Chapter 6

Integral solutions of Maxwell's equations

6.1 Vector Kirchoff solution: method of Stratton and Chu

One of the most powerful tools for the analysis of electromagnetics problems is the integral solution to Maxwell's equations formulated by Stratton and Chu [187, 188]. These authors used the vector Green's theorem to solve for $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ in much the same way as is done in static fields with the scalar Green's theorem. An alternative approach is to use the Lorentz reciprocity theorem of § 4.10.2, as done by Fradin [74]. The reciprocity approach allows the identification of terms arising from surface discontinuities, which must be added to the result obtained from the other approach [187].

6.1.1 The Stratton–Chu formula

Consider an isotropic, homogeneous medium occupying a bounded region V in space. The medium is described by permeability $\tilde{\mu}(\omega)$, permittivity $\tilde{\epsilon}(\omega)$, and conductivity $\tilde{\sigma}(\omega)$. The region V is bounded by a surface S, which can be multiply-connected so that S is the union of several surfaces $S_1, \ldots, S_N$ as shown in Figure 6.1; these are used to exclude unknown sources and to formulate the *vector Huygens principle*. Impressed electric and magnetic sources may thus reside both inside and outside V.

We wish to solve for the electric and magnetic fields at a point $\mathbf{r}$ within V. To do this we employ the Lorentz reciprocity theorem (4.173), written here using the frequency-domain fields as an integral over primed coordinates:

$$-\oint_S \left[\tilde{\mathbf{E}}_a(\mathbf{r}',\omega) \times \tilde{\mathbf{H}}_b(\mathbf{r}',\omega) - \tilde{\mathbf{E}}_b(\mathbf{r}',\omega) \times \tilde{\mathbf{H}}_a(\mathbf{r}',\omega)\right] \cdot \hat{\mathbf{n}}' dS' =$$

$$\int_V \left[\tilde{\mathbf{E}}_b(\mathbf{r}',\omega) \cdot \tilde{\mathbf{J}}_a(\mathbf{r}',\omega) - \tilde{\mathbf{E}}_a(\mathbf{r}',\omega) \cdot \tilde{\mathbf{J}}_b(\mathbf{r}',\omega) - \quad (6.1)$$

$$\tilde{\mathbf{H}}_b(\mathbf{r}',\omega) \cdot \tilde{\mathbf{J}}_{ma}(\mathbf{r}',\omega) + \tilde{\mathbf{H}}_a(\mathbf{r}',\omega) \cdot \tilde{\mathbf{J}}_{mb}(\mathbf{r}',\omega)\right] dV'. \quad (6.2)$$

Note that the negative sign on the left arises from the definition of $\hat{\mathbf{n}}$ as the *inward* normal to V as shown in Figure 6.1. We place an electric Hertzian dipole at the point $\mathbf{r} = \mathbf{r}_p$ where we wish to compute the field, and set $\tilde{\mathbf{E}}_b = \tilde{\mathbf{E}}_p$ and $\tilde{\mathbf{H}}_b = \tilde{\mathbf{H}}_p$ in the reciprocity theorem, where $\tilde{\mathbf{E}}_p$ and $\tilde{\mathbf{H}}_p$ are the fields produced by the dipole (5.88)–(5.89):

$$\tilde{\mathbf{H}}_p(\mathbf{r},\omega) = j\omega\nabla \times [\tilde{\mathbf{p}}G(\mathbf{r}|\mathbf{r}_p;\omega)], \quad (6.3)$$

$$\tilde{\mathbf{E}}_p(\mathbf{r},\omega) = \frac{1}{\tilde{\epsilon}^c}\nabla \times \left(\nabla \times [\tilde{\mathbf{p}}G(\mathbf{r}|\mathbf{r}_p;\omega)]\right). \quad (6.4)$$

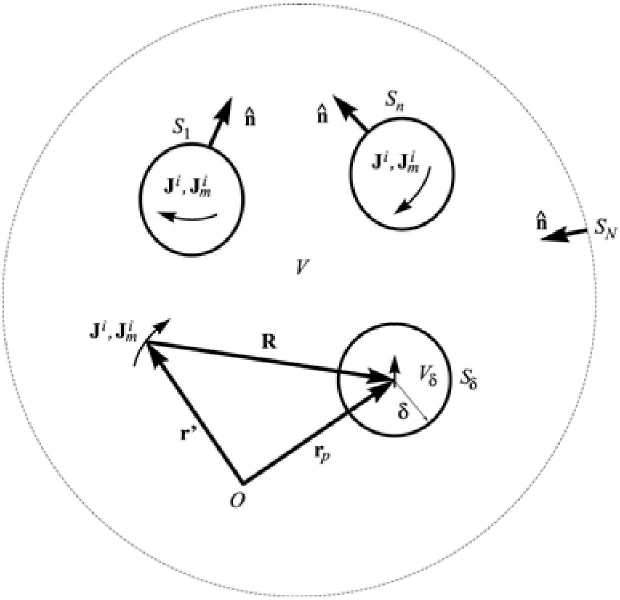

Figure 6.1: Geometry used to derive the Stratton–Chu formula.

We also let $\tilde{\mathbf{E}}_a = \tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}_a = \tilde{\mathbf{H}}$, where $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ are the fields produced by the impressed sources $\tilde{\mathbf{J}}_a = \tilde{\mathbf{J}}^i$ and $\tilde{\mathbf{J}}_{ma} = \tilde{\mathbf{J}}^i_m$ within V that we wish to find at $\mathbf{r} = \mathbf{r}_p$. Since the dipole fields are singular at $\mathbf{r} = \mathbf{r}_p$, we must exclude the point $\mathbf{r}_p$ with a small spherical surface S_δ surrounding the volume V_δ as shown in Figure 6.1. Substituting these fields into (6.2) we obtain

$$-\oint_{S+S_\delta} \left[\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}_p - \tilde{\mathbf{E}}_p \times \tilde{\mathbf{H}}\right] \cdot \hat{\mathbf{n}}'\, dS' = \int_{V-V_\delta} \left[\tilde{\mathbf{E}}_p \cdot \tilde{\mathbf{J}}^i - \tilde{\mathbf{H}}_p \cdot \tilde{\mathbf{J}}^i_m\right] dV'. \tag{6.5}$$

A useful identity involves the spatially-constant vector $\tilde{\mathbf{p}}$ and the Green's function $G(\mathbf{r}'|\mathbf{r}_p)$:

$$\begin{aligned}\nabla' \times \left[\nabla' \times (G\tilde{\mathbf{p}})\right] &= \nabla'[\nabla' \cdot (G\tilde{\mathbf{p}})] - \nabla'^2(G\tilde{\mathbf{p}}) \\ &= \nabla'[\nabla' \cdot (G\tilde{\mathbf{p}})] - \tilde{\mathbf{p}}\nabla'^2 G \\ &= \nabla'(\tilde{\mathbf{p}} \cdot \nabla' G) + \tilde{\mathbf{p}}k^2 G,\end{aligned} \tag{6.6}$$

where we have used $\nabla'^2 G = -k^2 G$ for $\mathbf{r}' \neq \mathbf{r}_p$.

We begin by computing the terms on the left side of (6.5). We suppress the $\mathbf{r}'$ dependence of the fields and also the dependencies of $G(\mathbf{r}'|\mathbf{r}_p)$. Substituting from (6.3) we have

$$\oint_{S+S_\delta} [\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}_p] \cdot \hat{\mathbf{n}}'\, dS' = j\omega \oint_{S+S_\delta} \left[\tilde{\mathbf{E}} \times \nabla' \times (G\tilde{\mathbf{p}})\right] \cdot \hat{\mathbf{n}}'\, dS'.$$

Using $\hat{\mathbf{n}}' \cdot [\tilde{\mathbf{E}} \times \nabla' \times (G\tilde{\mathbf{p}})] = \hat{\mathbf{n}}' \cdot [\tilde{\mathbf{E}} \times (\nabla' G \times \tilde{\mathbf{p}})] = (\hat{\mathbf{n}}' \times \tilde{\mathbf{E}}) \cdot (\nabla' G \times \tilde{\mathbf{p}})$ we can write

$$\oint_{S+S_\delta} [\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}_p] \cdot \hat{\mathbf{n}}'\, dS' = j\omega\tilde{\mathbf{p}} \cdot \oint_{S+S_\delta} [\hat{\mathbf{n}}' \times \tilde{\mathbf{E}}] \times \nabla' G\, dS'.$$

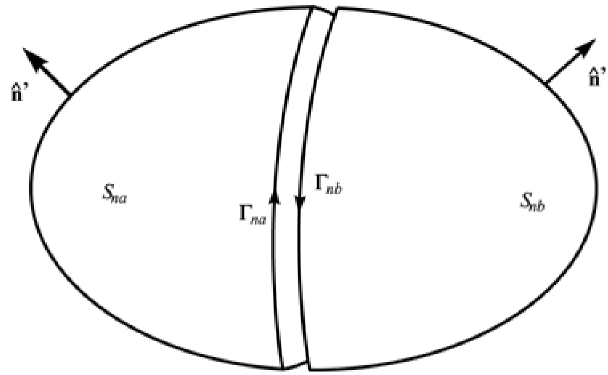

Figure 6.2: Decomposition of surface S_n to isolate surface field discontinuity.

Next we examine

$$\oint_{S+S_\delta} [\tilde{\mathbf{E}}_p \times \tilde{\mathbf{H}}] \cdot \hat{\mathbf{n}}'\, dS' = -\frac{1}{\tilde{\epsilon}^c} \oint_{S+S_\delta} \left[\tilde{\mathbf{H}} \times \nabla' \times \nabla' \times (G\tilde{\mathbf{p}})\right] \cdot \hat{\mathbf{n}}'\, dS'.$$

Use of (6.6) along with the identity (B.43) gives

$$\oint_{S+S_\delta} [\tilde{\mathbf{E}}_p \times \tilde{\mathbf{H}}] \cdot \hat{\mathbf{n}}'\, dS' = -\frac{1}{\tilde{\epsilon}^c} \oint_{S+S_\delta} \Big\{(\tilde{\mathbf{H}} \times \tilde{\mathbf{p}}) k^2 G - \\ - \nabla' \times \left[(\tilde{\mathbf{p}} \cdot \nabla' G)\tilde{\mathbf{H}}\right] + (\tilde{\mathbf{p}} \cdot \nabla' G)(\nabla' \times \tilde{\mathbf{H}})\Big\} \cdot \hat{\mathbf{n}}'\, dS'.$$

We would like to use Stokes's theorem on the second term of the right-hand side. Since the theorem is not valid for surfaces on which $\tilde{\mathbf{H}}$ has discontinuities, we break the closed surfaces in Figure 6.1 into open surfaces whose boundary contours isolate the discontinuities as shown in Figure 6.2. Then we may write

$$\oint_{S_n = S_{na} + S_{nb}} \hat{\mathbf{n}}' \cdot \nabla' \times \left[(\tilde{\mathbf{p}} \cdot \nabla' G)\tilde{\mathbf{H}}\right] dS' = \oint_{\Gamma_{na} + \Gamma_{nb}} \mathbf{dl}' \cdot \tilde{\mathbf{H}}(\tilde{\mathbf{p}} \cdot \nabla' G).$$

For surfaces not containing discontinuities of $\tilde{\mathbf{H}}$ the two contour integrals provide equal and opposite contributions and this term vanishes. Thus the left-hand side of (6.5) is

$$-\oint_{S+S_\delta} \left[\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}_p - \tilde{\mathbf{E}}_p \times \tilde{\mathbf{H}}\right] \cdot \hat{\mathbf{n}}'\, dS' = \\ -\frac{1}{\tilde{\epsilon}^c} \tilde{\mathbf{p}} \cdot \left\{ \oint_{S+S_\delta} \left[j\omega\tilde{\epsilon}^c (\hat{\mathbf{n}}' \times \tilde{\mathbf{E}}) \times \nabla' G + k^2 (\hat{\mathbf{n}}' \times \tilde{\mathbf{H}}) G + \hat{\mathbf{n}}' \cdot (\tilde{\mathbf{J}}^i + j\omega\tilde{\epsilon}^c \tilde{\mathbf{E}}) \nabla' G \right] dS' \right.$$

where we have substituted $\tilde{\mathbf{J}}^i + j\omega\tilde{\epsilon}^c\tilde{\mathbf{E}}$ for $\nabla' \times \tilde{\mathbf{H}}$ and used $(\tilde{\mathbf{H}} \times \tilde{\mathbf{p}}) \cdot \hat{\mathbf{n}}' = \tilde{\mathbf{p}} \cdot (\hat{\mathbf{n}}' \times \tilde{\mathbf{H}})$.

Now consider the right-hand side of (6.5). Substituting from (6.4) we have

$$\int_{V-V_\delta} \tilde{\mathbf{E}}_p \cdot \tilde{\mathbf{J}}^i\, dV' = \frac{1}{\tilde{\epsilon}^c} \int_{V-V_\delta} \tilde{\mathbf{J}}^i \cdot \left[\nabla' \times \nabla' \times (\tilde{\mathbf{p}} G)\right] dV'.$$

Using (6.6) and (B.42), we have

$$\int_{V-V_\delta} \tilde{\mathbf{E}}_p \cdot \tilde{\mathbf{J}}^i\, dV' = \frac{1}{\tilde{\epsilon}^c} \int_{V-V_\delta} \left\{ k^2 (\tilde{\mathbf{p}} \cdot \tilde{\mathbf{J}}^i) G + \nabla' \cdot [\tilde{\mathbf{J}}^i (\tilde{\mathbf{p}} \cdot \nabla' G)] - (\tilde{\mathbf{p}} \cdot \nabla' G) \nabla' \cdot \tilde{\mathbf{J}}^i \right\} dV'.$$

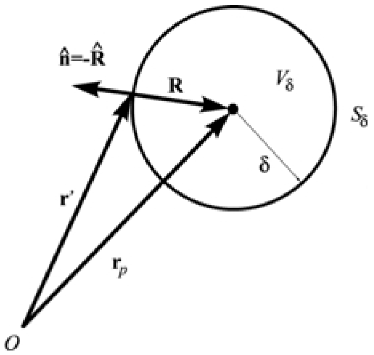

Figure 6.3: Geometry of surface integral used to extract $\mathbf{E}$ at $\mathbf{r}_p$.

Replacing $\nabla' \cdot \tilde{\mathbf{J}}^i$ with $-j\omega\tilde{\rho}^i$ from the continuity equation and using the divergence theorem on the second term on the right-hand side, we then have

$$\int_{V-V_\delta} \tilde{\mathbf{E}}_p \cdot \tilde{\mathbf{J}}^i \, dV' = \frac{1}{\tilde{\epsilon}^c}\tilde{\mathbf{p}} \cdot \left[\int_{V-V_\delta} (k^2 \tilde{\mathbf{J}}^i G + j\omega\tilde{\rho}^i \nabla' G) \, dV' - \oint_{S+S_\delta} (\hat{\mathbf{n}}' \cdot \tilde{\mathbf{J}}^i) \nabla' G \, dS' \right].$$

Lastly we examine

$$\int_{V-V_\delta} \tilde{\mathbf{H}}_p \cdot \tilde{\mathbf{J}}^i_m \, dV' = j\omega \int_{V-V_\delta} \tilde{\mathbf{J}}^i_m \cdot \nabla' \times (G\tilde{\mathbf{p}}) \, dV'.$$

Use of $\tilde{\mathbf{J}}^i_m \cdot \nabla' \times (G\tilde{\mathbf{p}}) = \tilde{\mathbf{J}}^i_m \cdot (\nabla' G \times \tilde{\mathbf{p}}) = \tilde{\mathbf{p}} \cdot (\tilde{\mathbf{J}}^i_m \times \nabla' G)$ gives

$$\int_{V-V_\delta} \tilde{\mathbf{H}}_p \cdot \tilde{\mathbf{J}}^i_m \, dV' = j\omega\tilde{\mathbf{p}} \cdot \int_{V-V_\delta} \tilde{\mathbf{J}}^i_m \times \nabla' G \, dV'.$$

We now substitute all terms into (6.5) and note that each term involves a dot product with $\tilde{\mathbf{p}}$. Since $\tilde{\mathbf{p}}$ is arbitrary we have

$$-\oint_{S+S_\delta} \left[(\hat{\mathbf{n}}' \times \tilde{\mathbf{E}}) \times \nabla' G + (\hat{\mathbf{n}}' \cdot \tilde{\mathbf{E}}) \nabla' G - j\omega\tilde{\mu}(\hat{\mathbf{n}}' \times \tilde{\mathbf{H}}) G \right] dS' +$$
$$+ \frac{1}{j\omega\tilde{\epsilon}^c} \oint_{\Gamma_a + \Gamma_b} (\mathbf{dl}' \cdot \tilde{\mathbf{H}}) \nabla' G = \int_{V-V_\delta} \left[-\tilde{\mathbf{J}}^i_m \times \nabla' G + \frac{\tilde{\rho}^i}{\tilde{\epsilon}^c} \nabla' G - j\omega\tilde{\mu}\tilde{\mathbf{J}}^i G \right] dV'.$$

The electric field may be extracted from the above expression by letting the radius of the excluding volume V_δ recede to zero. We first consider the surface integral over S_δ. Examining Figure 6.3 we see that $R = |\mathbf{r}_p - \mathbf{r}'| = \delta$, $\hat{\mathbf{n}}' = -\hat{\mathbf{R}}$, and

$$\nabla' G(\mathbf{r}'|\mathbf{r}_p) = \frac{d}{dR}\left(\frac{e^{-jkR}}{4\pi R}\right) \nabla' R = \hat{\mathbf{R}} \left(\frac{1 + jk\delta}{4\pi\delta^2}\right) e^{-jk\delta} \approx \frac{\hat{\mathbf{R}}}{\delta^2} \quad \text{as } \delta \to 0.$$

Assuming $\tilde{\mathbf{E}}$ is continuous at $\mathbf{r}' = \mathbf{r}_p$ we can write

$$-\lim_{\delta \to 0} \oint_{S_\delta} \left[(\hat{\mathbf{n}}' \times \tilde{\mathbf{E}}) \times \nabla' G + (\hat{\mathbf{n}}' \cdot \tilde{\mathbf{E}}) \nabla' G - j\omega\tilde{\mu}(\hat{\mathbf{n}}' \times \tilde{\mathbf{H}}) G \right] dS' =$$
$$\lim_{\delta \to 0} \int_\Omega \frac{1}{4\pi} \left[(\hat{\mathbf{R}} \times \tilde{\mathbf{E}}) \times \frac{\hat{\mathbf{R}}}{\delta^2} + (\hat{\mathbf{R}} \cdot \tilde{\mathbf{E}}) \frac{\hat{\mathbf{R}}}{\delta^2} - j\omega\tilde{\mu}(\hat{\mathbf{R}} \times \tilde{\mathbf{H}}) \frac{1}{\delta} \right] \delta^2 \, d\Omega =$$
$$\lim_{\delta \to 0} \int_\Omega \frac{1}{4\pi} \left[-(\hat{\mathbf{R}} \cdot \tilde{\mathbf{E}})\hat{\mathbf{R}} + (\hat{\mathbf{R}} \cdot \hat{\mathbf{R}})\tilde{\mathbf{E}} + (\hat{\mathbf{R}} \cdot \tilde{\mathbf{E}})\hat{\mathbf{R}} \right] d\Omega = \tilde{\mathbf{E}}(\mathbf{r}_p).$$

Here we have used $\int_\Omega d\Omega = 4\pi$ for the total solid angle subtending the sphere S_δ. Finally, assuming that the volume sources are continuous, the volume integral over V_δ vanishes and we have

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = \int_V \left(-\tilde{\mathbf{J}}_m^i \times \nabla' G + \frac{\tilde{\rho}^i}{\tilde{\epsilon}^c} \nabla' G - j\omega\tilde{\mu}\tilde{\mathbf{J}}^i G \right) dV' + \\ + \sum_{n=1}^{N} \int_{S_n} \left[(\hat{\mathbf{n}}' \times \tilde{\mathbf{E}}) \times \nabla' G + (\hat{\mathbf{n}}' \cdot \tilde{\mathbf{E}}) \nabla' G - j\omega\tilde{\mu}(\hat{\mathbf{n}}' \times \tilde{\mathbf{H}}) G \right] dS' - \\ - \sum_{n=1}^{N} \frac{1}{j\omega\tilde{\epsilon}^c} \oint_{\Gamma_{na}+\Gamma_{nb}} (\mathbf{dl}' \cdot \tilde{\mathbf{H}}) \nabla' G. \tag{6.7}$$

A similar formula for $\tilde{\mathbf{H}}$ can be derived by placing a magnetic dipole of moment $\tilde{\mathbf{p}}_m$ at $\mathbf{r} = \mathbf{r}_p$ and proceeding as above. This leads to

$$\tilde{\mathbf{H}}(\mathbf{r},\omega) = \int_V \left(\tilde{\mathbf{J}}^i \times \nabla' G + \frac{\tilde{\rho}_m^i}{\tilde{\mu}} \nabla' G - j\omega\tilde{\epsilon}^c \tilde{\mathbf{J}}_m^i G \right) dV' + \\ + \sum_{n=1}^{N} \int_{S_n} \left[(\hat{\mathbf{n}}' \times \tilde{\mathbf{H}}) \times \nabla' G + (\hat{\mathbf{n}}' \cdot \tilde{\mathbf{H}}) \nabla' G + j\omega\tilde{\epsilon}^c(\hat{\mathbf{n}}' \times \tilde{\mathbf{E}}) G \right] dS' + \\ + \sum_{n=1}^{N} \frac{1}{j\omega\tilde{\mu}} \oint_{\Gamma_{na}+\Gamma_{nb}} (\mathbf{dl}' \cdot \tilde{\mathbf{E}}) \nabla' G. \tag{6.8}$$

We can also obtain this expression by substituting (6.7) into Faraday's law.

6.1.2 The Sommerfeld radiation condition

In § 5.2.2 we found that if the potentials are not to be influenced by effects that are infinitely removed, then they must obey a radiation condition. We can make the same argument about the fields from (6.7) and (6.8). Let us allow one of the excluding surfaces, say S_N, to recede to infinity (enclosing all of the sources as it expands). As $S_N \to \infty$ any contributions from the fields on this surface to the fields at $\mathbf{r}$ should vanish.

Letting S_N be a sphere centered at the origin, we note that $\hat{\mathbf{n}}' = -\hat{\mathbf{r}}'$ and that as $r' \to \infty$

$$G(\mathbf{r}|\mathbf{r}';\omega) = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \approx \frac{e^{-jkr'}}{4\pi r'},$$

$$\nabla' G(\mathbf{r}|\mathbf{r}';\omega) = \hat{\mathbf{R}} \left(\frac{1+jkR}{4\pi R^2} \right) e^{-jkR} \approx -\hat{\mathbf{r}}' \left(\frac{1+jkr'}{r'} \right) \frac{e^{-jkr'}}{4\pi r'}.$$

Substituting these expressions into (6.7) we find that

$$\lim_{S_N \to S_\infty} \oint_{S_N} \left[(\hat{\mathbf{n}}' \times \tilde{\mathbf{E}}) \times \nabla' G + (\hat{\mathbf{n}}' \cdot \tilde{\mathbf{E}}) \nabla' G - j\omega\tilde{\mu}(\hat{\mathbf{n}}' \times \tilde{\mathbf{H}}) G \right] dS' \\ \approx \lim_{r' \to \infty} \int_0^{2\pi} \int_0^{\pi} \left\{ \left[(\hat{\mathbf{r}}' \times \tilde{\mathbf{E}}) \times \hat{\mathbf{r}}' + (\hat{\mathbf{r}}' \cdot \tilde{\mathbf{E}}) \hat{\mathbf{r}}' \right] \left(\frac{1+jkr'}{r'} \right) \right. \\ \left. + j\omega\tilde{\mu}(\hat{\mathbf{r}}' \times \tilde{\mathbf{H}}) \right\} \frac{e^{-jkr'}}{4\pi r'} r'^2 \sin\theta' \, d\theta' \, d\phi' \\ \approx \lim_{r' \to \infty} \int_0^{2\pi} \int_0^{\pi} \left\{ r' \left[jk\tilde{\mathbf{E}} + j\omega\tilde{\mu}(\hat{\mathbf{r}}' \times \tilde{\mathbf{H}}) \right] + \tilde{\mathbf{E}} \right\} \frac{e^{-jkr'}}{4\pi} \sin\theta' \, d\theta' \, d\phi'.$$

Since this gives the contribution to the field in V from the fields on the surface receding to infinity, we expect that this term should be zero. If the medium has loss, then the exponential term decays and drives the contribution to zero. For a lossless medium the contributions are zero if

$$\lim_{r\to\infty} r\tilde{\mathbf{E}}(\mathbf{r},\omega) < \infty, \tag{6.9}$$

$$\lim_{r\to\infty} r\left[\eta\hat{\mathbf{r}}\times\tilde{\mathbf{H}}(\mathbf{r},\omega)+\tilde{\mathbf{E}}(\mathbf{r},\omega)\right] = 0. \tag{6.10}$$

To accompany (6.8) we also have

$$\lim_{r\to\infty} r\tilde{\mathbf{H}}(\mathbf{r},\omega) < \infty, \tag{6.11}$$

$$\lim_{r\to\infty} r\left[\eta\tilde{\mathbf{H}}(\mathbf{r},\omega)-\hat{\mathbf{r}}\times\tilde{\mathbf{E}}(\mathbf{r},\omega)\right] = 0. \tag{6.12}$$

We refer to (6.9) and (6.11) as the *finiteness conditions*, and to (6.10) and (6.12) as the *Sommerfeld radiation condition*, for the electromagnetic field. They show that far from the sources the fields must behave as a wave TEM to the r-direction. We shall see in § 6.2 that the waves are in fact *spherical TEM waves*.

6.1.3 Fields in the excluded region: the extinction theorem

The Stratton–Chu formula provides a solution for the field within the region V, external to the excluded regions. An interesting consequence of this formula, and one that helps us identify the equivalence principle, is that it gives the null result $\tilde{\mathbf{H}} = \tilde{\mathbf{E}} = 0$ when evaluated at points within the excluded regions.

We can show this by considering two cases. In the first case we do *not* exclude the particular region V_m, but do exclude the remaining regions V_n, $n \neq m$. Then the electric field everywhere outside the remaining excluded regions (including at points within V_m) is, by (6.7),

$$\begin{aligned}
\tilde{\mathbf{E}}(\mathbf{r},\omega) = &\int_{V+V_m}\left(-\tilde{\mathbf{J}}_m^i\times\nabla' G+\frac{\tilde{\rho}^i}{\tilde{\epsilon}^c}\nabla' G-j\omega\tilde{\mu}\tilde{\mathbf{J}}^i G\right)dV' + \\
&+\sum_{n\neq m}\int_{S_n}\left[(\hat{\mathbf{n}}'\times\tilde{\mathbf{E}})\times\nabla' G+(\hat{\mathbf{n}}'\cdot\tilde{\mathbf{E}})\nabla' G-j\omega\tilde{\mu}(\hat{\mathbf{n}}'\times\tilde{\mathbf{H}})G\right]dS' - \\
&-\sum_{n\neq m}\frac{1}{j\omega\tilde{\epsilon}^c}\oint_{\Gamma_{na}+\Gamma_{nb}}(\mathbf{dl}'\cdot\tilde{\mathbf{H}})\nabla' G, \qquad \mathbf{r}\in V+V_m.
\end{aligned}$$

In the second case we apply the Stratton–Chu formula only to V_m, and exclude all other regions. We incur a sign change on the surface and line integrals compared to the first case because the normal is now directed oppositely. By (6.7) we have

$$\begin{aligned}
\tilde{\mathbf{E}}(\mathbf{r},\omega) = &\int_{V_m}\left(-\tilde{\mathbf{J}}_m^i\times\nabla' G+\frac{\tilde{\rho}^i}{\tilde{\epsilon}^c}\nabla' G-j\omega\tilde{\mu}\tilde{\mathbf{J}}^i G\right)dV' - \\
&-\int_{S_m}\left[(\hat{\mathbf{n}}'\times\tilde{\mathbf{E}})\times\nabla' G+(\hat{\mathbf{n}}'\cdot\tilde{\mathbf{E}})\nabla' G-j\omega\tilde{\mu}(\hat{\mathbf{n}}'\times\tilde{\mathbf{H}})G\right]dS' + \\
&+\frac{1}{j\omega\tilde{\epsilon}^c}\oint_{\Gamma_{na}+\Gamma_{nb}}(\mathbf{dl}'\cdot\tilde{\mathbf{H}})\nabla' G, \qquad \mathbf{r}\in V_m.
\end{aligned}$$

Each of the expressions for $\tilde{\mathbf{E}}$ is equally valid for points within V_m. Upon subtraction we get

$$
\begin{aligned}
0 = &\int_V \left(-\tilde{\mathbf{J}}_m^i \times \nabla' G + \frac{\tilde{\rho}^i}{\tilde{\epsilon}^c} \nabla' G - j\omega\tilde{\mu}\tilde{\mathbf{J}}^i G \right) dV' + \\
&+ \sum_{n=1}^{N} \int_{S_n} \left[(\hat{\mathbf{n}}' \times \tilde{\mathbf{E}}) \times \nabla' G + (\hat{\mathbf{n}}' \cdot \tilde{\mathbf{E}}) \nabla' G - j\omega\tilde{\mu}(\hat{\mathbf{n}}' \times \tilde{\mathbf{H}}) G \right] dS' - \\
&- \sum_{n=1}^{N} \frac{1}{j\omega\tilde{\epsilon}^c} \oint_{\Gamma_{na}+\Gamma_{nb}} (\mathbf{dl}' \cdot \tilde{\mathbf{H}}) \nabla' G, \qquad \mathbf{r} \in V_m.
\end{aligned}
$$

This expression is exactly the Stratton–Chu formula (6.7) evaluated at points within the excluded region V_m. The treatment of $\tilde{\mathbf{H}}$ is analogous and is left as an exercise. Since we may repeat this for any excluded region, we find that the Stratton–Chu formula returns the null field when evaluated at points outside V. This is sometimes referred to as the *vector Ewald–Oseen extinction theorem* [90]. We must emphasize that the fields within the excluded regions are *not* generally equal to zero; the Stratton–Chu formula merely returns this result when evaluated there.

6.2 Fields in an unbounded medium

Two special cases of the Stratton–Chu formula are important because of their application to antenna theory. The first is that of sources radiating into an unbounded region. The second involves a bounded region with all sources excluded. We shall consider the former here and the latter in § 6.3.

Assuming that there are no bounding surfaces in (6.7) and (6.8), except for one surface that has been allowed to recede to infinity and therefore provides no surface contribution, we find that the electromagnetic fields in unbounded space are given by

$$
\begin{aligned}
\tilde{\mathbf{E}} &= \int_V \left(-\tilde{\mathbf{J}}_m^i \times \nabla' G + \frac{\tilde{\rho}^i}{\tilde{\epsilon}^c} \nabla' G - j\omega\tilde{\mu}\tilde{\mathbf{J}}^i G \right) dV', \\
\tilde{\mathbf{H}} &= \int_V \left(\tilde{\mathbf{J}}^i \times \nabla' G + \frac{\tilde{\rho}_m^i}{\tilde{\mu}} \nabla' G - j\omega\tilde{\epsilon}^c \tilde{\mathbf{J}}_m^i G \right) dV'.
\end{aligned}
$$

We can view the right-hand sides as superpositions of the fields present in the cases where (1) electric sources are present exclusively, and (2) magnetic sources are present exclusively. With $\tilde{\rho}_m^i = 0$ and $\tilde{\mathbf{J}}_m^i = 0$ we find that

$$
\tilde{\mathbf{E}} = \int_V \left(\frac{\tilde{\rho}^i}{\tilde{\epsilon}^c} \nabla' G - j\omega\tilde{\mu}\tilde{\mathbf{J}}^i G \right) dV', \tag{6.13}
$$

$$
\tilde{\mathbf{H}} = \int_V \tilde{\mathbf{J}}^i \times \nabla' G \, dV'. \tag{6.14}
$$

Using $\nabla' G = -\nabla G$ we can write

$$
\begin{aligned}
\tilde{\mathbf{E}}(\mathbf{r}, \omega) &= -\nabla \int_V \frac{\tilde{\rho}^i(\mathbf{r}', \omega)}{\tilde{\epsilon}^c(\omega)} G(\mathbf{r}|\mathbf{r}'; \omega) \, dV' - j\omega \int_V \tilde{\mu}(\omega) \tilde{\mathbf{J}}^i(\mathbf{r}', \omega) G(\mathbf{r}|\mathbf{r}'; \omega) \, dV' \\
&= -\nabla \tilde{\phi}_e(\mathbf{r}, \omega) - j\omega \tilde{\mathbf{A}}_e(\mathbf{r}, \omega),
\end{aligned}
$$

where

$$\tilde{\phi}_e(\mathbf{r},\omega) = \int_V \frac{\tilde{\rho}^i(\mathbf{r}',\omega)}{\tilde{\epsilon}^c(\omega)} G(\mathbf{r}|\mathbf{r}';\omega)\,dV',$$

$$\tilde{\mathbf{A}}_e(\mathbf{r},\omega) = \int_V \tilde{\mu}(\omega)\tilde{\mathbf{J}}^i(\mathbf{r}',\omega)G(\mathbf{r}|\mathbf{r}';\omega)\,dV', \tag{6.15}$$

are the electric scalar and vector potential functions introduced in § 5.2. Using $\tilde{\mathbf{J}}^i \times \nabla' G = -\tilde{\mathbf{J}}^i \times \nabla G = \nabla \times (\tilde{\mathbf{J}}^i G)$ we have

$$\tilde{\mathbf{H}}(\mathbf{r},\omega) = \frac{1}{\tilde{\mu}(\omega)} \nabla \times \int_V \tilde{\mu}(\omega)\tilde{\mathbf{J}}^i(\mathbf{r}',\omega)G(\mathbf{r}|\mathbf{r}';\omega)\,dV'$$

$$= \frac{1}{\tilde{\mu}(\omega)} \nabla \times \tilde{\mathbf{A}}_e(\mathbf{r},\omega). \tag{6.16}$$

These expressions for the fields are identical to those of (5.56) and (5.57), and thus the integral formula for the electromagnetic fields produces a result identical to that obtained using potential relations. Similarly, with $\tilde{\rho}^i = 0, \tilde{\mathbf{J}}^i = 0$ we have

$$\tilde{\mathbf{E}} = -\int_V \tilde{\mathbf{J}}^i_m \times \nabla' G\,dV',$$

$$\tilde{\mathbf{H}} = \int_V \left(\frac{\tilde{\rho}^i_m}{\tilde{\mu}} \nabla' G - j\omega\tilde{\epsilon}^c \tilde{\mathbf{J}}^i_m G \right) dV',$$

or

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = -\frac{1}{\tilde{\epsilon}^c(\omega)} \nabla \times \tilde{\mathbf{A}}_h(\mathbf{r},\omega),$$

$$\tilde{\mathbf{H}}(\mathbf{r},\omega) = -\nabla\tilde{\phi}_h(\mathbf{r},\omega) - j\omega\tilde{\mathbf{A}}_h(\mathbf{r},\omega),$$

where

$$\tilde{\phi}_h(\mathbf{r},\omega) = \int_V \frac{\tilde{\rho}^i_m(\mathbf{r}',\omega)}{\tilde{\mu}(\omega)} G(\mathbf{r}|\mathbf{r}';\omega)\,dV',$$

$$\tilde{\mathbf{A}}_h(\mathbf{r},\omega) = \int_V \tilde{\epsilon}^c(\omega)\tilde{\mathbf{J}}^i_m(\mathbf{r}',\omega)G(\mathbf{r}|\mathbf{r}';\omega)\,dV',$$

are the magnetic scalar and vector potentials introduced in § 5.2.

6.2.1 The far-zone fields produced by sources in unbounded space

Many antennas may be analyzed in terms of electric currents and charges radiating in unbounded space. Since antennas are used to transmit information over great distances, the fields far from the sources are often of most interest. Assume that the sources are contained within a sphere of radius r_s centered at the origin. We define the *far zone* of the sources to consist of all observation points satisfying both $r \gg r_s$ (and thus $r \gg r'$) and $kr \gg 1$. For points in the far zone we may approximate the unit vector $\hat{\mathbf{R}}$ directed from the sources to the observation point by the unit vector $\hat{\mathbf{r}}$ directed from the origin to the observation point. We may also approximate

$$\nabla' G = \frac{d}{dR}\left(\frac{e^{-jkR}}{4\pi R}\right)\nabla' R = \hat{\mathbf{R}}\left(\frac{1+jkR}{R}\right)\frac{e^{-jkR}}{4\pi R} \approx \hat{\mathbf{r}} jk \frac{e^{-jkR}}{4\pi R} = \hat{\mathbf{r}} jkG. \tag{6.17}$$

Using this we can obtain expressions for $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ in the far zone of the sources. The approximation (6.17) leads directly to

$$\tilde{\rho}^i \nabla' G \approx \left[j\frac{\nabla' \cdot \tilde{\mathbf{J}}^i}{\omega} \right] (\hat{\mathbf{r}} jkG) = -\frac{k}{\omega}\hat{\mathbf{r}} \left[\nabla' \cdot (G\tilde{\mathbf{J}}^i) - \tilde{\mathbf{J}}^i \cdot \nabla' G \right].$$

Substituting this into (6.13), again using (6.17) and also using the divergence theorem, we have

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) \approx -\int_V j\omega\tilde{\mu} \left[\tilde{\mathbf{J}}^i - \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \tilde{\mathbf{J}}^i) \right] G\, dV' + \hat{\mathbf{r}}\frac{k}{\omega\tilde{\epsilon}^c} \oint_S (\hat{\mathbf{n}}' \cdot \tilde{\mathbf{J}}^i) G\, dS',$$

where the surface S surrounds the volume V that contains the impressed sources. If we let this volume slightly exceed that needed to contain the sources, then we do not change the value of the volume integral above; however, the surface integral vanishes since $\hat{\mathbf{n}}' \cdot \tilde{\mathbf{J}}^i = 0$ everywhere on the surface. Using $\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \tilde{\mathbf{J}}^i) = \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \tilde{\mathbf{J}}^i) - \tilde{\mathbf{J}}^i$ we then obtain the far-zone expression

$$\begin{aligned} \tilde{\mathbf{E}}(\mathbf{r}, \omega) &\approx j\omega\hat{\mathbf{r}} \times \left[\hat{\mathbf{r}} \times \int_V \tilde{\mu}(\omega)\tilde{\mathbf{J}}^i(\mathbf{r}', \omega) G(\mathbf{r}|\mathbf{r}'; \omega)\, dV' \right] \\ &= j\omega\hat{\mathbf{r}} \times \left[\hat{\mathbf{r}} \times \tilde{\mathbf{A}}_e(\mathbf{r}, \omega) \right], \end{aligned}$$

where $\tilde{\mathbf{A}}_e$ is the electric vector potential. The far-zone electric field has no r-component, and it is often convenient to write

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) \approx -j\omega\tilde{\mathbf{A}}_{eT}(\mathbf{r}, \omega) \tag{6.18}$$

where $\tilde{\mathbf{A}}_{eT}$ is the vector component of $\tilde{\mathbf{A}}_e$ transverse to the r-direction:

$$\tilde{\mathbf{A}}_{eT} = -\hat{\mathbf{r}} \times \left[\hat{\mathbf{r}} \times \tilde{\mathbf{A}}_e \right] = \tilde{\mathbf{A}}_e - \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \tilde{\mathbf{A}}_e) = \hat{\boldsymbol{\theta}}\tilde{A}_{e\theta} + \hat{\boldsymbol{\phi}}\tilde{A}_{e\phi}.$$

We can approximate the magnetic field in a similar fashion. Noting that $\tilde{\mathbf{J}}^i \times \nabla' G = \tilde{\mathbf{J}}^i \times (jk\hat{\mathbf{r}}G)$ we have

$$\begin{aligned} \tilde{\mathbf{H}}(\mathbf{r}, \omega) &\approx -j\frac{k}{\tilde{\mu}(\omega)}\hat{\mathbf{r}} \times \int_V \tilde{\mu}(\omega)\tilde{\mathbf{J}}^i(\mathbf{r}', \omega) G(\mathbf{r}|\mathbf{r}', \omega)\, dV' \\ &\approx -\frac{1}{\eta} j\omega\hat{\mathbf{r}} \times \tilde{\mathbf{A}}_e(\mathbf{r}, \omega). \end{aligned}$$

With this we have

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = -\eta\hat{\mathbf{r}} \times \tilde{\mathbf{H}}(\mathbf{r}, \omega), \qquad \tilde{\mathbf{H}}(\mathbf{r}, \omega) = \frac{\hat{\mathbf{r}} \times \tilde{\mathbf{E}}(\mathbf{r}, \omega)}{\eta},$$

in the far zone.

To simplify the computations involved, we often choose to approximate the vector potential in the far zone. Noting that

$$R = \sqrt{(\mathbf{r} - \mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')} = \sqrt{r^2 + r'^2 - 2(\mathbf{r} \cdot \mathbf{r}')}$$

and remembering that $r \gg r'$ for $\mathbf{r}$ in the far zone, we can use the leading terms of a binomial expansion of the square root to get

$$R = r\sqrt{1 - \frac{2(\hat{\mathbf{r}}\cdot\mathbf{r}')}{r} + \left(\frac{r'}{r}\right)^2} \approx r\sqrt{1 - \frac{2(\hat{\mathbf{r}}\cdot\mathbf{r}')}{r}} \approx r\left[1 - \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r}\right]$$
$$\approx r - \hat{\mathbf{r}}\cdot\mathbf{r}'. \tag{6.19}$$

Thus the Green's function may be approximated as

$$G(\mathbf{r}|\mathbf{r}';\omega) \approx \frac{e^{-jkr}}{4\pi r} e^{jk\hat{\mathbf{r}}\cdot\mathbf{r}'}. \tag{6.20}$$

Here we have kept the approximation (6.19) intact in the phase of G but have used $1/R \approx 1/r$ in the amplitude of G. We must keep a more accurate approximation for the phase since $k(\hat{\mathbf{r}}\cdot\mathbf{r}')$ may be an appreciable fraction of a radian. We thus have the far-zone approximation for the vector potential

$$\tilde{\mathbf{A}}_e(\mathbf{r},\omega) \approx \tilde{\mu}(\omega)\frac{e^{-jkr}}{4\pi r}\int_V \tilde{\mathbf{J}}^i(\mathbf{r}',\omega)e^{jk\hat{\mathbf{r}}\cdot\mathbf{r}'}\,dV',$$

which we may use in computing (6.18).

Let us summarize the expressions for computing the far-zone fields:

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = -j\omega\left[\hat{\boldsymbol{\theta}}\tilde{A}_{e\theta}(\mathbf{r},\omega) + \hat{\boldsymbol{\phi}}\tilde{A}_{e\phi}(\mathbf{r},\omega)\right], \tag{6.21}$$

$$\tilde{\mathbf{H}}(\mathbf{r},\omega) = \frac{\hat{\mathbf{r}}\times\tilde{\mathbf{E}}(\mathbf{r},\omega)}{\eta}, \tag{6.22}$$

$$\tilde{\mathbf{A}}_e(\mathbf{r},\omega) = \frac{e^{-jkr}}{4\pi r}\tilde{\mu}(\omega)\tilde{\mathbf{a}}_e(\theta,\phi,\omega), \tag{6.23}$$

$$\tilde{\mathbf{a}}_e(\theta,\phi,\omega) = \int_V \tilde{\mathbf{J}}^i(\mathbf{r}',\omega)e^{jk\hat{\mathbf{r}}\cdot\mathbf{r}'}\,dV'. \tag{6.24}$$

Here $\tilde{\mathbf{a}}_e$ is called the *directional weighting function*. This function is independent of r and describes the angular variation, or *pattern*, of the fields.

In the far zone $\tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}$, $\hat{\mathbf{r}}$ are mutually orthogonal. Because of this, and because the fields vary as e^{-jkr}/r, the electromagnetic field in the far zone takes the form of a spherical TEM wave, which is consistent with the Sommerfeld radiation condition.

Power radiated by time-harmonic sources in unbounded space. In § 5.2.1 we defined the power radiated by a time-harmonic source in unbounded space as the total time-average power passing through a sphere of very large radius. We found that for a Hertzian dipole the radiated power could be computed from the far-zone fields through

$$P_{av} = \lim_{r\to\infty}\int_0^{2\pi}\int_0^{\pi}\mathbf{S}_{av}\cdot\hat{\mathbf{r}}r^2\sin\theta\,d\theta\,d\phi$$

where

$$\mathbf{S}_{av} = \frac{1}{2}\operatorname{Re}\left\{\check{\mathbf{E}}\times\check{\mathbf{H}}^*\right\}$$

is the time-average Poynting vector. By superposition this holds for any localized source. Assuming a lossless medium and using phasor notation to describe the time-harmonic

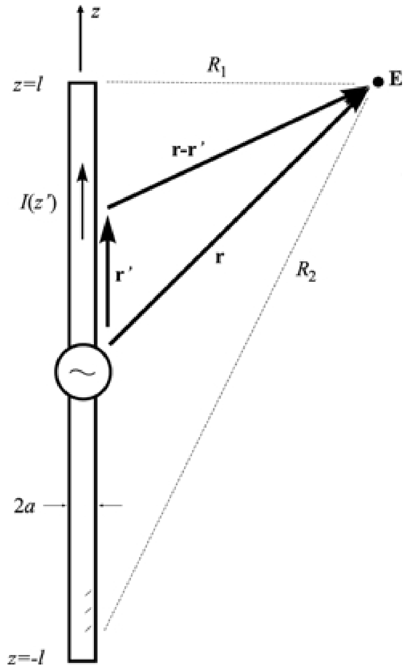

Figure 6.4: Dipole antenna in a lossless unbounded medium.

fields we have, by (6.22),

$$\mathbf{S}_{av} = \frac{1}{2}\,\text{Re}\left\{\frac{\check{\mathbf{E}} \times (\hat{\mathbf{r}} \times \check{\mathbf{E}}^*)}{\eta}\right\} = \hat{\mathbf{r}}\frac{\check{\mathbf{E}} \cdot \check{\mathbf{E}}^*}{2\eta}.$$

Substituting from (6.21), we can also write $\mathbf{S}_{av}$ in terms of the directional weighting function as

$$\mathbf{S}_{av} = \hat{\mathbf{r}}\frac{\check{\omega}^2}{2\eta}\left(\check{A}_{e\theta}\check{A}_{e\theta}^* + \check{A}_{e\phi}\check{A}_{e\phi}^*\right) = \hat{\mathbf{r}}\frac{k^2\eta}{(4\pi r)^2}\left(\frac{1}{2}\check{a}_{e\theta}\check{a}_{e\theta}^* + \frac{1}{2}\check{a}_{e\phi}\check{a}_{e\phi}^*\right). \tag{6.25}$$

We note that $\mathbf{S}_{av}$ describes the variation of the power density with θ, ϕ, and is thus sometimes used as a descriptor of the *power pattern* of the sources.

Example of a current source radiating into an unbounded medium: the dipole antenna. A common type of antenna consists of a thin wire of length $2l$ and radius a, fed at the center by a voltage generator as shown in Figure 6.4. The generator induces an impressed current on the surface of the wire which in turn radiates an electromagnetic wave. For very thin wires ($a \ll \lambda$, $a \ll l$) embedded in a lossless medium, the current may be accurately approximated using a standing-wave distribution:

$$\tilde{\mathbf{J}}^i(\mathbf{r}, \omega) = \hat{\mathbf{z}}\tilde{I}(\omega)\sin\left[k(l - |z|)\right]\delta(x)\delta(y). \tag{6.26}$$

We may compute the field produced by the dipole antenna by first finding the vector potential from (6.15) and then calculating the magnetic field from (6.16). The electric field may then be found by the use of Ampere's law.

We assume a lossless medium with parameters μ, ϵ. Substituting the current expression into (6.15) and integrating over x and y we find that

$$\tilde{\mathbf{A}}_e(\mathbf{r}, \omega) = \hat{\mathbf{z}} \frac{\mu \tilde{I}}{4\pi} \int_{-l}^{l} \sin k(l - |z'|) \frac{e^{-jkR}}{R} \, dz' \tag{6.27}$$

where $R = \sqrt{(z - z')^2 + \rho^2}$ and $\rho^2 = x^2 + y^2$. Using (6.16) we have

$$\tilde{\mathbf{H}} = \nabla \times \frac{1}{\mu} \tilde{\mathbf{A}}_e = -\hat{\boldsymbol{\phi}} \frac{1}{\mu} \frac{\partial \tilde{A}_{ez}}{\partial \rho}.$$

Writing the sine function in (6.27) in terms of exponentials, we then have

$$\tilde{H}_\phi = j \frac{\tilde{I}}{8\pi} \left[e^{jkl} \int_{-l}^{0} \frac{\partial}{\partial \rho} \frac{e^{-jk(R - z')}}{R} \, dz' - e^{-jkl} \int_{-l}^{0} \frac{\partial}{\partial \rho} \frac{e^{-jk(R + z')}}{R} \, dz' + \right.$$
$$\left. + \, e^{jkl} \int_{0}^{l} \frac{\partial}{\partial \rho} \frac{e^{-jk(R + z')}}{R} \, dz' - e^{-jkl} \int_{0}^{l} \frac{\partial}{\partial \rho} \frac{e^{-jk(R - z')}}{R} \, dz' \right].$$

Noting that

$$\frac{\partial}{\partial \rho} \frac{e^{-jk(R \pm z')}}{R} = \pm \rho \frac{\partial}{\partial z'} \frac{e^{-jk(R \pm z')}}{R \left[R \mp (z - z') \right]} = -\rho \frac{1 + jkR}{R^3} e^{-jk(R \pm z')}$$

we can write

$$\tilde{H}_\phi = j \frac{\tilde{I} \rho}{8\pi} \left[-e^{jkl} \left. \frac{e^{-jk(R - z')}}{R \left[R + (z - z') \right]} \right|_{-l}^{0} - e^{-jkl} \left. \frac{e^{-jk(R + z')}}{R \left[R - (z - z') \right]} \right|_{-l}^{0} + \right.$$
$$\left. + \, e^{jkl} \left. \frac{e^{-jk(R + z')}}{R \left[R - (z - z') \right]} \right|_{0}^{l} + e^{-jkl} \left. \frac{e^{-jk(R - z')}}{R \left[R + (z - z') \right]} \right|_{0}^{l} \right].$$

Collecting terms and simplifying we get

$$\tilde{H}_\phi(\mathbf{r}, \omega) = j \frac{\tilde{I}(\omega)}{4\pi \rho} \left[e^{-jkR_1} + e^{-jkR_2} - (2 \cos kl) e^{-jkr} \right] \tag{6.28}$$

where $R_1 = \sqrt{\rho^2 + (z - l)^2}$ and $R_2 = \sqrt{\rho^2 + (z + l)^2}$. For points external to the dipole the source current is zero and thus

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = \frac{1}{j\omega\epsilon} \nabla \times \tilde{\mathbf{H}}(\mathbf{r}, \omega) = \frac{1}{j\omega\epsilon} \left\{ -\hat{\boldsymbol{\rho}} \frac{\partial}{\partial z} \tilde{H}_\phi(\mathbf{r}, \omega) + \hat{\mathbf{z}} \frac{1}{\rho} \frac{\partial}{\partial \rho} [\rho \tilde{H}_\phi(\mathbf{r}, \omega)] \right\}.$$

Performing the derivatives we have

$$\tilde{E}_\rho(\mathbf{r}, \omega) = j \frac{\eta \tilde{I}(\omega)}{4\pi} \left[\frac{z - l}{\rho} \frac{e^{-jkR_1}}{R_1} + \frac{z + l}{\rho} \frac{e^{-jkR_2}}{R_2} - \frac{z}{\rho} (2 \cos kl) \frac{e^{-jkr}}{r} \right], \tag{6.29}$$

$$\tilde{E}_z(\mathbf{r}, \omega) = -j \frac{\eta \tilde{I}(\omega)}{4\pi} \left[\frac{e^{-jkR_1}}{R_1} + \frac{e^{-jkR_2}}{R_2} - (2 \cos kl) \frac{e^{-jkr}}{r} \right]. \tag{6.30}$$

The work of specializing these expressions for points in the far zone is left as an exercise. Instead, we shall use the general far-zone expressions (6.21)–(6.24). Substituting (6.26) into (6.24) and carrying out the x and y integrals we have the directional weighting function

$$\tilde{a}_e(\theta, \phi, \omega) = \int_{-l}^{l} \hat{\mathbf{z}}\tilde{I}(\omega) \sin k(l - |z'|) e^{jkz' \cos\theta} \, dz'.$$

Writing the sine functions in terms of exponentials we have

$$\tilde{a}_e(\theta, \phi, \omega) = \frac{\hat{\mathbf{z}}\tilde{I}(\omega)}{2j} \left[e^{jkl} \int_0^l e^{jkz'(\cos\theta - 1)} \, dz' - e^{-jkl} \int_0^l e^{jkz'(\cos\theta + 1)} \, dz' + \right.$$
$$\left. + \, e^{jkl} \int_{-l}^{0} e^{jkz'(\cos\theta + 1)} - e^{-jkl} \int_{-l}^{0} e^{jkz'(\cos\theta - 1)} \right].$$

Carrying out the integrals and simplifying, we obtain

$$\tilde{a}_e(\theta, \phi, \omega) = \hat{\mathbf{z}} \frac{2\tilde{I}(\omega)}{k} \frac{F(\theta, kl)}{\sin\theta}$$

where

$$F(\theta, kl) = \frac{\cos(kl\cos\theta) - \cos kl}{\sin\theta}$$

is called the *radiation function.* Using $\hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$ we find that

$$\tilde{a}_{e\theta}(\theta, \phi, \omega) = -\frac{2\tilde{I}(\omega)}{k} F(\theta, kl), \qquad \tilde{a}_{e\phi}(\theta, \phi, \omega) = 0.$$

Thus we have from (6.23) and (6.21) the electric field

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = \hat{\boldsymbol{\theta}} \frac{j\eta\tilde{I}(\omega)}{2\pi} \frac{e^{-jkr}}{r} F(\theta, kl) \tag{6.31}$$

and from (6.22) the magnetic field

$$\tilde{\mathbf{H}}(\mathbf{r}, \omega) = \hat{\boldsymbol{\phi}} \frac{j\tilde{I}(\omega)}{2\pi} \frac{e^{-jkr}}{r} F(\theta, kl). \tag{6.32}$$

We see that the radiation function contains all of the angular dependence of the field and thus describes the pattern of the dipole. When the dipole is short compared to a wavelength we may approximate the radiation function as

$$F(\theta, kl \ll 1) \approx \frac{1 - \frac{1}{2}(kl\cos\theta)^2 - 1 + \frac{1}{2}(kl)^2}{\sin\theta} = \frac{1}{2}(kl)^2 \sin\theta. \tag{6.33}$$

So a short dipole antenna has the same pattern as a Hertzian dipole, whose far-zone electric field is (5.93).

We may also calculate the radiated power for time-harmonic fields. The time-average Poynting vector for the far-zone fields is, from (6.25),

$$\mathbf{S}_{av} = \hat{\mathbf{r}}\eta \frac{|\check{I}|^2}{8\pi^2 r^2} F^2(\theta, kl),$$

and thus the radiated power is

$$P_{av} = \eta \frac{|\check{I}|^2}{4\pi} \int_0^{\pi} F^2(\theta, kl) \sin\theta \, d\theta.$$

This expression cannot be computed in closed form. For a short dipole we may use (6.33) to approximate the power, but the result is somewhat misleading since the current on a short dipole is much smaller than $\tilde{I}$. A better measure of the strength of the current is its value at the center, or *feedpoint*, of the dipole. This *input current* is by (6.26) merely $\tilde{I}_0(\omega) = \tilde{I}(\omega)\sin(kl)$. Using this we find

$$P_{av} \approx \eta \frac{|\check{I}_0|^2}{4\pi}\frac{1}{4}(kl)^2 \int_0^\pi \sin^3\theta \, d\theta = \eta\frac{\pi}{3}|\check{I}_0|^2 \left(\frac{l}{\lambda}\right)^2.$$

This is exactly 1/4 of the power radiated by a Hertzian dipole of the same length and current amplitude (5.95). The factor of 1/4 comes from the difference between the current of the dipole antenna, which is zero at each end, and the current on the Hertzian dipole, which is constant across the length of the antenna. It is more common to use a dipole antenna that is a half wavelength long ($2l = \lambda/2$), since it is then nearly resonant. With this we have through numerical integration the free-space radiated power

$$P_{av} = \eta_0 \frac{|\check{I}_0|^2}{4\pi}\int_0^\pi \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\, d\theta = 36.6|\check{I}_0|^2$$

and the radiation resistance

$$R_r = \frac{2P_{av}}{|\check{I}(z=0)|^2} = \frac{2P_{av}}{|\check{I}_0|^2} = 73.2 \ \Omega.$$

6.3 Fields in a bounded, source-free region

In § 6.2 we considered the first important special case of the Stratton–Chu formula: sources in an unbounded medium. We now consider the second important special case of a bounded, source-free region. This case has important applications to the study of microwave antennas and, in its scalar form, to the study of the diffraction of light.

6.3.1 The vector Huygens principle

We may derive the formula for a bounded, source-free region of space by specializing the general Stratton–Chu formulas. We assume that all sources of the fields are within the excluded regions and thus set the sources to zero within V. From (6.7)–(6.8) we have

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = \sum_{n=1}^{N}\int_{S_n}\left[(\hat{\mathbf{n}}'\times\tilde{\mathbf{E}})\times\nabla' G + (\hat{\mathbf{n}}'\cdot\tilde{\mathbf{E}})\nabla' G - j\omega\tilde{\mu}(\hat{\mathbf{n}}'\times\tilde{\mathbf{H}})G\right] dS' - \\ - \sum_{n=1}^{N}\frac{1}{j\omega\tilde{\epsilon}^c}\oint_{\Gamma_{na}+\Gamma_{nb}}(\mathbf{dl}'\cdot\tilde{\mathbf{H}})\nabla' G, \qquad (6.34)$$

and

$$\tilde{\mathbf{H}}(\mathbf{r},\omega) = \sum_{n=1}^{N}\int_{S_n}\left[(\hat{\mathbf{n}}'\times\tilde{\mathbf{H}})\times\nabla' G + (\hat{\mathbf{n}}'\cdot\tilde{\mathbf{H}})\nabla' G + j\omega\tilde{\epsilon}^c(\hat{\mathbf{n}}'\times\tilde{\mathbf{E}})G\right] dS' +$$

$$+\sum_{n=1}^{N}\frac{1}{j\omega\tilde{\mu}}\oint_{\Gamma_{na}+\Gamma_{nb}}(\mathbf{dl}'\cdot\tilde{\mathbf{E}})\nabla' G. \tag{6.35}$$

This is known as the *vector Huygens principle* after the Dutch physicist C. Huygens, who formulated his "secondary source concept" to explain the propagation of light. According to his idea, published in *Traité de la lumière* in 1690, points on a propagating wavefront are secondary sources of spherical waves that add together in just the right way to produce the field on any successive wavefront. We can interpret (6.34) and (6.35) in much the same way. The field at each point within V, where there are no sources, can be imagined to arise from spherical waves emanated from every point on the surface bounding V. The amplitudes of these waves are determined by the values of the fields on the boundaries. Thus, we may consider the boundary fields to be equivalent to secondary sources of the fields within V. We will expand on this concept below by introducing the concept of equivalence and identifying the specific form of the secondary sources.

6.3.2 The Franz formula

The vector Huygens principle as derived above requires secondary sources for the fields within V that involve both the tangential and normal components of the fields on the bounding surface. Since only tangential components are required to guarantee uniqueness within V, we seek an expression involving only $\hat{\mathbf{n}}\times\tilde{\mathbf{H}}$ and $\hat{\mathbf{n}}\times\tilde{\mathbf{E}}$. Physically, the normal component of the field is equivalent to a secondary charge source on the surface while the tangential component is equivalent to a secondary current source. Since charge and current are related by the continuity equation, specification of the normal component is superfluous.

To derive a version of the vector Huygens principle that omits the normal fields we take the curl of (6.35) to get

$$\nabla\times\tilde{\mathbf{H}}(\mathbf{r},\omega)=\sum_{n=1}^{N}\nabla\times\oint_{S_n}(\hat{\mathbf{n}}'\times\tilde{\mathbf{H}})\times\nabla' G\,dS'+\sum_{n=1}^{N}\oint_{S_n}\nabla\times\left[(\hat{\mathbf{n}}'\cdot\tilde{\mathbf{H}})\nabla' G\right]dS'+$$
$$+\sum_{n=1}^{N}\nabla\times\oint_{S_n}j\omega\tilde{\epsilon}^c(\hat{\mathbf{n}}'\times\tilde{\mathbf{E}})G\,dS'+\sum_{n=1}^{N}\frac{1}{j\omega\tilde{\mu}}\oint_{\Gamma_{na}+\Gamma_{nb}}\nabla\times\left[(\mathbf{dl}'\cdot\tilde{\mathbf{E}})\nabla' G\right]dS'. \tag{6.36}$$

Now, using $\nabla' G=-\nabla G$ and employing the vector identity (B.43) we can show that

$$\nabla\times\left[f(\mathbf{r}')\nabla' G(\mathbf{r}|\mathbf{r}')\right]=-f(\mathbf{r}')\left\{\nabla\times\left[\nabla G(\mathbf{r}|\mathbf{r}')\right]\right\}+\left[\nabla G(\mathbf{r}|\mathbf{r}')\right]\times\nabla f(\mathbf{r}')=0,$$

since $\nabla\times\nabla G=0$ and $\nabla f(\mathbf{r}')=0$. This implies that the second and fourth terms of (6.36) are zero. The first term can be modified using

$$\nabla\times\left\{\left[\hat{\mathbf{n}}'\times\tilde{\mathbf{H}}(\mathbf{r}')\right]G(\mathbf{r}|\mathbf{r}')\right\}=G(\mathbf{r}|\mathbf{r}')\nabla\times\left[\hat{\mathbf{n}}'\times\tilde{\mathbf{H}}(\mathbf{r}')\right]-\left[\hat{\mathbf{n}}'\times\tilde{\mathbf{H}}(\mathbf{r}')\right]\times\nabla G(\mathbf{r}|\mathbf{r}')$$
$$=\left[\hat{\mathbf{n}}'\times\tilde{\mathbf{H}}(\mathbf{r}')\right]\times\nabla' G(\mathbf{r}|\mathbf{r}'),$$

giving

$$\nabla\times\tilde{\mathbf{H}}(\mathbf{r},\omega)=\sum_{n=1}^{N}\nabla\times\oint_{S_n}\nabla\times\left[(\hat{\mathbf{n}}'\times\tilde{\mathbf{H}})G\right]dS'+\sum_{n=1}^{N}\nabla\times\oint_{S_n}j\omega\tilde{\epsilon}^c(\hat{\mathbf{n}}'\times\tilde{\mathbf{E}})G\,dS'.$$

Finally, using Ampere's law $\nabla \times \tilde{\mathbf{H}} = j\omega\tilde{\epsilon}^c\tilde{\mathbf{E}}$ in the source free region V, and taking the curl in the first term outside the integral, we have

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = \sum_{n=1}^{N} \nabla \times \nabla \times \oint_{S_n} \frac{1}{j\omega\tilde{\epsilon}^c}(\hat{\mathbf{n}}' \times \tilde{\mathbf{H}})G\, dS' + \sum_{n=1}^{N} \nabla \times \oint_{S_n} (\hat{\mathbf{n}}' \times \tilde{\mathbf{E}})G\, dS'. \tag{6.37}$$

Similarly

$$\tilde{\mathbf{H}}(\mathbf{r}, \omega) = -\sum_{n=1}^{N} \nabla \times \nabla \times \oint_{S_n} \frac{1}{j\omega\tilde{\mu}}(\hat{\mathbf{n}}' \times \tilde{\mathbf{E}})G\, dS' + \sum_{n=1}^{N} \nabla \times \oint_{S_n} (\hat{\mathbf{n}}' \times \tilde{\mathbf{H}})G\, dS'. \tag{6.38}$$

These expressions together constitute the *Franz formula* for the vector Huygens principle [192].

6.3.3 Love's equivalence principle

Love's equivalence principle allows us to identify the equivalent Huygens sources for the fields within a bounded, source-free region V. It then allows us to replace a problem in the bounded region with an "equivalent" problem in unbounded space where the source-excluding surfaces are replaced by equivalent sources. The field produced by both the real and the equivalent sources gives a field in V identical to that of the original problem. This is particularly useful since we know how to compute the fields within an unbounded region by employing potential functions.

We identify the equivalent sources by considering the electric and magnetic Hertzian potentials produced by electric and magnetic current sources. Consider an impressed electric surface current $\tilde{\mathbf{J}}_s^{eq}$ and a magnetic surface current $\tilde{\mathbf{J}}_{ms}^{eq}$ flowing on the closed surface S in a homogeneous, isotropic medium with permeability $\tilde{\mu}(\omega)$ and complex permittivity $\tilde{\epsilon}^c(\omega)$. These sources produce

$$\tilde{\mathbf{\Pi}}_e(\mathbf{r}, \omega) = \oint_S \frac{\tilde{\mathbf{J}}_s^{eq}(\mathbf{r}', \omega)}{j\omega\tilde{\epsilon}^c(\omega)} G(\mathbf{r}|\mathbf{r}'; \omega)\, dS', \tag{6.39}$$

$$\tilde{\mathbf{\Pi}}_h(\mathbf{r}, \omega) = \oint_S \frac{\tilde{\mathbf{J}}_{ms}^{eq}(\mathbf{r}', \omega)}{j\omega\tilde{\mu}(\omega)} G(\mathbf{r}|\mathbf{r}'; \omega)\, dS', \tag{6.40}$$

which in turn can be used to find

$$\tilde{\mathbf{E}} = \nabla \times (\nabla \times \tilde{\mathbf{\Pi}}_e) - j\omega\tilde{\mu}\nabla \times \tilde{\mathbf{\Pi}}_h,$$
$$\tilde{\mathbf{H}} = j\omega\tilde{\epsilon}^c\nabla \times \tilde{\mathbf{\Pi}}_e + \nabla \times (\nabla \times \tilde{\mathbf{\Pi}}_h).$$

Upon substitution we find that

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = \nabla \times \nabla \times \oint_S \frac{1}{j\omega\tilde{\epsilon}^c}\left[\tilde{\mathbf{J}}_s^{eq} G\right] dS' + \nabla \times \oint_S [-\tilde{\mathbf{J}}_{ms}^{eq}]G\, dS',$$
$$\tilde{\mathbf{H}}(\mathbf{r}, \omega) = -\nabla \times \nabla \times \oint_S \frac{1}{j\omega\tilde{\mu}}\left[-\tilde{\mathbf{J}}_{ms}^{eq} G\right] dS' + \nabla \times \oint_S \tilde{\mathbf{J}}_s^{eq} G\, dS'.$$

These are identical to the Franz equations (6.37) and (6.38) if we identify

$$\tilde{\mathbf{J}}_s^{eq} = \hat{\mathbf{n}} \times \tilde{\mathbf{H}}, \qquad \tilde{\mathbf{J}}_{ms}^{eq} = -\hat{\mathbf{n}} \times \tilde{\mathbf{E}}. \tag{6.41}$$

These are the equivalent source densities for the Huygens principle.

We now state *Love's equivalence principle* [39]. Consider the fields within a homogeneous, source-free region V with parameters $(\tilde{\epsilon}^c, \tilde{\mu})$ bounded by a surface S. We know how to compute the fields using the Franz formula and the surface fields. Now consider a second problem in which the same surface S exists in an unbounded medium with identical parameters. If the surface carries the equivalent sources (6.41) then the electromagnetic fields within V calculated using the Hertzian potentials (6.39) and (6.40) are identical to those of the first problem, while the fields calculated outside V are zero. We see that this must be true since the Franz formulas and the field/potential formulas are identical, and the Franz formula (since it was derived from the Stratton–Chu formula) gives the null field outside V. The two problems are *equivalent* in the sense that they produce identical fields within V.

The fields produced by the equivalent sources obey the appropriate boundary conditions across S. From (2.194) and (2.195) we have the boundary conditions

$$\hat{\mathbf{n}} \times (\tilde{\mathbf{H}}_1 - \tilde{\mathbf{H}}_2) = \tilde{\mathbf{J}}_s,$$
$$\hat{\mathbf{n}} \times (\tilde{\mathbf{E}}_1 - \tilde{\mathbf{E}}_2) = -\tilde{\mathbf{J}}_{ms}.$$

Here $\hat{\mathbf{n}}$ points inward to V, $(\tilde{\mathbf{E}}_1, \tilde{\mathbf{H}}_1)$ are the fields within V, and $(\tilde{\mathbf{E}}_2, \tilde{\mathbf{H}}_2)$ are the fields within the excluded region. If the fields produced by the equivalent sources within the excluded region are zero, then the fields must obey

$$\hat{\mathbf{n}} \times \tilde{\mathbf{H}}_1 = \tilde{\mathbf{J}}_s^{eq},$$
$$\hat{\mathbf{n}} \times \tilde{\mathbf{E}}_1 = -\tilde{\mathbf{J}}_{ms}^{eq},$$

which is true by the definition of $(\tilde{\mathbf{J}}_s^{eq}, \tilde{\mathbf{J}}_{sm}^{eq})$.

Note that we can extend the equivalence principle to the case where the media are different internal to V than external to V. See Chen [29].

With the equivalent sources identified we may compute the electromagnetic field in V using standard techniques. Specifically, we may use the Hertzian potentials as shown above or, since the Hertzian potentials are a simple remapping of the vector potentials, we may use (5.60) and (5.61) to write

$$\tilde{\mathbf{E}} = -j\frac{\omega}{k^2}\left[\nabla(\nabla \cdot \tilde{\mathbf{A}}_e) + k^2\tilde{\mathbf{A}}_e\right] - \frac{1}{\tilde{\epsilon}^c}\nabla \times \tilde{\mathbf{A}}_h,$$
$$\tilde{\mathbf{H}} = -j\frac{\omega}{k^2}\left[\nabla(\nabla \cdot \tilde{\mathbf{A}}_h) + k^2\tilde{\mathbf{A}}_h\right] + \frac{1}{\tilde{\mu}}\nabla \times \tilde{\mathbf{A}}_e,$$

where

$$\tilde{\mathbf{A}}_e(\mathbf{r}, \omega) = \oint_S \tilde{\mu}(\omega)\tilde{\mathbf{J}}_s^{eq}(\mathbf{r}', \omega)G(\mathbf{r}|\mathbf{r}'; \omega)\, dS' \tag{6.42}$$

$$= \oint_S \tilde{\mu}(\omega)[\hat{\mathbf{n}}' \times \tilde{\mathbf{H}}(\mathbf{r}', \omega)]G(\mathbf{r}|\mathbf{r}'; \omega)\, dS', \tag{6.43}$$

$$\tilde{\mathbf{A}}_h(\mathbf{r}, \omega) = \oint_S \tilde{\epsilon}^c(\omega)\tilde{\mathbf{J}}_{ms}^{eq}(\mathbf{r}', \omega)G(\mathbf{r}|\mathbf{r}'; \omega)\, dS' \tag{6.44}$$

$$= \oint_S \tilde{\epsilon}^c(\omega)[-\hat{\mathbf{n}}' \times \tilde{\mathbf{E}}(\mathbf{r}', \omega)]G(\mathbf{r}|\mathbf{r}'; \omega)\, dS'. \tag{6.45}$$

At points where the source is zero we can write the fields in the alternative form

$$\tilde{\mathbf{E}} = -j\frac{\omega}{k^2}\nabla \times \nabla \times \tilde{\mathbf{A}}_e - \frac{1}{\tilde{\epsilon}^c}\nabla \times \tilde{\mathbf{A}}_h, \tag{6.46}$$

$$\tilde{\mathbf{H}} = -j\frac{\omega}{k^2}\nabla \times \nabla \times \tilde{\mathbf{A}}_h + \frac{1}{\tilde{\mu}}\nabla \times \tilde{\mathbf{A}}. \tag{6.47}$$

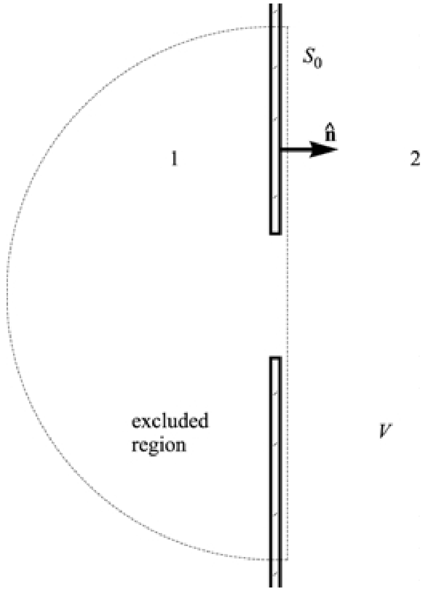

Figure 6.5: Geometry for problem of an aperture in a perfectly conducting ground screen illuminated by an impressed source.

By superposition, if there are volume sources within V we merely add the fields due to these sources as computed from the potential functions.

6.3.4 The Schelkunoff equivalence principle

With Love's equivalence principle we create an equivalent problem by replacing an excluded region by equivalent electric and magnetic sources. These require knowledge of both the tangential electric and magnetic fields over the bounding surface. However, the uniqueness theorem says that only one of either the tangential electric or the tangential magnetic fields need be specified to make the fields within V unique. Thus we may wonder whether it is possible to formulate an equivalent problem that involves only tangential $\tilde{\mathbf{E}}$ or tangential $\tilde{\mathbf{H}}$. It is indeed possible, as shown by Schelkunoff [39, 169].

When we use the equivalent sources to form the equivalent problem, we know that they produce a null field within the excluded region. Thus we may form a different equivalent problem by filling the excluded region with a perfect conductor, and keeping the same equivalent sources. The boundary conditions across S are not changed, and thus by the uniqueness theorem the fields within V are not altered. However, the manner in which we must compute the fields within V is changed. We can no longer use formulas for the fields produced by sources in free space, but must use formulas for fields produced by sources in the vicinity of a conducting body. In general this can be difficult since it requires the formation of a new Green's function that satisfies the boundary condition over the conducting body (which could possess a peculiar shape). Fortunately, we showed in § 4.10.2 that an electric source adjacent and tangential to a perfect electric conductor produces no field, hence we need not consider the equivalent electric sources $(\hat{\mathbf{n}} \times \tilde{\mathbf{H}})$ when computing the fields in V. Thus, in our new equivalent problem we need the single tangential field $-\hat{\mathbf{n}} \times \tilde{\mathbf{E}}$. This is the *Schelkunoff equivalence principle.*

There is one situation in which it is relatively easy to use the Schelkunoff equivalence. Consider a perfectly conducting ground screen with an aperture in it, as shown in Figure 6.5. We assume that the aperture has been illuminated in some way by an electromagnetic wave produced by sources in region 1 so that there are both fields within the aperture and electric current flowing on the region-2 side of the screen due to diffraction from the edges of the aperture. We wish to compute the fields in region 2. We can create an equivalent problem by placing a planar surface S_0 adjacent to the screen, but slightly offset into region 2, and then closing the surface at infinity so that all of the screen plus region 1 is excluded. Then we replace region 1 with homogeneous space and place on S_0 the equivalent currents $\tilde{\mathbf{J}}_s^{eq} = \hat{\mathbf{n}} \times \tilde{\mathbf{H}}$, $\tilde{\mathbf{J}}_{ms}^{eq} = -\hat{\mathbf{n}} \times \tilde{\mathbf{E}}$, where $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{E}}$ are the fields on S_0 in the original problem. We note that over the portion of S_0 adjacent to the screen $\tilde{\mathbf{J}}_{ms}^{eq} = 0$ since $\hat{\mathbf{n}} \times \tilde{\mathbf{E}} = 0$, but that $\tilde{\mathbf{J}}_s^{eq} \neq 0$. From the equivalent currents we can compute the fields in region 2 using the potential functions. However, it is often difficult to determine $\tilde{\mathbf{J}}_s^{eq}$ over the conducting surface. If we apply Schelkunoff's equivalence, we can formulate a second equivalent problem in which we place into region 1 a perfect conductor. Then we have the equivalent source currents $\tilde{\mathbf{J}}_s^{eq}$ and $\tilde{\mathbf{J}}_{ms}^{eq}$ adjacent and tangential to a perfect conductor. By the image theorem of § 5.1.1 we can replace this problem by yet another equivalent problem in which the conductor is replaced by the images of $\tilde{\mathbf{J}}_s^{eq}$ and $\tilde{\mathbf{J}}_{ms}^{eq}$ in homogeneous space. Since the image of the tangential electric current $\tilde{\mathbf{J}}_s^{eq}$ is oppositely directed, the fields of the electric current and its image cancel. Since the image of the magnetic current is in the same direction as $\tilde{\mathbf{J}}_{ms}^{eq}$, the fields produced by the magnetic current and its image add. We also note that $\tilde{\mathbf{J}}_{ms}^{eq}$ is nonzero only over the aperture (since $\hat{\mathbf{n}} \times \tilde{\mathbf{E}} = 0$ on the screen), and thus the field in region 1 can be found from

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = -\frac{1}{\tilde{\epsilon}^c(\omega)} \nabla \times \tilde{\mathbf{A}}_h(\mathbf{r}, \omega),$$

where

$$\tilde{\mathbf{A}}_h(\mathbf{r}, \omega) = \int_{S_0} \tilde{\epsilon}^c(\omega)[-2\hat{\mathbf{n}}' \times \tilde{\mathbf{E}}_{ap}(\mathbf{r}', \omega)] G(\mathbf{r}|\mathbf{r}'; \omega)\, dS'$$

and $\tilde{\mathbf{E}}_{ap}$ is the electric field in the aperture in the original problem. We shall present an example in the next section.

6.3.5 Far-zone fields produced by equivalent sources

The equivalence principle is useful for analyzing antennas with complicated source distributions. The sources may be excluded using a surface S, and then a knowledge of the fields over S (found, for example, by estimation or measurement) can be used to compute the fields external to the antenna. Here we describe how to compute these fields in the far zone.

Given that $\tilde{\mathbf{J}}_s^{eq} = \hat{\mathbf{n}} \times \tilde{\mathbf{H}}$ and $\tilde{\mathbf{J}}_{ms}^{eq} = -\hat{\mathbf{n}} \times \tilde{\mathbf{E}}$ are the equivalent sources on S, we may compute the fields using the potentials (6.43) and (6.45). Using (6.20) these can be approximated in the far zone ($r \gg r'$, $kr \gg 1$) as

$$\tilde{\mathbf{A}}_e(\mathbf{r}, \omega) = \tilde{\mu}(\omega) \frac{e^{-jkr}}{4\pi r} \tilde{a}_e(\theta, \phi, \omega),$$
$$\tilde{\mathbf{A}}_h(\mathbf{r}, \omega) = \tilde{\epsilon}^c(\omega) \frac{e^{-jkr}}{4\pi r} \tilde{a}_h(\theta, \phi, \omega), \tag{6.48}$$

where

$$\tilde{a}_e(\theta, \phi, \omega) = \oint_S \tilde{\mathbf{J}}_s^{eq}(\mathbf{r}', \omega) e^{jk\hat{\mathbf{r}}\cdot\mathbf{r}'}\, dS',$$

$$\tilde{a}_h(\theta,\phi,\omega) = \oint_S \tilde{\mathbf{J}}_{sm}^{eq}(\mathbf{r}',\omega) e^{jk\hat{\mathbf{r}}\cdot\mathbf{r}'}\, dS', \tag{6.49}$$

are the directional weighting functions.

To compute the fields from the potentials we must apply the curl operator. So we must evaluate

$$\nabla \times \left[\frac{e^{-jkr}}{r}\mathbf{V}(\theta,\phi)\right] = \frac{e^{-jkr}}{r}\nabla \times \mathbf{V}(\theta,\phi) + \nabla\left(\frac{e^{-jkr}}{r}\right) \times \mathbf{V}(\theta,\phi).$$

The curl of $\mathbf{V}$ is proportional to $1/r$ in spherical coordinates, hence the first term on the right is proportional to $1/r^2$. Since we are interested in the far-zone fields, this term can be discarded in favor of $1/r$-type terms. Using

$$\nabla\left(\frac{e^{-jkr}}{r}\right) = -\hat{\mathbf{r}}\left(\frac{1+jkr}{r}\right)\frac{e^{-jkr}}{r} \approx -\hat{\mathbf{r}}jk\frac{e^{-jkr}}{r}, \qquad kr \gg 1,$$

we have

$$\nabla \times \left[\frac{e^{-jkr}}{r}\mathbf{V}(\theta,\phi)\right] \approx -jk\hat{\mathbf{r}} \times \left[\frac{e^{-jkr}}{r}\mathbf{V}(\theta,\phi)\right].$$

Using this approximation we also establish

$$\nabla \times \nabla \times \left[\frac{e^{-jkr}}{r}\mathbf{V}(\theta,\phi)\right] \approx -k^2\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \left[\frac{e^{-jkr}}{r}\mathbf{V}(\theta,\phi)\right] = k^2\frac{e^{-jkr}}{r}\mathbf{V}_T(\theta,\phi)$$

where $\mathbf{V}_T = \mathbf{V} - \hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{V})$ is the vector component of $\mathbf{V}$ transverse to the r-direction.

With these formulas we can approximate (6.46) and (6.47) as

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = -j\omega\tilde{\mathbf{A}}_{eT}(\mathbf{r},\omega) + \frac{jk}{\tilde{\epsilon}^c(\omega)}\hat{\mathbf{r}} \times \tilde{\mathbf{A}}_h(\mathbf{r},\omega), \tag{6.50}$$

$$\tilde{\mathbf{H}}(\mathbf{r},\omega) = -j\omega\tilde{\mathbf{A}}_{hT}(\mathbf{r},\omega) - \frac{jk}{\tilde{\mu}(\omega)}\hat{\mathbf{r}} \times \tilde{\mathbf{A}}_e(\mathbf{r},\omega).$$

Note that

$$\hat{\mathbf{r}} \times \tilde{\mathbf{E}} = -j\omega\hat{\mathbf{r}} \times \tilde{\mathbf{A}}_{eT} + \frac{jk}{\tilde{\epsilon}^c}\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \tilde{\mathbf{A}}_h.$$

Since $\hat{\mathbf{r}} \times \tilde{\mathbf{A}}_{eT} = \hat{\mathbf{r}} \times \tilde{\mathbf{A}}_e$ and $\hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \tilde{\mathbf{A}}_h = -\tilde{\mathbf{A}}_{hT}$, we have

$$\hat{\mathbf{r}} \times \tilde{\mathbf{E}} = \eta\left[-j\omega\tilde{\mathbf{A}}_{hT} - \frac{jk}{\tilde{\mu}}\hat{\mathbf{r}} \times \tilde{\mathbf{A}}_e\right] = \eta\tilde{\mathbf{H}}.$$

Thus

$$\tilde{\mathbf{H}} = \frac{\hat{\mathbf{r}} \times \tilde{\mathbf{E}}}{\eta}$$

and the electromagnetic field in the far zone is a TEM spherical wave, as expected.

Example of fields produced by equivalent sources: an aperture antenna. As an example of calculating the fields in a bounded region from equivalent sources, let us find the far-zone field in free space produced by a rectangular waveguide opening into a perfectly-conducting ground screen of infinite extent as shown in Figure 6.6. For simplicity assume the waveguide propagates a pure TE_{10} mode, and that all higher-order

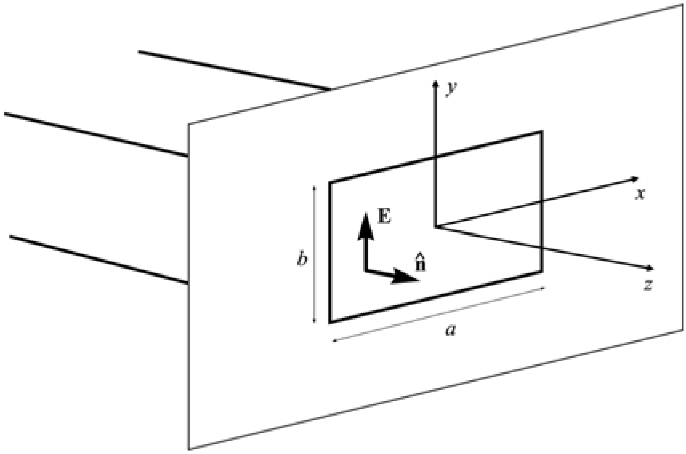

Figure 6.6: Aperture antenna consisting of a rectangular waveguide opening into a conducting ground screen of infinite extent.

modes excited when the guided wave is reflected at the aperture may be ignored. Thus the electric field in the aperture S_0 is

$$\tilde{\mathbf{E}}_a(x, y) = \hat{\mathbf{y}} E_0 \cos\left(\frac{\pi}{a}x\right).$$

We may compute the far-zone field using the Schelkunoff equivalence principle of § 6.3.4. We exclude the region $z < 0^+$ using a planar surface S which we close at infinity. We then fill the region $z < 0$ with a perfect conductor. By the image theory the equivalent electric sources on S cancel while the equivalent magnetic sources double. Since the only nonzero magnetic sources are on S_0 (since $\hat{\mathbf{n}} \times \tilde{\mathbf{E}} = 0$ on the screen), we have the equivalent problem of the source

$$\tilde{\mathbf{J}}_{ms}^{eq} = -2\hat{\mathbf{n}} \times \tilde{\mathbf{E}}_a = 2\hat{\mathbf{x}} E_0 \cos\left(\frac{\pi}{a}x\right)$$

on S_0 in free space, where the equivalence holds for $z > 0$.

We may find the far-zone field created by this equivalent current by first computing the directional weighting function (6.49). Since

$$\hat{\mathbf{r}} \cdot \mathbf{r}' = \hat{\mathbf{r}} \cdot (x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}) = x' \sin\theta\cos\phi + y' \sin\theta\sin\phi,$$

we find that

$$\begin{aligned} \tilde{a}_h(\theta, \phi, \omega) &= \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \hat{\mathbf{x}} 2E_0 \cos\left(\frac{\pi}{a}x'\right) e^{jkx'\sin\theta\cos\phi} e^{jky'\sin\theta\sin\phi}\, dx'\, dy' \\ &= \hat{\mathbf{x}} 4\pi E_0 ab \frac{\cos \pi X}{\pi^2 - 4(\pi X)^2} \frac{\sin \pi Y}{\pi Y} \end{aligned}$$

where

$$X = \frac{a}{\lambda}\sin\theta\cos\phi, \qquad Y = \frac{b}{\lambda}\sin\theta\sin\phi.$$

Here λ is the free-space wavelength. By (6.50) the electric field is

$$\tilde{\mathbf{E}} = \frac{jk_0}{\epsilon_0} \hat{\mathbf{r}} \times \tilde{\mathbf{A}}_h$$

where $\tilde{\mathbf{A}}_h$ is given in (6.48). Using

$$\hat{\mathbf{r}} \times \hat{\mathbf{x}} = \hat{\boldsymbol{\phi}} \cos\theta \cos\phi + \hat{\boldsymbol{\theta}} \sin\phi$$

we find that

$$\tilde{\mathbf{E}} = jk_0 ab E_0 \frac{e^{-jkr}}{r} \left(\hat{\boldsymbol{\theta}} \sin\phi + \hat{\boldsymbol{\phi}} \cos\theta \cos\phi\right) \frac{\cos(\pi X)}{\pi^2 - 4(\pi X)^2} \frac{\sin(\pi Y)}{\pi Y}.$$

The magnetic field is merely $\tilde{\mathbf{H}} = (\hat{\mathbf{r}} \times \tilde{\mathbf{E}})/\eta$.

6.4 Problems

6.1 Beginning with the Lorentz reciprocity theorem, derive (6.8).

6.2 Obtain (6.8) by substitution of (6.7) into Faraday's law.

6.3 Show that (6.8) returns the null result when evaluated within the excluded regions.

6.4 Show that under the condition $kr \gg 1$ the formula for the magnetic field of a dipole antenna (6.28) reduces to (6.32), while the formulas for the electric fields (6.29) and (6.30) reduce to (6.31).

6.5 Consider the dipole antenna shown in Figure 6.4. Instead of a standing-wave current distribution, assume the antenna carries a *traveling-wave current distribution*

$$\tilde{\mathbf{J}}^i(\mathbf{r}, \omega) = \hat{\mathbf{z}} \tilde{I}(\omega) e^{-jk|z|} \delta(x) \delta(y), \qquad -l \le z \le l.$$

Find the electric and magnetic fields at all points away from the current distribution. Specialize the result for $kr \gg 1$.

6.6 A circular loop of thin wire has radius a and lies in the $z = 0$ plane in free space. A current is induced on the wire with the density

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\boldsymbol{\phi}} \tilde{I}(\omega) \cos\left[k_0 a(\pi - |\phi|)\right] \delta(r - a) \frac{\delta(\theta - \pi/2)}{r}, \qquad |\phi| \le \pi.$$

Compute the far-zone fields produced by this loop antenna. Specialize your results for the electrically-small case of $k_0 a \ll 1$. Compute the time-average power radiated by, and the radiation resistance of, the electrically-small loop.

6.7 Consider a plane wave with the fields

$$\tilde{\mathbf{E}} = \tilde{E}_0 \hat{\mathbf{x}} e^{-jkz}, \qquad \tilde{\mathbf{H}} = \frac{\tilde{E}_0}{\eta} \hat{\mathbf{y}} e^{-jkz},$$

normally incident from $z < 0$ on a square aperture of side a in a PEC ground screen at $z = 0$. Assume that the field in the aperture is identical to the field of the plane wave with the screen absent (this is called the *Kirchhoff approximation*). Compute the far-zone electromagnetic fields for $z > 0$.

6.8 Consider a coaxial cable of inner radius a and outer radius b, opening into a PEC ground plane at $z = 0$. Assume that only the TEM wave exists in the line and that no higher-order modes are created when the wave reflects from the aperture. Compute the far-zone electric and magnetic fields of this aperture antenna.