5, where we find that $\nabla \cdot \mathbf{A}$ must also be specified for uniqueness of $\mathbf{A}$.

The vector potential can be used to develop a simple formula for the magnetic flux passing through an open surface S:

$$\Psi_m = \int_S \mathbf{B} \cdot \mathbf{dS} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{dS} = \oint_\Gamma \mathbf{A} \cdot \mathbf{dl}, \tag{3.128}$$

where Γ is the contour bounding S.

In the linear isotropic case where $\mathbf{B} = \mu \mathbf{H}$ we can find a partial differential equation for $\mathbf{A}$ by substituting (3.127) into (3.120). Using (B.43) we have

$$\nabla \times \left[\frac{1}{\mu(\mathbf{r})} \nabla \times \mathbf{A}(\mathbf{r}) \right] = \mathbf{J}(\mathbf{r}),$$

hence

$$\frac{1}{\mu(\mathbf{r})} \nabla \times [\nabla \times \mathbf{A}(\mathbf{r})] - [\nabla \times \mathbf{A}(\mathbf{r})] \times \nabla \left(\frac{1}{\mu(\mathbf{r})} \right) = \mathbf{J}(\mathbf{r}).$$

In a homogeneous region we have

$$\nabla \times (\nabla \times \mathbf{A}) = \mu \mathbf{J} \tag{3.129}$$

or

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} \tag{3.130}$$

by (B.47). As mentioned above we must eventually specify $\nabla \cdot \mathbf{A}$. Although the choice is arbitrary, certain selections make the computation of $\mathbf{A}$ both mathematically tractable and physically meaningful. The "Coulomb gauge condition" $\nabla \cdot \mathbf{A} = 0$ reduces (3.130) to

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}. \tag{3.131}$$

The vector potential concept can also be applied to the Maxwell–Boffi magnetostatic equations

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \nabla \times \mathbf{M}), \tag{3.132}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{3.133}$$

By (3.133) we may still define $\mathbf{A}$ through (3.127). Substituting this into (3.132) we have, under the Coulomb gauge,

$$\nabla^2 \mathbf{A} = -\mu_0 [\mathbf{J} + \mathbf{J}_M] \tag{3.134}$$

where $\mathbf{J}_M = \nabla \times \mathbf{M}$ is the magnetization current density.

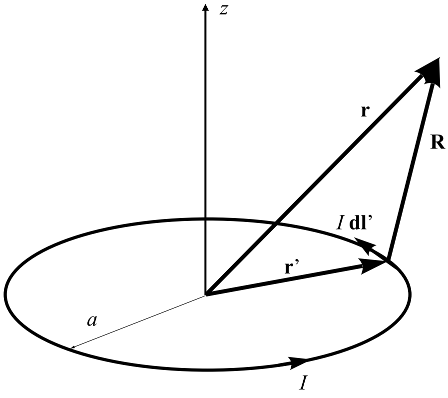

Figure 3.18: Circular loop of wire.

The differential equations (3.131) and (3.134) are vector versions of Poisson's equation, and may be solved quite easily for unbounded space by decomposing the vector source into rectangular components. For instance, dotting (3.131) with $\hat{\mathbf{x}}$ we find that

$$\nabla^2 A_x = -\mu J_x.$$

This scalar version of Poisson's equation has solution

$$A_x(\mathbf{r}) = \frac{\mu}{4\pi}\int_V \frac{J_x(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV'$$

in unbounded space. Repeating this for each component and assembling the results, we obtain the solution for the vector potential in an unbounded homogeneous medium:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV'. \tag{3.135}$$

Any surface sources can be easily included through a surface integral:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV' + \frac{\mu}{4\pi}\int_S \frac{\mathbf{J}_s(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dS'. \tag{3.136}$$

In unbounded free space containing materials represented by $\mathbf{M}$, we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')+\mathbf{J}_M(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV' + \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{J}_s(\mathbf{r}')+\mathbf{J}_{Ms}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV' \tag{3.137}$$

where $\mathbf{J}_{Ms} = -\hat{\mathbf{n}} \times \mathbf{M}$ is the surface density of magnetization current as described in (3.153). It may be verified directly from (3.137) that $\nabla \cdot \mathbf{A} = 0$.

Field of a circular loop. Consider a circular loop of line current of radius a in unbounded space (Figure 3.18). Using $\mathbf{J}(\mathbf{r}') = I\hat{\boldsymbol{\phi}}'\delta(z')\delta(\rho'-a)$ and noting that $\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$ and $\mathbf{r}' = a\hat{\boldsymbol{\rho}}'$, we can write (3.136) as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu I}{4\pi}\int_0^{2\pi} \hat{\boldsymbol{\phi}}' \frac{a\,d\phi'}{\left[\rho^2 + a^2 + z^2 - 2a\rho\cos(\phi-\phi')\right]^{1/2}}.$$

Because $\hat{\boldsymbol{\phi}}' = -\hat{\mathbf{x}}\cos\phi' + \hat{\mathbf{y}}\sin\phi'$ we find that

$$\mathbf{A}(\mathbf{r}) = \frac{\mu I a}{4\pi}\hat{\boldsymbol{\phi}} \int_0^{2\pi} \frac{\cos\phi'}{\left[\rho^2 + a^2 + z^2 - 2a\rho\cos\phi'\right]^{1/2}}\, d\phi'.$$

We put the integral into standard form by setting $\phi' = \pi - 2x$:

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu I a}{4\pi}\hat{\boldsymbol{\phi}} \int_{-\pi/2}^{\pi/2} \frac{1 - 2\sin^2 x}{\left[\rho^2 + a^2 + z^2 + 2a\rho(1 - 2\sin^2 x)\right]^{1/2}} 2\, dx.$$

Letting

$$k^2 = \frac{4a\rho}{(a+\rho)^2 + z^2}, \qquad F^2 = (a+\rho)^2 + z^2,$$

we have

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu I a}{4\pi}\hat{\boldsymbol{\phi}}\frac{4}{F} \int_0^{\pi/2} \frac{1 - 2\sin^2 x}{[1 - k^2\sin^2 x]^{1/2}}\, dx.$$

Then, since

$$\frac{1 - 2\sin^2 x}{[1 - k^2\sin^2 x]^{1/2}} = \frac{k^2 - 2}{k^2}[1 - k^2\sin^2 x]^{-1/2} + \frac{2}{k^2}[1 - k^2\sin^2 x]^{1/2},$$

we have

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\frac{\mu I}{\pi k}\sqrt{\frac{a}{\rho}}\left[\left(1 - \frac{1}{2}k^2\right)K(k^2) - E(k^2)\right]. \tag{3.138}$$

Here

$$K(k^2) = \int_0^{\pi/2} \frac{du}{[1 - k^2\sin^2 u]^{1/2}}, \qquad E(k^2) = \int_0^{\pi/2} [1 - k^2\sin^2 u]^{1/2}\, du,$$

are complete elliptic integrals of the first and second kinds, respectively.

We have $k^2 \ll 1$ when the observation point is far from the loop ($r^2 = \rho^2 + z^2 \gg a^2$). Using the expansions [47]

$$K(k^2) = \frac{\pi}{2}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \cdots\right], \qquad E(k^2) = \frac{\pi}{2}\left[1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \cdots\right],$$

in (3.138) and keeping the first nonzero term, we find that

$$\mathbf{A}(\mathbf{r}) \approx \hat{\boldsymbol{\phi}}\frac{\mu I}{4\pi r^2}(\pi a^2)\sin\theta. \tag{3.139}$$

Defining the *magnetic dipole moment* of the loop as

$$\mathbf{m} = \hat{\mathbf{z}} I \pi a^2,$$

we can write (3.139) as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}. \tag{3.140}$$

Generalization to an arbitrarily-oriented circular loop with center located at $\mathbf{r}_0$ is accomplished by writing $\mathbf{m} = \hat{\mathbf{n}} I A$ where A is the loop area and $\hat{\mathbf{n}}$ is normal to the loop in the right-hand sense. Then

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\mathbf{m}\times\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3}.$$

We shall find, upon investigating the general multipole expansion of **A** below, that this holds for any planar loop.

The magnetic field of the loop can be found by direct application of (3.127). For the case $r^2 \gg a^2$ we take the curl of (3.139) and find that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu}{4\pi}\frac{m}{r^3}(\hat{\mathbf{r}}\, 2\cos\theta + \hat{\boldsymbol{\theta}}\,\sin\theta). \tag{3.141}$$

Comparison with (3.93) shows why we often refer to a small loop as a *magnetic dipole*. But (3.141) is approximate, and since there are no magnetic monopoles we cannot construct an exact magnetic analogue to the electric dipole. On the other hand, we shall find below that the multipole expansion of a finite-extent steady current begins with the dipole term (since the current must form closed loops). We may regard small loops as the elemental units of steady current from which all other currents may be constructed.

3.3.2 Multipole expansion

It is possible to derive a general multipole expansion for **A** analogous to (3.94). But the vector nature of **A** requires that we use vector spherical harmonics, hence the result is far more complicated than (3.94). A simpler approach yields the first few terms and requires only the Taylor expansion of $1/R$. Consider a steady current localized near the origin and contained within a sphere of radius r_m. We substitute the expansion (3.89) into (3.135) to obtain

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int_V \mathbf{J}(\mathbf{r}')\left[\left.\frac{1}{R}\right|_{\mathbf{r}'=0} + (\mathbf{r}'\cdot\nabla')\left.\frac{1}{R}\right|_{\mathbf{r}'=0} + \frac{1}{2}(\mathbf{r}'\cdot\nabla')^2\left.\frac{1}{R}\right|_{\mathbf{r}'=0} + \cdots\right] dV', \tag{3.142}$$

which we view as

$$\mathbf{A}(\mathbf{r}) = \mathbf{A}^{(0)}(\mathbf{r}) + \mathbf{A}^{(1)}(\mathbf{r}) + \mathbf{A}^{(2)}(\mathbf{r}) + \cdots.$$

The first term is merely

$$\mathbf{A}^{(0)}(\mathbf{r}) = \frac{\mu}{4\pi r}\int_V \mathbf{J}(\mathbf{r}')\,dV' = \frac{\mu}{4\pi r}\sum_{i=1}^{3}\hat{\mathbf{x}}_i\int_V J_i(\mathbf{r}')\,dV'$$

where $(x, y, z) = (x_1, x_2, x_3)$. However, by (3.26) each of the integrals is zero and we have

$$\mathbf{A}^{(0)}(\mathbf{r}) = 0;$$

the leading term in the multipole expansion of **A** for a general steady current distribution vanishes.

Using (3.91) we can write the second term as

$$\mathbf{A}^{(1)}(\mathbf{r}) = \frac{\mu}{4\pi r^3}\int_V \mathbf{J}(\mathbf{r}')\sum_{i=1}^{3} x_i x_i'\,dV' = \frac{\mu}{4\pi r^3}\sum_{j=1}^{3}\hat{\mathbf{x}}_j\sum_{i=1}^{3} x_i\int_V x_i' J_j(\mathbf{r}')\,dV'. \tag{3.143}$$

By adding the null relation (3.28) we can write

$$\int_V x_i' J_j\,dV' = \int_V x_i' J_j\,dV' + \int_V [x_i' J_j + x_j' J_i]\,dV' = 2\int_V x_i' J_j\,dV' + \int_V x_j' J_i\,dV'$$

or

$$\int_V x_i' J_j\,dV' = \frac{1}{2}\int_V [x_i' J_j - x_j' J_i]\,dV'. \tag{3.144}$$

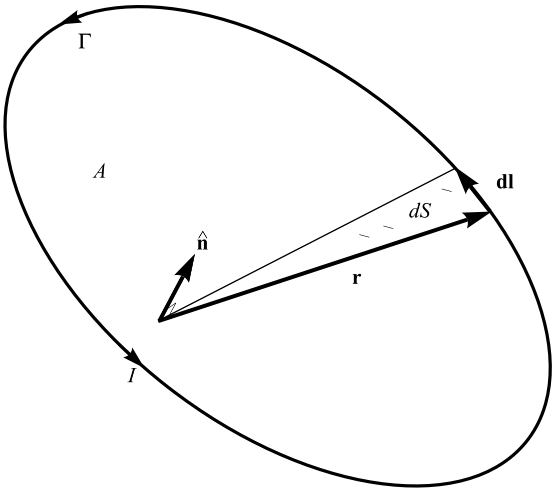

Figure 3.19: A planar wire loop.

By this and (3.143) the second term in the multipole expansion is

$$\mathbf{A}^{(1)}(\mathbf{r}) = \frac{\mu}{4\pi r^3}\frac{1}{2}\int_V \sum_{j=1}^{3}\hat{\mathbf{x}}_j \sum_{i=1}^{3} x_i[x_i' J_j - x_j' J_i]\,dV' = -\frac{\mu}{4\pi r^3}\frac{1}{2}\int_V \mathbf{r}\times[\mathbf{r}'\times\mathbf{J}(\mathbf{r}')]\,dV'.$$

Defining the *dipole moment vector*

$$\mathbf{m} = \frac{1}{2}\int_V \mathbf{r}\times\mathbf{J}(\mathbf{r})\,dV \tag{3.145}$$

we have

$$\mathbf{A}^{(1)}(\mathbf{r}) = \frac{\mu}{4\pi}\mathbf{m}\times\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = -\frac{\mu}{4\pi}\mathbf{m}\times\nabla\frac{1}{r}. \tag{3.146}$$

This is the *dipole moment potential* for the steady current **J**. Since steady currents of finite extent consist of loops, the dipole component is generally the first nonzero term in the expansion of **A**. Higher-order components may be calculated, but extension of (3.142) beyond the dipole term is quite tedious and will not be attempted.

As an example let us compute the dipole moment of the planar but otherwise arbitrary loop shown in Figure 3.19. Specializing (3.145) for a line current we have

$$\mathbf{m} = \frac{I}{2}\oint_\Gamma \mathbf{r}\times\mathbf{dl}.$$

Examining Figure 3.19, we see that

$$\frac{1}{2}\mathbf{r}\times\mathbf{dl} = \hat{\mathbf{n}}\,dS$$

where dS is the area of the sector swept out by **r** as it moves along **dl**, and $\hat{\mathbf{n}}$ is the normal to the loop in the right-hand sense. Thus

$$\mathbf{m} = \hat{\mathbf{n}} I A \tag{3.147}$$

where A is the area of the loop.

Physical interpretation of M in a magnetic material. In (3.137) we presented an expression for the vector potential produced by a magnetized material in terms of equivalent magnetization surface and volume currents. Suppose a magnetized medium is separated into volume regions with bounding surfaces across which the permeability is discontinuous. With $\mathbf{J}_M = \nabla \times \mathbf{M}$ and $\mathbf{J}_{Ms} = -\hat{\mathbf{n}} \times \mathbf{M}$ we obtain

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV' + \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{J}_s(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dS' + \\ + \sum_i \frac{\mu_0}{4\pi}\left[\int_{V_i} \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV' + \int_{S_i} \frac{-\hat{\mathbf{n}}' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dS'\right]. \tag{3.148}$$

Here $\hat{\mathbf{n}}$ points outward from region V_i. Using the curl theorem on the fourth term and employing the vector identity (B.43), we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV' + \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{J}_s(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dS' + \\ + \sum_i \left[\frac{\mu_0}{4\pi}\int_{V_i} \mathbf{M}(\mathbf{r}') \times \nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dV'\right]. \tag{3.149}$$

But $\nabla'(1/R) = \hat{\mathbf{R}}/R^2$, hence the third term is a sum of integrals of the form

$$\frac{\mu_0}{4\pi}\int_{V_i} \mathbf{M}(\mathbf{r}') \times \frac{\hat{\mathbf{R}}}{R^2}\,dV'.$$

Comparison with (3.146) shows that this integral represents a volume superposition of dipole moments where $\mathbf{M}$ is a volume density of magnetic dipole moments. Hence a magnetic material with permeability μ is equivalent to a volume distribution of magnetic dipoles in free space. As with our interpretation of the polarization vector in a dielectric, we base this conclusion only on Maxwell's equations and the assumption of a linear, isotropic relationship between $\mathbf{B}$ and $\mathbf{H}$.

3.3.3 Boundary conditions for the magnetostatic field

The boundary conditions found for the dynamic magnetic field remain valid in the magnetostatic case. Hence

$$\hat{\mathbf{n}}_{12} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \tag{3.150}$$

and

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0, \tag{3.151}$$

where $\hat{\mathbf{n}}_{12}$ points into region 1 from region 2. Since the magnetostatic curl and divergence equations are independent, so are the boundary conditions (3.150) and (3.151). We can also write (3.151) in terms of equivalent sources by (3.118):

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{H}_1 - \mathbf{H}_2) = \rho_{Ms1} + \rho_{Ms2}, \tag{3.152}$$

where $\rho_{Ms} = \hat{\mathbf{n}} \cdot \mathbf{M}$ is called the *equivalent magnetization surface charge density.* Here $\hat{\mathbf{n}}$ points outward from the material body.

For a linear, isotropic material described by $\mathbf{B} = \mu\mathbf{H}$, equation (3.150) becomes

$$\hat{\mathbf{n}}_{12} \times \left(\frac{\mathbf{B}_1}{\mu_1} - \frac{\mathbf{B}_2}{\mu_2}\right) = \mathbf{J}_s.$$

With (3.118) we can also write (3.150) as

$$\hat{\mathbf{n}}_{12} \times (\mathbf{B}_1 - \mathbf{B}_2) = \mu_0 (\mathbf{J}_s + \mathbf{J}_{Ms1} + \mathbf{J}_{Ms2}) \tag{3.153}$$

where $\mathbf{J}_{Ms} = -\hat{\mathbf{n}} \times \mathbf{M}$ is the equivalent magnetization surface current density.

We may also write the boundary conditions in terms of the scalar or vector potential. Using $\mathbf{H} = -\nabla\Phi_m$, we can write (3.150) as

$$\Phi_{m1}(\mathbf{r}) = \Phi_{m2}(\mathbf{r}) \tag{3.154}$$

provided that the surface current $\mathbf{J}_s = 0$. As was the case with (3.36), the possibility of an additive constant here is generally ignored. To write (3.151) in terms of Φ_m we first note that $\mathbf{B}/\mu_0 - \mathbf{M} = -\nabla\Phi_m$; substitution into (3.151) gives

$$\frac{\partial \Phi_{m1}}{\partial n} - \frac{\partial \Phi_{m2}}{\partial n} = -\rho_{Ms1} - \rho_{Ms2} \tag{3.155}$$

where the normal derivative is taken in the direction of $\hat{\mathbf{n}}_{12}$. For a linear isotropic material where $\mathbf{B} = \mu\mathbf{H}$ we have

$$\mu_1 \frac{\partial \Phi_{m1}}{\partial n} = \mu_2 \frac{\partial \Phi_{m2}}{\partial n}. \tag{3.156}$$

Note that (3.154) and (3.156) are independent.

Boundary conditions on $\mathbf{A}$ may be derived using the approach of § 2.8.2. Consider Figure 2.6. Here the surface may carry either an electric surface current $\mathbf{J}_s$ or an equivalent magnetization current $\mathbf{J}_{Ms}$, and thus may be a surface of discontinuity between differing magnetic media. If we integrate $\nabla \times \mathbf{A}$ over the volume regions V_1 and V_2 and add the results we find that

$$\int_{V_1} \nabla \times \mathbf{A}\, dV + \int_{V_2} \nabla \times \mathbf{A}\, dV = \int_{V_1+V_2} \mathbf{B}\, dV.$$

By the curl theorem

$$\int_{S_1+S_2} \hat{\mathbf{n}} \times \mathbf{A}\, dS + \int_{S_{10}} -\hat{\mathbf{n}}_{10} \times \mathbf{A}_1\, dS + \int_{S_{20}} -\hat{\mathbf{n}}_{20} \times \mathbf{A}_2\, dS = \int_{V_1+V_2} \mathbf{B}\, dV$$

where $\mathbf{A}_1$ is the field on the surface S_{10} and $\mathbf{A}_2$ is the field on S_{20}. As $\delta \to 0$ the surfaces S_1 and S_2 combine to give S. Also S_{10} and S_{20} coincide, as do the normals $\hat{\mathbf{n}}_{10} = -\hat{\mathbf{n}}_{20} = \hat{\mathbf{n}}_{12}$. Thus

$$\int_S (\hat{\mathbf{n}} \times \mathbf{A})\, dS - \int_V \mathbf{B}\, dV = \int_{S_{10}} \hat{\mathbf{n}}_{12} \times (\mathbf{A}_1 - \mathbf{A}_2)\, dS. \tag{3.157}$$

Now let us integrate over the entire volume region V including the surface of discontinuity. This gives

$$\int_S (\hat{\mathbf{n}} \times \mathbf{A})\, dS - \int_V \mathbf{B}\, dV = 0,$$

and for agreement with (3.157) we must have

$$\hat{\mathbf{n}}_{12} \times (\mathbf{A}_1 - \mathbf{A}_2) = 0. \tag{3.158}$$

A similar development shows that

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{A}_1 - \mathbf{A}_2) = 0. \tag{3.159}$$

Therefore $\mathbf{A}$ is continuous across a surface carrying electric or magnetization current.

3.3.4 Uniqueness of the magnetostatic field

Because the uniqueness conditions established for the dynamic field do not apply to magnetostatics, we begin with the magnetostatic field equations. Consider a region of space V bounded by a surface S. There may be source currents and magnetic materials both inside and outside V. Assume $(\mathbf{B}_1, \mathbf{H}_1)$ and $(\mathbf{B}_2, \mathbf{H}_2)$ are solutions to the magnetostatic field equations with source $\mathbf{J}$. We seek conditions under which $\mathbf{B}_1 = \mathbf{B}_2$ and $\mathbf{H}_1 = \mathbf{H}_2$.

The difference field $\mathbf{H}_0 = \mathbf{H}_2 - \mathbf{H}_1$ obeys $\nabla \times \mathbf{H}_0 = 0$. Using (B.44) we examine the quantity

$$\nabla \cdot (\mathbf{A}_0 \times \mathbf{H}_0) = \mathbf{H}_0 \cdot (\nabla \times \mathbf{A}_0) - \mathbf{A}_0 \cdot (\nabla \times \mathbf{H}_0) = \mathbf{H}_0 \cdot (\nabla \times \mathbf{A}_0)$$

where $\mathbf{A}_0$ is defined by $\mathbf{B}_0 = \mathbf{B}_2 - \mathbf{B}_1 = \nabla \times \mathbf{A}_0 = \nabla \times (\mathbf{A}_2 - \mathbf{A}_1)$. Integrating over V we obtain

$$\oint_S (\mathbf{A}_0 \times \mathbf{H}_0) \cdot \mathbf{dS} = \int_V \mathbf{H}_0 \cdot (\nabla \times \mathbf{A}_0)\, dV = \int_V \mathbf{H}_0 \cdot \mathbf{B}_0\, dV.$$

Then, since $(\mathbf{A}_0 \times \mathbf{H}_0) \cdot \hat{\mathbf{n}} = -\mathbf{A}_0 \cdot (\hat{\mathbf{n}} \times \mathbf{H}_0)$, we have

$$-\oint_S \mathbf{A}_0 \cdot (\hat{\mathbf{n}} \times \mathbf{H}_0) dS = \int_V \mathbf{H}_0 \cdot \mathbf{B}_0\, dV. \tag{3.160}$$

If $\mathbf{A}_0 = 0$ or $\hat{\mathbf{n}} \times \mathbf{H}_0 = 0$ everywhere on S, or $\mathbf{A}_0 = 0$ on part of S and $\hat{\mathbf{n}} \times \mathbf{H}_0 = 0$ on the remainder, then

$$\int_V \mathbf{H}_0 \cdot \mathbf{B}_0\, dS = 0. \tag{3.161}$$

So $\mathbf{H}_0 = 0$ or $\mathbf{B}_0 = 0$ by arbitrariness of V. Assuming $\mathbf{H}$ and $\mathbf{B}$ are linked by the constitutive relations, we have $\mathbf{H}_1 = \mathbf{H}_2$ and $\mathbf{B}_1 = \mathbf{B}_2$. The fields within V are unique provided that $\mathbf{A}$, the tangential component of $\mathbf{H}$, or some combination of the two, is specified over the bounding surface S.

One other condition will cause the left-hand side of (3.160) to vanish. If S recedes to infinity then, provided that the potential functions vanish sufficiently fast, the condition (3.161) still holds and uniqueness is guaranteed. Equation (3.135) shows that $\mathbf{A} \sim 1/r$ as $\mathbf{r} \to \infty$, hence $\mathbf{B}, \mathbf{H} \sim 1/r^2$. So uniqueness is ensured by the specification of $\mathbf{J}$ in unbounded space.

3.3.5 Integral solution for the vector potential

We have used the scalar Green's theorem to find a solution for the electrostatic potential within a region V in terms of the source charge in V and the values of the potential and its normal derivative on the boundary surface S. Analogously, we may find $\mathbf{A}$ within V in terms of the source current in V and the values of $\mathbf{A}$ and its derivatives on S. The vector relationship between $\mathbf{B}$ and $\mathbf{A}$ complicates the derivation somewhat, requiring Green's second identity for vector fields.

Let $\mathbf{P}$ and $\mathbf{Q}$ be continuous with continuous first and second derivatives throughout V and on S. The divergence theorem shows that

$$\int_V \nabla \cdot [\mathbf{P} \times (\nabla \times \mathbf{Q})]\, dV = \int_S [\mathbf{P} \times (\nabla \times \mathbf{Q})] \cdot \mathbf{dS}.$$

By virtue of (B.44) we have

$$\int_V [(\nabla \times \mathbf{Q}) \cdot (\nabla \times \mathbf{P}) - \mathbf{P} \cdot (\nabla \times \{\nabla \times \mathbf{Q}\})] \, dV = \int_S [\mathbf{P} \times (\nabla \times \mathbf{Q})] \cdot \mathbf{dS}.$$

We now interchange **P** and **Q** and subtract the result from the above, obtaining

$$\int_V [\mathbf{Q} \cdot (\nabla \times \{\nabla \times \mathbf{P}\}) - \mathbf{P} \cdot (\nabla \times \{\nabla \times \mathbf{Q}\})] \, dV = \int_S [\mathbf{P} \times (\nabla \times \mathbf{Q}) - \mathbf{Q} \times (\nabla \times \mathbf{P})] \cdot \mathbf{dS}. \tag{3.162}$$

Note that $\hat{\mathbf{n}}$ points outward from V. This is *Green's second identity for vector fields.*

Now assume that V contains a magnetic material of uniform permeability μ and set

$$\mathbf{P} = \mathbf{A}(\mathbf{r}'), \qquad \mathbf{Q} = \frac{\mathbf{c}}{R},$$

in (3.162) written in terms of primed coordinates. Here **c** is a constant vector, nonzero but otherwise arbitrary. We first examine the volume integral terms. Note that

$$\nabla' \times (\nabla' \times \mathbf{Q}) = \nabla' \times \left(\nabla' \times \frac{\mathbf{c}}{R}\right) = -\nabla'^2 \left(\frac{\mathbf{c}}{R}\right) + \nabla' \left[\nabla' \cdot \left(\frac{\mathbf{c}}{R}\right)\right].$$

By (B.162) and (3.58) we have

$$\nabla'^2 \left(\frac{\mathbf{c}}{R}\right) = \frac{1}{R}\nabla'^2 \mathbf{c} + \mathbf{c}\nabla'^2 \left(\frac{1}{R}\right) + 2\left(\nabla' \frac{1}{R} \cdot \nabla'\right)\mathbf{c} = \mathbf{c}\nabla'^2 \left(\frac{1}{R}\right) = -\mathbf{c}4\pi\delta(\mathbf{r} - \mathbf{r}'),$$

hence

$$\mathbf{P} \cdot [\nabla' \times (\nabla' \times \mathbf{Q})] = 4\pi \mathbf{c} \cdot \mathbf{A}\delta(\mathbf{r} - \mathbf{r}') + \mathbf{A} \cdot \nabla' \left[\nabla' \cdot \left(\frac{\mathbf{c}}{R}\right)\right].$$

Since $\nabla \cdot \mathbf{A} = 0$ the second term on the right-hand side can be rewritten using (B.42):

$$\nabla' \cdot (\psi \mathbf{A}) = \mathbf{A} \cdot (\nabla' \psi) + \psi \nabla' \cdot \mathbf{A} = \mathbf{A} \cdot (\nabla' \psi).$$

Thus

$$\mathbf{P} \cdot [\nabla' \times (\nabla' \times \mathbf{Q})] = 4\pi \mathbf{c} \cdot \mathbf{A}\delta(\mathbf{r} - \mathbf{r}') + \nabla' \cdot \left[\mathbf{A}\left\{\mathbf{c} \cdot \nabla' \left(\frac{1}{R}\right)\right\}\right],$$

where we have again used (B.42). The other volume integral term can be found by substituting from (3.129):

$$\mathbf{Q} \cdot [\nabla' \times (\nabla' \times \mathbf{P})] = \mu \frac{1}{R} \mathbf{c} \cdot \mathbf{J}(\mathbf{r}').$$

Next we investigate the surface integral terms. Consider

$$\begin{aligned}
\hat{\mathbf{n}}' \cdot \left[\mathbf{P} \times (\nabla' \times \mathbf{Q})\right] &= \hat{\mathbf{n}}' \cdot \left\{\mathbf{A} \times \left[\nabla' \times \left(\frac{\mathbf{c}}{R}\right)\right]\right\} \\
&= \hat{\mathbf{n}}' \cdot \left\{\mathbf{A} \times \left[\frac{1}{R}\nabla' \times \mathbf{c} - \mathbf{c} \times \nabla' \left(\frac{1}{R}\right)\right]\right\} \\
&= -\hat{\mathbf{n}}' \cdot \left\{\mathbf{A} \times \left[\mathbf{c} \times \nabla' \left(\frac{1}{R}\right)\right]\right\}.
\end{aligned}$$

This can be put in slightly different form by the use of (B.8). Note that

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= \mathbf{A} \cdot [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] \\ &= (\mathbf{C} \times \mathbf{D}) \cdot (\mathbf{A} \times \mathbf{B}) \\ &= \mathbf{C} \cdot [\mathbf{D} \times (\mathbf{A} \times \mathbf{B})],\end{aligned}$$

hence

$$\hat{\mathbf{n}}' \cdot \left[\mathbf{P} \times (\nabla' \times \mathbf{Q})\right] = -\mathbf{c} \cdot \left[\nabla' \left(\frac{1}{R}\right) \times (\hat{\mathbf{n}}' \times \mathbf{A})\right].$$

The other surface term is given by

$$\hat{\mathbf{n}}' \cdot [\mathbf{Q} \times (\nabla' \times \mathbf{P})] = \hat{\mathbf{n}}' \cdot \left[\frac{\mathbf{c}}{R} \times (\nabla' \times \mathbf{A})\right] = \hat{\mathbf{n}}' \cdot \left(\frac{\mathbf{c}}{R} \times \mathbf{B}\right) = -\frac{\mathbf{c}}{R} \cdot (\hat{\mathbf{n}}' \times \mathbf{B}).$$

We can now substitute each of the terms into (3.162) and obtain

$$\begin{aligned}\mu\mathbf{c} \cdot \int_V \frac{\mathbf{J}(\mathbf{r}')}{R}\, dV' - 4\pi\mathbf{c} \cdot \int_V \mathbf{A}(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')\, dV' - \mathbf{c} \cdot \oint_S [\hat{\mathbf{n}}' \cdot \mathbf{A}(\mathbf{r}')]\nabla' \left(\frac{1}{R}\right) dS' \\ = -\mathbf{c} \cdot \oint_S \nabla' \left(\frac{1}{R}\right) \times [\hat{\mathbf{n}}' \times \mathbf{A}(\mathbf{r}')]\, dS' + \mathbf{c} \cdot \oint_S \frac{1}{R}\hat{\mathbf{n}}' \times \mathbf{B}(\mathbf{r}')\, dS'.\end{aligned}$$

Since $\mathbf{c}$ is arbitrary we can remove the dot products to obtain a vector equation. Then

$$\begin{aligned}\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{R}\, dV' - \frac{1}{4\pi} \oint_S \Bigg\{ [\hat{\mathbf{n}}' \times \mathbf{A}(\mathbf{r}')] \times \nabla' \left(\frac{1}{R}\right) + \\ + \frac{1}{R}\hat{\mathbf{n}}' \times \mathbf{B}(\mathbf{r}') + [\hat{\mathbf{n}}' \cdot \mathbf{A}(\mathbf{r}')]\nabla' \left(\frac{1}{R}\right) \Bigg\}\, dS'.\end{aligned} \tag{3.163}$$

We have expressed $\mathbf{A}$ in a closed region in terms of the sources within the region and the values of $\mathbf{A}$ and $\mathbf{B}$ on the surface. While uniqueness requires specification of *either* $\mathbf{A}$ or $\hat{\mathbf{n}} \times \mathbf{B}$ on S, the expression (3.163) includes *both* quantities. This is similar to (3.56) for electrostatic fields, which required both the scalar potential and its normal derivative.

The reader may be troubled by the fact that we require $\mathbf{P}$ and $\mathbf{Q}$ to be somewhat well behaved, then proceed to involve the singular function $\mathbf{c}/R$ and integrate over the singularity. We choose this approach to simplify the presentation; a more rigorous approach which excludes the singular point with a small sphere also gives (3.163). This approach was used in § 3.2.4 to establish (3.58). The interested reader should see Stratton [187] for details on the application of this technique to obtain (3.163).

It is interesting to note that as $S \to \infty$ the surface integral vanishes since $\mathbf{A} \sim 1/r$ and $\mathbf{B} \sim 1/r^2$, and we recover (3.135). Moreover, (3.163) returns the null result when evaluated at points outside S (see Stratton [187]). We shall see this again when studying the integral solutions for electrodynamic fields in § 6.1.3.

Finally, with

$$\mathbf{Q} = \nabla' \left(\frac{1}{R}\right) \times \mathbf{c}$$

we can find an integral expression for $\mathbf{B}$ within an enclosed region, representing a generalization of the Biot–Savart law (Problem 3.20). However, this case will be covered in the more general development of § 6.1.1.

The Biot–Savart law. We can obtain an expression for $\mathbf{B}$ in unbounded space by performing the curl operation directly on the vector potential:

$$\mathbf{B}(\mathbf{r}) = \nabla \times \frac{\mu}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV' = \frac{\mu}{4\pi}\int_V \nabla \times \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV'.$$

Using (B.43) and $\nabla \times \mathbf{J}(\mathbf{r}') = 0$, we have

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu}{4\pi}\int_V \mathbf{J} \times \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,dV'.$$

The *Biot–Savart law*

$$\mathbf{B}(\mathbf{r}) = \frac{\mu}{4\pi}\int_V \mathbf{J}(\mathbf{r}') \times \frac{\hat{\mathbf{R}}}{R^2}\,dV' \tag{3.164}$$

follows from (3.57).

For the case of a line current we can replace $\mathbf{J}\,dV'$ by $I\mathbf{dl}'$ and obtain

$$\mathbf{B}(\mathbf{r}) = I\frac{\mu}{4\pi}\int_\Gamma \mathbf{dl}' \times \frac{\hat{\mathbf{R}}}{R^2}. \tag{3.165}$$

For an infinitely long line current on the z-axis we have

$$\mathbf{B}(\mathbf{r}) = I\frac{\mu}{4\pi}\int_{-\infty}^{\infty} \hat{\mathbf{z}} \times \frac{\hat{\mathbf{z}}(z-z') + \hat{\boldsymbol{\rho}}\rho}{[(z-z')^2+\rho^2]^{3/2}}\,dz' = \hat{\boldsymbol{\phi}}\frac{\mu I}{2\pi\rho}. \tag{3.166}$$

This same result follows from taking $\nabla \times \mathbf{A}$ after direct computation of $\mathbf{A}$, or from direct application of the large-scale form of Ampere's law.

3.3.6 Force and energy

Ampere force on a system of currents. If a steady current $\mathbf{J}(\mathbf{r})$ occupying a region V is exposed to a magnetic field, the force on the moving charge is given by the Lorentz force law

$$d\mathbf{F}(\mathbf{r}) = \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}). \tag{3.167}$$

This can be integrated to give the total force on the current distribution:

$$\mathbf{F} = \int_V \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\,dV. \tag{3.168}$$

It is apparent that the charge flow comprising a steady current must be constrained in some way, or the Lorentz force will accelerate the charge and destroy the steady nature of the current. This constraint is often provided by a conducting wire.

As an example, consider an infinitely long wire of circular cross-section centered on the z-axis in free space. If the wire carries a total current I uniformly distributed over the cross-section, then within the wire $\mathbf{J} = \hat{\mathbf{z}}I/(\pi a^2)$ where a is the wire radius. The resulting field can be found through direct integration using (3.164), or by the use of symmetry and either (3.118) or (3.120). Since $\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}B_\phi(\rho)$, equation (3.118) shows that

$$\int_0^{2\pi} B_\phi(\rho)\rho\,d\phi = \begin{cases} \frac{\mu_0 I}{a^2}\rho^2, & \rho \le a \\ \mu_0 I, & \rho \ge a. \end{cases}$$

Thus

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \hat{\boldsymbol{\phi}}\, \mu_0 I \rho / 2\pi a^2, & \rho \le a, \\ \hat{\boldsymbol{\phi}}\, \mu_0 I / 2\pi \rho, & \rho \ge a. \end{cases} \tag{3.169}$$

The force density within the wire,

$$d\mathbf{F} = \mathbf{J} \times \mathbf{B} = -\hat{\boldsymbol{\rho}} \frac{\mu_0 I^2 \rho}{2\pi^2 a^4},$$

is directed inward and tends to compress the wire. Integration over the wire volume gives $\mathbf{F} = 0$ because

$$\int_0^{2\pi} \hat{\boldsymbol{\rho}}\, d\phi = 0;$$

however, a section of the wire may experience a net force. For instance, we can compute the force on one half of the wire split down its axis by using $\hat{\boldsymbol{\rho}} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi$ to obtain $F_x = 0$ and

$$F_y = -\frac{\mu_0 I^2}{2\pi^2 a^4} \int dz \int_0^a \rho^2 \, d\rho \int_0^\pi \sin\phi \, d\phi = -\frac{\mu_0 I^2}{3\pi^2 a} \int dz.$$

The force per unit length

$$\frac{\mathbf{F}}{l} = -\hat{\mathbf{y}} \frac{\mu_0 I^2}{3\pi^2 a} \tag{3.170}$$

is directed toward the other half as expected.

If the wire takes the form of a loop carrying current I, then (3.167) becomes

$$d\mathbf{F}(\mathbf{r}) = I\, \mathbf{dl}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) \tag{3.171}$$

and the total force acting is

$$\mathbf{F} = I \oint_\Gamma \mathbf{dl}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}).$$

We can write the force on $\mathbf{J}$ in terms of the current producing $\mathbf{B}$. Assuming this latter current $\mathbf{J}'$ occupies region V', the Biot–Savart law (3.164) yields

$$\mathbf{F} = \frac{\mu}{4\pi} \int_V \mathbf{J}(\mathbf{r}) \times \int_{V'} \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, dV' \, dV. \tag{3.172}$$

This can be specialized to describe the force between line currents. Assume current 1, following a path Γ_1 along the direction $\mathbf{dl}$, carries current I_1, while current 2, following path Γ_2 along the direction $\mathbf{dl}'$, carries current I_2. Then the force on current 1 is

$$\mathbf{F}_1 = I_1 I_2 \frac{\mu}{4\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} \mathbf{dl} \times \left(\mathbf{dl}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right).$$

This equation, known as *Ampere's force law*, can be written in a better form for computational purposes. We use (B.7) and $\nabla(1/R)$ from (3.57):

$$\mathbf{F}_1 = I_1 I_2 \frac{\mu}{4\pi} \oint_{\Gamma_2} \mathbf{dl}' \oint_{\Gamma_1} \mathbf{dl} \cdot \nabla' \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) - I_1 I_2 \frac{\mu}{4\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} (\mathbf{dl} \cdot \mathbf{dl}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{3.173}$$

The first term involves an integral of a perfect differential about a closed path, producing a null result. Thus

$$\mathbf{F}_1 = -I_1 I_2 \frac{\mu}{4\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} (\mathbf{dl} \cdot \mathbf{dl}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{3.174}$$

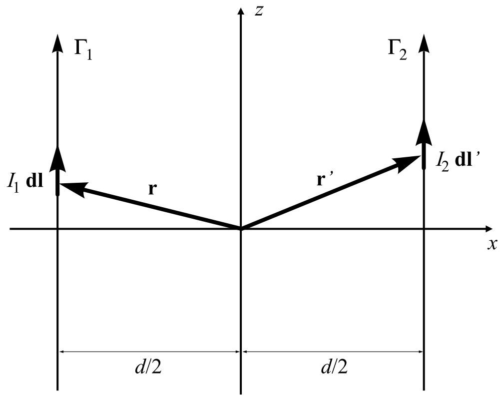

Figure 3.20: Parallel, current carrying wires.

As a simple example, consider parallel wires separated by a distance d (Figure 3.20). In this case

$$\mathbf{F}_1 = -I_1 I_2 \frac{\mu}{4\pi} \int \left[\int_{-\infty}^{\infty} \frac{-d\hat{\mathbf{x}} + (z-z')\hat{\mathbf{z}}}{[d^2 + (z-z')^2]^{3/2}} \, dz' \right] dz = I_1 I_2 \frac{\mu}{2\pi d} \hat{\mathbf{x}} \int dz$$

so the force per unit length is

$$\frac{\mathbf{F}_1}{l} = \hat{\mathbf{x}} I_1 I_2 \frac{\mu}{2\pi d}. \tag{3.175}$$

The force is attractive if $I_1 I_2 \geq 0$ (i.e., if the currents flow in the same direction).

Maxwell's stress tensor. The magnetostatic version of the stress tensor can be obtained from (2.288) by setting $\mathbf{E} = \mathbf{D} = 0$:

$$\bar{\mathbf{T}}_m = \frac{1}{2}(\mathbf{B} \cdot \mathbf{H})\bar{\mathbf{I}} - \mathbf{BH}. \tag{3.176}$$

The total magnetic force on the current in a region V surrounded by surface S is given by

$$\mathbf{F}_m = -\oint_S \bar{\mathbf{T}}_m \cdot \mathbf{dS} = \int_V \mathbf{f}_m \, dV$$

where $\mathbf{f}_m = \mathbf{J} \times \mathbf{B}$ is the magnetic force volume density.

Let us compute the force between two parallel wires carrying identical currents in free space (let $I_1 = I_2 = I$ in Figure 3.20) and compare the result with (3.175). The force on the wire at $x = -d/2$ can be computed by integrating $\bar{\mathbf{T}}_m \cdot \hat{\mathbf{n}}$ over the yz-plane with $\hat{\mathbf{n}} = \hat{\mathbf{x}}$. Using (3.166) we see that in this plane the total magnetic field is

$$\mathbf{B} = -\hat{\mathbf{x}} \mu_0 \frac{I}{\pi} \frac{y}{y^2 + d^2/4}.$$

Therefore

$$\bar{\mathbf{T}}_m \cdot \hat{\mathbf{n}} = \frac{1}{2} B_x \frac{B_x}{\mu_0} \hat{\mathbf{x}} - \hat{\mathbf{x}} B_x \frac{B_x}{\mu_0} = -\mu_0 \frac{I^2}{2\pi^2} \frac{y^2}{[y^2 + d^2/4]^2} \hat{\mathbf{x}}$$

and by integration

$$\mathbf{F}_1 = \mu_0 \frac{I^2}{2\pi^2} \hat{\mathbf{x}} \int dz \int_{-\infty}^{\infty} \frac{y^2}{[y^2 + d^2/4]^2} \, dy = I^2 \frac{\mu_0}{2\pi d} \hat{\mathbf{x}} \int dz.$$

The resulting force per unit length agrees with (3.175) when $I_1 = I_2 = I$.

Torque in a magnetostatic field. The torque exerted on a current-carrying conductor immersed in a magnetic field plays an important role in many engineering applications. If a rigid body is exposed to a force field of volume density $d\mathbf{F}(\mathbf{r})$, the torque on that body about a certain origin is given by

$$\mathbf{T} = \int_V \mathbf{r} \times d\mathbf{F} \, dV \tag{3.177}$$

where integration is performed over the body and $\mathbf{r}$ extends from the origin of torque. If the force arises from the interaction of a current with a magnetostatic field, then $d\mathbf{F} = \mathbf{J} \times \mathbf{B}$ and

$$\mathbf{T} = \int_V \mathbf{r} \times (\mathbf{J} \times \mathbf{B}) \, dV. \tag{3.178}$$

For a line current we can replace $\mathbf{J} \, dV$ with $I \mathbf{dl}$ to obtain

$$\mathbf{T} = I \int_\Gamma \mathbf{r} \times (\mathbf{dl} \times \mathbf{B}).$$

If $\mathbf{B}$ is uniform then by (B.7) we have

$$\mathbf{T} = \int_V [\mathbf{J}(\mathbf{r} \cdot \mathbf{B}) - \mathbf{B}(\mathbf{r} \cdot \mathbf{J})] \, dV.$$

The second term can be written as

$$\int_V \mathbf{B}(\mathbf{r} \cdot \mathbf{J}) \, dV = \mathbf{B} \sum_{i=1}^{3} \int_V x_i J_i \, dV = 0$$

where $(x_1, x_2, x_3) = (x, y, z)$, and where we have employed (3.27). Thus

$$\mathbf{T} = \int_V \mathbf{J}(\mathbf{r} \cdot \mathbf{B}) \, dV = \sum_{j=1}^{3} \hat{\mathbf{x}}_j \int_V J_j \sum_{i=1}^{3} x_i B_i \, dV = \sum_{i=1}^{3} B_i \sum_{j=1}^{3} \hat{\mathbf{x}}_j \int_V J_j x_i \, dV.$$

We can replace the integral using (3.144) to get

$$\mathbf{T} = \frac{1}{2} \int_V \sum_{j=1}^{3} \hat{\mathbf{x}}_j \sum_{i=1}^{3} B_i [x_i J_j - x_j J_i] \, dV = -\frac{1}{2} \int_V \mathbf{B} \times (\mathbf{r} \times \mathbf{J}) \, dV.$$

Since $\mathbf{B}$ is uniform we have, by (3.145),

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \tag{3.179}$$

where $\mathbf{m}$ is the dipole moment. For a planar loop we can use (3.147) to obtain

$$\mathbf{T} = I A \hat{\mathbf{n}} \times \mathbf{B}.$$

Joule's law. In § 2.9.5 we showed that when a moving charge interacts with an electric field in a volume region V, energy is transferred between the field and the charge. If the source of that energy is outside V, the energy is carried into V as an energy flux over the boundary surface S. The energy balance described by Poynting's theorem (3.299) also holds for static fields supported by steady currents: we must simply recognize that we have no time-rate of change of stored energy. Thus

$$-\int_V \mathbf{J} \cdot \mathbf{E}\, dV = \oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{dS}. \tag{3.180}$$

The term

$$P = -\int_V \mathbf{J} \cdot \mathbf{E}\, dV \tag{3.181}$$

describes the rate at which energy is supplied to the fields by the current within V; we have $P > 0$ if there are sources within V that result in energy transferred to the fields, and $P < 0$ if there is energy transferred to the currents. The latter case occurs when there are conducting materials in V. Within these conductors

$$P = -\int_V \sigma \mathbf{E} \cdot \mathbf{E}\, dV. \tag{3.182}$$

Here $P < 0$; energy is transferred from the fields to the currents, and from the currents into heat (i.e., into lattice vibrations via collisions). Equation (3.182) is called *Joule's law*, and the transfer of energy from the fields into heat is *Joule heating*. Joule's law is the power relationship for a conducting material.

An important example involves a straight section of conducting wire having circular cross-section. Assume a total current I is uniformly distributed over the cross-section of the wire, and that the wire is centered on the z-axis and extends between the planes $z = 0, L$. Let the potential difference between the ends be V. Using (3.169) we see that at the surface of the wire

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{I}{2\pi a}, \qquad \mathbf{E} = \hat{\mathbf{z}} \frac{V}{L}.$$

The corresponding Poynting flux $\mathbf{E} \times \mathbf{H}$ is $-\hat{\boldsymbol{\rho}}$-directed, implying that energy flows into wire volume through the curved side surface. We can verify (3.180):

$$-\int_V \mathbf{J} \cdot \mathbf{E}\, dV = \int_0^L \int_0^{2\pi} \int_0^a \hat{\mathbf{z}} \frac{I}{\pi a^2} \cdot \hat{\mathbf{z}} \frac{V}{L} \rho\, d\rho\, d\phi\, dz = -IV,$$

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{dS} = \int_0^{2\pi} \int_0^L \left(-\hat{\boldsymbol{\rho}} \frac{IV}{2\pi a L} \right) \cdot \hat{\boldsymbol{\rho}} a\, d\phi\, dz = -IV.$$

Stored magnetic energy. We have shown that the energy stored in a static charge distribution may be regarded as the "assembly energy" required to bring charges from infinity against the Coulomb force. By proceeding very slowly with this assembly, we are able to avoid any complications resulting from the motion of the charges.

Similarly, we may equate the energy stored in a steady current distribution to the energy required for its assembly from current filaments[6] brought in from infinity. However, the calculation of assembly energy is more complicated in this case: moving a current

[6]Recall that a flux tube of a vector field is bounded by streamlines of the field. A current filament is a flux tube of current having vanishingly small, but nonzero, cross-section.

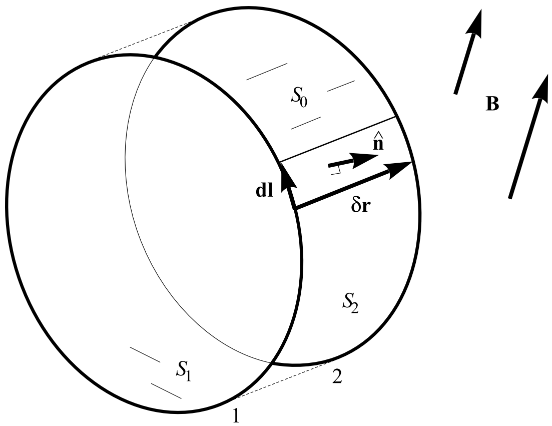

Figure 3.21: Calculation of work to move a filamentary loop in an applied magnetic field.

filament into the vicinity of existing filaments changes the total magnetic flux passing through the existing loops, regardless of how slowly we assemble the filaments. As described by Faraday's law, this change in flux must be associated with an induced emf, which will tend to change the current flowing in the filament (and any existing filaments) unless energy is expended to keep the current constant (by the application of a battery emf in the opposite direction). We therefore regard the assembly energy as consisting of two parts: (1) the energy required to bring a filament with *constant* current from infinity against the Ampere force, and (2) the energy required to keep the current in this filament, and any existing filaments, constant. We ignore the energy required to keep the steady current flowing through an isolated loop (i.e., the energy needed to overcome Joule losses).

We begin by computing the amount of energy required to bring a filament with current I from infinity to a given position within an applied magnetostatic field $\mathbf{B}(\mathbf{r})$. In this first step we assume that the field is supported by localized sources, hence vanishes at infinity, and that it will not be altered by the motion of the filament. The force on each small segment of the filament is given by Ampere's force law (3.171), and the total force is found by integration. Suppose an external agent displaces the filament incrementally from a starting position 1 to an ending position 2 along a vector $\delta\mathbf{r}$ as shown in Figure 3.21. The work required is

$$\delta W = -(I\,\mathbf{dl} \times \mathbf{B}) \cdot \delta\mathbf{r} = (I\,\mathbf{dl} \times \delta\mathbf{r}) \cdot \mathbf{B}$$

for each segment of the wire. Figure 3.21 shows that $\mathbf{dl} \times \delta\mathbf{r}$ describes a small patch of surface area between the starting and ending positions of the filament, hence $-(\mathbf{dl}\times\delta\mathbf{r})\cdot\mathbf{B}$ is the *outward* flux of $\mathbf{B}$ through the patch. Integrating over all segments comprising the filament, we obtain

$$\Delta W = I \oint_\Gamma (\mathbf{dl} \times \delta\mathbf{r}) \cdot \mathbf{B} = -I \int_{S_0} \mathbf{B} \cdot \mathbf{dS}$$

for the total work required to displace the entire filament through $\delta\mathbf{r}$; here the surface S_0 is described by the superposition of all patches. If S_1 and S_2 are the surfaces bounded by the filament in its initial and final positions, respectively, then S_1, S_2, and S_0 taken together form a closed surface. The outward flux of $\mathbf{B}$ through this surface is

$$\oint_{S_0+S_1+S_2} \mathbf{B} \cdot \mathbf{dS} = 0$$

so that

$$\Delta W = -I \int_{S_0} \mathbf{B} \cdot \mathbf{dS} = I \int_{S_1+S_2} \mathbf{B} \cdot \mathbf{dS}$$

where $\hat{\mathbf{n}}$ is outward from the closed surface. Finally, let $\Psi_{1,2}$ be the flux of $\mathbf{B}$ through $S_{1,2}$ in the direction determined by $\mathbf{dl}$ and the right-hand rule. Then

$$\Delta W = -I(\Psi_2 - \Psi_1) = -I\Delta\Psi. \tag{3.183}$$

Now suppose that the initial position of the filament is at infinity. We bring the filament into a final position within the field $\mathbf{B}$ through a succession of small displacements, each requiring work (3.183). By superposition over all displacements, the total work is $W = -I(\Psi - \Psi_\infty)$ where Ψ_∞ and Ψ are the fluxes through the filament in its initial and final positions, respectively. However, since the source of the field is localized, we know that $\mathbf{B}$ is zero at infinity. Therefore $\Psi_\infty = 0$ and

$$W = -I\Psi = -I \int_S \mathbf{B} \cdot \hat{\mathbf{n}}\, dS \tag{3.184}$$

where $\hat{\mathbf{n}}$ is determined from $\mathbf{dl}$ in the right-hand sense.

Now let us find the work required to position two current filaments in a field-free region of space, starting with both filaments at infinity. Assume filament 1 carries current I_1 and filament 2 carries current I_2, and that we hold these currents constant as we move the filaments into position. We can think of assembling these filaments in two ways: by placing filament 1 first, or by placing filament 2 first. In either case, placing the first filament requires no work since (3.184) is zero. The work required to place the second filament is $W_1 = -I_1\Psi_1$ if filament 2 is placed first, where Ψ_1 is the flux passing through filament 1 in its final position, caused by the presence of filament 2. If filament 1 is placed first, the work required is $W_2 = -I_2\Psi_2$. Since the work cannot depend on which loop is placed first, we have $W_1 = W_2 = W$ where we can use either $W = -I_1\Psi_1$ or $W = -I_2\Psi_2$. It is even more convenient, as we shall see, to average these values and use

$$W = -\frac{1}{2}\left(I_1\Psi_1 + I_2\Psi_2\right). \tag{3.185}$$

We must determine the energy required to keep the currents constant as we move the filaments into position. When moving the first filament into place there is no induced emf, since no applied field is yet present. However, when moving the second filament into place we will change the flux linked by *both* the first and second loops. This change of flux will induce an emf in each of the loops, and this will change the current. To keep the current constant we must supply an opposing emf. Let dW_{emf}/dt be the rate of work required to keep the current constant. Then by (3.153) and (3.181) we have

$$\frac{dW_{\text{emf}}}{dt} = -\int_V \mathbf{J} \cdot \mathbf{E}\, dV = -I \int \mathbf{E} \cdot \mathbf{dl} = -I\frac{d\Psi}{dt}.$$

Integrating, we find the total work ΔW required to keep the current constant in either loop as the flux through the loop is changed by an amount $\Delta\Psi$:

$$\Delta W_{emf} = I\Delta\Psi.$$

So the total work required to keep I_1 constant as the loops are moved from infinity (where the flux is zero) to their final positions is $I_1\Psi_1$. Similarly, a total work $I_2\Psi_2$ is required to keep I_2 constant during the same process. Adding these to (3.185), the work required to position the loops, we obtain the complete assembly energy

$$W = \frac{1}{2}\left(I_1\Psi_1 + I_2\Psi_2\right)$$

for two filaments. The extension to N filaments is

$$W_m = \frac{1}{2}\sum_{n=1}^{N} I_n\Psi_n. \tag{3.186}$$

Consequently, the energy of a single current filament is

$$W_m = \frac{1}{2}I\Psi. \tag{3.187}$$

We may interpret this as the "assembly energy" required to bring the single loop into existence by bringing vanishingly small loops (magnetic dipoles) in from infinity. We may also interpret it as the energy required to establish the current in this single filament against the back emf. That is, if we establish I by slowly increasing the current from zero in N small steps $\Delta I = I/N$, an energy $\Psi_n\Delta I$ will be required at each step. Since Ψ_n increases proportionally to I, we have

$$W_m = \sum_{n=1}^{N}\frac{I}{N}\left[(n-1)\frac{\Psi}{N}\right]$$

where Ψ is the flux when the current is fully established. Since $\sum_{n=1}^{N}(n-1) = N(N-1)/2$ we obtain

$$W_m = \frac{1}{2}I\Psi \tag{3.188}$$

as $N \to \infty$.

A volume current $\mathbf{J}$ can be treated as though it were composed of N current filaments. Equations (3.128) and (3.186) give

$$W_m = \frac{1}{2}\sum_{n=1}^{N} I_n \oint_{\Gamma_n}\mathbf{A}\cdot\mathbf{dl}.$$

Since the total current is

$$I = \int_{CS}\mathbf{J}\cdot\mathbf{dS} = \sum_{n=1}^{N} I_n$$

where CS denotes the cross-section of the steady current, we have as $N \to \infty$

$$W_m = \frac{1}{2}\int_V \mathbf{A}\cdot\mathbf{J}\,dV. \tag{3.189}$$

Alternatively, using (3.135), we may write

$$W_m = \frac{1}{2}\int_V\int_V \frac{\mathbf{J}(\mathbf{r})\cdot\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV\,dV'.$$

Note the similarity between (3.189) and (3.86). We now manipulate (3.189) into a form involving only the electromagnetic fields. By Ampere's law

$$W_m = \frac{1}{2}\int_V \mathbf{A}\cdot(\nabla\times\mathbf{H})\,dV.$$

Using (B.44) and the divergence theorem we can write

$$W_m = \frac{1}{2}\oint_S (\mathbf{H}\times\mathbf{A})\cdot\mathbf{dS} + \frac{1}{2}\int_V \mathbf{H}\cdot(\nabla\times\mathbf{A})\,dV.$$

We now let S expand to infinity. This does not change the value of W_m since we do not enclose any more current; however, since $\mathbf{A}\sim\ 1/r$ and $\mathbf{H}\sim 1/r^2$, the surface integral vanishes. Thus, remembering that $\nabla\times\mathbf{A}=\mathbf{B}$, we have

$$W_m = \frac{1}{2}\int_{V_\infty}\mathbf{H}\cdot\mathbf{B}\,dV \tag{3.190}$$

where V_∞ denotes all of space.

Although we do not provide a derivation, (3.190) is also valid within linear materials. For nonlinear materials, the total energy required to build up a magnetic field from $\mathbf{B}_1$ to $\mathbf{B}_2$ is

$$W_m = \frac{1}{2}\int_{V_\infty}\left[\int_{\mathbf{B}_1}^{\mathbf{B}_2}\mathbf{H}\cdot d\mathbf{B}\right]dV. \tag{3.191}$$

This accounts for the work required to drive a ferromagnetic material through its hysteresis loop. Readers interested in a complete derivation of (3.191) should consult Stratton [187].

As an example, consider two thin-walled, coaxial, current-carrying cylinders having radii a, b $(b > a)$. The intervening region is a linear magnetic material having permeability μ. Assume that the inner and outer conductors carry total currents I in the $\pm z$ directions, respectively. From the large-scale form of Ampere's law we find that

$$\mathbf{H} = \begin{cases} 0, & \rho \le a, \\ \hat{\phi}\, I/2\pi\rho, & a \le \rho \le b, \\ 0, & \rho > b, \end{cases} \tag{3.192}$$

hence by (3.190)

$$W_m = \frac{1}{2}\int dz\int_0^{2\pi}\int_a^b \frac{\mu I^2}{(2\pi\rho)^2}\rho\,d\rho\,d\phi,$$

and the stored energy is

$$\frac{W_m}{l} = \mu\frac{I^2}{4\pi}\ln\left(\frac{b}{a}\right) \tag{3.193}$$

per unit length.

Suppose instead that the inner cylinder is solid and that current is spread uniformly throughout. Then the field between the cylinders is still given by (3.192) but within the inner conductor we have

$$\mathbf{H} = \hat{\phi}\frac{I\rho}{2\pi a^2}$$

by (3.169). Thus, to (3.193) we must add the energy

$$\frac{W_{m,\text{inside}}}{l} = \frac{1}{2}\int_0^{2\pi}\int_0^a \frac{\mu_0 I^2\rho^2}{(2\pi a^2)^2}\rho\,d\rho\,d\phi = \frac{\mu_0 I^2}{16\pi}$$

stored within the solid wire. The result is

$$\frac{W_m}{l} = \frac{\mu_0 I^2}{4\pi}\left[\mu_r \ln\left(\frac{b}{a}\right) + \frac{1}{4}\right].$$

3.3.7 Magnetic field of a permanently magnetized body

We now have the tools necessary to compute the magnetic field produced by a permanent magnet (a body with permanent magnetization **M**). As an example, we shall find the field due to a uniformly magnetized sphere in three different ways: by computing the vector potential integral and taking the curl, by computing the scalar potential integral and taking the gradient, and by finding the scalar potential using separation of variables and applying the boundary condition across the surface of the sphere.

Consider a magnetized sphere of radius a, residing in free space and having permanent magnetization

$$\mathbf{M}(\mathbf{r}) = M_0\hat{\mathbf{z}}.$$

The equivalent magnetization current and charge densities are given by

$$\mathbf{J}_M = \nabla \times \mathbf{M} = 0, \tag{3.194}$$

$$\mathbf{J}_{Ms} = -\hat{\mathbf{n}} \times \mathbf{M} = -\hat{\mathbf{r}} \times M_0\hat{\mathbf{z}} = M_0\hat{\boldsymbol{\phi}}\sin\theta, \tag{3.195}$$

and

$$\rho_M = -\nabla \cdot \mathbf{M} = 0, \tag{3.196}$$

$$\rho_{Ms} = \hat{\mathbf{n}} \cdot \mathbf{M} = \hat{\mathbf{r}} \cdot M_0\hat{\mathbf{z}} = M_0\cos\theta. \tag{3.197}$$

The vector potential is produced by the equivalent magnetization surface current. Using (3.137) we find that

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{J}_{Ms}}{|\mathbf{r}-\mathbf{r}'|}\,dS' = \frac{\mu_0}{4\pi}\int_{-\pi}^{\pi}\int_0^{\pi}\frac{M_0\hat{\boldsymbol{\phi}}'\sin\theta'}{|\mathbf{r}-\mathbf{r}'|}\sin\theta'\,d\theta'\,d\phi'.$$

Since $\hat{\boldsymbol{\phi}}' = -\hat{\mathbf{x}}\sin\phi' + \hat{\mathbf{y}}\cos\phi'$, the rectangular components of $\mathbf{A}$ are

$$\left\{\begin{matrix} -A_x \\ A_y \end{matrix}\right\} = \frac{\mu_0}{4\pi}\int_{-\pi}^{\pi}\int_0^{\pi}\frac{M_0 \begin{matrix}\sin\phi' \\ \cos\phi'\end{matrix}\sin\theta'}{|\mathbf{r}-\mathbf{r}'|}a^2\sin\theta'\,d\theta'\,d\phi'. \tag{3.198}$$

The integrals are most easily computed via the spherical harmonic expansion (E.200) for the inverse distance $|\mathbf{r}-\mathbf{r}'|^{-1}$:

$$\left\{\begin{matrix} -A_x \\ A_y \end{matrix}\right\} = \mu_0 M_0 a^2 \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{Y_{nm}(\theta,\phi)}{2n+1}\frac{r_<^n}{r_>^{n+1}}\int_{-\pi}^{\pi}\int_0^{\pi}\begin{matrix}\sin\phi' \\ \cos\phi'\end{matrix}\sin^2\theta' Y_{nm}^*(\theta',\phi')\,d\theta'\,d\phi'.$$

Because the ϕ' variation is $\sin\phi'$ or $\cos\phi'$, all terms in the sum vanish except $n = 1$, $m = \pm 1$. Since

$$Y_{1,-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-j\phi}, \qquad Y_{1,1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{j\phi},$$

we have

$$\begin{Bmatrix} -A_x \\ A_y \end{Bmatrix} = \mu_0 M_0 \frac{a^2}{3}\frac{r_<}{r_>^2}\frac{3}{8\pi}\sin\theta \int_0^{\pi} \sin^3\theta'\, d\theta' \cdot$$
$$\cdot \left[e^{-j\phi} \int_{-\pi}^{\pi} \frac{\sin\phi'}{\cos\phi'} e^{j\phi'}\, d\phi' + e^{j\phi} \int_{-\pi}^{\pi} \frac{\sin\phi'}{\cos\phi'} e^{-j\phi'}\, d\phi' \right].$$

Carrying out the integrals we find that

$$\begin{Bmatrix} -A_x \\ A_y \end{Bmatrix} = \mu_0 M_0 \frac{a^2}{3}\frac{r_<}{r_>^2}\sin\theta \begin{Bmatrix} \sin\phi \\ \cos\phi \end{Bmatrix}$$

or

$$\mathbf{A} = \mu_0 M_0 \frac{a^2}{3}\frac{r_<}{r_>^2}\sin\theta\hat{\boldsymbol{\phi}}.$$

Finally, $\mathbf{B} = \nabla \times \mathbf{A}$ gives

$$\mathbf{B} = \begin{cases} \frac{2\mu_0 M_0}{3}\hat{\mathbf{z}}, & r < a, \\ \frac{\mu_0 M_0 a^3}{3r^3}\left(\hat{\mathbf{r}}\, 2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right), & r > a. \end{cases} \tag{3.199}$$

Hence $\mathbf{B}$ within the sphere is uniform and in the same direction as $\mathbf{M}$, while $\mathbf{B}$ outside the sphere has the form of the magnetic dipole field with moment

$$m = \left(\frac{4}{3}\pi a^3\right) M_0.$$

We can also compute $\mathbf{B}$ by first finding the scalar potential through direct computation of the integral (3.126). Substituting for ρ_{Ms} from (3.197), we have

$$\Phi_m(\mathbf{r}) = \frac{1}{4\pi}\int_S \frac{\rho_{Ms}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, dS' = \frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^{\pi} \frac{M_0\cos\theta'}{|\mathbf{r}-\mathbf{r}'|}\sin\theta'\, d\theta'\, d\phi'.$$

This integral has the form of (3.100) with $f(\theta) = M_0\cos\theta$. Thus, from (3.102),

$$\Phi_m(\mathbf{r}) = M_0 \frac{a^2}{3}\cos\theta \frac{r_<}{r_>^2}. \tag{3.200}$$

The magnetic field $\mathbf{H}$ is then

$$\mathbf{H} = -\nabla\Phi_m = \begin{cases} -\frac{M_0}{3}\hat{\mathbf{z}}, & r < a, \\ \frac{M_0 a^3}{3r^3}\left(\hat{\mathbf{r}}\, 2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right), & r > a. \end{cases}.$$

Inside the sphere $\mathbf{B}$ is given by $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, while outside the sphere it is merely $\mathbf{B} = \mu_0\mathbf{H}$. These observations lead us again to (3.199).

Since the scalar potential obeys Laplace's equation both inside and outside the sphere, as a last approach to the problem we shall write Φ_m in terms of the separation of variables solution discussed in § A.4. We can repeat our earlier arguments for the dielectric sphere in an impressed electric field (§ 3.2.10). Copying equations (3.109) and (3.110), we can write for $r \le a$

$$\Phi_{m1}(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta), \tag{3.201}$$

and for $r \geq a$

$$\Phi_{m2}(r, \theta) = \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta). \tag{3.202}$$

The boundary condition (3.154) at $r = a$ requires that

$$\sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta) = \sum_{n=0}^{\infty} B_n a^{-(n+1)} P_n(\cos\theta);$$

upon application of the orthogonality of the Legendre functions, this becomes

$$A_n a^n = B_n a^{-(n+1)}. \tag{3.203}$$

We can write (3.155) as

$$-\frac{\partial \Phi_{m1}}{\partial r} + \frac{\partial \Phi_{m2}}{\partial r} = -\rho_{Ms}$$

so that at $r = a$

$$-\sum_{n=0}^{\infty} A_n n a^{n-1} P_n(\cos\theta) - \sum_{n=0}^{\infty} B_n(n+1) a^{-(n+2)} P_n(\cos\theta) = -M_0 \cos\theta.$$

After application of orthogonality this becomes

$$A_1 + 2B_1 a^{-3} = M_0, \tag{3.204}$$

$$n a^{n-1} A_n = -(n+1) B_n a^{-(n+2)}, \qquad n \neq 1. \tag{3.205}$$

Solving (3.203) and (3.204) simultaneously for $n = 1$ we find that

$$A_1 = \frac{M_0}{3}, \qquad B_1 = \frac{M_0}{3} a^3.$$

We also see that (3.203) and (3.205) are inconsistent unless $A_n = B_n = 0$, $n \neq 1$. Substituting these results into (3.201) and (3.202), we have

$$\Phi_m = \begin{cases} \frac{M_0}{3} r \cos\theta, & r \leq a, \\ \frac{M_0}{3} \frac{a^3}{r^2} \cos\theta, & r \geq a, \end{cases}$$

which is (3.200).

3.3.8 Bodies immersed in an impressed magnetic field: magnetostatic shielding

A highly permeable enclosure can provide partial shielding from external magnetostatic fields. Consider a spherical shell of highly permeable material (Figure 3.22); assume it is immersed in a uniform impressed field $\mathbf{H}_0 = H_0 \hat{\mathbf{z}}$. We wish to determine the internal field and the factor by which it is reduced from the external applied field. Because there are no sources (the applied field is assumed to be created by sources far removed), we may use magnetic scalar potentials to represent the fields everywhere. We may represent the scalar potentials using a separation of variables solution to Laplace's equation, with a contribution only from the $n = 1$ term in the series. In region 1 we have both scattered

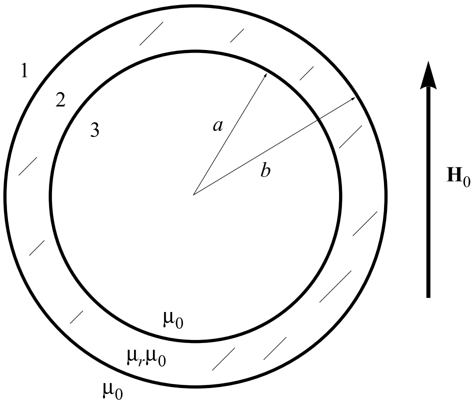

Figure 3.22: Spherical shell of magnetic material.

and applied potentials, where the applied potential is just $\Phi_0 = -H_0 z = -H_0 r \cos\theta$, since $\mathbf{H}_0 = -\nabla\Phi_0 = H_0\hat{\mathbf{z}}$. We have

$$\Phi_1(\mathbf{r}) = A_1 r^{-2}\cos\theta - H_0 r\cos\theta, \tag{3.206}$$

$$\Phi_2(\mathbf{r}) = (B_1 r^{-2} + C_1 r)\cos\theta, \tag{3.207}$$

$$\Phi_3(\mathbf{r}) = D_1 r\cos\theta. \tag{3.208}$$

We choose (3.109) for the scattered potential in region 1 so that it decays as $\mathbf{r} \to \infty$, and (3.110) for the scattered potential in region 3 so that it remains finite at $r = 0$. In region 2 we have no restrictions and therefore include both contributions. The coefficients A_1, B_1, C_1, D_1 are found by applying the appropriate boundary conditions at $r = a$ and $r = b$. By continuity of the scalar potential across each boundary we have

$$A_1 b^{-2} - H_0 b = B_1 b^{-2} + C_1 b,$$

$$B_1 a^{-2} + C_1 a = D_1 a.$$

By (3.156), the quantity $\mu\partial\Phi/\partial r$ is also continuous at $r = a$ and $r = b$; this gives two more equations:

$$\mu_0(-2A_1 b^{-3} - H_0) = \mu(-2B_1 b^{-3} + C_1),$$

$$\mu(-2B_1 a^{-3} + C_1) = \mu_0 D_1.$$

Simultaneous solution yields

$$D_1 = -\frac{9\mu_r}{K}H_0$$

where

$$K = (2 + \mu_r)(1 + 2\mu_r) - 2(a/b)^3(\mu_r - 1)^2.$$

Substituting this into (3.208) and using $\mathbf{H} = -\nabla\Phi_m$, we find that

$$\mathbf{H} = \kappa H_0\hat{\mathbf{z}}$$

within the enclosure, where $\kappa = 9\mu_r/K$. This field is uniform and, since $\kappa < 1$ for $\mu_r > 1$, it is weaker than the applied field. For $\mu_r \gg 1$ we have $K \approx 2\mu_r^2[1 - (a/b)^3]$. Denoting the shell thickness by $\Delta = b - a$, we find that $K \approx 6\mu_r^2 \Delta/a$ when $\Delta/a \ll 1$. Thus

$$\kappa = \frac{3}{2}\frac{1}{\mu_r \frac{\Delta}{a}}$$

describes the coefficient of shielding for a highly permeable spherical enclosure, valid when $\mu_r \gg 1$ and $\Delta/a \ll 1$. A shell for which $\mu_r = 10,000$ and $a/b = 0.99$ can reduce the enclosure field to 0.15% of the applied field.

3.4 Static field theorems

3.4.1 Mean value theorem of electrostatics

The average value of the electrostatic potential over a sphere is equal to the potential at the center of the sphere, provided that the sphere encloses no electric charge. To see this, write

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon}\int_V \frac{\rho(\mathbf{r}')}{R}\,dV' + \frac{1}{4\pi}\oint_S \left[-\Phi(\mathbf{r}')\frac{\hat{\mathbf{R}}}{R^2} + \frac{\nabla'\Phi(\mathbf{r}')}{R}\right]\cdot \mathbf{dS}';$$

put $\rho \equiv 0$ in V, and use the obvious facts that if S is a sphere centered at point $\mathbf{r}$ then (1) R is constant on S and (2) $\hat{\mathbf{n}}' = -\hat{\mathbf{R}}$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi R^2}\oint_S \Phi(\mathbf{r}')\,dS' - \frac{1}{4\pi R}\oint_S \mathbf{E}(\mathbf{r}')\cdot \mathbf{dS}'.$$

The last term vanishes by Gauss's law, giving the desired result.

3.4.2 Earnshaw's theorem

It is impossible for a charge to rest in stable equilibrium under the influence of electrostatic forces alone. This is an easy consequence of the mean value theorem of electrostatics, which precludes the existence of a point where Φ can assume a maximum or a minimum.

3.4.3 Thomson's theorem

Static charge on a system of perfect conductors distributes itself so that the electric stored energy is a minimum. Figure 3.23 shows a system of n conducting bodies held at potentials $\Phi_1, \ldots, \Phi_n$. Suppose the potential field associated with the actual distribution of charge on these bodies is Φ, giving

$$W_e = \frac{\epsilon}{2}\int_V \mathbf{E}\cdot\mathbf{E}\,dV = \frac{\epsilon}{2}\int_V \nabla\Phi\cdot\nabla\Phi\,dV$$

for the actual stored energy. Now assume a slightly different charge distribution, resulting in a new potential $\Phi' = \Phi + \delta\Phi$ that satisfies the same boundary conditions (i.e., assume $\delta\Phi = 0$ on each conducting body). The stored energy associated with this hypothetical situation is

$$W_e' = W_e + \delta W_e = \frac{\epsilon}{2}\int_V \nabla(\Phi + \delta\Phi)\cdot\nabla(\Phi + \delta\Phi)\,dV$$

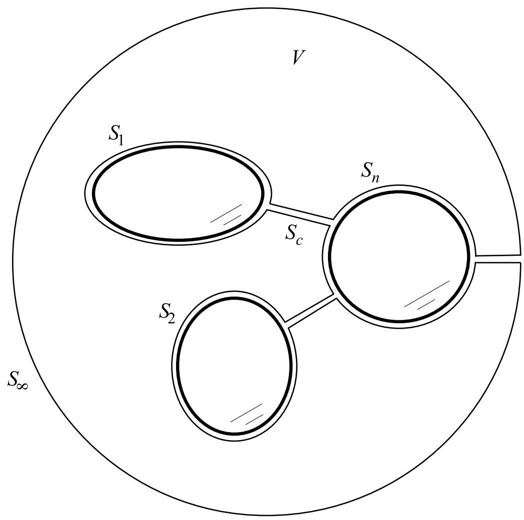

Figure 3.23: System of conductors used to derive Thomson's theorem.

so that

$$\delta W_e = \epsilon \int_V \nabla\Phi \cdot \nabla(\delta\Phi)\, dV + \frac{\epsilon}{2} \int_V |\nabla(\delta\Phi)|^2\, dV;$$

Thomson's theorem will be proved if we can show that

$$\int_V \nabla\Phi \cdot \nabla(\delta\Phi)\, dV = 0, \tag{3.209}$$

because then we shall have

$$\delta W_e = \frac{\epsilon}{2} \int_V |\nabla(\delta\Phi)|^2\, dV \geq 0.$$

To establish (3.209), we use Green's first identity

$$\int_V (\nabla u \cdot \nabla v + u\nabla^2 v)\, dV = \oint_S u\nabla v \cdot \mathbf{dS}$$

with $u = \delta\Phi$ and $v = \Phi$:

$$\int_V \nabla\Phi \cdot \nabla(\delta\Phi)\, dV = \oint_S \delta\Phi\, \nabla\Phi \cdot \mathbf{dS}.$$

Here S is composed of (1) the exterior surfaces S_k ($k = 1, \ldots, n$) of the n bodies, (2) the surfaces S_c of the "cuts" that are introduced in order to keep V a simply-connected region (a condition for the validity of Green's identity), and (3) the sphere S_∞ of very large radius r. Thus

$$\int_V \nabla\Phi \cdot \nabla(\delta\Phi)\, dV = \sum_{k=1}^{n} \int_{S_k} \delta\Phi\, \nabla\Phi \cdot \mathbf{dS} + \int_{S_c} \delta\Phi\, \nabla\Phi \cdot \mathbf{dS} + \int_{S_\infty} \delta\Phi\, \nabla\Phi \cdot \mathbf{dS}.$$

The first term on the right vanishes because $\delta\Phi = 0$ on each S_k. The second term vanishes because the contributions from opposite sides of each cut cancel (note that $\hat{\mathbf{n}}$ occurs in pairs that are oppositely directed). The third term vanishes because $\Phi \sim 1/r$, $\nabla\Phi \sim 1/r^2$, and $dS \sim r^2$ where $r \to \infty$ for points on S_∞.

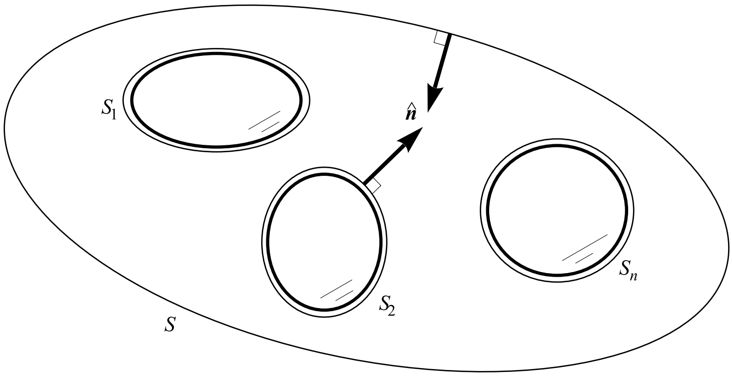

Figure 3.24: System of conductors used to derive Green's reciprocation theorem.

3.4.4 Green's reciprocation theorem

Consider a system of n conducting bodies as in Figure 3.24. An associated mathematical surface S_t consists of the exterior surfaces $S_1, \ldots, S_n$ of the n bodies, taken together with a surface S that enclosed all of the bodies. Suppose Φ and Φ' are electrostatic potentials produced by two distinct distributions of stationary charge over the set of conductors. Then $\nabla^2\Phi = 0 = \nabla^2\Phi'$ and Green's second identity gives

$$\oint_{S_t} \left(\Phi \frac{\partial \Phi'}{\partial n} - \Phi' \frac{\partial \Phi}{\partial n} \right) dS = 0$$

or

$$\sum_{k=1}^{n} \int_{S_k} \Phi \frac{\partial \Phi'}{\partial n} \, dS + \int_{S} \Phi \frac{\partial \Phi'}{\partial n} \, dS = \sum_{k=1}^{n} \int_{S_k} \Phi' \frac{\partial \Phi}{\partial n} \, dS + \int_{S} \Phi' \frac{\partial \Phi}{\partial n} \, dS.$$

Now let S be a sphere of very large radius R so that at points on S we have

$$\Phi, \Phi' \sim \frac{1}{R}, \qquad \frac{\partial \Phi}{\partial n}, \frac{\partial \Phi'}{\partial n} \sim \frac{1}{R^2}, \qquad dS \sim R^2;$$

as $R \to \infty$ then,

$$\sum_{k=1}^{n} \int_{S_k} \Phi \frac{\partial \Phi'}{\partial n} \, dS = \sum_{k=1}^{n} \int_{S_k} \Phi' \frac{\partial \Phi}{\partial n} \, dS.$$

Furthermore, the conductors are equipotentials so that

$$\sum_{k=1}^{n} \Phi_k \int_{S_k} \frac{\partial \Phi'}{\partial n} \, dS = \sum_{k=1}^{n} \Phi'_k \int_{S_k} \frac{\partial \Phi}{\partial n} \, dS$$

and we therefore have

$$\sum_{k=1}^{n} q'_k \Phi_k = \sum_{k=1}^{n} q_k \Phi'_k \tag{3.210}$$

where the kth conductor ($k = 1, \ldots, n$) has potential Φ_k when it carries charge q_k, and has potential Φ'_k when it carries charge q'_k. This is *Green's reciprocation theorem.* A classic application is to determine the charge induced on a grounded conductor by

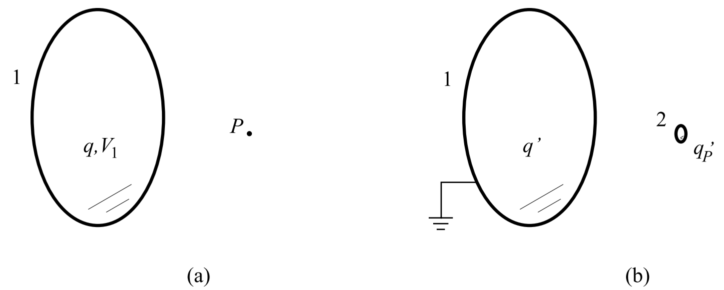

Figure 3.25: Application of Green's reciprocation theorem. (a) The "unprimed situation" permits us to determine the potential V_P at point P produced by a charge q placed on body 1. Here V_1 is the potential of body 1. (b) In the "primed situation" we ground body 1 and induce a charge q' by bringing a point charge q'_P into proximity.

a nearby point charge. This is accomplished as follows. Let the conducting body of interest be designated as body 1, and model the nearby point charge q_P as a very small conducting body designated as body 2 and located at point P in space. Take

$$q_1 = q, \qquad q_2 = 0, \qquad \Phi_1 = V_1, \qquad \Phi_2 = V_P,$$

and

$$q'_1 = q', \qquad q'_2 = q'_P, \qquad \Phi'_1 = 0, \qquad \Phi'_2 = V'_P,$$

giving the two situations shown in Figure 3.25. Substitution into Green's reciprocation theorem

$$q'_1\Phi_1 + q'_2\Phi_2 = q_1\Phi'_1 + q_2\Phi'_2$$

gives $q'V_1 + q'_P V_P = 0$ so that

$$q' = -q'_P V_P / V_1. \tag{3.211}$$

3.5 Problems

3.1 The z-axis carries a line charge of nonuniform density $\rho_l(z)$. Show that the electric field in the plane $z = 0$ is given by

$$\mathbf{E}(\rho, \phi) = \frac{1}{4\pi\epsilon}\left[\hat{\boldsymbol{\rho}}\rho\int_{-\infty}^{\infty}\frac{\rho_l(z')\,dz'}{(\rho^2 + z'^2)^{3/2}} - \hat{\mathbf{z}}\int_{-\infty}^{\infty}\frac{\rho_l(z')z'\,dz'}{(\rho^2 + z'^2)^{3/2}}\right].$$

Compute $\mathbf{E}$ when $\rho_l = \rho_0 \operatorname{sgn}(z)$, where $\operatorname{sgn}(z)$ is the signum function (A.6).

3.2 The ring $\rho = a$, $z = 0$, carries a line charge of nonuniform density $\rho_l(\phi)$. Show that the electric field at an arbitrary point on the z-axis is given by

$$\mathbf{E}(z) = \frac{-a^2}{4\pi\epsilon(a^2 + z^2)^{3/2}}\left[\hat{\mathbf{x}}\int_0^{2\pi}\rho_l(\phi')\cos\phi'\,d\phi' + \hat{\mathbf{y}}\int_0^{2\pi}\rho_l(\phi')\sin\phi'\,d\phi'\right] +$$
$$+\,\hat{\mathbf{z}}\frac{az}{4\pi\epsilon(a^2 + z^2)^{3/2}}\int_0^{2\pi}\rho_l(\phi')\,d\phi'.$$

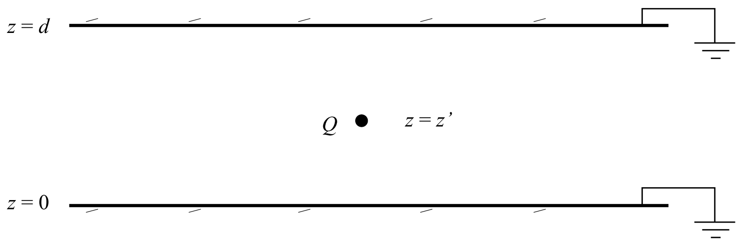

Figure 3.26: Geometry for computing Green's function for parallel plates.

Compute $\mathbf{E}$ when $\rho_l(\phi) = \rho_0 \sin\phi$. Repeat for $\rho_l(\phi) = \rho_0 \cos^2\phi$.

3.3 The plane $z = 0$ carries a surface charge of nonuniform density $\rho_s(\rho, \phi)$. Show that at an arbitrary point on the z-axis the rectangular components of $\mathbf{E}$ are given by

$$E_x(z) = -\frac{1}{4\pi\epsilon}\int_0^\infty \int_0^{2\pi} \frac{\rho_s(\rho', \phi')\,\rho'^2 \cos\phi'\,d\phi'\,d\rho'}{(\rho'^2 + z^2)^{3/2}},$$

$$E_y(z) = -\frac{1}{4\pi\epsilon}\int_0^\infty \int_0^{2\pi} \frac{\rho_s(\rho', \phi')\,\rho'^2 \sin\phi'\,d\phi'\,d\rho'}{(\rho'^2 + z^2)^{3/2}},$$

$$E_z(z) = \frac{z}{4\pi\epsilon}\int_0^\infty \int_0^{2\pi} \frac{\rho_s(\rho', \phi')\,\rho'\,d\phi'\,d\rho'}{(\rho'^2 + z^2)^{3/2}}.$$

Compute $\mathbf{E}$ when $\rho_s(\rho, \phi) = \rho_0 U(\rho - a)$ where $U(\rho)$ is the unit step function (A.5). Repeat for $\rho_s(\rho, \phi) = \rho_0[1 - U(\rho - a)]$.

3.4 The sphere $r = a$ carries a surface charge of nonuniform density $\rho_s(\theta)$. Show that the electric intensity at an arbitrary point on the z-axis is given by

$$\mathbf{E}(z) = \hat{\mathbf{z}}\frac{a^2}{2\epsilon}\int_0^\pi \frac{\rho_s(\theta')(z - a\cos\theta')\sin\theta'\,d\theta'}{(a^2 + z^2 - 2az\cos\theta')^{3/2}}.$$

Compute $\mathbf{E}(z)$ when $\rho_s(\theta) = \rho_0$, a constant. Repeat for $\rho_s(\theta) = \rho_0\cos^2\theta$.

3.5 Beginning with the postulates for the electrostatic field

$$\nabla \times \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{D} = \rho,$$

use the technique of § 2.8.2 to derive the boundary conditions (3.32)–(3.33).

3.6 A material half space of permittivity ϵ_1 occupies the region $z > 0$, while a second material half space of permittivity ϵ_2 occupies $z < 0$. Find the polarization surface charge densities and compute the total induced polarization charge for a point charge Q located at $z = h$.

3.7 Consider a point charge between two grounded conducting plates as shown in Figure 3.26. Write the Green's function as the sum of primary and secondary terms and apply the boundary conditions to show that the secondary Green's function is

$$G^s(\mathbf{r}|\mathbf{r}') = \frac{1}{(2\pi)^2}\int_{-\infty}^\infty \int_{-\infty}^\infty \left[-e^{-k_\rho(d-z)}\frac{\sinh k_\rho z'}{\sinh k_\rho d} - e^{-k_\rho z}\frac{\sinh k_\rho(d - z')}{\sinh k_\rho d}\right]\frac{e^{-j\mathbf{k}_\rho\cdot\mathbf{r}'}}{2k_\rho}\,d^2k_\rho. \tag{3.212}$$

3.8 Use the expansion

$$\frac{1}{\sinh k_\rho d} = \operatorname{csch} k_\rho d = 2\sum_{n=0}^{\infty} e^{-(2n+1)k_\rho d}$$

to show that the secondary Green's function for parallel conducting plates (3.212) may be written as an infinite sequence of images of the primary point charge. Identify the geometrical meaning of each image term.

3.9 Find the Green's functions for a dielectric slab of thickness d placed over a perfectly conducting ground plane located at $z = 0$.

3.10 Find the Green's functions for a dielectric slab of thickness $2d$ immersed in free space and centered on the $z = 0$ plane. Compare to the Green's function found in Problem 3.9.

3.11 Referring to the system of Figure 3.9, find the charge density on the surface of the sphere and integrate to show that the total charge is equal to the image charge.

3.12 Use the method of Green's functions to find the potential inside a conducting sphere for ρ inside the sphere.

3.13 Solve for the total potential and electric field of a grounded conducting sphere centered at the origin within a uniform impressed electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$. Find total charge induced on the sphere.

3.14 Consider a spherical cavity of radius a centered at the origin within a homogeneous dielectric material of permittivity $\epsilon = \epsilon_0\epsilon_r$. Solve for total potential and electric field inside the cavity in the presence of an impressed field $\mathbf{E} = E_0\hat{\mathbf{z}}$. Show that the field in the cavity is stronger than the applied field, and explain this using polarization surface charge.

3.15 Find the field of a point charge Q located at $z = d$ above a perfectly conducting ground plane at $z = 0$. Use the boundary condition to find the charge density on the plane and integrate to show that the total charge is $-Q$. Integrate Maxwell's stress tensor over the surface of the ground plane and show that the force on the ground plane is the same as the force on the image charge found from Coulomb's law.

3.16 Consider in free space a point charge $-q$ at $\mathbf{r} = \mathbf{r}_0 + \mathbf{d}$, a point charge $-q$ at $\mathbf{r} = \mathbf{r}_0 - \mathbf{d}$, and a point charge $2q$ at $\mathbf{r}_0$. Find the first three multipole moments and the resulting potential produced by this charge distribution.

3.17 A spherical charge distribution of radius a in free space has the density

$$\rho(\mathbf{r}) = \frac{Q}{\pi a^3}\cos 2\theta.$$

Compute the multipole moments for the charge distribution and find the resulting potential. Find a suitable arrangement of point charges that will produce the same potential field for $r > a$ as produced by the spherical charge.

3.18 Compute the magnetic flux density $\mathbf{B}$ for the circular wire loop of Figure 3.18 by (a) using the Biot–Savart law (3.165), and (b) computing the curl of (3.138).

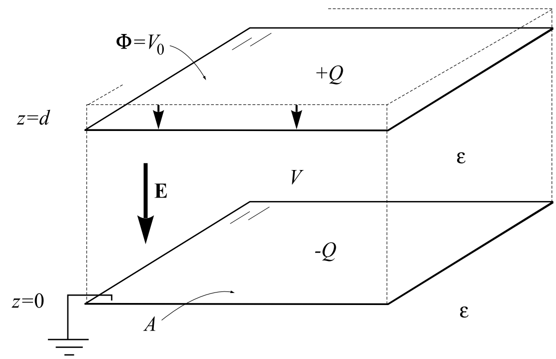

Figure 3.27: Parallel plate capacitor.

3.19 Two circular current-carrying wires are arranged coaxially along the z-axis. Loop 1 has radius a_1, carries current I_1, and is centered in the $z = 0$ plane. Loop 2 has radius a_2, carries current I_2, and is centered in the $z = d$ plane. Find the force between the loops.

3.20 Choose $\mathbf{Q} = \nabla'\left(\frac{1}{R}\right) \times \mathbf{c}$ in (3.162) and derive the following expression for $\mathbf{B}$:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu}{4\pi}\int_V \mathbf{J}(\mathbf{r}') \times \nabla'\left(\frac{1}{R}\right) dV' - \\ - \frac{1}{4\pi}\oint_S \left[[\hat{\mathbf{n}}' \times \mathbf{B}(\mathbf{r}')] \times \nabla'\left(\frac{1}{R}\right) + [\hat{\mathbf{n}}' \cdot \mathbf{B}(\mathbf{r}')]\nabla'\left(\frac{1}{R}\right)\right] dS',$$

where $\hat{\mathbf{n}}$ is the normal vector outward from V. Compare to the Stratton–Chu formula (6.8).

3.21 Compute the curl of (3.163) to obtain the integral expression for $\mathbf{B}$ given in Problem 3.20. Compare to the Stratton–Chu formula (6.8).

3.22 Obtain (3.170) by integration of Maxwell's stress tensor over the xz-plane.

3.23 Consider two thin conducting parallel plates embedded in a region of permittivity ϵ (Figure 3.27). The bottom plate is connected to ground, and we apply an excess charge $+Q$ to the top plate (and thus $-Q$ is drawn onto the bottom plate.) Neglecting fringing, (a) solve Laplace's equation to show that

$$\Phi(z) = \frac{Q}{A\epsilon} z.$$

Use (3.87) to show that

$$W = \frac{Q^2 d}{2A\epsilon}.$$

(b) Verify W using (3.88). (c) Use $\mathbf{F} = -\hat{\mathbf{z}} dW/dz$ to show that the force on the top plate is

$$\mathbf{F} = -\hat{\mathbf{z}}\frac{Q^2}{2A\epsilon}.$$

(d) Verify $\mathbf{F}$ by integrating Maxwell's stress tensor over a closed surface surrounding the top plate.

3.24 Consider two thin conducting parallel plates embedded in a region of permittivity ϵ (Figure 3.27). The bottom plate is connected to ground, and we apply a potential V_0 to the top plate using a battery. Neglecting fringing, (a) solve Laplace's equation to show that

$$\Phi(z) = \frac{V_0}{d}z.$$

Use (3.87) to show that

$$W = \frac{V_0^2 A\epsilon}{2d}.$$

(b) Verify W using (3.88). (c) Use $\mathbf{F} = -\hat{\mathbf{z}}dW/dz$ to show that the force on the top plate is

$$\mathbf{F} = -\hat{\mathbf{z}}\frac{V_0^2 A\epsilon}{2d^2}.$$

(d) Verify $\mathbf{F}$ by integrating Maxwell's stress tensor over a closed surface surrounding the top plate.

3.25 A group of N perfectly conducting bodies is arranged in free space. Body n is held at potential V_n with respect to ground, and charge Q_n is induced upon its surface. By linearity we may write

$$Q_m = \sum_{n=1}^{N} c_{mn} V_n$$

where the c_{mn} are called the *capacitance coefficients*. Using Green's reciprocation theorem, demonstrate that $c_{mn} = c_{nm}$. Hint: Use (3.210). Choose one set of voltages so that $V_k = 0$, $k \neq n$, and place V_n at some potential, say $V_n = V_0$, producing the set of charges $\{Q_k\}$. For the second set choose $V_k' = 0$, $k \neq m$, and $V_m = V_0$, producing $\{Q_k'\}$.

3.26 For the set of conductors of Problem 3.25, show that we may write

$$Q_m = C_{mm} V_m + \sum_{k \neq m} C_{mk}(V_m - V_k)$$

where

$$C_{mn} = -c_{mn}, \quad m \neq n, \qquad C_{mm} = \sum_{k=1}^{N} c_{mk}.$$

Here C_{mm}, called the *self capacitance*, describes the interaction between the mth conductor and ground, while C_{mn}, called the *mutual capacitance*, describes the interaction between the mth and nth conductors.

3.27 For the set of conductors of Problem 3.25, show that the stored electric energy is given by

$$W = \frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N} c_{mn} V_n V_m.$$

3.28 A group of N wires is arranged in free space as shown in Figure 3.28. Wire n carries a steady current I_n, and a flux Ψ_n passes through the surface defined by its contour Γ_n. By linearity we may write

$$\Psi_m = \sum_{n=1}^{N} L_{mn} I_n$$

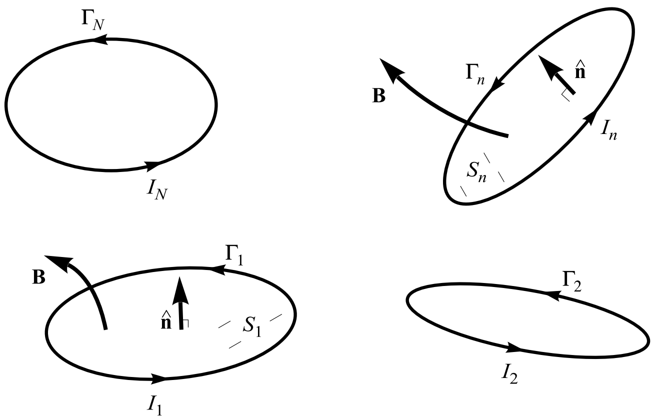

Figure 3.28: A system of current-carrying wires.

where the L_{mn} are called the *coefficients of inductance.* Derive *Neumann's formula*

$$L_{mn} = \frac{\mu_0}{4\pi} \oint_{\Gamma_n} \oint_{\Gamma_m} \frac{\mathbf{dl} \cdot \mathbf{dl}'}{|\mathbf{r} - \mathbf{r}'|},$$

and thereby demonstrate the reciprocity relation $L_{mn} = L_{nm}$.

3.29 For the group of wires shown in Figure 3.28, show that the stored magnetic energy is given by

$$W = \frac{1}{2} \sum_{m=1}^{N} \sum_{n=1}^{N} L_{mn} I_n I_m.$$

3.30 Prove the *minimum heat generation theorem*: steady electric currents distribute themselves in a conductor in such a way that the dissipated power is a minimum. Hint: Let $\mathbf{J}$ be the actual distribution of current in a conducting body, and let the power it dissipates be P. Let $\mathbf{J}' = \mathbf{J} + \delta\mathbf{J}$ be any other current distribution, and let the power it dissipates be $P' = P + \delta P$. Show that

$$\delta P = \frac{1}{2} \int_V \frac{1}{\sigma} |\delta\mathbf{J}|^2 \, dV \geq 0.$$