Chapter 5

Field decompositions and the EM potentials

5.1 Spatial symmetry decompositions

Spatial symmetry can often be exploited to solve electromagnetics problems. For analytic solutions, symmetry can be used to reduce the number of boundary conditions that must be applied. For computer solutions the storage requirements can be reduced. Typical symmetries include rotation about a point or axis, and reflection through a plane, along an axis, or through a point. We shall consider the common case of reflection through a plane. Reflections through the origin and through an axis will be treated in the exercises.

Note that spatial symmetry decompositions may be applied even if the sources and fields possess no spatial symmetry. As long as the boundaries and material media are symmetric, the sources and fields may be decomposed into constituents that individually mimic the symmetry of the environment.

5.1.1 Planar field symmetry

Consider a region of space consisting of linear, isotropic, time-invariant media having material parameters $\epsilon(\mathbf{r})$, $\mu(\mathbf{r})$, and $\sigma(\mathbf{r})$. The electromagnetic fields $(\mathbf{E}, \mathbf{H})$ within this region are related to their impressed sources $(\mathbf{J}^i, \mathbf{J}_m^i)$ and their secondary sources $\mathbf{J}^s = \sigma\mathbf{E}$ through Maxwell's curl equations:

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu\frac{\partial H_x}{\partial t} - J_{mx}^i, \tag{5.1}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu\frac{\partial H_y}{\partial t} - J_{my}^i, \tag{5.2}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu\frac{\partial H_z}{\partial t} - J_{mz}^i, \tag{5.3}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \epsilon\frac{\partial E_x}{\partial t} + \sigma E_x + J_x^i, \tag{5.4}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \epsilon\frac{\partial E_y}{\partial t} + \sigma E_y + J_y^i, \tag{5.5}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \epsilon\frac{\partial E_z}{\partial t} + \sigma E_z + J_z^i. \tag{5.6}$$

We assume the material constants are symmetric about some plane, say $z = 0$. Then

$$\epsilon(x, y, -z) = \epsilon(x, y, z),$$
$$\mu(x, y, -z) = \mu(x, y, z),$$
$$\sigma(x, y, -z) = \sigma(x, y, z).$$

That is, with respect to z the material constants are even functions. We further assume that the boundaries and boundary conditions, which guarantee uniqueness of solution, are also symmetric about the $z = 0$ plane. Then we define two cases of reflection symmetry.

Conditions for even symmetry. We claim that if the sources obey

$$J_x^i(x, y, z) = J_x^i(x, y, -z), \qquad J_{mx}^i(x, y, z) = -J_{mx}^i(x, y, -z),$$
$$J_y^i(x, y, z) = J_y^i(x, y, -z), \qquad J_{my}^i(x, y, z) = -J_{my}^i(x, y, -z),$$
$$J_z^i(x, y, z) = -J_z^i(x, y, -z), \qquad J_{mz}^i(x, y, z) = J_{mz}^i(x, y, -z),$$

then the fields obey

$$E_x(x, y, z) = E_x(x, y, -z), \qquad H_x(x, y, z) = -H_x(x, y, -z),$$
$$E_y(x, y, z) = E_y(x, y, -z), \qquad H_y(x, y, z) = -H_y(x, y, -z),$$
$$E_z(x, y, z) = -E_z(x, y, -z), \qquad H_z(x, y, z) = H_z(x, y, -z).$$

The electric field shares the symmetry of the electric source: components parallel to the $z = 0$ plane are even in z, and the component perpendicular is odd. The magnetic field shares the symmetry of the magnetic source: components parallel to the $z = 0$ plane are odd in z, and the component perpendicular is even.

We can verify our claim by showing that the symmetric fields and sources obey Maxwell's equations. At an arbitrary point $z = a > 0$ equation (5.1) requires

$$\left.\frac{\partial E_z}{\partial y}\right|_{z=a} - \left.\frac{\partial E_y}{\partial z}\right|_{z=a} = -\mu|_{z=a} \left.\frac{\partial H_x}{\partial t}\right|_{z=a} - J_{mx}^i|_{z=a}.$$

By the assumed symmetry condition on source and material constant we get

$$\left.\frac{\partial E_z}{\partial y}\right|_{z=a} - \left.\frac{\partial E_y}{\partial z}\right|_{z=a} = -\mu|_{z=-a} \left.\frac{\partial H_x}{\partial t}\right|_{z=a} + J_{mx}^i|_{z=-a}.$$

If our claim holds regarding the field behavior, then

$$\left.\frac{\partial E_z}{\partial y}\right|_{z=-a} = -\left.\frac{\partial E_z}{\partial y}\right|_{z=a},$$
$$\left.\frac{\partial E_y}{\partial z}\right|_{z=-a} = -\left.\frac{\partial E_y}{\partial z}\right|_{z=a},$$
$$\left.\frac{\partial H_x}{\partial t}\right|_{z=-a} = -\left.\frac{\partial H_x}{\partial t}\right|_{z=a},$$

and we have

$$-\left.\frac{\partial E_z}{\partial y}\right|_{z=-a} + \left.\frac{\partial E_y}{\partial z}\right|_{z=-a} = \mu|_{z=-a} \left.\frac{\partial H_x}{\partial t}\right|_{z=-a} + J_{mx}^i|_{z=-a}.$$

So this component of Faraday's law is satisfied. With similar reasoning we can show that the symmetric sources and fields satisfy (5.2)–(5.6) as well.

Conditions for odd symmetry. We can also show that if the sources obey

$$
\begin{aligned}
J_x^i(x,y,z) &= -J_x^i(x,y,-z), & J_{mx}^i(x,y,z) &= J_{mx}^i(x,y,-z),\\
J_y^i(x,y,z) &= -J_y^i(x,y,-z), & J_{my}^i(x,y,z) &= J_{my}^i(x,y,-z),\\
J_z^i(x,y,z) &= J_z^i(x,y,-z), & J_{mz}^i(x,y,z) &= -J_{mz}^i(x,y,-z),
\end{aligned}
$$

then the fields obey

$$
\begin{aligned}
E_x(x,y,z) &= -E_x^i(x,y,-z), & H_x(x,y,z) &= H_x(x,y,-z),\\
E_y(x,y,z) &= -E_y(x,y,-z), & H_y(x,y,z) &= H_y(x,y,-z),\\
E_z(x,y,z) &= E_z(x,y,-z), & H_z(x,y,z) &= -H_z(x,y,-z).
\end{aligned}
$$

Again the electric field has the same symmetry as the electric source. However, in this case components parallel to the $z = 0$ plane are odd in z and the component perpendicular is even. Similarly, the magnetic field has the same symmetry as the magnetic source. Here components parallel to the $z = 0$ plane are even in z and the component perpendicular is odd.

Field symmetries and the concept of source images. In the case of odd symmetry the electric field parallel to the $z = 0$ plane is an odd function of z. If we assume that the field is also continuous across this plane, then the electric field tangential to $z = 0$ must vanish: the condition required at the surface of a perfect electric conductor (PEC). We may regard the problem of sources above a perfect conductor in the $z = 0$ plane as *equivalent* to the problem of sources odd about this plane, as long as the sources in both cases are identical for $z > 0$. We refer to the source in the region $z < 0$ as the *image* of the source in the region $z > 0$. Thus the image source $(\mathbf{J}^I, \mathbf{J}_m^I)$ obeys

$$
\begin{aligned}
J_x^I(x,y,-z) &= -J_x^i(x,y,z), & J_{mx}^I(x,y,-z) &= J_{mx}^i(x,y,z),\\
J_y^I(x,y,-z) &= -J_y^i(x,y,z), & J_{my}^I(x,y,-z) &= J_{my}^i(x,y,z),\\
J_z^I(x,y,-z) &= J_z^i(x,y,z), & J_{mz}^I(x,y,-z) &= -J_{mz}^i(x,y,z).
\end{aligned}
$$

That is, parallel components of electric current image in the opposite direction, and the perpendicular component images in the same direction; parallel components of the magnetic current image in the same direction, while the perpendicular component images in the opposite direction.

In the case of even symmetry, the magnetic field parallel to the $z = 0$ plane is odd, and thus the magnetic field tangential to the $z = 0$ plane must be zero. We therefore have an equivalence between the problem of a source above a plane of perfect magnetic conductor (PMC) and the problem of sources even about that plane. In this case we identify image sources that obey

$$
\begin{aligned}
J_x^I(x,y,-z) &= J_x^i(x,y,z), & J_{mx}^I(x,y,-z) &= -J_{mx}^i(x,y,z),\\
J_y^I(x,y,-z) &= J_y^i(x,y,z), & J_{my}^I(x,y,-z) &= -J_{my}^i(x,y,z),\\
J_z^I(x,y,-z) &= -J_z^i(x,y,z), & J_{mz}^I(x,y,-z) &= J_{mz}^i(x,y,z).
\end{aligned}
$$

Parallel components of electric current image in the same direction, and the perpendicular component images in the opposite direction; parallel components of magnetic current image in the opposite direction, and the perpendicular component images in the same direction.

In the case of odd symmetry, we sometimes say that an "electric wall" exists at $z = 0$. The term "magnetic wall" can be used in the case of even symmetry. These terms are particularly common in the description of waveguide fields.

Symmetric field decomposition. Field symmetries may be applied to arbitrary source distributions through a symmetry decomposition of the sources and fields. Consider the general impressed source distributions $(\mathbf{J}^i, \mathbf{J}_m^i)$. The source set

$$
\begin{aligned}
J_x^{ie}(x, y, z) &= \frac{1}{2}\left[J_x^i(x, y, z) + J_x^i(x, y, -z)\right],\\
J_y^{ie}(x, y, z) &= \frac{1}{2}\left[J_y^i(x, y, z) + J_y^i(x, y, -z)\right],\\
J_z^{ie}(x, y, z) &= \frac{1}{2}\left[J_z^i(x, y, z) - J_z^i(x, y, -z)\right],\\
J_{mx}^{ie}(x, y, z) &= \frac{1}{2}\left[J_{mx}^i(x, y, z) - J_{mx}^i(x, y, -z)\right],\\
J_{my}^{ie}(x, y, z) &= \frac{1}{2}\left[J_{my}^i(x, y, z) - J_{my}^i(x, y, -z)\right],\\
J_{mz}^{ie}(x, y, z) &= \frac{1}{2}\left[J_{mz}^i(x, y, z) + J_{mz}^i(x, y, -z)\right],
\end{aligned}
$$

is clearly of even symmetric type while the source set

$$
\begin{aligned}
J_x^{io}(x, y, z) &= \frac{1}{2}\left[J_x^i(x, y, z) - J_x^i(x, y, -z)\right],\\
J_y^{io}(x, y, z) &= \frac{1}{2}\left[J_y^i(x, y, z) - J_y^i(x, y, -z)\right],\\
J_z^{io}(x, y, z) &= \frac{1}{2}\left[J_z^i(x, y, z) + J_z^i(x, y, -z)\right],\\
J_{mx}^{io}(x, y, z) &= \frac{1}{2}\left[J_{mx}^i(x, y, z) + J_{mx}^i(x, y, -z)\right],\\
J_{my}^{io}(x, y, z) &= \frac{1}{2}\left[J_{my}^i(x, y, z) + J_{my}^i(x, y, -z)\right],\\
J_{mz}^{io}(x, y, z) &= \frac{1}{2}\left[J_{mz}^i(x, y, z) - J_{mz}^i(x, y, -z)\right],
\end{aligned}
$$

is of the odd symmetric type. Since $\mathbf{J}^i = \mathbf{J}^{ie} + \mathbf{J}^{io}$ and $\mathbf{J}_m^i = \mathbf{J}_m^{ie} + \mathbf{J}_m^{io}$, we can decompose any source into constituents having, respectively, even and odd symmetry with respect to a plane. The source with even symmetry produces an even field set, while the source with odd symmetry produces an odd field set. The total field is the sum of the fields from each field set.

Planar symmetry for frequency-domain fields. The symmetry conditions introduced above for the time-domain fields also hold for the frequency-domain fields. Because both the conductivity and permittivity must be even functions, we combine their effects and require the complex permittivity to be even. Otherwise the field symmetries and source decompositions are identical.

Example of symmetry decomposition: line source between conducting planes. Consider a z-directed electric line source $\tilde{I}_0$ located at $y = h, x = 0$ between conducting planes at $y = \pm d$, $d > h$. The material between the plates has permeability $\tilde{\mu}(\omega)$ and complex permittivity $\tilde{\epsilon}^c(\omega)$. We decompose the source into one of even symmetric type with line sources $\tilde{I}_0/2$ located at $y = \pm h$, and one of odd symmetric type with a line source $\tilde{I}_0/2$ located at $y = h$ and a line source $-\tilde{I}_0/2$ located at $y = -h$. We solve each of these problems by exploiting the appropriate symmetry, and superpose the results to find the solution to the original problem.

For the even-symmetric case, we begin by using (4.407) to represent the impressed field:

$$\tilde{E}_z^i(x, y, \omega) = -\frac{\omega\tilde{\mu}\frac{\tilde{I}_0(\omega)}{2}}{2\pi} \int_{-\infty+j\Delta}^{\infty+j\Delta} \frac{e^{-jk_y|y-h|} + e^{-jk_y|y+h|}}{2k_y} e^{-jk_x x}\, dk_x.$$

For $y > h$ this becomes

$$\tilde{E}_z^i(x, y, \omega) = -\frac{\omega\tilde{\mu}\frac{\tilde{I}_0(\omega)}{2}}{2\pi} \int_{-\infty+j\Delta}^{\infty+j\Delta} \frac{2\cos k_y h}{2k_y} e^{-jk_y y} e^{-jk_x x}\, dk_x.$$

The secondary (scattered) field consists of waves propagating in both the $\pm y$-directions:

$$\tilde{E}_z^s(x, y, \omega) = \frac{1}{2\pi} \int_{-\infty+j\Delta}^{\infty+j\Delta} \left[A^+(k_x, \omega)e^{-jk_y y} + A^-(k_x, \omega)e^{jk_y y}\right] e^{-jk_x x}\, dk_x. \tag{5.7}$$

The impressed field is even about $y = 0$. Since the total field $E_z = E_z^i + E_z^s$ must be even in y (E_z is parallel to the plane $y = 0$), the scattered field must also be even. Thus, $A^+ = A^-$ and the total field is for $y > h$

$$\tilde{E}_z(x, y, \omega) = \frac{1}{2\pi} \int_{-\infty+j\Delta}^{\infty+j\Delta} \left[2A^+(k_x, \omega)\cos k_y y - \omega\tilde{\mu}\frac{\tilde{I}_0(\omega)}{2}\frac{2\cos k_y h}{2k_y} e^{-jk_y y}\right] e^{-jk_x x}\, dk_x.$$

Now the electric field must obey the boundary condition $\tilde{E}_z = 0$ at $y = \pm d$. However, since $\tilde{E}_z$ is even the satisfaction of this condition at $y = d$ automatically implies its satisfaction at $y = -d$. So we set

$$\frac{1}{2\pi} \int_{-\infty+j\Delta}^{\infty+j\Delta} \left[2A^+(k_x, \omega)\cos k_y d - \omega\tilde{\mu}\frac{\tilde{I}_0(\omega)}{2}\frac{2\cos k_y h}{2k_y} e^{-jk_y d}\right] e^{-jk_x x}\, dk_x = 0$$

and invoke the Fourier integral theorem to get

$$A^+(k_x, \omega) = \omega\tilde{\mu}\frac{\tilde{I}_0(\omega)}{2}\frac{\cos k_y h}{2k_y}\frac{e^{-jk_y d}}{\cos k_y d}.$$

The total field for this case is

$$\tilde{E}_z(x, y, \omega) = -\frac{\omega\tilde{\mu}\frac{\tilde{I}_0(\omega)}{2}}{2\pi} \int_{-\infty+j\Delta}^{\infty+j\Delta} \left[\frac{e^{-jk_y|y-h|} + e^{-jk_y|y+h|}}{2k_y} - \right.$$
$$\left. - \frac{2\cos k_y h}{2k_y}\frac{e^{-jk_y d}}{\cos k_y d}\cos k_y y\right] e^{-jk_x x}\, dk_x.$$

For the odd-symmetric case the impressed field is

$$\tilde{E}_z^i(x, y, \omega) = -\frac{\omega\tilde{\mu}\frac{\tilde{I}_0(\omega)}{2}}{2\pi} \int_{-\infty+j\Delta}^{\infty+j\Delta} \frac{e^{-jk_y|y-h|} - e^{-jk_y|y+h|}}{2k_y} e^{-jk_x x}\, dk_x,$$

which for $y > h$ is

$$\tilde{E}_z^i(x, y, \omega) = -\frac{\omega\tilde{\mu}\frac{\tilde{I}_0(\omega)}{2}}{2\pi} \int_{-\infty+j\Delta}^{\infty+j\Delta} \frac{2j \sin k_y h}{2k_y} e^{-jk_y y} e^{-jk_x x}\, dk_x.$$

The scattered field has the form of (5.7) but must be odd. Thus $A^+ = -A^-$ and the total field for $y > h$ is

$$\tilde{E}_z(x, y, \omega) = \frac{1}{2\pi} \int_{-\infty+j\Delta}^{\infty+j\Delta} \left[2jA^+(k_x, \omega) \sin k_y y - \omega\tilde{\mu}\frac{\tilde{I}_0(\omega)}{2} \frac{2j \sin k_y h}{2k_y} e^{-jk_y y}\right] e^{-jk_x x}\, dk_x.$$

Setting $\tilde{E}_z = 0$ at $z = d$ and solving for A^+ we find that the total field for this case is

$$\tilde{E}_z(x, y, \omega) = -\frac{\omega\tilde{\mu}\frac{\tilde{I}_0(\omega)}{2}}{2\pi} \int_{-\infty+j\Delta}^{\infty+j\Delta} \left[\frac{e^{-jk_y|y-h|} - e^{-jk_y|y+h|}}{2k_y} - \right.$$
$$\left. - \frac{2j \sin k_y h}{2k_y} \frac{e^{-jk_y d}}{\sin k_y d} \sin k_y y\right] e^{-jk_x x}\, dk_x.$$

Adding the fields for the two cases we find that

$$\tilde{E}_z(x, y, \omega) = -\frac{\omega\tilde{\mu}\tilde{I}_0(\omega)}{2\pi} \int_{-\infty+j\Delta}^{\infty+j\Delta} \frac{e^{-jk_y|y-h|}}{2k_y} e^{-jk_x x}\, dk_x +$$
$$+\frac{\omega\tilde{\mu}\tilde{I}_0(\omega)}{2\pi} \int_{-\infty+j\Delta}^{\infty+j\Delta} \left[\frac{\cos k_y h \cos k_y y}{\cos k_y d} + j\frac{\sin k_y h \sin k_y y}{\sin k_y d}\right] \frac{e^{-jk_y d}}{2k_y} e^{-jk_x x}\, dk_x, \tag{5.8}$$

which is a superposition of impressed and scattered fields.

5.2 Solenoidal–lamellar decomposition

We now discuss the decomposition of a general vector field into a *lamellar* component having zero curl and a *solenoidal* component having zero divergence. This is known as a *Helmholtz decomposition.* If $\mathbf{V}$ is any vector field then we wish to write

$$\mathbf{V} = \mathbf{V}_s + \mathbf{V}_l, \tag{5.9}$$

where $\mathbf{V}_s$ and $\mathbf{V}_l$ are the solenoidal and lamellar components of $\mathbf{V}$. Formulas expressing these components in terms of $\mathbf{V}$ are obtained as follows. We first write $\mathbf{V}_s$ in terms of a "vector potential" $\mathbf{A}$ as

$$\mathbf{V}_s = \nabla \times \mathbf{A}. \tag{5.10}$$

This is possible by virtue of (B.49). Similarly, we write $\mathbf{V}_l$ in terms of a "scalar potential" ϕ as

$$\mathbf{V}_l = \nabla\phi. \tag{5.11}$$

To obtain a formula for $\mathbf{V}_l$ we take the divergence of (5.9) and use (5.11) to get

$$\nabla \cdot \mathbf{V} = \nabla \cdot \mathbf{V}_l = \nabla \cdot \nabla\phi = \nabla^2\phi.$$

The result,

$$\nabla^2\phi = \nabla \cdot \mathbf{V},$$

may be regarded as Poisson's equation for the unknown ϕ. This equation is solved in Chapter 3. By (3.61) we have

$$\phi(\mathbf{r}) = -\int_V \frac{\nabla' \cdot \mathbf{V}(\mathbf{r}')}{4\pi R}\, dV',$$

where $R = |\mathbf{r} - \mathbf{r}'|$, and we have

$$\mathbf{V}_l(\mathbf{r}) = -\nabla \int_V \frac{\nabla' \cdot \mathbf{V}(\mathbf{r}')}{4\pi R}\, dV'. \tag{5.12}$$

Similarly, a formula for $\mathbf{V}_s$ can be obtained by taking the curl of (5.9) to get

$$\nabla \times \mathbf{V} = \nabla \times \mathbf{V}_s.$$

Substituting (5.10) we have

$$\nabla \times \mathbf{V} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}.$$

We may choose any value we wish for $\nabla \cdot \mathbf{A}$, since this does not alter $\mathbf{V}_s = \nabla \times \mathbf{A}$. (We discuss such "gauge transformations" in greater detail later in this chapter.) With $\nabla \cdot \mathbf{A} = 0$ we obtain

$$-\nabla \times \mathbf{V} = \nabla^2\mathbf{A}.$$

This is Poisson's equation for each rectangular component of $\mathbf{A}$; therefore

$$\mathbf{A}(\mathbf{r}) = \int_V \frac{\nabla' \times \mathbf{V}(\mathbf{r}')}{4\pi R}\, dV',$$

and we have

$$\mathbf{V}_s(\mathbf{r}) = \nabla \times \int_V \frac{\nabla' \times \mathbf{V}(\mathbf{r}')}{4\pi R}\, dV'.$$

Summing the results we obtain the Helmholtz decomposition

$$\mathbf{V} = \mathbf{V}_l + \mathbf{V}_s = -\nabla \int_V \frac{\nabla' \cdot \mathbf{V}(\mathbf{r}')}{4\pi R}\, dV' + \nabla \times \int_V \frac{\nabla' \times \mathbf{V}(\mathbf{r}')}{4\pi R}\, dV'. \tag{5.13}$$

Identification of the electromagnetic potentials. Let us write the electromagnetic fields as a general superposition of solenoidal and lamellar components:

$$\mathbf{E} = \nabla \times \mathbf{A}_E + \nabla\phi_E, \tag{5.14}$$

$$\mathbf{B} = \nabla \times \mathbf{A}_B + \nabla\phi_B. \tag{5.15}$$

One possible form of the potentials $\mathbf{A}_E$, $\mathbf{A}_B$, ϕ_E, and ϕ_B appears in (5.13). However, because $\mathbf{E}$ and $\mathbf{B}$ are related by Maxwell's equations, the potentials should be related to the sources. We can determine the explicit relationship by substituting (5.14) and (5.15) into Ampere's and Faraday's laws. It is most convenient to analyze the relationships using superposition of the cases for which $\mathbf{J}_m = 0$ and $\mathbf{J} = 0$.

With $\mathbf{J}_m = 0$ Faraday's law is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{5.16}$$

Since $\nabla \times \mathbf{E}$ is solenoidal, $\mathbf{B}$ must be solenoidal and thus $\nabla \phi_B = 0$. This implies that $\phi_B = 0$, which is equivalent to the auxiliary Maxwell equation $\nabla \cdot \mathbf{B} = 0$. Now, substitution of (5.14) and (5.15) into (5.16) gives

$$\nabla \times [\nabla \times \mathbf{A}_E + \nabla \phi_E] = -\frac{\partial}{\partial t} [\nabla \times \mathbf{A}_B] .$$

Using $\nabla \times (\nabla \phi_E) = 0$ and combining the terms we get

$$\nabla \times \left[\nabla \times \mathbf{A}_E + \frac{\partial \mathbf{A}_B}{\partial t} \right] = 0,$$

hence

$$\nabla \times \mathbf{A}_E = -\frac{\partial \mathbf{A}_B}{\partial t} + \nabla \xi .$$

Substitution into (5.14) gives

$$\mathbf{E} = -\frac{\partial \mathbf{A}_B}{\partial t} + [\nabla \phi_E + \nabla \xi] .$$

Combining the two gradient functions together, we see that we can write both $\mathbf{E}$ and $\mathbf{B}$ in terms of two potentials:

$$\mathbf{E} = -\frac{\partial \mathbf{A}_e}{\partial t} - \nabla \phi_e, \tag{5.17}$$

$$\mathbf{B} = \nabla \times \mathbf{A}_e, \tag{5.18}$$

where the negative sign on the gradient term is introduced by convention.

Gauge transformations and the Coulomb gauge. We pay a price for the simplicity of using only two potentials to represent $\mathbf{E}$ and $\mathbf{B}$. While $\nabla \times \mathbf{A}_e$ is definitely solenoidal, $\mathbf{A}_e$ itself may not be: because of this (5.17) may not be a decomposition into solenoidal and lamellar components. However, a corollary of the Helmholtz theorem states that a vector field is uniquely specified only when *both* its curl and divergence are specified. Here there is an ambiguity in the representation of $\mathbf{E}$ and $\mathbf{B}$; we may remove this ambiguity and define $\mathbf{A}_e$ uniquely by requiring that

$$\nabla \cdot \mathbf{A}_e = 0. \tag{5.19}$$

Then $\mathbf{A}_e$ is solenoidal and the decomposition (5.17) is solenoidal–lamellar. This requirement on $\mathbf{A}_e$ is called the *Coulomb gauge*.

The ambiguity implied by the non-uniqueness of $\nabla \cdot \mathbf{A}_e$ can also be expressed by the observation that a transformation of the type

$$\mathbf{A}_e \to \mathbf{A}_e + \nabla \Gamma, \tag{5.20}$$

$$\phi_e \to \phi_e - \frac{\partial \Gamma}{\partial t}, \tag{5.21}$$

leaves the expressions (5.17) and (5.18) unchanged. This is called a *gauge transformation*, and the choice of a certain Γ alters the specification of $\nabla \cdot \mathbf{A}_e$. Thus we may begin with the Coulomb gauge as our baseline, and allow any alteration of $\mathbf{A}_e$ according to (5.20) as long as we augment $\nabla \cdot \mathbf{A}_e$ by $\nabla \cdot \nabla\Gamma = \nabla^2\Gamma$.

Once $\nabla \cdot \mathbf{A}_e$ is specified, the relationship between the potentials and the current $\mathbf{J}$ can be found by substitution of (5.17) and (5.18) into Ampere's law. At this point we assume media that are linear, homogeneous, isotropic, and described by the time-invariant parameters μ, ϵ, and σ. Writing $\mathbf{J} = \mathbf{J}^i + \sigma\mathbf{E}$ we have

$$\frac{1}{\mu}\nabla \times (\nabla \times \mathbf{A}_e) = \mathbf{J}^i - \sigma\frac{\partial \mathbf{A}_e}{\partial t} - \sigma\nabla\phi_e - \epsilon\frac{\partial^2 \mathbf{A}_e}{\partial t^2} - \epsilon\frac{\partial}{\partial t}\nabla\phi_e. \tag{5.22}$$

Taking the divergence of both sides of (5.22) we get

$$0 = \nabla \cdot \mathbf{J}^i - \sigma\frac{\partial}{\partial t}\nabla \cdot \mathbf{A} - \sigma\nabla \cdot \nabla\phi_e - \epsilon\frac{\partial^2}{\partial t^2}\nabla \cdot \mathbf{A}_e - \epsilon\frac{\partial}{\partial t}\nabla \cdot \nabla\phi_e. \tag{5.23}$$

Then, by substitution from the continuity equation and use of (5.19) along with $\nabla\cdot\nabla\phi_e = \nabla^2\phi_e$ we obtain

$$\frac{\partial}{\partial t}\left(\rho^i + \epsilon\nabla^2\phi_e\right) = -\sigma\nabla^2\phi_e.$$

For a lossless medium this reduces to

$$\nabla^2\phi_e = -\rho^i/\epsilon \tag{5.24}$$

and we have

$$\phi_e(\mathbf{r}, t) = \int_V \frac{\rho^i(\mathbf{r}', t)}{4\pi\epsilon R}\,dV'. \tag{5.25}$$

We can obtain an equation for $\mathbf{A}_e$ by expanding the left-hand side of (5.22) to get

$$\nabla(\nabla \cdot \mathbf{A}_e) - \nabla^2\mathbf{A}_e = \mu\mathbf{J}^i - \sigma\mu\frac{\partial \mathbf{A}_e}{\partial t} - \sigma\mu\nabla\phi_e - \mu\epsilon\frac{\partial^2 \mathbf{A}_e}{\partial t^2} - \mu\epsilon\frac{\partial}{\partial t}\nabla\phi_e, \tag{5.26}$$

hence

$$\nabla^2\mathbf{A}_e - \mu\epsilon\frac{\partial^2 \mathbf{A}_e}{\partial t^2} = -\mu\mathbf{J}^i + \sigma\mu\frac{\partial \mathbf{A}_e}{\partial t} + \sigma\mu\nabla\phi_e + \mu\epsilon\frac{\partial}{\partial t}\nabla\phi_e$$

under the Coulomb gauge. For lossless media this becomes

$$\nabla^2\mathbf{A}_e - \mu\epsilon\frac{\partial^2 \mathbf{A}_e}{\partial t^2} = -\mu\mathbf{J}^i + \mu\epsilon\frac{\partial}{\partial t}\nabla\phi_e. \tag{5.27}$$

Observe that the left-hand side of (5.27) is solenoidal (since the Laplacian term came from the curl-curl, and $\nabla \cdot \mathbf{A}_e = 0$), while the right-hand side contains a general vector field $\mathbf{J}^i$ and a lamellar term. We might expect the $\nabla\phi_e$ term to cancel the lamellar portion of $\mathbf{J}^i$, and this does happen [91]. By (5.12) and the continuity equation we can write the lamellar component of the current as

$$\mathbf{J}_l^i(\mathbf{r}, t) = -\nabla\int_V \frac{\nabla' \cdot \mathbf{J}^i(\mathbf{r}', t)}{4\pi R}\,dV' = \frac{\partial}{\partial t}\nabla\int_V \frac{\rho^i(\mathbf{r}', t)}{4\pi R}\,dV' = \epsilon\frac{\partial}{\partial t}\nabla\phi_e.$$

Thus (5.27) becomes

$$\nabla^2\mathbf{A}_e - \mu\epsilon\frac{\partial^2 \mathbf{A}_e}{\partial t^2} = -\mu\mathbf{J}_s^i. \tag{5.28}$$

Therefore the vector potential $\mathbf{A}_e$, which describes the solenoidal portion of both $\mathbf{E}$ and $\mathbf{B}$, is found from just the solenoidal portion of the current. On the other hand, the scalar potential, which describes the lamellar portion of $\mathbf{E}$, is found from ρ^i which arises from $\nabla \cdot \mathbf{J}^i$, the lamellar portion of the current.

From the perspective of field computation, we see that the introduction of potential functions has reoriented the solution process from dealing with two coupled first-order partial differential equations (Maxwell's equations), to two uncoupled second-order equations (the potential equations (5.24) and (5.28)). The decoupling of the equations is often worth the added complexity of dealing with potentials, and, in fact, is the solution technique of choice in such areas as radiation and guided waves. It is worth pausing for a moment to examine the form of these equations. We see that the scalar potential obeys Poisson's equation with the solution (5.25), while the vector potential obeys the wave equation. As a wave, the vector potential must propagate away from the source with finite velocity. However, the solution for the scalar potential (5.25) shows no such behavior. In fact, any change to the charge distribution instantaneously permeates all of space. This apparent violation of Einstein's postulate shows that we must be careful when interpreting the physical meaning of the potentials. Once the computations (5.17) and (5.18) are undertaken, we find that both $\mathbf{E}$ and $\mathbf{B}$ behave as waves, and thus propagate at finite velocity. Mathematically, the conundrum can be resolved by realizing that individually the solenoidal and lamellar components of current must occupy all of space, even if their sum, the actual current $\mathbf{J}^i$, is localized [91].

The Lorentz gauge. A different choice of gauge condition can allow both the vector and scalar potentials to act as waves. In this case $\mathbf{E}$ may be written as a sum of two terms: one purely solenoidal, and the other a superposition of lamellar and solenoidal parts.

Let us examine the effect of choosing the *Lorentz gauge*

$$\nabla \cdot \mathbf{A}_e = -\mu\epsilon \frac{\partial \phi_e}{\partial t} - \mu\sigma\phi_e. \tag{5.29}$$

Substituting this expression into (5.26) we find that the gradient terms cancel, giving

$$\nabla^2 \mathbf{A}_e - \mu\sigma \frac{\partial \mathbf{A}_e}{\partial t} - \mu\epsilon \frac{\partial^2 \mathbf{A}_e}{\partial t^2} = -\mu \mathbf{J}^i. \tag{5.30}$$

For lossless media

$$\nabla^2 \mathbf{A}_e - \mu\epsilon \frac{\partial^2 \mathbf{A}_e}{\partial t^2} = -\mu \mathbf{J}^i, \tag{5.31}$$

and (5.23) becomes

$$\nabla^2 \phi_e - \mu\epsilon \frac{\partial^2 \phi_e}{\partial t^2} = -\frac{\rho^i}{\epsilon}. \tag{5.32}$$

For lossy media we have obtained a second-order differential equation for $\mathbf{A}_e$, but ϕ_e must be found through the somewhat cumbersome relation (5.29). For lossless media the coupled Maxwell equations have been decoupled into two second-order equations, one involving $\mathbf{A}_e$ and one involving ϕ_e. Both (5.31) and (5.32) are wave equations, with $\mathbf{J}^i$ as the source for $\mathbf{A}_e$ and ρ^i as the source for ϕ_e. Thus the expected finite-velocity wave nature of the electromagnetic fields is also manifested in each of the potential functions. The drawback is that, even though we can still use (5.17) and (5.18), the expression for $\mathbf{E}$ is no longer a decomposition into solenoidal and lamellar components. Nevertheless, the choice of the Lorentz gauge is very popular in the study of radiated and guided waves.

The Hertzian potentials. With a little manipulation and the introduction of a new notation, we can maintain the wave nature of the potential functions and still provide a decomposition into purely lamellar and solenoidal components. In this analysis we shall assume lossless media only.

When we chose the Lorentz gauge to remove the arbitrariness of the divergence of the vector potential, we established a relationship between $\mathbf{A}_e$ and ϕ_e. Thus we should be able to write both the electric and magnetic fields in terms of a single potential function. From the Lorentz gauge we can write ϕ_e as

$$\phi_e(\mathbf{r}, t) = -\frac{1}{\mu\epsilon}\int_{-\infty}^{t} \nabla \cdot \mathbf{A}_e(\mathbf{r}, t)\, dt.$$

By (5.17) and (5.18) we can thus write the EM fields as

$$\mathbf{E} = \frac{1}{\mu\epsilon}\nabla\int_{-\infty}^{t} \nabla \cdot \mathbf{A}_e dt - \frac{\partial \mathbf{A}_e}{\partial t}, \tag{5.33}$$

$$\mathbf{B} = \nabla \times \mathbf{A}_e. \tag{5.34}$$

The integro-differential representation of $\mathbf{E}$ in (5.33) is somewhat clumsy in appearance. We can make it easier to manipulate by defining the *Hertzian potential*

$$\mathbf{\Pi}_e = \frac{1}{\mu\epsilon}\int_{-\infty}^{t} \mathbf{A}_e\, dt.$$

In differential form

$$\mathbf{A}_e = \mu\epsilon\frac{\partial \mathbf{\Pi}_e}{dt}. \tag{5.35}$$

With this, (5.33) and (5.34) become

$$\mathbf{E} = \nabla(\nabla \cdot \mathbf{\Pi}_e) - \mu\epsilon\frac{\partial^2}{\partial t^2}\mathbf{\Pi}_e, \tag{5.36}$$

$$\mathbf{B} = \mu\epsilon\nabla \times \frac{\partial \mathbf{\Pi}_e}{\partial t}. \tag{5.37}$$

An equation for $\mathbf{\Pi}_e$ in terms of the source current can be found by substituting (5.35) into (5.31):

$$\mu\epsilon\frac{\partial}{\partial t}\left(\nabla^2\mathbf{\Pi}_e - \mu\epsilon\frac{\partial^2}{\partial t^2}\mathbf{\Pi}_e\right) = -\mu\mathbf{J}^i.$$

Let us define

$$\mathbf{J}^i = \frac{\partial \mathbf{P}^i}{\partial t}. \tag{5.38}$$

For general impressed current sources (5.38) is just a convenient notation. However, we can conceive of an *impressed polarization current* that is independent of $\mathbf{E}$ and defined through the relation $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} + \mathbf{P}^i$. Then (5.38) has a physical interpretation as described in (2.119). We now have

$$\nabla^2\mathbf{\Pi}_e - \mu\epsilon\frac{\partial^2}{\partial t^2}\mathbf{\Pi}_e = -\frac{1}{\epsilon}\mathbf{P}^i, \tag{5.39}$$

which is a wave equation for $\mathbf{\Pi}_e$. Thus the Hertzian potential has the same wave behavior as the vector potential under the Lorentz gauge.

We can use (5.39) to perform one final simplification of the EM field representation. By the vector identity $\nabla(\nabla \cdot \mathbf{\Pi}) = \nabla \times (\nabla \times \mathbf{\Pi}) + \nabla^2\mathbf{\Pi}$ we get

$$\nabla(\nabla \cdot \mathbf{\Pi}_e) = \nabla \times (\nabla \times \mathbf{\Pi}_e) - \frac{1}{\epsilon}\mathbf{P}^i + \mu\epsilon\frac{\partial^2}{\partial t^2}\mathbf{\Pi}_e.$$

Substituting this into (5.36) we obtain

$$\mathbf{E} = \nabla \times (\nabla \times \mathbf{\Pi}_e) - \frac{\mathbf{P}^i}{\epsilon}, \tag{5.40}$$

$$\mathbf{B} = \mu\epsilon\nabla \times \frac{\partial \mathbf{\Pi}_e}{\partial t}. \tag{5.41}$$

Let us examine these closely. We know that $\mathbf{B}$ is solenoidal since it is written as the curl of another vector (this is also clear from the auxiliary Maxwell equation $\nabla \cdot \mathbf{B} = 0$). The first term in the expression for $\mathbf{E}$ is also solenoidal. So the lamellar part of $\mathbf{E}$ must be contained within the source term $\mathbf{P}^i$. If we write $\mathbf{P}^i$ in terms of its lamellar and solenoidal components by using

$$\mathbf{J}_s^i = \frac{\partial \mathbf{P}_s^i}{\partial t}, \qquad \mathbf{J}_l^i = \frac{\partial \mathbf{P}_l^i}{\partial t},$$

then (5.40) becomes

$$\mathbf{E} = \left[\nabla \times (\nabla \times \mathbf{\Pi}_e) - \frac{\mathbf{P}_s^i}{\epsilon}\right] - \frac{\mathbf{P}_l^i}{\epsilon}. \tag{5.42}$$

So we have again succeeded in dividing $\mathbf{E}$ into lamellar and solenoidal components.

Potential functions for magnetic current. We can proceed as above to derive the field–potential relationships when $\mathbf{J}^i = 0$ but $\mathbf{J}_m^i \neq 0$. We assume a homogeneous, lossless, isotropic medium with permeability μ and permittivity ϵ, and begin with Faraday's and Ampere's laws

$$\nabla \times \mathbf{E} = -\mathbf{J}_m^i - \frac{\partial \mathbf{B}}{\partial t}, \tag{5.43}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}. \tag{5.44}$$

We write $\mathbf{H}$ and $\mathbf{D}$ in terms of two potential functions $\mathbf{A}_h$ and ϕ_h as

$$\mathbf{H} = -\frac{\partial \mathbf{A}_h}{\partial t} - \nabla\phi_h,$$

$$\mathbf{D} = -\nabla \times \mathbf{A}_h,$$

and the differential equation for the potentials is found by substitution into (5.43):

$$\nabla \times (\nabla \times \mathbf{A}_h) = \epsilon\mathbf{J}_m^i - \mu\epsilon\frac{\partial^2 \mathbf{A}_h}{\partial t^2} - \mu\epsilon\frac{\partial}{\partial t}\nabla\phi_h. \tag{5.45}$$

Taking the divergence of this equation and substituting from the magnetic continuity equation we obtain

$$\mu\epsilon\frac{\partial^2}{\partial t^2}\nabla \cdot \mathbf{A}_h + \mu\epsilon\frac{\partial}{\partial t}\nabla^2\phi_h = -\epsilon\frac{\partial \rho_m^i}{\partial t}.$$

Under the Lorentz gauge condition

$$\nabla \cdot \mathbf{A}_h = -\mu\epsilon \frac{\partial \phi_h}{\partial t}$$

this reduces to

$$\nabla^2 \phi_h - \mu\epsilon \frac{\partial^2 \phi_h}{\partial t^2} = -\frac{\rho_m^i}{\mu}.$$

Expanding the curl-curl operation in (5.45) we have

$$\nabla(\nabla \cdot \mathbf{A}_h) - \nabla^2 \mathbf{A}_h = \epsilon \mathbf{J}_m^i - \mu\epsilon \frac{\partial^2 \mathbf{A}_h}{\partial t^2} - \mu\epsilon \frac{\partial}{\partial t} \nabla \phi_h,$$

which, upon substitution of the Lorentz gauge condition gives

$$\nabla^2 \mathbf{A}_h - \mu\epsilon \frac{\partial^2 \mathbf{A}_h}{\partial t^2} = -\epsilon \mathbf{J}_m^i. \tag{5.46}$$

We can also derive a Hertzian potential for the case of magnetic current. Letting

$$\mathbf{A}_h = \mu\epsilon \frac{\partial \mathbf{\Pi}_h}{\partial t} \tag{5.47}$$

and employing the Lorentz condition we have

$$\mathbf{D} = -\mu\epsilon \nabla \times \frac{\partial \mathbf{\Pi}_h}{\partial t},$$
$$\mathbf{H} = \nabla(\nabla \cdot \mathbf{\Pi}_h) - \mu\epsilon \frac{\partial^2 \mathbf{\Pi}_h}{\partial t^2}.$$

The wave equation for $\mathbf{\Pi}_h$ is found by substituting (5.47) into (5.46) to give

$$\frac{\partial}{\partial t}\left[\nabla^2 \mathbf{\Pi}_h - \mu\epsilon \frac{\partial^2 \mathbf{\Pi}_h}{\partial t^2}\right] = -\frac{1}{\mu}\mathbf{J}_m^i. \tag{5.48}$$

Defining $\mathbf{M}^i$ through

$$\mathbf{J}_m^i = \mu \frac{\partial \mathbf{M}^i}{\partial t},$$

we write the wave equation as

$$\nabla^2 \mathbf{\Pi}_h - \mu\epsilon \frac{\partial^2 \mathbf{\Pi}_h}{\partial t^2} = -\mathbf{M}^i.$$

We can think of $\mathbf{M}^i$ as a convenient way of representing $\mathbf{J}_m^i$, or we can conceive of an *impressed magnetization current* that is independent of $\mathbf{H}$ and defined through $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M} + \mathbf{M}^i)$. With the help of (5.48) we can also write the fields as

$$\mathbf{H} = \nabla \times (\nabla \times \mathbf{\Pi}_h) - \mathbf{M}_i,$$
$$\mathbf{D} = -\mu\epsilon \nabla \times \frac{\partial \mathbf{\Pi}_h}{\partial t}.$$

Summary of potential relations for lossless media. When both electric and magnetic sources are present, we may superpose the potential representations derived above. We assume a homogeneous, lossless medium with time-invariant parameters μ and ϵ. For the scalar/vector potential representation we have

$$\mathbf{E} = -\frac{\partial \mathbf{A}_e}{\partial t} - \nabla\phi_e - \frac{1}{\epsilon}\nabla \times \mathbf{A}_h, \tag{5.49}$$

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A}_e - \frac{\partial \mathbf{A}_h}{\partial t} - \nabla\phi_h. \tag{5.50}$$

Here the potentials satisfy the wave equations

$$\left(\nabla^2 - \mu\epsilon\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix} \mathbf{A}_e \\ \phi_e \end{Bmatrix} = \begin{Bmatrix} -\mu\mathbf{J}^i \\ -\frac{\rho^i}{\epsilon} \end{Bmatrix}, \tag{5.51}$$

$$\left(\nabla^2 - \mu\epsilon\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix} \mathbf{A}_h \\ \phi_h \end{Bmatrix} = \begin{Bmatrix} -\epsilon\mathbf{J}_m^i \\ -\frac{\rho_m^i}{\mu} \end{Bmatrix},$$

and are linked by the Lorentz conditions

$$\nabla \cdot \mathbf{A}_e = -\mu\epsilon\frac{\partial\phi_e}{\partial t},$$

$$\nabla \cdot \mathbf{A}_h = -\mu\epsilon\frac{\partial\phi_h}{\partial t}.$$

We also have the Hertz potential representation

$$\begin{aligned}\mathbf{E} &= \nabla(\nabla\cdot\mathbf{\Pi}_e) - \mu\epsilon\frac{\partial^2\mathbf{\Pi}_e}{\partial t^2} - \mu\nabla\times\frac{\partial\mathbf{\Pi}_h}{\partial t} \\ &= \nabla\times(\nabla\times\mathbf{\Pi}_e) - \frac{\mathbf{P}^i}{\epsilon} - \mu\nabla\times\frac{\partial\mathbf{\Pi}_h}{\partial t},\end{aligned} \tag{5.52}$$

$$\begin{aligned}\mathbf{H} &= \epsilon\nabla\times\frac{\partial\mathbf{\Pi}_e}{\partial t} + \nabla(\nabla\cdot\mathbf{\Pi}_h) - \mu\epsilon\frac{\partial^2\mathbf{\Pi}_h}{\partial t^2} \\ &= \epsilon\nabla\times\frac{\partial\mathbf{\Pi}_e}{\partial t} + \nabla\times(\nabla\times\mathbf{\Pi}_h) - \mathbf{M}_i.\end{aligned} \tag{5.53}$$

The Hertz potentials satisfy the wave equations

$$\left(\nabla^2 - \mu\epsilon\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix} \mathbf{\Pi}_e \\ \mathbf{\Pi}_h \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{\epsilon}\mathbf{P}^i \\ -\mathbf{M}^i \end{Bmatrix}.$$

Potential functions for the frequency-domain fields. In the frequency domain it is much easier to handle lossy media. Consider a lossy, isotropic, homogeneous medium described by the frequency-dependent parameters $\tilde{\mu}$, $\tilde{\epsilon}$, and $\tilde{\sigma}$. Maxwell's curl equations are

$$\nabla \times \tilde{\mathbf{E}} = -\tilde{\mathbf{J}}_m^i - j\omega\tilde{\mu}\tilde{\mathbf{H}}, \tag{5.54}$$

$$\nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}}^i + j\omega\tilde{\epsilon}^c\tilde{\mathbf{E}}. \tag{5.55}$$

Here we have separated the primary and secondary currents through $\tilde{\mathbf{J}} = \tilde{\mathbf{J}}^i + \tilde{\sigma}\tilde{\mathbf{E}}$, and used the complex permittivity $\tilde{\epsilon}^c = \tilde{\epsilon} + \tilde{\sigma}/j\omega$. As with the time-domain equations we introduce the potential functions using superposition. If $\tilde{\mathbf{J}}_m^i = 0$ and $\tilde{\mathbf{J}}^i \neq 0$ then we may introduce the electric potentials through the relationships

$$\tilde{\mathbf{E}} = -\nabla\tilde{\phi}_e - j\omega\tilde{\mathbf{A}}_e, \tag{5.56}$$

$$\tilde{\mathbf{H}} = \frac{1}{\tilde{\mu}}\nabla\times\tilde{\mathbf{A}}_e. \tag{5.57}$$

Assuming the Lorentz condition

$$\nabla\cdot\tilde{\mathbf{A}}_e = -j\omega\tilde{\mu}\tilde{\epsilon}^c\tilde{\phi}_e,$$

we find that upon substitution of (5.56)–(5.57) into (5.54)–(5.55) the potentials must obey the Helmholtz equation

$$\left(\nabla^2 + k^2\right)\begin{Bmatrix}\tilde{\phi}_e\\ \tilde{\mathbf{A}}_e\end{Bmatrix} = \begin{Bmatrix}-\tilde{\rho}^i/\tilde{\epsilon}^c\\ -\tilde{\mu}\tilde{\mathbf{J}}^i\end{Bmatrix}.$$

If $\tilde{\mathbf{J}}_m^i \neq 0$ and $\tilde{\mathbf{J}}^i = 0$ then we may introduce the magnetic potentials through

$$\tilde{\mathbf{E}} = -\frac{1}{\tilde{\epsilon}^c}\nabla\times\tilde{\mathbf{A}}_h, \tag{5.58}$$

$$\tilde{\mathbf{H}} = -\nabla\tilde{\phi}_h - j\omega\tilde{\mathbf{A}}_h. \tag{5.59}$$

Assuming

$$\nabla\cdot\tilde{\mathbf{A}}_h = -j\omega\tilde{\mu}\tilde{\epsilon}^c\tilde{\phi}_h,$$

we find that upon substitution of (5.58)–(5.59) into (5.54)–(5.55) the potentials must obey

$$\left(\nabla^2 + k^2\right)\begin{Bmatrix}\tilde{\phi}_h\\ \tilde{\mathbf{A}}_h\end{Bmatrix} = \begin{Bmatrix}-\tilde{\rho}_m^i/\tilde{\mu}\\ -\tilde{\epsilon}^c\tilde{\mathbf{J}}_m^i\end{Bmatrix}.$$

When both electric and magnetic sources are present, we use superposition:

$$\tilde{\mathbf{E}} = -\nabla\tilde{\phi}_e - j\omega\tilde{\mathbf{A}}_e - \frac{1}{\tilde{\epsilon}^c}\nabla\times\tilde{\mathbf{A}}_h,$$

$$\tilde{\mathbf{H}} = \frac{1}{\tilde{\mu}}\nabla\times\tilde{\mathbf{A}}_e - \nabla\tilde{\phi}_h - j\omega\tilde{\mathbf{A}}_h.$$

Using the Lorentz conditions we can also write the fields in terms of the vector potentials alone:

$$\tilde{\mathbf{E}} = -\frac{j\omega}{k^2}\nabla(\nabla\cdot\tilde{\mathbf{A}}_e) - j\omega\tilde{\mathbf{A}}_e - \frac{1}{\tilde{\epsilon}^c}\nabla\times\tilde{\mathbf{A}}_h, \tag{5.60}$$

$$\tilde{\mathbf{H}} = \frac{1}{\tilde{\mu}}\nabla\times\tilde{\mathbf{A}}_e - \frac{j\omega}{k^2}\nabla(\nabla\cdot\tilde{\mathbf{A}}_h) - j\omega\tilde{\mathbf{A}}_h. \tag{5.61}$$

We can also define Hertzian potentials for the frequency-domain fields. When $\tilde{\mathbf{J}}_m^i = 0$ and $\tilde{\mathbf{J}}^i \neq 0$ we let

$$\tilde{\mathbf{A}}_e = j\omega\tilde{\mu}\tilde{\epsilon}^c\tilde{\mathbf{\Pi}}_e$$

and find

$$\tilde{\mathbf{E}} = \nabla(\nabla\cdot\tilde{\mathbf{\Pi}}_e) + k^2\tilde{\mathbf{\Pi}}_e = \nabla\times(\nabla\times\tilde{\mathbf{\Pi}}_e) - \frac{\tilde{\mathbf{J}}^i}{j\omega\tilde{\epsilon}^c} \tag{5.62}$$

and

$$\tilde{\mathbf{H}} = j\omega\tilde{\epsilon}^c \nabla \times \tilde{\mathbf{\Pi}}_e. \tag{5.63}$$

Here $\tilde{\mathbf{J}}^i$ can represent either an impressed electric current source or an impressed polarization current source $\tilde{\mathbf{J}}^i = j\omega\tilde{\mathbf{P}}^i$. The electric Hertzian potential obeys

$$(\nabla^2 + k^2)\tilde{\mathbf{\Pi}}_e = -\frac{\tilde{\mathbf{J}}^i}{j\omega\tilde{\epsilon}^c}. \tag{5.64}$$

When $\tilde{\mathbf{J}}_m^i \neq 0$ and $\tilde{\mathbf{J}}^i = 0$ we let

$$\tilde{\mathbf{A}}_h = j\omega\tilde{\mu}\tilde{\epsilon}^c\tilde{\mathbf{\Pi}}_h$$

and find

$$\tilde{\mathbf{E}} = -j\omega\tilde{\mu}\nabla \times \tilde{\mathbf{\Pi}}_h \tag{5.65}$$

and

$$\tilde{\mathbf{H}} = \nabla(\nabla \cdot \tilde{\mathbf{\Pi}}_h) + k^2\tilde{\mathbf{\Pi}}_h = \nabla \times (\nabla \times \tilde{\mathbf{\Pi}}_h) - \frac{\tilde{\mathbf{J}}_m^i}{j\omega\tilde{\mu}}. \tag{5.66}$$

Here $\tilde{\mathbf{J}}_m^i$ can represent either an impressed magnetic current source or an impressed magnetization current source $\tilde{\mathbf{J}}_m^i = j\omega\tilde{\mu}\tilde{\mathbf{M}}^i$. The magnetic Hertzian potential obeys

$$(\nabla^2 + k^2)\tilde{\mathbf{\Pi}}_h = -\frac{\tilde{\mathbf{J}}_m^i}{j\omega\tilde{\mu}}. \tag{5.67}$$

When both electric and magnetic sources are present we have by superposition

$$\begin{aligned}\tilde{\mathbf{E}} &= \nabla(\nabla \cdot \tilde{\mathbf{\Pi}}_e) + k^2\tilde{\mathbf{\Pi}}_e - j\omega\tilde{\mu}\nabla \times \tilde{\mathbf{\Pi}}_h \\ &= \nabla \times (\nabla \times \tilde{\mathbf{\Pi}}_e) - \frac{\tilde{\mathbf{J}}^i}{j\omega\tilde{\epsilon}^c} - j\omega\tilde{\mu}\nabla \times \tilde{\mathbf{\Pi}}_h\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathbf{H}} &= j\omega\tilde{\epsilon}^c\nabla \times \tilde{\mathbf{\Pi}}_e + \nabla(\nabla \cdot \tilde{\mathbf{\Pi}}_h) + k^2\tilde{\mathbf{\Pi}}_h \\ &= j\omega\tilde{\epsilon}^c\nabla \times \tilde{\mathbf{\Pi}}_e + \nabla \times (\nabla \times \tilde{\mathbf{\Pi}}_h) - \frac{\tilde{\mathbf{J}}_m^i}{j\omega\tilde{\mu}}.\end{aligned}$$

5.2.1 Solution for potentials in an unbounded medium: the retarded potentials

Under the Lorentz condition each of the potential functions obeys the wave equation. This equation can be solved using the method of Green's functions to determine the potentials, and the electromagnetic fields can therefore be determined. We now examine the solution for an unbounded medium. Solutions for bounded regions are considered in § 5.2.2.

Consider a linear operator $\mathcal{L}$ that operates on a function of $\mathbf{r}$ and t. If we wish to solve the equation

$$\mathcal{L}\{\psi(\mathbf{r}, t)\} = S(\mathbf{r}, t), \tag{5.68}$$

we first solve

$$\mathcal{L}\{G(\mathbf{r}, t|\mathbf{r}', t')\} = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$$

and determine the Green's function G for the operator $\mathcal{L}$. Provided that S resides within V we have

$$\begin{aligned}\mathcal{L}\left\{\int_V\int_{-\infty}^{\infty} S(\mathbf{r}',t')G(\mathbf{r},t|\mathbf{r}',t')\,dt'\,dV'\right\} &= \int_V\int_{-\infty}^{\infty} S(\mathbf{r}',t')\,\mathcal{L}\{G(\mathbf{r},t|\mathbf{r}',t')\}\,dt'\,dV' \\ &= \int_V\int_{-\infty}^{\infty} S(\mathbf{r}',t')\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\,dt'\,dV' \\ &= S(\mathbf{r},t),\end{aligned}$$

hence

$$\psi(\mathbf{r},t) = \int_V\int_{-\infty}^{\infty} S(\mathbf{r}',t')G(\mathbf{r},t|\mathbf{r}',t')\,dt'\,dV' \tag{5.69}$$

by comparison with (5.68).

We can also apply this idea in the frequency domain. The solution to

$$\mathcal{L}\{\tilde{\psi}(\mathbf{r},\omega)\} = \tilde{S}(\mathbf{r},\omega) \tag{5.70}$$

is

$$\tilde{\psi}(\mathbf{r},\omega) = \int_V \tilde{S}(\mathbf{r}',\omega)G(\mathbf{r}|\mathbf{r}';\omega)\,dV'$$

where the Green's function G satisfies

$$\mathcal{L}\{G(\mathbf{r}|\mathbf{r}';\omega)\} = \delta(\mathbf{r}-\mathbf{r}').$$

Equation (5.69) is the basic superposition integral that allows us to find the potentials in an infinite, unbounded medium. We note that if the medium is bounded then we must use Green's theorem to include the effects of sources that reside external to the boundaries. These are manifested in terms of the values of the potentials on the boundaries in the same manner as with the static potentials in Chapter 3. In order to determine whether (5.69) is the unique solution to the wave equation, we must also examine the behavior of the fields on the boundary as the boundary recedes to infinity. In the frequency domain we find that an additional "radiation condition" is required to ensure uniqueness.

The retarded potentials in the time domain. Consider an unbounded, homogeneous, lossy, isotropic medium described by parameters μ, ϵ, σ. In the time domain the vector potential $\mathbf{A}_e$ satisfies (5.30). The scalar components of $\mathbf{A}_e$ must obey

$$\nabla^2 A_{e,n}(\mathbf{r},t) - \mu\sigma\frac{\partial A_{e,n}(\mathbf{r},t)}{\partial t} - \mu\epsilon\frac{\partial^2 A_{e,n}(\mathbf{r},t)}{\partial t^2} = -\mu J_n^i(\mathbf{r},t), \qquad n = x,y,z.$$

We may write this in the form

$$\left(\nabla^2 - \frac{2\Omega}{v^2}\frac{\partial}{\partial t} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\psi(\mathbf{r},t) = -S(\mathbf{r},t) \tag{5.71}$$

where $\psi = A_{e,n}$, $v^2 = 1/\mu\epsilon$, $\Omega = \sigma/2\epsilon$, and $S = \mu J_n^i$. The solution is

$$\psi(\mathbf{r},t) = \int_V\int_{-\infty}^{\infty} S(\mathbf{r}',t')G(\mathbf{r},t|\mathbf{r}',t')\,dt'\,dV' \tag{5.72}$$

where G satisfies

$$\left(\nabla^2 - \frac{2\Omega}{v^2}\frac{\partial}{\partial t} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}, t|\mathbf{r}', t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{5.73}$$

In § A.1 we find that

$$G(\mathbf{r}, t|\mathbf{r}', t') = e^{-\Omega(t-t')}\frac{\delta(t-t'-R/v)}{4\pi R} + \\ + \frac{\Omega^2}{4\pi v}e^{-\Omega(t-t')}\frac{I_1\left(\Omega\sqrt{(t-t')^2-(R/v)^2}\right)}{\Omega\sqrt{(t-t')^2-(R/v)^2}}, \quad t-t' > \frac{R}{v},$$

where $R = |\mathbf{r}-\mathbf{r}'|$. For lossless media where $\sigma = 0$ this becomes

$$G(\mathbf{r}, t|\mathbf{r}', t') = \frac{\delta(t-t'-R/v)}{4\pi R}$$

and thus

$$\psi(\mathbf{r}, t) = \int_V \int_{-\infty}^{\infty} S(\mathbf{r}', t')\frac{\delta(t-t'-R/v)}{4\pi R}\, dt'\, dV' \\ = \int_V \frac{S(\mathbf{r}', t-R/v)}{4\pi R}\, dV'. \tag{5.74}$$

For lossless media, the scalar potentials and all rectangular components of the vector potentials obey the same wave equation. Thus we have, for instance, the solutions to (5.51):

$$\mathbf{A}_e(\mathbf{r}, t) = \frac{\mu}{4\pi}\int_V \frac{\mathbf{J}^i(\mathbf{r}', t-R/v)}{R}\, dV',$$
$$\phi_e(\mathbf{r}, t) = \frac{1}{4\pi\epsilon}\int_V \frac{\rho^i(\mathbf{r}', t-R/v)}{R}\, dV'.$$

These are called the *retarded potentials* since their values at time t are determined by the values of the sources at an earlier (or retardation) time $t - R/v$. The retardation time is determined by the propagation velocity v of the potential waves.

The fields are determined by the potentials:

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\frac{1}{4\pi\epsilon}\int_V \frac{\rho^i(\mathbf{r}', t-R/v)}{R}\, dV' - \frac{\partial}{\partial t}\frac{\mu}{4\pi}\int_V \frac{\mathbf{J}^i(\mathbf{r}', t-R/v)}{R}\, dV',$$
$$\mathbf{H}(\mathbf{r}, t) = \nabla\times\frac{1}{4\pi}\int_V \frac{\mathbf{J}^i(\mathbf{r}', t-R/v)}{R}\, dV'.$$

The derivatives may be brought inside the integrals, but some care must be taken when the observation point $\mathbf{r}$ lies within the source region. In this case the integrals must be performed in a principal value sense by excluding a small volume around the observation point. We discuss this in more detail below for the frequency-domain fields. For details regarding this procedure in the time domain the reader may see Hansen [81].

The retarded potentials in the frequency domain. Consider an unbounded, homogeneous, isotropic medium described by $\tilde{\mu}(\omega)$ and $\tilde{\epsilon}^c(\omega)$. If $\tilde{\psi}(\mathbf{r}, \omega)$ represents a scalar potential or any rectangular component of a vector or Hertzian potential then it must satisfy

$$(\nabla^2 + k^2)\tilde{\psi}(\mathbf{r}, \omega) = -\tilde{S}(\mathbf{r}, \omega) \tag{5.75}$$

where $k = \omega(\tilde{\mu}\tilde{\epsilon}^c)^{1/2}$. This Helmholtz equation has the form of (5.70) and thus

$$\tilde{\psi}(\mathbf{r}, \omega) = \int_V \tilde{S}(\mathbf{r}', \omega) G(\mathbf{r}|\mathbf{r}'; \omega)\, dV'$$

where

$$(\nabla^2 + k^2) G(\mathbf{r}|\mathbf{r}'; \omega) = -\delta(\mathbf{r} - \mathbf{r}'). \tag{5.76}$$

This is equation (A.46) and its solution, as given by (A.49), is

$$G(\mathbf{r}|\mathbf{r}'; \omega) = \frac{e^{-jkR}}{4\pi R}. \tag{5.77}$$

Here we use $v^2 = 1/\tilde{\mu}\tilde{\epsilon}$ and $\Omega = \tilde{\sigma}/2\epsilon$ in (A.47):

$$k = \frac{1}{v}\sqrt{\omega^2 - j2\omega\Omega} = \omega\sqrt{\tilde{\mu}\left(\tilde{\epsilon} - j\frac{\tilde{\sigma}}{\omega}\right)} = \omega\sqrt{\tilde{\mu}\tilde{\epsilon}^c}.$$

The solution to (5.75) is therefore

$$\tilde{\psi}(\mathbf{r}, \omega) = \int_V \tilde{S}(\mathbf{r}', \omega) \frac{e^{-jkR}}{4\pi R}\, dV'. \tag{5.78}$$

When the medium is lossless, the potential must also satisfy the *radiation condition*

$$\lim_{r\to\infty} r\left(\frac{\partial}{\partial r} + jk\right)\tilde{\psi}(\mathbf{r}) = 0 \tag{5.79}$$

to guarantee uniqueness of solution. In § 5.2.2 we shall show how this requirement arises from the solution within a bounded region. For a uniqueness proof for the Helmholtz equation, the reader may consult Chew [33].

We may use (5.78) to find that

$$\tilde{\mathbf{A}}_e(\mathbf{r}, \omega) = \frac{\tilde{\mu}}{4\pi}\int_V \tilde{\mathbf{J}}^i(\mathbf{r}', \omega)\frac{e^{-jkR}}{R}\, dV'. \tag{5.80}$$

Comparison with (5.74) shows that in the frequency domain, time retardation takes the form of a phase shift. Similarly,

$$\tilde{\phi}(\mathbf{r}, \omega) = \frac{1}{4\pi\tilde{\epsilon}^c}\int_V \tilde{\rho}^i(\mathbf{r}', \omega)\frac{e^{-jkR}}{R}\, dV'. \tag{5.81}$$

The electric and magnetic dyadic Green's functions. The frequency-domain electromagnetic fields may be found for electric sources from the electric vector potential using (5.60) and (5.61):

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = -j\omega\tilde{\mu}(\omega)\int_V \tilde{\mathbf{J}}^i(\mathbf{r}', \omega) G(\mathbf{r}|\mathbf{r}'; \omega)\, dV' - \frac{j\omega\tilde{\mu}(\omega)}{k^2}\nabla\nabla\cdot\int_V \tilde{\mathbf{J}}^i(\mathbf{r}', \omega) G(\mathbf{r}|\mathbf{r}'; \omega)\, dV',$$

$$\tilde{\mathbf{H}} = \nabla\times\int_V \tilde{\mathbf{J}}^i(\mathbf{r}', \omega) G(\mathbf{r}|\mathbf{r}'; \omega)\, dV'. \tag{5.82}$$

As long as the observation point $\mathbf{r}$ does not lie within the source region we may take the derivatives inside the integrals. Using

$$\nabla \cdot \left[\tilde{\mathbf{J}}^i(\mathbf{r}', \omega)G(\mathbf{r}|\mathbf{r}'; \omega)\right] = \tilde{\mathbf{J}}^i(\mathbf{r}', \omega) \cdot \nabla G(\mathbf{r}|\mathbf{r}'; \omega) + G(\mathbf{r}|\mathbf{r}'; \omega)\nabla \cdot \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$
$$= \nabla G(\mathbf{r}|\mathbf{r}'; \omega) \cdot \tilde{\mathbf{J}}^i(\mathbf{r}', \omega)$$

we have

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = -j\omega\tilde{\mu}(\omega) \int_V \left\{\tilde{\mathbf{J}}^i(\mathbf{r}', \omega)G(\mathbf{r}|\mathbf{r}'; \omega) + \frac{1}{k^2}\nabla\left[\nabla G(\mathbf{r}|\mathbf{r}'; \omega) \cdot \mathbf{J}^i(\mathbf{r}', \omega)\right]\right\} dV'.$$

This can be written more compactly as

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = -j\omega\tilde{\mu}(\omega) \int_V \bar{\mathbf{G}}_e(\mathbf{r}|\mathbf{r}'; \omega) \cdot \tilde{\mathbf{J}}^i(\mathbf{r}', \omega)\, dV'$$

where

$$\bar{\mathbf{G}}_e(\mathbf{r}|\mathbf{r}'; \omega) = \left[\bar{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right] G(\mathbf{r}|\mathbf{r}'; \omega) \tag{5.83}$$

is called the *electric dyadic Green's function*. Using

$$\nabla \times [\tilde{\mathbf{J}}^i G] = \nabla G \times \tilde{\mathbf{J}}^i + G\nabla \times \tilde{\mathbf{J}}^i = \nabla G \times \tilde{\mathbf{J}}^i$$

we have for the magnetic field

$$\tilde{\mathbf{H}}(\mathbf{r}, \omega) = \int_V \nabla G(\mathbf{r}|\mathbf{r}'; \omega) \times \tilde{\mathbf{J}}^i(\mathbf{r}', \omega)\, dV'.$$

Now, using the dyadic identity (B.15) we may show that

$$\tilde{\mathbf{J}}^i \times \nabla G = (\tilde{\mathbf{J}}^i \times \nabla G) \cdot \bar{\mathbf{I}} = (\nabla G \times \bar{\mathbf{I}}) \cdot \mathbf{J}^i.$$

So

$$\tilde{\mathbf{H}}(\mathbf{r}, \omega) = -\int_V \bar{\mathbf{G}}_m(\mathbf{r}|\mathbf{r}'; \omega) \cdot \tilde{\mathbf{J}}^i(\mathbf{r}', \omega)\, dV'$$

where

$$\bar{\mathbf{G}}_m(\mathbf{r}|\mathbf{r}'; \omega) = \nabla G(\mathbf{r}|\mathbf{r}'; \omega) \times \bar{\mathbf{I}} \tag{5.84}$$

is called the *magnetic dyadic Green's function*.

Proceeding similarly for magnetic sources (or using duality) we have

$$\tilde{\mathbf{H}}(\mathbf{r}) = -j\omega\tilde{\epsilon}^c \int_V \bar{\mathbf{G}}_e(\mathbf{r}|\mathbf{r}'; \omega) \cdot \tilde{\mathbf{J}}^i_m(\mathbf{r}', \omega)\, dV',$$
$$\tilde{\mathbf{E}}(\mathbf{r}) = \int_V \bar{\mathbf{G}}_m(\mathbf{r}|\mathbf{r}'; \omega) \cdot \tilde{\mathbf{J}}^i_m(\mathbf{r}', \omega)\, dV'.$$

When both electric and magnetic sources are present we simply use superposition and add the fields.

When the observation point lies within the source region, we must be much more careful about how we formulate the dyadic Green's functions. In (5.82) we encounter the integral

$$\int_V \tilde{\mathbf{J}}^i(\mathbf{r}', \omega)G(\mathbf{r}|\mathbf{r}'; \omega)\, dV'.$$

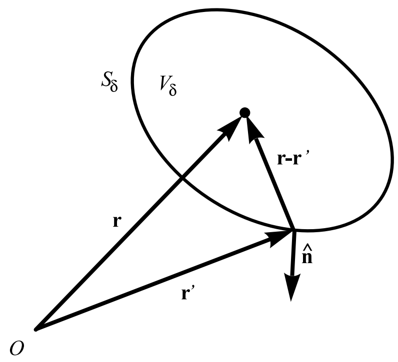

Figure 5.1: Geometry of excluded region used to compute the electric field within a source region.

If $\mathbf{r}$ lies within the source region then G is singular since $R \to 0$ when $\mathbf{r} \to \mathbf{r}'$. However, the integral converges and the potentials exist within the source region. While we run into trouble when we pass both derivatives in the operator $\nabla\nabla\cdot$ through the integral and allow them to operate on G, since differentiation of G increases the order of the singularity, we may safely take one derivative of G.

Even when we allow one derivative on G we must be careful in how we compute the integral. We exclude the point $\mathbf{r}$ by surrounding it with a small volume element V_δ as shown in Figure 5.1 and write

$$\nabla\nabla \cdot \int_V \tilde{\mathbf{J}}^i(\mathbf{r}',\omega) G(\mathbf{r}|\mathbf{r}';\omega)\, dV' =$$
$$\lim_{V_\delta \to 0} \int_{V - V_\delta} \nabla \left[\nabla G(\mathbf{r}|\mathbf{r}';\omega) \cdot \tilde{\mathbf{J}}^i(\mathbf{r}',\omega)\right] dV' + \lim_{V_\delta \to 0} \nabla \int_{V_\delta} \nabla G(\mathbf{r}|\mathbf{r}';\omega) \cdot \tilde{\mathbf{J}}^i(\mathbf{r}',\omega)\, dV'.$$

The first integral on the right-hand side is called the *principal value integral* and is usually abbreviated

$$\text{P.V.} \int_V \nabla \left[\nabla G(\mathbf{r}|\mathbf{r}';\omega) \cdot \tilde{\mathbf{J}}^i(\mathbf{r}',\omega)\right] dV'.$$

It converges to a value dependent on the shape of the excluded region V_δ, as does the second integral. However, the sum of these two integrals produces a unique result. Using $\nabla G = -\nabla' G$, the identity $\nabla' \cdot (\tilde{\mathbf{J}} G) = \tilde{\mathbf{J}} \cdot \nabla' G + G \nabla' \cdot \tilde{\mathbf{J}}$, and the divergence theorem, we can write

$$-\int_{V_\delta} \nabla' G(\mathbf{r}|\mathbf{r}';\omega) \cdot \tilde{\mathbf{J}}^i(\mathbf{r}',\omega)\, dV' =$$
$$-\oint_{S_\delta} G(\mathbf{r}|\mathbf{r}';\omega) \tilde{\mathbf{J}}^i(\mathbf{r}',\omega) \cdot \hat{\mathbf{n}}'\, dS' + \int_{V_\delta} G(\mathbf{r}|\mathbf{r}';\omega) \nabla' \cdot \tilde{\mathbf{J}}^i(\mathbf{r}',\omega)\, dV'$$

where S_δ is the surface surrounding V_δ. By the continuity equation the second integral on the right-hand side is proportional to the scalar potential produced by the charge within V_δ, and thus vanishes as $V_\delta \to 0$. The first term is proportional to the field at $\mathbf{r}$ produced by surface charge on S_δ, which results in a value proportional to $\mathbf{J}^i$. Thus

$$\lim_{V_\delta \to 0} \nabla \int_{V_\delta} \nabla G(\mathbf{r}|\mathbf{r}';\omega) \cdot \tilde{\mathbf{J}}^i(\mathbf{r}',\omega)\, dV' = -\lim_{V_\delta \to 0} \nabla \oint_{S_\delta} G(\mathbf{r}|\mathbf{r}';\omega) \tilde{\mathbf{J}}^i(\mathbf{r}',\omega) \cdot \hat{\mathbf{n}}'\, dS'$$
$$= -\bar{\mathbf{L}} \cdot \tilde{\mathbf{J}}^i(\mathbf{r},\omega), \qquad (5.85)$$

so

$$\nabla\nabla\cdot\int_V \tilde{\mathbf{J}}^i(\mathbf{r}',\omega)G(\mathbf{r}|\mathbf{r}';\omega)\,dV' = P.V.\int_V \nabla\left[\nabla G(\mathbf{r}|\mathbf{r}';\omega)\cdot\tilde{\mathbf{J}}^i(\mathbf{r}',\omega)\right]dV' - \bar{\mathbf{L}}\cdot\tilde{\mathbf{J}}^i(\mathbf{r},\omega).$$

Here $\bar{\mathbf{L}}$ is usually called the *depolarizing dyadic* [113]. Its value depends on the shape of V_δ, as considered below.

We may now write

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = -j\omega\tilde{\mu}(\omega)\,\text{P.V.}\int_V \bar{\mathbf{G}}_e(\mathbf{r}|\mathbf{r}';\omega)\cdot\tilde{\mathbf{J}}(\mathbf{r}',\omega)\,dV' - \frac{1}{j\omega\tilde{\epsilon}^c(\omega)}\bar{\mathbf{L}}\cdot\tilde{\mathbf{J}}^i(\mathbf{r},\omega). \tag{5.86}$$

We may also incorporate both terms into a single dyadic Green's function using the notation

$$\bar{\mathbf{G}}(\mathbf{r}|\mathbf{r}';\omega) = \text{P.V.}\,\bar{\mathbf{G}}_e(\mathbf{r}|\mathbf{r}';\omega) - \frac{1}{k^2}\bar{\mathbf{L}}\delta(\mathbf{r}-\mathbf{r}').$$

Hence when we compute

$$\begin{aligned}\tilde{\mathbf{E}}(\mathbf{r},\omega) &= -j\omega\tilde{\mu}(\omega)\int_V \bar{\mathbf{G}}(\mathbf{r}|\mathbf{r}';\omega)\cdot\tilde{\mathbf{J}}^i(\mathbf{r}',\omega)\,dV' \\ &= -j\omega\tilde{\mu}(\omega)\int_V\left[\text{P.V.}\,\bar{\mathbf{G}}_e(\mathbf{r}|\mathbf{r}';\omega) - \frac{1}{k^2}\bar{\mathbf{L}}\delta(\mathbf{r}-\mathbf{r}')\right]\cdot\tilde{\mathbf{J}}^i(\mathbf{r}',\omega)\,dV'\end{aligned}$$

we reproduce (5.86). That is, the symbol P.V. on G_e indicates that a principal value integral must be performed.

Our final task is to compute $\bar{\mathbf{L}}$ from (5.85). When we remove the excluded region from the principal value computation we leave behind a hole in the source region. The contribution to the field at $\mathbf{r}$ by the sources in the excluded region is found from the scalar potential produced by the surface distribution $\hat{\mathbf{n}}\cdot\mathbf{J}^i$. The value of this *correction term* depends on the shape of the excluding volume. However, the correction term always adds to the principal value integral to give the true field at $\mathbf{r}$, regardless of the shape of the volume. So we must always match the shape of the excluded region used to compute the principal value integral with that used to compute the correction term so that the true field is obtained. Note that as $V_\delta \to 0$ the phase factor in the Green's function becomes insignificant, and the values of the current on the surface approach the value at $\mathbf{r}$ (assuming $\mathbf{J}^i$ is continuous at $\mathbf{r}$). Thus we may write

$$\lim_{V_\delta\to 0}\nabla\oint_{S_\delta}\frac{\tilde{\mathbf{J}}^i(\mathbf{r},\omega)\cdot\hat{\mathbf{n}}'}{4\pi|\mathbf{r}-\mathbf{r}'|}\,dS' = \bar{\mathbf{L}}\cdot\tilde{\mathbf{J}}^i(\mathbf{r},\omega).$$

This has the form of a static field integral. For a spherical excluded region we may compute the above quantity quite simply by assuming the current to be uniform throughout V_δ and by aligning the current with the z-axis and placing the center of the sphere at the origin. We then compute the integral at a point $\mathbf{r}$ within the sphere, take the gradient, and allow $\mathbf{r}\to 0$. We thus have for a sphere

$$\lim_{V_\delta\to 0}\nabla\oint_S\frac{\tilde{J}^i\cos\theta'}{4\pi|\mathbf{r}-\mathbf{r}'|}\,dS' = \bar{\mathbf{L}}\cdot[\hat{\mathbf{z}}\tilde{J}^i(\mathbf{r},\omega)].$$

This integral has been computed in § 3.2.7 with the result given by (3.103). Using this we find

$$\lim_{V_\delta\to 0}\left[\nabla\left(\frac{1}{3}\tilde{J}^i z\right)\right]\Bigg|_{\mathbf{r}=0} = \hat{\mathbf{z}}\frac{\tilde{J}^i}{3} = \bar{\mathbf{L}}\cdot[\hat{\mathbf{z}}\tilde{J}^i(\mathbf{r},\omega)]$$

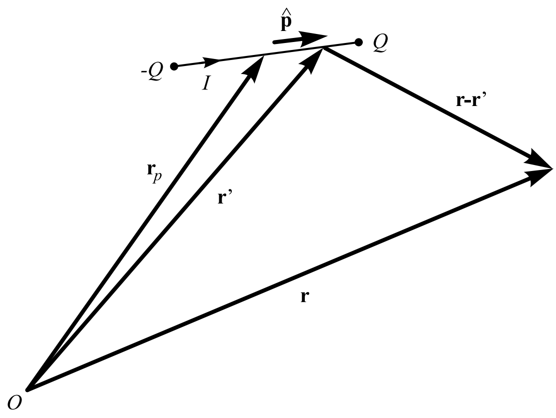

Figure 5.2: Geometry of an electric Hertzian dipole.

and thus

$$\bar{\mathbf{L}} = \frac{1}{3}\bar{\mathbf{I}}.$$

We leave it as an exercise to show that for a cubical excluding volume the depolarizing dyadic is also $\bar{\mathbf{L}} = \bar{\mathbf{I}}/3$. Values for other shapes may be found in Yaghjian [215].

The theory of dyadic Green's functions is well developed and there exist techniques for their construction under a variety of conditions. For an excellent overview the reader may see Tai [192].

Example of field calculation using potentials: the Hertzian dipole. Consider a short line current of length $l \ll \lambda$ at position $\mathbf{r}_p$, oriented along a direction $\hat{\mathbf{p}}$ in a medium with constitutive parameters $\tilde{\mu}(\omega)$, $\tilde{\epsilon}^c(\omega)$, as shown in Figure 5.2. We assume that the frequency-domain current $\tilde{I}(\omega)$ is independent of position, and therefore this *Hertzian dipole* must be terminated by point charges

$$\tilde{Q}(\omega) = \pm\frac{\tilde{I}(\omega)}{j\omega}$$

as required by the continuity equation. The electric vector potential produced by this short current element is

$$\tilde{\mathbf{A}}_e = \frac{\tilde{\mu}}{4\pi}\int_\Gamma \tilde{I}\hat{\mathbf{p}}\frac{e^{-jkR}}{R}\,dl'.$$

At observation points far from the dipole (compared to its length) such that $|\mathbf{r}-\mathbf{r}_p| \gg l$ we may approximate

$$\frac{e^{-jkR}}{R} \approx \frac{e^{-jk|\mathbf{r}-\mathbf{r}_p|}}{|\mathbf{r}-\mathbf{r}_p|}.$$

Then

$$\tilde{\mathbf{A}}_e = \hat{\mathbf{p}}\tilde{\mu}\tilde{I}G(\mathbf{r}|\mathbf{r}_p;\omega)\int_\Gamma dl' = \hat{\mathbf{p}}\tilde{\mu}\tilde{I}lG(\mathbf{r}|\mathbf{r}_p;\omega). \tag{5.87}$$

Note that we obtain the same answer if we let the current density of the dipole be

$$\tilde{\mathbf{J}} = j\omega\tilde{\mathbf{p}}\delta(\mathbf{r}-\mathbf{r}_p)$$

where $\tilde{\mathbf{p}}$ is the *dipole moment* defined by

$$\tilde{\mathbf{p}} = \tilde{Q}l\hat{\mathbf{p}} = \frac{\tilde{I}l}{j\omega}\hat{\mathbf{p}}.$$

That is, we consider a Hertzian dipole to be a "point source" of electromagnetic radiation. With this notation we have

$$\tilde{\mathbf{A}}_e = \tilde{\mu}\int_V \left[j\omega\tilde{\mathbf{p}}\delta(\mathbf{r}' - \mathbf{r}_p)\right] G(\mathbf{r}|\mathbf{r}';\omega)\, dV' = j\omega\tilde{\mu}\tilde{\mathbf{p}}G(\mathbf{r}|\mathbf{r}_p;\omega),$$

which is identical to (5.87). The electromagnetic fields are then

$$\tilde{\mathbf{H}}(\mathbf{r},\omega) = j\omega\nabla\times[\tilde{\mathbf{p}}G(\mathbf{r}|\mathbf{r}_p;\omega)], \tag{5.88}$$

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = \frac{1}{\tilde{\epsilon}^c}\nabla\times\nabla\times[\tilde{\mathbf{p}}G(\mathbf{r}|\mathbf{r}_p;\omega)]. \tag{5.89}$$

Here we have obtained $\tilde{\mathbf{E}}$ from $\tilde{\mathbf{H}}$ outside the source region by applying Ampere's law. By duality we may obtain the fields produced by a magnetic Hertzian dipole of moment

$$\tilde{\mathbf{p}}_m = \frac{\tilde{I}_m l}{j\omega}\hat{\mathbf{p}}$$

located at $\mathbf{r} = \mathbf{r}_p$ as

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = -j\omega\nabla\times[\tilde{\mathbf{p}}_m G(\mathbf{r}|\mathbf{r}_p;\omega)],$$

$$\tilde{\mathbf{H}}(\mathbf{r},\omega) = \frac{1}{\tilde{\mu}}\nabla\times\nabla\times[\tilde{\mathbf{p}}_m G(\mathbf{r}|\mathbf{r}_p;\omega)].$$

We can learn much about the fields produced by localized sources by considering the simple case of a Hertzian dipole aligned along the z-axis and centered at the origin. Using $\hat{\mathbf{p}} = \hat{\mathbf{z}}$ and $\mathbf{r}_p = 0$ in (5.88) we find that

$$\tilde{\mathbf{H}}(\mathbf{r},\omega) = j\omega\nabla\times\left[\hat{\mathbf{z}}\frac{\tilde{I}}{j\omega}l\frac{e^{-jkr}}{4\pi r}\right] = \hat{\boldsymbol{\phi}}\frac{1}{4\pi}\tilde{I}l\left[\frac{1}{r^2} + j\frac{k}{r}\right]\sin\theta e^{-jkr}. \tag{5.90}$$

By Ampere's law

$$\begin{aligned}\tilde{\mathbf{E}}(\mathbf{r},\omega) &= \frac{1}{j\omega\tilde{\epsilon}^c}\nabla\times\tilde{\mathbf{H}}(\mathbf{r},\omega)\\ &= \hat{\mathbf{r}}\frac{\eta}{4\pi}\tilde{I}l\left[\frac{2}{r^2} - j\frac{2}{kr^3}\right]\cos\theta e^{-jkr} + \hat{\boldsymbol{\theta}}\frac{\eta}{4\pi}\tilde{I}l\left[j\frac{k}{r} + \frac{1}{r^2} - j\frac{1}{kr^3}\right]\sin\theta e^{-jkr}.\end{aligned} \tag{5.91}$$

The fields involve various inverse powers of r, with the $1/r$ and $1/r^3$ terms 90° out-of-phase from the $1/r^2$ term. Some terms dominate the field close to the source, while others dominate far away. The terms that dominate near the source[1] are called the *near-zone* or *induction-zone fields*:

$$\tilde{\mathbf{H}}^{NZ}(\mathbf{r},\omega) = \hat{\boldsymbol{\phi}}\frac{\tilde{I}l}{4\pi}\frac{e^{-jkr}}{r^2}\sin\theta,$$

$$\tilde{\mathbf{E}}^{NZ}(\mathbf{r},\omega) = -j\eta\frac{\tilde{I}l}{4\pi}\frac{e^{-jkr}}{kr^3}\left[2\hat{\mathbf{r}}\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right].$$

[1]Note that we still require $r \gg l$.

We note that $\tilde{\mathbf{H}}^{NZ}$ and $\tilde{\mathbf{E}}^{NZ}$ are 90° out-of-phase. Also, the electric field has the same spatial dependence as the field of a static electric dipole. The terms that dominate far from the source are called the *far-zone* or *radiation fields*:

$$\tilde{\mathbf{H}}^{FZ}(\mathbf{r},\omega) = \hat{\boldsymbol{\phi}}\frac{jk\tilde{I}l}{4\pi}\frac{e^{-jkr}}{r}\sin\theta, \tag{5.92}$$

$$\tilde{\mathbf{E}}^{FZ}(\mathbf{r},\omega) = \hat{\boldsymbol{\theta}}\eta\frac{jk\tilde{I}l}{4\pi}\frac{e^{-jkr}}{r}\sin\theta. \tag{5.93}$$

The far-zone fields are in-phase and in fact form a TEM spherical wave with

$$\tilde{\mathbf{H}}^{FZ} = \frac{\hat{\mathbf{r}}\times\tilde{\mathbf{E}}^{FZ}}{\eta}. \tag{5.94}$$

We speak of the time-average power *radiated* by a time-harmonic source as the integral of the time-average power density over a very large sphere. Thus *radiated power* is the power delivered by the sources to infinity. If the dipole is situated within a lossy medium, all of the time-average power delivered by the sources is dissipated by the medium. If the medium is lossless then all the time-average power is delivered to infinity. Let us compute the power radiated by a time-harmonic Hertzian dipole immersed in a lossless medium. Writing (5.90) and (5.91) in terms of phasors we have the complex Poynting vector

$$\begin{aligned}\mathbf{S}^c(\mathbf{r}) &= \check{\mathbf{E}}(\mathbf{r})\times\check{\mathbf{H}}^*(\mathbf{r})\\ &= \hat{\boldsymbol{\theta}}\eta\left(\frac{|\check{I}|l}{4\pi}\right)^2 j\frac{2}{kr^5}\left[k^2r^2+1\right]\cos\theta\sin\theta + \hat{\mathbf{r}}\eta\left(\frac{|\check{I}|l}{4\pi}\right)^2\frac{k^2}{r^2}\left[1-j\frac{1}{k^3r^5}\right]\sin^2\theta.\end{aligned}$$

We notice that the θ-component of $\mathbf{S}^c$ is purely imaginary and gives rise to no time-average power flux. This component falls off as $1/r^3$ for large r and produces no net flux through a sphere with radius $r\to\infty$. Additionally, the angular variation $\sin\theta\cos\theta$ integrates to zero over a sphere. In contrast, the r-component has a real part that varies as $1/r^2$ and as $\sin^2\theta$. Hence we find that the total time-average power passing through a sphere expanding to infinity is nonzero:

$$\begin{aligned}P_{av} &= \lim_{r\to\infty}\int_0^{2\pi}\int_0^{\pi}\frac{1}{2}\,\mathrm{Re}\left\{\hat{\mathbf{r}}\eta\left(\frac{|\check{I}|l}{4\pi}\right)^2\frac{k^2}{r^2}\sin^2\theta\right\}\cdot\hat{\mathbf{r}}r^2\sin\theta\,d\theta\,d\phi\\ &= \eta\frac{\pi}{3}|\check{I}|^2\left(\frac{l}{\lambda}\right)^2\end{aligned} \tag{5.95}$$

where $\lambda = 2\pi/k$ is the wavelength in the lossless medium. This is the power radiated by the Hertzian dipole. The power is proportional to $|\check{I}|^2$ as it is in a circuit, and thus we may define a *radiation resistance*

$$R_r = \frac{2P_{av}}{|\check{I}|^2} = \eta\frac{2\pi}{3}\left(\frac{l}{\lambda}\right)^2$$

that represents the resistance of a lumped element that would absorb the same power as radiated by the Hertzian dipole when presented with the same current. We also note that the power radiated by a Hertzian dipole (and, in fact, by any source of finite extent) may

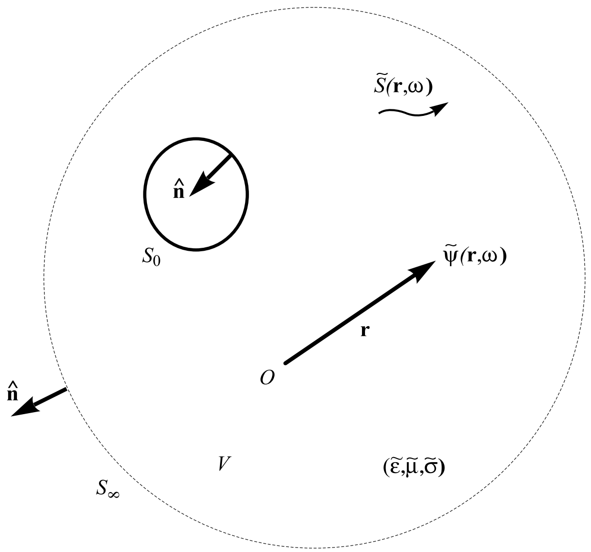

Figure 5.3: Geometry for solution to the frequency-domain Helmholtz equation.

be calculated directly from its far-zone fields. In fact, from (5.94) we have the simple formula for the time-average power density in lossless media

$$\mathbf{S}_{av} = \frac{1}{2}\,\mathrm{Re}\left\{\check{\mathbf{E}}^{FZ} \times \check{\mathbf{H}}^{FZ*}\right\} = \hat{\mathbf{r}}\frac{1}{2}\frac{|\check{\mathbf{E}}^{FZ}|^2}{\eta}.$$

The dipole field is the first term in a general expansion of the electromagnetic fields in terms of the multipole moments of the sources. Either a Taylor expansion or a spherical-harmonic expansion may be used. The reader may see Papas [141] for details.

5.2.2 Solution for potential functions in a bounded medium

In the previous section we solved for the frequency-domain potential functions in an unbounded region of space. Here we shall extend the solution to a bounded region and identify the physical meaning of the radiation condition (5.79).

Consider a bounded region of space V containing a linear, homogeneous, isotropic medium characterized by $\tilde{\mu}(\omega)$ and $\tilde{\epsilon}^c(\omega)$. As shown in Figure 5.3 we decompose the multiply-connected boundary into a closed "excluding surface" S_0 and a closed "encompassing surface" S_∞ that we shall allow to expand outward to infinity. S_0 may consist of more than one closed surface and is often used to exclude unknown sources from V. We wish to solve the Helmholtz equation (5.75) for $\tilde{\psi}$ within V in terms of the sources within V and the values of $\tilde{\psi}$ on S_0. The actual sources of $\tilde{\psi}$ lie entirely with S_∞ but may lie partly, or entirely, within S_0.

We solve the Helmholtz equation in much the same way that we solved Poisson's equation in § 3.2.4. We begin with Green's second identity, written in terms of the source point (primed) variables and applied to the region V:

$$\int_V [\psi(\mathbf{r}',\omega)\nabla'^2 G(\mathbf{r}|\mathbf{r}';\omega) - G(\mathbf{r}|\mathbf{r}';\omega)\nabla'^2\psi(\mathbf{r}',\omega)]\, dV' =$$

$$\oint_{S_0+S_\infty}\left[\psi(\mathbf{r}',\omega)\frac{\partial G(\mathbf{r}|\mathbf{r}';\omega)}{\partial n'} - G(\mathbf{r}|\mathbf{r}';\omega)\frac{\partial \psi(\mathbf{r}',\omega)}{\partial n'}\right] dS'.$$

We note that $\hat{\mathbf{n}}$ points outward from V, and G is the Green's function (5.77). By inspection, this Green's function obeys the reciprocity condition

$$G(\mathbf{r}|\mathbf{r}';\omega) = G(\mathbf{r}'|\mathbf{r};\omega)$$

and satisfies

$$\nabla^2 G(\mathbf{r}|\mathbf{r}';\omega) = \nabla'^2 G(\mathbf{r}|\mathbf{r}';\omega).$$

Substituting $\nabla'^2\tilde{\psi} = -k^2\tilde{\psi} - \tilde{S}$ from (5.75) and $\nabla'^2 G = -k^2 G - \delta(\mathbf{r}-\mathbf{r}')$ from (5.76) we get

$$\tilde{\psi}(\mathbf{r},\omega) = \int_V \tilde{S}(\mathbf{r}',\omega) G(\mathbf{r}|\mathbf{r}';\omega)\, dV' - \\ - \oint_{S_0+S_\infty}\left[\tilde{\psi}(\mathbf{r}',\omega)\frac{\partial G(\mathbf{r}|\mathbf{r}';\omega)}{\partial n'} - G(\mathbf{r}|\mathbf{r}';\omega)\frac{\partial \tilde{\psi}(\mathbf{r}',\omega)}{\partial n'}\right] dS'.$$

Hence $\tilde{\psi}$ within V may be written in terms of the sources within V and the values of $\tilde{\psi}$ and its normal derivative over $S_0 + S_\infty$. The surface contributions account for sources excluded by S_0.

Let us examine the integral over S_∞ more closely. If we let S_∞ recede to infinity, we expect no contribution to the potential at $\mathbf{r}$ from the fields on S_∞. Choosing a sphere centered at the origin, we note that $\hat{\mathbf{n}}' = \hat{\mathbf{r}}'$ and that as $r' \to \infty$

$$G(\mathbf{r}|\mathbf{r}';\omega) = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \approx \frac{e^{-jkr'}}{4\pi r'},$$

$$\frac{\partial G(\mathbf{r}|\mathbf{r}';\omega)}{\partial n'} = \hat{\mathbf{n}}' \cdot \nabla' G(\mathbf{r}|\mathbf{r}';\omega) \approx \frac{\partial}{\partial r'}\frac{e^{-jkr'}}{4\pi r'} = -(1+jkr')\frac{e^{-jkr'}}{4\pi r'}.$$

Substituting these, we find that as $r' \to \infty$

$$\oint_{S_\infty}\left[\tilde{\psi}\frac{\partial G}{\partial n'} - G\frac{\partial \tilde{\psi}}{\partial n'}\right] dS' \approx \int_0^{2\pi}\int_0^{\pi}\left[-\frac{1+jkr'}{r'^2}\tilde{\psi} - \frac{1}{r'}\frac{\partial \tilde{\psi}}{\partial r'}\right]\frac{e^{-jkr'}}{4\pi} r'^2 \sin\theta'\, d\theta'\, d\phi' \\ \approx -\int_0^{2\pi}\int_0^{\pi}\left[\tilde{\psi} + r'\left(jk\tilde{\psi} + \frac{\partial \tilde{\psi}}{\partial r'}\right)\right]\frac{e^{-jkr}}{4\pi}\sin\theta'\, d\theta'\, d\phi'.$$

Since this gives the contribution to the field in V from the fields on the surface receding to infinity, we expect that this term should be zero. If the medium has loss, then the exponential term decays and drives the contribution to zero. For a lossless medium the contribution is zero if

$$\lim_{r\to\infty} \tilde{\psi}(\mathbf{r},\omega) = 0, \tag{5.96}$$

$$\lim_{r\to\infty} r\left[jk\tilde{\psi}(\mathbf{r},\omega) + \frac{\partial \tilde{\psi}(\mathbf{r},\omega)}{\partial r}\right] = 0. \tag{5.97}$$

This is called the *radiation condition* for the Helmholtz equation. It is also called the *Sommerfeld radiation condition* after the German physicist A. Sommerfeld. Note that we have not derived this condition: we have merely postulated it. As with all postulates it is subject to experimental verification.

The radiation condition implies that for points far from the source the potentials behave as spherical waves:

$$\tilde{\psi}(\mathbf{r},\omega) \sim \frac{e^{-jkr}}{r}, \qquad r \to \infty.$$

Substituting this into (5.96) and (5.97) we find that the radiation condition is satisfied.

With $S_\infty \to \infty$ we have

$$\tilde{\psi}(\mathbf{r},\omega) = \int_V \tilde{S}(\mathbf{r}',\omega) G(\mathbf{r}|\mathbf{r}';\omega)\, dV' - \\ - \oint_{S_0} \left[\tilde{\psi}(\mathbf{r}',\omega) \frac{\partial G(\mathbf{r}|\mathbf{r}';\omega)}{\partial n'} - G(\mathbf{r}|\mathbf{r}';\omega) \frac{\partial \tilde{\psi}(\mathbf{r}',\omega)}{\partial n'} \right] dS',$$

which is the expression for the potential within an infinite medium having source-excluding regions. As $S_0 \to 0$ we obtain the expression for the potential in an unbounded medium:

$$\tilde{\psi}(\mathbf{r},\omega) = \int_V \tilde{S}(\mathbf{r}',\omega) G(\mathbf{r}|\mathbf{r}';\omega)\, dV',$$

as expected.

The time-domain equation (5.71) may also be solved (at least for the lossless case) in a bounded region of space. The interested reader should see Pauli [143] for details.

5.3 Transverse–longitudinal decomposition

We have seen that when only electric sources are present, the electromagnetic fields in a homogeneous, isotropic region can be represented by a single vector potential $\mathbf{\Pi}_e$. Similarly, when only magnetic sources are present, the fields can be represented by a single vector potential $\mathbf{\Pi}_h$. Hence two vector potentials may be used to represent the field if both electric and magnetic sources are present.

We may also represent the electromagnetic field in a homogeneous, isotropic region using two scalar functions and the sources. This follows naturally from another important field decomposition: a splitting of each field vector into (1) a component along a certain pre-chosen constant direction, and (2) a component transverse to this direction. Depending on the geometry of the sources, it is possible that only one of these components will be present. A special case of this decomposition, the *TE–TM field decomposition*, holds for a source-free region and will be discussed in the next section.

5.3.1 Transverse–longitudinal decomposition in terms of fields

Consider a direction defined by a constant unit vector $\hat{\mathbf{u}}$. We define the *longitudinal component* of $\mathbf{A}$ as $\hat{\mathbf{u}}A_u$ where

$$A_u = \hat{\mathbf{u}} \cdot \mathbf{A},$$

and the *transverse component* of $\mathbf{A}$ as

$$\mathbf{A}_t = \mathbf{A} - \hat{\mathbf{u}}A_u.$$

We may thus decompose any vector into a sum of longitudinal and transverse parts. An important consequence of Maxwell's equations is that the transverse fields may be written entirely in terms of the longitudinal fields and the sources. This holds in both the time and frequency domains; we derive the decomposition in the frequency domain and leave the derivation of the time-domain expressions as exercises. We begin by decomposing the operators in Maxwell's equations into longitudinal and transverse components. We note that

$$\frac{\partial}{\partial u} \equiv \hat{\mathbf{u}} \cdot \nabla$$

and define a *transverse del operator* as

$$\nabla_t \equiv \nabla - \hat{\mathbf{u}} \frac{\partial}{\partial u}.$$

Using these basic definitions, the identities listed in Appendix B may be derived. We shall find it helpful to express the vector curl and Laplacian operations in terms of their longitudinal and transverse components. Using (B.93) and (B.96) we find that the transverse component of the curl is given by

$$\begin{aligned}(\nabla \times \mathbf{A})_t &= -\hat{\mathbf{u}} \times \hat{\mathbf{u}} \times (\nabla \times \mathbf{A}) \\ &= -\hat{\mathbf{u}} \times \hat{\mathbf{u}} \times (\nabla_t \times \mathbf{A}_t) - \hat{\mathbf{u}} \times \hat{\mathbf{u}} \times \left(\hat{\mathbf{u}} \times \left[\frac{\partial \mathbf{A}_t}{\partial u} - \nabla_t A_u \right] \right). \end{aligned} \tag{5.98}$$

The first term in the right member is zero by property (B.91). Using (B.7) we can replace the second term by

$$-\hat{\mathbf{u}} \left\{ \hat{\mathbf{u}} \cdot \left(\hat{\mathbf{u}} \times \left[\frac{\partial \mathbf{A}_t}{\partial u} - \nabla_t A_u \right] \right) \right\} + (\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}) \left(\hat{\mathbf{u}} \times \left[\frac{\partial \mathbf{A}_t}{\partial u} - \nabla_t A_u \right] \right).$$

The first of these terms is zero since

$$\hat{\mathbf{u}} \cdot \left(\hat{\mathbf{u}} \times \left[\frac{\partial \mathbf{A}_t}{\partial u} - \nabla_t A_u \right] \right) = \left[\frac{\partial \mathbf{A}_t}{\partial u} - \nabla_t A_u \right] \cdot (\hat{\mathbf{u}} \times \hat{\mathbf{u}}) = 0,$$

hence

$$(\nabla \times \mathbf{A})_t = \hat{\mathbf{u}} \times \left[\frac{\partial \mathbf{A}_t}{\partial u} - \nabla_t A_u \right]. \tag{5.99}$$

The longitudinal part is then, by property (B.80), merely the difference between the curl and its transverse part, or

$$\hat{\mathbf{u}} \, (\hat{\mathbf{u}} \cdot \nabla \times \mathbf{A}) = \nabla_t \times \mathbf{A}_t. \tag{5.100}$$

A similar set of steps gives the transverse component of the Laplacian as

$$(\nabla^2 \mathbf{A})_t = \left[\nabla_t (\nabla_t \cdot \mathbf{A}_t) + \frac{\partial^2 \mathbf{A}_t}{\partial u^2} - \nabla_t \times \nabla_t \times \mathbf{A}_t \right], \tag{5.101}$$

and the longitudinal part as

$$\hat{\mathbf{u}} \left(\hat{\mathbf{u}} \cdot \nabla^2 \mathbf{A} \right) = \hat{\mathbf{u}} \nabla^2 A_u. \tag{5.102}$$

Verification is left as an exercise.

Now we are ready to give a longitudinal–transverse decomposition of the fields in a lossy, homogeneous, isotropic region in terms of the direction $\hat{\mathbf{u}}$. We write Maxwell's equations as

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\tilde{\mu}\tilde{\mathbf{H}}_t - j\omega\tilde{\mu}\hat{\mathbf{u}}\tilde{H}_u - \tilde{\mathbf{J}}^i_{mt} - \hat{\mathbf{u}}\tilde{J}^i_{mu}, \tag{5.103}$$

$$\nabla \times \tilde{\mathbf{H}} = j\omega\tilde{\epsilon}^c\tilde{\mathbf{E}}_t + j\omega\tilde{\epsilon}^c\hat{\mathbf{u}}\tilde{E}_u + \tilde{\mathbf{J}}^i_t + \hat{\mathbf{u}}\tilde{J}^i_u, \tag{5.104}$$

where we have split the right-hand sides into longitudinal and transverse parts. Then, using (5.99) and (5.100), we can equate the transverse and longitudinal parts of each equation to obtain

$$\nabla_t \times \tilde{\mathbf{E}}_t = -j\omega\tilde{\mu}\hat{\mathbf{u}}\tilde{H}_u - \hat{\mathbf{u}}\tilde{J}^i_{mu}, \tag{5.105}$$

$$-\hat{\mathbf{u}} \times \nabla_t\tilde{E}_u + \hat{\mathbf{u}} \times \frac{\partial\tilde{\mathbf{E}}_t}{\partial u} = -j\omega\tilde{\mu}\tilde{\mathbf{H}}_t - \tilde{\mathbf{J}}^i_{mt}, \tag{5.106}$$

$$\nabla_t \times \tilde{\mathbf{H}}_t = j\omega\tilde{\epsilon}^c\hat{\mathbf{u}}\tilde{E}_u + \hat{\mathbf{u}}\tilde{J}^i_u, \tag{5.107}$$

$$-\hat{\mathbf{u}} \times \nabla_t\tilde{H}_u + \hat{\mathbf{u}} \times \frac{\partial\tilde{\mathbf{H}}_t}{\partial u} = j\omega\tilde{\epsilon}^c\tilde{\mathbf{E}}_t + \tilde{\mathbf{J}}^i_t. \tag{5.108}$$

We shall isolate the transverse fields in terms of the longitudinal fields. Forming the cross product of $\hat{\mathbf{u}}$ and the partial derivative of (5.108) with respect to u, we have

$$-\hat{\mathbf{u}} \times \hat{\mathbf{u}} \times \nabla_t\frac{\partial\tilde{H}_u}{\partial u} + \hat{\mathbf{u}} \times \hat{\mathbf{u}} \times \frac{\partial^2\tilde{\mathbf{H}}_t}{\partial u^2} = j\omega\tilde{\epsilon}^c\hat{\mathbf{u}} \times \frac{\partial\tilde{\mathbf{E}}_t}{\partial u} + \hat{\mathbf{u}} \times \frac{\partial\tilde{\mathbf{J}}^i_t}{\partial u}.$$

Using (B.7) and (B.80) we find that

$$\nabla_t\frac{\partial\tilde{H}_u}{\partial u} - \frac{\partial^2\tilde{\mathbf{H}}_t}{\partial u^2} = j\omega\tilde{\epsilon}^c\hat{\mathbf{u}} \times \frac{\partial\mathbf{E}_t}{\partial u} + \hat{\mathbf{u}} \times \frac{\partial\tilde{\mathbf{J}}^i_t}{\partial u}. \tag{5.109}$$

Multiplying (5.106) by $j\omega\tilde{\epsilon}^c$ we have

$$-j\omega\tilde{\epsilon}^c\hat{\mathbf{u}} \times \nabla_t\tilde{E}_u + j\omega\tilde{\epsilon}^c\hat{\mathbf{u}} \times \frac{\partial\tilde{\mathbf{E}}_t}{\partial u} = \omega^2\tilde{\mu}\tilde{\epsilon}^c\tilde{\mathbf{H}}_t - j\omega\tilde{\epsilon}^c\tilde{\mathbf{J}}^i_{mt}. \tag{5.110}$$

We now add (5.109) to (5.110) and eliminate $\tilde{\mathbf{E}}_t$ to get

$$\left(\frac{\partial^2}{\partial u^2} + k^2\right)\tilde{\mathbf{H}}_t = \nabla_t\frac{\partial\tilde{H}_u}{\partial u} - j\omega\tilde{\epsilon}^c\hat{\mathbf{u}} \times \nabla_t\tilde{E}_u + j\omega\tilde{\epsilon}^c\tilde{\mathbf{J}}^i_{mt} - \hat{\mathbf{u}} \times \frac{\partial\tilde{\mathbf{J}}^i_t}{\partial u}. \tag{5.111}$$

This one-dimensional Helmholtz equation can be solved to find the transverse magnetic field from the longitudinal components of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$. Similar steps lead to a formula for the transverse component of $\tilde{\mathbf{E}}$:

$$\left(\frac{\partial^2}{\partial u^2} + k^2\right)\tilde{\mathbf{E}}_t = \nabla_t\frac{\partial\tilde{E}_u}{\partial u} + j\omega\tilde{\mu}\hat{\mathbf{u}} \times \nabla_t\tilde{H}_u + \hat{\mathbf{u}} \times \frac{\partial\tilde{\mathbf{J}}^i_{mt}}{\partial u} + j\omega\tilde{\mu}\tilde{\mathbf{J}}^i_t. \tag{5.112}$$

We find the longitudinal components from the wave equation for $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$. Recall that the fields satisfy

$$(\nabla^2 + k^2)\tilde{\mathbf{E}} = \frac{1}{\tilde{\epsilon}^c}\nabla\tilde{\rho}^i + j\omega\tilde{\mu}\tilde{\mathbf{J}}^i + \nabla \times \tilde{\mathbf{J}}^i_m,$$

$$(\nabla^2 + k^2)\tilde{\mathbf{H}} = \frac{1}{\tilde{\mu}}\nabla\tilde{\rho}^i_m + j\omega\tilde{\epsilon}^c\tilde{\mathbf{J}}^i_m - \nabla \times \tilde{\mathbf{J}}^i.$$

Splitting the vectors into longitudinal and transverse parts, and using (5.100) and (5.102), we equate the longitudinal components of the wave equations to obtain

$$\left(\nabla^2 + k^2\right) \tilde{E}_u = \frac{1}{\tilde{\epsilon}^c} \frac{\partial \tilde{\rho}^i}{\partial u} + j\omega\tilde{\mu}\tilde{J}_u^i + \nabla_t \times \tilde{\mathbf{J}}_{mt}^i, \tag{5.113}$$

$$\left(\nabla^2 + k^2\right) \tilde{H}_u = \frac{1}{\tilde{\mu}} \frac{\partial \tilde{\rho}_m^i}{\partial u} + j\omega\tilde{\epsilon}^c\tilde{J}_{mu}^i - \nabla_t \times \tilde{\mathbf{J}}_t^i. \tag{5.114}$$

We note that if $\tilde{\mathbf{J}}_m^i = \tilde{\mathbf{J}}_t^i = 0$, then $\tilde{H}_u = 0$ and the fields are TM to the u-direction; these fields may be determined completely from $\tilde{E}_u$. Similarly, if $\tilde{\mathbf{J}}^i = \tilde{\mathbf{J}}_{mt}^i = 0$, then $\tilde{E}_u = 0$ and the fields are TE to the u-direction; these fields may be determined completely from $\tilde{H}_u$. These properties are used in § 4.11.7, where the fields of electric and magnetic line sources aligned along the z-direction are assumed to be purely TM_z or TE_z, respectively.

5.4 TE–TM decomposition

5.4.1 TE–TM decomposition in terms of fields

A particularly useful field decomposition results if we specialize to a source-free region. With $\tilde{\mathbf{J}}^i = \tilde{\mathbf{J}}_m^i = 0$ in (5.111)–(5.112) we obtain

$$\left(\frac{\partial^2}{\partial u^2} + k^2\right) \tilde{\mathbf{H}}_t = \nabla_t \frac{\partial \tilde{H}_u}{\partial u} - j\omega\tilde{\epsilon}^c \hat{\mathbf{u}} \times \nabla_t \tilde{E}_u, \tag{5.115}$$

$$\left(\frac{\partial^2}{\partial u^2} + k^2\right) \tilde{\mathbf{E}}_t = \nabla_t \frac{\partial \tilde{E}_u}{\partial u} + j\omega\tilde{\mu} \hat{\mathbf{u}} \times \nabla_t \tilde{H}_u. \tag{5.116}$$

Setting the sources to zero in (5.113) and (5.114) we get

$$\left(\nabla^2 + k^2\right) \tilde{E}_u = 0,$$
$$\left(\nabla^2 + k^2\right) \tilde{H}_u = 0.$$

Hence the longitudinal field components are solutions to the homogeneous Helmholtz equation, and the transverse components are specified solely in terms of the longitudinal components. The electromagnetic field is completely specified by the two scalar fields $\tilde{E}_u$ and $\tilde{H}_u$ (and, of course, appropriate boundary values).

We can use superposition to simplify the task of solving (5.115)–(5.116). Since each equation has two forcing terms on the right-hand side, we can solve the equations using one forcing term at a time, and add the results. That is, let $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{H}}_1$ be the solutions to (5.115)–(5.116) with $\tilde{E}_u = 0$, and $\tilde{\mathbf{E}}_2$ and $\tilde{\mathbf{H}}_2$ be the solutions with $\tilde{H}_u = 0$. This results in a decomposition

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_1 + \tilde{\mathbf{E}}_2, \tag{5.117}$$

$$\tilde{\mathbf{H}} = \tilde{\mathbf{H}}_1 + \tilde{\mathbf{H}}_2, \tag{5.118}$$

with

$$\tilde{\mathbf{E}}_1 = \tilde{\mathbf{E}}_{1t}, \qquad \tilde{\mathbf{H}}_1 = \tilde{\mathbf{H}}_{1t} + \tilde{H}_{1u}\hat{\mathbf{u}},$$
$$\tilde{\mathbf{H}}_2 = \tilde{\mathbf{H}}_{2t}, \qquad \tilde{\mathbf{E}}_2 = \tilde{\mathbf{E}}_{2t} + \tilde{E}_{2u}\hat{\mathbf{u}}.$$

Because $\tilde{\mathbf{E}}_1$ has no u-component, $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{H}}_1$ are termed *transverse electric* (or *TE*) to the u-direction; $\tilde{\mathbf{H}}_2$ has no u-component, and $\tilde{\mathbf{E}}_2$ and $\tilde{\mathbf{H}}_2$ are termed *transverse magnetic* (or *TM*) to the u-direction.[2] We see that in a source-free region any electromagnetic field can be decomposed into a set of two fields that are TE and TM, respectively, to some fixed u-direction. This is useful when solving boundary value (e.g., waveguide and scattering) problems where information about external sources is easily specified using the values of the fields on the boundary of the source-free region. In that case $\tilde{E}_u$ and $\tilde{H}_u$ are determined by solving the homogeneous wave equation in an appropriate coordinate system, and the other field components are found from (5.115)–(5.116). Often the boundary conditions can be satisfied by the TM fields or the TE fields alone. This simplifies the analysis of many types of EM systems.

5.4.2 TE–TM decomposition in terms of Hertzian potentials

We are free to represent $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ in terms of scalar fields other than $\tilde{E}_u$ and $\tilde{H}_u$. In doing so, it is helpful to retain the wave nature of the solution so that a meaningful physical interpretation is still possible; we thus use Hertzian potentials since they obey the wave equation.

For the TM case let $\tilde{\mathbf{\Pi}}_h = 0$ and $\tilde{\mathbf{\Pi}}_e = \hat{\mathbf{u}}\tilde{\Pi}_e$. Setting $\tilde{\mathbf{J}}^i = 0$ in (5.64) we have

$$(\nabla^2 + k^2)\tilde{\mathbf{\Pi}}_e = 0.$$

Since $\tilde{\mathbf{\Pi}}_e$ is purely longitudinal, we can use (B.99) to obtain the scalar Helmholtz equation for $\tilde{\Pi}_e$:

$$(\nabla^2 + k^2)\tilde{\Pi}_e = 0. \tag{5.119}$$

Once $\tilde{\Pi}_e$ has been found by solving this wave equation, the fields can be found by using (5.62)–(5.63) with $\tilde{\mathbf{J}}^i = 0$:

$$\tilde{\mathbf{E}} = \nabla \times (\nabla \times \tilde{\mathbf{\Pi}}_e), \tag{5.120}$$

$$\tilde{\mathbf{H}} = j\omega\tilde{\epsilon}^c \nabla \times \tilde{\mathbf{\Pi}}_e. \tag{5.121}$$

We can evaluate $\tilde{\mathbf{E}}$ by noting that $\tilde{\mathbf{\Pi}}_e$ is purely longitudinal. Use of property (B.98) gives

$$\nabla \times \nabla \times \tilde{\mathbf{\Pi}}_e = \nabla_t \frac{\partial \tilde{\Pi}_e}{\partial u} - \hat{\mathbf{u}}\nabla_t^2 \tilde{\Pi}_e.$$

Then, by property (B.97),

$$\nabla \times \nabla \times \tilde{\mathbf{\Pi}}_e = \nabla_t \frac{\partial \tilde{\Pi}_e}{\partial u} - \hat{\mathbf{u}}\left[\nabla^2 \tilde{\Pi}_e - \frac{\partial^2 \tilde{\Pi}_e}{\partial u^2}\right].$$

By (5.119) then,

$$\tilde{\mathbf{E}} = \nabla_t \frac{\partial \tilde{\Pi}_e}{\partial u} + \hat{\mathbf{u}}\left(\frac{\partial^2}{\partial u^2} + k^2\right)\tilde{\Pi}_e. \tag{5.122}$$

The field $\tilde{\mathbf{H}}$ can be found by noting that $\tilde{\mathbf{\Pi}}_e$ is purely longitudinal. Use of property (B.96) in (5.121) gives

$$\tilde{\mathbf{H}} = -j\omega\tilde{\epsilon}^c \hat{\mathbf{u}} \times \nabla_t \tilde{\Pi}_e. \tag{5.123}$$

[2]Some authors prefer to use the terminology *E mode* in place of TM, and *H mode* in place of TE, indicating the presence of a u-directed electric or magnetic field component.

Similar steps can be used to find the TE representation. Substitution of $\tilde{\mathbf{\Pi}}_e = 0$ and $\tilde{\mathbf{\Pi}}_h = \hat{\mathbf{u}}\tilde{\Pi}_h$ into (5.65)–(5.66) gives the fields

$$\tilde{\mathbf{E}} = j\omega\tilde{\mu}\hat{\mathbf{u}} \times \nabla_t \tilde{\Pi}_h, \tag{5.124}$$

$$\tilde{\mathbf{H}} = \nabla_t \frac{\partial \tilde{\Pi}_h}{\partial u} + \hat{\mathbf{u}}\left(\frac{\partial^2}{\partial u^2} + k^2\right)\tilde{\Pi}_h, \tag{5.125}$$

while $\tilde{\Pi}_h$ must satisfy

$$(\nabla^2 + k^2)\tilde{\Pi}_h = 0. \tag{5.126}$$

Hertzian potential representation of TEM fields. An interesting situation occurs when a field is both TE and TM to a particular direction. Such a field is said to be *transverse electromagnetic* (or *TEM*) to that direction. Unfortunately, with $\tilde{E}_u = \tilde{H}_u = 0$ we cannot use (5.115) or (5.116) to find the transverse field components. It turns out that a single scalar potential function is sufficient to represent the field, and we may use either $\tilde{\Pi}_e$ or $\tilde{\Pi}_h$.

For the TM case, equations (5.122) and (5.123) show that we can represent the electromagnetic fields completely with $\tilde{\Pi}_e$. Unfortunately (5.122) has a longitudinal component, and thus cannot describe a TEM field. But if we require that $\tilde{\Pi}_e$ obey the additional equation

$$\left(\frac{\partial^2}{\partial u^2} + k^2\right)\tilde{\Pi}_e = 0, \tag{5.127}$$

then both $\mathbf{E}$ and $\mathbf{H}$ are transverse to u and thus describe a TEM field. Since $\tilde{\Pi}_e$ must also obey

$$\left(\nabla^2 + k^2\right)\tilde{\Pi}_e = 0,$$

using (B.7) we can write (5.127) as

$$\nabla_t^2 \tilde{\Pi}_e = 0.$$

Similarly, for the TE case we found that the EM fields were completely described in (5.124) and (5.125) by $\tilde{\Pi}_h$. In this case $\tilde{\mathbf{H}}$ has a longitudinal component. Thus, if we require

$$\left(\frac{\partial^2}{\partial u^2} + k^2\right)\tilde{\Pi}_h = 0, \tag{5.128}$$

then both $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ are purely transverse to u and again describe a TEM field. Equation (5.128) is equivalent to

$$\nabla_t^2 \tilde{\Pi}_h = 0.$$

We can therefore describe a TEM field using either $\tilde{\Pi}_e$ or $\tilde{\Pi}_h$, since a TEM field is both TE and TM to the longitudinal direction. If we choose $\tilde{\Pi}_e$ we can use (5.122) and (5.123) to obtain the expressions

$$\tilde{\mathbf{E}} = \nabla_t \frac{\partial \tilde{\Pi}_e}{\partial u}, \tag{5.129}$$

$$\tilde{\mathbf{H}} = -j\omega\tilde{\epsilon}^c \hat{\mathbf{u}} \times \nabla_t \tilde{\Pi}_e, \tag{5.130}$$

where $\tilde{\Pi}_e$ must obey

$$\nabla_t^2 \tilde{\Pi}_e = 0, \qquad \left(\frac{\partial^2}{\partial u^2} + k^2\right)\tilde{\Pi}_e = 0. \tag{5.131}$$

If we choose $\tilde{\Pi}_h$ we can use (5.124) and (5.125) to obtain

$$\tilde{\mathbf{E}} = j\omega\tilde{\mu}\hat{\mathbf{u}} \times \nabla_t \tilde{\Pi}_h, \tag{5.132}$$

$$\tilde{\mathbf{H}} = \nabla_t \frac{\partial \tilde{\Pi}_h}{\partial u}, \tag{5.133}$$

where $\tilde{\Pi}_h$ must obey

$$\nabla_t^2 \tilde{\Pi}_h = 0, \qquad \left(\frac{\partial^2}{\partial u^2} + k^2\right) \tilde{\Pi}_h = 0. \tag{5.134}$$

5.4.3 Application: hollow-pipe waveguides

A classic application of the TE–TM decomposition is to the calculation of waveguide fields. Consider a hollow pipe with PEC walls, aligned along the z-axis. The inside is filled with a homogeneous, isotropic material of permeability $\tilde{\mu}(\omega)$ and complex permittivity $\tilde{\epsilon}^c(\omega)$, and the guide cross-sectional shape is assumed to be independent of z. We assume that a current source exists somewhere within the waveguide, creating waves that either propagate or evanesce away from the source. If the source is confined to the region $-d < z < d$ then each of the regions $z > d$ and $z < -d$ is source-free and we may decompose the fields there into TE and TM sets. Such a waveguide is a good candidate for TE–TM analysis because the TE and TM fields independently satisfy the boundary conditions at the waveguide walls. This is not generally the case for certain other guided-wave structures such as fiber optic cables and microstrip lines.

We may represent the fields either in terms of the longitudinal fields $\tilde{E}_z$ and $\tilde{H}_z$, or in terms of the Hertzian potentials. We choose the Hertzian potentials. For TM fields we choose $\tilde{\mathbf{\Pi}}_e = \hat{\mathbf{z}}\tilde{\Pi}_e$, $\tilde{\mathbf{\Pi}}_h = 0$; for TE fields we choose $\tilde{\mathbf{\Pi}}_h = \hat{\mathbf{z}}\tilde{\Pi}_h$, $\tilde{\mathbf{\Pi}}_e = 0$. Both of the potentials must obey the same Helmholtz equation:

$$\left(\nabla^2 + k^2\right) \tilde{\Pi}_z = 0, \tag{5.135}$$

where $\tilde{\Pi}_z$ represents either $\tilde{\Pi}_e$ or $\tilde{\Pi}_h$. We seek a solution to this equation using the separation of variables technique, and assume the product solution

$$\tilde{\Pi}_z(\mathbf{r}, \omega) = \tilde{Z}(z, \omega)\tilde{\psi}(\boldsymbol{\rho}, \omega),$$

where $\boldsymbol{\rho}$ is the transverse position vector ($\mathbf{r} = \hat{\mathbf{z}}z + \boldsymbol{\rho}$). Substituting the trial solution into (5.135) and writing

$$\nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial z^2}$$

we find that

$$\frac{1}{\tilde{\psi}(\boldsymbol{\rho}, \omega)} \nabla_t^2 \tilde{\psi}(\boldsymbol{\rho}, \omega) + k^2 = -\frac{1}{Z(z, \omega)} \frac{\partial^2}{\partial z^2} Z(z, \omega).$$

Because the left-hand side of this expression has positional dependence only on $\boldsymbol{\rho}$ while the right-hand side has dependence only on z, we must have both sides equal to a constant, say k_z^2. Then

$$\frac{\partial^2 Z}{\partial z^2} + k_z^2 Z = 0,$$

which is an ordinary differential equation with the solutions

$$Z = e^{\mp j k_z z}.$$

We also have

$$\nabla_t^2 \tilde{\psi}(\boldsymbol{\rho}, \omega) + k_c^2 \tilde{\psi}(\boldsymbol{\rho}, \omega) = 0, \tag{5.136}$$

where $k_c = k^2 - k_z^2$ is called the *cutoff wavenumber.* The solution to this equation depends on the geometry of the waveguide cross-section and whether the field is TE or TM.

The fields may be computed from the Hertzian potentials using $u = z$ in (5.122)–(5.123) and (5.124)–(5.125). Because the fields all contain the common term $e^{\mp jk_z z}$, we define the field quantities $\tilde{\mathbf{e}}$ and $\tilde{\mathbf{h}}$ through

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = \tilde{\mathbf{e}}(\boldsymbol{\rho}, \omega) e^{\mp jk_z z}, \qquad \tilde{\mathbf{H}}(\mathbf{r}, \omega) = \tilde{\mathbf{h}}(\boldsymbol{\rho}, \omega) e^{\mp jk_z z}.$$

Then, substituting $\tilde{\Pi}_e = \tilde{\psi}_e e^{\mp jk_z z}$, we have for TM fields

$$\tilde{\mathbf{e}} = \mp jk_z \nabla_t \tilde{\psi}_e + \hat{\mathbf{z}} k_c^2 \tilde{\psi}_e,$$
$$\tilde{\mathbf{h}} = -j\omega \tilde{\epsilon}^c \hat{\mathbf{z}} \times \nabla_t \tilde{\psi}_e.$$

Because we have a simple relationship between the transverse parts of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$, we may also write the fields as

$$\tilde{e}_z = k_c^2 \tilde{\psi}_e, \tag{5.137}$$
$$\tilde{\mathbf{e}}_t = \mp jk_z \nabla_t \tilde{\psi}_e, \tag{5.138}$$
$$\tilde{\mathbf{h}}_t = \pm Y_e (\hat{\mathbf{z}} \times \tilde{\mathbf{e}}_t). \tag{5.139}$$

Here

$$Y_e = \frac{\omega \tilde{\epsilon}^c}{k_z}$$

is the complex *TM wave admittance.* For TE fields we have with $\tilde{\Pi}_h = \tilde{\psi}_h e^{\mp jk_z z}$

$$\tilde{\mathbf{e}} = j\omega \tilde{\mu} \hat{\mathbf{z}} \times \nabla_t \tilde{\psi}_h,$$
$$\tilde{\mathbf{h}} = \mp jk_z \nabla_t \tilde{\psi}_h + \hat{\mathbf{z}} k_c^2 \tilde{\psi}_h,$$

or

$$\tilde{h}_z = k_c^2 \tilde{\psi}_h, \tag{5.140}$$
$$\tilde{\mathbf{h}}_t = \mp jk_z \nabla_t \tilde{\psi}_h, \tag{5.141}$$
$$\tilde{\mathbf{e}}_t = \mp Z_h (\hat{\mathbf{z}} \times \tilde{\mathbf{h}}_t). \tag{5.142}$$

Here

$$Z_h = \frac{\omega \tilde{\mu}}{k_z}$$

is the *TM wave impedance.*

Modal solutions for the transverse field dependence. Equation (5.136) describes the transverse behavior of the waveguide fields. When coupled with an appropriate boundary condition, this homogeneous equation has an infinite spectrum of discrete solutions called *eigenmodes* or simply *modes.* Each mode has associated with it a real *eigenvalue* k_c that is dependent on the cross-sectional shape of the waveguide, but independent of frequency and homogeneous material parameters. We number the modes so that $k_c = k_{cn}$ for the nth mode. The amplitude of each modal solution depends on the excitation source within the waveguide.

The appropriate boundary conditions can be found by employing the condition that for both TM and TE fields the tangential component of $\tilde{\mathbf{E}}$ must be zero on the waveguide walls: $\hat{\mathbf{n}} \times \tilde{\mathbf{E}} = 0$, where $\hat{\mathbf{n}}$ is the unit inward normal to the waveguide wall. For TM fields we have $\tilde{E}_z = 0$ and thus

$$\tilde{\psi}_e(\boldsymbol{\rho}, \omega) = 0, \qquad \boldsymbol{\rho} \in \Gamma, \tag{5.143}$$

where Γ is the contour describing the waveguide boundary. For TE fields we have $\hat{\mathbf{n}} \times \tilde{\mathbf{E}}_t = 0$, or

$$\hat{\mathbf{n}} \times (\hat{\mathbf{z}} \times \nabla_t \tilde{\psi}_h) = 0.$$

Using

$$\hat{\mathbf{n}} \times (\hat{\mathbf{z}} \times \nabla_t \tilde{\psi}_h) = \hat{\mathbf{z}}(\hat{\mathbf{n}} \cdot \nabla_t \tilde{\psi}_h) - (\hat{\mathbf{n}} \cdot \hat{\mathbf{z}}) \nabla_t \tilde{\psi}_h$$

and noting that $\hat{\mathbf{n}} \cdot \hat{\mathbf{z}} = 0$, we have the boundary condition

$$\hat{\mathbf{n}} \cdot \nabla_t \tilde{\psi}_h(\boldsymbol{\rho}, \omega) = \frac{\partial \tilde{\psi}_h(\boldsymbol{\rho}, \omega)}{\partial n} = 0, \qquad \boldsymbol{\rho} \in \Gamma. \tag{5.144}$$

The wave nature of the waveguide fields. We have seen that all waveguide field components, for both TE and TM modes, vary as $e^{\mp j k_{zn} z}$. Here $k_{zn}^2 = k^2 - k_{cn}^2$ is the *propagation constant* of the nth mode. Letting

$$k_z = \beta - j\alpha$$

we thus have

$$\tilde{\mathbf{E}}, \tilde{\mathbf{H}} \sim e^{\mp j\beta z} e^{\mp \alpha z}.$$

For $z > d$ we choose the minus sign so that we have a wave propagating away from the source; for $z < -d$ we choose the plus sign.

When the guide is filled with a good dielectric we may assume $\tilde{\mu} = \mu$ is real and independent of frequency and use (4.254) to show that

$$\begin{aligned} k_z = \beta - j\alpha &= \sqrt{\left[\omega^2 \mu \epsilon' - k_c^2\right] - j\omega^2 \mu \epsilon' \tan \delta_c} \\ &= \sqrt{\mu \epsilon'} \sqrt{\omega^2 - \omega_c^2} \sqrt{1 - j \frac{\tan \delta_c}{1 - (\omega_c/\omega)^2}} \end{aligned}$$

where δ_c is the loss tangent (4.253) and where

$$\omega_c = \frac{k_c}{\sqrt{\mu \epsilon'}}$$

is called the *cutoff frequency*. Under the condition

$$\frac{\tan \delta_c}{1 - (\omega_c/\omega)^2} \ll 1 \tag{5.145}$$

we may approximate the square root using the first two terms of the binomial series to show that

$$\beta - j\alpha \approx \sqrt{\mu \epsilon'} \sqrt{\omega^2 - \omega_c^2} \left[1 - j\frac{1}{2} \frac{\tan \delta_c}{1 - (\omega_c/\omega)^2}\right]. \tag{5.146}$$

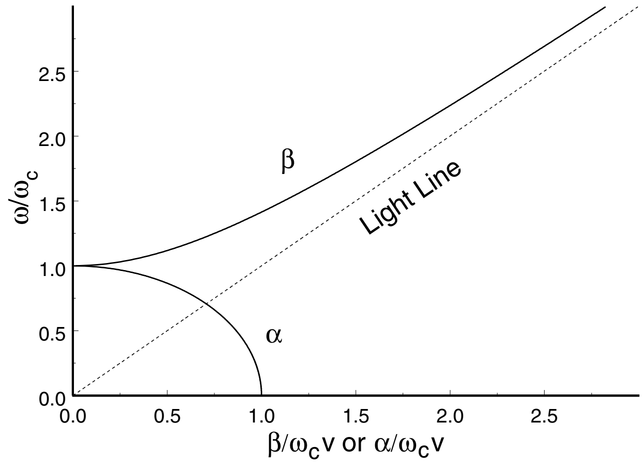

Figure 5.4: Dispersion plot for a hollow-pipe waveguide. Light line computed using $v = 1/\sqrt{\mu\epsilon}$.

Condition (5.145) requires that ω be sufficiently removed from ω_c, either by having $\omega > \omega_c$ or $\omega < \omega_c$. When $\omega > \omega_c$ we say that the frequency is *above cutoff* and find from (5.146) that

$$\beta \approx \omega\sqrt{\mu\epsilon'}\sqrt{1 - \omega_c^2/\omega^2}, \qquad \alpha \approx \frac{\omega^2\mu\epsilon'}{2\beta}\tan\delta_c.$$

Here $\alpha \ll \beta$ and the wave propagates down the waveguide with relatively little loss. When $\omega < \omega_c$ we say that the waveguide is *cut off* or that the frequency is *below cutoff* and find that

$$\alpha \approx \omega\sqrt{\mu\epsilon'}\sqrt{\omega_c^2/\omega^2 - 1}, \qquad \beta \approx \frac{\omega^2\mu\epsilon'}{2\alpha}\tan\delta_c.$$

In this case the wave has a very small phase constant and a very large rate of attenuation. For frequencies near ω_c there is an abrupt but continuous transition between these two types of wave behavior.

When the waveguide is filled with a lossless material having permittivity ϵ and permeability μ, the transition across the cutoff frequency is discontinuous. For $\omega > \omega_c$ we have

$$\beta = \omega\sqrt{\mu\epsilon}\sqrt{1 - \omega_c^2/\omega^2}, \qquad \alpha = 0,$$

and the wave propagates without loss. For $\omega < \omega_c$ we have

$$\alpha = \omega\sqrt{\mu\epsilon}\sqrt{\omega_c^2/\omega^2 - 1}, \qquad \beta = 0,$$

and the wave is evanescent. The dispersion diagram shown in Figure 5.4 clearly shows the abrupt cutoff phenomenon. We can compute the phase and group velocities of the wave above cutoff just as we did for plane waves:

$$v_p = \frac{\omega}{\beta} = \frac{v}{\sqrt{1 - \omega_c^2/\omega^2}},$$

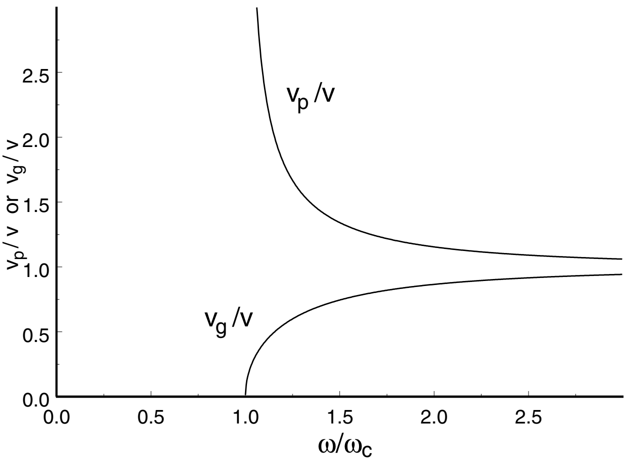

Figure 5.5: Phase and group velocity for a hollow-pipe waveguide.

$$v_g = \frac{d\omega}{d\beta} = v\sqrt{1 - \omega_c^2/\omega^2}, \tag{5.147}$$

where $v = 1/\sqrt{\mu\epsilon}$. Note that $v_g v_p = v^2$. We show later that v_g is the velocity of energy transport within a lossless guide. We also see that as $\omega \to \infty$ we have $v_p \to v$ and $v_g \to v$. More interestingly, as $\omega \to \omega_c$ we find that $v_p \to \infty$ and $v_g \to 0$. This is shown graphically in Figure 5.5.

We may also speak of the *guided wavelength* of a monochromatic wave propagating with frequency $\check{\omega}$ in a waveguide. We define this wavelength as

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1 - \omega_c^2/\check{\omega}^2}} = \frac{\lambda}{\sqrt{1 - \lambda^2/\lambda_c^2}}.$$

Here

$$\lambda = \frac{2\pi}{\check{\omega}\sqrt{\mu\epsilon}}, \qquad \lambda_c = \frac{2\pi}{k_c}.$$

Orthogonality of waveguide modes. The modal fields in a closed-pipe waveguide obey several orthogonality relations. Let $(\check{\mathbf{E}}_n, \check{\mathbf{H}}_n)$ be the time-harmonic electric and magnetic fields of one particular waveguide mode (TE or TM), and let $(\check{\mathbf{E}}_m, \check{\mathbf{H}}_m)$ be the fields of a different mode (TE or TM). One very useful relation states that for a waveguide containing lossless materials

$$\int_{CS} \hat{\mathbf{z}} \cdot \left(\check{\mathbf{e}}_n \times \check{\mathbf{h}}_m^*\right) \, dS = 0, \qquad m \neq n, \tag{5.148}$$

where CS is the guide cross-section. This is used to establish that the total power carried by a wave is the sum of the powers carried by individual modes (see below).

Other important relationships include the orthogonality of the longitudinal fields,

$$\int_{CS} \check{E}_{zm}\check{E}_{zn}\,dS = 0, \qquad m \neq n, \tag{5.149}$$

$$\int_{CS} \check{H}_{zm}\check{H}_{zn}\,dS = 0, \qquad m \neq n, \tag{5.150}$$

and the orthogonality of transverse fields,

$$\int_{CS} \check{\mathbf{E}}_{tm}\cdot\check{\mathbf{E}}_{tn}\,dS = 0, \qquad m \neq n,$$

$$\int_{CS} \check{\mathbf{H}}_{tm}\cdot\check{\mathbf{H}}_{tn}\,dS = 0, \qquad m \neq n.$$

These may also be combined to give an orthogonality relation for the complete fields:

$$\int_{CS} \check{\mathbf{E}}_{m}\cdot\check{\mathbf{E}}_{n}\,dS = 0, \qquad m \neq n, \tag{5.151}$$

$$\int_{CS} \check{\mathbf{H}}_{m}\cdot\check{\mathbf{H}}_{n}\,dS = 0, \qquad m \neq n. \tag{5.152}$$

For proofs of these relations the reader should see Collin [39].

Power carried by time-harmonic waves in lossless waveguides. The power carried by a time-harmonic wave propagating down a waveguide is defined as the time-average Poynting flux passing through the guide cross-section. Thus we may write

$$P_{av} = \frac{1}{2}\int_{CS} \mathrm{Re}\left\{\check{\mathbf{E}}\times\check{\mathbf{H}}^*\right\}\cdot\hat{\mathbf{z}}\,dS.$$

The field within the guide is assumed to be a superposition of all possible waveguide modes. For waves traveling in the $+z$-direction this implies

$$\check{\mathbf{E}} = \sum_m \left(\check{\mathbf{e}}_{tm} + \hat{\mathbf{z}}\check{e}_{zm}\right)e^{-jk_{zm}z}, \qquad \check{\mathbf{H}} = \sum_n \left(\check{\mathbf{h}}_{tn} + \hat{\mathbf{z}}\check{h}_{zn}\right)e^{-jk_{zn}z}.$$

Substituting we have

$$\begin{aligned} P_{av} &= \frac{1}{2}\mathrm{Re}\left\{\int_{CS}\left[\sum_m \left(\check{\mathbf{e}}_{tm} + \hat{\mathbf{z}}\check{e}_{zm}\right)e^{-jk_{zm}z} \times \sum_n \left(\check{\mathbf{h}}^*_{tn} + \hat{\mathbf{z}}\check{h}^*_{zn}\right)e^{jk^*_{zn}z}\right]\cdot\hat{\mathbf{z}}\,dS\right\} \\ &= \frac{1}{2}\mathrm{Re}\left\{\sum_m\sum_n e^{-j(k_{zm}-k^*_{zn})z}\int_{CS}\hat{\mathbf{z}}\cdot\left(\check{\mathbf{e}}_{tm}\times\check{\mathbf{h}}^*_{tn}\right)\,dS\right\}. \end{aligned}$$

By (5.148) we have

$$P_{av} = \frac{1}{2}\mathrm{Re}\left\{\sum_n e^{-j(k_{zn}-k^*_{zn})z}\int_{CS}\hat{\mathbf{z}}\cdot\left(\check{\mathbf{e}}_{tn}\times\check{\mathbf{h}}^*_{tn}\right)\,dS\right\}.$$

For modes propagating in a lossless guide $k_{zn} = \beta_{zn}$. For modes that are cut off $k_{zn} = -j\alpha_{zn}$. However, we find below that terms in this series representing modes that are cut off are zero. Thus

$$P_{av} = \sum_n \frac{1}{2}\mathrm{Re}\left\{\int_{CS}\hat{\mathbf{z}}\cdot\left(\check{\mathbf{e}}_{tn}\times\check{\mathbf{h}}^*_{tn}\right)\,dS\right\} = \sum_n P_{n,av}.$$

Hence for waveguides filled with lossless media the total time-average power flow is given by the superposition of the individual modal powers.

Simple formulas for the individual modal powers in a lossless guide may be obtained by substituting the expressions for the fields. For TM modes we use (5.138) and (5.139) to get

$$P_{av} = \frac{1}{2}\,\mathrm{Re}\left\{|k_z|^2 Y_e^* e^{-j(k_z-k_z^*)}\int_{CS}\hat{\mathbf{z}}\cdot\left(\nabla_t\check{\psi}_e\times[\hat{\mathbf{z}}\times\nabla_t\check{\psi}_e^*]\right)\,dS\right\}$$
$$= \frac{1}{2}|k_z|^2\,\mathrm{Re}\left\{Y_e^*\right\}e^{-j(k_z-k_z^*)}\int_{CS}\nabla_t\check{\psi}_e\cdot\nabla_t\check{\psi}_e^*\,dS.$$

Here we have used (B.7) and $\hat{\mathbf{z}}\cdot\nabla_t\check{\psi}_e = 0$. This expression can be simplified by using the two-dimensional version of Green's first identity (B.29):

$$\int_S(\nabla_t a\cdot\nabla_t b + a\nabla_t^2 b)\,dS = \oint_\Gamma a\frac{\partial b}{\partial n}\,dl.$$

Using $a = \check{\psi}_e$ and $b = \check{\psi}_e^*$ and integrating over the waveguide cross-section we have

$$\int_{CS}(\nabla_t\check{\psi}_e\cdot\nabla_t\check{\psi}_e^* + \check{\psi}_e\nabla^2\check{\psi}_e^*)\,dS = \oint_\Gamma\check{\psi}_e\frac{\partial\check{\psi}_e^*}{\partial n}\,dl.$$

Substituting $\nabla_t^2\check{\psi}_e^* = -k_c^2\check{\psi}_e^*$ and remembering that $\check{\psi}_e = 0$ on Γ we reduce this to

$$\int_{CS}\nabla_t\check{\psi}_e\cdot\nabla_t\check{\psi}_e^*\,dS = k_c^2\int_{CS}\check{\psi}_e\check{\psi}_e^*\,dS. \qquad (5.153)$$

Thus the power is

$$P_{av} = \frac{1}{2}\,\mathrm{Re}\left\{Y_e^*\right\}|k_z|^2 k_c^2 e^{-j(k_z-k_z^*)z}\int_{CS}\check{\psi}_e\check{\psi}_e^*\,dS.$$

For modes above cutoff we have $k_z = \beta$ and $Y_e = \omega\epsilon/k_z = \omega\epsilon/\beta$. The power carried by these modes is thus

$$P_{av} = \frac{1}{2}\omega\epsilon\beta k_c^2\int_{CS}\check{\psi}_e\check{\psi}_e^*\,dS. \qquad (5.154)$$

For modes below cutoff we have $k_z = -j\alpha$ and $Y_e = j\omega\epsilon/\alpha$. Thus $\mathrm{Re}\{Y_e^*\} = 0$ and $P_{av} = 0$. For frequencies below cutoff the fields are evanescent and do not carry power in the manner of propagating waves.

For TE modes we may proceed similarly and show that

$$P_{av} = \frac{1}{2}\omega\mu\beta k_c^2\int_{CS}\check{\psi}_h\check{\psi}_h^*\,dS. \qquad (5.155)$$

The details are left as an exercise.

Stored energy in a waveguide and the velocity of energy transport. Consider a source-free section of lossless waveguide bounded on its two ends by the cross-sectional surfaces CS_1 and CS_2. Setting $\check{\mathbf{J}}^i = \check{\mathbf{J}}^c = 0$ in (4.156) we have

$$\frac{1}{2}\oint_S(\check{\mathbf{E}}\times\check{\mathbf{H}}^*)\cdot\mathbf{dS} = 2j\omega\int_V[\langle w_e\rangle - \langle w_m\rangle]\,dV,$$

where V is the region of the guide between CS_1 and CS_2. The right-hand side represents the difference between the total time-average stored electric and magnetic energies. Thus

$$2j\omega\left[\langle W_e\rangle - \langle W_m\rangle\right] = \frac{1}{2}\int_{CS_1} -\hat{\mathbf{z}}\cdot(\check{\mathbf{E}}\times\check{\mathbf{H}}^*)\,dS + \frac{1}{2}\int_{CS_2}\hat{\mathbf{z}}\cdot(\check{\mathbf{E}}\times\check{\mathbf{H}}^*)\,dS - \frac{1}{2}\int_{S_{\text{cond}}}(\check{\mathbf{E}}\times\check{\mathbf{H}}^*)\cdot\mathbf{dS},$$

where S_{cond} indicates the conducting walls of the guide and $\hat{\mathbf{n}}$ points into the guide. For a propagating mode the first two terms on the right-hand side cancel since with no loss $\check{\mathbf{E}}\times\check{\mathbf{H}}^*$ is the same on CS_1 and CS_2. The third term is zero since $(\check{\mathbf{E}}\times\check{\mathbf{H}}^*)\cdot\hat{\mathbf{n}} = (\hat{\mathbf{n}}\times\check{\mathbf{E}})\cdot\check{\mathbf{H}}^*$, and $\hat{\mathbf{n}}\times\check{\mathbf{E}} = 0$ on the waveguide walls. Thus we have

$$\langle W_e\rangle = \langle W_m\rangle$$

for any section of a lossless waveguide.

We may compute the time-average stored magnetic energy in a section of lossless waveguide of length l as

$$\langle W_m\rangle = \frac{\mu}{4}\int_0^l\int_{CS}\check{\mathbf{H}}\cdot\check{\mathbf{H}}^*\,dS\,dz.$$

For propagating TM modes we can substitute (5.139) to find

$$\langle W_m\rangle/l = \frac{\mu}{4}(\beta Y_e)^2\int_{CS}(\hat{\mathbf{z}}\times\nabla_t\check{\psi}_e)\cdot(\hat{\mathbf{z}}\times\nabla_t\check{\psi}_e^*)\,dS.$$

Using

$$(\hat{\mathbf{z}}\times\nabla_t\check{\psi}_e)\cdot(\hat{\mathbf{z}}\times\nabla_t\check{\psi}_e^*) = \hat{\mathbf{z}}\cdot\left[\nabla_t\check{\psi}_e^*\times(\hat{\mathbf{z}}\times\nabla_t\check{\psi}_e)\right] = \nabla_t\check{\psi}_e\cdot\nabla_t\check{\psi}_e^*$$

we have

$$\langle W_m\rangle/l = \frac{\mu}{4}(\beta Y_e)^2\int_{CS}\nabla_t\check{\psi}_e\cdot\nabla_t\check{\psi}_e^*\,dS.$$

Finally, using (5.153) we have the stored energy per unit length for a propagating TM mode:

$$\langle W_m\rangle/l = \langle W_e\rangle/l = \frac{\mu}{4}(\omega\epsilon)^2k_c^2\int_{CS}\check{\psi}_e\check{\psi}_e^*\,dS.$$

Similarly we may show that for a TE mode

$$\langle W_e\rangle/l = \langle W_m\rangle/l = \frac{\epsilon}{4}(\omega\mu)^2k_c^2\int_{CS}\check{\psi}_h\check{\psi}_h^*\,dS.$$

The details are left as an exercise.

As with plane waves in (4.261) we may describe the velocity of energy transport as the ratio of the Poynting flux density to the total stored energy density:

$$\mathbf{S}_{av} = \langle w_T\rangle\mathbf{v}_e.$$

For TM modes this energy velocity is

$$v_e = \frac{\frac{1}{2}\omega\epsilon\beta k_c^2\check{\psi}_e\check{\psi}_e^*}{2\frac{\mu}{4}(\omega\epsilon)^2k_c^2\check{\psi}_e\check{\psi}_e^*} = \frac{\beta}{\omega\mu\epsilon} = v\sqrt{1-\omega_c^2/\omega^2},$$

which is identical to the group velocity (5.147). This is also the case for TE modes, for which

$$v_e = \frac{\frac{1}{2}\omega\mu\beta k_c^2\check{\psi}_h\check{\psi}_h^*}{2\frac{\epsilon}{4}(\omega\mu)^2k_c^2\check{\psi}_h\check{\psi}_h^*} = \frac{\beta}{\omega\mu\epsilon} = v\sqrt{1-\omega_c^2/\omega^2}.$$

Example: fields of a rectangular waveguide. Consider a rectangular waveguide with a cross-section occupying $0 \le x \le a$ and $0 \le y \le b$. The material within the guide is assumed to be a lossless dielectric of permittivity ϵ and permeability μ. We seek the modal fields within the guide.

Both TE and TM fields exist within the guide. In each case we must solve the differential equation

$$\nabla_t^2 \tilde{\psi} + k_c^2 \tilde{\psi} = 0.$$

A product solution in rectangular coordinates may be sought using the separation of variables technique (§ A.4). We find that

$$\tilde{\psi}(x, y, \omega) = [A_x \sin k_x x + B_x \cos k_x x]\left[A_y \sin k_y y + B_y \cos k_y y\right]$$

where $k_x^2 + k_y^2 = k_c^2$. This solution is easily verified by substitution.

For TM modes the solution is subject to the boundary condition (5.143):

$$\tilde{\psi}_e(\boldsymbol{\rho}, \omega) = 0, \qquad \boldsymbol{\rho} \in \Gamma.$$

Applying this at $x = 0$ and $y = 0$ we find $B_x = B_y = 0$. Applying the boundary condition at $x = a$ we then find $\sin k_x a = 0$ and thus

$$k_x = \frac{n\pi}{a}, \qquad n = 1, 2, \ldots.$$

Note that $n = 0$ corresponds to the trivial solution $\tilde{\psi}_e = 0$. Similarly, from the condition at $y = b$ we find that

$$k_y = \frac{m\pi}{b}, \qquad m = 1, 2, \ldots.$$

Thus

$$\tilde{\psi}_e(x, y, \omega) = A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right).$$

From (5.137)–(5.139) we find that the fields are

$$\tilde{E}_z = k_{c_{nm}}^2 A_{nm}\left[\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\right] e^{\mp jk_z z},$$

$$\tilde{\mathbf{E}}_t = \mp jk_z A_{nm}\left[\hat{\mathbf{x}}\frac{n\pi}{a}\cos\frac{n\pi x}{a}\sin\frac{m\pi y}{b} + \hat{\mathbf{y}}\frac{m\pi}{b}\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b}\right] e^{\mp jk_z z},$$

$$\tilde{\mathbf{H}}_t = jk_z Y_e A_{nm}\left[\hat{\mathbf{x}}\frac{m\pi}{b}\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b} - \hat{\mathbf{y}}\frac{n\pi}{a}\cos\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\right] e^{\mp jk_z z}.$$

Here

$$Y_e = \frac{1}{\eta\sqrt{1 - \omega_{c_{nm}}^2/\omega^2}}$$

with $\eta = (\mu\epsilon)^{1/2}$.

Each combination of m, n describes a different field pattern and thus a different mode, designated TM_{nm}. The cutoff wavenumber of the TM_{nm} mode is

$$k_{c_{nm}} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}, \qquad m, n = 1, 2, 3, \ldots$$

and the cutoff frequency is

$$\omega_{c_{nm}} = v\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}, \qquad m, n = 1, 2, 3, \ldots$$

where $v = 1/(\mu\epsilon)^{1/2}$. Thus the TM_{11} mode has the lowest cutoff frequency of any TM mode. There is a range of frequencies for which this is the only propagating TM mode.

For TE modes the solution is subject to

$$\hat{\mathbf{n}} \cdot \nabla_t \tilde{\psi}_h(\boldsymbol{\rho}, \omega) = \frac{\partial \tilde{\psi}_h(\boldsymbol{\rho}, \omega)}{\partial n} = 0, \qquad \boldsymbol{\rho} \in \Gamma.$$

At $x = 0$ we have

$$\frac{\partial \tilde{\psi}_h}{\partial x} = 0$$

leading to $A_x = 0$. At $y = 0$ we have

$$\frac{\partial \tilde{\psi}_h}{\partial y} = 0$$

leading to $A_y = 0$. At $x = a$ we require $\sin k_x a = 0$ and thus

$$k_x = \frac{n\pi}{a}, \qquad n = 0, 1, 2, \ldots.$$

Similarly, from the condition at $y = b$ we find

$$k_y = \frac{m\pi}{b}, \qquad m = 0, 1, 2, \ldots.$$

The case $n = m = 0$ is not allowed since it produces the trivial solution. Thus

$$\tilde{\psi}_h(x, y, \omega) = B_{nm} \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right), \qquad m, n = 0, 1, 2, \ldots, \quad m + n > 0.$$

From (5.140)–(5.142) we find that the fields are

$$\begin{aligned}
\tilde{H}_z &= k_{c_{nm}}^2 B_{nm} \left[\cos\frac{n\pi x}{a} \cos\frac{m\pi y}{b}\right] e^{\mp jk_z z}, \\
\tilde{\mathbf{H}}_t &= \pm jk_z B_{nm} \left[\hat{\mathbf{x}}\frac{n\pi}{a} \sin\frac{n\pi x}{a} \cos\frac{m\pi y}{b} + \hat{\mathbf{y}}\frac{m\pi}{b} \cos\frac{n\pi x}{a} \sin\frac{m\pi y}{b}\right] e^{\mp jk_z z}, \\
\tilde{\mathbf{E}}_t &= jk_z Z_h B_{nm} \left[\hat{\mathbf{x}}\frac{m\pi}{b} \cos\frac{n\pi x}{a} \sin\frac{m\pi y}{b} - \hat{\mathbf{y}}\frac{n\pi}{a} \sin\frac{n\pi x}{a} \cos\frac{m\pi y}{b}\right] e^{\mp jk_z z}.
\end{aligned}$$

Here

$$Z_h = \frac{\eta}{\sqrt{1 - \omega_{c_{nm}}^2/\omega^2}}.$$

In this case the modes are designated TE_{nm}. The cutoff wavenumber of the TE_{nm} mode is

$$k_{c_{nm}} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}, \qquad m, n = 0, 1, 2, \ldots, \quad m + n > 0$$

and the cutoff frequency is

$$\omega_{c_{nm}} = v\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}, \qquad m, n = 0, 1, 2, \ldots, \quad m + n > 0$$

where $v = 1/(\mu\epsilon)^{1/2}$. Modes having the same cutoff frequency are said to be *degenerate*. This is the case with the TE and TM modes. However, the field distributions differ and thus the modes are distinct. Note that we may also have degeneracy among the TE or TM modes. For instance, if $a = b$ then the cutoff frequency of the TE_{nm} mode is identical to that of the TE_{mn} mode. If $a \geq b$ then the TE_{10} mode has the lowest cutoff frequency and is termed the *dominant* mode in a rectangular guide. There is a finite band of frequencies in which this is the only mode propagating (although the bandwidth is small if $a \approx b$.)

Calculation of the time-average power carried by propagating TE and TM modes is left as an exercise.

5.4.4 TE–TM decomposition in spherical coordinates

It is not necessary for the longitudinal direction to be constant to achieve a TE–TM decomposition. It is possible, for instance, to represent the electromagnetic field in terms of components either TE or TM to the radial direction of spherical coordinates. This may be shown using a procedure identical to that used for the longitudinal–transverse decomposition in rectangular coordinates. We carry out the decomposition in the frequency domain and leave the time-domain decomposition as an exercise.

TE–TM decomposition in terms of the radial fields. Consider a source-free region of space filled with a homogeneous, isotropic material described by parameters $\tilde{\mu}(\omega)$ and $\tilde{\epsilon}^c(\omega)$. We substitute the spherical coordinate representation of the curl into Faraday's and Ampere's laws with source terms $\tilde{\mathbf{J}}$ and $\tilde{\mathbf{J}}_m$ set equal to zero. Equating vector components we have, in particular,

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial \tilde{E}_r}{\partial \phi} - \frac{\partial}{\partial r}(r\tilde{E}_\phi)\right] = -j\omega\tilde{\mu}\tilde{H}_\theta \tag{5.156}$$

and

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(r\tilde{H}_\theta) - \frac{\partial \tilde{H}_r}{\partial \theta}\right] = j\omega\tilde{\epsilon}^c\tilde{E}_\phi. \tag{5.157}$$

We seek to isolate the transverse components of the fields in terms of the radial components. Multiplying (5.156) by $j\omega\tilde{\epsilon}^c r$ we get

$$j\omega\tilde{\epsilon}^c\frac{1}{\sin\theta}\frac{\partial \tilde{E}_r}{\partial \phi} - j\omega\tilde{\epsilon}^c\frac{\partial (r\tilde{E}_\phi)}{\partial r} = k^2 r\tilde{H}_\theta;$$

next, multiplying (5.157) by r and then differentiating with respect to r we get

$$\frac{\partial^2}{\partial r^2}(r\tilde{H}_\theta) - \frac{\partial^2 \tilde{H}_r}{\partial\theta\,\partial r} = j\omega\tilde{\epsilon}^c\frac{\partial (r\tilde{E}_\phi)}{\partial r}.$$

Subtracting these two equations and rearranging, we obtain

$$\left(\frac{\partial^2}{\partial r^2} + k^2\right)(r\tilde{H}_\theta) = j\omega\tilde{\epsilon}^c\frac{1}{\sin\theta}\frac{\partial \tilde{E}_r}{\partial \phi} + \frac{\partial^2 \tilde{H}_r}{\partial r\,\partial\theta}.$$

This is a one-dimensional wave equation for the product of r with the transverse field component $\tilde{H}_\theta$. Similarly

$$\left(\frac{\partial^2}{\partial r^2} + k^2\right)(r\tilde{H}_\phi) = -j\omega\tilde{\epsilon}^c\frac{\partial \tilde{E}_r}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial^2 \tilde{H}_r}{\partial r\,\partial\phi},$$

and

$$\left(\frac{\partial^2}{\partial r^2}+k^2\right)(r\tilde{E}_\phi) = \frac{1}{\sin\theta}\frac{\partial^2\tilde{E}_r}{\partial\phi\partial r} + j\omega\tilde{\mu}\frac{\partial\tilde{H}_r}{\partial\theta}, \tag{5.158}$$

$$\left(\frac{\partial^2}{\partial r^2}+k^2\right)(r\tilde{E}_\theta) = \frac{\partial^2\tilde{E}_r}{\partial\theta\partial r} + j\omega\tilde{\mu}\frac{1}{\sin\theta}\frac{\partial\tilde{H}_r}{\partial\phi}. \tag{5.159}$$

Hence we can represent the electromagnetic field in a source-free region in terms of the two scalar quantities $\tilde{E}_r$ and $\tilde{H}_r$. Superposition allows us to solve the TE case with $\tilde{E}_r = 0$ and the TM case with $\tilde{H}_r = 0$, and combine the results for the general expansion of the field.

TE–TM decomposition in terms of potential functions. If we allow the vector potential (or Hertzian potential) to have only an r-component, then the resulting fields are TE or TM to the r-direction. Unfortunately, this scalar component does not satisfy the Helmholtz equation. If we wish to use a potential component that satisfies the Helmholtz equation then we must discard the Lorentz condition and choose a different relationship between the vector and scalar potentials.

1. TM fields. To generate fields TM to r we recall that the electromagnetic fields may be written in terms of electric vector and scalar potentials as

$$\tilde{\mathbf{E}} = -j\omega\tilde{\mathbf{A}}_e - \nabla\phi_e, \tag{5.160}$$

$$\tilde{\mathbf{B}} = \nabla\times\tilde{\mathbf{A}}_e. \tag{5.161}$$

In a source-free region we have by Ampere's law

$$\tilde{\mathbf{E}} = \frac{1}{j\omega\tilde{\mu}\tilde{\epsilon}^c}\nabla\times\tilde{\mathbf{B}} = \frac{1}{j\omega\tilde{\mu}\tilde{\epsilon}^c}\nabla\times(\nabla\times\tilde{\mathbf{A}}_e).$$

Here $\tilde{\phi}_e$ and $\tilde{\mathbf{A}}_e$ must satisfy a differential equation that may be derived by examining

$$\nabla\times(\nabla\times\tilde{\mathbf{E}}) = -j\omega\nabla\times\tilde{\mathbf{B}} = -j\omega(j\omega\tilde{\mu}\tilde{\epsilon}^c\tilde{\mathbf{E}}) = k^2\tilde{\mathbf{E}},$$

where $k^2 = \omega^2\tilde{\mu}\tilde{\epsilon}^c$. Substitution from (5.160) gives

$$\nabla\times\left(\nabla\times[-j\omega\tilde{\mathbf{A}}_e - \nabla\tilde{\phi}_e]\right) = k^2[-j\omega\tilde{\mathbf{A}}_e - \nabla\tilde{\phi}_e]$$

or

$$\nabla\times(\nabla\times\tilde{\mathbf{A}}_e) - k^2\tilde{\mathbf{A}}_e = \frac{k^2}{j\omega}\nabla\tilde{\phi}_e. \tag{5.162}$$

We are still free to specify $\nabla\cdot\tilde{\mathbf{A}}_e$.

At this point let us examine the effect of choosing a vector potential with only an r-component: $\tilde{\mathbf{A}}_e = \hat{\mathbf{r}}\tilde{A}_e$. Since

$$\nabla\times(\hat{\mathbf{r}}\tilde{A}_e) = \frac{\hat{\boldsymbol{\theta}}}{r\sin\theta}\frac{\partial\tilde{A}_e}{\partial\phi} - \frac{\hat{\boldsymbol{\phi}}}{r}\frac{\partial\tilde{A}_e}{\partial\theta} \tag{5.163}$$

we see that $\mathbf{B} = \nabla\times\tilde{\mathbf{A}}_e$ has no r-component. Since

$$\nabla\times(\nabla\times\tilde{\mathbf{A}}_e) = -\frac{\hat{\mathbf{r}}}{r\sin\theta}\left[\frac{1}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\tilde{A}_e}{\partial\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial^2\tilde{A}_e}{\partial\phi^2}\right] + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial^2\tilde{A}_e}{\partial r\partial\theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial^2\tilde{A}_e}{\partial r\partial\phi}$$

we see that $\tilde{\mathbf{E}} \sim \nabla \times (\nabla \times \tilde{\mathbf{A}}_e)$ has all three components. This choice of $\tilde{\mathbf{A}}_e$ produces a field TM to the r-direction. We need only choose $\nabla \cdot \tilde{\mathbf{A}}_e$ so that the resulting differential equation is convenient to solve. Substituting the above expressions into (5.162) we find that

$$-\frac{\hat{\mathbf{r}}}{r\sin\theta}\left[\frac{1}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\tilde{A}_e}{\partial\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial^2\tilde{A}_e}{\partial\phi^2}\right]+\frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial^2\tilde{A}_e}{\partial r\partial\theta}+\frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial^2\tilde{A}_e}{\partial r\partial\phi}-\hat{\mathbf{r}}k^2\tilde{A}_e =$$
$$\hat{\mathbf{r}}\frac{k^2}{j\omega}\frac{\partial\tilde{\phi}_e}{\partial r}+\frac{\hat{\boldsymbol{\theta}}}{r}\frac{k^2}{j\omega}\frac{\partial\tilde{\phi}_e}{\partial\theta}+\frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{k^2}{j\omega}\frac{\partial\tilde{\phi}_e}{\partial\phi}. \tag{5.164}$$

Since $\nabla \cdot \tilde{\mathbf{A}}_e$ only involves the derivatives of $\tilde{\mathbf{A}}_e$ with respect to r, we may specify $\nabla \cdot \tilde{\mathbf{A}}_e$ indirectly through

$$\tilde{\phi}_e = \frac{j\omega}{k^2}\frac{\partial\tilde{A}_e}{\partial r}.$$

With this (5.164) becomes

$$\frac{1}{r\sin\theta}\left[\frac{1}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\tilde{A}_e}{\partial\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial^2\tilde{A}_e}{\partial\phi^2}\right]+k^2\tilde{A}_e+\frac{\partial^2\tilde{A}_e}{\partial r^2}=0.$$

Using

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{\tilde{A}_e}{r}\right)\right]=\frac{\partial^2\tilde{A}_e}{\partial r^2}$$

we can write the differential equation as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial(\tilde{A}_e/r)}{\partial r}\right]+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial(\tilde{A}_e/r)}{\partial\theta}\right]+\frac{1}{r^2\sin^2\theta}\frac{\partial^2(\tilde{A}_e/r)}{\partial\phi^2}+k^2\frac{\tilde{A}_e}{r}=0.$$

The first three terms of this expression are precisely the Laplacian of $\tilde{A}_e/r$. Thus we have

$$(\nabla^2+k^2)\left(\frac{\tilde{A}_e}{r}\right)=0 \tag{5.165}$$

and the quantity $\tilde{A}_e/r$ satisfies the homogeneous Helmholtz equation.

The TM fields generated by the vector potential $\tilde{\mathbf{A}}_e = \hat{\mathbf{r}}\tilde{A}_e$ may be found by using (5.160) and (5.161). From (5.160) we have the electric field

$$\tilde{\mathbf{E}} = -j\omega\tilde{\mathbf{A}}_e - \nabla\tilde{\phi}_e = -j\omega\hat{\mathbf{r}}\tilde{A}_e - \nabla\left(\frac{j\omega}{k^2}\frac{\partial\tilde{A}_e}{\partial r}\right).$$

Expanding the gradient we have the field components

$$\tilde{E}_r = \frac{1}{j\omega\tilde{\mu}\tilde{\epsilon}^c}\left(\frac{\partial^2}{\partial r^2}+k^2\right)\tilde{A}_e, \tag{5.166}$$

$$\tilde{E}_\theta = \frac{1}{j\omega\tilde{\mu}\tilde{\epsilon}^c}\frac{1}{r}\frac{\partial^2\tilde{A}_e}{\partial r\partial\theta}, \tag{5.167}$$

$$\tilde{E}_\phi = \frac{1}{j\omega\tilde{\mu}\tilde{\epsilon}^c}\frac{1}{r\sin\theta}\frac{\partial^2\tilde{A}_e}{\partial r\partial\phi}. \tag{5.168}$$

The magnetic field components are found using (5.161) and (5.163):

$$\tilde{H}_\theta = \frac{1}{\tilde{\mu}} \frac{1}{r \sin\theta} \frac{\partial \tilde{A}_e}{\partial \phi}, \tag{5.169}$$

$$\tilde{H}_\phi = -\frac{1}{\tilde{\mu}} \frac{1}{r} \frac{\partial \tilde{A}_e}{\partial \theta}. \tag{5.170}$$

2. TE fields. To generate fields TE to r we recall that the electromagnetic fields in a source-free region may be written in terms of magnetic vector and scalar potentials as

$$\tilde{\mathbf{H}} = -j\omega\tilde{\mathbf{A}}_h - \nabla\phi_h, \tag{5.171}$$

$$\tilde{\mathbf{D}} = -\nabla \times \tilde{\mathbf{A}}_h. \tag{5.172}$$

In a source-free region we have from Faraday's law

$$\tilde{\mathbf{H}} = \frac{1}{-j\omega\tilde{\mu}\tilde{\epsilon}^c}\nabla \times \tilde{\mathbf{D}} = \frac{1}{j\omega\tilde{\mu}\tilde{\epsilon}^c}\nabla \times (\nabla \times \tilde{\mathbf{A}}_h).$$

Here $\tilde{\phi}_h$ and $\tilde{\mathbf{A}}_h$ must satisfy a differential equation that may be derived by examining

$$\nabla \times (\nabla \times \tilde{\mathbf{H}}) = j\omega\nabla \times \tilde{\mathbf{D}} = j\omega\tilde{\epsilon}^c(-j\omega\tilde{\mu}\tilde{\mathbf{H}}) = k^2\tilde{\mathbf{H}},$$

where $k^2 = \omega^2\tilde{\mu}\tilde{\epsilon}^c$. Substitution from (5.171) gives

$$\nabla \times \left(\nabla \times [-j\omega\tilde{\mathbf{A}}_h - \nabla\tilde{\phi}_h]\right) = k^2[-j\omega\tilde{\mathbf{A}}_h - \nabla\tilde{\phi}_h]$$

or

$$\nabla \times (\nabla \times \tilde{\mathbf{A}}_h) - k^2\tilde{\mathbf{A}}_h = \frac{k^2}{j\omega}\nabla\tilde{\phi}_h. \tag{5.173}$$

Choosing $\tilde{\mathbf{A}}_h = \hat{\mathbf{r}}\tilde{A}_h$ and

$$\tilde{\phi}_h = \frac{j\omega}{k^2}\frac{\partial \tilde{A}_h}{\partial r}$$

we find, as with the TM fields,

$$(\nabla^2 + k^2)\left(\frac{\tilde{A}_h}{r}\right) = 0. \tag{5.174}$$

Thus the quantity $\tilde{A}_h/r$ obeys the Helmholtz equation.

We can find the TE fields using (5.171) and (5.172). Substituting we find that

$$\tilde{H}_r = \frac{1}{j\omega\tilde{\mu}\tilde{\epsilon}^c}\left(\frac{\partial^2}{\partial r^2} + k^2\right)\tilde{A}_h, \tag{5.175}$$

$$\tilde{H}_\theta = \frac{1}{j\omega\tilde{\mu}\tilde{\epsilon}^c}\frac{1}{r}\frac{\partial^2 \tilde{A}_h}{\partial r \partial \theta}, \tag{5.176}$$

$$\tilde{H}_\phi = \frac{1}{j\omega\tilde{\mu}\tilde{\epsilon}^c}\frac{1}{r\sin\theta}\frac{\partial^2 \tilde{A}_h}{\partial r \partial \phi}, \tag{5.177}$$

$$\tilde{E}_\theta = -\frac{1}{\tilde{\epsilon}^c}\frac{1}{r\sin\theta}\frac{\partial \tilde{A}_h}{\partial \phi}, \tag{5.178}$$

$$\tilde{E}_\phi = \frac{1}{\tilde{\epsilon}^c}\frac{1}{r}\frac{\partial \tilde{A}_h}{\partial \theta}. \tag{5.179}$$

Example of spherical TE–TM decomposition: a plane wave. Consider a uniform plane wave propagating in the z-direction in a lossless, homogeneous material of permittivity ϵ and permeability μ, such that its electromagnetic field is

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = \hat{\mathbf{x}}\tilde{E}_0(\omega)e^{-jkz} = \hat{\mathbf{x}}\tilde{E}_0(\omega)e^{-jkr\cos\theta},$$
$$\tilde{\mathbf{H}}(\mathbf{r},\omega) = \hat{\mathbf{y}}\frac{\tilde{E}_0(\omega)}{\eta}e^{-jkz} = \hat{\mathbf{x}}\frac{\tilde{E}_0(\omega)}{\eta}e^{-jkr\cos\theta}.$$

We wish to represent this field in terms of the superposition of a field TE to r and a field TM to r. We first find the potential functions $\tilde{\mathbf{A}}_e = \hat{\mathbf{r}}\tilde{A}_e$ and $\tilde{\mathbf{A}}_h = \hat{\mathbf{r}}\tilde{A}_h$ that represent the field. Then we may use (5.166)–(5.170) and (5.175)–(5.179) to find the TE and TM representations.

From (5.166) we see that $\tilde{A}_e$ is related to $\tilde{E}_r$, where $\tilde{E}_r$ is given by

$$\tilde{E}_r = \tilde{E}_0 \sin\theta\cos\phi e^{-jkr\cos\theta} = \frac{\tilde{E}_0\cos\phi}{jkr}\frac{\partial}{\partial\theta}\left[e^{-jkr\cos\theta}\right].$$

We can separate the r and θ dependences of the exponential function by using the identity (E.101). Since $j_n(-z) = (-1)^n j_n(z) = j^{-2n} j_n(z)$ we have

$$e^{-jkr\cos\theta} = \sum_{n=0}^{\infty} j^{-n}(2n+1)j_n(kr)P_n(\cos\theta).$$

Using

$$\frac{\partial P_n(\cos\theta)}{\partial\theta} = \frac{\partial P_n^0(\cos\theta)}{\partial\theta} = P_n^1(\cos\theta)$$

we thus have

$$\tilde{E}_r = -\frac{j\tilde{E}_0\cos\phi}{kr}\sum_{n=1}^{\infty} j^{-n}(2n+1)j_n(kr)P_n^1(\cos\theta).$$

Here we start the sum at $n = 1$ since $P_0^1(x) = 0$. We can now identify the vector potential as

$$\frac{\tilde{A}_e}{r} = \frac{\tilde{E}_0 k}{\omega}\cos\phi\sum_{n=1}^{\infty}\frac{j^{-n}(2n+1)}{n(n+1)}j_n(kr)P_n^1(\cos\theta) \tag{5.180}$$

since by direct differentiation we have

$$\begin{aligned}\tilde{E}_r &= \frac{1}{j\omega\tilde{\mu}\tilde{\epsilon}^c}\left(\frac{\partial^2}{\partial r^2}+k^2\right)\tilde{A}_e \\ &= \frac{\tilde{E}_0 k}{j\omega^2\tilde{\mu}\tilde{\epsilon}^c}\cos\phi\sum_{n=1}^{\infty}\frac{j^{-n}(2n+1)}{n(n+1)}P_n^1(\cos\theta)\left(\frac{\partial^2}{\partial r^2}+k^2\right)[rj_n(kr)] \\ &= -\frac{j\tilde{E}_0\cos\phi}{kr}\sum_{n=1}^{\infty} j^{-n}(2n+1)j_n(kr)P_n^1(\cos\theta),\end{aligned}$$

which satisfies (5.166). Here we have used the defining equation of the spherical Bessel functions (E.15) to show that

$$\begin{aligned}\left(\frac{\partial^2}{\partial r^2}+k^2\right)[rj_n(kr)] &= r\frac{\partial^2}{\partial r^2}j_n(kr) + 2\frac{\partial}{\partial r}j_n(kr) + k^2 r j_n(kr) \\ &= k^2 r\left[\frac{\partial^2}{\partial(kr)^2} + \frac{2}{kr}\frac{\partial}{\partial(kr)}\right]j_n(kr) + k^2 r j_n(kr) \\ &= -k^2 r\left[1 - \frac{n(n+1)}{(kr)^2}\right]j_n(kr) + k^2 r j_n(kr) = \frac{n(n+1)}{r}j_n(kr).\end{aligned}$$

We note immediately that $\tilde{A}_e/r$ satisfies the Helmholtz equation (5.165) since it has the form of the separation of variables solution (D.113).

We may find the vector potential $\tilde{\mathbf{A}}_h = \hat{\mathbf{r}}\tilde{A}_h$ in the same manner. Noting that

$$\tilde{H}_r = \frac{\tilde{E}_0}{\eta}\sin\theta\sin\phi e^{-jkr\cos\theta} = \frac{\tilde{E}_0\sin\phi}{\eta jkr}\frac{\partial}{\partial\theta}\left[e^{-jkr\cos\theta}\right]$$
$$= \frac{1}{j\omega\tilde{\mu}\tilde{\epsilon}^c}\left(\frac{\partial^2}{\partial r^2} + k^2\right)\tilde{A}_h,$$

we have the potential

$$\frac{\tilde{A}_h}{r} = \frac{\tilde{E}_0 k}{\eta\omega}\sin\phi\sum_{n=1}^{\infty}\frac{j^{-n}(2n+1)}{n(n+1)}j_n(kr)P_n^1(\cos\theta). \tag{5.181}$$

We may now compute the transverse components of the TM field using (5.167)–(5.170). For convenience, let us define a new function $\hat{J}_n$ by

$$\hat{J}_n(x) = xj_n(x).$$

Then we may write

$$\tilde{E}_r = -\frac{j\tilde{E}_0\cos\phi}{(kr)^2}\sum_{n=1}^{\infty}j^{-n}(2n+1)\hat{J}_n(kr)P_n^1(\cos\theta), \tag{5.182}$$

$$\tilde{E}_\theta = \frac{j\tilde{E}_0}{kr}\sin\theta\cos\phi\sum_{n=1}^{\infty}a_n\hat{J}_n'(kr)P_n^{1'}(\cos\theta), \tag{5.183}$$

$$\tilde{E}_\phi = \frac{j\tilde{E}_0}{kr\sin\theta}\sin\phi\sum_{n=1}^{\infty}a_n\hat{J}_n'(kr)P_n^1(\cos\theta), \tag{5.184}$$

$$\tilde{H}_\theta = -\frac{\tilde{E}_0}{kr\eta\sin\theta}\sin\phi\sum_{n=1}^{\infty}a_n\hat{J}_n(kr)P_n^1(\cos\theta), \tag{5.185}$$

$$\tilde{H}_\phi = \frac{\tilde{E}_0}{kr\eta}\sin\theta\cos\phi\sum_{n=1}^{\infty}a_n\hat{J}_n(kr)P_n^{1'}(\cos\theta). \tag{5.186}$$

Here

$$\hat{J}_n'(x) = \frac{d}{dx}\hat{J}_n(x) = \frac{d}{dx}[xj_n(x)] = xj_n'(x) + j_n(x)$$

and

$$a_n = \frac{j^{-n}(2n+1)}{n(n+1)}. \tag{5.187}$$

Similarly, we have the TE fields from (5.176)–(5.179):

$$\tilde{H}_r = -\frac{j\tilde{E}_0\sin\phi}{\eta(kr)^2}\sum_{n=1}^{\infty}j^{-n}(2n+1)\hat{J}_n(kr)P_n^1(\cos\theta), \tag{5.188}$$

$$\tilde{H}_\theta = j\frac{\tilde{E}_0}{\eta kr}\sin\theta\sin\phi\sum_{n=1}^{\infty}a_n\hat{J}_n'(kr)P_n^{1'}(\cos\theta), \tag{5.189}$$

$$\tilde{H}_\phi = -j\frac{\tilde{E}_0}{\eta kr\sin\theta}\cos\phi\sum_{n=1}^{\infty}a_n\hat{J}_n'(kr)P_n^1(\cos\theta), \tag{5.190}$$

$$\tilde{E}_\theta = -\frac{\tilde{E}_0}{kr\sin\theta}\cos\phi\sum_{n=1}^{\infty} a_n \hat{J}_n(kr) P_n^1(\cos\theta), \tag{5.191}$$

$$\tilde{E}_\phi = -\frac{\tilde{E}_0}{kr}\sin\theta\sin\phi\sum_{n=1}^{\infty} a_n \hat{J}_n(kr) P_n^{1'}(\cos\theta). \tag{5.192}$$

The total field is then the sum of the TE and TM components.

Example of spherical TE–TM decomposition: scattering by a conducting sphere. Consider a PEC sphere of radius a centered at the origin and imbedded in a homogeneous, isotropic material having parameters $\tilde{\mu}$ and $\tilde{\epsilon}^c$. The sphere is illuminated by a plane wave incident along the z-axis with the fields

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = \hat{\mathbf{x}}\tilde{E}_0(\omega)e^{-jkz} = \hat{\mathbf{x}}\tilde{E}_0(\omega)e^{-jkr\cos\theta},$$

$$\tilde{\mathbf{H}}(\mathbf{r},\omega) = \hat{\mathbf{y}}\frac{\tilde{E}_0(\omega)}{\eta}e^{-jkz} = \hat{\mathbf{x}}\frac{\tilde{E}_0(\omega)}{\eta}e^{-jkr\cos\theta}.$$

We wish to find the field scattered by the sphere.

The boundary condition that determines the scattered field is that the total (incident plus scattered) electric field tangential to the sphere must be zero. We saw in the previous example that the incident electric field may be written as the sum of a field TE to the r-direction and a field TM to the r-direction. Since the region external to the sphere is source-free, we may also represent the scattered field as a sum of TE and TM fields. These may be found from the functions $\tilde{A}_e^s$ and $\tilde{A}_h^s$, which obey the Helmholtz equations (5.165) and (5.174). The general solution to the Helmholtz equation may be found using the separation of variables technique in spherical coordinates, as shown in § A.4, and is given by

$$\left\{\begin{matrix} \tilde{A}_e^s/r \\ \tilde{A}_h^s/r \end{matrix}\right\} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_{nm}Y_{nm}(\theta,\phi)h_n^{(2)}(kr).$$

Here Y_{nm} is the spherical harmonic and we have chosen the spherical Hankel function $h_n^{(2)}$ as the radial dependence since it represents the expected outward-going wave behavior of the scattered field. Since the incident field generated by the potentials (5.180) and (5.181) exactly cancels the field generated by $\tilde{A}_e^s$ and $\tilde{A}_h^s$ on the surface of the sphere, by orthogonality the scattered potential must have ϕ and θ dependencies that match those of the incident field. Thus

$$\frac{\tilde{A}_e^s}{r} = \frac{\tilde{E}_0 k}{\omega}\cos\phi\sum_{n=1}^{\infty} b_n h_n^{(2)}(kr)P_n^1(\cos\theta),$$

$$\frac{\tilde{A}_h^s}{r} = \frac{\tilde{E}_0 k}{\eta\omega}\sin\phi\sum_{n=1}^{\infty} c_n h_n^{(2)}(kr)P_n^1(\cos\theta),$$

where b_n and c_n are constants to be determined by the boundary conditions. By superposition the total field may be computed from the total potentials, which are the sum of the incident and scattered potentials. These are given by

$$\frac{\tilde{A}_e^t}{r} = \frac{\tilde{E}_0 k}{\omega}\cos\phi\sum_{n=1}^{\infty}\left[a_n j_n(kr) + b_n h_n^{(2)}(kr)\right]P_n^1(\cos\theta),$$

$$\frac{\tilde{A}_h^t}{r} = \frac{\tilde{E}_0 k}{\eta\omega}\sin\phi\sum_{n=1}^{\infty}\left[a_n j_n(kr) + c_n h_n^{(2)}(kr)\right]P_n^1(\cos\theta),$$

where a_n is given by (5.187).

The total transverse electric field is found by superposing the TE and TM transverse fields found from the total potentials. We have already computed the transverse incident fields and may easily generalize these results to the total potentials. By (5.183) and (5.191) we have

$$\tilde{E}_\theta^t(a) = \frac{j\tilde{E}_0}{ka}\sin\theta\cos\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}_n'(ka) + b_n\hat{H}_n^{(2)\prime}(ka)\right]P_n^{1\prime}(\cos\theta) -$$
$$- \frac{\tilde{E}_0}{ka\sin\theta}\cos\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}_n(ka) + c_n\hat{H}_n^{(2)}(ka)\right]P_n^1(\cos\theta) = 0,$$

where

$$\hat{H}_n^{(2)}(x) = xh_n^{(2)}(x).$$

By (5.184) and (5.192) we have

$$\tilde{E}_\phi^t(a) = \frac{j\tilde{E}_0}{ka\sin\theta}\sin\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}_n'(ka) + b_n\hat{H}_n^{(2)\prime}(ka)\right]P_n^1(\cos\theta) -$$
$$- \frac{\tilde{E}_0}{ka}\sin\theta\sin\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}_n(ka) + c_n\hat{H}_n^{(2)}(ka)\right]P_n^{1\prime}(\cos\theta) = 0.$$

These two sets of equations are satisfied by the conditions

$$b_n = -\frac{\hat{J}_n'(ka)}{\hat{H}_n^{(2)\prime}(ka)}a_n, \qquad c_n = -\frac{\hat{J}_n(ka)}{\hat{H}_n^{(2)}(ka)}a_n.$$

We can now write the scattered electric fields as

$$\tilde{\mathbf{E}}_r^s = -j\tilde{E}_0\cos\phi\sum_{n=1}^{\infty}b_n\left[\hat{H}_n^{(2)\prime\prime}(kr) + \hat{H}_n^{(2)}(kr)\right]P_n^1(\cos\theta),$$
$$\tilde{\mathbf{E}}_\theta^s = \frac{\tilde{E}_0}{kr}\cos\phi\sum_{n=1}^{\infty}\left[jb_n\sin\theta\hat{H}_n^{(2)\prime}(kr)P_n^{1\prime}(\cos\theta) - c_n\frac{1}{\sin\theta}\hat{H}_n^{(2)}(kr)P_n^1(\cos\theta)\right],$$
$$\tilde{\mathbf{E}}_\phi^s = \frac{\tilde{E}_0}{kr}\sin\phi\sum_{n=1}^{\infty}\left[jb_n\frac{1}{\sin\theta}\hat{H}_n^{(2)\prime}(kr)P_n^1(\cos\theta) - c_n\sin\theta\hat{H}_n^{(2)}(kr)P_n^{1\prime}(\cos\theta)\right].$$

Let us approximate the scattered field for observation points far from the sphere. We may approximate the spherical Hankel functions using (E.68) as

$$\hat{H}_n^{(2)}(z) = zh_n^{(2)}(z) \approx j^{n+1}e^{-jz}, \qquad \hat{H}_n^{(2)\prime}(z) \approx j^n e^{-jz}, \qquad \hat{H}_n^{(2)\prime\prime}(z) \approx -j^{n+1}e^{-jz}.$$

Substituting these we find that $\tilde{E}_r \to 0$ as expected for the far-zone field, while

$$\tilde{E}_\theta^s \approx \tilde{E}_0\frac{e^{-jkr}}{kr}\cos\phi\sum_{n=1}^{\infty}j^{n+1}\left[b_n\sin\theta P_n^{1\prime}(\cos\theta) - c_n\frac{1}{\sin\theta}P_n^1(\cos\theta)\right],$$
$$\tilde{E}_\phi^s \approx \tilde{E}_0\frac{e^{-jkr}}{kr}\sin\phi\sum_{n=1}^{\infty}j^{n+1}\left[b_n\frac{1}{\sin\theta}P_n^1(\cos\theta) - c_n\sin\theta P_n^{1\prime}(\cos\theta)\right].$$

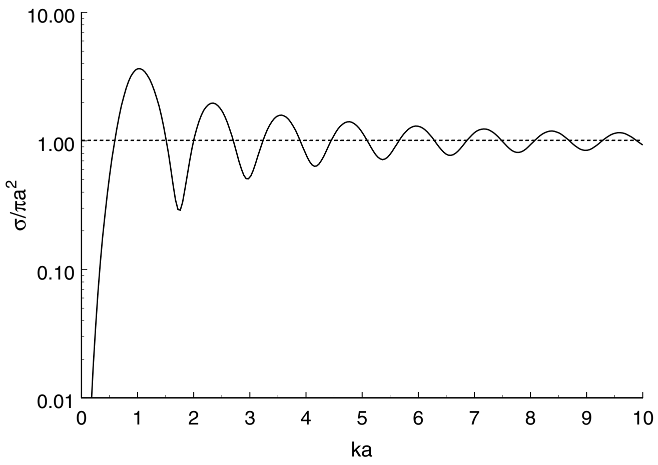

Figure 5.6: Monostatic radar cross-section of a conducting sphere.

From the far-zone fields we can compute the *radar cross-section* (RCS) or *echo area* of the sphere, which is defined by

$$\sigma = \lim_{r\to\infty}\left(4\pi r^2 \frac{|\tilde{\mathbf{E}}^s|^2}{|\tilde{\mathbf{E}}^i|^2}\right). \tag{5.193}$$

Carrying units of m^2, this quantity describes the relative energy density of the scattered field normalized by the distance from the scattering object. Figure 5.6 shows the RCS of a conducting sphere in free space for the *monostatic* case: when the observation direction is aligned with the direction of the incident wave (i.e., $\theta = \pi$), also called the *backscatter* direction. At low frequencies the RCS is proportional to λ^{-4}; this is the range of *Rayleigh scattering*, showing that higher-frequency light scatters more strongly from microscopic particles in the atmosphere (explaining why the sky is blue) [19]. At high frequencies the result approaches that of geometrical optics, and the RCS becomes the interception area of the sphere, πa^2. This is the region of *optical scattering*. Between these two regions lies the *resonance region*, or the region of *Mie scattering*, named for G. Mie who in 1908 published the first rigorous solution for scattering by a sphere (followed soon after by Debye in 1909).

Several interesting phenomena of sphere scattering are best examined in the time domain. We may compute the temporal scattered field by taking the inverse transform of the frequency-domain field. Figure 5.7 shows $E_\theta(t)$ computed in the backscatter direction ($\theta = \pi$) when the incident field waveform $E_0(t)$ is a gaussian pulse and the sphere is in free space. Two distinct features are seen in the scattered field waveform. The first is a sharp pulse almost duplicating the incident field waveform, but of opposite polarity. This is the *specular reflection* produced when the incident field first contacts the sphere and begins to induce a current on the sphere surface. The second feature, called the *creeping wave*, occurs at a time approximately $(2+\pi)a/c$ seconds after the

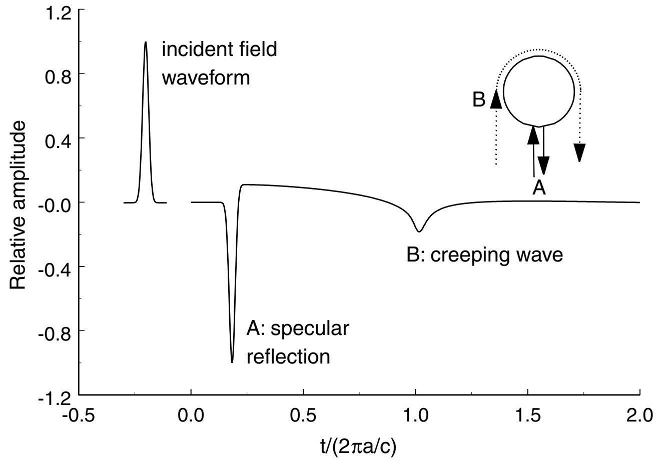

Figure 5.7: Time-domain field back-scattered by a conducting sphere.

specular reflection. This represents the field radiated back along the incident direction by a wave of current excited by the incident field at the tangent point, which travels around the sphere at approximately the speed of light in free space. Although this wave continues to traverse the sphere, its amplitude is reduced so significantly by radiation damping that only a single feature is seen.

5.5 Problems

5.1 Verify that the fields and sources obeying even planar reflection symmetry obey the component Maxwell's equations (5.1)–(5.6). Repeat for fields and sources obeying odd planar reflection symmetry.

5.2 We wish to investigate reflection symmetry through the origin in a homogeneous medium. Under what conditions on magnetic field, magnetic current density, and electric current density are we guaranteed that

$$E_x(x, y, z) = E_x(-x, -y, -z),$$
$$E_y(x, y, z) = E_y(-x, -y, -z),$$
$$E_z(x, y, z) = E_z(-x, -y, -z)?$$

5.3 We wish to investigate reflection symmetry through an axis in a homogeneous medium. Under what conditions on magnetic field, magnetic current density, and electric current density are we guaranteed that

$$E_x(x, y, z) = -E_x(-x, -y, z),$$

$$E_y(x, y, z) = -E_y(-x, -y, z),$$
$$E_z(x, y, z) = E_z(-x, -y, z)?$$

5.4 Consider an electric Hertzian dipole located on the z-axis at $z = h$. Show that if the dipole is parallel to the plane $z = 0$, then adding an oppositely-directed dipole of the same strength at $z = -h$ produces zero electric field tangential to the plane. Also show that if the dipole is z-directed, then adding another z-directed dipole at $z = -h$ produces zero electric field tangential to the $z = 0$ plane. Since the field for $z > 0$ is unaltered in each case if we place a PEC in the $z = 0$ plane, we establish that tangential components of electric current image in the opposite direction while vertical components image in the same direction.

5.5 Consider a z-directed electric line source $\tilde{I}_0$ located at $y = h, x = 0$ between conducting planes at $y = \pm d$, $d > h$. The material between the plates has permeability $\tilde{\mu}(\omega)$ and complex permittivity $\tilde{\epsilon}^c(\omega)$. Write the impressed and scattered fields in terms of Fourier transforms and apply the boundary conditions at $z = \pm d$ to determine the electric field between the plates. Show that the result is identical to the expression (5.8) obtained using symmetry decomposition, which required the boundary condition to be applied only on the top plate.

5.6 Consider a z-directed electric line source $\tilde{I}_0$ located at $y = h, x = 0$ in free space above a dielectric slab occupying $-d < y < d$, $d < h$. The slab has permeability μ_0 and permittivity ϵ. Decompose the source into even and odd constituents and solve for the electric field everywhere using the Fourier transform approach. Describe how you would use the even and odd solutions to solve the problem of a dielectric slab located on top of a PEC ground plane.

5.7 Consider an unbounded, homogeneous, isotropic medium described by permeability $\tilde{\mu}(\omega)$ and complex permittivity $\tilde{\epsilon}^c(\omega)$. Assuming there are magnetic sources present, but no electric sources, show that the fields may be written as

$$\tilde{\mathbf{H}}(\mathbf{r}) = -j\omega\tilde{\epsilon}^c \int_V \bar{\mathbf{G}}_e(\mathbf{r}|\mathbf{r}'; \omega) \cdot \tilde{\mathbf{J}}^i_m(\mathbf{r}', \omega)\, dV',$$
$$\tilde{\mathbf{E}}(\mathbf{r}) = \int_V \bar{\mathbf{G}}_m(\mathbf{r}|\mathbf{r}'; \omega) \cdot \tilde{\mathbf{J}}^i_m(\mathbf{r}', \omega)\, dV',$$

where $\bar{\mathbf{G}}_e$ is given by (5.83) and $\bar{\mathbf{G}}_m$ is given by (5.84).

5.8 Show that for a cubical excluding volume the depolarizing dyadic is $\bar{\mathbf{L}} = \bar{\mathbf{I}}/3$.

5.9 Compute the depolarizing dyadic for a cylindrical excluding volume with height and diameter both $2a$, and with the limit taken as $a \to 0$. Show that $\bar{\mathbf{L}} = 0.293\bar{\mathbf{I}}$.

5.10 Show that the spherical wave function

$$\tilde{\psi}(\mathbf{r}, \omega) = \frac{e^{-jkr}}{4\pi r}$$

obeys the radiation conditions (5.96) and (5.97).

5.11 Verify that the transverse component of the Laplacian of $\mathbf{A}$ is

$$(\nabla^2\mathbf{A})_t = \left[\nabla_t(\nabla_t \cdot \mathbf{A}_t) + \frac{\partial^2 \mathbf{A}_t}{\partial u^2} - \nabla_t \times \nabla_t \times \mathbf{A}_t\right].$$

Verify that the longitudinal component of the Laplacian of **A** is

$$\hat{\mathbf{u}}\left(\hat{\mathbf{u}}\cdot\nabla^2\mathbf{A}\right) = \hat{\mathbf{u}}\nabla^2 A_u.$$

5.12 Verify the identities (B.82)–(B.93).

5.13 Verify the identities (B.94)–(B.98).

5.14 Derive the formula (5.112) for the transverse component of the electric field.

5.15 The longitudinal/transverse decomposition can be performed beginning with the time-domain Maxwell's equations. Show that for a homogeneous, lossless, isotropic region described by permittivity ϵ and permeability μ the longitudinal fields obey the wave equations

$$\left(\frac{\partial^2}{\partial u^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{H}_t = \nabla_t\frac{\partial H_u}{\partial u} - \epsilon\hat{\mathbf{u}}\times\nabla_t\frac{\partial E_u}{\partial t} + \epsilon\frac{\partial \mathbf{J}_{mt}}{\partial t} - \hat{\mathbf{u}}\times\frac{\partial \mathbf{J}_t}{\partial u},$$
$$\left(\frac{\partial^2}{\partial u^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}_t = \nabla_t\frac{\partial E_u}{\partial u} + \mu\hat{\mathbf{u}}\times\nabla_t\frac{\partial H_u}{\partial t} + \hat{\mathbf{u}}\times\frac{\partial \mathbf{J}_{mt}}{\partial u} + \mu\frac{\partial \mathbf{J}_t}{\partial t}.$$

Also show that the transverse fields may be found from the longitudinal fields by solving

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial}{\partial t^2}\right)E_u = \frac{1}{\epsilon}\frac{\partial\rho}{\partial u} + \mu\frac{\partial J_u}{\partial t} + \nabla_t\times\mathbf{J}_{mt},$$
$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial}{\partial t^2}\right)H_u = \frac{1}{\mu}\frac{\partial\rho_m}{\partial u} + \epsilon\frac{\partial J_{mu}}{\partial t} - \nabla_t\times\mathbf{J}_t.$$

Here $v = 1/\sqrt{\mu\epsilon}$.

5.16 Consider a homogeneous, lossless, isotropic region of space described by permittivity ϵ and permeability μ. Beginning with the source-free time-domain Maxwell equations in rectangular coordinates, choose z as the longitudinal direction and show that the TE–TM decomposition is given by

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)E_y = \frac{\partial^2 E_z}{\partial z\partial y} + \mu\frac{\partial^2 H_z}{\partial x\partial t}, \tag{5.194}$$
$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)E_x = \frac{\partial^2 E_z}{\partial x\partial z} - \mu\frac{\partial^2 H_z}{\partial y\partial t}, \tag{5.195}$$
$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)H_y = -\epsilon\frac{\partial^2 E_z}{\partial x\partial t} + \frac{\partial^2 H_z}{\partial y\partial z}, \tag{5.196}$$
$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)H_x = \epsilon\frac{\partial^2 E_z}{\partial y\partial t} + \frac{\partial^2 H_z}{\partial x\partial z}, \tag{5.197}$$

with

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)E_z = 0, \tag{5.198}$$
$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)H_z = 0. \tag{5.199}$$

Here $v = 1/\sqrt{\mu\epsilon}$.

5.17 Consider the case of TM fields in the time domain. Show that for a homogeneous, isotropic, lossless medium with permittivity ϵ and permeability μ the fields may be derived from a single Hertzian potential $\mathbf{\Pi}_e(\mathbf{r}, t) = \hat{\mathbf{u}}\tilde{\Pi}_e(\mathbf{r}, t)$ that satisfies the wave equation

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\Pi_e = 0$$

and that the fields are

$$\mathbf{E} = \nabla_t \frac{\partial \Pi_e}{\partial u} + \hat{\mathbf{u}}\left(\frac{\partial^2}{\partial u^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\Pi_e, \qquad \mathbf{H} = -\epsilon\hat{\mathbf{u}} \times \nabla_t \frac{\partial \Pi_e}{\partial t}.$$

5.18 Consider the case of TE fields in the time domain. Show that for a homogeneous, isotropic, lossless medium with permittivity ϵ and permeability μ the fields may be derived from a single Hertzian potential $\mathbf{\Pi}_h(\mathbf{r}, t) = \hat{\mathbf{u}}\tilde{\Pi}_h(\mathbf{r}, t)$ that satisfies the wave equation

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\Pi_h = 0$$

and that the fields are

$$\mathbf{E} = \mu\hat{\mathbf{u}} \times \nabla_t \frac{\partial \Pi_h}{\partial t}, \qquad \mathbf{H} = \nabla_t \frac{\partial \Pi_h}{\partial u} + \hat{\mathbf{u}}\left(\frac{\partial^2}{\partial u^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\Pi_h.$$

5.19 Show that in the time domain TEM fields may be written for a homogeneous, isotropic, lossless medium with permittivity ϵ and permeability μ in terms of a Hertzian potential $\mathbf{\Pi}_e = \hat{\mathbf{u}}\Pi_e$ that satisfies

$$\nabla_t^2 \Pi_e = 0$$

and that the fields are

$$\mathbf{E} = \nabla_t \frac{\partial \Pi_e}{\partial u}, \qquad \mathbf{H} = -\epsilon\hat{\mathbf{u}} \times \nabla_t \frac{\partial \Pi_e}{\partial t}.$$

5.20 Show that in the time domain TEM fields may be written for a homogeneous, isotropic, lossless medium with permittivity ϵ and permeability μ in terms of a Hertzian potential $\mathbf{\Pi}_h = \hat{\mathbf{u}}\Pi_h$ that satisfies

$$\nabla_t^2 \Pi_h = 0$$

and that the fields are

$$\mathbf{E} = \mu\hat{\mathbf{u}} \times \nabla_t \frac{\partial \Pi_h}{\partial t}, \qquad \mathbf{H} = \nabla_t \frac{\partial \Pi_h}{\partial u}.$$

5.21 Consider a TEM plane-wave field of the form

$$\tilde{\mathbf{E}} = \hat{\mathbf{x}}\tilde{E}_0 e^{-jkz}, \qquad \tilde{\mathbf{H}} = \hat{\mathbf{y}}\frac{\tilde{E}_0}{\eta}e^{-jkz},$$

where $k = \omega\sqrt{\mu\epsilon}$ and $\eta = \sqrt{\mu/\epsilon}$. Show that:

(a) $\tilde{\mathbf{E}}$ may be obtained from $\tilde{\mathbf{H}}$ using the equations for a field that is TE_y;
(b) $\tilde{\mathbf{H}}$ may be obtained from $\tilde{\mathbf{E}}$ using the equations for a field that is TM_x;
(c) $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ may be obtained from the potential $\tilde{\mathbf{\Pi}}_h = \hat{\mathbf{y}}(\tilde{E}_0/k^2\eta)e^{-jkz}$;

(d) $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ may be obtained from the potential $\tilde{\mathbf{\Pi}}_e = \hat{\mathbf{x}}(\tilde{E}_0/k^2)e^{-jkz}$;
(e) $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ may be obtained from the potential $\tilde{\mathbf{\Pi}}_e = \hat{\mathbf{z}}(j\tilde{E}_0 x/k)e^{-jkz}$;
(f) $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ may be obtained from the potential $\tilde{\mathbf{\Pi}}_h = \hat{\mathbf{z}}(j\tilde{E}_0 y/k\eta)e^{-jkz}$.

5.22 Prove the orthogonality relationships (5.149) and (5.150) for the longitudinal fields in a lossless waveguide. *Hint*: Substitute $a = \check{\psi}_e$ and $b = \check{\psi}_h$ into Green's second identity (B.30) and apply the boundary conditions for TE and TM modes.

5.23 Verify the waveguide orthogonality conditions (5.151)-(5.152) by substituting the field expressions for a rectangular waveguide.

5.24 Show that the time-average power carried by a propagating TE mode in a lossless waveguide is given by

$$P_{av} = \frac{1}{2}\omega\mu\beta k_c^2 \int_{CS} \check{\psi}_h \check{\psi}_h^* \, dS.$$

5.25 Show that the time-average stored energy per unit length for a propagating TE mode in a lossless waveguide is

$$\langle W_e \rangle / l = \langle W_m \rangle / l = \frac{\epsilon}{4}(\omega\mu)^2 k_c^2 \int_{CS} \check{\psi}_h \check{\psi}_h^* \, dS.$$

5.26 Consider a waveguide of circular cross-section aligned on the z-axis and filled with a lossless material having permittivity ϵ and permeability μ. Solve for both the TE and TM fields within the guide. List the first ten modes in order by cutoff frequency.

5.27 Consider a propagating TM mode in a lossless rectangular waveguide. Show that the time-average power carried by the propagating wave is

$$P_{av_{nm}} = \frac{1}{2}\omega\epsilon\beta_{nm} k_{c_{nm}}^2 |A_{nm}|^2 \frac{ab}{4}.$$

5.28 Consider a propagating TE mode in a lossless rectangular waveguide. Show that the time-average power carried by the propagating wave is

$$P_{av_{nm}} = \frac{1}{2}\omega\mu\beta_{nm} k_{c_{nm}}^2 |B_{nm}|^2 \frac{ab}{4}.$$

5.29 Consider a homogeneous, lossless region of space characterized by permeability μ and permittivity ϵ. Beginning with the time-domain Maxwell equations, show that the θ and ϕ components of the electromagnetic fields can be written in terms of the radial components. From this give the TE_r–TM_r field decomposition.

5.30 Consider the formula for the radar cross-section of a PEC sphere (5.193). Show that for the monostatic case the RCS becomes

$$\sigma = \frac{\lambda^2}{4\pi} \left| \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)}{\hat{H}_n^{(2)\prime}(ka)\hat{H}_n^{(2)}(ka)} \right|^2.$$

5.31 Beginning with the monostatic formula for the RCS of a conducting sphere given in Problem 5.30, use the small-argument approximation to the spherical Hankel functions to show that the RCS is proportional to λ^{-4} when $ka \ll 1$.

5.32 Beginning with the monostatic formula for the RCS of a conducting sphere given in Problem 5.30, use the large-argument approximation to the spherical Hankel functions to show that the RCS approaches the interception area of the sphere, πa^2, as $ka \to \infty$.

5.33 A material sphere of radius a has permittivity ϵ and permeability μ. The sphere is centered at the origin and illuminated by a plane wave traveling in the z-direction with the fields

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = \hat{\mathbf{x}}\tilde{E}_0(\omega)e^{-jkz}, \qquad \tilde{\mathbf{H}}(\mathbf{r}, \omega) = \hat{\mathbf{y}}\frac{\tilde{E}_0(\omega)}{\eta}e^{-jkz}.$$

Find the fields internal and external to the sphere.