P • A • R • T 3

CIRCUITS AND FUNCTIONS

SECTION 10

FILTERS AND ATTENUATORS

To make communication systems, radar systems, and the like work, we need filters and attenuators. Filters are basic electronic building blocks that are passive and active, analog and digital. They basically allow us to condition electrical signals in order to accomplish the elements of most of our complex electrical systems in use today.

Active filters have led to the elimination of elements that prohibited miniaturization. Active filters easily fit into today's microcircuits and systems.

Digital filters represent a class of filters that do not have the limitations of analog filters. Although they require much care and attention in how they are designed and how they will be used, they most certainly have given us a significantly larger set of applications some of which could never be handled by analog filters. Because of the nature of digital filters, we need to fully understand the problems with phase.

In this section we look at the basic principles behind all of these filters. In Chap. 10.7, we look at attenuators that are used in matching impedances, critical in high-frequency systems. C.A.

In This Section:

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CHAPTER 10.1
IDEAL FILTERS AND APPROXIMATIONS

Edwin C. Jones, Jr., Harry W. Hale

INTRODUCTION

Campbell and Wagner independently developed the concept of an electrical wave filter in 1915. The continuation of this work proceeded along two paths, *image-parameter filter design* and *insertion-loss filter design*. The latter technique is now dominant.

Insertion-loss filter design requires the designer to specify an appropriate network response as a part of a transfer function. The part might be magnitude, phase, or delay. Norton, Foster, Cauer, Bode, and Darlington developed procedures to determine the complete transfer function and to synthesize the network. Digital computers have made the techniques practical, and now the design of these filters is largely a matter of looking up element values in computer-generated tables and modifying them in a routine fashion. Virtually all techniques use a low-pass prototype and derive other filters from this prototype.

This chapter describes insertion loss techniques, develops common transfer functions, and gives structures and element values for passive networks. It also extends use of the transfer functions to several types of active networks.

Throughout, the notation is that of the Laplace transformation. The general frequency variable is $s = \sigma + j\omega$, where ω is the signal frequency in rad/s.

INSERTION-LOSS FILTER DESIGN

Figure 10.1.1 shows two general networks, often called two-ports. One has a current source, the other, a voltage source. The design procedure consists of these steps:

1. Determine filter specifications
2. Approximate the specifications with a network function part that will lead to a realizable circuit. Typical magnitude functions are

$$|G_{LS}(j\omega)| = \left|\frac{V_L}{V_S}(j\omega)\right| \tag{1}$$

Chapters 10.1 to 10.5 and Chap. 10.7 were contributed by Edwin C. Jones Jr. and Harry W. Hale. Portions were adapted from Fink and Beaty (eds.), "Standard Handbook for Electrical Engineers," 13th ed., McGraw-Hill, 1993. Chapter 10.6 was contributed by Arthur B. Williams and Fred J. Taylor, author's of the "Electronic Filter Design Handbook," 3rd ed., McGraw-Hill, 1995, from which portions were adapted.

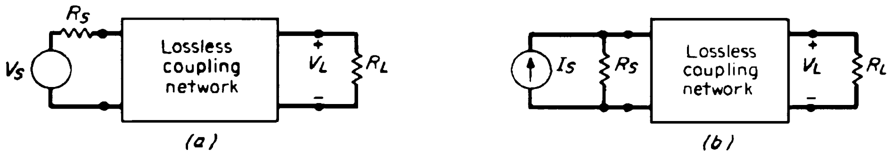

FIGURE 10.1.1 General form of network: (a) with voltage source; (b) with current source.

$$|Z_{LS}(j\omega)| = \left|\frac{V_L}{I_S}(j\omega)\right| \tag{2}$$

for networks of Fig. 10.1.1*a* and *b*, respectively. (For convenience, subsequent references to G_{LS} omit subscripts, and the discussion applies to Z_{LS} except where noted.) The phase function

$$\theta(\omega) = \arg G(j\omega) \tag{3}$$

is often the specification.

3. Synthesize the coupling network of Fig. 10.1.1*a* or *b*. Darlington* showed that this network may be lossless, and often that it is a ladder network, an attractive circuit structure.

CLASSIFICATION OF IDEAL FILTERS

It is convenient to have ideal filter characteristics for discussion purposes. Four are ideal magnitude characteristics, and one is for phase. They are:

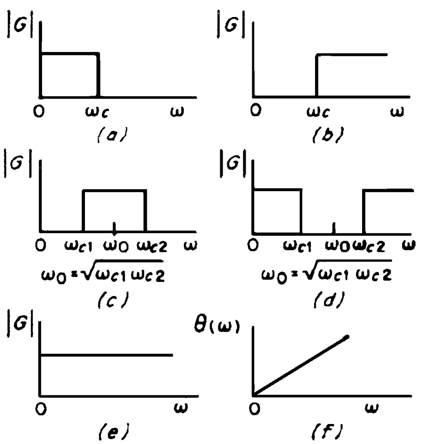

FIGURE 10.1.2 Ideal-filter characteristics: (*a*) low-pass; (*b*) high-pass; (*c*) bandpass; (*d*) band-elimination; (*e*) all-pass (magnitude); (*f*) all-pass (phase).

1. The *low-pass* filter transmits without attenuation or loss signal frequencies from zero to a cutoff frequency and stops all signal frequencies higher than the cutoff.
2. The *high-pass* filter stops all signal frequencies below its cutoff, and transmits without attenuation signal frequencies above the cutoff.
3. The *bandpass* filter passes all signal frequencies between the lower and upper cutoff frequencies, and stops all signal frequencies outside this range.
4. The *bandstop* (or reject) filter stops signal frequencies between its lower and upper cutoff frequencies, and transmits all signal frequencies outside this range.
5. The *all-pass* filter transmits all signal frequencies. It produces a predictable phase shift.

These ideal filter characteristics are shown in Fig. 10.1.2.

*In many cases, explicit bibliographic references are omitted. However, the bibliography entries are in about the same order as the topics that are discussed.

THE LOW-PASS PROTOTYPE

Synthesis techniques lead to low-pass prototypes, for which extensive tables exist. Modifications of these prototypes enable design of the three remaining filters. The technique is to make $R_L = 1$ and $\omega_C = 1$, to transform specifications to the low-pass domain, and to determine the prototype. Transformations of the low-pass prototype to the needed domain lead to the final network.

THE APPROXIMATION PROBLEM

The ideal low-pass filter is not realizable. A basic problem is to select a transfer function magnitude $|G(j\omega)|$ that *approximates* the ideal characteristic and results in a practical network. Figure 10.1.3 illustrates the concept, and defines the limits on the frequency response. The approximation for $|G(j\omega)|$ must lie within the shaded region.

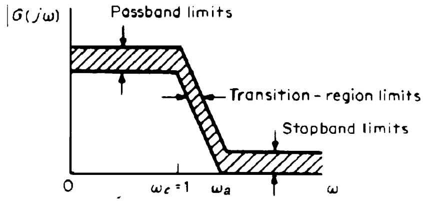

FIGURE 10.1.3 Limits on frequency response.

Often it is easier to work with the magnitude-squared function, which is of the form

$$|G(j\omega)|^2 = K^2 \frac{A(\omega^2)}{D(\omega^2)} \tag{4}$$

The numerator and denominator must be even, nonnegative polynomials, and other restrictions apply. Another form for Eq. (4) is

$$|G(j\omega)|^2 = K^2 \frac{N(\omega^2)}{N(\omega^2)+M(\omega^2)} = K^2 \frac{1}{1+\dfrac{M(\omega^2)}{N(\omega^2)}} \tag{5}$$

A common technique is to let $N(\omega^2) = 1$. Equation (5) then becomes

$$|G(j\omega)|^2 = \frac{K^2}{1+M(\omega^2)} \tag{6}$$

A requirement for the polynomial $M(\omega^2)$ is that

$$|M(\omega^2)| \begin{cases} \ll 1 & \omega \ll 1 \\ \gg 1 & \omega \gg 1 \end{cases}$$

which leads to all-pole approximations.

Low-pass filters based on Eq. (6) are ladder structures with inductors as series elements, and capacitors as shunt. The more general form of Eq. (5) leads to a filter that is a ladder with the series elements having parallel combinations of inductors and capacitors, while the shunt elements are series combinations of inductors and capacitors. The Inverse Chebyshev and elliptic approximations lead to such filters.

K of Eq. (4) is not arbitrary. The requirement is that, in terms of Fig. 10.1.1,

$$|G(0)| = \frac{R_L}{R_S + R_L} \tag{7}$$

for a voltage source, and

$$|Z(0)| = \frac{R_S R_L}{R_S + R_L} \tag{8}$$

for a current source. K must satisfy this constraint.

TRANSFER FUNCTION CONSTRUCTION

It is necessary to construct a complete transfer function from the approximation. The complete transfer function allows computation of time-domain responses. Several active realizations require them. The processes show construction of $G(s)$, in terms of Eq. (6). The technique for $Z(s)$ is identical. The Inverse Chebyshev shows an extension for the case of Eq. (5), where the nonconstant numerator imposes additional considerations.

From a suitable $G(\omega^2)$, use analytic continuation to obtain

$$|G(j\omega)|^2_{\omega^2=-s^2} = G(s)G(-s) = \frac{K^2}{1+M(-s^2)} \tag{9}$$

Assign the left-half-plane poles of $G(s)G(-s)$ to $G(s)$ and the right-half-plane poles to $G(-s)$, respectively, to ensure stability. The result is

$$G(s) = \frac{K}{\prod_{j=1}^{n}(s-s_j)} \tag{10}$$

where s_j are the left-half-plane poles.

The following sections describe six different approximations, five to the ideal low-pass filter and one to the ideal all-pass or time delay filter. Two approximations (the Inverse Chebyshev, and elliptic) are examples of Eq. (5), while the others all are examples of Eq. (6). Explicit equations give the poles for some approximations. Examples show use of these equations. Numerical solutions give the poles for other approximations. Tables show the roots, tabulated in terms of linear and quadratic factors.

THE BUTTERWORTH APPROXIMATION

The nth Butterworth approximation is

$$|G(j\omega)| = \frac{K}{(1+\omega^{2n})^{1/2}} \tag{11}$$

Figure 10.1.4 shows the general form of the function. The first $2n - 1$ derivatives of $|G(j\omega)|$ are zero, and the magnitude of the function decreases monotonically with ω. Figure 10.1.4 also shows that increasing n improves the approximation in both the pass and stop bands, at the price of a more complex network. The appropriate n is the smallest value that will meet frequency domain specifications, as suggested in Fig. 10.1.5.

If the specifications require that

$$|G(j\omega)| \le A \qquad \omega_a \le \omega < \infty \tag{12}$$

then the value of n required is the smallest integer satisfying the inequality

$$A \ge \frac{K}{(1+\omega_a^{2n})^{1/2}} \tag{13}$$

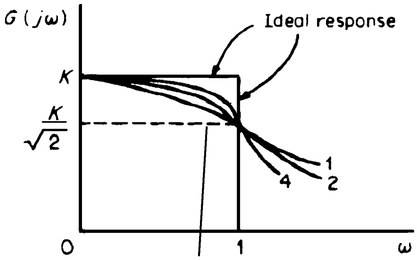

FIGURE 10.1.4 The Butterworth approximation.

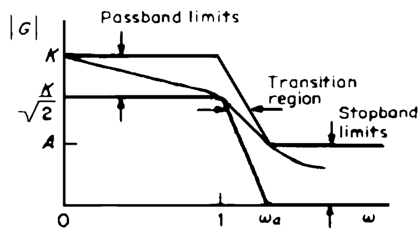

FIGURE 10.1.5 Determination of the minimum value of n for the Butterworth approximation.

In terms of attenuation, the frequency response is

$$\alpha = 20 \log \frac{|G(j\omega)|}{|G(0)|} = 10 \log (1 + \omega^{2n}) \quad \text{dB} \tag{14}$$

At $\omega = 1$,

$$\alpha = 10 \log 2 = 3.0103 \text{ dB} \tag{15}$$

This is a common interpretation of the cutoff frequency, frequently called the half-power or "3-dB" frequency. A more general specification of minimum attenuation $\alpha_{\min}$ (dB) for $\omega \ge \omega_a$ and a maximum attenuation $\alpha_{\max}$ for $\omega \le \omega_b$ leads to a solution for n of

$$n = \log \frac{(10^{\alpha_{\min}/10} - 1)}{(10^{\alpha_{\max}/10} - 1)} \Bigg/ 2 \log \left(\frac{\omega_a}{\omega_b} \right) \tag{16}$$

with the next larger integer being chosen.

The stable poles lie on a unit circle in the left half s-plane, and are given by $s_k = \sigma_k + j\omega_k$, where

$$\sigma_k = -\sin \frac{2k-1}{2n}\pi \qquad \omega_k = \cos \frac{2k-1}{2n}\pi \qquad k = 1, 2, \ldots, n \tag{17}$$

Since complex poles occur in conjugate pairs, it is convenient to combine them into quadratic factors. When $n = 3$, Eq. (17) gives

$$G(s) = (s + 1)(s^2 + s + 1) \tag{18}$$

and for $n = 4$,

$$G(s) = (s^2 + 1.84776s + 1)(s^2 + 0.76537s + 1) \tag{19}$$

THE CHEBYSHEV APPROXIMATION

The nth order Chebyshev approximation is

$$|G(j\omega)| = \frac{K}{[1 + \epsilon^2 C_n^2(\omega)]^{1/2}} \tag{20}$$

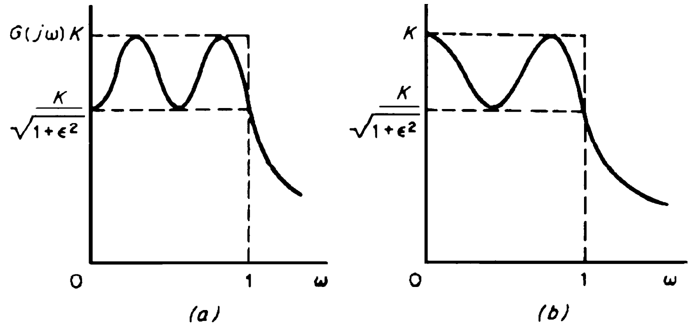

FIGURE 10.1.6 The Chebyshev approximation: (*a*) n even; (*b*) n odd.

where $C_n(\omega)$ is the nth-order Chebyshev polynomial and ϵ is a real constant less than 1. Specifically,

$$C_n(\omega) = \begin{cases} \cos(n \cos^{-1}(\omega)) & \text{for } 0 < \omega \le 1 \\ \cosh(n \cosh^{-1}(\omega)) & \textit{for } 1 < \omega \end{cases} \tag{21, 22}$$

The polynomials in the form of Eqs. (21) and (22) are convenient for calculations. To show a conventional polynomial appearance, note that $C_0(\omega) = 1$ and $C_1(\omega) = \omega$. Use the recursion formula, which is derived from a trigonometric identity,

$$C_{n+1}(\omega) = 2\omega C_n(\omega) - C_{n-1}(\omega) \tag{23}$$

to develop higher-order polynomials. For example,

$$C_5(\omega) = 16\omega^5 - 20\omega^3 + 5\omega \tag{24}$$

Figure 10.1.6 shows the frequency response for the Chebyshev approximation for n even and odd. In either case there are n half cycles (from maximum to minimum and the reverse) in the interval $0 \le \omega \le 1$. One effect of the relative minimum at $\omega = 0$ when n is even is that there are combinations of R_S, R_L, and ϵ that are not realizable. In particular, the even-order Chebyshev approximation is unrealizable for any value of ϵ when $R_S = R_L$. The modified Chebyshev approximation that leads to realizable networks in this case.

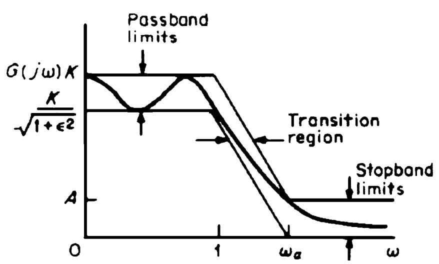

FIGURE 10.1.7 Identification of ϵ and n.

Two parameters, ϵ and n, define a particular Chebyshev approximation. Figure 10.1.7 defines the parameters for n odd (the results also apply to n even). The *ripple* factor ϵ controls the passband limits, while both ϵ and n control the stopband. The ratio of upper to lower passband limits is $(1 + \epsilon^2)^{1/2}$, and the value of this ratio is sufficient to determine ϵ. The logarithmic function

$$10 \log(1 + \epsilon^2) \quad (\text{dB}) \tag{25}$$

describes this ratio. Thus, a Chebyshev approximation with $\epsilon = 0.7648$ has a 2.0 dB ripple. Equation (26) relates the ripple R, in dB, and ϵ, the ripple factor

$$\epsilon = \sqrt{10^{R/10} - 1} \tag{26}$$

With ϵ known and a specification that

$$G(j\omega) \le A \quad \text{for} \quad \omega_a \le \omega < \infty \tag{27}$$

The attenuation requirement in the stopband leads to the inequality

$$A \geq \frac{K}{[1 + \epsilon^2 C_n^2(\omega_a)]^{1/2}} \tag{28}$$

From Eq. (28) and the known value of ϵ,

$$C_n(\omega_a) = \frac{(K^2 - A^2)^{1/2}}{A\epsilon} \tag{29}$$

This leads to the requirement on n, the order of the filter,

$$n = \frac{\cosh^{-1} C_n(\omega_a)}{\cosh^{-1} \omega_a} \tag{30}$$

with the next larger integer value being chosen.

The *half-power frequency* is a commonly used figure of merit for a filter. At this frequency, the power transmission to the load is 50 percent of the maximum, corresponding to a reduction of 3.0103 (or approximately 3) dB. Equation (31) gives the half-power frequency.

$$\omega_{hp} = \omega_{3dB} = \cosh\left(\frac{1}{n}\cosh^{-1}\frac{1}{\epsilon}\right) \tag{31}$$

To find the stable or left-half s-plane poles of $G(s)$, substitute ω-s/j and set the denominator of Eq. (20) to zero. From this,

$$C_n\left(\frac{s}{j}\right) = \cos\left(n\cos^{-1}\frac{s}{j}\right) = \pm\frac{j}{\epsilon} \tag{32}$$

This equation is a complex function, and the complex solution is

$$\sigma_k = -\sin u_k \sinh v \tag{33}$$

$$\omega_k = \cos u_k \cosh v \tag{34}$$

where

$$u_k = \frac{2k-1}{2n}\pi \tag{35}$$

$$k = 1, 2, 3, \ldots, n \tag{36}$$

$$v = \frac{1}{n}\sinh^{-1}\frac{1}{\epsilon} \tag{37}$$

Since complex roots occur in conjugate pairs, it is convenient to combine such pairs into quadratic factors. For example, with a ripple of 0.50 dB and third order,

$$G(s) = (s + 0.62646)(s^2 + 0.62646s + 1.14245) \tag{38}$$

while for a ripple of 1.0 dB and fourth order,

$$G(s) = (s^2 + 0.27907s + 0.98650)(s^2 + 0.67374s + 0.27940) \tag{39}$$

COMPARISON OF BUTTERWORTH AND CHEBYSHEV APPROXIMATIONS

A good comparison of the Butterworth and Chebyshev responses is possible when they have the same half-power frequency. In general, this requires a normalization of the Chebyshev approximation to a 3.0103 dB bandwidth before comparison. This occurs naturally when $\epsilon = 1$. Figure 10.1.8 shows the two-frequency responses for third order filters. Analysis shows that:

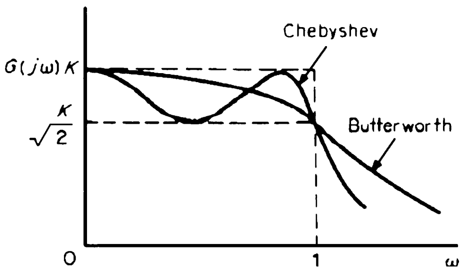

FIGURE 10.1.8 Butterworth and Chebyshev approximations for $n = 3$ and $\epsilon = 1$.

1. The Butterworth approximation is superior at and near $\omega = 0$. Of all polynomial approximations, it has the highest number of zero derivatives at the origin.
2. The Chebyshev approximation is superior at and near the cutoff frequency or passband edge.
3. The Chebyshev approximation is superior in the stopband.
4. The Chebyshev approximation sacrifices smoothness in the passband.

THE MODIFIED CHEBYSHEV APPROXIMATION

With passive networks, even-ordered Chebyshev networks with equal source and load resistances are unrealizable, because of the relative minimum at zero frequency in the frequency response. Saal has shown a modification of the Chebyshev polynomials that leads to even-ordered polynomials with a zero at $\omega = 0$, relative maxima and minima of +1 and −1 within the interval $0 < \omega \leq 1$, and that are monotonically increasing for $\omega > 1$. These polynomials are given in Table 10.1.1.

For a given n, the modified Chebyshev polynomials have the property that

$$C_n(\omega) > \bar{C}_n(\omega) > C_{n-1}(\omega) \tag{40}$$

for $\omega > 1$. Thus a low-pass approximation based on these polynomials for even n will be better in the stopband than that using the Chebyshev polynomial of order $n - 1$, but not as good as that using the regular Chebyshev polynomial of order n. However, it will, in passive networks, permit equal source and load resistances.

Figure 10.1.9 compares $C_n(\omega)$ and $\bar{C}_n(\omega)$ by plotting the ratio as a function of ω.

Table 10.1.2 gives the poles of the transfer function $G(s)$, in terms of quadratic factors, for n = 2, 4, 6, 8, and 10, and for ripples of 0.01, 0.10, 0.50, 1.00, and 3.00 dB. These poles assume that the end of the ripple band is the passband edge at $\omega = 1$. Table 10.1.11 gives the corresponding half-power frequencies.

TABLE 10.1.1 Modified Chebyshev Polynomials of Even Order

n	$C_n(\omega)$
2	ω^2
4	$5.82842713\omega^4 - 4.82842713\omega^2$
6	$25.9903811\omega^6 - 36.1865335\omega^4 + 11.1961524\omega^2$
8	$109.597711\omega^8 - 210.522708\omega^6$
	$+ 122.034355\omega^4 - 20.109358\omega^2$
10	$452.344415\omega^{10} - 1102.492675\omega^8 + 926.297276\omega^6$
	$- 306.717773\omega^1 + 31.568758\omega^2$

THE LEGENDRE–PAPOULIS APPROXIMATION

Papoulis derived an approximation using Legendre polynomials ($L_n(\omega^2)$) of the first kind. The nth order approximation is

$$|G(j\omega)| = \frac{1}{\sqrt{1+L_n(\omega^2)}} \tag{41}$$

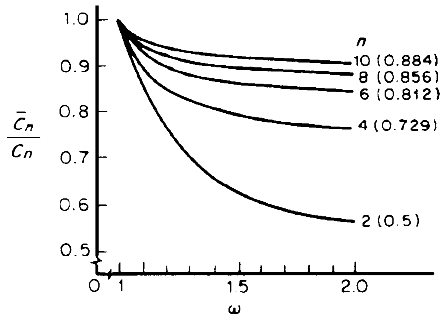

FIGURE 10.1.9 Ratio of modified Chebyshev polynomial to Chebyshev polynomial. Limiting value as ω approaches infinity is in parentheses for each value of n.

Table 10.1.3 shows the polynomials, and Table 10.1.4 shows the denominator roots (poles) for this function.

This approximation has three important properties:

1. $L_n(\omega^2)$ increases monotonically, and thus $|G(j\omega)|$ decreases monotonically.
2. $L_n(0) = 0$ and $L_n(1) = 1$, which means that $|G(0)| = 1$ and $|G(1)| = 0.50$.
3. Of all polynomials satisfying 1 and 2, $L_n(\omega^2)$ has the largest derivative at $\omega = 1$, and thus this filter has the steepest cutoff characteristic.

Figure 10.1.10 compares the Legendre–Papoulis approximation with the Butterworth, and Chebyshev (3 dB) approximations for $n = 3$.

The selection of n follows the same general procedure as for the Butterworth approximation. To satisfy the specification

$$A \geq |G(j\omega_a)| = 1/\sqrt{1+L_n(\omega^2)} \tag{42}$$

or

$$L_n(\omega_a^2) \geq (1/A^2) - 1 \tag{43}$$

evaluate successively higher-ordered polynomials of Table 10.1.3 at ω_a.

THE BESSEL APPROXIMATION

The ideal all pass characteristics of Fig. 10.1.2.

$$|G(j\omega)| = K \tag{44}$$

and

$$\theta(j\omega) = -T\omega \tag{45}$$

implies that the output is a scaled (by K) replica of the input, delayed in time by T s. This notion leads to the definition of group delay, which is the negative derivative of the phase function

$$-\frac{\partial}{\partial\omega}\theta(j\omega) = \tau_d(\omega) \tag{46}$$

In the ideal case, the group delay is a constant T.

TABLE 10.1.2 Quadratic Factors Giving Poles of Modified Chebyshev Transfer Functions

n	Ripple, dB	$[G(s)]^{-1}$
2	0.01	$s^2 + 6.4541054s + 20.8277385$
	0.10	$s^2 + 3.6200009s + 6.5522033$
	0.50	$s^2 + 2.3928122s + 2.8627752$
	1.00	$s^2 + 1.9825371s + 1.9652267$
	3.00	$s^2 + 1.4158936s + 1.0023773$
4	0.01	$s^2 + 2.2680440s + 1.6170715$
		$s^2 + 0.9235542s + 2.2098435$
	0.10	$s^2 + 1.5452285s + 0.7991383$
		$s^2 + 0.6059475s + 1.4067406$
	0.50	$s^2 + 1.1191759s + 0.4523364$
		$s^2 + 0.4086395s + 1.0858612$
	1.00	$s^2 + 0.9486848s + 0.3398608$
		$s^2 + 0.3272401s + 0.9921108$
	3.00	$s^2 + 0.6845838s + 0.1932927$
		$s^2 + 0.2013994s + 0.8897428$
6	0.01	$s^2 + 1.4118889s + 0.5893450$
		$s^2 + 1.0058571s + 0.9699759$
		$s^2 + 0.3640166s + 1.4018414$
	0.10	$s^2 + 0.9995998s + 0.3173248$
		$s^2 + 0.6819299s + 0.6966158$
		$s^2 + 0.2448454s + 1.1404529$
	0.50	$s^2 + 0.7362729s + 0.1875012$
		$s^2 + 0.4662515s + 0.5727968$
		$s^2 + 0.1663135s + 1.0255810$
	1.00	$s^2 + 0.6267633s + 0.1428406$
		$s^2 + 0.3748110s + 0.5343433$
		$s^2 + 0.1333343s + 0.9906678$
	3.00	$s^2 + 0.4534072s + 0.0825380$
		$s^2 + 0.2315851s + 0.4909174$
		$s^2 + 0.0820877s + 0.9518235$
8	0.01	$s^2 + 1.0335548s + 0.3097540$
		$s^2 + 0.8491714s + 0.5489161$
		$s^2 + 0.5591534s + 0.9283901$
		$s^2 + 0.1954150s + 1.2038909$
	0.10	$s^2 + 0.7417790s + 0.1718214$
		$s^2 + 0.5828511s + 0.4083754$
		$s^2 + 0.3806989s + 0.7943820$
		$s^2 + 0.1328521s + 1.0725554$
	0.50	$s^2 + 0.5496051s + 0.1030188$
		$s^2 + 0.4007060s + 0.3416991$
		$s^2 + 0.2600622s + 0.7328336$
		$s^2 + 0.0906706s + 1.0125586$
	1.00	$s^2 + 0.4685593s + 0.0788470$
		$s^2 + 0.3226466s + 0.3205446$
		$s^2 + 0.2088595s + 0.7137508$
		$s^2 + 0.0727952s + 0.9940103$
	3.00	$s^2 + 0.3392712s + 0.0458028$
		$s^2 + 0.1997071s + 0.2963792$
		$s^2 + 0.1288387s + 0.6922959$
		$s^2 + 0.0448874s + 0.9731910$
10	0.01	$s^2 + 0.8175750s + 0.1921666$
		$s^2 + 0.7133475s + 0.3528340$

TABLE 10.1.2 Quadratic Factors Giving Poles of Modified Chebyshev Transfer Functions (*Continued*)

n	Ripple, dB	$[G(s)]^{-1}$
		$s^2 + 0.5571617s + 0.6426718$
		$s^2 + 0.3555630s + 0.9397252$
		$s^2 + 0.1222554s + 1.1244354$
	0.10	$s^2 + 0.5905036s + 0.1080675$
		$s^2 + 0.4925060s + 0.2665100$
		$s^2 + 0.3816272s + 0.5602769$
		$s^2 + 0.2431732s + 0.8592520$
		$s^2 + 0.0835800s + 1.0446869$
	0.50	$s^2 + 0.4387194s + 0.0652281$
		$s^2 + 0.3394908s + 0.2246543$
		$s^2 + 0.2614400s + 0.5216326$
		$s^2 + 0.1664374s + 0.8217378$
		$s^2 + 0.0571933s + 1.0075592$
	1.00	$s^2 + 0.3742846s + 0.0500294$
		$s^2 + 0.2735736s + 0.2112433$
		$s^2 + 0.2101536s + 0.5095395$
		$s^2 + 0.1337446s + 0.8100361$
		$s^2 + 0.0459557s + 0.9959854$
	3.00	$s^2 + 0.2711265s + 0.0291328$
		$s^2 + 0.1694797s + 0.1958431$
		$s^2 + 0.1297678s + 0.4958770$
		$s^2 + 0.0825542s + 0.7968406$
		$s^2 + 0.0283639s + 0.9829386$

A series of Bessel polynomials provides a transfer function that yields a maximally flat approximation to the ideal delay. The form of the function that gives a group delay of 1 s at $\omega = 0$ is

$$G(s) = \frac{Kb_0}{s^n + b_{n-1}s^{n-1} + \cdots + bs + b_0} \tag{47}$$

A recursion formula relates these polynomials

$$B_n = (2n - 1)B_{n-1} + s^2 B_{n-2} \tag{48}$$

and Table 10.1.5 gives the coefficients of the first eight Bessel polynomials. Table 10.1.6 gives the pole locations for $G(s)$ in terms of quadratic factors.

TABLE 10.1.3 The Polynomials $L_n(\omega^2)$

n	
2	ω^4
3	$3\omega^6 - 3\omega^4 + \omega^2$
4	$6\omega^8 - 8\omega^6 + 3\omega^4$
5	$20\omega^{10} - 40\omega^8 + 28\omega^6 - 8\omega^4 + \omega^2$
6	$50\omega^{12} - 120\omega^{10} + 105\omega^8 - 40\omega^6 + 6\omega^4$
7	$175\omega^{14} - 525\omega^{12} + 615\omega^{10} - 355\omega^8 + 105\omega^6 - 15\omega^4 + \omega^2$
8	$490\omega^{16} - 1680\omega^{14} + 2310\omega^{12} - 1624\omega^{10} + 615\omega^8 - 120\omega^6 + 10\omega^4$
9	$1764\omega^{18} - 7056\omega^{16} + 11704\omega^{14} - 10416\omega^{12} + 5376\omega^{10} - 1624\omega^8 + 276\omega^6 - 24\omega^4 + \omega^2$
10	$5292\omega^{20} - 23520\omega^{18} + 44100\omega^{16} - 45360\omega^{14} + 27860\omega^{12} - 10416\omega^{10} + 2310\omega^8 - 280\omega^6 + 15\omega^4$

TABLE 10.1.4 Linear and Quadratic Factors for Poles of Legendre-Papoulis Approximation

n	$[G(s)]^{-1}$	n	$[G(s)]^{-1}$
2	$s^2 + 1.4142136s + 1.0000000$	8	$s^2 + 0.1378844s + 0.9808397$
3	$s + 0.6203318$		$s^2 + 0.3885518s + 0.7179832$
	$s^2 + 0.6903712s + 0.9307119$		$s^2 + 0.6005680s + 0.3828971$
4	$s^2 + 0.4633774s + 0.9476701$		$s^2 + 0.7343526s + 0.1675357$
	$s^2 + 1.0994868s + 0.4307915$	9	$s + 0.3256878$
5	$s + 0.4680899$		$s^2 + 0.1101944s + 0.9844435$
	$s^2 + 0.3071734s + 0.9608963$		$s^2 + 0.3145676s + 0.7666498$
	$s^2 + 0.7762796s + 0.4971406$		$s^2 + 0.4971058s + 0.4635058$
6	$s^2 + 0.2303854s + 0.9696012$		$s^2 + 0.6187708s + 0.2089807$
	$s^2 + 0.6179218s + 0.5828947$	10	$s^2 + 0.0918020s + 0.9869313$
	$s^2 + 0.8778030s + 0.2502256$		$s^2 + 0.2650376s + 0.8012497$
7	$s + 0.3821033$		$s^2 + 0.4283460s + 0.5282527$
	$s^2 + 0.1724170s + 0.9764158$		$s^2 + 0.5548108s + 0.2702425$
	$s^2 + 0.4748794s + 0.6621299$		$s^2 + 0.6344130s + 0.1217699$
	$s^2 + 0.6984636s + 0.3060005$		

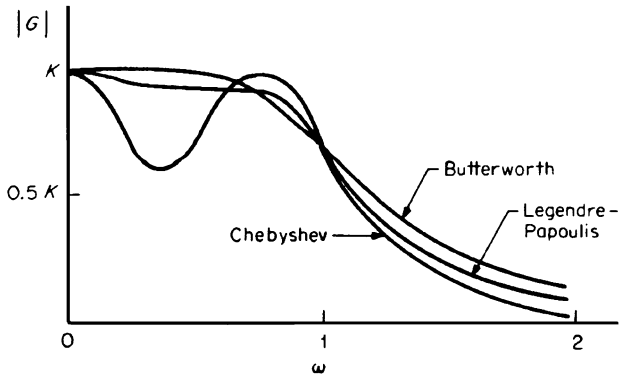

FIGURE 10.1.10 Comparison of Butterworth, Legendre-Papoulis, and Chebyshev ($\epsilon = 1$) approximations for $n = 3$.

TABLE 10.1.5 Coefficients of the Bessel Polynomials

n	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7
1	1							
2	3	3						
3	15	15	6					
4	105	105	45	10				
5	945	945	420	105	15			
6	10,395	10,395	4,725	1,260	210	21		
7	135,135	135,135	62,370	17,325	3,150	378	28	
8	2,027,025	2,027,025	945,945	270,270	51,975	6,930	630	36

TABLE 10.1.6 Linear and Quadratic Factors for Poles of Bessel Approximation

n	$[G(s)]^{-1}$
2	$s^2 + 3s + 3$
3	$s + 2.322185$ $s^2 + 3.677814s + 6.459432$
4	$s^2 + 5.792422s + 9.140133$ $s^2 + 4.207578s + 11.487799$
5	$s + 3.646739$ $s^2 + 6.703912s + 14.272476$ $s^2 + 4.649348s + 18.156314$
6	$s^2 + 8.496718s + 18.801128$ $s^2 + 7.471416s + 20.852819$ $s^2 + 5.031864s + 26.514025$
7	$s + 4.971787$ $s^2 + 9.516582s + 25.666449$ $s^2 + 8.140278s + 28.936544$ $s^2 + 5.371354s + 36.596784$
8	$s^2 + 11.175772s + 31.977224$ $s^2 + 10.409682s + 33.934741$ $s^2 + 8.736578s + 38.569256$ $s^2 + 5.677968s + 48.432015$
9	$s + 6.297019$ $s^2 + 12.258736s + 40.589268$ $s^2 + 11.208844s + 43.646648$ $s^2 + 9.276880s + 49.788507$ $s^2 + 5.958522s + 62.041443$
10	$s^2 + 13.844090s + 48.667550$ $s^2 + 13.230582s + 50.582362$ $s^2 + 11.935056s + 54.839151$ $s^2 + 9.772440s + 62.625584$ $s^2 + 6.217832s + 77.442692$

Figure 10.1.11 shows the general forms of the delay and magnitude functions for this approximation. The magnitude behavior is that of a low-pass function, and Table 10.1.11 gives the resulting half-power frequencies.

The parameter n is selected to satisfy either a minimum group delay requirement or a minimum magnitude (or the equivalent maximum attenuation) requirement at some frequency ω_a, that is,

$$\tau_d(\omega_a) \geq T_a \tag{49}$$

or

$$|G(j\omega_a)| \geq A \tag{50}$$

with the more stringent condition being chosen. Figure 10.1.12 shows the attenuation versus ω and the group delay versus ω for $n = 2$ to 8, and can be used to select n.

THE STEP RESPONSE

The primary design consideration for a filter is its frequency response, but often the step response is of interest. Computation of the step response requires computation of the inverse Laplace transform of the transfer function

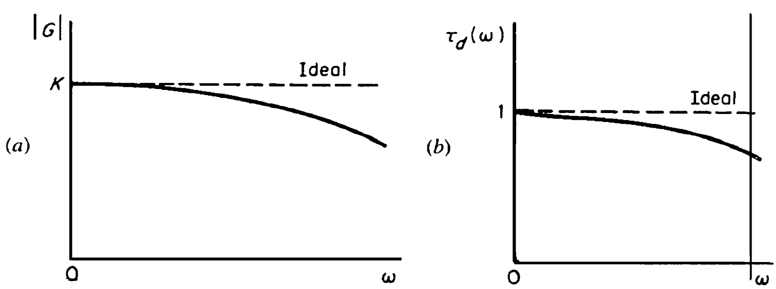

FIGURE 10.1.11 (*a*) Magnitude and (*b*) group delay for maximally flat group-delay approximation.

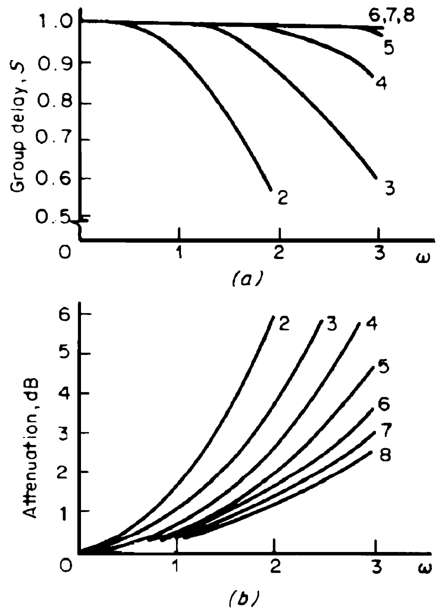

FIGURE 10.1.12 Group delay and attenuation for the Bessel approximation.

multiplied by $1/s$, readily done with modern computers and suitable software. Three figures of merit commonly characterize the step response. Figure 10.1.13 defines these, which are *overshoot, rise time*, and *delay time*. While the general form of Fig. 10.1.13 is common, the odd order, high ripple Chebyshev and modified Chebyshev responses have unusual features, as suggested in Figs. 10.1.14 and 10.1.15. Often several relative maxima occur before the peak value. Tables 10.1.7, 10.1.8, and 10.1.9 give the percent overshoot, rise time, and delay time, respectively, for the all pole approximations described.

THE PROTOTYPE FILTER NETWORKS

Passive networks that realize all pole functions are lossless ladder networks terminated in resistance at both ends. Figure 10.1.16 shows four networks classified by the type of transfer function, Z_{LS} or G_{LS}, being realized

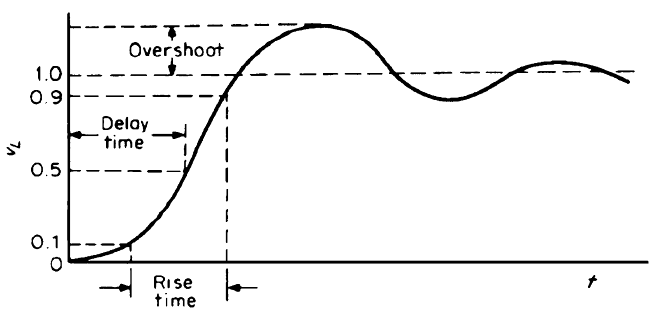

FIGURE 10.1.13 General form of the step response and definitions of figures of merit.

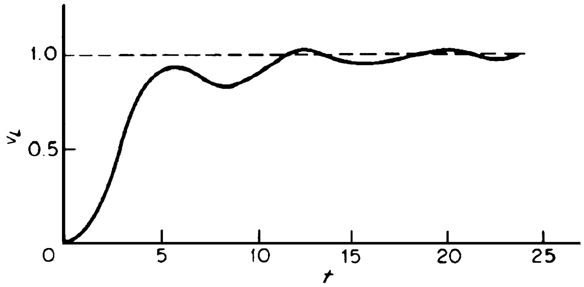

FIGURE 10.1.14 Step response of third-order Chebyshev approximation with a 3-dB ripple.

and whether n is odd or even. The networks of Figs. 10.1.16*c* and *d* are duals of those in Figs. 10.1.16*a* and *b*, respectively. An application of source transformations to the networks of Fig. 10.1.16 yields the four networks of Fig. 10.1.17.

For a given type of source and value of n there are two networks to choose from. When the source is a current source and n is odd, Figs. 10.1.16*b* and 10.1.17*d* apply. In the first case, the first and last lossless elements are capacitors; in the second case, they are inductors.

For a given type of approximation and order of filter, numerical element values apply to four different networks. For example, a Butterworth approximation with $n = 2$ and R_S or $G_S = 1/2$ according to the specific networking being used, yields the four networks of Fig. 10.1.18. Here, the numerical values for the lossless elements are the same when they are taken in order from the source end. This fact makes it possible to construct a table of element values, which uses as parameters for entry (1) the type of approximation, including the ripple width in dB, Eq. (25), for Chebyshev approximations; (2) the value of n; and (3) the value of R_S or G_S, according to the particular network for Figs. 10.1.16 and 10.1.17 that is being used. Table 10.1.10 is such a table.

THE TABLE OF ELEMENT VALUES

Table 10.1.10 gives element values, numbered in order from the source end as in Figs. 10.1.16 and 10.1.17, for the five all-pole approximations, $n = 2$ to 10, and for R_S or G_S equal to 0 and 1. Five different ripple widths are tabulated for the Chebyshev approximations.

The synthesis process for determining the element values in Table 10.1.10 often involves a choice of location for the zeros of the reflection coefficient. In this table when choices were necessary, the networks have left-half s-plane zeros.

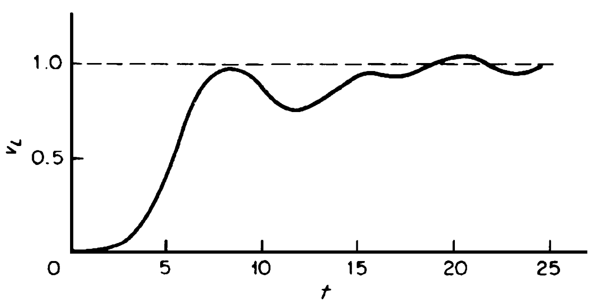

FIGURE 10.1.15 Step response of fifth-order Chebyshev approximation with a 3-dB ripple.

TABLE 10.1.7 Percent Overshoot

				Chebyshev, %				Modified Chebyshev, %			
Order	Bessel, %	Butterworth, %	Legendre-Papoulis, %	3.0 dB	1.0 dB	0.5 dB	0.1 dB	3.0 dB	1.0 dB	0.5 dB	0.1 dB
2	0.4	4.3	4.3	27.2	14.6	10.7	6.7	4.3	4.3	4.3	4.3
3	0.7	8.1	7.5	2.7	6.4	8.9	10.2				
4	0.8	10.8	11.2	35.7	21.9	18.1	14.5	8.0	10.5	12.1	12.9
5	0.8	12.8	13.3	1.7	10.2	13.2	15.2				
6	0.6	14.3	15.2	38.7	24.9	21.2	18.0	8.5	12.8	15.1	16.5
7	0.5	15.4	16.4	2.0	12.1	15.3	17.7				
8	0.3	16.3	17.6	40.4	26.5	23.0	19.8	8.7	14.0	16.6	18.5
9	0.2	17.1	18.3	3.1	13.3	16.7	19.1				
10	0.1	17.8	19.1	41.5	27.6	24.1	21.0	9.5	14.8	17.6	19.7

TABLE 10.1.8 Rise Time for Polynomial Filters

				Chebyshev				Modified Chebyshev			
Order	Bessel	Butterworth	Legendre-Papoulis	3.0 dB	1.0 dB	0.5 dB	0.1 dB	3.0 dB	1.0 dB	0.5 dB	0.1 dB
2	1.58	2.2	2.2	1.7	1.6	1.5	1.1	2.2	1.5	1.3	0.8
3	1.25	2.3	2.5	3.2	2.4	2.2	1.7				
4	1.04	2.4	2.7	2.5	2.5	2.4	2.2	3.2	2.6	2.4	2.1
5	0.91	2.6	2.9	3.7	3.0	2.8	2.5				
6	0.82	2.7	3.0	2.9	3.0	2.9	2.8	3.7	3.2	3.0	2.7
7	0.75	2.8	3.2	4.0	3.4	3.3	3.0				
8	0.69	2.9	3.4	3.2	3.3	3.3	3.2	4.1	3.6	3.4	3.2
9	0.64	3.0	3.5	4.3	3.8	3.6	3.4				
10	0.60	3.1	3.6	3.5	3.6	3.6	3.5	4.4	3.9	3.7	3.5

TABLE 10.1.9 Delay Times for Polynomial Filters

Order	Bessel	Butterworth	Legendre-Papoulis	Chebyshev				Modified Chebyshev			
				3.0 dB	1.0 dB	0.5 dB	0.1 dB	3.0 dB	1.0 dB	0.5 dB	0.1 dB
2	0.90	1.4	1.4	1.5	1.2	1.1	0.8	1.4	1.0	0.8	0.6
3	0.96	2.1	2.4	3.0	2.4	2.2	1.7				
4	0.98	2.8	3.3	3.6	3.3	3.2	2.7	3.9	3.3	3.1	2.6
5	0.99	3.5	4.2	5.2	4.6	4.3	3.8				
6	0.99	4.2	5.2	5.6	5.4	5.2	4.8	6.1	5.5	5.2	4.7
7	1.00	4.8	6.2	7.3	6.7	6.4	5.8				
8	1.00	5.5	7.1	7.7	7.5	7.3	6.8	8.2	7.6	7.3	6.8
9	1.00	6.2	8.1	9.4	8.8	8.5	7.9				
10	1.00	6.8	9.0	9.7	9.6	9.4	8.9	10.3	9.7	9.4	6.9

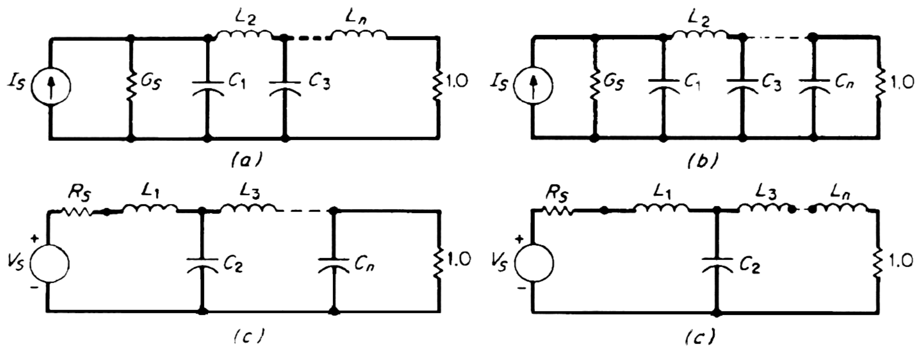

FIGURE 10.1.16 Four filter networks classified according to type of transfer function and n even or odd: (*a*) Z_{LS}, n even; (*b*) Z_{LS}, n odd; (*c*) G_{LS}, n even; (*d*) G_{LS}, n odd.

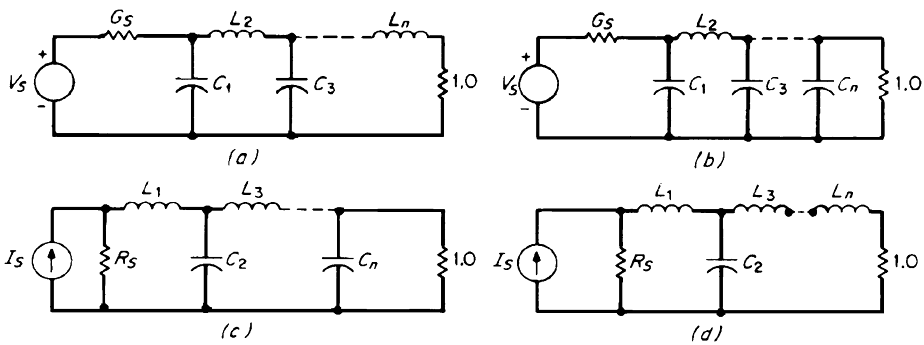

FIGURE 10.1.17 Filter networks derived from those in Fig. 10.1.16 by source transformations: (*a*) G_{LS}, n even; (*b*) G_{LS}, n odd; (*c*) Z_{LS}, n even; (*d*) Z_{LS}, n odd.

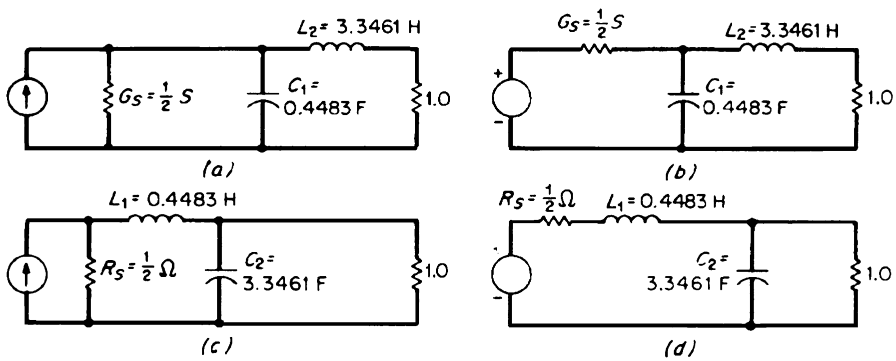

FIGURE 10.1.18 Four networks resulting from a Butterworth approximation with $n = 2$ and R_S or G_S (as appropriate to the type of network) equal to $1/2$.

TABLE 10.1.10 Low-Pass Filter Circuit Element Values*
(Table 10.1.11 gives half-power frequencies for Bessel and Chebyshev circuits, the half-power frequency is 1.000 rad/s for Butterworth and Legendre-Papoulis; empty spaces indicate unrealizable networks.)

Filter type	R_s or G_s \ Element number	Second-order network $n = 2$		Third-order network $n = 3$			Fourth-order network $n = 4$				Fifth-order network $n = 5$				
		1	2	1	2	3	1	2	3	4	1	2	3	4	5
Bessel	0	1.0000	0.3333	0.8333	0.4800	0.1667	0.7101	0.4627	0.2899	0.1000	0.6231	0.4215	0.3103	0.1948	0.0667
	1	1.5774	0.4227	1.2550	0.5528	0.1922	1.0598	0.5116	0.3181	0.1104	0.9303	0.4577	0.3312	0.2089	0.0718
Butterworth	0	1.4142	0.7071	1.5000	1.3333	0.5000	1.5307	1.5772	1.0824	0.3827	1.5451	1.6944	1.3820	0.8944	0.3090
	1	1.4142	1.4142	1.0000	2.0000	1.0000	0.7654	1.8478	1.8478	0.7654	0.6180	1.6180	2.0000	1.6180	0.6180
Legendre-Papoulis	0	1.4142	0.7071	1.5909	1.4270	0.7629	1.6120	1.6616	1.4292	0.6399	1.6372	1.7509	1.7358	1.3945	0.6445
	1	1.4142	1.4142	2.1801	1.3538	1.1737	1.5645	1.9584	1.4769	1.0826	1.9990	1.5395	2.0673	1.4780	0.9512
Chebyshev 0.01 dB	0	0.4274	0.2244	0.7997	0.7634	0.3146	1.0421	1.1547	0.8945	0.3564	1.1978	1.3902	1.2739	0.9576	0.3782
	1			0.6292	0.9703	0.6292					0.7563	1.3049	1.5773	1.3049	0.7563
Chebyshev 0.10 dB	0	0.7159	0.4215	1.0895	1.0864	0.5158	1.2453	1.4576	1.1994	0.5544	1.3759	1.5924	1.5562	1.2490	0.5734
	1			1.0316	1.1474	1.0316					1.1468	1.3712	1.9750	1.3712	1.1468
Chebyshev 0.50 dB	0	0.9403	0.7014	1.3465	1.3001	0.7981	1.3138	1.7279	1.3916	0.8352	1.5388	1.6426	1.8142	1.4291	0.8529
	1			1.5963	1.0967	1.5963					1.7058	1.2296	2.5408	1.2296	1.7058
Chebyshev 1.0 dB	0	0.9957	0.9110	1.5088	1.3332	1.0118	1.2817	1.9093	1.4126	1.0495	1.6652	1.5908	1.9938	1.4441	1.0674
	1			2.0236	0.9941	2.0236					2.1349	1.0911	3.0009	1.0911	2.1349
Chebyshev 3.0 dB	0	0.9109	1.5506	2.0302	1.1739	1.6744	1.0578	2.5272	1.2292	1.7195	2.1489	1.3016	2.6224	1.2502	1.7406
	1			3.3487	0.7117	3.3487					3.4813	0.7619	4.5375	0.7619	3.4813
Modified Chebyshev 0.01 dB	1	0.3099	0.3099				0.6266	1.1938	1.1938	0.6266					
Modified Chebyshev 0.10 dB	1	0.5525	0.5525				0.9297	1.4346	1.4346	0.9297					
0.50 dB	1	0.8358	0.8358				1.3091	1.5415	1.5415	1.3091					
1.0 dB	1	1.0088	1.0088				1.5675	1.5537	1.5537	1.5675					
3.0 dB	1	1.4125	1.4125				2.2574	1.5107	1.5107	2.2574					

(Continued)

TABLE 10.1.10 Low-Pass Filter Circuit Element Values* (*Continued*)

Filter type	Element number / R_s or G_s	Sixth-order network $n = 6$						Seventh-order network $n = 7$						
		1	2	3	4	5	6	1	2	3	4	5	6	7
Bessel	0	0.5595	0.3821	0.3005	0.2246	0.1400	0.0476	0.5111	0.3487	0.2827	0.2288	0.1704	0.1055	0.0357
	1	0.8376	0.4116	0.3158	0.2364	0.1480	0.0505	0.7677	0.3744	0.2944	0.2378	0.1778	0.1104	0.0375
Butterworth	0	1.5529	1.7593	1.5529	1.2016	0.7579	0.2588	1.5576	1.7988	1.6588	1.3972	1.0550	0.6560	0.2225
	1	0.5176	1.4142	1.9319	1.9319	1.4142	0.5176	0.4450	1.2470	1.8019	2.0000	1.8019	1.2470	0.4450
Legendre-	0	1.6348	1.8088	1.8223	1.6795	1.3486	0.5793	1.6391	1.8312	1.8911	1.7988	1.6845	1.3290	0.5787
Papoulis	1	1.5763	1.9040	1.7442	1.9857	1.4852	0.9160	1.8640	1.5895	2.1506	1.7270	1.9394	1.4770	0.8394
Chebyshev	0	1.2931	1.5400	1.4922	1.3319	0.9925	0.3907	1.3615	1.6303	1.6290	1.5412	1.3650	1.0137	0.3985
0.01 dB	1							0.7969	1.3924	1.7481	1.6331	1.7481	1.3924	0.7969
Chebyshev	0	1.4035	1.7236	1.6749	1.5999	1.2752	0.5841	1.4745	1.7395	1.7987	1.7107	1.6236	1.2908	0.5906
0.10 dB	1							1.1812	1.4228	2.0967	1.5734	2.0967	1.4228	1.1812
Chebyshev	0	1.4042	1.9018	1.7101	1.8494	1.4483	0.8627	1.5983	1.7252	1.9713	1.7369	1.8677	1.4595	0.8686
0.50 dB	1							1.7373	1.2582	2.6383	1.3443	2.6383	1.2582	1.7373
Chebyshev	0	1.3457	2.0491	1.6507	2.0270	1.4601	1.0773	1.7120	1.6488	2.1194	1.6735	2.0438	1.4692	1.0833
1.0 dB	1							2.1666	1.1115	3.0936	1.1735	3.0936	1.1115	2.1666
Chebyshev	0	1.0876	2.6309	1.3455	2.6578	1.2606	1.7522	2.1828	1.3281	2.7143	1.3613	2.6752	1.2665	1.7593
3.0 dB	1							3.5185	0.7722	4.6390	0.8038	4.6390	0.7722	3.5185
Modified 0.01 dB	1	0.7190	1.3728	1.6006	1.6006	1.3728	0.7190							
Chebyshev 0.10 dB	1	1.0382	1.5163	1.7892	1.7892	1.5163	1.0382							
0.50 dB	1	1.4611	1.5085	1.9333	1.9333	1.5085	1.4611							
1.0 dB	1	1.7623	1.4528	2.0089	2.0089	1.4528	1.7623							
3.0 dB	1	2.6073	1.2711	2.1729	2.1729	1.2711	2.6073							

TABLE 10.1.10 Low-Pass Filter Circuit Element Values* (*Continued*)

Filter type	R_s or G_s \ Element number	Eight-order network $n = 8$: 1	2	3	4	5	6	7	8	Ninth-order network $n = 9$: 1	2	3	4	5	6	7	8	9
Bessel	0	0.4732	0.3212	0.2639	0.2227	0.1806	0.1338	0.0823	0.0278	0.4424	0.2986	0.2465	0.2129	0.1811	0.1463	0.1077	0.0660	0.0222
	1	0.7125	0.3446	0.2735	0.2297	0.1867	0.1387	0.0855	0.0289	0.6678	0.3203	0.2547	0.2184	0.1859	0.1506	0.1112	0.0682	0.0230
Butterworth	0	1.5607	1.8246	1.7287	1.5283	1.2588	0.9371	0.5776	0.1951	1.5628	1.8424	1.7772	1.6202	1.4037	1.1408	0.8414	0.5155	0.1736
	1	0.3902	1.1111	1.6629	1.9616	1.9616	1.6629	1.1111	0.3902	0.3473	1.0000	1.5321	1.8794	2.0000	1.8794	1.5321	1.0000	0.3473
Legendre-	0	1.6345	1.8542	1.9102	1.8673	1.8019	1.6437	1.2869	0.5372	1.6341	1.8625	1.9349	1.8961	1.8815	1.7860	1.6397	1.2740	0.5358
Papoulis	1	1.5564	1.8501	1.8411	2.0515	1.7672	1.9115	1.4688	0.8205	1.7645	1.6134	2.1585	1.7816	2.0662	1.7755	1.8674	1.4555	0.7695
Chebyshev	0	1.4036	1.6971	1.7097	1.6708	1.5697	1.3858	1.0275	0.4036	1.4392	1.7371	1.7701	1.7457	1.6949	1.5878	1.3998	1.0370	0.4072
0.01 dB	1									0.8145	1.4271	1.8044	1.7125	1.9058	1.7125	1.8044	1.4271	0.8145
Chebyshev	0	1.4660	1.8163	1.8070	1.8302	1.7302	1.6380	1.3008	0.5949	1.5182	1.7991	1.8814	1.8343	1.8473	1.7423	1.6476	1.3076	0.5978
0.10 dB	1									1.1957	1.4426	2.1346	1.6167	2.2054	1.6167	2.1346	1.4426	1.1957
Chebyshev	0	1.4379	1.9571	1.7838	1.9980	1.7508	1.8786	1.4666	0.8725	1.6238	1.7571	2.0203	1.8055	2.0116	1.7591	1.8856	1.4714	0.8752
0.50 dB	1									1.7504	1.2690	2.6678	1.3673	2.7239	1.3673	2.6678	1.2690	1.7504
Chebyshev	0	1.3691	2.0922	1.7021	2.1453	1.6850	2.0537	1.4751	1.0872	1.7317	1.6707	2.1574	1.7213	2.1582	1.6918	2.0601	1.4790	1.0899
1.0 dB	1									2.1797	1.1192	3.1214	1.1897	3.1746	1.1897	3.1214	1.1192	2.1797
Chebyshev	0	1.0982	2.6618	1.3687	2.7436	1.3690	2.6852	1.2701	1.7638	2.1970	1.3380	2.7413	1.3827	2.7576	1.3733	2.6915	1.2726	1.7670
3.0 dB	1									3.5339	0.7760	4.6691	0.8118	4.7270	0.8118	4.6691	0.7760	3.5339
Modified 0.01 dB	1	0.7584	1.4274	1.7122	1.7503	1.7503	1.7122	1.4274	0.7584									
Chebyshev 0.10 dB	1	1.0880	1.5265	1.9029	1.8301	1.8301	1.9029	1.5265	1.0880									
0.50 dB	1	1.5372	1.4679	2.1033	1.8436	1.8436	2.1033	1.4679	1.5372									
1.0 dB	1	1.8642	1.3833	2.2357	1.8319	1.8319	2.2357	1.3833	1.8642									
3.0 dB	1	2.8062	1.1473	2.5784	1.7813	1.7813	2.5784	1.1473	2.8062									

(*Continued*)

TABLE 10.1.10 Low-Pass Filter Circuit Element Values* (*Continued*)

Filter type	R_s or G_s \ Element number	1	2	3	4	5	6	7	8	9	10
		Tenth-order network $n = 10$									
Bessel	0	0.4170	0.2797	0.2311	0.2021	0.1770	0.1504	0.1209	0.0886	0.0541	0.0182
	1	0.6305	0.3002	0.2384	0.2066	0.1808	0.1539	0.1240	0.0911	0.0556	0.0187
Butterworth	0	1.5643	1.8552	1.8121	1.6869	1.5100	1.2921	1.0406	0.7626	0.4654	0.1564
	1	0.3129	0.9080	1.4142	1.7820	1.9754	1.9754	1.7820	1.4142	0.9080	0.3129
Legendre-Papoulis	0	1.6298	1.8741	1.9405	1.9223	1.9102	1.8629	1.7785	1.6082	1.2386	0.5065
	1	1.5286	1.8122	1.8953	2.0409	1.8453	2.0327	1.7839	1.8537	1.4454	0.7575
Chebyshev 0.01 dB	0	1.4598	1.7731	1.8054	1.8020	1.7664	1.7106	1.6003	1.4096	1.0439	0.4098
	1										
Chebyshev 0.10 dB	0	1.4964	1.8585	1.8600	1.9068	1.8489	1.8579	1.7503	1.6542	1.3124	0.6000
	1										
Chebyshev 0.50 dB	0	1.4539	1.9816	1.8119	2.0432	1.8165	2.0197	1.7645	1.8905	1.4748	0.8771
	1										
Chebyshev 1.0 dB	0	1.3801	2.1111	1.7215	2.1803	1.7307	2.1658	1.6962	2.0645	1.4817	1.0918
	1										
Chebyshev 3.0 dB	0	1.1032	2.6753	1.3774	2.7682	1.3893	2.7655	1.3761	2.6958	1.2744	1.7692
	1										
Modified Chebyshev 0.01 dB	1	0.7795	1.4503	1.7608	1.7903	1.8495	1.8495	1.7903	1.7608	1.4503	0.7795
0.10 dB	1	1.1165	1.5249	1.9621	1.8183	1.9346	1.9346	1.8183	1.9621	1.5249	1.1165
0.50 dB	1	1.5832	1.4380	2.2072	1.7663	2.0029	2.0029	1.7663	2.2072	1.4380	1.5832
1.0 dB	1	1.9273	1.3374	2.3826	1.7118	2.0410	2.0410	1.7118	2.3826	1.3374	1.9273
3.0 dB	1	2.9356	1.0731	2.8661	1.5644	2.1269	2.1269	1.5644	2.8661	1.0731	2.9356

*Values for the Legendre-Papoulis approximation by permission from A. Zverev, "Handbook of Filter Synthesis," John Wiley & Sons, 1967.

TABLE 10.1.11 Half-Power Frequencies for Various Chebyshev and Bessel Filters*

	Order of filter								
Type of filter	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
Modified Chebyshev 0.01 dB	4.563742		1.532784		1.212468		1.114760		1.072036
Modified Chebyshev 0.10 dB	2.559727		1.246004		1.099300		1.053929		1.033948
Modified Chebyshev 0.50 dB	1.691974		1.108290		1.043913		1.023911		1.015073
Modified Chebyshev 1.00 dB	1.401865		1.061830		1.025105		1.013681		1.008628
Modified Chebyshev 3.00 dB	1.001188		1.000174		1.000071		1.000039		1.000024
Bessel half-power frequency	1.3617	1.7557	2.1140	2.4274	2.7034	2.9517	3.1797	3.3917	3.5910
Delay at half-power frequency	0.8090	0.9349	0.9819	0.9960	0.9993	0.9999	0.9999	1.0000	1.0000

*The Chebyshev filter prototypes have the ripple specified at 1 rad/s. The Bessel filter has a delay of 1 s at very low frequency. The second figure given is the delay at the half-power frequency.

Table 10.1.10 assumes:

1. $R_L = 1$.
2. $\omega_{hp} = 1$ for the Butterworth and Legendre-Papoulis approximations.
3. $\omega = 1$ is the end of the ripple band for the Chebyshev approximations.
4. The group delay at $\omega = 0$ is 1 s for the Bessel approximation.

The half-power frequencies for the (regular) Chebyshev approximation are found from Eq. (31), and, for the modified Chebyshev and Bessel approximations, are tabulated in Table 10.1.11.

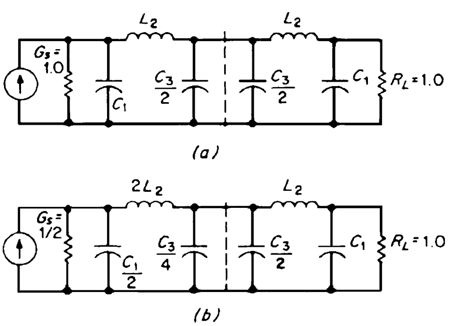

FIGURE 10.1.19 The scaling of symmetrical networks to change the value of G_S.

The two values of R_S or G_S, 0 and 1, used in Table 10.1.10, represent the two most commonly encountered situations. Zverev and Weinberg give tables for additional values of R_S or G_S.

When n is odd, $G_S = R_S = 1$, and the approximation is either Butterworth or (regular) Chebyshev, the networks are symmetrical; that is, $C_1 = C_n$, $L_2 = L_{n-1}$, etc. In these cases, there is a technique to realize other values of R_S or G. Consider Fig. 10.1.16*b* with $n = 5$, redrawn as Fig. 10.1.19*a* to emphasize the symmetry. Scale the half of the network containing G_S, and recombine the two center elements. Making $G_S = {}^1\!/_2$ leads to the network of Fig. 10.1.19*b*, where the two parts of the center capacitor are shown separately before recombination. This procedure leads to right-half s-plane zeros of the reflection coefficient.

EXPLICIT BUTTERWORTH AND CHEBYSHEV FORMULAS

Closed-form solutions for doubly terminated Butterworth and Chebyshev filters were derived by Takahasi and restated by Humpherys.[8] They assume that the zeros of the reflection coefficient lie in the left half plane, and so give different networks than the symmetrical network procedure. This gives the designer a choice in cases where both techniques may be applicable.

The Butterworth formulas make use of the poles of the transfer functions and require also a factor λ that relates the load and source resistances. The source resistance is assumed to be 1.0, and the formulas are

$$s_i = 2\sin(\pi i/2\pi) \tag{51}$$

$$c_i = 2\cos(\pi i/2\pi) \tag{52}$$

$$\lambda = -\left(\frac{R_L - 1}{R_L + 1}\right)^{1/n} \tag{53}$$

when the first reactive element is a shunt capacitor, and

$$\lambda = -\left(\frac{G_L - 1}{G_L + 1}\right)^{1/n} \tag{54}$$

when the first reactive element is a series inductor. Recursive equations give the element values

$$C_1 = \frac{s_1}{1-\lambda} \tag{55}$$

$$C_n = \frac{s_1}{(1+\lambda)R_L} \qquad n \text{ odd} \tag{56}$$

$$L_n = \frac{s_1 R_L}{1+\lambda} \qquad n \text{ even} \tag{57}$$

$$C_{2m-1}L_{2m} = \frac{s_{4m-1}s_{4m+1}}{1-\lambda c_{4m-2}+\lambda^2} \tag{58}$$

$$L_{2m-1}C_{2m+1} = \frac{s_{4m-1}s_{4m+1}}{1-\lambda c_{4m}+\lambda^2}$$

$$\text{where } m = \begin{cases} 1,2,\ldots,(n-1)/2 & n \text{ odd} \\ 1,2,\ldots,n/2 & n \text{ even} \end{cases}$$

When the first element is a series inductor, the roles of L and C are interchanged. An example follows. Develop the prototype network for third-order Butterworth, $R_L = 3$, $R_s = 1$. It follows that

$$\lambda = -0.7937 \tag{59}$$

$$s_1 = 1.0000 = s_5 \tag{60}$$

$$s_2 = 1.7321 = s_4 \tag{61}$$

$$s_3 = 2.0000 \tag{62}$$

$$c_1 = 1.7321 \tag{63}$$

$$c_2 = 1.0000 \tag{64}$$

$$c_3 = 0.0000 \tag{65}$$

$$c_4 = 1.0000 \tag{66}$$

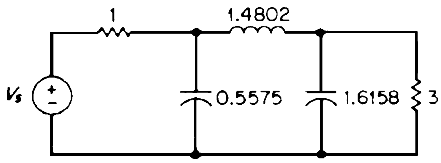

FIGURE 10.1.20 Prototype Butterworth network, $n = 3$, $R_S = 1$, $R_L = 3$.

$$C_1 = \frac{1.0000}{1-(-0.7937)} = 0.5575 \tag{67}$$

$$L_2 = \frac{s_1 s_3}{1-(-0.7937)(1.0000)+(0.7937)^2}\,\frac{1}{C_1} = 1.4802 \tag{68}$$

$$C_3 = \frac{s_3 s_5}{1-(-0.7937)(-1.0000)+(0.7937)^2}\,\frac{1}{L_2} = 1.6158 \tag{69}$$

The prototype network thus becomes that of Fig. 10.1.20.

The results are similar if the equation for C_n is used first and the network is developed from the load end; this serves as a check on the work.

The Chebyshev equations are similar. Define the following terms:

$$\epsilon = \sqrt{10^{r/10}-1} \qquad \begin{aligned} \epsilon &= \text{ripple factor} \\ r &= \text{ripple, dB} \end{aligned} \tag{70}$$

$$A = \begin{cases} 4R_L R_S/(R_L+R_S)^2 & n \text{ odd} \\ 4(1+\epsilon^2)R_L R_S/(R_L+R_S)^2 \le 1 & n \text{ even} \end{cases} \tag{71a, 71b}$$

$$s_1 = 2\sin(\pi i/2n) \tag{72}$$

$$c_1 = 2\cos(\pi i/2n) \tag{73}$$

$$k = \left(\frac{1}{\epsilon} + \sqrt{\frac{1}{\epsilon^2}+1}\right)^{1/n} \tag{74}$$

$$h = -\left(\sqrt{\frac{1-A}{\epsilon^2}} + \sqrt{\frac{1-A}{\epsilon^2}+1}\right)^{1/n} \tag{75}$$

$$k' = k - 1/k \tag{76}$$

$$h' = h - 1/h \tag{77}$$

Recursive equations give the element values

$$C_1 = \frac{2s_1/R_s}{k'-h'} \tag{78}$$

$$C_n = \frac{2s_1/R_L}{k'+h'} \qquad n \text{ odd} \tag{79}$$

$$L_n = \frac{2s_1 R_L}{k'-h'} \qquad n \text{ even} \tag{80}$$

$$C_{2m-1}L_{2m} = \frac{4s_{4m-3}s_{4m-1}}{k'^2 - c_{2i}k'h' + h'^2 + s_{2i}^2} \tag{81}$$

$$L_{2m}C_{2m+1} = \frac{4S_{4m-1}S_{4m+1}}{k'^2 - c_{2i}k'h' + h'^2 + s_{2i}^2} \tag{82}$$

$$\text{where } m = \begin{cases} 1,2,\ldots,(n-1)/2 & n \text{ odd} \\ 1,2,\ldots,n/2 & n \text{ even} \end{cases}$$

When the first element is a series inductor, the roles of L and C are interchanged, while G_s and G_L are substituted for R_S and R_L. As an example, develop the prototype when $R_S = 3$, $R_L = 1$, 0.5 dB ripple, and a fourth-order network is needed. It follows that

$$\epsilon = \sqrt{10^{0.05} - 1} = 0.3493 \tag{83}$$

$$A = 0.75[1 + (0.3493)^2] = 0.8415 \tag{84}$$

$$s_1 = 0.7654 = s_7 \tag{85}$$

$$s_2 = 1.4142 = s_6 \tag{86}$$

$$s_3 = 1.8478 = s_5 \tag{87}$$

$$s_4 = 2.0000 \tag{88}$$

$$c_0 = 2.0000 \tag{89}$$

$$c_2 = 1.4142 \tag{90}$$

$$c_4 = 0.0000 \tag{91}$$

$$c_6 = -1.4142 \tag{92}$$

$$k = 1.5582 \tag{93}$$

$$h = -1.2766 \tag{94}$$

$$k' = 0.9164 \tag{95}$$

$$h' = -0.4933 \tag{96}$$

$$C_1 = \frac{2(0.7654)}{3[0.9164 - (-0.4933)]} = 0.3620 \tag{97}$$

$$L_2 = \frac{4s_1s_3}{C_1(k'^2 - c_2k'h' + h'^2 + s_2^2)} = 4.1985 \tag{98}$$

$$C_3 = \frac{4s_3s_5}{L_2(k'^2 - c_4k'h' + h'^2 + s_4^2)} = 0.6399 \tag{99}$$

$$L_4 = \frac{4s_5s_7}{C_3(h'^2 - c_6k'h' + h'^2 + s_6^2)} = 3.6172 \tag{100}$$

The prototype network is shown in Fig. 10.1.21.

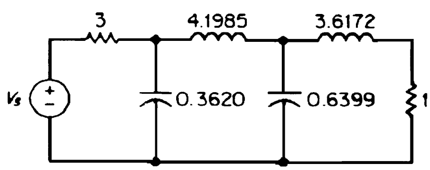

FIGURE 10.1.21 Prototype Chebyshev network, $R_S = 3$, $R_L = 1$, $n = 4$, 0.5 dB ripple.

THE INVERSE CHEBYSHEV APPROXIMATION

The Inverse Chebyshev approximation uses the Chebyshev polynomials in an approximation that decreases monotonically in the passband, and has ripples in the stopband. Specifically,

$$|G_{IC}(\omega)| = \sqrt{\frac{\epsilon^2 C_n^2(1/\omega)}{1+\epsilon^2 C_n^2(1/\omega)}} \tag{101}$$

where Eqs. (21) and (22) define C_n. Figure 10.1.22 shows this approximation and defines appropriate design criteria. K_s is the minimum attenuation in the stopband; K_p is the maximum attenuation in the passband, and it occurs at $\omega \geq \omega_p$. The edge of the stopband is at $\omega_a = 1$, which requires scaling of the specifications differently from the previous approximations.

When K_s is expressed in dB,

$$\epsilon = \sqrt{\frac{1}{10^{0.1k_s} - 1}} \tag{102}$$

Equation (103) gives the order n_{IC} required to meet the specifications, with the next larger integer being chosen.

$$n_{IC} = \frac{\cosh^{-1}\sqrt{\dfrac{(10^{0.1k_s} - 1)}{(10^{0.1k_p} - 1)}}}{\cosh^{-1}(1/\omega_p)} \tag{103}$$

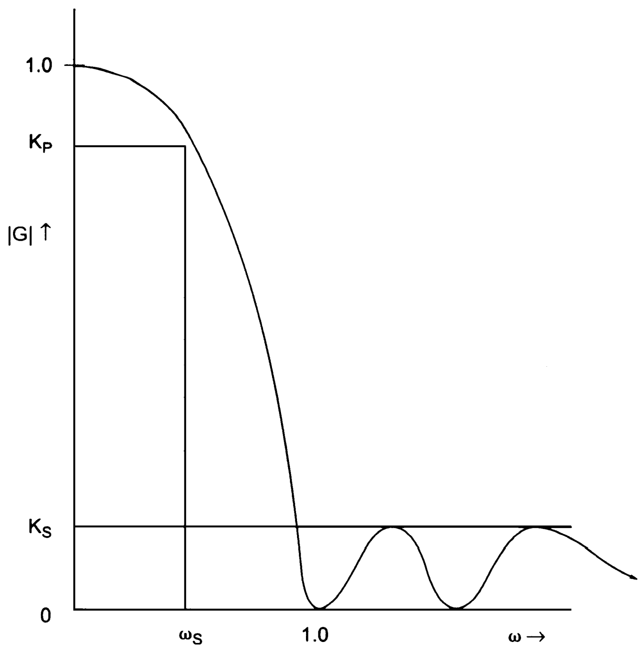

FIGURE 10.1.22 The Inverse Chebyshev low-pass approximation. A fifth-order function is illustrated.

In general, the order of inverse Chebyshev filter required to meet a set of specifications is identical with the order required for a conventional Chebyshev filter. It has improved step response and phase response characteristics when compared with the (regular) Chebyshev approximation, and a more complex network realization.

The stopband ripples and the zero magnitudes at finite frequencies in the stopband mean that the Inverse Chebyshev filter is not an *all-pole* approximation as the preceding have been, but rather it has roots in the numerator as well as in the denominator. Explicit formulas for these roots exist.

The numerator roots, or zeros, are

$$z_k = \alpha_k + j\beta_k \tag{104}$$

where

$$\alpha_k = 0 \quad \beta_k = \frac{1}{\cos u_k} \qquad k = 1, 2, 3, \dots, n \tag{105}$$

and

$$u_k = \frac{2k-1}{2n}\pi \tag{106}$$

The denominator factors, or poles, are

$$p_k = \frac{1}{\sigma_k + j\omega_k} \qquad k = 1, 2, 3, \dots n \tag{107}$$

where

$$\sigma_k = -\sin u_k \sinh v \quad \omega_k = \cos u_k \cosh v \tag{108}$$

and

$$v = \frac{1}{n}\sinh^{-1}\frac{1}{\epsilon} \tag{109}$$

and Eq. (106) gives u_k.

For example, assume that, using scaled frequency parameters, a designer needs a filter with a maximum passband attenuation K_p of 2.0 dB at frequencies below $\omega_p = 0.58$ rad/s, and a minimum stopband attenuation K_s of 40 dB above 1.0 rad/s. Equation (103) shows that a fifth-order filter is adequate. Application of Eqs. (104) through (109) shows that

$$G(s) = \frac{0.0500(s^2 + 1.1056)(s^2 + 2.8944)}{(s + 0.7878)(s^2 + 1.0496s + 0.5110)(s^2 + 0.3118s + 0.3975)} \tag{110}$$

where the numerator constant makes $G(0) = 1.0000$.

THE ELLIPTIC FILTER

The five approximations thus far described are of a class known as *all-pole approximations* because the magnitude-squared function of Eq. (6) has no finite zeros. This concentration of the zeros at infinity usually gives more attenuation than required at the higher frequencies and less in the vicinity of the cutoff frequency. If the more general form of Eq. (5) is used for the magnitude-squared function, some of the zeros can be placed

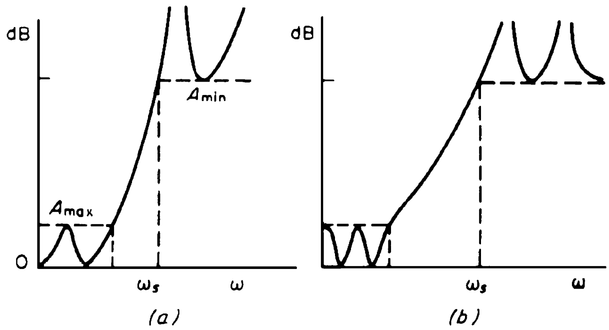

FIGURE 10.1.23 The elliptic approximation for (*a*) $n = 3$ and (*b*) $n = 4$.

close to the stopband edge frequency, with an improvement in the cutoff rate. Specifically, if

$$|G(j\omega)|^2 = K^2 \frac{1}{1+\epsilon^2 R_n^2(\omega)} \tag{111}$$

where

$$R_n(\omega) = \begin{cases} r\dfrac{\omega \displaystyle\prod_{i=1}^{(n-1)/2} (\omega^2 - \omega_{pi}^2)}{\displaystyle\prod_{i=1}^{(n-1)/2} (\omega^2 - \omega_{si}^2)} & n \text{ odd} \qquad (112) \\[2ex] r\dfrac{\displaystyle\prod_{i=1}^{n/2} (\omega^2 - \omega_{pi}^2)}{\displaystyle\prod_{i=1}^{n/2} (\omega^2 - \omega_{si}^2)} & n \text{ even} \qquad (113) \end{cases}$$

r being a multiplicative constant, the approximation can be made equiripple in both the passband and the stopband. Such an approximation is called an *elliptic approximation* because elliptic functions are used in its determination. Figure 10.1.23 shows such an approximation in a plot of attenuation versus frequency for $n = 3$ and $n = 4$, illustrative of the general characteristics for even and odd n, respectively.

Since the even-order elliptic approximation does not have infinite attenuation at infinite frequency, it cannot be realized by an *LC* ladder. A modified even-order elliptic approximation can be obtained by a frequency transformation. This transformation sacrifices some stopband attenuation to shift a pole of attenuation to ∞. A further modification of the even-order approximation can be used to make the attenuation zero to zero frequency. This permits an *LC* ladder realization to be equally terminated.

The elliptic approximation is characterized by four parameters.

1. The passband ripple A_{max} or, equivalently, the reflection coefficient ρ, related to A_{max} by $A_{max} = -10 \log (1 - \rho^2)$
2. The order n
3. The minimum stopband attenuation A_{min}
4. The stopband edge frequency ω_s or, equivalently, the modular angle θ[6]

Any three of these four parameters can be independently specified.

As for the all-pole approximations, tabulations of pole-zero locations and tables of element values for normalized low-pass filters based on the elliptic approximation are possible. Since three parameters are required to specify in elliptic approximation, these tabulations are voluminous.

DELAY EQUALIZATION

The ideal low-pass filter characteristic would have a constant group delay as well as a constant amplitude throughout the passband so that the various frequency components of a signal would arrive at the output unattenuated and with the proper phase relationship. Deviations from constant amplitude and group delay produce amplitude and phase distortion, respectively. While phase distortion is not a problem in many applications, it is significant when pulse transmission is involved.

With the exception of the Bessel approximation, the various approximations have focused on the amplitude characteristic. While the Bessel approximation has a good group-delay characteristic, it has a significantly poorer amplitude characteristic. One approach to the problem of obtaining filters with good amplitude *and* group-delay characteristics is through the use of delay equalizers. This consists of using one of the approximations with desirable amplitude characteristics in conjunction with an all-pass function that does not affect the amplitude characteristic but modifies the group delay beneficially.

The first-order all-pass function is

$$G(s) = \frac{s-\delta}{s+\delta} \tag{114}$$

This function has an emplitude of 1 for all frequencies and a group-delay function

$$\tau_d(\omega) = \frac{2}{\delta}\frac{1}{1+(\omega/\delta)^2} \tag{115}$$

The specific properties of the group-delay function are controlled by the selection of δ.

The second-order all-pass function is

$$G(s) = \frac{(s-\delta)^2+\beta^2}{(s+\delta)^2+\beta^2} \tag{116}$$

with

$$|G(j\omega)| = \mathrm{I} \tag{117}$$

and

$$\tau_d(\omega) = \frac{4\delta(\omega^2+\delta^2+\beta^2)}{(\delta^2+\beta^2-\omega^2)^2+4\delta^2\omega^2} \tag{118}$$

The specific properties of this group-delay function are controlled by the selection of δ and β.

Except for some simple situations, the problem of delay equalization is best approached by using a computer to optimize the group delay in some sense. Blinchikoff and Zverev[11] give an excellent discussion of a least-squares optimization as well as the basic principles of delay equalization.

TABLES

Many authors have published extensive tables of prototype element values for filters. Table 10.1.12 lists some of the readily available tables, with comments.

TABLE 10.1.12 Books Containing Element Values for Filters

First-named author	Reference number in references and bibliography	Length, pages	Comment
Biey	33	624	Cauer (elliptic and multicritical-pole equal-ripple rational (MCPER) functions
Biey	34	561	Active Cauer and MCPER functions
Christian	35	310	Nomographs, poles, and zeros for Butterworth, Chebyshev, inverted Chebyshev, and Cauer filters
Craig	36	197	Nomographs and tables for lossy Butterworth and Chebyshev filters
Genesio	37	598	Constants for digital representations of Butterworth and Chebyshev filters
Hansell	38	203	Passive filter parameters, attenuation and phase data
Johnson	39	244	60 pages of tables. Design formulas
Moschytz	21	316	Calculator and computer programs
Saal	31	662	Element value tables. Design formulas
Weinberg	7	692	70 pages of element value tables. Design formulas
Wetherhold	40	9 22	Tables using standard capacitor sizes for 50-Ω, 5- and 7-element Chebyshev, and for 5-element elliptic filters
Williams	41 42	540	116 pages of tables. Design techniques and formulas for active networks
Zverev	6	576	About 150 pages of tables for Butterworth, Chebyshev, Bessel, linear phase, Gaussian, Legendre, Cauer (elliptic) filters. Design formulas. Crystal filters