MECHANICS OF MATERIALS

MECHANICS OF MATERIALS

Irving J. Levinson

Professor of Mechanical Engineering
Lawrence Institute of Technology
Southfield, Michigan

PRENTICE-HALL OF INDIA (PRIVATE) LTD.

New Delhi, 1965

Rs. 12.00

PRENTICE-HALL OF INDIA (PRIVATE) LTD. New Delhi

Reprinted in India by special arrangement with Prentice-Hall, Inc., Englewood Cliffs, N.J., U.S.A.

Printed by G. D. Makhija at the India Offset Press, Delhi, and published by Prentice-Hall of India (Private) Ltd., New Delhi.

1074-E

To my daughter
JUDITH CAROL

PREFACE

After years of intensive study, painstaking research, and daily experience in an area of learning, an urge develops to update basic theories to contemporary situations and to benefit others with the judgements developed and with their practical applications to present-day problems. As a result, a book is written.

The purpose of this book is to present the useable and practical principles of mechanics of materials through basic mathematics. There are professions and areas of learning where intensive and advanced concentration in mathematics is not required in the preparatory curricula. Yet, there is a pressing and immediate need for an understanding of the realistic principles of mechanics of materials.

The author wishes to express his appreciation for the help, suggestions, and encouragement given by his many associates. He particularly wishes to thank Dr. Stephen Kalmar, Professor Wilson Daugherty, and Mr. Robert Garofalo for the many hours spent in reading the manuscript and for their extremely constructive suggestions.

Lastly, the author wishes to thank his wife, not only for typing the manuscript many times, but for retaining her composure through the innumerable birthdays and anniversaries which were forgotten during the preparation of this book.

I.J.L.

CONTENTS

6. *Deflection of Beams* 149

7. *Combined Loading* 187

8. *Welded, Bolted, and Riveted Connections* 233

9. *Columns* 257

10. *Experimental Stress Analysis* 277

MECHANICS OF MATERIALS

CHAPTER 1

Concept of Stress

Mechanics of materials is a science which relates the physical properties of matter to the geometric properties of form. It is a science born of mechanics and developed through intuition, reasoning, and experimentation. In its many applications it provides a "first step" in the design of bodies and structures which must resist deformation as they transmit force from one point to another. The purpose of this book is to offer an understanding of the vocabulary and the basic concepts of the subject of *mechanics of materials.*

1-1 Internal Reactions

The rigid body emphasized throughout the study of engineering mechanics provided a means to an end in the analysis of action and reaction of forces. The mere fact that such a body had never existed in nature in no way influenced computations, since only forces acting on the body and not within the body were involved. Consider, for example, two bars that each support two 100 lb weights, as shown in Fig. 1-1. By rigid-body analysis the two situations are equivalent; a 200 lb reactive force is required in both cases to balance the external load. From the point of view of deformation, however, the two bars are far from equal: every particle of matter in the first bar helps, equally, to support the total weight, and every particle of matter, therefore, deforms equally. In contrast, forces in the second bar vary with position, as the free-body diagrams indicate. The deformation in the two segments would, therefore, be different.

It will be apparent, as the theory is developed, that many of the concepts considered valid in rigid-body mechanics do not apply to deformable bodies since *forces and moments that act within such bodies depend upon the locations* *as well as the magnitudes of the external forces and moments.* This simply means that couples can not be transferred from place to place nor forces moved, at will, along their lines of action without changing the character of the deformation.

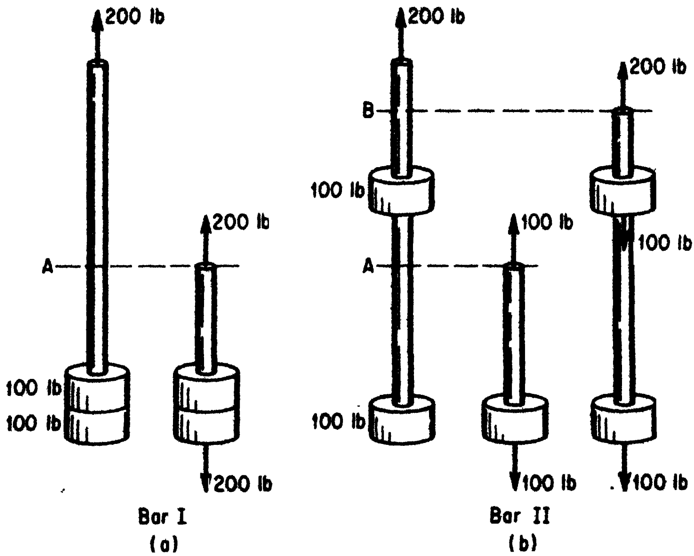


Fig. 1-1

Internal reactions can be classified into five distinct groups: tensile or compressive forces, shear forces, bearing forces, moments, and torques. Typical force systems that illustrate each type of internal reaction are shown in Fig. 1-2. Although the illustrations depict a single reaction in each instance, the possibility exists that these reactions can act in combination in response to more sophisticated external force systems. The solved examples that follow will serve as a guide to the method of finding internal reactions.

Example 1. The crank shown in Fig. 1-3(a) is rigidly fastened to a support at A. Determine the internal forces, moments, and torques that act within the member on a plane normal to the axis at point B.

Solution: A free-body diagram of a portion of the crank cut at B is drawn as shown in Fig. 1-3(b). Three reactive components are necessary to maintain equilibrium: the force F_y, a moment M_y, and a torque T_x. Numerically,

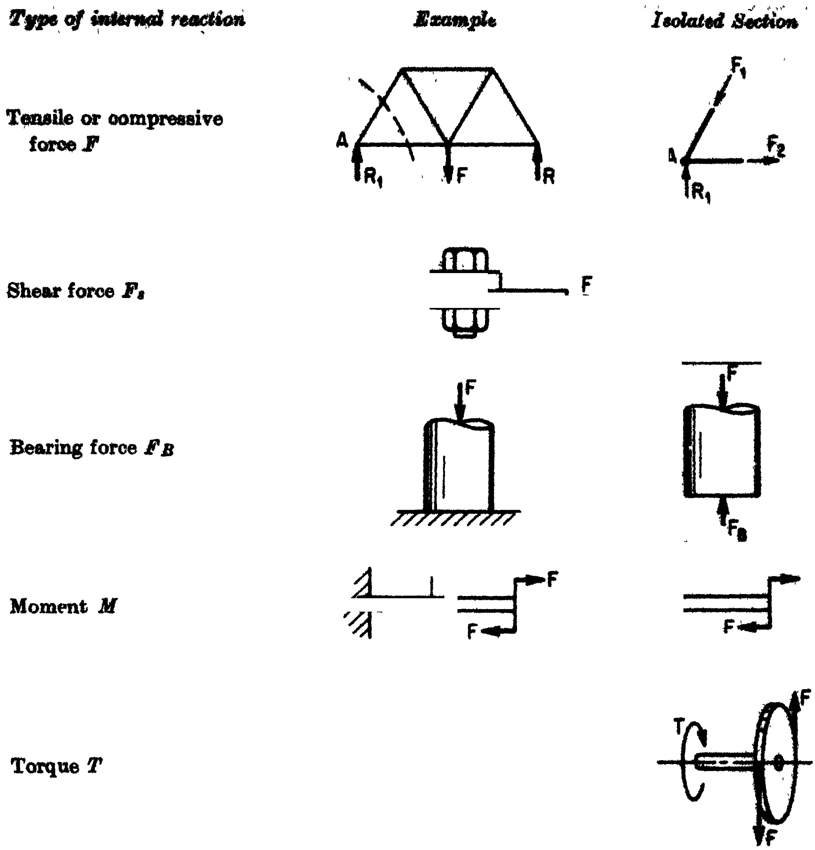

Type of internal reaction
Example
Isolated Section
Tensile or compressive force F
A
R_1
F
R
F_1
F_2
A
R_1
Shear force F_s
F
Bearing force F_B
F
F
F_B
Moment M
F
F
F
Torque T
T
F
F

Fig. 1-2

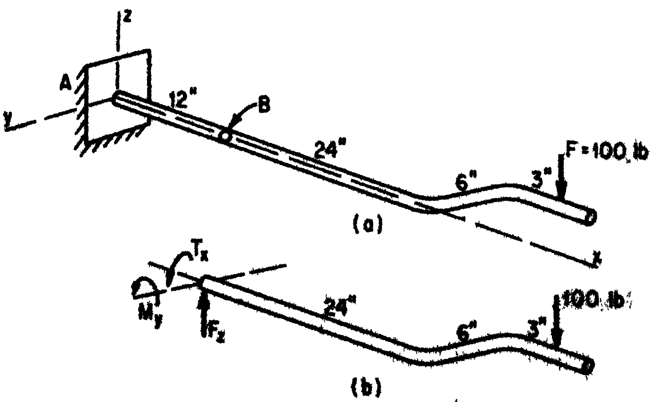

z
A
y
12"
B
24"
F = 100 lb
6"
3"
(a)
x
T_x
M_y
F_z
24"
100 lb
6"
3"
(b)

Fig. 1-3

these are equal to

$$\sum F_z = 0$$

$$F_z - 100 = 0$$

$$F_z = 100 \text{ lb}$$

$$\sum M_y = 0$$

$$M_y - 100(24 + 3) = 0$$

$$M_y = 2700 \text{ lb in.}$$

and

$$\sum M_x = 0$$

$$T_x - 100(6) = 0$$

$$T_x = 600 \text{ lb in.}$$

Example 2. The column of Fig. 1-4(a) supports a uniformly distributed horizontal load and a concentrated vertical load as shown. Determine the reactions within the column on a plane perpendicular to the axis at A.

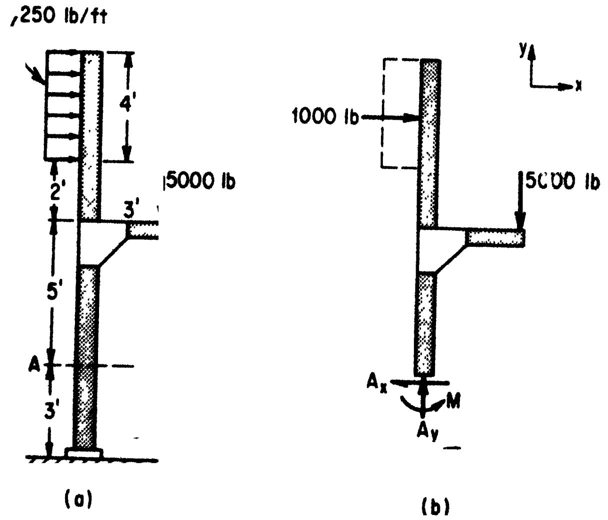


Fig. 1-4

Solution: Three reactions at A are exposed by the free-body diagram pictured in Fig. 1-4(b). These reactions consist of a shear force A_x, a compressive force A_y, and a moment M perpendicular to the x–y plane. The equations of static equilibrium are used to evaluate these reactions.

$$\sum F_x = 0$$

$$A_x = -1000 \text{ lb}$$

$$\sum F_y = 0$$

$$A_y = 5000 \text{ lb}$$

$$\sum M_{xy} = 0$$

$$M - 250(4)9 - 5000(3) = 0$$

$$M = 24{,}000 \text{ lb in.}$$

Example 3. The post pictured in Fig. 1-5(a) supports a compressive load of $P = 5$ kips. Find the internal force components that act normal and tangent to plane $ABCD$, which cuts the post at an angle of 30 deg with the vertical.

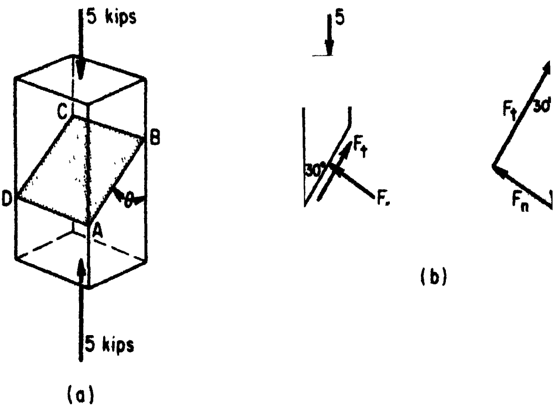


Fig. 1-5

Solution: The free-body diagram and the accompanying closed polygon of forces are shown in Fig. 1-5(b). Solving the vector diagram for the desired forces gives

$$F_n = 5 \sin 30° = 5(0.5)$$

$$F_n = 2.5 \text{ kips}$$

and

$$F_t = 5 \cos 30° = 5(0.866)$$

$$F_t = 4.33 \text{ kips}$$

1-2 Stress

Stress, like pressure, is a term used in mechanics of materials to describe the *intensity of a force*—the quantity of force that acts on a unit of area. To say, for instance, that a particular piece of steel can withstand a tensile stress of 60,000 psi simply means that every square inch of cross-sectional area can support 60,000 lb in tension. The inference in the preceding statement is that every portion of the area supports an equal share of the load; the stress, in other words, is *uniform.*

If, for some reason, the stress varies from point to point over a given area within a body, the stress is referred to as *non-uniform stress.* It will be shown, as the theory is developed, that there are many more cases of non-uniform stress than of uniform stress.

1-3 *Axial, Shear, and Bearing Stresses*

Stress can be classified in accord with the internal reaction that produces it. An axial tensile or compressive force, as shown in Fig. 1-6(a), produces a tensile or compressive stress. This type of stress can be thought of as being caused by a longitudinal "push" or "pull." Mathematically, the average value

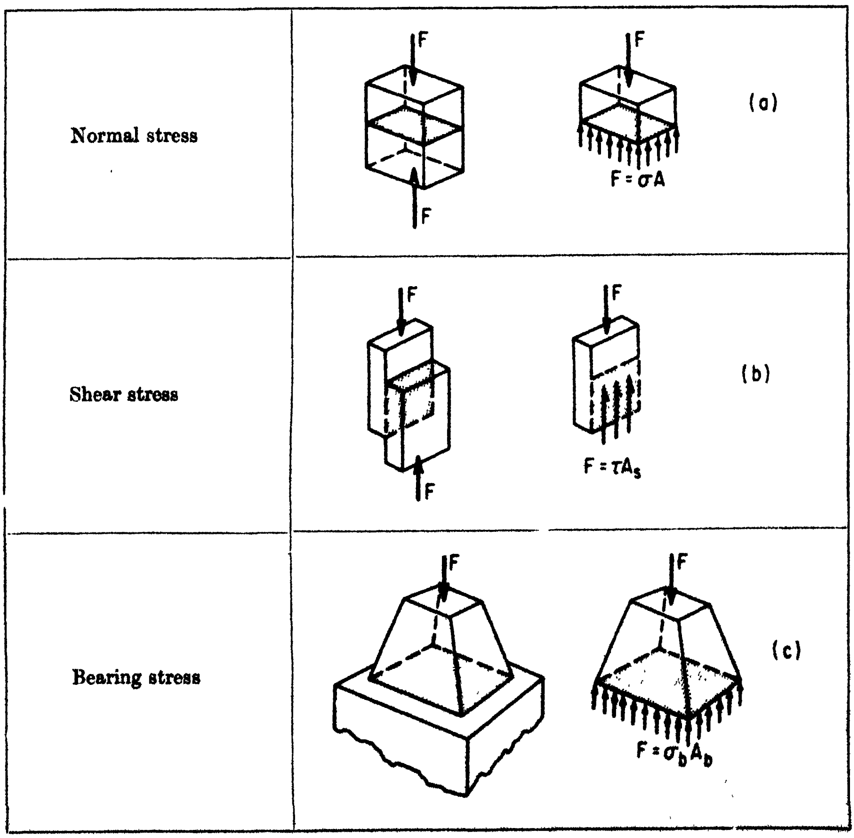


Fig. 1-6. Simple uniform stress

of the axial stress, represented by the Greek letter σ, is the ratio of the force to the area:

$$\sigma = \frac{F}{A} \tag{1-1}$$

Shear stress, the second type of stress illustrated, is caused by a force that acts at right angles to the axis of the member. This form of stress acts parallel to the cross-sectional area and has an average value of

$$\frac{F}{A_s} \tag{1-2}$$

where the Greek letter τ represents shear stress, F the force causing shear, and A_s the area being sheared.

The third fundamental type of stress, the bearing stress, is actually a pressure, since it represents the intensity of force between a body and its support. Like the previous two stresses, the average bearing stress is defined in terms of force and area:

$$\sigma_b = \frac{F}{A_b} \tag{1-3}$$

Example 4. The clevis shown in Fig. 1-7(a) supports a load $P = 5$ tons. Determine: (a) the tensile stress in the circular section at A; (b) the tensile stress in the rectangular section at B; (c) the shear stress in the bolt.

Solution:

Parts (a) and (b). The tensile stress in both the circular section and the rectangular section is given by Eq. (1-1).

For the circular section:

$$\sigma = \frac{P}{-} = \frac{5(2000)}{\frac{\pi}{4}\left(\frac{3}{4}\right)^2} = 22{,}600 \text{ psi tension}$$

For the rectangular section:

$$\sigma = \frac{P}{A} = \frac{5(2000)}{0.5(2)} = 10{,}000 \text{ psi tension}$$

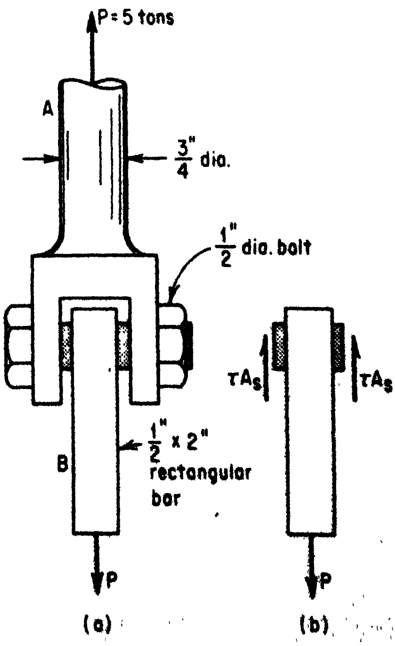


Fig. 1-7

Part (c). The bolt is in *double shear*; two transverse areas, one to the right and the other to the left of the rectangular bar, help support the load. This is illustrated in Fig. 1-7(b). The shear stress, found through Eq. (1-2), is

$$\tau = \frac{P}{A_s} = \frac{5(2000)}{2\left(\frac{\pi}{4}\right)\left(\frac{1}{2}\right)^2} = 25{,}500 \text{ psi}$$

Example 5. Angle clips welded to the column pictured in Fig. 1-8(a) support the loads indicated. The column, an 8 in. wide-flanged section weighing 35 lb per ft (8 WF 35), has a cross-sectional area of 10.3 sq in. Determine the axial stress at sections A, B, and C, and the bearing stress between the plate and the pedestal. Neglect, in each instance, the weight of the column.

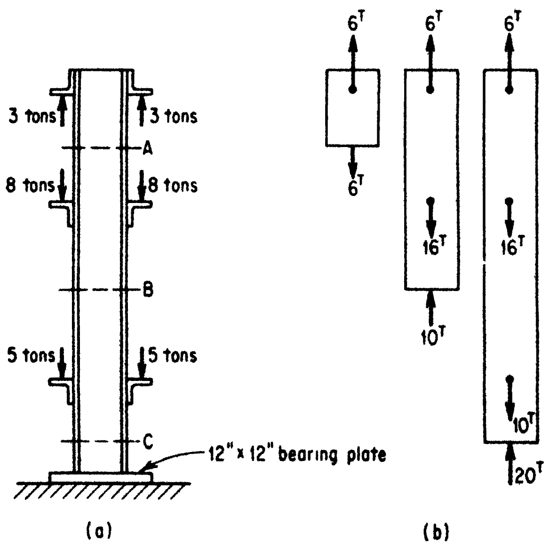


Fig. 1-8

Solution: Three free-body diagrams, each showing the force required to maintain internal equilibrium, are drawn as illustrated in Fig. 1-8(b). Eq. (1-1) is used to compute the stress at each section.

At A: $\sigma_A = \dfrac{P}{A} = \dfrac{6(2000)}{10.3} = 1170$ psi tension

At B: $\sigma_B = \dfrac{P}{A} = \dfrac{10(2000)}{10.3} = 1940$ psi compression

At C: $\sigma_C = \dfrac{P}{A} = \dfrac{20(2000)}{10.3} = 3880$ psi compression

The bearing stress, equivalent to the pressure between the bearing plate and the pedestal, is

$$\sigma_b = \frac{P}{A} = \frac{20(2000)}{12 \times 12} = 278 \text{ psi}$$

1-4 Stresses on Oblique Planes

Situations often arise in design when consideration must be given to stresses that act within a body on planes other than transverse planes. Consider a bar of cross-sectional area A, Fig. 1-9(a), to be acted upon by a tensile

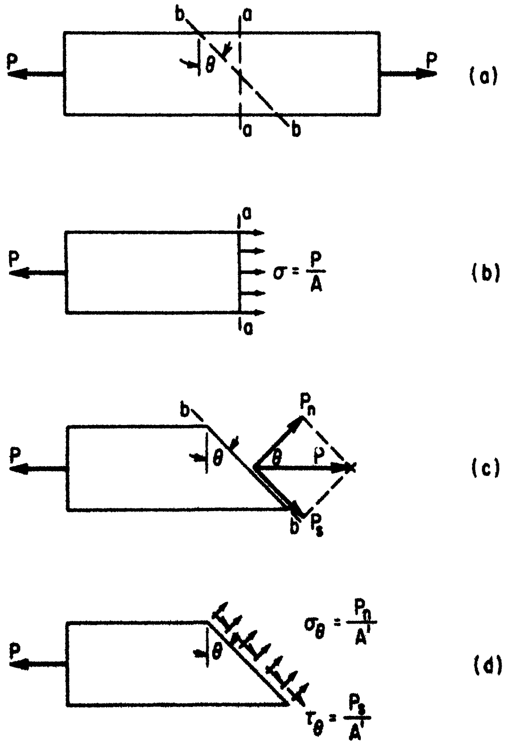


Fig. 1-9

force P. The transverse stress, that on plane a–a, is equal to P/A. By selecting an oblique plane b–b inclined as shown in Fig. 1-9(c), a free-body can be drawn with the force P resolved into a component P_n normal to the plane and a component P_s parallel to the plane. Regardless of the inclination of the oblique plane, the vector sum of P_n and P_s must always be equal to the force P. These components, in terms of θ and P are

$$P_n = P \cos \theta$$

and

$$P_s = P \sin \theta$$

The stresses associated with these components, Fig. 1-9(d), can be found by dividing each by the magnitude of the oblique area A', which in terms of the transverse area A is

$$A' = \frac{A}{\cos \theta}$$

The resulting normal stress and shear stress are, respectively,

$$\sigma_\theta = \frac{P_n}{A'} = \frac{P \cos \theta}{\dfrac{A}{\cos \theta}} = \frac{P}{A} \cos^2 \theta \tag{1-4}$$

and

$$\tau_\theta = \frac{P_s}{A'} = \frac{P \sin \theta}{\dfrac{A}{\cos \theta}} = \frac{P}{A} \sin \theta \cos \theta$$

Since

$$\sin \theta \cos \theta = \frac{\sin 2\theta}{2}$$

the shear stress can be written

$$\tau_\theta = \frac{P}{2A} \sin 2\theta \tag{1-5}$$

By inspection, the normal stress is a maximum on the transverse plane, $\theta = 0$, and a minimum on an axial plane, $\theta = 90$ deg.

$$\sigma_{\theta_{max}} = \frac{P}{A} \cos 0° = \frac{P}{A}$$

and

$$\sigma_{\theta_{min}} = \frac{P}{A} \cos 90° = 0$$

Similarly, the shear stress is a maximum at $2\theta = \pm 90$ deg. or $\theta = \pm 45$ deg, and a minimum at $2\theta = 0$ deg.

$$\tau_{\theta_{max}} = \frac{P}{2A} \sin (2 \times 45°) = \frac{P}{2A}$$

and

$$\tau_{\theta_{min}} = \frac{P}{2A} \sin (2 \times 0°) = 0$$

Example 6. The rectangular bar of Fig. 1-10 is made of two pieces of steel solidly welded along a plane inclined at 30 deg. as shown. Determine: (a) the maximum safe load P if the normal stress and the shearing stress are not to exceed 8000 psi and 4000 psi respectively; (b) the ratio of the design load to the tensile strength of the bar based on a working stress of 25,000 psi in tension.

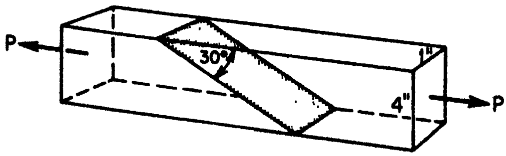


Fig. 1-10

Solution:

Part (a). Two values of the load P must be computed: one based on the allowable normal stress and the other on the allowable shearing stress. The smaller of these two values will be the design load P.
From Eq. (1-4)

$$= \frac{P}{A}\cos^2\theta$$

Solving for P gives

$$P = \frac{\sigma_\theta A}{\cos^2\theta} = \frac{8000(4 \times 1)}{(\cos 60°)^2} = 128{,}000 \text{ lb}$$

Note: The angle θ in Eqs. (1-4) and (1-5) lies between a perpendicular drawn to the inclined surface and the axis of the bar.

Next, a value of the allowable load P' is computed on the basis of the maximum allowable shearing stress of 4000 psi; thus

$$\tau_\theta = \frac{P'}{2A}\sin 2\theta$$

$$P' = \frac{2\tau_\theta A}{\sin 2\theta}$$

where

$$P' = \frac{2\tau_\theta A}{\sin 2\theta} = \frac{2(4000)\,(4 \times 1)}{\sin 120°} = 37{,}000 \text{ lb}$$

Therefore the design load, the lesser value of P, is 37,000 lb.

Part (b). The tensile strength of the bar, based on a working stress of 25,000 psi is

$$P_{ts} = \sigma A = 25{,}000(4 \times 1) = 100{,}000 \text{ lb}$$

and the ratio of the allowable load to the strength of the bar is, therefore,

$$\frac{P'}{P_{ts}} = \frac{37{,}000}{100{,}000} = 0.37$$

This numerical value is called the *efficiency of the joint.*

1-5 Geometric Stress Concentration

An abrupt change in the geometry of a structural member, such as that caused by a hole, a notch, or a groove, results in a non-uniform stress pattern, as illustrated in Fig. 1-11. The maximum stress in each instance occurs at the boundary of the geometric discontinuity, and failure, particularly under dynamic loading, begins at these points of high localized stress.

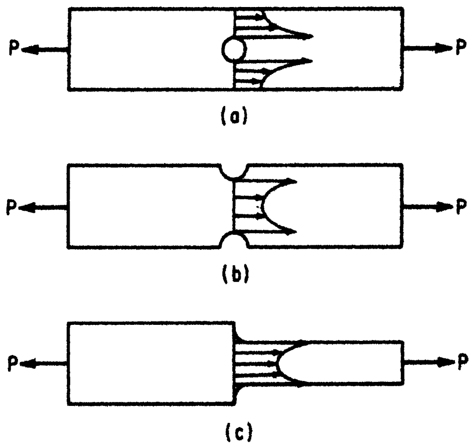


Fig. 1-11

The problem of geometric stress concentration has attracted both the experimenter and the theorizer; their labors have produced data that enable one to predict with fair precision the maximum stress that will occur under a given condition. Data are usually presented in the form of curves that cover a large range of variables for a given type of geometrical discontinuity. Five typical curves[1] for members in tension are shown in Figs. 1-12, 1-13, 1-14, 1-15, and 1-16. In each a stress concentration factor k is given in terms of the critical dimensions of the member. The actual stress, then, in terms

[1] For a complete discussion of stress concentration, see Charles Lipson, G. C. Noll, and L. S. Clock, *Stress and Strength of Manufactured Parts* (New York: McGraw-Hill Book Company, Inc., 1950).

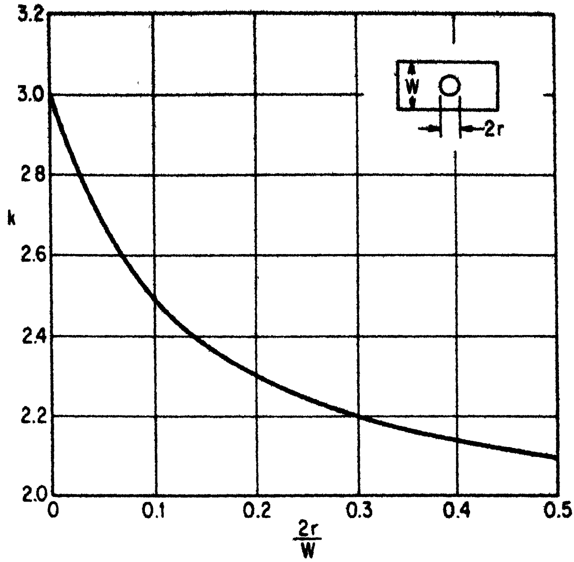


Fig. 1-12

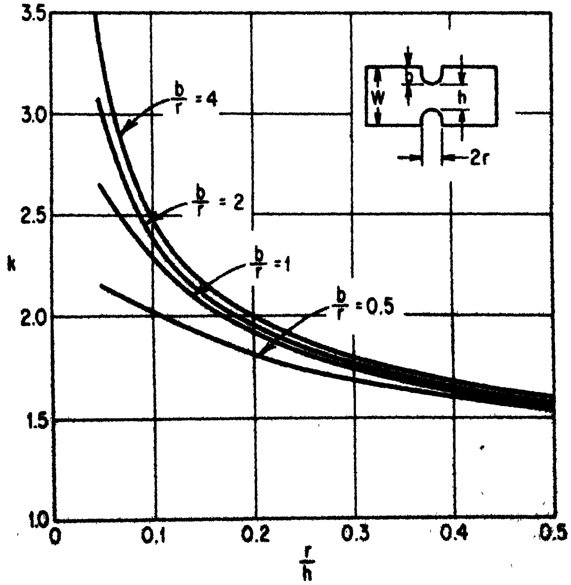


Fig. 1-13

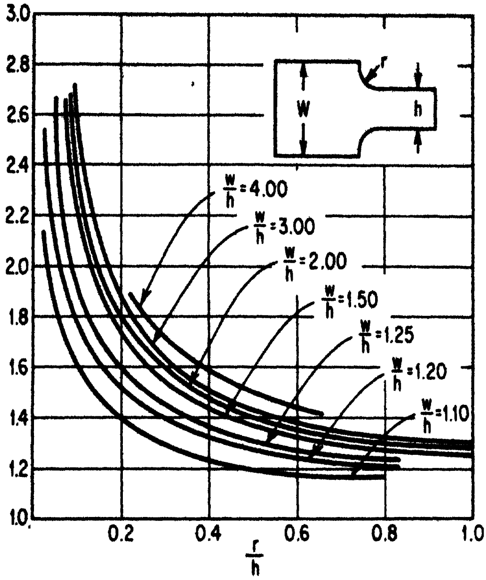


Fig. 1-14

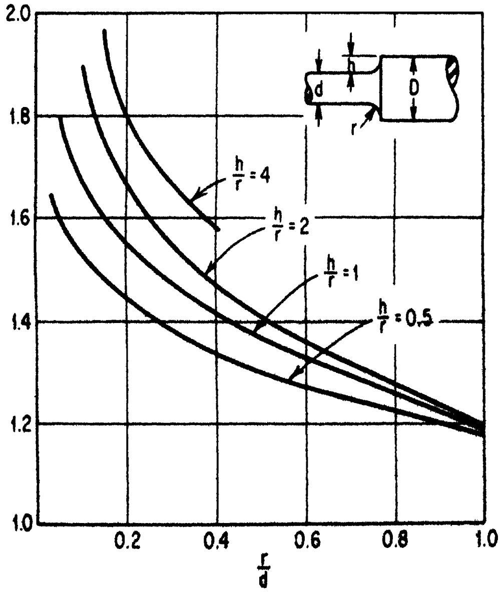


Fig. 1-15

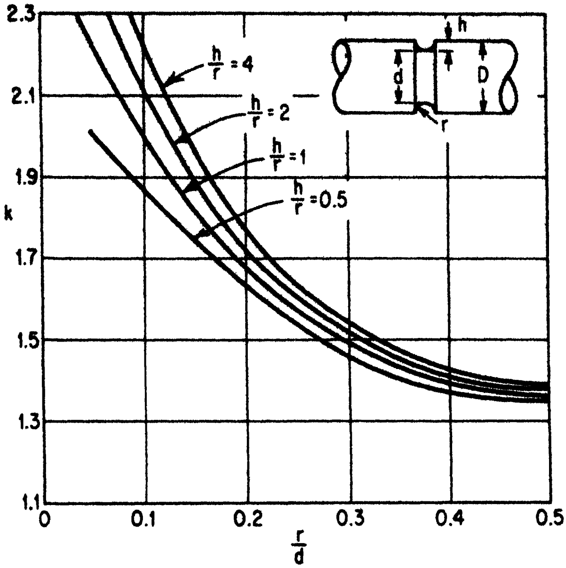


Fig. 1-16

of k and the average stress P/A is

$$\sigma_{max} = k\sigma_{avg} = k\frac{P}{A} \tag{1-6}$$

Example 7. Determine the safe axial load P for the rectangular bar shown in Fig. 1-17 if the tensile stress in the bar is not to exceed 15,000 psi.

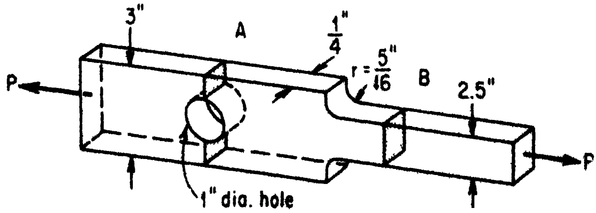


Fig. 1-17

Solution: Two regions of geometric stress concentration are present in the bar; one near the hole at A and the other near the fillet at B. An allowable load must be determined for each section and the least value selected as the design load.

Section A. The average stress on a transverse plane through the hole is

$$\sigma_{avg} = \frac{P}{A} = \frac{P}{(3-1)\frac{1}{4}} = 2P \text{ psi}$$

The stress concentration factor k from Fig. 1-12 is dependent upon the ratio of the diameter of the hole to the width of the plate.

$$\frac{2r}{W} = \frac{1}{3} = 0.333$$

The data—the given allowable stress, the computed average stress, and the approximate value of k from the graph—are substituted into Eq. (1-6).

$$\sigma_{max} = k\sigma_{avg}$$

$$15{,}000 = 2.18(2P)$$

Solving for P gives

$$P = \frac{15{,}000}{2(2.18)} = 3440 \text{ lb}$$

Section B. The average stress at B, the region of the fillet, is

$$\sigma_{avg} = \frac{P}{A} = \frac{P}{2.5(\frac{1}{4})} = 1.6P$$

The ratios r/h and W/h are needed to determine k from the graph of Fig. 1-14.

$$\frac{r}{h} = \frac{\frac{5}{16}}{2.5} = 0.125$$

and

$$\frac{W}{h} = \frac{3}{2.5} = 1.2$$

The approximate value of k, read from the graph, is 1.7.
Substitution of these data into Eq. (1-6) gives a second critical value of P; thus

$$\sigma_{max} = k\sigma_{avg}$$

$$15{,}000 = 1.7(1.6P)$$

where

$$P = \frac{15{,}000}{1.7(1.6)} = 5510 \text{ lb}$$

The least of the two values of P is the safe load; thus, $P = 3440$ lb, where the critical section is at the hole.

1-6 *Stresses in Thin-Walled Cylindrical Vessels*

As is often the case in the study of mechanics of materials, certain simplifying assumptions are made that enable one to arrive at a reasonably correct answer to what might be an exceedingly complex problem. Such is the case with thin-walled cylinders—cylinders whose wall thicknesses are less than $\frac{1}{20}$ of their diameters. The simplifying assumption in this instance is that the stress in the walls of the cylinder is uniform. Consider the closed cylindrical vessel subjected to an internal pressure p, as shown in Fig.

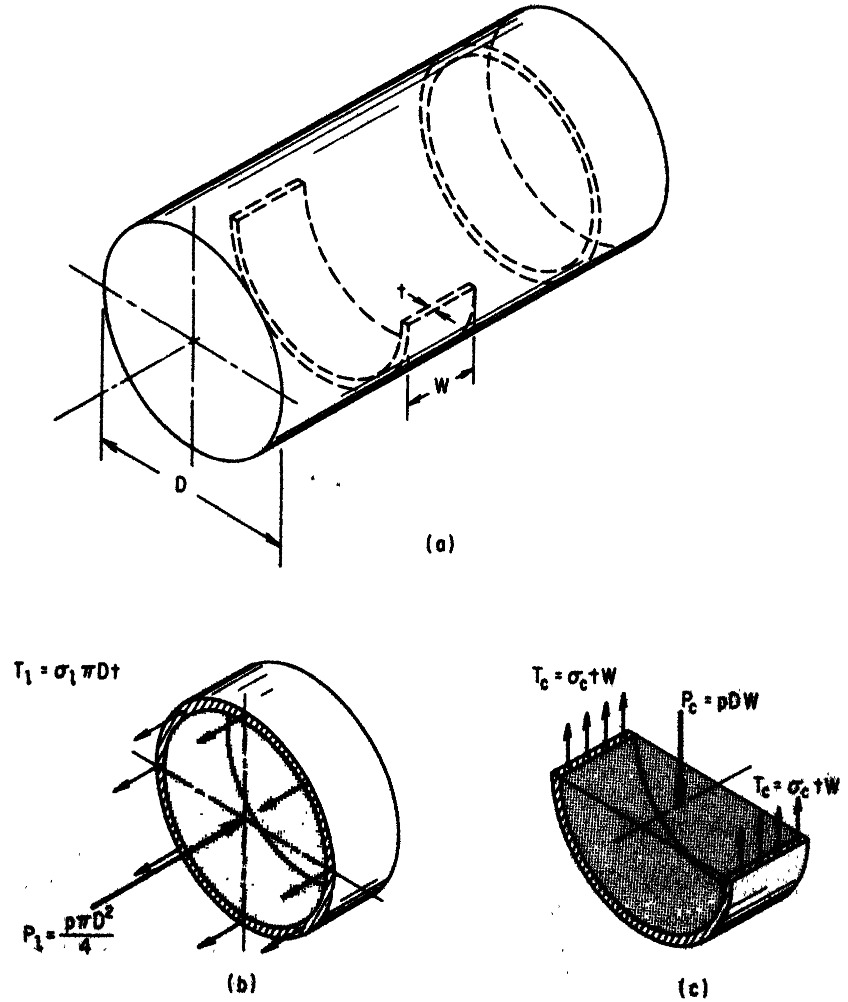


Fig. 1-18

1-18(a). A free-body diagram of a portion of the cylinder cut about a circumferential line, Fig. 1-18(b) indicates an external pressure force P_l to act on the closed end and an internal axial tensile force T_l to act within the wall. To maintain equilibrium in the longitudinal direction, the tensile force and the pressure force must be equal.

$$T_l = P_l = p\frac{\pi D^2}{4}$$

The tensile force in terms of the average longitudinal stress across severed circumferential area is

$$T_l = \sigma_l A = \sigma_l \pi D t$$

Equating Eqs. (a) and (b) gives

$$\sigma_l \pi D t = p\frac{\pi D^2}{4}$$

or

$$\sigma_l = p\frac{D}{4t} \tag{1-7}$$

To arrive at an expression for the circumferential stress, a second free-body is sketched as shown in Fig. 1-18(c). This is drawn to include the portion of the pressurized gas or liquid contained by this segment of the cylinder. The forces in the vertical direction include a pressure force P_c equal to the product of the internal pressure p and the area WD, and tensile forces on each of the two severed areas. To maintain equilibrium, then,

$$2T_c = P_c = pWD \tag{c}$$

The tensile force T_c in terms of the circumferential stress on the severed area is

$$T_c = \sigma_c A = \sigma_c W t \tag{d}$$

Combining Eqs. (c) and (d) gives

$$2\sigma_c W t = pWD$$

or

$$\sigma_c = \frac{pD}{2t} \tag{1-8}$$

Since the circumferential stress is twice the longitudinal stress, the cylinder would fail, burst, or explode, by splitting longitudinally.

An expression for the stress in a spherical shell subjected to an internal pressure p can be found by an approach similar to that used for the cylinder. A hemispherical free-body which includes the pressurized liquid or gas is drawn as shown in Fig. 1-19. To maintain equilibrium, the tensile force in the wall must just equal the pressure force required to maintain the matter within the hemisphere.

$$T = p\frac{\pi D^2}{4} \tag{e}$$

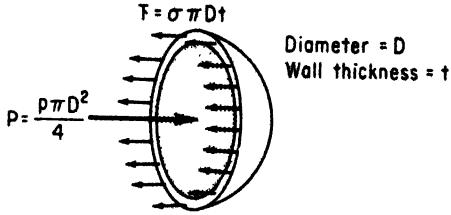


Fig. 1-19

The tensile force, in terms of the product of the stress and the severed area, is substituted in Eq. (e):

$$\sigma \pi D t = \frac{p\pi D^2}{4}$$

Simplifying gives

$$\sigma = \frac{pD}{4t} \tag{1-9}$$

Example 8. A tank, 16 ft in diameter and 24 ft high, is filled to the top with a liquid weighing 160 lb per cu ft. If the tank is fabricated from three 8-ft-wide steel rings, as shown in Fig. 1-20(a), determine the proper thickness of each ring for the most economical construction. Assume that the tensile stress in the steel is not to exceed 5000 psi.

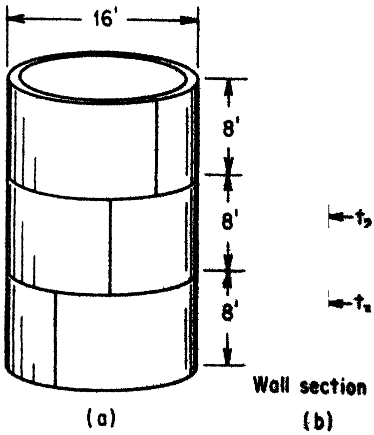


Fig. 1-20

Solution: The pressure within the tank varies directly with the depth h.

$$p = \gamma h$$

where γ is the specific weight of the fluid:

$$\gamma = 160\frac{\text{lb}}{\text{ft}^3} \times \frac{1\ \text{ft}^3}{1728\ \text{in.}^3} \quad 0.0926\ \text{lb/in.}^3$$

Thus, the pressures at the 8 ft, 16 ft, and 24 ft marks are

$$p_8 = 0.0926(8 \times 12) = 9.24\ \text{psi}$$

$$p_{16} = 0.0926(16 \times 12) = 18.5 \text{ psi}$$

$$p_{24} = 0.0926(24 \times 12) = 27.7 \text{ psi}$$

The required thicknesses from Eq. (1-8) are

$$t_8 = \frac{pD}{2\sigma_\theta} = \frac{9.24(8 \times 12)}{2(5000)} \quad 0.177 \text{ in.}$$

$$t_{16} = \frac{18.5(16 \times 12)}{2(5000)} = 0.355 \text{ in.}$$

$$t_{24} = \frac{27.7(24 \times 12)}{2(5000)} = 0.532 \text{ in}$$

1-7 Working Stress and Factor of Safety

In several of the problems that have served as examples the terms *working stress* and *allowable stress* have been used. These are values of stress that provide a margin of safety in the design. The need for this safety margin is apparent for many reasons: stress, itself, is seldom uniform; materials lack the homogeneous properties theoretically assigned to them; abnormal loads might occur; manufacturing processes often impart dangerous stresses within the component. These and other factors make it necessary to select working stresses substantially below those known to cause failure.

The phrase *factor of safety* is a term generally defined as the ratio of the stress necessary to produce failure to the working stress. Under this definition, a factor of safety of 3 would mean that the load could be increased 3 times before failure would occur.

PROBLEMS

1-1. A force $F = 100$ lb is applied to the hook as shown in Fig. P1-1. Determine the internal reactions at a section normal to the axis at point A.

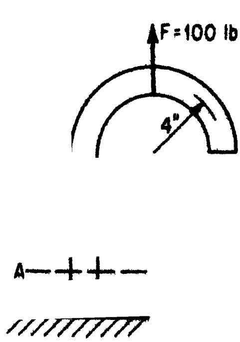


Fig. P1-1

1-2. Determine the internal reactions on a plane normal to the axis of the lamp post shown in Fig. P1-2. The arm and lamp have a combined weight of 1000 lb and a center of gravity at *G*, and the column weighs 120 lb per ft.

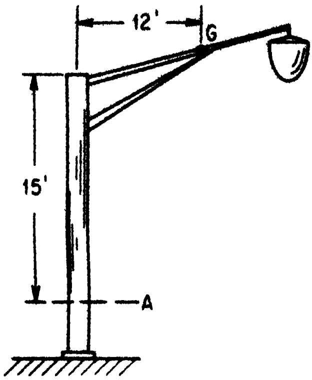


Fig. P1-2

1-3. The reaction at *A* to the wedge driven into the split ring shown in Fig. P1-3 is a moment of 1800 lb in. and a normal force. Determine the magnitude of this normal force.

1-4. Compute the reaction of the support at *A* on the cantilever beam illustrated in Fig. P1-4.

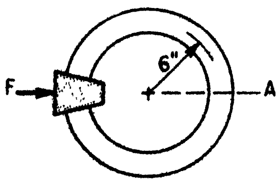


Fig. P1-3

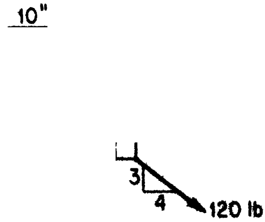


Fig. P1-4

1-5. The simply supported beam is subjected to a concentrated load and a distributed load, as shown in Fig. P1-5. Determine the reactions within the beam at points *B* and *C*.

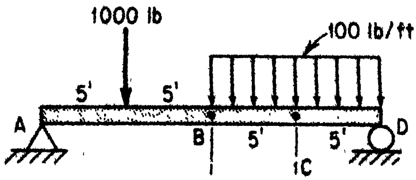


Fig. P1-5

1-6. Solve Prob. 1-5 with the distributed load assumed to be concentrated at point C, and account for any difference in the answers to the two problems.

1-7. The beam illustrated in Fig. P1-7 is subjected to two forces. Find the internal reactions at section B.

1-8. Write an equation for the moment within the beam, Fig. P1-8, as a function of x.

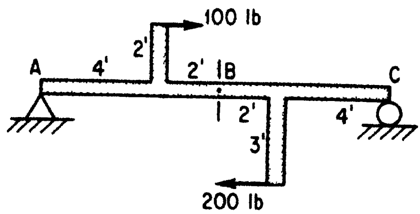


Fig. P1-7

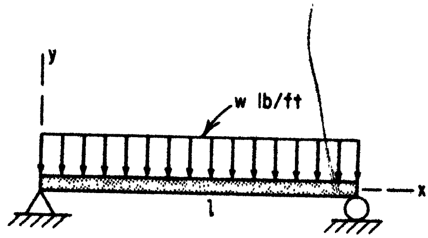


Fig. P1-8

1-9. A concrete beam weighing 150 lb per lin ft is hoisted by the cable arrangement shown in Fig. P1-9. Determine the internal reactions that act on a plane perpendicular to the axis of the beam at its midpoint.

1-10. Solve Prob. 1-9, assuming that a single cable at the midpoint is used to hoist the beam.

1-11. One half of a laminated roof truss is illustrated in Fig. P1-11. If the truss is subjected to the external forces indicated, determine the internal reactions at section C.

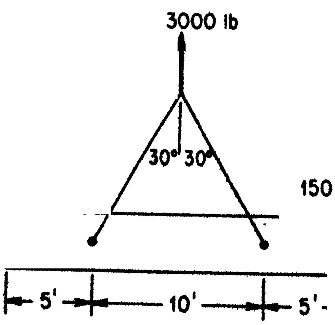


Fig. P1-9

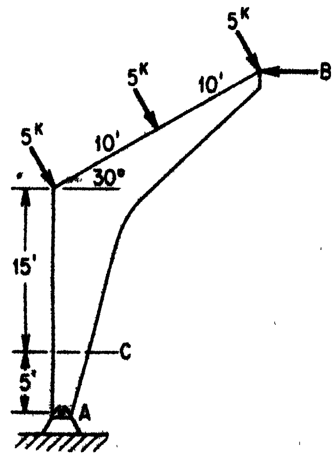


Fig. P1-11

1-12. The 90 deg outrigger arm shown in Fig. P1-12 supports a 500 lb load. Determine the internal reactions within the arm at section A.

1-13. The torque arm supports two loads, as illustrated in Fig. P1-13. Find the internal reactions at a section normal to the axis of the bar at A.

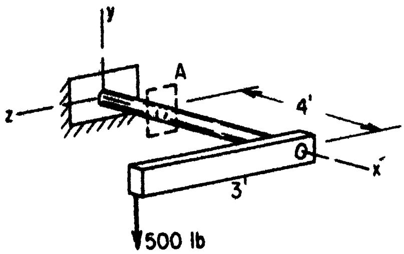


Fig. P1-12

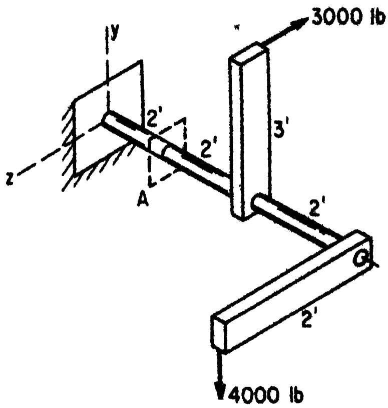


Fig. P1-13

1-14. The steel column of Fig. P1-14 weighs 150 lb per ft and carries a load $W = 2$ kips. Write an equation for the tensile force F in the column as a function of y and sketch the curve of this function. Let the ordinate represent F and the abscissa y.

1-15. The traveling crane supports a load W as shown in Fig. P1-15. Write an equation for the moment M and the shear force V in the beam as a function of the distance x.

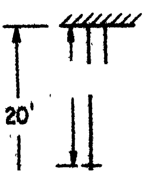


Fig. P1-14

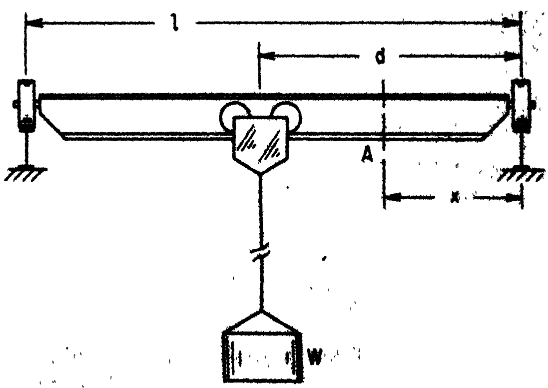


Fig. P1-15

1-16. The jack-shaft of Fig. P1-16 is subjected to the belt tensions shown. Determine the components of the reaction within the shaft at a section normal to the axis at C.

1-17. The circular bar illustrated in Fig. P1-17 is subjected to an axial tension force as shown. Determine the components of force normal and tangent to a plane inclined at 45 deg with the axis of the bar.

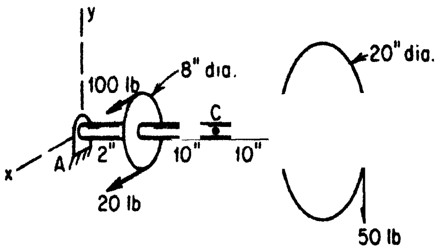


Fig. P1-16

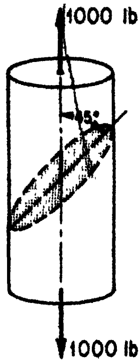


Fig. P1-17

1-18. Determine the components of force, normal and tangential, that act on the plane inclined at 30 deg to the axis of the tie-bar AB, as shown in Fig. P1-18.

1-19. The plate illustrated in Fig. P1-19 is subjected to the forces shown. Determine the normal and tangential components of force that act on the diagonal plane.

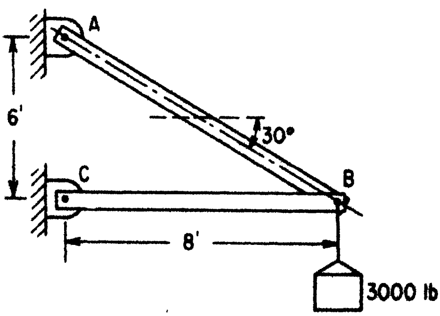


Fig. P1-18

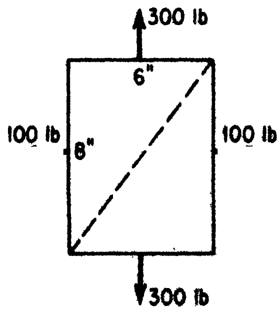


Fig. P1-19

1-20. A uniform plate 1 in. thick is subjected to the edge pressure shown in Fig. P1-20. Find the normal and tangential components of force that act on a diagonal plane.

1-21. A plate 4 in. thick is subjected to a concentrated force on its vertical edges and a uniformly distributed load on its horizontal edges, as illustrated in Fig. P1-21. Determine the normal and tangential components of force that act on a diagonal plane.

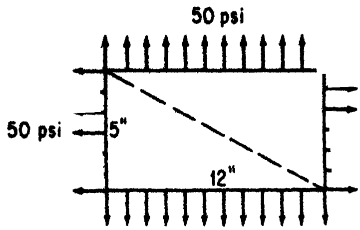


Fig. P1-20

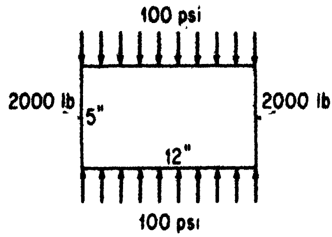


Fig. P1-21

1-22. The component of force F_t tangent to a plane inclined at an angle θ, as shown in Fig. P1-22, is 100 lb up and to the right when $\theta = 45°$, and 200 lb up and to the right when $\theta = 60$ deg. Find the forces F_x and F_y.

1-23. Determine the normal stress in a 2-in.-diameter cylindrical bar that supports an axial tensile force of 15 tons.

1-24. The truss illustrated in Fig. P1-24 supports a load of 25 tons. Determine the required cross-sectional areas of members *AB*, *BD*, and *AD*, if the stresses in tension and compression are not to exceed 20,000 psi and 15,000 psi respectively.

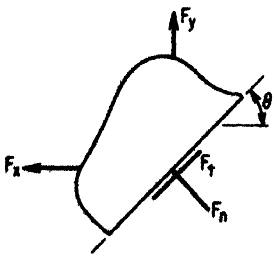


Fig. P1-22

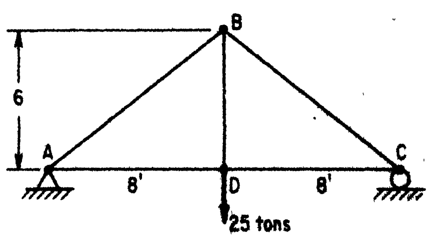


Fig. P1-24

1-25. Determine the required cross-sectional areas of members *AB*, *BC*, *CD*, and *BD* in Fig. P1-24, if the force of Prob. 1-24 acts horizontally to the right at pin *B*.

1-26. Determine the force *F* in Fig. P1-26 required to maintain equilibrium and the stresses in sections *AB*, *BC*, and *CD*. The bar has a uniform cross-sectional area of 0.5 in.^{2}

1-27. Determine the maximum force that must be exerted on a 1-in.-diameter punch that is used to shear a hole in a $\frac{3}{16}$-in.-thick steel plate, if the steel is known to fail in shear at a stress of 38,000 psi. What is the compressive stress in the punch when this force is applied?

1-28. A tensile stress of 5000 psi is developed in the straight shank of a $\frac{1}{2}$-in.-diameter bolt that is used to secure the two pieces of wood of Fig. P1-28. Determine the areas of the washers at A and B if the allowable bearing stress in the wood is 75 psi.

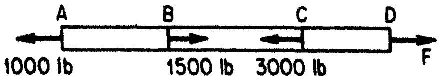


Fig. P1-26

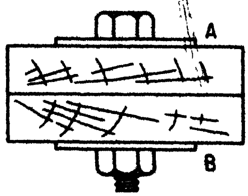


Fig. P1-28

1-29. The flanged pivot of Fig. P1-29 sustains a compressive stress of 5000 psi when it is acted upon by the force F. Determine the shearing stress between the flange and the cylinder, and the bearing stress between the flange and the support.

1-30. An adjustable ratchet bar used in machine setups is shown in Fig. P1-30. Find the shearing stress and the bearing stress developed in the teeth. Assume the teeth in contact to share the load equally.

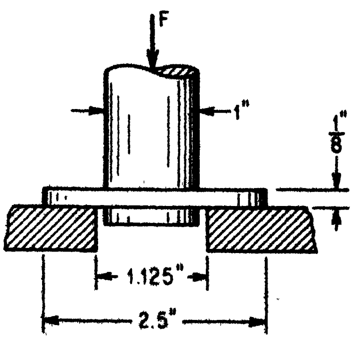


Fig. P1-29

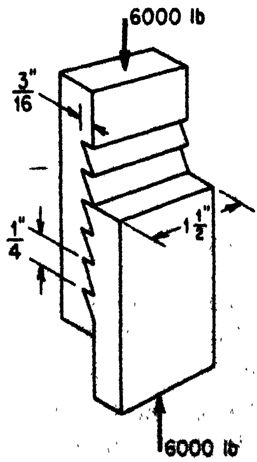


Fig. P1-30

1-31. Solve Prob. 1-30 if the rt one-six total load.

1-32. Determine the maximum torque that can be transmitted to the gear through the $2\frac{7}{16}$ in. keyed shaft shown in Fig. P1-32. The $\frac{3}{4}$ in. square key is $2\frac{1}{4}$ in. long and has a limiting shear strength of 10,000 psi.

1-33. A short column is made by welding an 8 WF 31 beam to two 10 WF 45 beams, as illustrated in Fig. P1-33. If the maximum allowable compressive stress in the beams is 10,000 psi, determine (a) the maximum value of P_1 if $P_2 = 100$ kips; (b) the maximum value of P_1 if $P_2 = 20$ kips. See Appendix B for the properties of these beams.

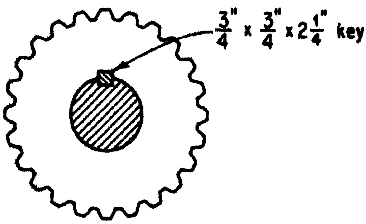


Fig. P1-32

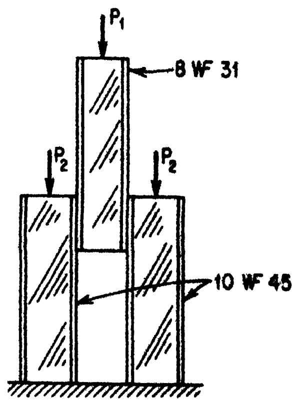


Fig. P1-33

1-34. Two materials are being considered for a machine component, a circular tension member 15 in. long that must sustain a axial tensile load of 10,000 lb. One material is a "super-strength" titanium alloy and the other is plain carbon steel. The working stress for the former is 100,000 psi and for the latter 30,000 psi. Compare the weights of the two members if the titanium alloy and the steel have specific weights of 0.165 lb/cu in. and 0.285 lb/cu in. respectively.

1-35. Two $\frac{1}{4}$-in.-diameter dowels in the upright *A* and two $\frac{1}{8}$-in. diameter dowels in the upright *B* support the horizontal beam shown in Fig. P1-35. Determine the maximum load *P* that the assembly will support if the shearing stress in the dowels is not to exceed 15,000 psi.

1-36. The column shown in Fig. P1-36 is fabricated by inserting one tube into another for a distance *d* and then brazing the two to make a homogeneous joint. The maximum allowable shear stress in the joint is 1000 psi, and the maximum compressive stress in each tube is 5000 psi. Find the inside diameter of the smaller tube, the outside diameter of the larger tube, and the distance d.

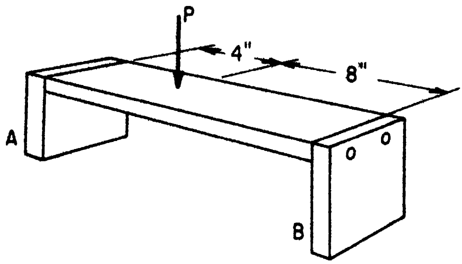


Fig. P1-35

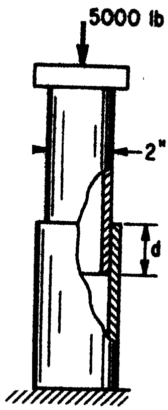


Fig. P1-36

1-37. Axial stresses in members A and B in Fig. P1-37 are 1200 psi compression and 5000 psi tension respectively. Determine the load P and the shearing stress in the pin at C.

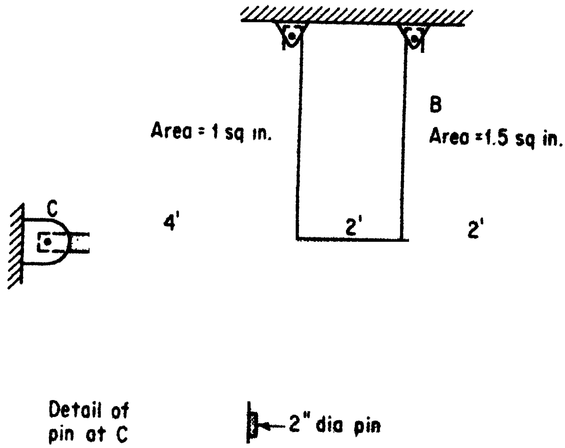


Fig. P1-37

1-38. The uniform weight $W = 10{,}000$ lb is supported by three tension members as shown in Fig. P1-38. The stress in C is twice the stress in A and one-half the stress in B; member B carries 40 per cent of the load. Find the cross-sectional area of each of the members if the maximum stress that any one member can withstand is 5000 psi.

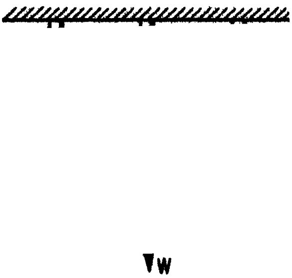


Fig. P1-38

1-39. Select the most economical (lightest) wide-flanged beams *A* and *B* that will serve as short columns for the structure shown in Fig. P1-39 if the compressive stress in the column is not to exceed 12,000 psi. Assume the horizontal member to be rigid (see Appendix B for data on wide-flanged sections).

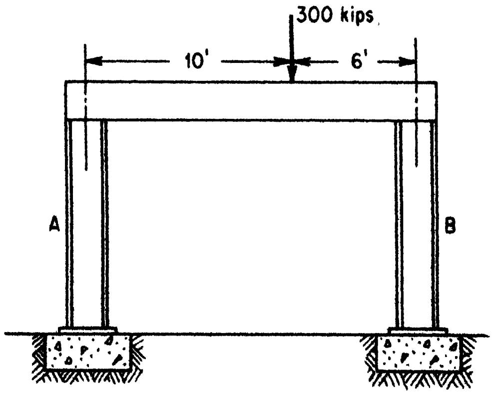


Fig. P1-39

1-40. A rectangular bar $\frac{1}{2}$ in. thick by 3 in. wide is subjected to a tensile force of 5000 lb. Determine the shear and normal stresses on a plane inclined at 30 deg with the axis of the bar. Solve the problem in two ways: first, by treating a section of the bar as a free-body and applying the equations of equilibrium; second, by the equations of Sec. 1-4.

1-41. A short cylindrical post having a diameter of 3 in. is known to with- [illegible] ly. What maximum axial load can be placed on the post?

1-42. A square tension member, 2 in. on edge, is known to have a shearing-stress component of 5000 psi on a plane inclined at 30 deg with the axis of the bar. Find the tensile load acting on the member and the normal stress on the inclined plane.

1-43. The maximum shearing stress that a particular tensile member can withstand is known to be one-third of its maximum tensile stress. If a load is gradually applied, will the member fail in tension or in shear?

1-44. Three $\frac{1}{2}$-in.-diameter birch dowels are used to secure the planks shown in Fig. P1-44. Determine the maximum load P that can be applied if the dowels have a strength of 1200 psi in shear.

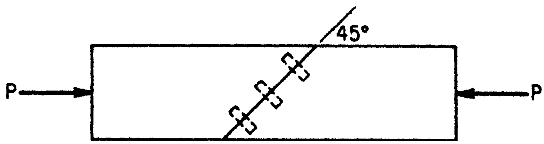


Fig. P1-44

1-45. Three boards 6 in. long are glued together, as illustrated in Fig. P1-45. If the maximum shearing stress parallel to the grain for birch, fir, and oak are 840 psi, 670 psi, and 1350 psi, respectively, determine the maximum allowable value of the load P.

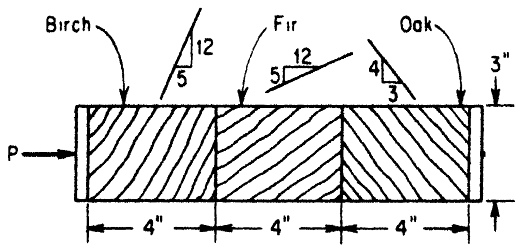


Fig. P1-45

1-46. Two 4 in. by 4 in. timbers 6 in. long support a load as shown in Fig. P1-46. If the maximum shearing stress parallel to the grain is 600 psi, determine the maximum value of P and its location x. Assume the horizontal member to be rigid and weightless.

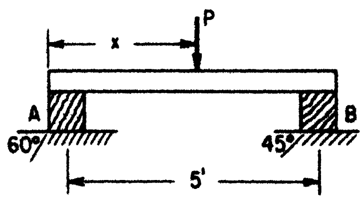


Fig. P1-46

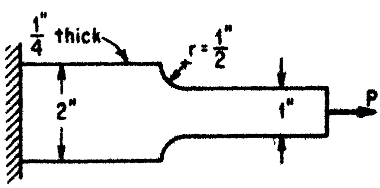


Fig. P1-47

1-47 through 1-51. Determine the maximum permissible static load P that may be applied to the tension members shown in the respective figures if the tensile stress at any section is not to exceed 15,000 psi.

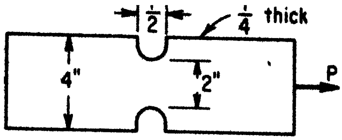


Fig. P1-48

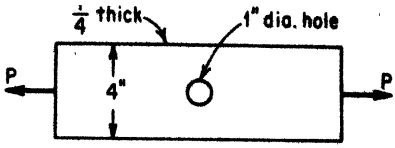


Fig. P1-49

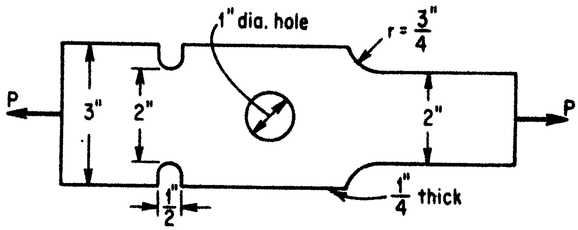


Fig. P1-50

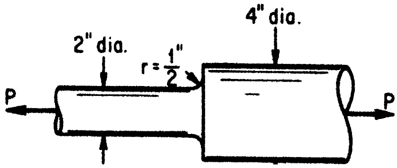


Fig. P1-51

1-52. Estimate the value of the radius of the fillet r, Fig. P1-52, in order that the stress in sections A and B will be the same.

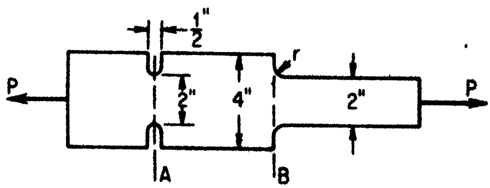


Fig. P1-52

1-53. A cylindrical pressure vessel 1 ft in diameter is to be designed to operate at a pressure of 250 psi. How thick must the wall be if the tensile stress is not to exceed 15,000 psi?

1-54. A 4-ft-diameter cylindrical pressure vessel is fabricated from 12 gage

(0.1046 in.) steel. Determine the maximum permissible pressure within the vessel if the tensile stress is not to exceed 10,000 psi.

1-55. The dimensional details of a 100 ton hydraulic jack are shown in Fig. P1-55. Determine the minimum wall thickness t if the cylinder is fabricated from high-strength steel with a permissible working stress of 40,000 psi in tension.

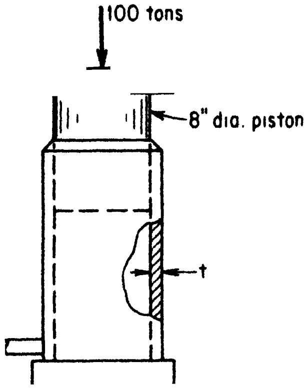


Fig. P1-55

1-56. Prepare a table that will indicate the safe working pressures for specially prepared thin-walled copper tubing whose inside diameters range from $\frac{1}{4}$ in. to 1 in. in $\frac{1}{8}$ in. increments. The wall thickness of the tube in each instance is equal to $\sqrt{D/40}$, where D is the inside diameter of the tube. Use a working stress based on a rupture strength of 45,000 psi with a factor of safety of five.

1-57. An inspection cover on a pressure vessel is secured with fifteen $\frac{1}{2}$-in.-diameter bolts, as shown in Fig. P1-57. Find the maximum stress in the bolts if the vessel is pressurized to 150 psi. The cross-sectional area at the thread root of each bolt is 0.126 sq. in.

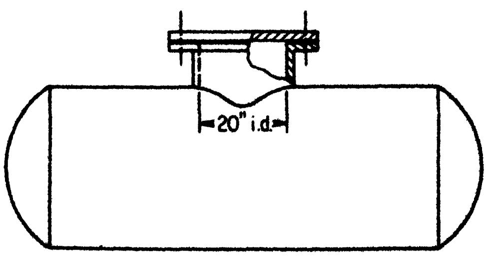


Fig. P1-57

1-58. An oil tank made of $\frac{1}{2}$ in. plate is 50 ft in diameter and 40 ft high. Determine the height of oil (specific gravity = 0.84) that will cause a circumferential stress of 5000 psi.

1-59. A spherical pressure vessel with an inside diameter of 50 in. is made of material having an allowable stress in tension of 10,000 psi. Determine the thickness of the wall if the vessel is to sustain a pressure of 200 psi.

1-60. A 10-ft-diameter spherical vessel is made by bolting two hemispherical sections together, as shown in Fig. P1-60. How many 1-in.-diameter bolts are required to secure the two halves if the internal pressure is 20 psi? Use a working stress of 15,000 psi and a cross-sectional area of 0.551 sq. in. for the bolts.

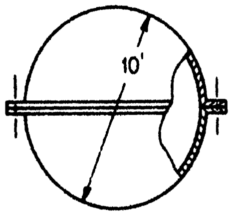


Fig. P1-60

CHAPTER 2

Concept of Strain

Most materials of construction deform under the action of loads according to a set pattern. They behave first elastically, then, as the load increases, plastically, until failure occurs. Although many designs can tolerate a certain amount of plastic deformation, emphasis in this chapter will be on the elastic character of materials.

2–1 Deformation and Strain

The terms *deformation* and *strain* have much the same meaning; they both are a measure of a change in a physical dimension. The first represents

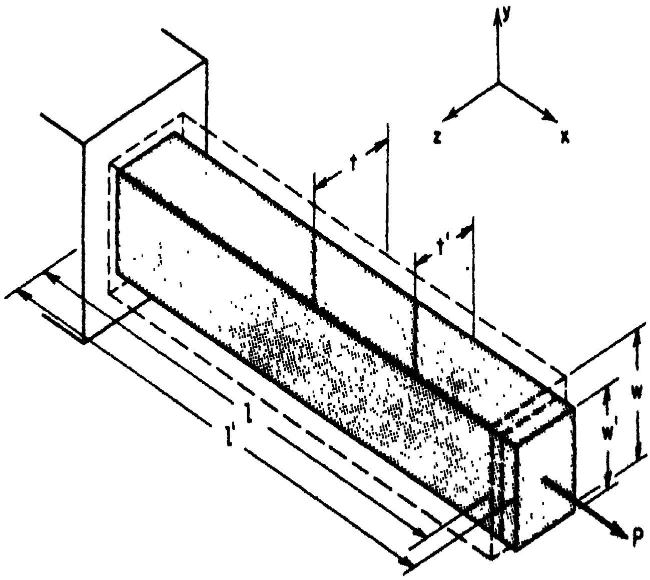


Fig. 2–1

a ***total change***, and the second a ***unit change***. Consider, for example, the rectangular bar illustrated in Fig. 2-1, which has the dimensions of t, w, and l. Under the action of an axial tensile load P the bar stretches to a length of l'; accompanying the stretch will be a shortening of the thickness and width to t' and w'. Deformations, in terms of an x-, y-, z-coordinate system, are

$$\begin{aligned} \delta_x &= l' - l \\ \delta_y &= w' \quad w \\ \delta_z &= t' - \end{aligned} \tag{2-1}$$

Each of the three terms carries the units of the particular dimension, and each represents a *total* or *net change*. The strain ϵ, a measure of the unit change δ/L along each of the coordinate axes, is defined as

$$\begin{aligned} \epsilon_x &= \frac{l' - l}{l} \\ \epsilon_y &= \frac{w' - w}{w} \\ & \frac{t' - t}{t} \end{aligned} \tag{2-2}$$

The equations clearly show that strain is a dimensionless number that represents a change in length per unit length; a strain of 0.1 can equally mean a change of 0.1 in. per inch of length, 0.1 ft per foot, or 0.1 yd per yard. It is also important to note that deformation and strain can be either positive or negative, depending on whether the particular dimension increases or decreases.

Example 1. A rod 10 ft long deforms 0.024 in. because of the action of an axial force. Determine the accompanying axial strain.

Solution: The strain given by Eq. 2-2 is

$$L \quad \frac{0.024}{10(12)} = 0.0002 \text{ in./in.}$$

Example 2. The midpoint M on the stretched horizontal wire drops to M' when weight W is suspended as shown in Fig. 2-2. Find the strain in the wire.

Solution: The change in length of wire is

$$\Delta MB \quad M'B - MB = \sqrt{(4 \times 12)^2 + 5^2} - (4 \times 12)$$

$$\sqrt{2329} - 48 = 0.26 \text{ in.}$$

The strain, by Eq. 2-2, is

$$\epsilon = \frac{\delta}{L} = \frac{0.26}{4 \times 12} = 0.00542 \text{ in./in.}$$

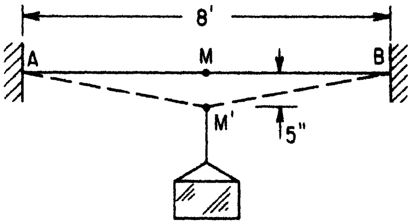


Fig. 2-2

2-2 Elasticity: The Relationship Between Stress and Strain

There is little doubt that the most important discovery in the science of mechanics of materials was that pertaining to the elastic character of materials. This discovery, made by an English scientist, Robert Hooke, in 1678, mathematically relates stress to strain. The relationship, known as Hooke's law, states that *in elastic materials stress and strain are proportional.* Hooke approached the problem experimentally; he applied weights to "springy bodies" and measured their deformations. Although techniques have improved, this experiment is still performed hundreds upon hundreds of times to determine the elastic and plastic properties of materials. Universal testing machines, similar to the one shown in Fig. 2-3, are now employed to apply precise loads at precise rates to standardized tensile or compressive specimens. A variety of devices for measuring and recording strain can be attached to the specimen to make an accurate plot of the variation of stress with strain. Typical of these strain-measuring devices, which are called *extensometers*, is the linear differential transformer shown attached to a specimen in Fig. 2-4. As the specimen stretches, the activating knife edge moves, which in turn moves the core of a differential transformer. This movement creates an alternating voltage which can be amplified to run recording instrumentation. Modern extensometers have the ability to measure strains to an accuracy of a millionth of an inch—a far cry from Robert Hooke's yardstick. Figure 2-5, a photograph of a mild steel tensile specimen, illustrates the type of failure typical of this material.

All materials deform differently under the action of tensile loads; some materials, such as mild steel, show a true proportionality between stress and

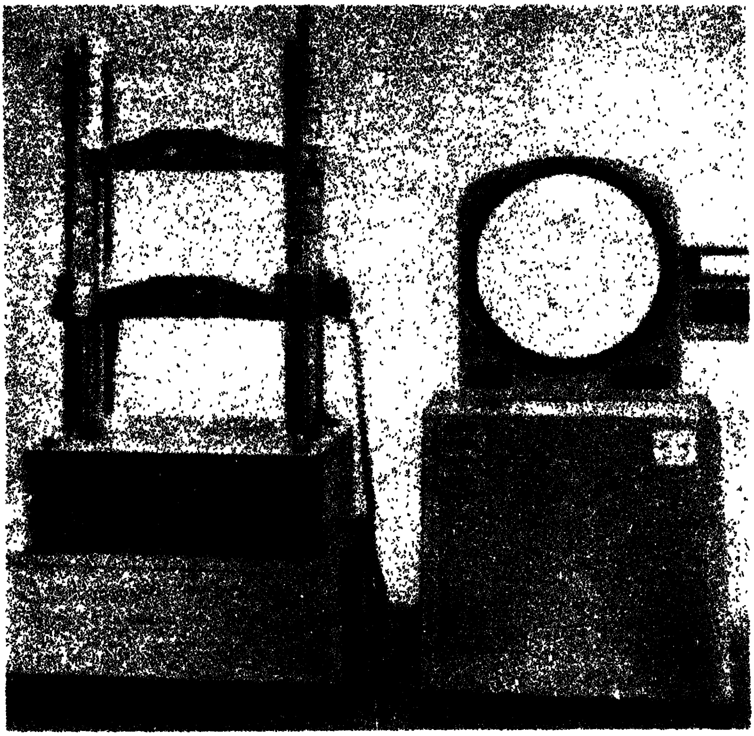


Fig. 2-3. Universal testing machine. Courtesy Tinius Olsen Testing Machine Company

strain up to a point, whereas others, like aluminum and copper, only approximate a proportionality. Typical plots of stress versus strain for several materials are shown in Fig. 2-6. Regardless of how the deformation progresses with load, a great deal of information can be obtained from the stress-strain diagram. Consider the curve shown in Fig. 2-7; stress in the units of pounds per square inch, computed on the basis of the original area, is represented as the ordinate in the curve, and strain in the units of inches per inch is the abscissa. As can be seen, the strain in this particular specimen increases uniformly with stress up to 66,000 psi. Beyond this value of stress, called the *proportional limit*, the material deforms at a non-uniform rate. The maximum stress obtained in the plot, 116,000 psi in this example, is called the *ultimate stress*, or in some instances the *ultimate strength*, or the *tensile strength*. The final point in the curve represents the *rupture stress* or *rupture strength* of the material. It must be remembered that the stress plotted in the curve is computed on the basis of the original cross-sectional area, which, by the time the specimen fails, may be reduced appreciably.

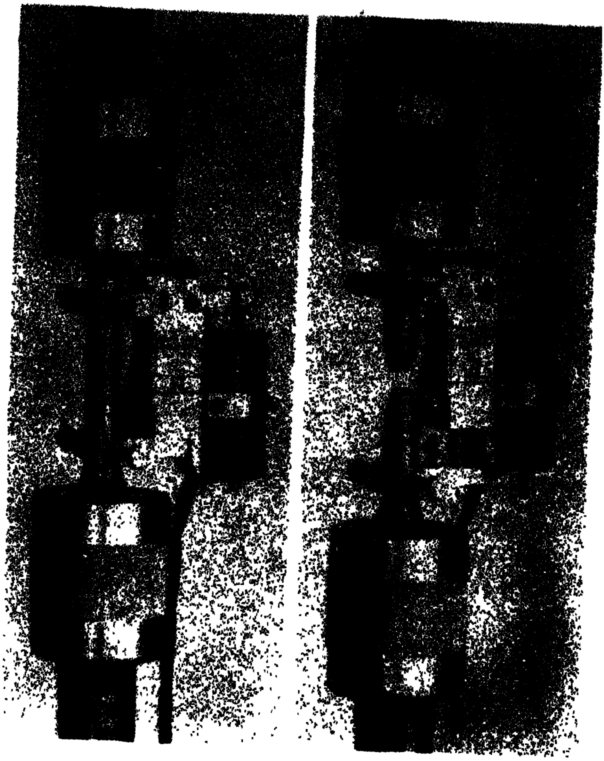


Fig. 2-4. Differential transformer extensometer which automatically separates when its measuring range has been exceeded. Courtesy Tinius Olsen Testing Machine Company.

For this reason the rupture strength obtained from the curve is a ficticious number, and little emphasis is placed on its value.

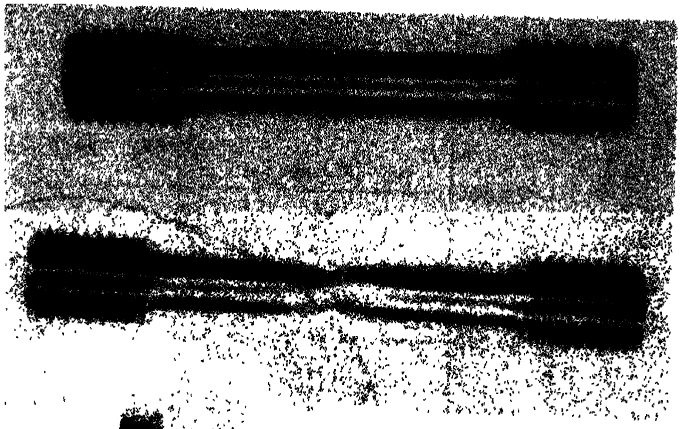
The slope of the elastic portion of the stress-strain curve, the modulus line in Fig. 2-7, is called the *modulus of elasticity* and the constant that represents the slope, Young's modulus E.

$$\frac{\text{Stress}}{\text{Strain}} = \text{Constant}$$

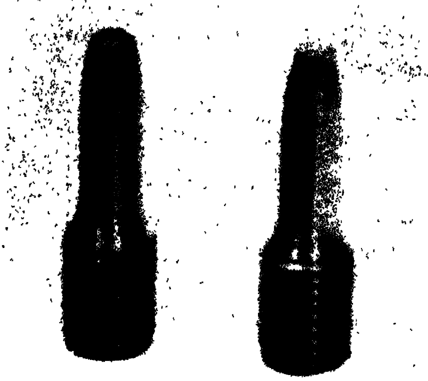
symbolically

$$\frac{\sigma}{\epsilon} = E \tag{2-3}$$

Another point of concern in the stress-strain diagram is the *yield strength*, or *yield point*, which is usually defined in terms of a specified amount of permanent deformation. A yield strength at 0.2 per cent offset, as illustrated in Fig. 2-7, is that value of stress which would cause a permanent deformation of 0.002 in. per in. with the load removed. This point is found by constructing a line parallel to the modulus line, offset by the specified amount as shown.



(b)



(a)

Fig. 2-5. Ductile steel tensile specimen: (a) showing the physical features of a tensile specimen before and after break, (b) cross-section view at the break.

Example 3. A tensile test on a steel specimen produced the stress-strain diagram illustrated in Fig. 2-7. Prior to testing, the specimen had a diameter of 0.505 in., and after rupture the diameter at the break was measured and found to be 0.425 in. Before testing, small punch marks 2 in. apart were made on the specimen, and, by placing the two portions of the specimen together after the test, the gage marks were found to be $2\frac{3}{8}$ in. apart. Determine the following: (a) modulus of elasticity, (b) proportional limit, (c) ultimate strength, (d) rupture strength based on the original area, (e) rupture strength based on the actual area, (f) yield strength at 0.02 per cent offset, (g) per cent elongation, and (h) per cent reduction in area.

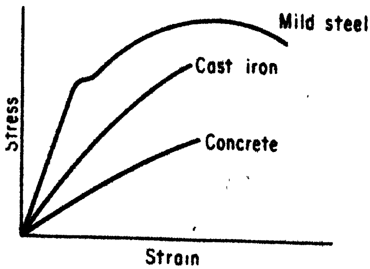


Fig. 2-6

Solution: (a) The slope of the modulus line, the ratio of a particular stress to the corresponding strain, is equal to the modulus of elasticity.

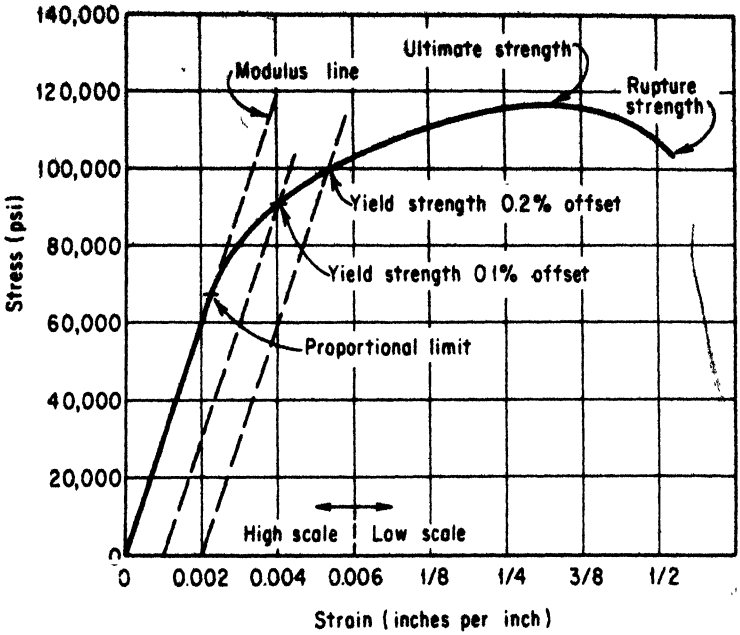


Fig. 2-7

$$E = \frac{30{,}000 \text{ psi}}{0.001 \text{ in./in.}} = 30 \times 10^6 \text{ psi}$$

(b) The proportional limit, that point where stress and strain cease to be proportional, is approximated from the graph; thus

$$\sigma_{pl} = 66{,}000 \text{ psi}$$

(c) The ultimate strength, the maximum point on the curve, is approximately

$$\sigma_{ult} = 116{,}000 \text{ psi}$$

(d) Since stress in the diagram is computed on the basis of the original cross-sectional area, the rupture strength is read directly from the curve

$$\sigma_{rup} = 103{,}000 \text{ psi}$$

(e) The load at failure P_f, computed in terms of the original area, is

$$P_f = \sigma A = 103{,}000 \frac{\pi}{4}(0.505)^2 = 20{,}600 \text{ lb}$$

This load divided by the actual area gives the true rupture strength:

$$\sigma_{rup'} = \frac{P}{\;} = \frac{20{,}600}{\pi(0.425)^2/4} \quad 145{,}000 \text{ psi}$$

(f) The yield strength at 0.02 per cent offset is found by constructing a line parallel to the modulus line, offset by 0.002 in. per in. at the zero stress point, as shown. The yield strength read from the graph is approximately 100,000 psi.

(g) and (h) The per cent elongation and per cent reduction in area, both measures of ductility, are, respectively,

$$\text{Per cent elongation} \qquad \frac{2.375 - 2}{} \times 100 = 18.8 \text{ per cent}$$

$$\text{Per cent reduction in area} \qquad \frac{\pi(0.505)^2/4 - \pi(0.425)^2/4}{\pi(0.505)^2/4} \times 100 = 9.5 \text{ per cent}$$

2-3 *The Equations of Elasticity*

Hooke's law provides a means of predicting with reasonable accuracy the elastic deformation in an axial loaded member. The mathematical expression for the law, in terms of the definitions of stress and strain, is

$$\frac{\sigma}{\epsilon} = \frac{\frac{P}{A}}{\frac{\delta}{L}} = E$$

Rearranging terms and solving for δ gives

$$\delta = \frac{PL}{EA} \tag{2-4}$$

Equation (2-4) expresses the important relationship among the variables deformation δ, axial load P, length L, cross-sectional area A, and modulus of elasticity E.

It must be remembered that Eq. (2-4) is valid *only* for axial loaded members of constant cross-sectional area which are not stressed beyond their proportional limit. Care must be exercised in the selection of appropriate dimensions for each of the terms. Most tables, like those of Appendix B, give values of the modulus of elasticity E, in pounds per square inch. To be dimensionally correct, therefore, the cross-sectional area A must be expressed in square inches and the load P in *pounds*; deformation δ will then assume the units assigned to the length L.

Example 4. A cylindrical steel bar which has a length of 15 in. is subjected to a tensile force of 5000 lb. Determine the required diameter of the bar if the stress is not to exceed 20,000 psi or the total elongation 0.005 in. $E = 30 \times 10^6$ psi.

Solution: Two restrictions are stated in the problem; the first places an upper limit on the stress, and the second places an upper limit on the deformation. To satisfy the first condition, the area must be

$$A = \frac{F}{\ } = \frac{5000}{20{,}000} = 0.25 \text{ sq in.}$$

The area, in terms of the deformation δ, by Eq. (2-4) is

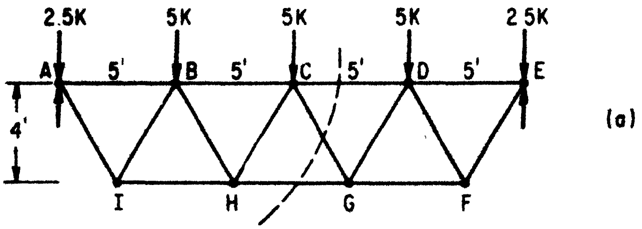
$$A = \frac{FL}{\delta E} \quad \frac{5000(15)}{0.005(30 \times 10^6)} \quad 0.50 \text{ sq in.}$$

To satisfy both conditions, the larger of the two areas must be used. The required diameter, then, is given by

$$A = \frac{\pi D^2}{4}$$

$$D \quad \sqrt{\frac{4A}{\pi}} = \sqrt{\frac{4(0.5)}{\pi}} = 0.798 \text{ in.}$$

Example 5. The roof truss shown in Fig. 2-8 is simply supported at A and E, and it in turn supports the panel loads as illustrated. Member GH is an annealed carbon steel tube with an outside diameter of 2 in. and a wall thickness of ⅛ in. Determine the deformation and stress in member GH and the factor of safety in the design based on the ultimate tensile strength. $E = 30 \times 10^6$ psi.



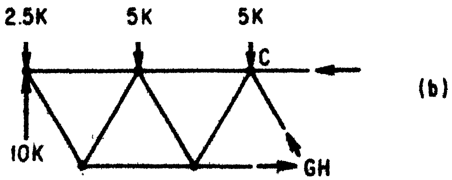


Fig. 2-8

Solution: Since the truss is symmetrical, the reactions A and E are each equal to one-half the total external load.

$$R_A = R_E \qquad 2.5 + 5 + 5 + 5 + 2.5 \qquad = 10 \text{ kips}$$

The force in member GH is found by sectioning the truss and taking moments about joint C in the free-body diagram shown in Fig. 2-8(b).

$$\sum M_C = 0$$

$$4GH + 5(5) + 2.5(10) - 10(10) = 0$$

$$GH = {}^{50}\!/_{4} = 12.5 \text{ kips (tension)}$$

The cross-sectional area of member GH is next computed.

$$A = \frac{\pi}{4}(2^2 - 1.75^2) = 0.736 \text{ in.}^2$$

Substitution of numerical data into Eq. (2-4) gives the desired deformation.

$$\delta = \frac{PL}{EA} = \frac{12{,}500(5 \times 12)}{30 \times 10^6(0.736)} \qquad 0.034 \text{ in.}$$

The second quantity desired, the stress, is given by

$$\sigma = \frac{P}{A} = \frac{12{,}500}{0.736} = 17{,}000 \text{ psi}$$

This value of stress is considerably less than the yield strength of 38,000 psi given in Appendix B, thus validating the use of Hooke's law in the prior step.

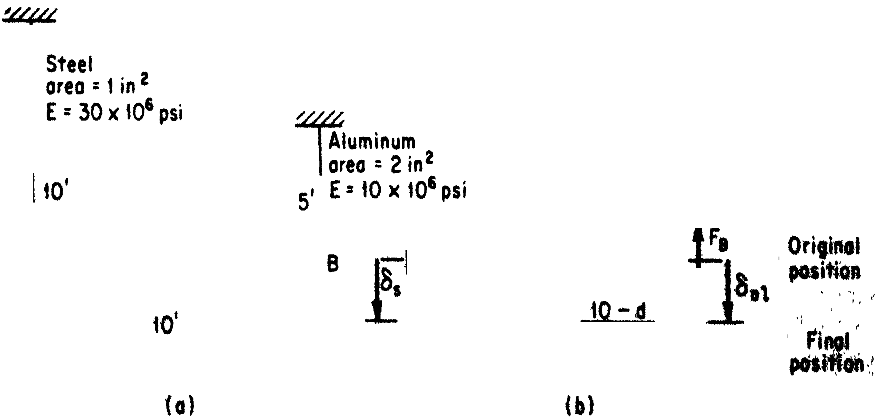


Fig. 2-9

Lastly, the safety factor in the design based on the tensile strength from Appendix B is

$$\text{S.F.} = \frac{\text{tensile strength}}{\text{working stress}}$$

$$= \frac{65{,}000}{17{,}000} = 3.82$$

Example 6. A rigid beam of negligible weight is supported in a horizontal position by two rods, as shown in Fig. 2-9(a). Where, with respect to end A, should the force P act if the beam is to remain horizontal?

Solution: The beam will remain horizontal only if both rods deform equally.

$$\delta_s = \delta_{al}$$

$$\left(\frac{FL}{EA}\right)_s = \left(\frac{FL}{EA}\right)_{al}$$

$$\frac{F_A(10)12}{30(10)^6(1)} = \frac{F_B(5)12}{10(10)^6(2)}$$

$$F_A = \tfrac{3}{4}F_B$$

This result is combined with the equations of static equilibrium to give the desired answer.

$$\sum F_y = 0$$

$$F_A + F_B = P$$

$$F_B(\tfrac{3}{4} + 1) = P$$

$$F_B = \tfrac{4}{7}P$$

$$\sum M_A = 0$$

$$Pd = \tfrac{4}{7}P(10)$$

$$d = \tfrac{40}{7} = 5.71 \text{ ft to right of } A$$

2-4 *Poisson's Ratio*

Experimentation has shown that an axial elongation is always accompanied by a lateral contraction, and for a given material strained within its elastic region the ratio of lateral strain to axial strain is a constant.

$$\frac{\text{lateral strain}}{\text{axial strain}} \qquad (2\text{-}5)$$

Discovered in the early nineteenth century by the French mathematician Poisson, the constant μ, called *Poisson's ratio*, varies between 0 and $\frac{1}{2}$ for all materials. Precise measurements indicate that for most metals it lies between $\frac{1}{4}$ and $\frac{1}{3}$.

Example 7. A load of 40,000 lb applied to a brass cylinder 15 in. long and 4 in. in diameter caused the length to decrease 0.0032 in. and the diameter to increase 0.00022 in. Find the modulus of elasticity E and Poisson's ratio μ of the brass.

Solution: The modulus of elasticity, the ratio of axial stress to axial strain, is

$$E = \frac{PL}{\delta A} = \frac{40{,}000(15)}{0.0032\pi(2)^2} = 15 \times 10^6 \text{ psi}$$

Poisson's ratio, the ratio of lateral strain to axial strain, is

$$\mu = \frac{\text{lateral strain}}{\text{axial strain}} = \frac{(0.00022)/4}{(0.0032)/15} = 0.26$$

2-5 *Thermal Strain*

Most materials expand when heated and contract when cooled. Careful measurements have shown that the ratio of strain E to temperature change ΔT is a constant.

$$\alpha = \frac{\text{strain}}{\text{change in temperature}} = \frac{\delta/L}{\Delta T}$$

Solving this equation for the deformation gives

$$\delta = \alpha L \Delta T \tag{2-6}$$

where α is called the *thermal expansion coefficient*. In American engineering practice the customary units of this constant are inches per inch per degree Fahrenheit. Steel, for example, has a thermal expansion coefficient of 6.5×10^{-6} in./in./°F, which simply means that for every degree Fahrenheit temperature change, the length, width, or breadth will change by 0.0000065 in. per in.

Problems in which both thermal deformation and stress deformation are involved are best treated by the method of *superposition*: by considering separately the deformations in the structure caused by thermal expansion or contraction and the deformations caused by axial forces. A geometric picture is obtained which will relate the strains, and this relationship, together with equations of static equilibrium, is used to solve the problem.

Example 8. A manganese bronze rod is fastened securely to support A, as shown in Fig. 2-10(a). The free end can move 0.01 in. before contacting the solid support B. Determine (a) the temperature increase necessary to cause the free end to just touch B, and (b) the stress in the rod if its temperature increases 50°F.

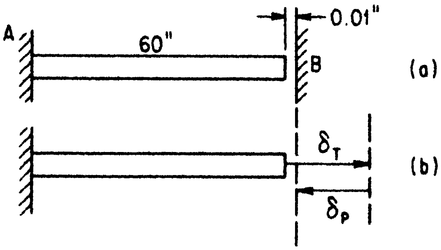


Fig. 2-10

Solution: The following data are obtained from Appendix B:

$$\alpha = 11.2 \times 10^{-6} \text{ in./in./°F}$$

$$E = 15 \times 10^{6} \text{ psi}$$

Part (a). The change in length as a function of temperature is

$$\delta_T = \alpha L \Delta T$$

$$0.01 = 11.2 \times 10^{-6}(60)\Delta T$$

Solving for the temperature change gives

$$\Delta T = \frac{0.01}{11.2 \times 10^{-6}(60)} = 14.9\text{°F}$$

Part (b). With wall B removed, Fig. 2-10(b), the bar could expand freely a distance equal to δ_T. To return the bar to its constrained position, the wall must exert a force sufficiently large to deform the bar an amount δ_P. From the geometry of the picture

$$\delta_T = \delta_P + 0.01$$

$$\alpha L \Delta T = \frac{PL}{EA} + 0.01 = \frac{\sigma L}{E} + 0.01$$

Substitution of data gives

$$11.2 \times 10^{-6}(60)50 \qquad \frac{60\sigma}{15 \times 10^{6}} + 0.01$$

$$4\sigma = 11.2(60)50 - 10^{4}$$

$$\sigma = 5900 \text{ psi}$$

Example 9. A 25 ton weight is supported by three columns, two steel and one manganese bronze, as shown in Fig. 2-11(a). Determine the temperature change necessary to just relieve the bronze column of all stress. The columns have equal cross-sectional areas of 2 sq in. and are initially the same length.

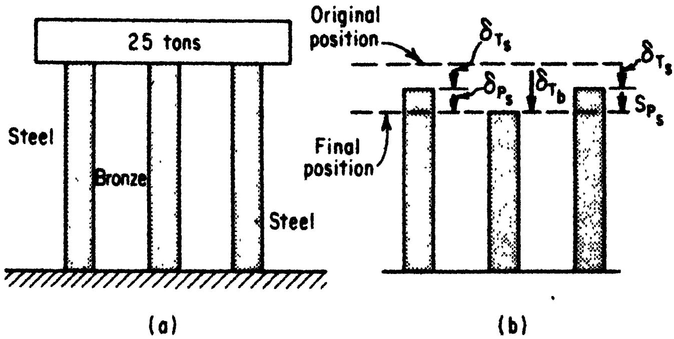


Fig. 2-11

Solution: The following data are obtained from Appendix B:

$$\alpha_s = 6.5 \times 10^{-6}\ \text{in./in./}^\circ\text{F}$$

$$\alpha_b = 11.2 \times 10^{-6}\ \text{in./in./}^\circ\text{F}$$

$$E_s = 30 \times 10^6\ \text{psi}$$

$$E_b = 15 \times 10^6\ \text{psi}$$

A sketch is drawn as shown in Fig. 2-11(b). With the weight removed and the temperature assumed to drop, the two steel columns contract a distance δ_{T_s} and the bronze column a distance δ_{T_b}. To be realistic, the latter deformation is sketched as a greater distance, since the coefficient of expansion of bronze is nearly twice that of steel. The weight is then replaced, and the two steel columns are allowed to deform a distance δ_{P_s}; this replacement positions all three columns at the same level. Since the steel alone supports the weight of 25 tons, each steel column supports half of the total load, or 25,000 lb.

$$\delta_{T_s} + \delta_{P_s} = \delta_{T_b}$$

$$(\alpha L \Delta T)_s + \left(\frac{PL}{EA}\right)_s = (\alpha L \Delta T)_b$$

Substitution of data gives

$$(6.5 \times 10^{-6})L\Delta T + \frac{25{,}000L}{30 \times 10^6(2)} = (11.2 \times 10^{-6})L\Delta T$$

Every term in the equation is a function of L; therefore,

$$\Delta T(10^{-6})\,(11.2 - 6.5) \qquad \frac{25{,}000}{30 \times 10^6(2)}$$

$$\Delta T = \frac{25{,}000}{30(2)\,(4.7)} \qquad 88.7°\text{F (drop)}$$

Example 10. A steel bar and an aluminum bar, each secured to a rigid support, are fastened at their free ends by a 1-in.-diameter pin, as shown in Fig. 2-12(a). Determine the shearing stress in the pin if the temperature drops 50°F. The bars are initially free of stress.

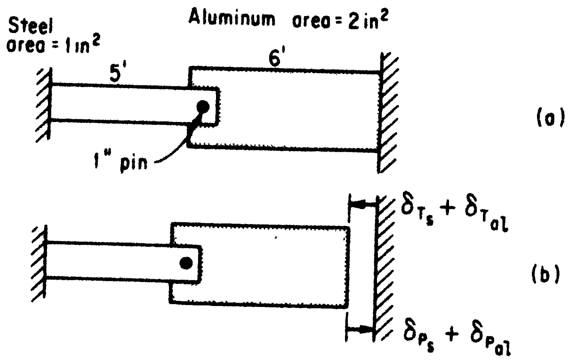


Fig. 2-12

Solution: The following data are taken from Appendix B:

$$\alpha_s = 6.5 \times 10^{-6}\text{ in./in./°F}$$

$$\alpha_{al} = 13.1 \times 10^{-6}\text{ in./in./°F}$$

$$E_s = 30 \times 10^6\text{ psi}$$

$$E_{al} = 10 \times 10^6\text{ psi}$$

With one constraint removed, the right-hand wall in this instance, the joined members would contract an amount equal to $\delta_{T_s} + \delta_{T_{al}}$, as the temperature drops. To return the members to their original length, the wall must exert a force that will cause the combination to stretch $\delta_{P_s} + \delta_{P_{al}}$.

$$\delta_{P_s} + \delta_{P_{al}} = \delta_{T_s} + \delta_{T_{al}}$$

$$\left(\frac{PL}{EA}\right)_s + \left(\frac{PL}{EA}\right)_{al} = (\alpha L \Delta T)_s + (\alpha L \Delta T)_{al}$$

The force P is common to both members; factoring and substituting data gives

$$\left[\frac{5(12)}{30\times 10^6(1)}+\frac{6(12)}{10\times 10^6(2)}\right] = 6.5(10^{-6})\,(5\times 12)50 + 13.1(10^{-6})\,(6\times 12)50$$

$$P(\tfrac{5}{30} + \tfrac{6}{20}) = [6.5(5) + 13.1(6)]\,50$$

$$0.467P = 5560$$

$$P = 11{,}900 \text{ lb}$$

Finally, the shear stress in the pin is

$$\tau = \frac{P}{A} = \frac{11{,}900}{\pi(0.5)^2} = 15{,}200 \text{ psi}$$

2-6 *Rigidity*

When shear forces act on a body, Fig. 2-13, the deformation that occurs is due to a sliding action between adjoining layers of matter, and as a result of this action the shape of the body, rather than its volume, changes. The measure of shearing strain is the ratio of the deformation δ_s to the length L:

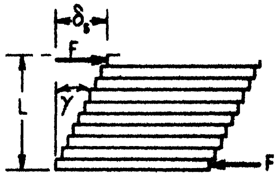


Fig. 2-13

$$\epsilon_s = \frac{\delta_s}{L}$$

Since δ_s is small, the strain can be expressed in terms of the angular displacement γ in radians:

$$\epsilon_s = \frac{\delta_s}{L} = \gamma \qquad (2\text{-}7)$$

Hooke's law relates the proportionality between shearing stress and shearing strain in terms of the constant G, called the ***modulus of rigidity***.

$$\frac{\tau}{\gamma} = \frac{FL}{\delta_s A_s} = G \qquad (2\text{-}8)$$

An important theoretical equation shows the three elastic constants E, G, and μ to be dependent upon one another for homogeneous materials.

$$G = \frac{E}{2(1+\mu)} \qquad (2\text{-}9)$$

Example 11. A brass specimen having a square cross section 2 in. on edge and a height of 3 in. is subjected to a force of 20,000 lb, first in shear and then in compression, as shown in Fig. 2-14. The following data are observed:

$$\delta_s = 0.00234 \text{ in.}$$

$$\delta_a = 0.00088 \text{ in.}$$

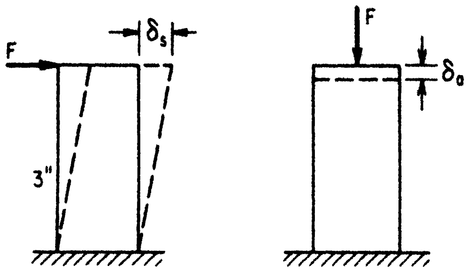


Fig. 2-14

Determine the modulus of elasticity, the modulus of rigidity, and Poisson's ratio.

Solution: Hooke's law is used to find the first two constants E and G.

$$E = \frac{FL}{\delta A} = \frac{20{,}000(3)}{0.00088(2 \times 2)} = 17 \times 10^6 \text{ psi}$$

and

$$G = \frac{FL}{\delta_s A_s} = \frac{20{,}000(3)}{0.00234(2 \times 2)} = 6.4 \times 10^6 \text{ psi}$$

Solving Eq. (2-9) for Poisson's ratio gives

$$G = \frac{E}{2(1 + \mu)}$$

$$\mu = \frac{E}{2G} - 1 = \frac{17 \times 10^6}{2(6.4 \times 10^6)} - 1 = 1.33 - 1 = 0.33$$

PROBLEMS

2-1. A cylindrical bar 20 in. long deforms 0.05 in. when an axial tensile force is applied. Find the axial strain in the bar.

2-2. The maximum permissible strain in a particular elevator cable is 0.0005 in./in. Determine the elongation in 200 ft of this cable.

2-3. Precise measurements on a bar 20 in. long, 3 in. wide, and $\frac{1}{2}$ in. thick, subjected to an axial tensile force, show its length to increase by 0.024 in. and its width and thickness to decrease by 0.0009 in. and by 0.00015 in. respectively. Find the three components of strain in the bar.

2-4. A brass cube 5 in. on edge is surrounded by a pressure force which strains each edge equally 0.0002 in./in. Determine the change in volume of the cube.

2-5. A cube of concrete measuring 10 in. on edge is surrounded by a high hydrostatic pressure which decreases its volume by 0.72 cu in. Find the strain in the cube.

2-6. Cable BD is fastened to the midpoint of a stretched horizontal wire, as shown in Fig. P2-6. With the weight suspended from the end of the cable, B moves to B' and D to D'. Find the displacement DD' if the strain in the horizontal wire is 0.005 in./in. and in the vertical cable 0.008 in./in.

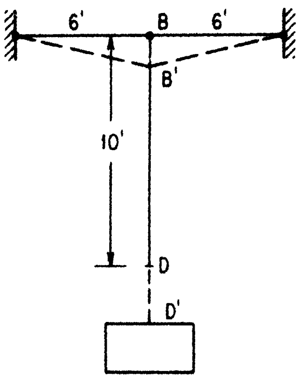


Fig. P2-6

2-7. The rigid beam AC shown in Fig. P2-7 is pinned at A and is held in the horizontal position by the flexible cable BC. Determine the vertical displacement of C if a weight causes the wire to strain 0.001 in./in.

2-8. The free ends of the vertical rods in Fig. P2-8 are initially at the same level. Find the inclination of the weight W with respect to the horizontal if rod A is strained 0.001 in./in. and rod B 0.004 in./in. while supporting the weight.

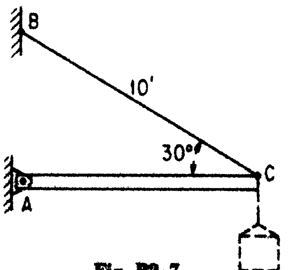


Fig. P2-7

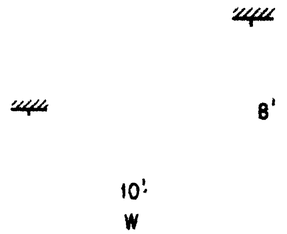


Fig. P2-8

2-9. Determine the required ratio of the strain in rod A to the strain in rod B if the weight in Prob. 2-8 is to remain horizontal.

2-10. A stress-strain diagram for an alloy tensile specimen is shown in Fig. P2-10. The following data are observed:

Initial diameter	=	0.505 in.
Diameter at break	=	0.408 in.
Initial gage length	=	2 in.
Final gage length	=	$2\frac{7}{8}$ in.

Find: (a) the modulus of elasticity, (b) the proportional limit, (c) the yield strength at 0.2 per cent offset, (d) the per cent elongation, (e) the per cent reduction in area, (f) the indicated rupture strength, and (g) the true rupture strength based on area at break.

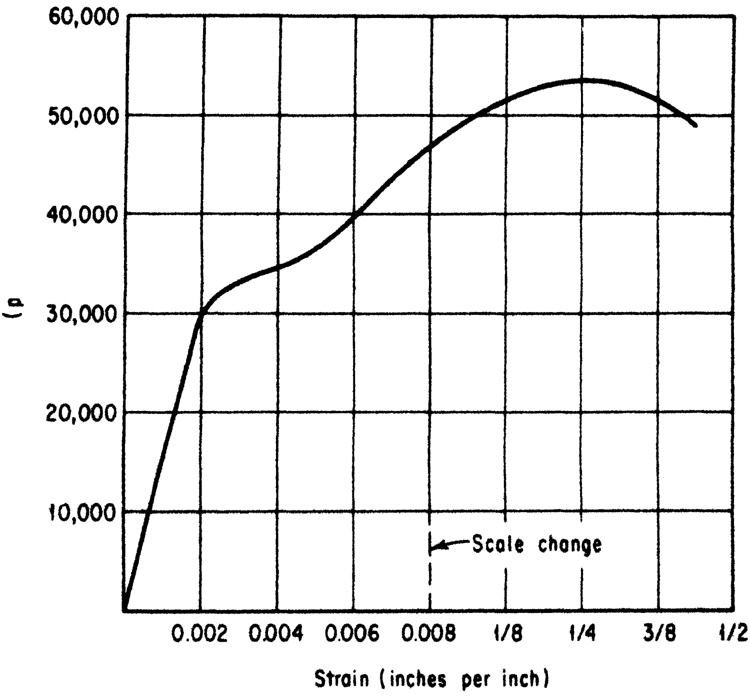


Fig. P2-10

2-11. The following data were obtained during a tensile test on a mild steel specimen having an initial diameter of 0.505 in. At failure the reduced diameter of the specimen was 0.305 in. Plot the data and determine (a) the modulus of elasticity, (b) the proportional limit, (c) the yield strength at 0.2 per cent offset, (d) the ultimate strength, (e) the per cent reduction in area, elongation, (g) the indicated strength at rupture, (h) the

Axial load (*lb*)	*Elongation in a 2 in. length* (*in.*)	*Axial load* (*lb*)	*Elongation in a 2 in. length* (*in.*)
0	0		
1640	0.00050	8040	0.00938
3140	0.00100	8060	0.0125
4580	0.00150	9460	0.050
6000	0.00200	12,000	0.125
7440	0.00250	13,260	0.225
8000	0.00300	13,580	0.325
7980	0.00375	13,460	0.475
7900	0.00500	13,220	0.535
8040	0.00624	9860	0.625

2-12. A 1-in.-diameter manganese bronze test specimen was subjected to an axial tensile load, and the following data were observed:

Gage length	10 in.
Final gage length	12.25 in.
Load at proportional limit	18,500 lb
Elongation at proportional limit	0.016 in.
Maximum load	55,000 lb
Load at rupture	42,000 lb
Diameter at rupture	0.845 in.

Find: (a) the modulus of elasticity, (b) the proportional limit, (c) the ultimate strength, (d) the per cent elongation, (e) the per cent reduction in area, (f) the indicated rupture strength, and (g) the true rupture strength.

2-13. Interest has been focused recently on the tensile properties of alloys at elevated temperatures. The following data were observed during a tensile test on aluminum alloy:

Temperature (°F)	*Ultimate strength* (psi)	*Yield strength* (0.2 per cent offset psi)	*Elongation* (per cent)
75	35,000	29,000	12
212	34,000	29,000	13
300	28,000	25,000	22
400	21,000	15,000	35
500	14,000	7,500	55
600	7,500	5,000	80
700	5,000	3,000	90

Prepare a single graph of four curves that will show (a) the ultimate strength, (b) the yield strength, (c) the per cent elongation, and (d) the safe working stress based upon 40 per cent of the ultimate strength, all as functions of temperature.

2-14. A straight steel bar having a cross-sectional area of 1 sq in. is suspended and loaded as shown in Fig. P2-14. Determine the total deformation of the bar. $E_s = 30 \times 10^6$ psi.

2-15. A 20 kip load is supported in tension by the composite steel and bronze column as shown in Fig. P2-15. Find the total elongation of the column. $E_s = 30 \times 10^6$ psi; $E_b = 17 \times 10^6$ psi.

2-16. An 8 WF 17 steel beam supports the two loads as shown in Fig. P2-16. Determine the force P and the deflection of the free end if the maximum stress in the beam is not to exceed 30,000 psi.

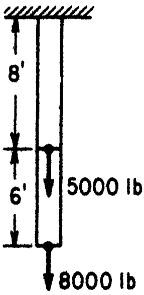


Fig. P2-14

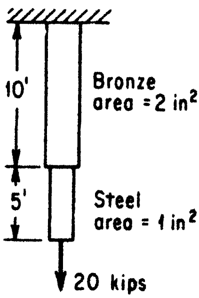


Fig. P2-15

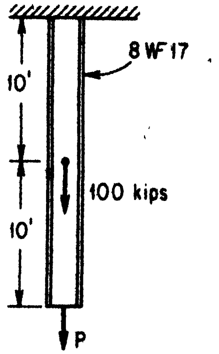


Fig. P2-16

2-17. The aluminum bar illustrated in Fig. P2-17 has a uniform cross-sectional area of 0.50 in.² Determine the total change in length of the bar if it is subjected to the axial forces shown. $E_{al} = 10 \times 10^6$ psi.

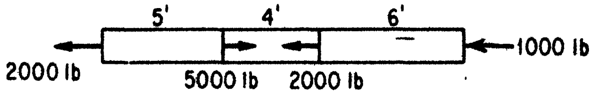


Fig. P2-17

2-18. The short steel tube of Fig. P2-18 is used as a post. Determine the allowable load P that can be applied, as shown, if the compressive stress is to be limited to 20,000 psi and the deformation to 0.010 in. $E_s = 30 \times 10^6$ psi.

2-19. Would it be possible, in Prob. 2-18, to select a wall thickness for the tube that would allow both maximum stress and maximum deformation to be satisfied simultaneously?

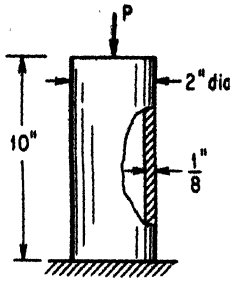


Fig. P2-18

2-20. A 10 ft structural-steel bar 1 in. by 4 in. in cross section is to support an axial tensile load. The allowable normal and shearing stresses are 20,000 psi and 8000 psi respectively, and the maximum permissible elongation is 0.05 in. Determine the safe load P. $E = 30 \times 10^6$ psi.

2-21. A concrete cylinder 6 in. in diameter and 5 ft high is to support an axial compressive load P. Find the maximum value of the load if the normal stress is not to exceed 1200 psi, the shearing stress 540 psi, and the deformation 0.03 in. $E_c = 2 \times 10^6$ psi.

2-22. A uniform weight W is supported by two bars as shown in Fig. P2-22. If the ends of the bars are initially at the same level and the weight is to remain horizontal, find the required cross-sectional area of the bronze bar. $E_b = 17 \times 10^6$ psi; $E_s = 30 \times 10^6$ psi.

2-23. Solve Prob. 2-22 if the center of gravity of the weight W is 6 ft to the right of cable A.

2-24. The steel post illustrated in Fig. P2-24 is fastened to non-yielding supports at A and C. Determine the stress in a section of the post 2 ft from A and in a section 8 ft from A. The bar has a cross-sectional area of ½ sq in.

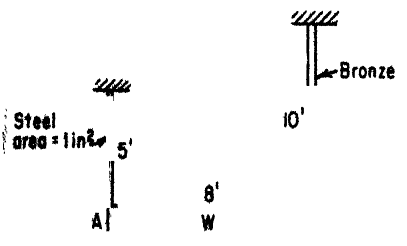


Fig. P2-22

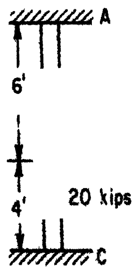


Fig. P2-24

2-25. Determine the stress in sections AB and BC in Prob. 2-24 if the top support yields 0.005 in. when the load is applied. Consider the support at C to remain rigid.

2-26. Determine the stress in the bronze portions and in the steel portions of the post shown in Fig. P2-26. Assume the supports at A and C to be non-yielding. $E_s = 30 \times 10^6$ psi; $E_b = 17 \times 10^6$ psi.

2-27. A bronze cylinder and an aluminum tube jointly support a 20 kip load, as shown in Fig. P2-27. Determine the stress in each material if prior to loading the cylinder and tube are the same length. $E_a = 10 \times 10^6$ psi; $E_b = 17 \times 10^6$ psi.

2-28. The column illustrated in Fig. P2-28 consists of a number of steel reinforcing rods imbedded in a concrete cylinder. Determine the cross-sectional area of the concrete and the steel and the stresses in each if the column deflects 0.01 in. while carrying a load of 100,000 lb. $E_s = 30 \times 10^6$ psi; $E_c = 2 \times 10^6$ psi.

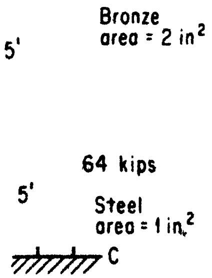


Fig. P2-26

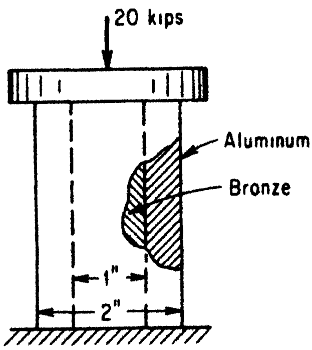


Fig. P2-27

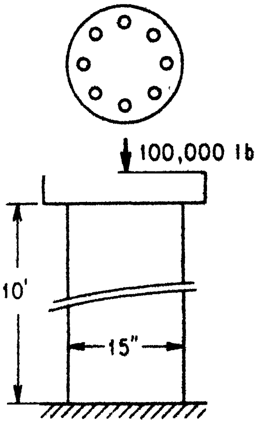


Fig. P2-28

2-29. Find the safe weight W that can be supported by the three rods shown in Fig. P2-29 if the maximum permissible stresses are $\sigma_s = 30{,}000$ psi, and $\sigma_b = 10{,}000$ psi. $E_s = 30 \times 10^6$ psi; $E_b = 17 \times 10^6$ psi.

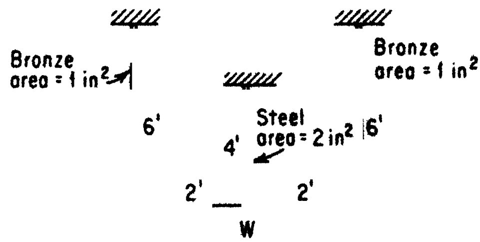


Fig. P2-29

2-30. Four steel tie-rods secure the end plates to the aluminum tube shown in Fig. P2-30. The ends of the rods are threaded with a standard ½ - 20 die, and each nut on the right plate is tightened one-half turn from the snug position. Determine the stress in the rods and in the tube. The thickness of the end plates may be considered negligible. $E_s = 30 \times 10^6$ psi; $E_a = 10 \times 10^6$ psi.

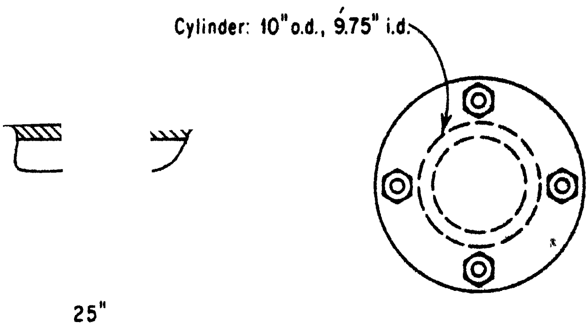


Fig. P2-30

2-31. Determine the greatest permissible weight W that can be supported as shown in Fig. P2-31, if the stresses in the steel and in the aluminum are not to exceed 20 ksi and 8 ksi respectively. Assume the pinned beam AB to be rigid and weightless. $E_s = 30 \times 10^6$ psi; $E_a = 10 \times 10^6$ psi.

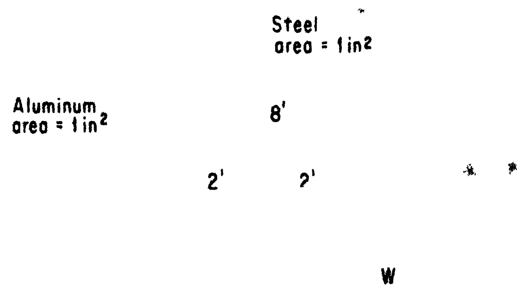


Fig. P2-31

2-32. Three rods, each with a cross-sectional area of 1 sq in., support a rigid and weightless beam, as shown in Fig. P2-32. Where, with respect to end A, should a weight W be placed if the beam is to remain horizontal? $E_s = 30 \times 10^6$ psi; $E_b = 15 \times 10^6$ psi; $E_a = 10 \times 10^6$ psi.

2-33. A square post is constructed of a piece of aluminum and a piece of brass placed side by side, as shown in Fig. P2-33. Find the position e of the load if each material is to be subjected to a uniform compressive stress. $E_a = 10 \times 10^6$ psi; $E_b = 15 \times 10^6$ psi.

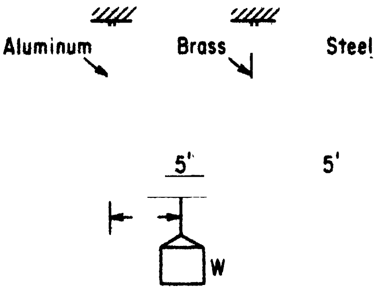


Fig. P2-32

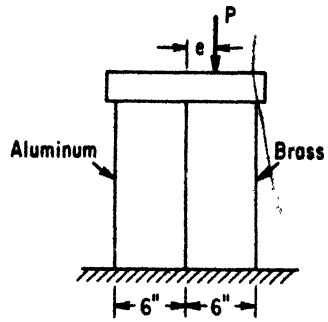


Fig. P2-33

2-34. Determine the decreases in diameter of a solid circular brass bar 2 in. in diameter that is subjected to a axial tensile force of 20,000 lb. $E_b = 15 \times 10^6$ psi; $\mu_b = 0.26$.

2-35. An axially loaded bar 2 in. by 2 in. by 60 in. long becomes 0.0006 in. narrower and 0.072 in. longer after loading. Determine Poisson's ratio of this material.

2-36. A cylindrical concrete test specimen 5 in. in diameter and 20 in. long is subjected to an axial compressive load of 270 kips. Precise measurements indicate the length to decrease by 0.138 in. and the circumference to increase by 0.0157 in. Determine the modulus of elasticity and Poisson's ratio of this material.

2-37. An axial force of 8000 lb was required to reach the proportional limit of an alloy specimen 0.505 in. in diameter. Careful measurements indicated the length to increase 0.0125 in. in a gage length of 10 in. and the diameter to decrease by 0.000156 in. Determine the proportional limit, the modulus of elasticity, and Poisson's ratio of this alloy.

2-38. The circular aluminum bar shown in Fig. P2-38 is to fit snugly in a steel collar when the bar is subjected to an axial compressive force of 10,000 lb. How much larger in diameter should the collar be? $E_a = 10 \times 10^6$ psi; $\mu_a = 0.33$.

2-39. A square aluminum bar 1 in. on edge by 40 in. long is subjected to an axial tensile force of 10,000 lb. Determine the change in length, the change in cross-sectional area, and the change in volume of the bar. $E_a = 10 \times 10^6$ psi; $\mu_a = 0.33$.

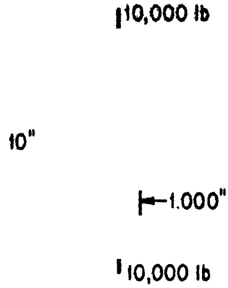


Fig. P2-38

2-40. A steel bridge spans a channel 2000 ft wide. What change in length would be caused by the change in summer and winter temperatures, if the range is $-20°F$ to 100°F? $\alpha_s = 6.5 \times 10^{-6}$ in./in./°F.

2-41. A circular hole 10.000 in. in diameter is cut in a sheet of aluminum at 0°F. By how much will the area of the hole change if the aluminum is heated to 100°F? $\alpha_a = 12.5 \times 10^{-6}$ in./in./°F.

2-42. What should be the separation between successive 40 ft steel rails to allow for expansion if they are laid when the temperature is 30°F and the maximum temperature reached is 120°F? $\alpha_s = 6.5 \times 10^{-6}$ in./in./°F.

2-43. The steel bar illustrated in Fig. P2-43 is held by the rigid supports at *A* and *B*. Determine the stress in the bar if the temperature drops 130°F. Assume the bar to be initially free of stress. $\alpha_s = 6.5 \times 10^{-6}$ in./in./°F.

2-44. The aluminum rod shown in Fig. P2-44 is held by a rigid support at *A*. If the temperature rises, the gap closes, and the rod will push against the rigid support at *B*. Determine the stress in the rod if the temperature increases by double the amount necessary to close the gap. $\alpha_a = 12.5 \times 10^{-6}$ in./in./°F. $E_a = 10 \times 10^6$ psi.

Fig. P2-43

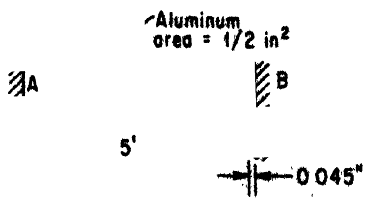


Fig. P2-44

2-45. An aluminum rod and a steel rod are fastened to non-yielding supports *A* and *B*, as shown in Fig. P2-45. Find the temperature change required to close the gap. $\alpha_s = 6.5 \times 10^{-6}$ in./in./°F; $\alpha_{al} = 12.5 \times 10^{-6}$ in./in./°F.

2-46. The composite steel and brass member shown in Fig. P2-46 is held, with no initial stress, by the rigid supports at A and B. Find the stress in each material if the temperature drops 100°F. $E_s = 30 \times 10^6$ psi; $E_b = 15 \times 10^6$ psi; $\alpha_s = 6.5 \times 10^{-6}$ in./in./°F; $\alpha_b = 10.4 \times 10^{-6}$ in./in./°F.

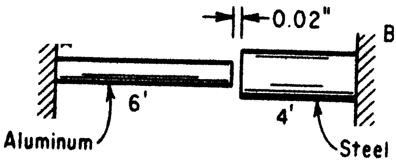


Fig. P2-45

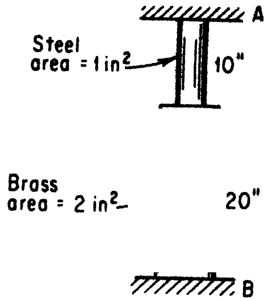


Fig. P2-46

2-47. The composite section is held by the rigid supports A and B as shown in Fig. P2-47. Determine the stress in each material if the temperature drops 60°F. Assume each of the materials to be initially free of stress (see Probs. 2-45 and 2-46 for the required data).

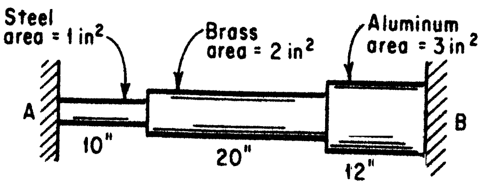


Fig. P2-47

2-48. The essential features of a temperature-actuated relay are shown in Fig. P2-48. The sensing device is a copper rod 15 in. long that is heated by a resistance coil. Through what temperature increment will the switch S be in the "open" position illustrated? $\alpha_c = 9.3 \times 10^{-6}$ in./in./°F.

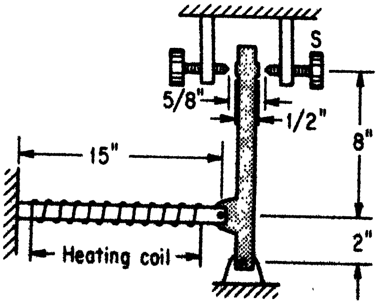


Fig. P2-48

2-49. A temperature-measuring device, Fig. P2-49, consists of an aluminum bar AB pinned to a light pointer AD. Determine the scale reading S in inches that would correspond to an incremental temperature change of 25°F. $\alpha_a = 12.5 \times 10^{-6}$ in./in./°F.

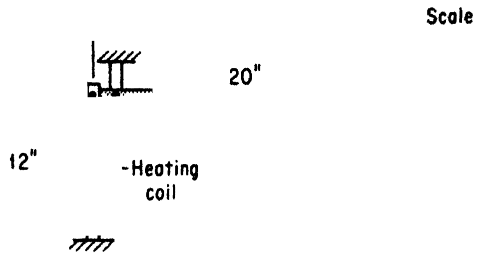


Fig. P2-49

2-50. The important features of a *differential dilatometer*, a device employed to measure the difference in linear expansion coefficients, are shown in Fig. P2-50. A light ray is deflected by mirror M, as block B rotates due to the differential expansion of the steel and aluminum rods. What equal change in temperature of both rods would cause a scale reading of $\frac{1}{4}$ in.? $\alpha_s = 6.5 \times 10^{-6}$ in./in./°F; $\alpha_a = 12.5 \times 10^{-6}$ in./in./°F.

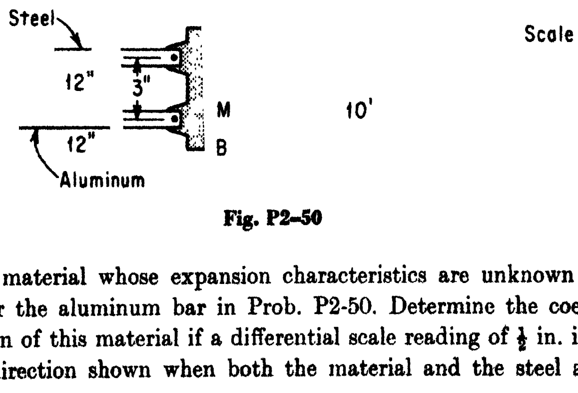


Fig. P2-50

2-51. A material whose expansion characteristics are unknown is substituted for the aluminum bar in Prob. P2-50. Determine the coefficient of expansion of this material if a differential scale reading of $\frac{1}{8}$ in. is recorded in the direction shown when both the material and the steel are heated 200°F.

2-52. Three rods of equal length support the 10,000 lb weight as shown in Fig. P2-52. What temperature change in the system would relieve the brass rod of all load? $E_s = 30 \times 10^6$ psi; $E_b = 15 \times 10^6$ psi; $\alpha_s = 6.5 \times 10^{-6}$ in./in./°F; $\alpha_b = 10.4 \times 10^{-6}$ in./in./°F.

2-53. A 4800 lb weight is supported by aluminum and steel rods, as shown in Fig. P2-53. What equal temperature change in both members will relieve the aluminum rod of all load? Assume the pinned beam AB to be rigid and weightless. $E_s = 30 \times 10^6$ psi; $E_a = 10 \times 10^6$ psi; $\alpha_s = 6.5 \times 10^{-6}$ in./in./°F; $\alpha_a = 12.5 \times 10^{-6}$ in./in./°F.

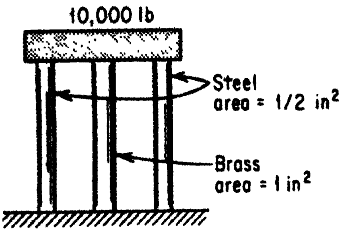


Fig. P2–52

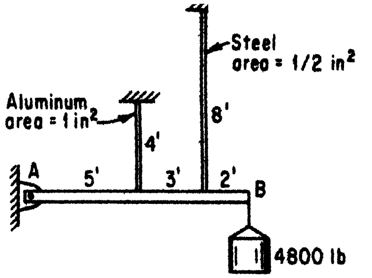


Fig. P2–53

2-54. The square column of Prob. P2-33 is shown again in Fig. P2-54. Determine the eccentricity e of a 200 kip load in order to maintain a uniform stress in each material if the temperature of both increases 120°F (see prior problems for required data).

2-55. A steel block 5 in. high. Fig. P2-55, is subjected to a shear force which deforms the block as shown. Determine the shearing strain.

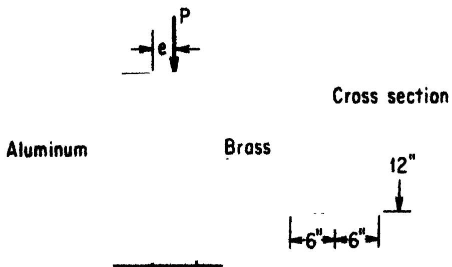


Fig. P2–54

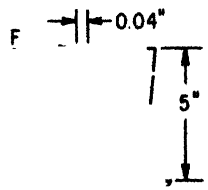


Fig. P2–55

2-56. If the block of Prob. P2-55 has a cross-sectional area of 2 sq in., what force would be required to cause a shearing deformation of 0.0005 in.? $G = 12 \times 10^6$ psi.

2-57. A brass cube 2 in. on edge is subjected to a shearing force of 20,000 lb. Determine the shearing strain that would accompany this force. $G = 6.4 \times 10^6$ psi.

2-58. Two steel blocks are deformed equally, as shown in Fig. P2-58, by rigid member AB. Determine the force P if one block has a cross-sectional area of 1 sq in. and the other a cross-sectonial area of $\frac{1}{2}$ sq in. $G = 12 \times 10^6$ psi.

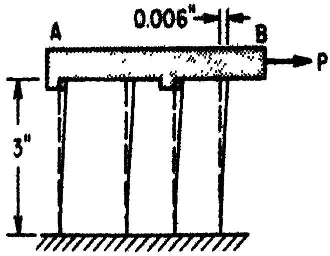


Fig. P2-58

2-59. Solve Prob. P2-58 if the larger block is aluminum and the smaller block is brass. $G_a = 3.8 \times 10^6$ psi; $G_b = 6.4 \times 10^6$ psi.

2-60. Determine Poisson's ratio of manganese bronze based on values of E and G given in Appendix B.

2-61. A particular metal has a Poisson's ratio of 0.28 and a modulus of elasticity of 10.3×10^6 psi. Find the modulus of rigidity.

2-62. What is the ratio of the shear modulus to the elastic modulus if Poisson's ratio is 0.25?

expressed as a function of torque, length, rigidity, and polar moment of inertia.

$$\theta = \frac{TL}{GJ} \tag{3-1}$$

An expression for maximum shearing stress is next obtained by regrouping the terms in the previous derivation.

$$\tau_{max} = G\gamma = G \cdot \frac{c\theta}{L} = G \cdot \frac{c}{L} \cdot \frac{TL}{GJ}$$

$$\tau_{max} = \frac{Tc}{J} \tag{3-2}$$

By replacing J by its appropriate value, detailed equations may be obtained for solid and hollow sections.

For solid circular sections:

$$J = \frac{\pi d^4}{32}$$

$$\theta = \frac{TL}{G} \frac{32}{\pi d^4} \tag{3-3}$$

$$\tau_{max} = \frac{Td/2}{\pi d^4/32} = \frac{16T}{\pi d^3} \tag{3-4}$$

For hollow circular sections:

$$J = \frac{\pi}{32}(d_o^4 - d_i^4)$$

$$\theta = \frac{TL}{G} \frac{32}{\pi(d_o^4 - d_i^4)} \tag{3-5}$$

$$\tau_{max} = \frac{16Td_o}{\pi(d_o^4 - d_i^4)} \tag{3-6}$$

The examples that follow illustrate the use of the equations of torsional stress and strain. Of particular interest are those problems which are statically indeterminate.

Example 1. A core drill consists of a hollow circular steel shaft 50 ft long with a 2 in. outside diameter and a 1.5 in. inside diameter. If a twisting moment of 10,000 lb in. acts on the shaft, find the maximum shearing stress and the angle of twist. The modulus of rigidity of steel is $G = 12 \times 10^6$ psi.

Solution: To compute the stress, numerical values are substituted into Eq. (3-6).

$$\tau_{max} = \frac{16Td_o}{\pi(d_o^4 - d_i^4)} = \frac{16(10{,}000)(2)}{\pi(2^4 - 1.5^4)}$$

$$9310 \text{ psi}$$

The angle of twist is computed by substituting numerical values into Eq. (3-5).

$$\theta = \frac{TL}{G}\frac{32}{\pi(d_o^4 - d_i^4)} \quad \frac{(10{,}000)\,(50 \times 12)32}{12 \times 10^6\pi(2^4 - 1.5^4)}$$

$$= 0.466 \text{ rad}$$

$$= 0.466(57.3) = 26.7 \text{ deg}$$

Example 2. Find the safe torque that may be applied to a solid steel shaft 4 in. in diameter. The working stress in shear is 5000 psi, and the allowable angle of twist per foot of length is 1/10 deg.

Solution: Two values of torque must be computed: one based on stress and the other on deformation. The smaller of the two will be the safe torque. By Eq. (3-4),

$$\tau_{max} \quad \frac{16T}{\pi d^3}$$

Where

$$\frac{\tau_{max}\pi d^3}{16} = \frac{5000\,\pi(4)^3}{16}$$

$$= 62{,}800 \text{ lb in.}$$

By Eq. (3-3),

$$\theta \quad \frac{TL}{G}\frac{32}{\pi d^4}$$

Solving for T gives

$$T = \frac{\theta G}{L}\frac{\pi d^4}{32} = \left(\frac{1}{10} \times \frac{\pi}{180}\right)\left(\frac{12 \times 10^6}{12}\right)\left(\frac{\pi 4^4}{32}\right)$$

$$= 43{,}800 \text{ lb in.}$$

The lesser of the two values, $T = 43{,}800$ lb in., is the permissible torque for this shaft.

Example 3. The composite shaft shown in Fig. 3-2 is held between rigid supports at A and C. Find the torque in section AB and the torque in section BC. $G_s = 12 \times 10^6$ psi; $G_b = 6.4 \times 10^6$ psi.

Solution: This problem is *statically indeterminate.* Only one equation of equilibrium $\sum M = 0$ can be written, and in this equation are the unknowns T_A and T_B. The sum of these two torques is the applied torque.

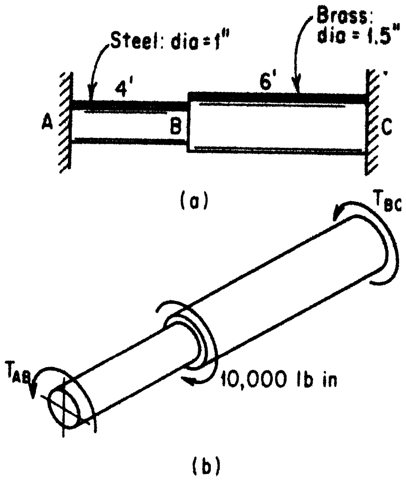


Fig. 3-2

$$T_A + T_B = 10{,}000 \qquad \text{(a)}$$

Since the torque acts at the junction of the two shafts, the twists in the two sections are equal. This equality provides a second equation involving T_A and T_B.

$$\theta_{AB} = \theta_{BC}$$

$$\left(\frac{TL}{G}\frac{32}{\pi d^4}\right)_{AB} = \left(\frac{TL}{G}\frac{32}{\pi d^4}\right)_{BC}$$

Like terms are canceled and numerical data substituted:

$$\frac{T_{AB}(4 \times 12)}{(12 \times 10^6)(1)^4} \qquad \frac{T_{BC}(6 \times 12)}{(6.4 \times 10^6)(1.5)^4}$$

$$T_{AB} = 0.547T_{BC} \qquad \text{(b)}$$

Equations (a) and (b) are then combined:

$$T_{BC}(0.547 + 1) = 10{,}000$$

$$T_{BC} = 6460 \text{ lb in.}$$

and

$$T_{AB} = (0.547)6460 = 3540 \text{ lb in.}$$

Example 4. Two steel shafts are gear-connected, as shown in Fig. 3-3. Find the angle of twist of end D if shaft CD is subjected to a twisting moment of 10,000 lb in. $G = 12 \times 10^6$ psi.

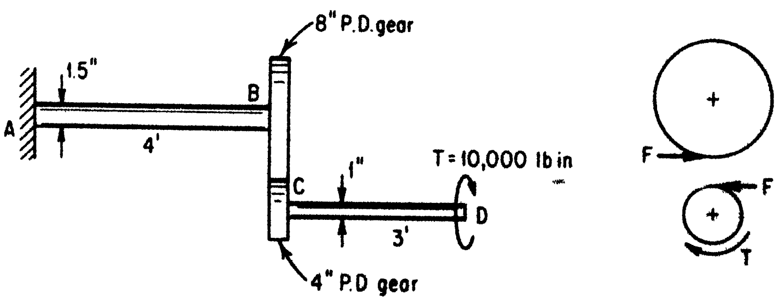


Fig. 3-3

Solution: The torque that acts in shaft AB is twice as great as that in shaft CD. This is disclosed by the free-body diagram of the two gears. The mutual contact force F can be computed in terms of the applied torque:

$$F_r = T_{CD}$$
$$F(2) = 10{,}000 \text{ lb in.}$$
$$F = 5000 \text{ lb}$$

The moment produced by this force in the larger gear is double, since it acts on a lever arm double in length.

$$T_{AB} = F_r' = 5000(4) = 20{,}000 \text{ lb in.}$$

A study of the geometry of motion also indicates the angle of rotation of gear C to be twice that of gear B; thus

$$\theta_C = 2\theta_B$$

The total rotation at D, then, is the angle of twist in shaft CD added to twice the angle of twist of shaft AB.

$$\theta_D = \theta_{CD} + 2\theta_{AB}$$

By Eq. (3-3),

$$\theta_D = \left(\frac{TL}{G} \times \frac{32}{\pi d^4}\right)_{CD} + 2\left(\frac{TL}{G} \times \frac{32}{\pi d^4}\right)_{AB}$$

$$\theta_D = \frac{10{,}000(3 \times 12)32}{(12 \times 10^6)\pi(1)^4} + 2\frac{(20{,}000)(4 \times 12)32}{(12 \times 10^6)\pi(1.5)^4}$$
$$= 0.628 \text{ rad or } 36 \text{ deg}$$

3-2 Power Transmission

There are many applications in which shafts must be designed to transfer power to various machines. Computations which involve the determination of proper shaft sizes can be easily made, since power and torque are related by the kinetic equation:

$$P = T\omega$$

In this relationship power P has the units of pound feet per second, torque T the units of pound feet, and angular velocity ω the units of radians per second. The equation can be rewritten in terms of horsepower (hp) and revolutions per minute (n), since

$$1 \text{ hp} = 550(12)\frac{\text{lb in.}}{\text{sec}}$$

and

$$\omega = \frac{2\pi n}{60}$$

thus

$$\text{hp} = \frac{Tn}{550(12)} \times \frac{2\pi}{60} = \frac{Tn}{63{,}000} \qquad (3\text{-}7)$$

Example 5. Design a solid steel shaft to transmit 250 hp at 1800 rpm. The maximum allowable shearing stress is 5000 psi.

Solution: By Eq. (3-7),

$$T = \frac{63{,}000(\text{hp})}{n} = \frac{63{,}000(250)}{1800} = 8750 \text{ lb in.}$$

The relationship between shearing stress and torque is given by Eq. (3-4).

$$\frac{16T}{\pi d^3}$$

Solving for the diameter d gives

$$d = \sqrt[3]{\frac{16T}{\pi\tau}} \quad \sqrt[3]{\frac{16(8750)}{\pi(5000)}} = 2.07 \text{ in.}$$

Example 6. A solid 2-in.-diameter steel line-shaft, Fig. 3-4(a), is driven by a 50 hp motor at a speed of 315 rpm. Power take-offs are situated at A and C. Determine the angle of twist of gear C relative to gear A. Assume $G_s = 12 \times 10^6$ psi.

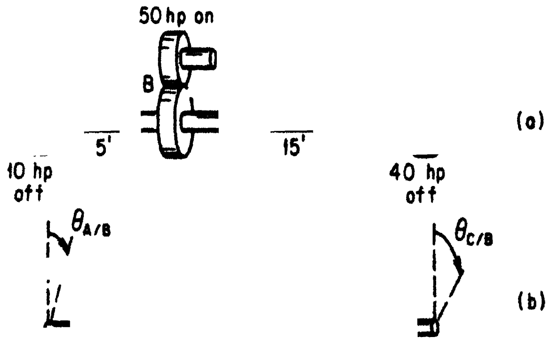


Fig. 3-4

Solution: The torque in segments AB and BC is computed by Eq. (3-7):

$$T_{AB} = \frac{63{,}000(\text{hp})}{n} = \frac{63{,}000(10)}{315} = 2000 \text{ lb in.}$$

$$T_{BC} = \frac{63{,}000(\text{hp})}{n} = \frac{63{,}000(40)}{315} = 8000 \text{ lb in.}$$

To determine the twist of gear C relative to gear A (written as $\theta_{C/A}$), imagine B to be fixed, Fig. 3-5(b), and compute the twists of gears A and C. Since these gears rotate in the same direction relative to B, their rotation relative to each other is the difference $\theta_{C/B} - \theta$

By Eq. (3-3),

$$\theta_{C/A} = \theta_{C/B} - \theta_{A/B}$$

$$\theta_{C/B} = \frac{32}{\pi d^4}\frac{TL}{G} = \frac{32(8000)\,(15 \times 12)}{\pi(2)^4(12 \times 10^6)} = 0.0764 \text{ rad}$$

$$= 0.0764(57.3) = 4.38 \text{ deg}$$

and

$$\theta_{A/B} = \frac{32}{\pi d^4} \times \frac{TL}{G} = \frac{32(2000)\,(5 \times 12)}{\pi(2)^4(12 \times 10^6)} = 0.00637 \text{ rad}$$

$$= 0.00637(57.3) = 0.365 \text{ deg}$$

Therefore,

$$\theta_{C/A} = 4.38 - 0.365 = 4.015 \text{ deg}$$

3-3 *Torsion Bars*

There are countless applications in the field of design in which torsion members can be employed as springs; the *torsion bar* is one of the many examples. The *spring constant*, or *spring rate*, as it is sometimes called, is the ratio of torque to twist expressed in pound inches per radian (lb in./rad). Since the twist in radians is in itself a function of torque, the spring constant k is equal to

$$k = \frac{T}{\theta} = \frac{T}{TL/GJ} = \frac{GJ}{L} \tag{3-8}$$

For solid sections and for hollow sections k has the detailed value of

$$k_s = \frac{\pi d^4}{32}\frac{G}{L} \tag{3-9}$$

and

$$k_h = \frac{\pi(d_o^4 - d_i^4)}{32}\frac{G}{L} \tag{3-10}$$

Two or more torsion bars can be coupled to form a spring more flexible than any of those combined, a *series combination*, or stiffer than any of those combined, a *parallel combination*.

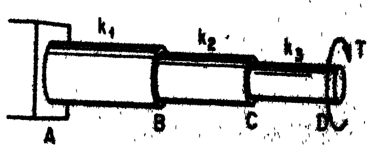


Fig. 3-8

Consider three torsion bars connected end to end, as shown in Fig. 3-5. The internal reaction in each segment is the applied torque T, and the angle of twist of D relative to A is the sum

$$\theta_{D/A} = \theta_{B/A} + \theta_{C/B} + \theta_{D/C}$$

Since θ for any segment is T/k, it follows that

$$\theta_{D/A} = \frac{T}{k_1} + \frac{T}{k_2} + \frac{T}{k_3}$$

Dividing by T gives

$$\frac{\theta_{D/A}}{T} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

The reciprocal of this expression is the equivalent constant k_e of the system

$$k_e = \frac{T}{\theta_{D/A}} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2} + \dfrac{1}{k_3}} \tag{3-11}$$

Thus, *torsion bars are in series if the same torque acts in each. The equivalent spring constant of a series system is equal to the reciprocal of the sum of the reciprocals of the individual spring constants.*

Example 7. In Fig. 3-5, let k_1, k_2, and k_3 be 6×10^6 lb in./rad, 3×10^6 lb in./rad, and 2×10^6 lb in./rad respectively. Find the equivalent spring constant of the system and the angle of twist $\theta_{D/A}$ if $T = 10{,}000$ lb in.

Solution: By Eq. (3-11),

$$k_e = \frac{1}{\dfrac{1}{k_1}+\dfrac{1}{k_2}+\dfrac{1}{k_3}} = \frac{1}{\dfrac{1}{6\times 10^6}+\dfrac{1}{3\times 10^6}+\dfrac{1}{2\times 10^6}} = \frac{10^6}{\dfrac{1}{6}+\dfrac{1}{3}+\dfrac{1}{2}}$$

$$= 10^6 \text{ lb in./rad}$$

and

$$\theta_{D/A} = \frac{T}{k_e} = \frac{10{,}000}{10^6} = 0.01 \text{ rad}$$

$$= 0.573 \text{ deg.}$$

Torsion bars are *parallel-connected* when their angular displacements are equal. The connected bars illustrated in Fig. 3-6 represent a parallel system; the bars share the torque while twisting equally. The applied torque T is the sum

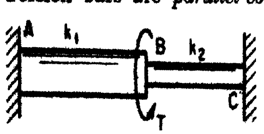


Fig. 3-6

$$T = T_{AB} + T_{BC}$$

The torques T_{AB} and T_{BC} can be expressed in terms of the spring constants k_1 and k_2:

$$T = k_1\theta + k_2\theta$$

Dividing by the angle of twist gives

$$\frac{T}{\theta} = k_1 + k_2$$

where T/θ is the equivalent spring constant k_e.

$$k_e = k_1 + k_2 \tag{3-12}$$

Formally stated: *Torsion bars are in parallel if they have equal angles of twist and an applied torque is apportioned between them; the equivalent spring constant of such a system is equal to the sum of the individual spring constants.*

Example 8. In Fig. 3-6, let $k_1 = 2 \times 10^6$ lb in./rad and $k_2 = 3 \times 10^6$ lb in./rad. Find the equivalent spring constant for the system and the angle of twist at B. Let the torque T equal 20,000 lb in.

Solution: By Eq. (3-12),

$$k_e = k_1 + k_2 = (2 + 3)10^6 = 5 \times 10^6 \text{ lb in./rad}$$

$$\theta_B = \frac{T}{k_e} = \frac{20{,}000}{5 \times 10^6} \quad 0.004 \text{ rad} = 0.229 \text{ deg}$$

3-4 Helical Springs

The torsion theory can be expanded to include a rather interesting and important application, the design of close-coiled helical springs. One of the primary functions of a spring, of course, is to store or release energy; this it must do without becoming overstressed. Consider the tension spring acted upon by a force P as shown in Fig. 3-7(a). It is presumed that the spring has been made by winding a helical coil of a solid circular wire of diameter d on a mandril of diameter D_m. The mean radius R of the coil, a term that appears in the derivation that follows, is

$$R = \frac{D_m + d}{2}$$

A free-body diagram of the upper portion of the spring, Fig. 3-7(b), indicates that both a shear force P and a torque $P \times R$ are required at the cut section to maintain equilibrium. Each of these two internal reactions results in a shearing stress. The first, $\tau_1 = P/A$ is distributed uniformly over the area as shown in Fig. 3-7(c), whereas the second $\tau_2 = Tc/J$ varies uniformly with distance from the center of the wire, Fig. 3-7(d). By superimposing the stress patterns, Fig. 3-7(e), it can be seen that the greatest stress in the spring occurs at the innermost portion of the coil; it is here that the stresses τ_1 and τ_2 add directly to give τ_{max}.

$$\tau_{max} = \tau_1 + \tau_2 = \frac{P}{A} + \frac{Tc}{J}$$

$$\frac{4P}{\pi d^2} + \frac{16PR}{\pi d^3} = \frac{16PR}{\pi d^3}\left(1 + \frac{d}{4R}\right) \qquad (3\text{-}13)$$

When the wire diameter is small compared to the mean radius, the term $d/4R$ can be neglected. In this case the stress in the spring is caused principally by the torsional load. In heavy coil springs, however, the term $d/4R$ *cannot* be ignored. In this instance the direct stress is appreciable.

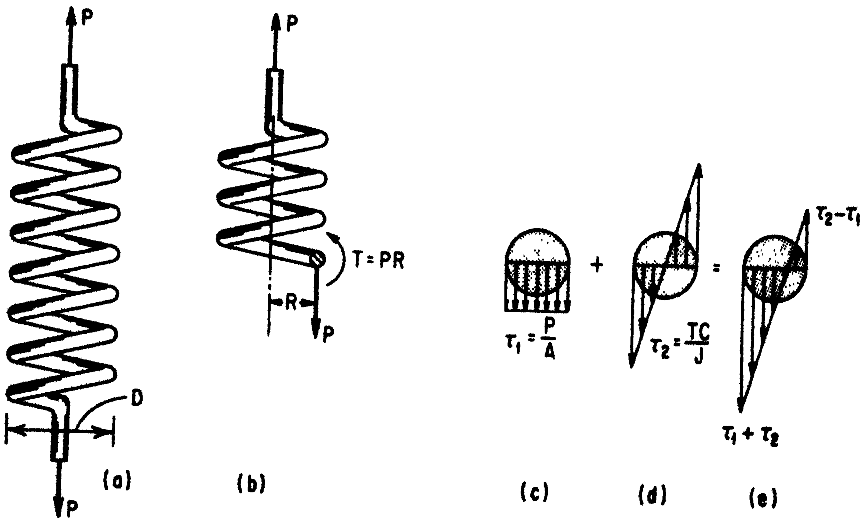


Fig. 3-7

Considerable liberty was taken in the derivation with the expression Tc/J. In theory this term applies only to straight circular bars; here it is used to describe the shearing stress in a curved wire. The derivation, therefore, is in error and particularly so when the springs are heavy and closely coiled. A corrected equation which gives a true picture of the maximum stress states that

$$\tau_{max} = K\left(\frac{16PR}{\pi d^3}\right) \qquad (3\text{-}14)$$

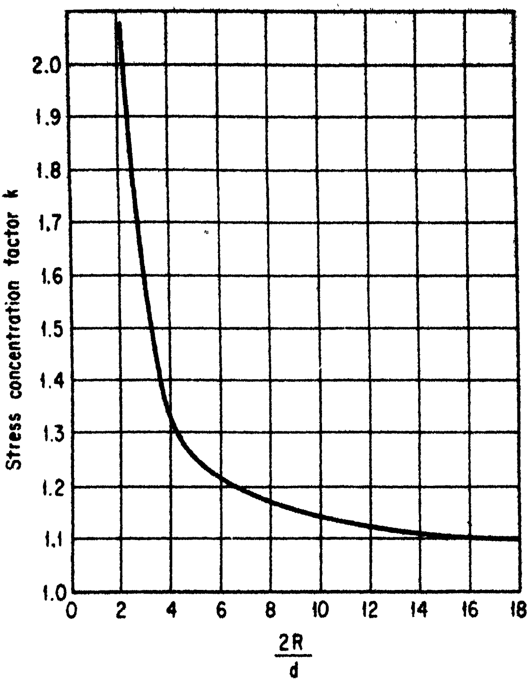


Fig. 3-8

Where K, *the stress concentration factor*,[1] is a function of the ratio $2R/d$. The value of K can be obtained from the graph of Fig. 3-8.

The deflection of a coil spring is caused, for the most part, by twisting rather than by direct shear. To obtain a relationship for the deflection, one can imagine that the spring, consisting of n coils of wire, is straightened into a shaft $2\pi Rn$ in length, as shown in Fig. 3-9. The deflection δ caused by the

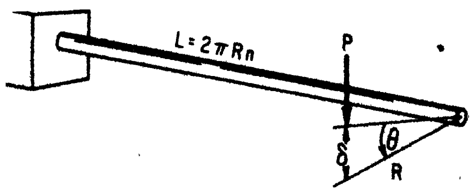


Fig. 3-9

[1] The constant K is known as *Wahl's correction factor*. For a complete discussion of spring design, see A. M. Wahl, *Mechanical Springs*, (Cleveland, Ohio: Penton Publishing Co., 1944).

rotation of the shaft is approximately

$$\delta = R\theta$$

and, since $\theta = TL/GJ$,

$$\delta = R \times \frac{TL}{GJ} = \frac{R \times PR \times 2\pi Rn}{G \times \pi d^4/32}$$

$$= \frac{64PR^3n}{Gd^4} \qquad (3\text{-}15)$$

A spring constant can now be defined in terms of force and deflection.

$$k = \frac{P}{\delta} = \frac{Gd^4}{64R^3n} \qquad (3\text{-}16)$$

A *series spring system* is one in which an equal force acts in each spring, and the total deflection is the sum of the individual deflections. For the system shown in Fig. 3-10(a), the total deflection is

$$\delta = \delta_1 + \delta_2 + \delta_3$$

where

$$\delta = \frac{F}{k_1} + \frac{F}{k_2} + \frac{F}{k_3}$$

Dividing both sides of the equation by F gives

$$\frac{\delta}{F} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

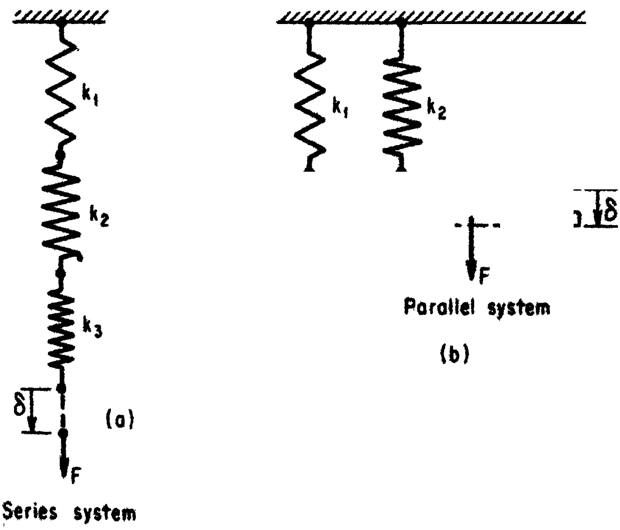


Fig. 3-10

Since δ/F is the reciprocal of an equivalent spring whose constant is k_e, it follows that

$$k_e = \frac{1}{\dfrac{1}{k_1}+\dfrac{1}{k_2}+\dfrac{1}{k_3}} \tag{3-17}$$

In a *parallel system* the springs deform equally as they each share a portion of the load. The equivalent constant in this case is the sum of the individual constants. To prove this, consider the system shown in Fig. 3-10(b). If the bar is to remain horizontal, each spring will support a share of the load.

$$F = F_1 + F_2 + F_3 = k_1\delta + k_2\delta + k_3\delta$$

Dividing this equation by the deflection δ gives an expression for the equivalent spring constant.

$$\frac{F}{\delta} = k_e = k_1 + k_2 + k_3 \tag{3-18}$$

In many instances the *equivalent spring* concept is a helpful aid in setting up and solving spring problems.

Example 9. A close-coiled helical spring, made of 10 coils of solid 1-in.-diameter steel wire, has a mean coil diameter of 4 in. Determine the maximum stress, corrected for stress concentration, and the elongation of the spring if it supports an axial load of 1200 lb.

Solution: The concentration factor K is found from the graph of Fig. 3-8.

$$\frac{2R}{d} = \frac{2(2)}{1} = 4$$

$$K = 1.33$$

To compute the stress, numerical data are substituted into Eq. (3-14).

$$\tau_{max} = K\left(\frac{16PR}{\pi d^3}\right) = \frac{1.33(16)1200(2)}{\pi(1)^3} = 16{,}300 \text{ psi}$$

The deflection, by Eq. (3-15), is

$$\delta = \frac{64PR^3n}{Gd^4} = \frac{64(1200)\,(2)^3(10)}{12 \times 10^6(1)^4} = 0.512 \text{ in.}$$

Example 10. Four identical steel springs are joined together, as shown in Fig. 3-11. Each spring has 10 coils of 0.2-in.-diameter wire wound to a mean radius of 2 in. Find the elongation δ if a 10 lb force F is applied as shown. Assume the bar to remain horizontal.

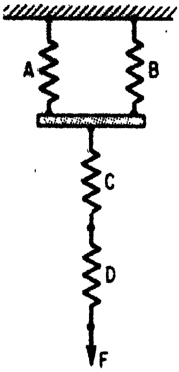


Fig. 3-11

Solution: The constant of each of the springs, by Eq. (3-16), is

$$k = \frac{Gd^4}{64R^3n} = \frac{12 \times 10^6(0.2)^4}{64(2)^3 10} = 3.75 \text{ lb per in.}$$

The parallel combination of springs A and B is series connected to springs C and D. The equivalent spring constant is

$$k_e = \frac{1}{\dfrac{1}{k+k} + \dfrac{1}{k} + \dfrac{1}{k}} \quad \frac{1}{\dfrac{1}{2k} + \dfrac{2}{k}} = \frac{2}{5}k = 0.4k$$

Thus, the system behaves as a single spring having a constant of 0.4 × 3.75. The deflection is simply F/k_e:

$$\delta = \frac{F}{k_e} = \frac{10}{0.4(3.75)} = 6.67 \text{ in.}$$

3-5 Shaft Couplings

There are many ways in which power can be transmitted directly from one shaft to another; the flanged and bolted coupling shown in Fig. 3-12 illustrates one very practical method. To analyze the strength requirements

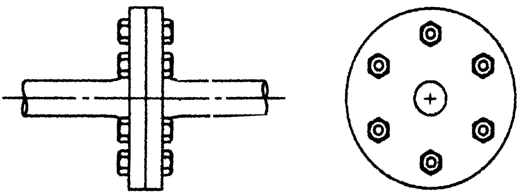


Fig. 3-12

of the coupling, several assumptions must be made. First, it is assumed that torque is transmitted by the bolts alone, and second, that the stress within the bolts is uniform. If more than one circle of bolts is used, it is also assumed that the stress in any circle varies directly with the radius.

To illustrate, consider a free-body diagram of one face of a coupling, Fig. 3-13. To retain equilibrium, each of the n bolts must help to balance the external torque T.

$$T = nFr \tag{a}$$

Since each bolt is in direct shear, the force F is

$$F = \tau A \tag{b}$$

where A represents the cross-sectional area of a single bolt. Combining Eqs. (a) and (b) and solving for the stress gives

$n\tau Ar$

$$\tau = \frac{T}{rAn} \tag{3-19}$$

If the bolts in a coupling are arranged in two or more concentric circles, the shear-

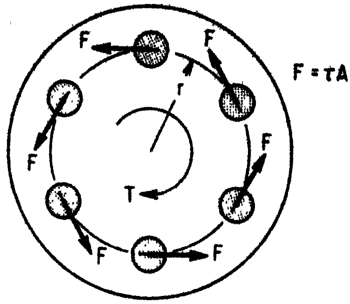


Fig. 3-13

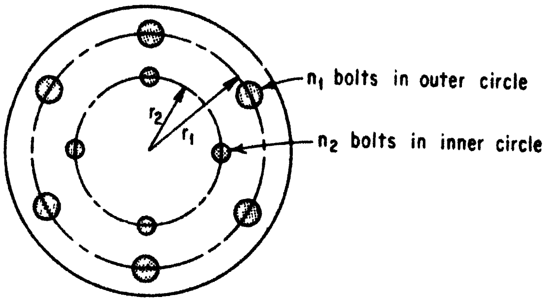


Fig. 3-14

ing stress decreases proportionally with the radius. In Fig. 3-14, n_1 bolts are arranged in circle r_1, and n_2 bolts in circle r_2. To maintain equilibrium, the internal and external torques must balance.

$$T = (rnA\tau)_1 + (rnA\tau)_2 \tag{a}$$

where

$$\frac{\tau_1}{r_1} = \frac{\tau_2}{r_2} \tag{b}$$

Equations (a) and (b) can be combined to give an expression for the maximum stress τ_1.

$$\tau_1 = \frac{T}{(rnA)_1 + \dfrac{r_2}{r_1}(rnA)_2} \tag{3-20}$$

It is interesting to note that if r_2 is set equal to zero, and the inner circle of bolts is thus eliminated, Eqs. (3-19) and (3-20) are identical.

Although Eq. (3-20) can prove to be a useful relationship, it is often simpler to substitute data directly into Eqs. (a) and (b).

Example 11. Ten $\frac{1}{2}$-in.-full-diameter bolts are arranged in two concentric circles in a flanged coupling similar to that shown in Fig. 3-14. Find the maximum horsepower that can be transmitted by the coupling if the shaft speed is 315 rpm and the maximum permissible shearing stress is 5000 psi. $r_1 = 8$ in. and $r_2 = 4$ in.

Solution: Shearing stress is directly proportional to the radii; hence, for the inner circle,

$$\tau_2 = \frac{r_2}{r_1}\tau_1 = \frac{4}{8}(5000) = 2500 \text{ psi}$$

To maintain equilibrium, the internal and external moments must balance.

$$T = (rna\tau)_1 + (rnA\tau)_2$$

$$= 8(6)\frac{\pi}{4}\left(\frac{1}{2}\right)^2 5000 + 4(4)\frac{\pi}{4}\left(\frac{1}{2}\right)^2 2500$$

$$= 55{,}000 \text{ lb in.}$$

The permissible horsepower, given by Eq. (3-7), is

$$\text{hp} = \frac{Tn}{63{,}000} = \frac{55{,}000(315)}{63{,}000} = 275 \text{ hp}$$

PROBLEMS

3-1. Determine the torque that can be applied to a 1-in.-diameter shaft if the shearing stress is not to exceed 8,000 psi.

3-2. A 2-in.-diameter steel shaft is subjected to a torque that produces a maximum shearing stress of 10,000 psi. Determine the angle of twist in 50 ft of this shaft. $G_s = 12 \times 10^6$ psi.

3-3. Determine the maximum shearing stress developed in a hollow shaft that is subjected to a torque of 20,000 lb in. The shaft has an outside diameter of 2 in. and an inside diameter of 1.5 in.

3-4. Compare the strengths of two shafts, one solid and the other hollow. Both shafts have an outside diameter of D, and the hollow shaft has an inside diameter of $D/2$.

3-5. A solid steel shaft 2 in. in diameter and 10 ft long is acted upon by torque that produces a shearing stress of 5,000 psi within the bar at a radius of $\frac{1}{2}$ in. Find the torque T and the angle of twist of the shaft. $G_s = 12 \times 10^6$ psi.

3-6. A solid steel shaft having a diameter of 3 in. twists through an angle of 5 deg in 20 ft of length because of the action of a torque. Determine the maximum shearing stress in the shaft. $G_s = 12 \times 10^6$ psi.

3-7. What is the minimum diameter of a solid steel shaft that will not twist through more than 2 deg in 10 ft of length when subjected to a torque of 10,000 lb in.? $G_s = 12 \times 10^6$ psi.

3-8. Determine the total angle of twist of the brass stepped shaft shown in Fig. P3-8. $G_b = 6.4 \times 10^6$ psi.

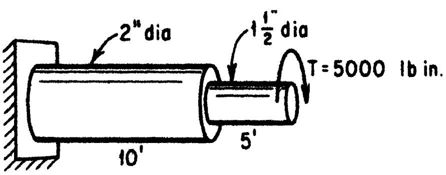


Fig. P3-8

3-9. The composite shaft shown in Fig. P3-9 is rigidly supported at A and at C. Determine the stress in the brass and in the steel if the shaft is subjected to a torque $T = 50{,}000$ lb in. $G_s = 12 \times 10^6$ psi; $G_b = 6.4 \times 10^6$ psi.

3-10. The composite shaft shown in Fig. P3-10 is rigidly supported at A and C. If the maximum permissible shearing stress in the brass is 8000 psi, and in the aluminum 4000 psi, find the ratio of lengths a/b so that each shaft is stressed to its limit when subjected to the torque T. $G_b = 6.4 \times 10^6$ psi; $G_a = 3.8 \times 10^6$ psi.

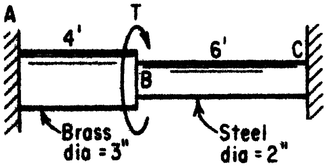


Fig. P3-9

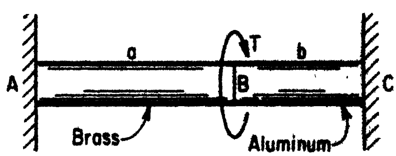


Fig. P3-10

3-11. A solid steel shaft is inserted in a hollow brass tube as shown in Fig. P3-11. The shaft and the tube are securely fastened at each end. Find the angle of twist of the coupled system if the applied torque is $T = 1000$ lb in. $G_b = 6.4 \times 10^6$ psi; $G_s = 12 \times 10^6$ psi.

3-12. Find the maximum permissible torque T in Prob. 3-11 if the shearing stress in the brass must not exceed 6000 psi and in the steel, 10,000 psi.

3-13. A torque of 2000 lb in. is applied to end B of the geared 1-in.-diameter shafts as shown in Fig. P3-13. Determine the torque at A necessary to maintain equilibrium and the angle of twist of end B with respect to end A. Assume the shafts to be supported by bearings to prevent bending. $G_b = 6.4 \times 10^6$ psi; $G_s = 12 \times 10^6$ psi.

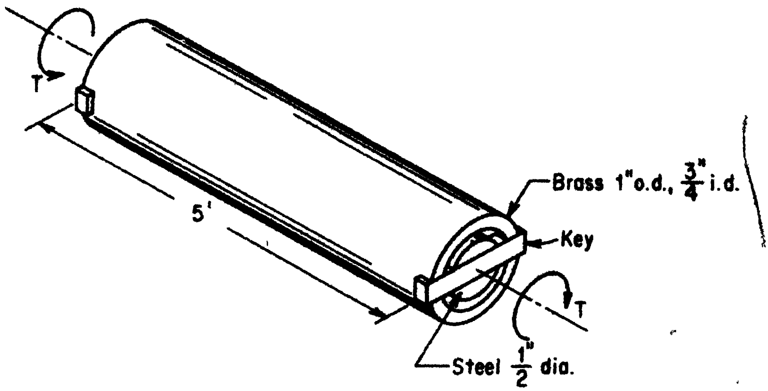


Fig. P3–11

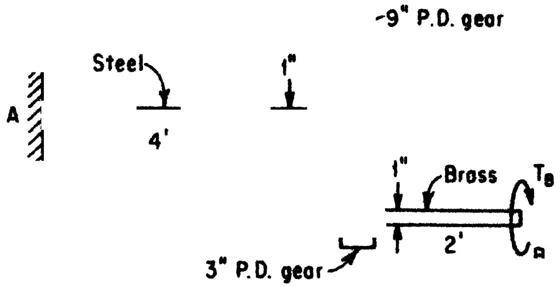


Fig. P3–13

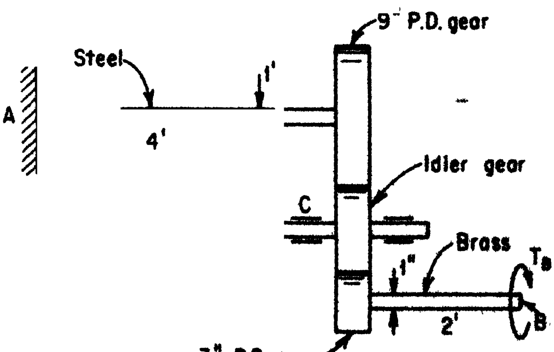


Fig. P3–14

3-14. Fig. P3-14 illustrates a shaft and gear arrangement similar to that of Prob. 3-13, except that an idler gear C is employed to change the direction of rotation. Determine the angle of twist of end B with respect to end A. A torque of 2000 lb in. is applied at B.

3-15. Shafts A, B, and C are interconnected by three identical bevel gears as shown in Fig. P3-15. Find the torques $T_{A'}$ and $T_{B'}$ supplied by the respective supports A' and B' if a torque of 25,000 lb in. acts as shown. The three shafts are steel and are supported to prevent bending.

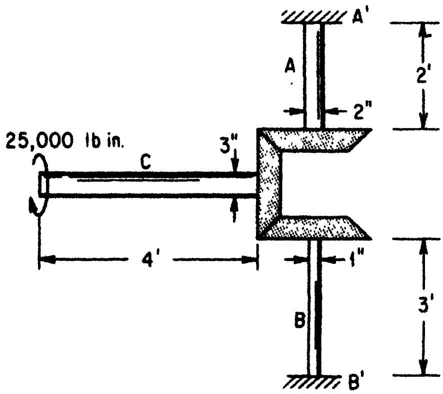


Fig. P3-15

3-16. Determine the maximum horsepower a solid steel shaft 2 in. in diameter can transmit at 200 rpm if the maximum permissible shearing stress is 10,000 psi.

3-17. A solid steel shaft is to transmit 50 hp at 500 rpm. Determine the necessary diameter if it is not to twist more than 2 deg per foot or be stressed more than 10,000 psi in shear. $G_s = 12 \times 10^6$ psi.

3-18. Determine the horsepower that a $\frac{1}{4}$-in.-diameter steel shaft can transmit at a speed of 10,000 rpm if the working stress in shear is 8000 psi. $G_s = 12 \times 10^6$ psi.

3-19. A hollow steel shaft with an 18 in. outside diameter and a 10 in. inside diameter is to transmit 15,000 hp. The shearing stress is not to exceed 8000 psi. Determine the permissible speed of the shaft. Is this speed a maximum or a minimum value?

3-20. A motor delivers 50 hp at 630 rpm to one end of a steel line-shaft 2 in. in diameter and 10 ft long. The power is used to drive two machines, one at the midpoint of the shaft consuming 30 hp and the other at the extreme end consuming 20 hp. Determine the maximum shearing stress in the shaft and the relative angle of twist between the two extreme ends of the shaft. $G_s = 12 \times 10^6$ psi.

3-21. A solid 2-in.-diameter steel shaft is driven at a speed of 210 rpm by a 60 hp motor, as shown in Fig. P3-21. Power take-offs are located at A and B. Determine the torsional stresses in shafts AB and BC, and the angle of twist of gear A relative to gear C. $G_s = 12 \times 10^6$ psi.

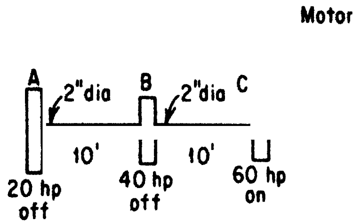


Fig. P3-21

3-22. A steel line-shaft is driven at a speed of 315 rpm by a 100 hp motor at C, as shown in Fig. P3-22. Determine the minimum diameters of the three shaft segments if the shearing stress in any segment is not to exceed 8000 psi.

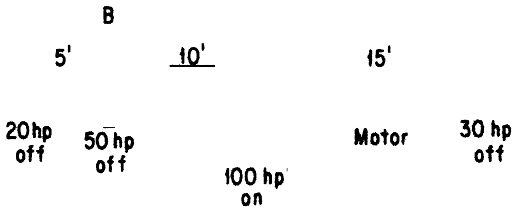


Fig. P3-22

3-23. Assume the entire line-shaft of Prob. 3-22 to be 2 in. in diameter, and compute the angular twist of gear A relative to gear D. $G_s = 12 \times 10^6$ psi.

3-24. Determine the torsional spring constant of a 2-in.-diameter steel shaft 10 ft long. $G_s = 12 \times 10^6$ psi.

3-25. Find the length of a 3-in.-diameter brass shaft that has a torsional spring constant of 2×10^6 lb in./rad. $G_b = 6.4 \times 10^6$ psi.

3-26. Compute a factor that will convert kip feet per degree to pound inches per radian.

3-27. Find the length of a 1-in.-diameter steel shaft that has the same spring constant as a 2-in.-diameter brass shaft 20 ft long. $G_s = 12 \times 10^6$ psi; $G_b = 6.4 \times 10^6$ psi.

3-28 through **3-33.** Determine the equivalent spring constants of the systems shown in Figs. P3-28–P3-33 and the angles of twist at the point of application of the torque if for each case $T = 10{,}000$ lb in.

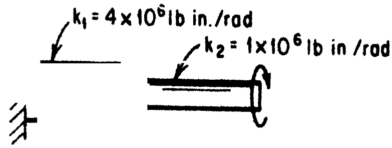


Fig. P3-28

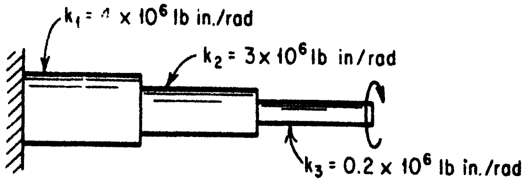


Fig. P3-29

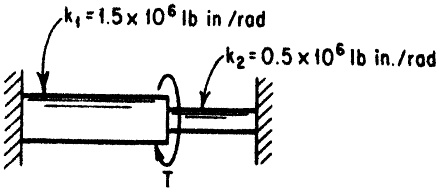


Fig. P3-30

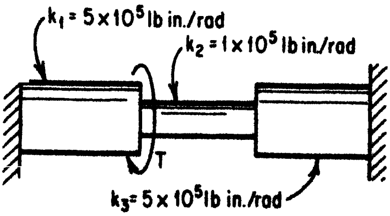


Fig. P3-31

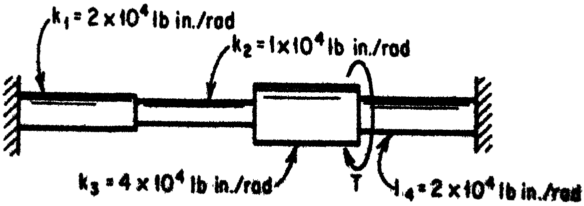


Fig. P3-32

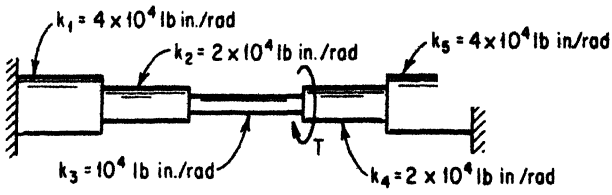


Fig. P3-33

3-34. A torsion bar having a spring constant k is cut into three equal lengths. What is the spring constant of each portion?

3-35. Solve Prob. 3-8 by the method of equivalent springs.

3-36. Solve Prob. 3-11 by the method of equivalent springs.

3-37. Solve Prob. 3-13 by the method of equivalent springs.

3-38. Solve Prob. 3-15 by the method of equivalent springs.

Problems 3-39 through 3-44 are problems in helical spring design in which certain data are given. Determine the unknown quantities for each problem. *Note:* Where the stress is one of the included quantities, use Eq. 3-13.

	P (lb)	R (in.)	d (in.)	n	G (psi $\times 10^6$)	τ_{max} (psi)	δ (in.)	k (lb/in.)
3-39.	100	3	1	10	12			
3-40.	10		$\frac{1}{2}$	10	6.4		2	
3-41.		3	$\frac{3}{4}$		12	25,000		100
3-42.	50	$\frac{1}{2}$	$\frac{1}{8}$		8			10
3-43.	31.4	1			12	20,000	2	
3-44.	31.4			20	6.4	10,000		40

3-45. A 10-turn helical compressive spring with $\frac{1}{2}$-in. steel wire and a 4-in. outside coil diameter is used to exert an axial force on a clutch plate. The free length of the spring is $8\frac{1}{2}$ in., and the compressed length is $5\frac{1}{2}$ in. Determine the maximum stress in the wire and the pressure exerted against the clutch plate. *Note:* Correct for stress concentration.

3-46. A helical spring whose constant is 100 lb per in. is cut into quarters, and the four pieces are then combined in parallel. Find the equivalent spring constant of the parallel combination.

3-47. A rigid horizontal bar of negligible weight is supported by two springs, as shown in Fig. P3-47. Determine the distance x in order that the bar remain horizontal after a load P is applied.

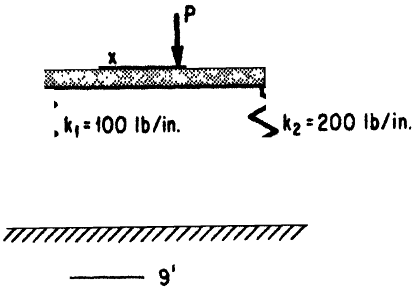


Fig. P3-47

3-48. The rigid bar AB, Fig. P3-48, weighs 400 lb and supports a load $P = 1000$ lb. If the free length of the springs are equal prior to loading, where should the load P be placed if the bar is to remain horizontal?

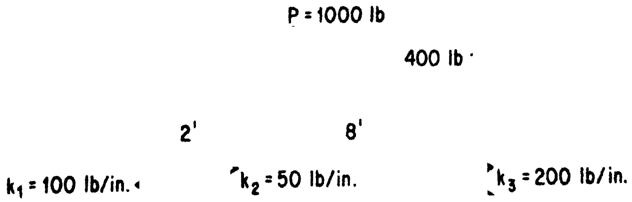


Fig. P3-48

3-49. Two springs, Fig. P3-49, are joined at C and then placed, unstretched, between the supports A and B. Find the reactions at the supports if a force of 600 lb acts as shown.

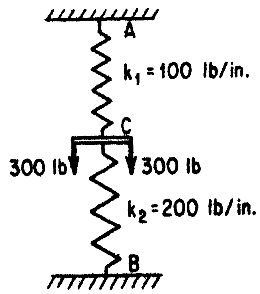


Fig. P3-49

3-50. Two springs, Fig. P3-50, each having a constant k, are supported taut, but unstressed, between A and B. The junction M moves 5 in. downward when a 100 lb force is applied slowly, as shown. Find the constant of each spring.

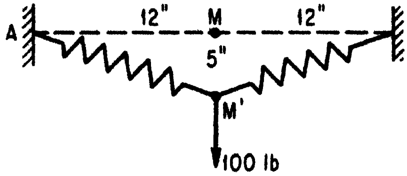


Fig. P3-50

3-51. Determine the horsepower that the flanged coupling of Example 11 is capable of transmitting if the inner circle of bolts is removed.

3-52. How many $\frac{3}{8}$-in. bolts arranged in a 5-in.-diameter bolt circle would be required to transmit 100 hp at 315 rpm through a flanged coupling? The maximum permissible shearing stress is 10,000 psi.

3-53. The coupling shown in Fig. P3-53 consists of two sprockets held together by a flexible roller chain. The chain has 32 pins, each with a diameter of $\frac{1}{2}$ in., located in a circle of 6 in. diameter. Determine the theoretical horsepower rating of this coupling at a speed of 315 rpm. The shearing stress in the pins is not to exceed 10,000 psi.

3-54. Determine the maximum shearing stress developed in the bolts of the coupling shown in Fig. P3-54. The coupling transmits 400 hp at 630 rpm, and all bolts have equal diameters of $\frac{3}{8}$ in.

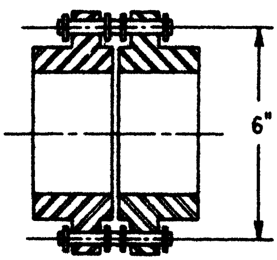


Fig. P3-53

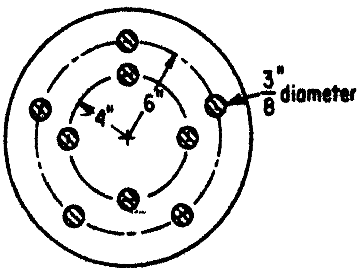


Fig. P3-54

3-55. Determine the number of bolts required in a coupling that joins two 2-in.-diameter shafts. A torque that acts on the shafts produces a shearing stress of 12,000 psi in each. The allowable shearing stress in the bolts is 10,000 psi, the diameter of the bolt circle is 8 in., and the bolts are $\frac{1}{2}$ in. in diameter.

CHAPTER 4

Shear and Moment in Beams

Structural members capable of sustaining loads normal to their axes are called *beams*. The term "beam" generally brings to mind large and massive objects employed principally in the building trades. Although this is as good a description as any, there are many types of beams that are used in machines and controls whose very function depends on their ability to bend without becoming overstressed; the leaf-spring is a typical example.

4-1 Classification of Beams and Loads

There are six fundamental methods of supporting beams; these are illustrated in Fig. 4-1. A beam that rests on two supports in a manner which offers, at most, one restraint along the beam axis is called a *simple beam*. A *cantilever beam* is held at one end only by a support which is capable of sustaining both moments and forces. As the name implies, an *overhanging* beam is one that extends beyond its supports at either or both ends. If the external forces and moments that support the beam can be found by the equations of statics alone, the beam is *statically determinate*, whereas beams that have more supports than are necessary to maintain equilibrium are *statically indeterminate*. Beams that are *propped*, *continually supported*, or *built-in* are examples of this latter class.

There are four fundamental types of loads which can act on a beam; these are illustrated in Fig. 4-2. The *concentrated load*, the simplest, acts at a single point on the beam, and the *distributed load*, which may be *uniform* or *non-uniform*, acts over a given length of the beam. A fourth type of loading occurs when the beam is subiected to a *pure moment*. Although these loads are illustrated as acting singly, the possibility exists, of course, that they can act in any combination on a single beam.

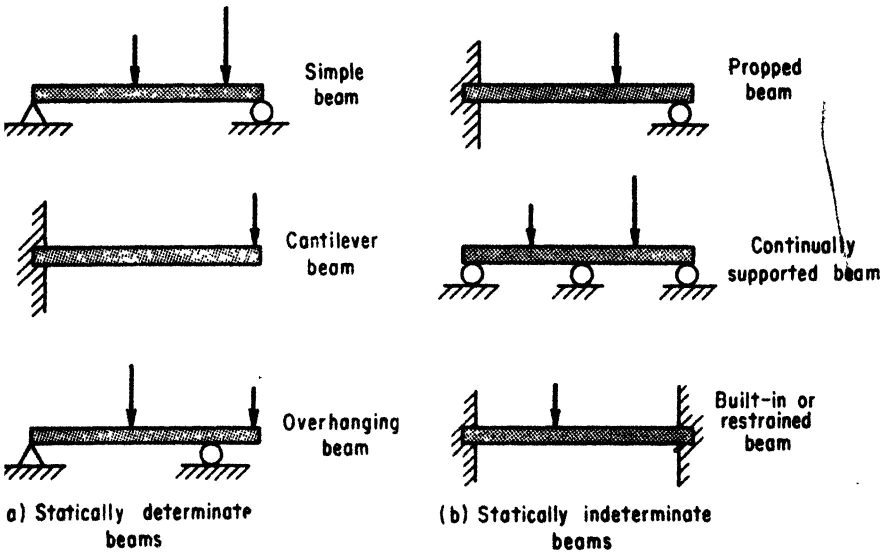


Fig. 4–1

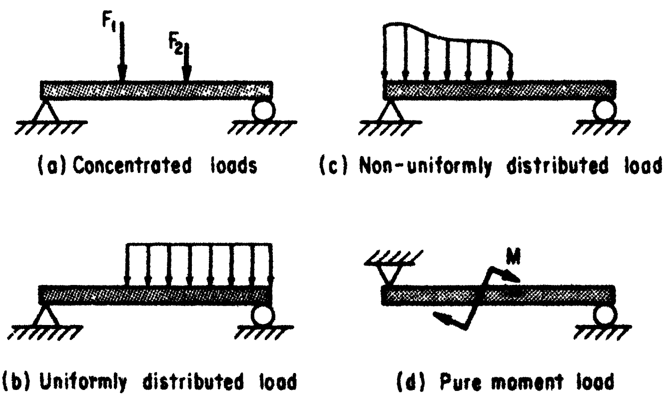


Fig. 4–2

4–2 Shear and Moment Diagrams

Stresses and deflections in beams are functions of the internal reactions, forces, and moments, and for this reason it is convenient to "map" these internal reactions and form diagrams that give a complete picture of the magnitudes and directions of the forces and moments that act throughout the beam. These graphs are called *shear* and *moment diagrams.*

Consider the simply supported beam shown in Fig. 4-3(a). The reactions at the supports, computed through the equations of static equilibrium, are Pb/l and Pa/l. If the beam is cut by a transverse plane at a distance x from

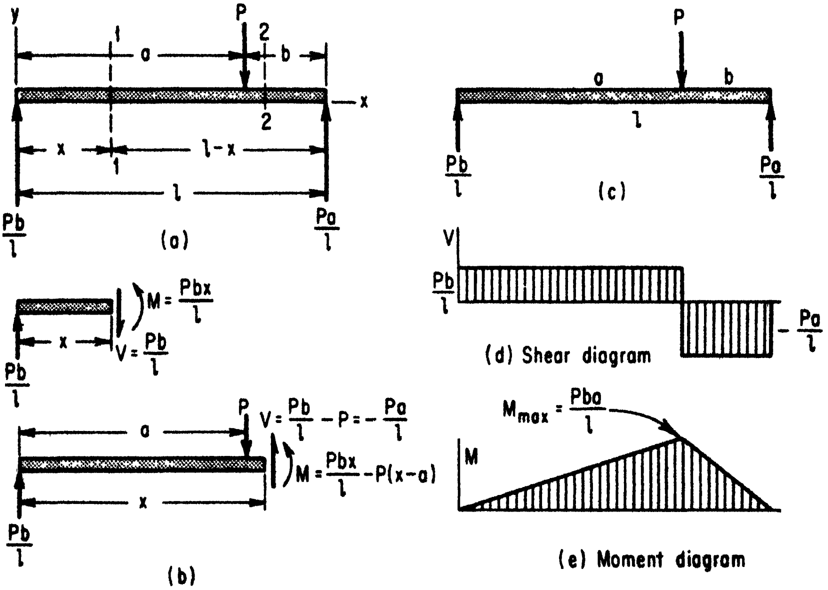


Fig. 4-3

the left end, a force V and a moment M are required to keep the severed section in equilibrium. These two internal reactions, shown in Fig. 4-3(b), are the *shear* V and the *moment* M at x. A force summation $\sum F_y = 0$ and a moment summation $\sum M = 0$ on the left-hand portion of the free-body will show V and M to be

$$V = \frac{Pb}{l}$$

and

$$M = \frac{Pb}{l}x$$

The shear remains unchanged between the limits of $x = 0$ and $x = a$, whereas the moment between these limits increases uniformly with the distance x. At $x = a$ the moment is a maximum and has a value of

$$M \quad \frac{Pba}{l}$$

If a second free-body is drawn, this time cut by a transverse plane to the right of load P, the shear and moment are

$$= \frac{Pb}{l} - P = \frac{P(b-l)}{l} = -\frac{Pa}{l}$$

and

$$M = \frac{Pb}{l}x \quad P(x-a) = \frac{Pa}{l}(l-x)$$

A study of the moment equation for the region $a < x < l$ indicates that x increases, M decreases, and that at $x = l$ the moment is zero.

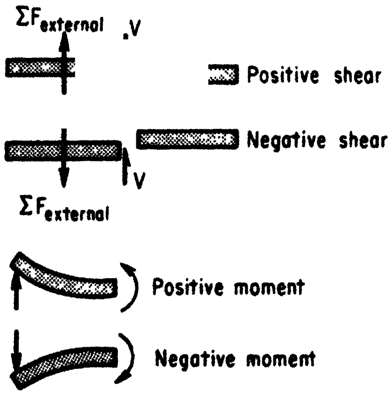


Fig. 4-4

The graphs, or "maps" of the she and the moment for all values of x between 0 and l are called the *shear* and *moment diagrams*. A uniform and consistent sign convention must be used in plotting these diagrams: *External forces that act upward on the left-hand free-body cause positive shear; positive moments are caused by forces and moments that tend to bend the beam in a way to cause its radius of curvature to be upward.* This sign convention is illustrated in Fig. 4-4.

The examples that follow will further illustrate the method of plotting shear and moment diagrams. It will be noted that the free-body diagram, sketched between segments of the beam where changes in loading occur, is a tremendous aid in writing mathematical relationships for shear and moment.

Example 1. Draw the shear and moment diagrams for the simply supported beam shown in Fig. 4-5(a) and determine the maximum values of V and M.

Solution: The reactions at the supports A and D are first computed:

$$\sum M_A = 0$$

$$12R_D - 6(200) - 10(300) = 0$$

$$R_D = 350 \text{ lb}$$

and

$$\sum M_D = 0$$

$$12R_A - 2(300) - 6(200)$$

$$R_A = 150 \text{ lb}$$

Check:

$$\sum F_y = 0$$

$$200 + 300 - 350 - 150 = 0$$

To plot the shear and the moment, free-body diagrams of three sections of the beams are required; these are illustrated in Figs. 4-5(b), (c), and (d).

The first free-body results when a transverse cut is made at section 1-1,

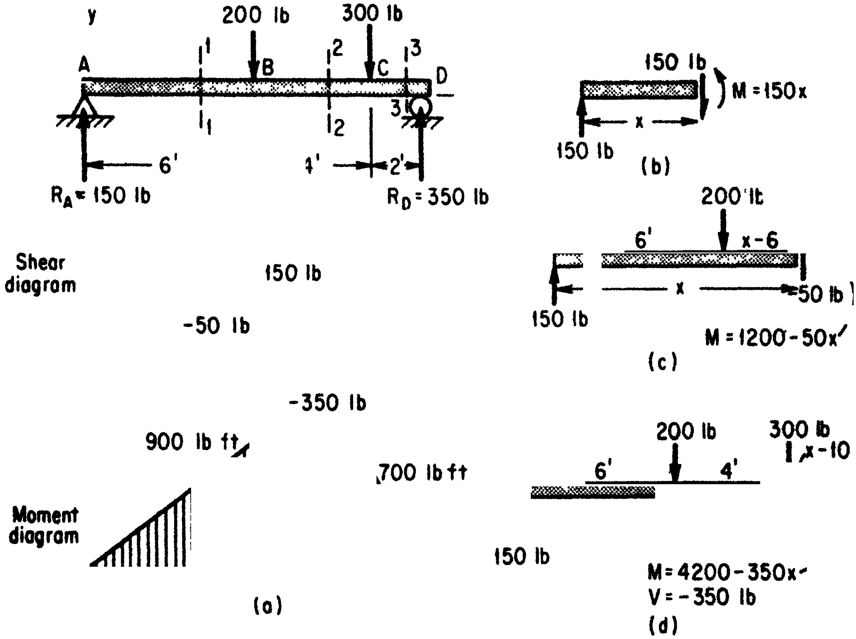


Fig. 4-5

between points A and B, at a distance x from A. The shear in this section is constant and equal to

$$V_{AB} = +150 \text{ lb}$$

whereas the moment increases uniformly as x increases:

$$M_{AB} = +150x \text{ lb ft}$$

At $x = 6$ ft, the moment is 150(6) = 900 lb ft.

The second free-body diagram, Fig. 4-5(c), is representative of a segment of the beam cut at section 2-2 between points B and C. The shear in this region is negative, since the sum of the external forces is downward.

$$V_{BC} = 150 - 200 = -50 \text{ lb}$$

The moment at the cut, an internal reaction required to maintain equilibrium, is

$$M_{BC} = 150x - 200(x - 6)$$
$$= 1200 - 50x$$

Thus, in the portion of the beam lying between $x = 6$ ft and $x = 10$ ft, the moment decreases uniformly as x becomes greater; the limiting value of the moment given by this expression occurs at $x = 10$ ft.

$$M = 1200 - 50(10) = 700 \text{ lb ft}$$

The third and final free-body, that of a section to the left of section 3-3, shows the shear and moment to be

$$V_{CD} = 150 - 200 - 300 = -350 \text{ lb}$$

and

$$M_{CD} = 150x - 200(x - 6) - 300(x - 10)$$
$$= 4200 - 350x$$

The shear is negative and constant in this segment, whereas the moment decreases with x; at $x = 12$ ft, the moment is zero.

$$M = 4200 - 350(12) = 0$$

When sketched, the shear and moment diagrams present a picture of the entire shear force and moment distribution in the beam. The greatest numerical values of V and M are

$$V = -350 \text{ lb}$$
$$M = +900 \text{ lb ft}$$

Example 2. Draw the shear and moment diagrams for the cantilever beam shown in Fig. 4-6(a).

Solution: The shear in a section of the beam between points A and B is found by inspection to be

$$V_{AB} = -200 \text{ lb}$$

and between points B and C

$$V_{BC} = -700 \text{ lb}$$

Equations for the moment in the beam as a function of x, measured to the right from A, are

$$M_{AB} = -200x$$

and

$$M_{BC} = -200x - 500(x - 4) = 2000 - 700x$$

The values of the moment at the *critical points* B and C are found by substitution; thus at $x = 4$ ft

$$M = -200(4) = -800 \text{ lb ft}$$

and at $x = 8$ ft

$$M = 2000 - 700(8) = -3600 \text{ lb ft}$$

When plotted, the shear and moment diagrams appear as shown in Fig. 4-6(b).

The general procedure for drawing shear and moment diagrams outlined in the previous paragraphs applies to any loading arrangement; it follows, however, that these diagrams become more complex as the loading becomes more involved. The examples that follow will illustrate the drawing of shear and moment diagrams for the more complex loading arrangements.

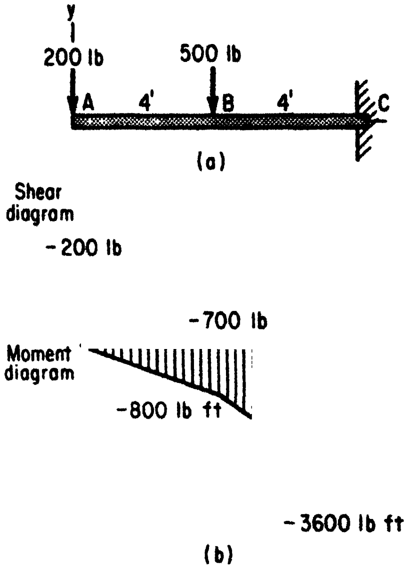


Fig. 4-6

Example 3. The cantilever beam in Fig. 4-7(a) is subjected to both a couple and a concentrated load as shown. Draw the shear and moment diagrams for the beam.

Solution: Two free-body diagrams are required, one which will describe the internal reactions in a section of the beam between points A and B, and another between points B and C. In the first free-body, Fig. 4-7(b), the shear is zero and the moment is constant and equal to +100 lb ft. The second free-body, Fig. 4-7(c), indicates the internal reactions to consist of both shear and moment; the former is constant, whereas the latter decreases with x.

$$V_{BC} = -200 \text{ lb}$$

and

$$M_{BC} = 100 - 200(x - 4)$$

$$900 - 200x$$

and at $x = 10$ ft

$$M = 900 - 200(10) \qquad 1100 \text{ lb ft}$$

The diagrams that descriptively picture the shear and moment in this beam are shown in Fig. 4-7(d).

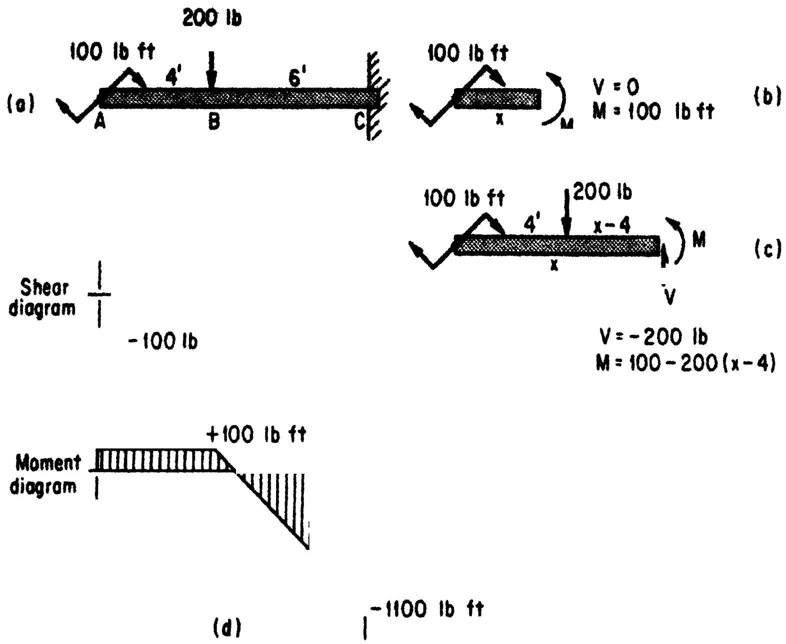


Fig. 4-7

Example 4. Draw the shear and moment diagrams for the beam shown in Fig. 4-8(a) and determine the maximum value of the moment.

Solution: The reactions are determined by a moment summation, first about A and then about C:

$$\sum M_A \quad 0$$

$$10R_C - 6(100)3 = 0$$

$$R_C = 180 \text{ lb}$$

and

$$\sum M_C = 0$$

$$10R_A - 6(100)7 = 0$$

$$R_A = 420 \text{ lb}$$

Check:

$$\sum F_y = 0$$

$$6(100) - 420 - 180 = ($$

A free-body diagram is next drawn, Fig. 4-8(b), for a portion of the beam having a length of x, where x assumes any value between zero and 6 ft. The external forces that act on this free-body are the upward reactions of 420 lb

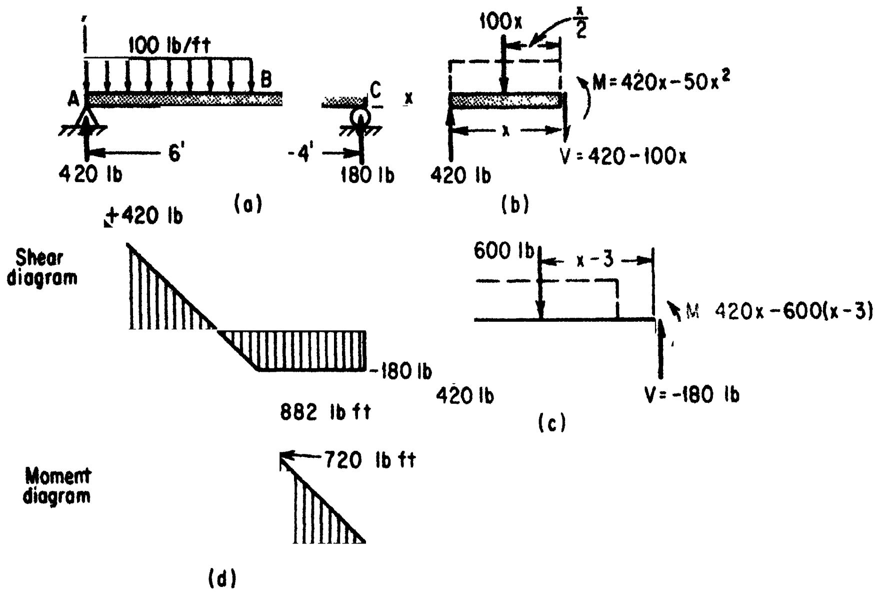


Fig. 4-8

at A and the downward weight force of $100x$ lb. The vertical shear at the transverse cut is, therefore,

$$V_{AB} = 420 - 100x$$

By this equation, the curve of the shear is shown to be a straight line sloping downward to the right; at a value of $x = 0$, the shear is $+ 420$ lb, and at a value of $x = 6$ ft, the shear is -180 lb.

The same two external forces that produced the vertical shear also tend to bend the beam. The sum of the moments of these two forces is equal to the internal moment at the cut:

$$M_{AB} = 420x - 100x\left(\frac{x}{2}\right) = 420x - 50x^2$$

Since the magnitude of the moment is a function of both x and x^2, the curve that represents the moment is a parabola. Careful plotting of this parabola will show *the maximum moment to occur at a point in the beam where the shear is zero.* This statement, which is true for any type of loading, places the maximum moment at $x = 4.2$ ft; thus

$$V_{AB} = 420 - 100x = 0$$

$$x = 4.2 \text{ ft}$$

The magnitude of the moment, evaluated at three points: $x = 0$, $x = 4.2$, and $x = 6$, is all that need be computed to sketch adequately the moment diagram between the limits of $x = 0$ and $x = 6$ ft.

$$\text{At } x = 0: \quad M = 420(0) - 50(0)^2 = 0$$

$$\text{At } x = 4.2: \quad M = 420(4.2) - 50(4.2)^2 = 882 \text{ lb ft}$$

$$\text{At } x = 6: \quad M = 420(6) - 50(6)^2 = 720 \text{ lb ft}$$

A free-body diagram for a portion of the beam to the left of a transver cut between B and C, Fig. 4-8(c), indicates the shear to be constant and negative:

$$V_{BC} = 420 - 600 = -180 \text{ lb}$$

and the moment to be

$$M_{BC} = 420x - 600(x - 3) = 1800 - 180x$$

Thus, the moment in the beam between the limits of $x = 6$ ft and $x = 10$ ft decreases uniformly to zero from an initial value of 720 lb ft.

4-3 *Relationships Between Load, Shear, and Moment*

There are several approaches one can use to construct shear and moment diagrams; the previous section illustrates the purely mathematical method. Equations are derived for the shear and moment in the beam as a function of a distance. On the basis of these equations, V and M are plotted from point to point in the beam. While this is surely a direct approach, there are

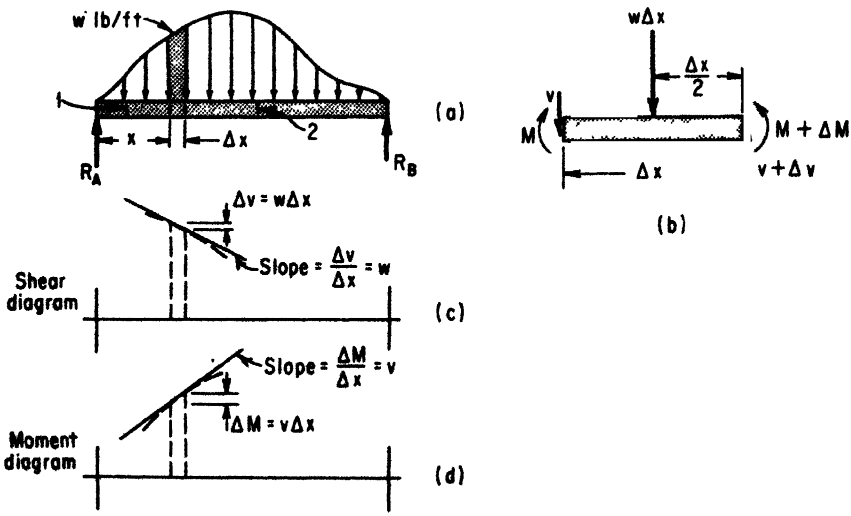


Fig. 4-9

several concepts that can be employed to reduce the problem to one of simple geometry.

Consider the purely arbitrary loading arrangement shown in Fig. 4-9(a). A non-uniform load is selected simply to illustrate a general situation. A section of the beam of infinitesimal length Δx is isolated, and a free-body diagram is drawn as shown in Fig. 4-9(b). On the left-hand face, the shear and moment are assumed to be simply V and M. At the midpoint of the element, a portion of load $w\Delta x$ acts downward as shown. To maintain equilibrium, the shear and the moment on the extreme right face differ from those on the left by small incremental values ΔV and ΔM.

The equations of static equilibrium require the resultant force and resultant moment both to be zero; thus

$$\sum F_y = 0$$

$$V + w\Delta x - (V + \Delta V) = 0$$

$$\Delta V = w\Delta x \tag{a}$$

and

$$\sum M = 0$$

$$M - V\Delta x - w\Delta x\left(\frac{\Delta x}{2}\right) + (M + \Delta M) = 0$$

$$\Delta M = V\Delta x \tag{b}$$

Since Δx is small, the term $w(\Delta x)^2/2$ approaches zero in value and is dropped from the computations. The physical interpretation of Eqs. (a) and (b) is the key to the method about to be described. The first equation states that the incremental change ΔV is equivalent to an area of a rectangle of height w and width Δx: the amount of external load on the elemental length of beam. *The difference in the value of the shear between points* 1 *and* 2 *in the beam is the sum of the incremental values* ΔV *or simply the area of the loading diagram between points* 1 *and* 2.

$$V_2 - V_1 = \sum \Delta V = \sum w\Delta x = \text{area of loading diagram between points 1 and 2} \tag{4-1}$$

Rearrangement of terms in Eq. (a) contributes a second important fact: *the slope of the shear diagram at a given point is equal to the magnitude of the load at that point.*

$$\frac{\Delta V}{\Delta x} = w \tag{4-2}$$

The same reasoning, applied to Eq. (b), gives

$$\Delta M = V\Delta x$$

$$M_2 - M_1 = \sum \Delta M = \sum V\Delta x = \text{area of shear diagram between points 1 and 2} \qquad (4\text{-}3)$$

The difference in the value of the moment between points 1 *and* 2 *in the beam is equal to the area enclosed by the shear diagram between these two points*; and

$$\frac{\Delta M}{\Delta x} = V \qquad (4\text{-}4)$$

The slope of the moment diagram at a point is equivalent to the magnitude of the shear at that point. This statement is most significant, since it establishes a simple way of determining the maximum moment. Mathematically, maxima and minima occur in a curve when its slope is either zero or undefined —simply stated, *maxima or minima in the moment diagram occur at those points where the shear passes through zero.*

Example 5. Use the relationship between loading, shear, and moment to sketch the shear and moment diagram for the beam shown in Fig. 4-10.

Solution: The reactions at A and D are found in the usual way.

$$\sum M_A = 0$$

$$10R_D = 100(5)2.5 + 1000(8)$$

$$R_D = 925 \text{ lb}$$

$$\sum M_D = 0$$

$$10R_A = 100(5)(7.5) + 1000(2)$$

$$R_A = 575 \text{ lb}$$

Check:

$$\sum F_y = 0$$

$$100(5) + 1000 - 925 - 575 = 0$$

The value of the shear at A is equal to the reaction $R_A = 575$ lb; it is an upward force and, therefore, positive. Next, the change in shear between points A and B is equal to the area of the loading diagram between these two points; thus

$$\Delta V = V_B - V_A = [\text{Area}_{AB}]_{\text{loading}}$$

$$V_B - 575 = -100(5)$$

$$V_B = 575 - 500 = +75 \text{ lb}$$

Note: The area representing the distributed load is negative since the load is downward.

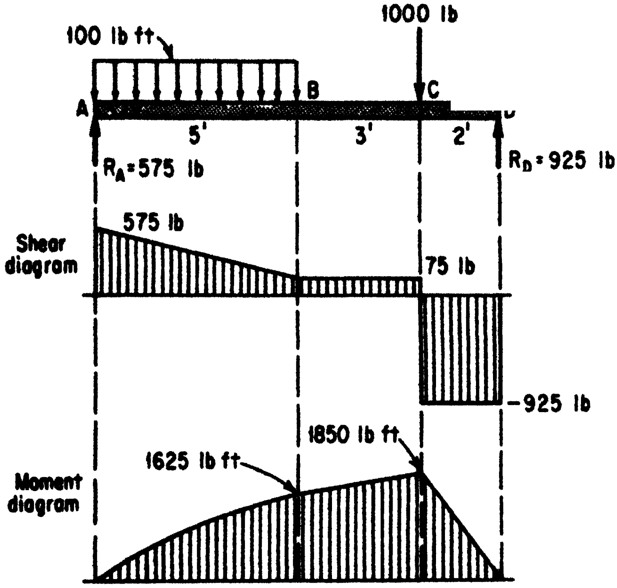


Fig. 4-10

A portion of the moment diagram can now be sketched. Since the slope of the moment curve is the magnitude of the shear, the moment diagram has a positive, but diminishing, slope between points A and B. At A the slope is $+ 575$ and at B, $+ 75$. The general shape of the curve is thus established.

Equation (4-3) is next used to find the value of the moment at B.

$$\Delta M = M_B - M_A = [\text{Area}_{AB}]_{\text{shear}}$$

$$M_B = \left(\frac{575 + 75}{2}\right)5 + 0 = 1625 \text{ lb ft}$$

Between points B and C, the shear is unchanged, since the area of loading is zero.

$$V_C - V_B = [\text{Area}_{BC}]_{\text{loading}} = 0$$

$$V_C = V_B = +\ 75 \text{ lb}$$

Since $\Delta w/\Delta x = 0$ between these two points, the shear is represented by a horizontal line.

The moment continues to increase, however, and the difference in moment between points B and C is the area under the shear diagram between these two points:

$$M_C - M_B = [\text{Area}_{BC}]_{\text{shear}}$$

$$M_C = 1625 + 75(3)$$

$$M_C = 1625 + 225 = 1850 \text{ lb ft}$$

Since the shear is constant and positive between B and C, the moment diagram is a straight line having a positive slope.

The concentrated 1000 lb load at C causes the abrupt change in the shear from + 75 lb to − 925 lb; no additional load acts on the beam until the reaction R_D is reached, which brings the shear to zero at D.

The moment diagram between C and D is a straight line having a negative slope, as shown. If the computations are correctly carried out, the moment at D for this beam should be zero.

$$\Delta M = M_D - M_C = [\text{Area}_{CD}]_{\text{shear}}$$

$$M_D - 1850 = -925(2)$$

$$M_D = 0$$

The shear passes through the zero axis at C, the point of maximum moment.

$$M_{\max} = +1850 \text{ lb ft, 8 ft to right of } A$$

Moment Diagrams by Superposition

There are computations involving the deflections of beams which require a knowledge of the properties of the areas enclosed by moment diagrams rather than the principal values. To aid in this sort of computation, moment diagrams are sometimes drawn "in parts;" each load is imagined to produce a separate moment diagram. The *superposition*, or sum, of these diagrams is equivalent to the diagram drawn in the usual manner.

Consider the four fundamental cantilever loadings and their respective moment diagrams shown in Fig. 4-11. These diagrams are simple curves: rectangles, triangles, and parabolas. Next, consider a cantilever beam subjected to a combined loading of three concentrated forces, as shown in Fig. 4-12(a). By considering the loads individually, three separate moment

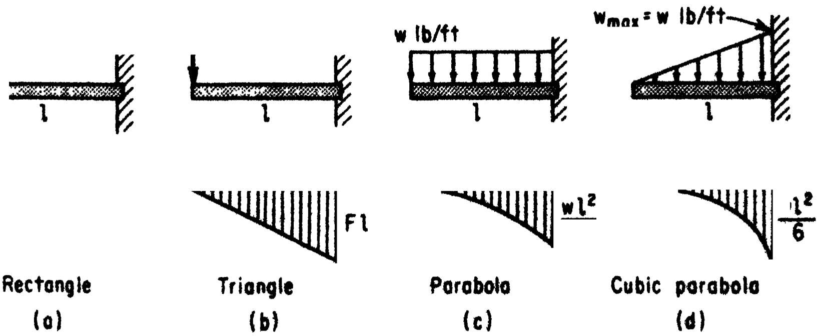


Fig. 4-11

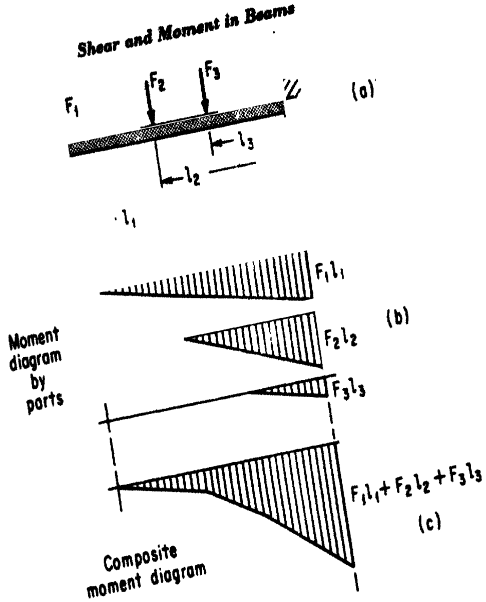


Fig. 4-12

diagrams could be constructed, as in Fig. 4-12(b). The composite moment diagram is actually the sum of these three parts, shown in Fig. 4-12(c). If areas and centroids of the moment diagrams are required, it is far simpler to work with the diagrams drawn in parts rather than in composite form.

Before we consider examples, it would be well to review some of the properties of simple areas: the rectangle, triangle, parabola, and cubic parabola. The areas and their centroids are listed in Table 4-1. The order in which these are listed offers a rather interesting memory aid: both the areas and their centroids become progressively smaller as the degree of the curve increases.

Example 6. A cantilever beam is subjected to a concentrated load of 100 lb and a distributed load of 50 lb per ft, as shown in Fig. 4-13 (a). Draw the moment diagram by parts and determine the moment of the area of this diagram, first about end B, and then about end A.

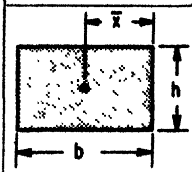
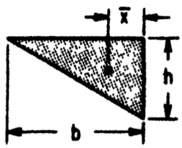
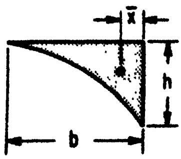
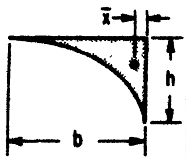
Solution: The moment diagram consists of two areas, a triangle and a parabola. The moment at B caused by the concentrated load is

$$M_1 = 100(8) = 800 \text{ lb ft}$$

and the moment caused by the distributed load is

$$M_2 = 500(4)\left(\tfrac{4}{2}\right) = 400 \text{ lb ft}$$

Table 4-1. Properties of Areas

		Area	*Centoid* $\bar{x}$
	Rectangle	bh	$b/2$
	Triangle	$\frac{1}{2}bh$	$b/3$
	Parabola	$\frac{1}{3}bh$	$b/4$
	Cubic Parabola	$\frac{1}{4}bh$	$b/5$

The moments of these areas about a vertical line drawn at the extreme right side of the diagram would be

$$\begin{aligned}\text{Area}_{AB}\bar{X}_B &= (A\bar{X})_1 + (A\bar{X})_2 \\ &= -[\tfrac{1}{2}(8)800]\tfrac{8}{3} - [\tfrac{1}{3}(4)400]\tfrac{4}{4} \\ &= -9070 \text{ lb ft}^3\end{aligned}$$

The sum of the moments of the areas about the extreme left end is

$$\begin{aligned}\text{Area}_{AB}\bar{X}_A &= (A\bar{X})_1 + (A\bar{X})_2 \\ &= -[\tfrac{1}{2}(8)(800)(\tfrac{2}{3} \times 8)] - [\tfrac{1}{3}(4)(400)(4 + \tfrac{3}{4} \times 4)] \\ &= -20{,}800 \text{ lb ft}^3\end{aligned}$$

The concept of *moments by parts* can be extended to all varieties of beams and modes of loading. Consider, for example, a simply supported beam, Fig. 4-14(a), acted upon by a concentrated load P. There are several ways of constructing a moment diagram by parts for this beam; if moments are drawn with respect to a reference line at the extreme left end, for instance, the diagram would appear as shown in Fig. 4-14(b). If a reference at the right end is selected, the diagram appears as shown in Fig. 4-14(c), and if an arbitrary line is drawn within the beam, the diagram appears as illustrated in Fig. 4-14(d). With care, a reference can usually be selected for any beam that will yield the simple areas described in Table 4-1.

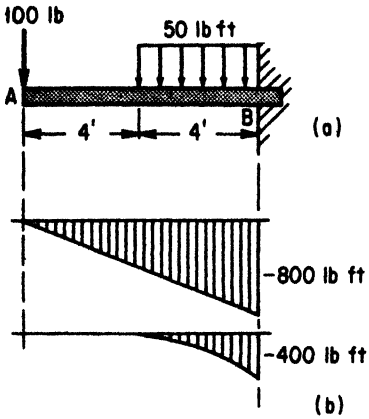


Fig. 4-13

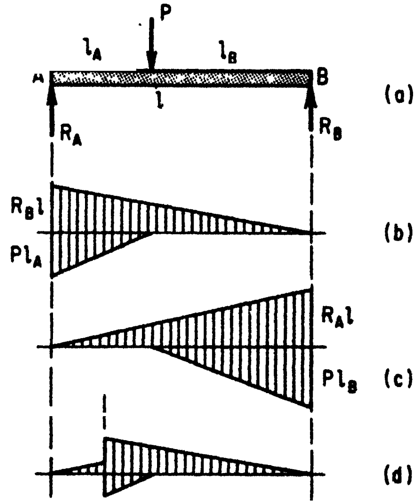


Fig. 4-14

Example 7. Draw the moment diagram by parts for the loading arrangement shown in Fig. 4-15(a). Select a reference line that passes through the left support, and find the moment of the area of the moment diagram about this support.

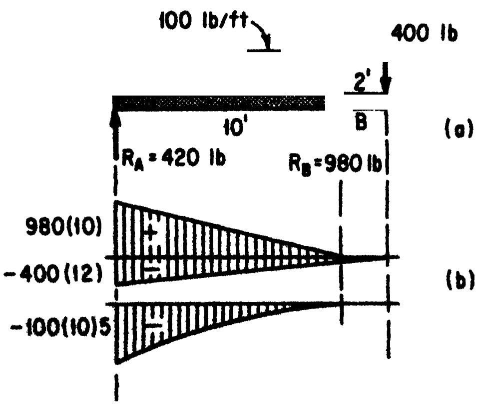


Fig. 4-15

Solution: The reactions R_A and R_B are computed:

$$\sum M_A = 0$$

$$100(10)5 - 10R_B + 400(12) = 0$$

$$R_B \quad 980 \text{ lb}$$

$$\sum M_B \quad 0$$

$$10R_A - 100(10)5 + 400(2)$$

$$R_A = 420 \text{ lb}$$

Check: $980 + 420 - 100(10) - 400$

There are three parts to the moment diagram: the positive portion caused by the upward reaction R_B, and two negative portions, one caused by the concentrated load and the other by the distributed load.

Written symbolically, the desired quantity is $\text{Area}_{AC}\,\overline{X}_A$; thus

$$\text{Area}_{AC}\overline{X}_A = \tfrac{1}{2}(10)9800(\tfrac{10}{3}) - \tfrac{1}{2}(12)4800(\tfrac{12}{3}) - \tfrac{1}{3}(10)5000(\tfrac{10}{4})$$
$$= 6470 \text{ lb ft}^3$$

PROBLEMS

4-1 through 4-14. Write the equations for the shear and moment as functions of x measured from the extreme left end of the beams shown in the respective figures, and then graph the shear and moment diagrams. Determine the maximum shear and maximum moment for each case. Neglect the weight of the beam in each instance.

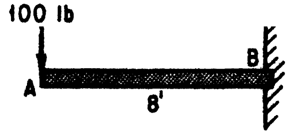


Fig. P4-1

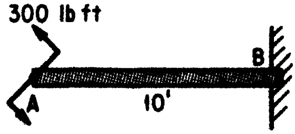


Fig. P4-2

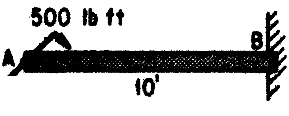


Fig. P4-3

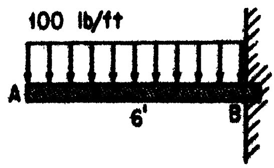


Fig. P4-4

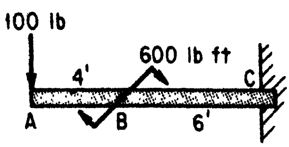


Fig. P4-5

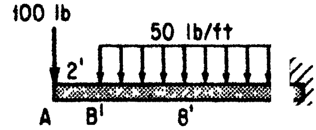


Fig. P4-6

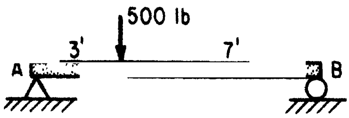


Fig. P4-7

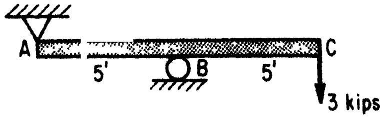


Fig. P4-8

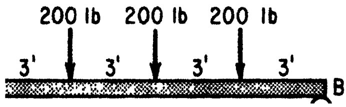


Fig. P4-9

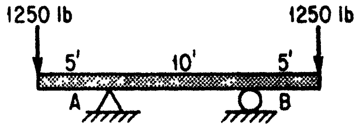


Fig. P4-10

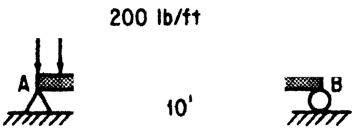


Fig. P4-11

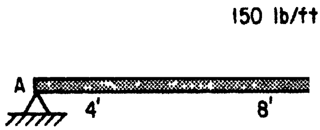


Fig. P4-12

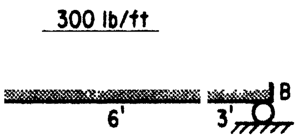


Fig. P4-13

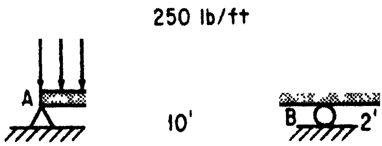


Fig. P4-14

4-15 through **4-25.** Without writing equations, sketch the shear and moment diagrams for the beams shown in the respective figures. Specify numerical values for all change of loading positions and at all points of zero shear, and determine the maximum moment and its location.

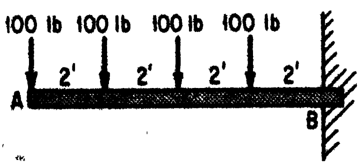


Fig. P4-15

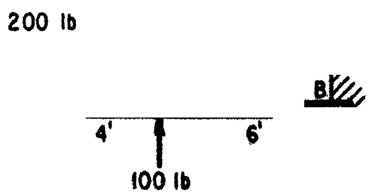


Fig. P4-16

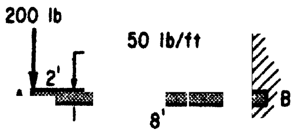


Fig. P4–17

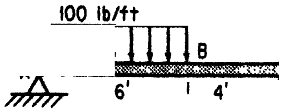


Fig. P4–18

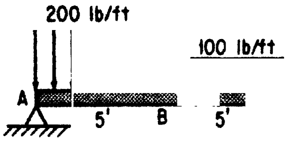


Fig. P4–19

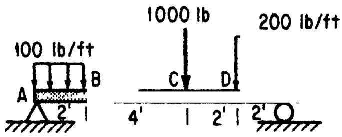


Fig. P4–20

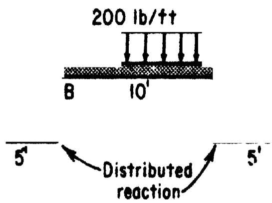


Fig. P4–21

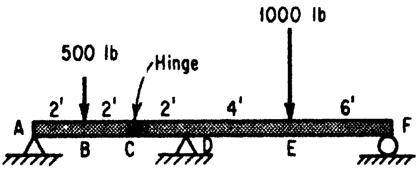


Fig. P4–22

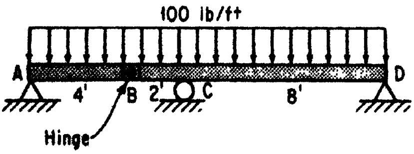


Fig. P4–23

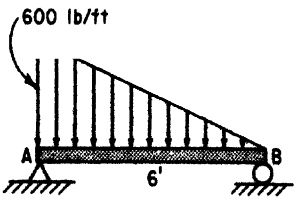


Fig. P4–24

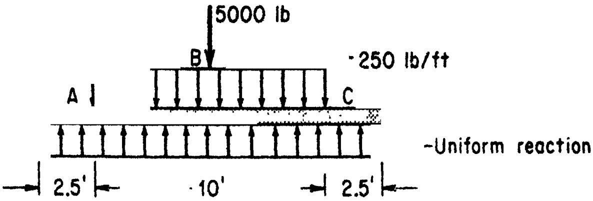


Fig. P4-25

4-26. Draw the moment diagram for the beam of Prob. 4-7 by the method of superposition. Select a reference line through A and find the moment of the area of the diagram about this reference line.

4-27. Use the method of superposition to draw the moment diagram for the beam of Prob. 4-9. Select a reference line through the right reaction and find the moment of the area of the diagram about this line.

4-28. Draw the moment diagram by the method of superposition for the beam of Prob. 4-11. Select a reference line through the right reaction and find the moment of the area of the diagram about this line.

4-29. Draw the moment diagram for the beam shown in Fig. P4-29 by the method of superposition. Select a reference line through the midspan M and find $\text{Area}_{MB}\bar{X}_B$.

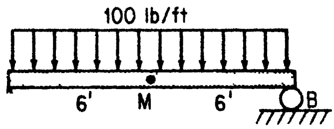


Fig. P4-29

4-30. Use the method of superposition to draw the moment diagram for the beam of Prob. 4-12. Select a reference line 4 ft to the right of the point A and determine $\text{Area}_{AB}\bar{X}_A$.

4-31. Employ the method of superposition to draw the moment diagram for the beam of Prob. 4-13. Select a reference line through the midspan and find the moment of the area of the diagram between the midspan and the right support about this support.

4-32. Find the moment of the area of the moment diagram, $\text{Area}_{AB}\bar{X}_B$, for the beam shown in Fig. P4-32.

4-33. Find the moment of the area of the moment diagram, $\text{Area}_{AC}\bar{X}_C$, for the beam shown in Fig. P4-33. *Hint:* add and subtract a distributed load between B and C and then use a line through C as a reference.

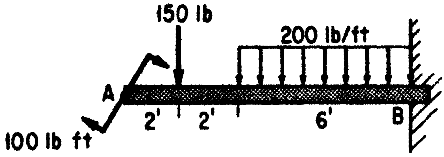


Fig. P4–32

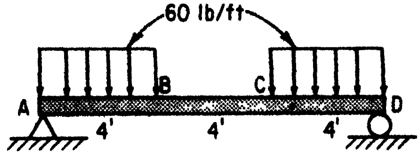


Fig. P4–33

4-34 through **4-38.** The respective figures represent the shear diagram for various beam loadings. Draw the beam together with its loading arrangement and the moment diagram for each.

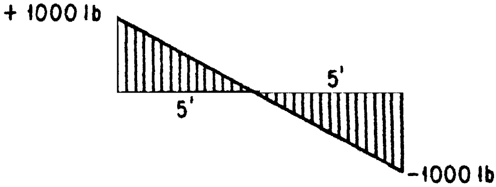


Fig. P4–34

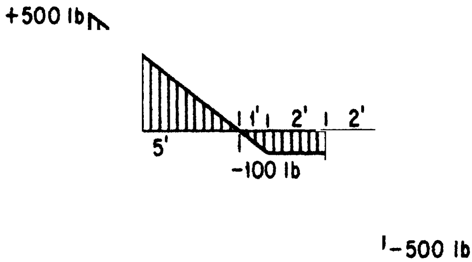


Fig. P4–35

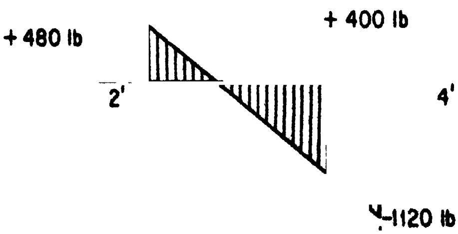


Fig. P4–36

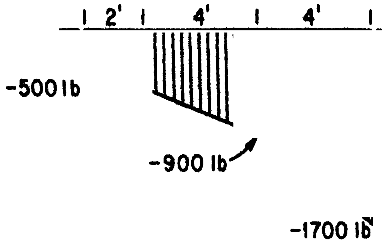


Fig. P4–37

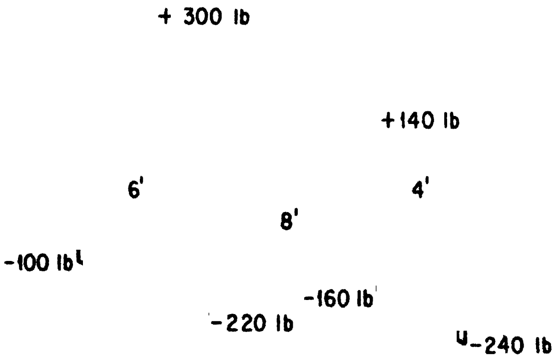


Fig. P4–38

4-39 through **4-43.** Moment diagrams by parts are illustrated in the respective figures for various beam loadings. Draw the loading arrangement for each.

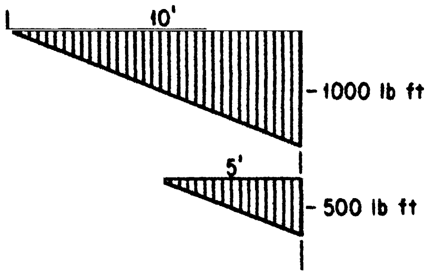


Fig. P4–39

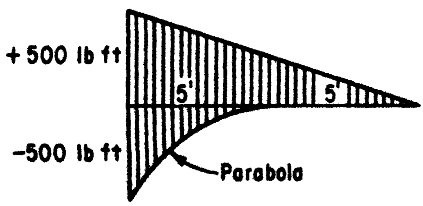


Fig. P4–40

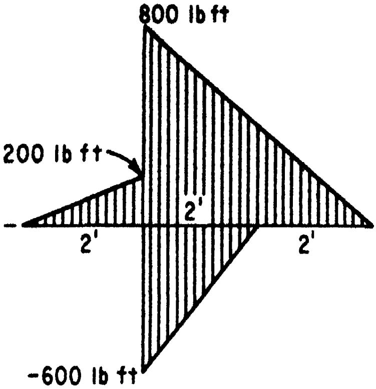


Fig. P4-41

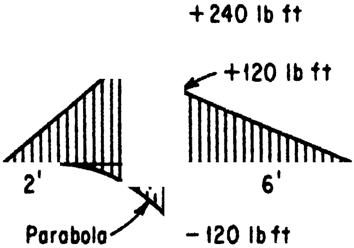


Fig. P4-42

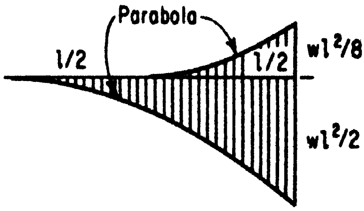


Fig. P4-43

CHAPTER 5

Stresses in Beams

Some of the earliest studies in mechanics of materials were concerned with the strength and deflection of beams. Galileo, in 1638, was one of the first to propose a theory of stress distribution caused by bending, which, in part, was the accepted approach for many years to follow. Errors, however, in the thinking of the day, were pointed out in 1773 by Coulomb, a French military engineer, and his concepts of stress distribution are now accepted as theoretically correct.

5–1 Tensile and Compressive Stresses Caused by Bending

Consider a simple beam, Fig. 5-1(a), acted upon by end moments; its bending is exaggerated. The beam is assumed (a) *to be initially straight and of constant cross section*, (b) *to be elastic and have equal moduli of elasticity in tension and compression*, (c) *to be homogeneous*, and (d) *to obey Hooke's law.* It is further assumed *that plane sections within the beam prior to bending remain plane.* To study the geometry of bending, a small segment of the beam is isolated as shown in Fig. 5-1(b). As the beam deforms, the top fibers contract and the bottom fibers elongate; the *neutral plane* of the beam is defined as *a plane whose length remains unchanged during the deformation.* The length of the segment of the neutral plane, Δx, can be expressed in terms of the radius of curvature ρ of the beam and the inclination $\Delta\theta$ of the transverse plane $a'b'$; thus

$$\Delta x = \rho\Delta\theta$$

At an arbitrary distance y from the neutral plane the deformation δ of a *fiber* within the beam is

$$\delta = y\Delta\theta$$

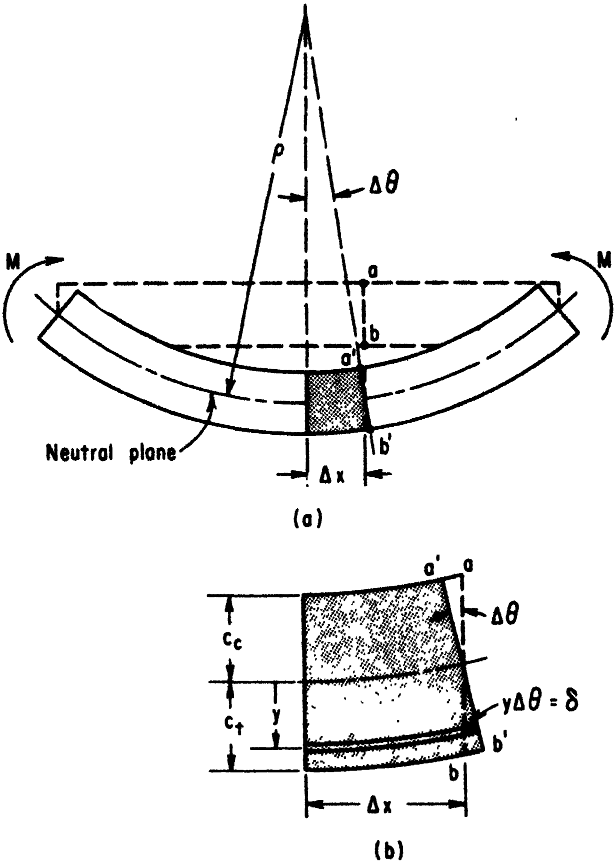


Fig. 5–1

The strain in this fiber, its change in length per unit length, is

$$\epsilon = \frac{\delta}{\Delta x} = \frac{y\Delta\theta}{\rho\Delta\theta} = \frac{y}{\rho}$$

Stress in terms of strain and the modulus of elasticity is, therefore,

$$\sigma = \epsilon E = \frac{Ey}{\rho} \tag{5-1}$$

The value of σ given by this equation is called the *bending*, or *flexural*, *stress* and is directly proportional to the distance y from the neutral plane. The fibers elongate when they lie on the convex side of the neutral plane; the maximum tensile stress, therefore, occurs at the outermost fiber and has a magnitude of

$$= \frac{Ec_t}{\quad} \tag{a}$$

where c_t is the distance y from the neutral plane to the convex face of the beam. On the concave face of the beam, the fibers contract and are, therefore, in compression; in this case the maximum compressive stress is

$$\sigma_c = \frac{Ec_c}{\rho} \tag{b}$$

Although these equations can be employed to find bending stresses, they are not convenient formulas to use, since the radius of curvature ρ is usually an unknown quantity. An equation must be obtained which will relate bending stress to the external moment and the geometric properties of the beam. This can be done by considering a segment of the beam whose internal reaction at a given transverse section is a moment M, as shown in

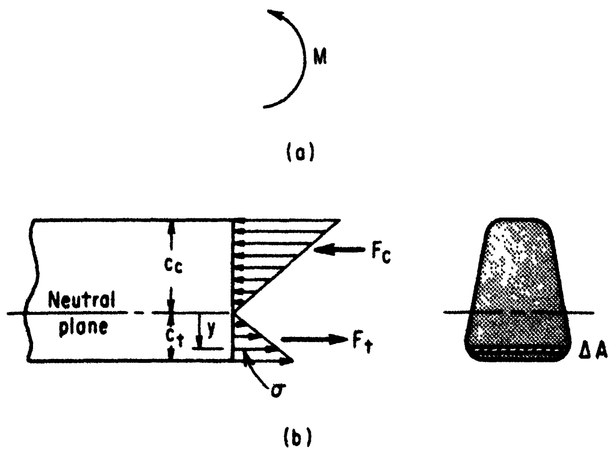


Fig. 5-2

Fig. 5-2(a). The stress distribution acting on the section, Fig. 5-2(b), varies directly with the distance from the neutral plane; in effect, the stress is a distributed horizontal load, partly negative and partly positive, whose resultant must be zero. Thus

$$\sum F_x = \sum \sigma \Delta A = 0 \tag{c}$$

where σ is an arbitrary stress acting on a cross-sectional element of area

ΔA. Replacing the value of stress by its equivalent from Eq. (5-1) gives

$$\sum \sigma \Delta A = \sum \frac{Ey}{\rho} \Delta A = \frac{E}{\rho} \sum y \Delta A = \frac{E\bar{y}\Delta A}{\rho} = 0 \tag{d}$$

The only factor in Eq. (d) that can possibly be zero is $\bar{y}$; this places the neutral axis of the beam at the centroid of its cross-sectional area.

To satisfy equilibrium, the internal and external moments at the section must be equal.

$$M = \sum \sigma y \Delta A \tag{e}$$

Expressing the stress in terms of its equivalent from Eq. (5-1) gives

$$M = \sum \frac{Ey}{\rho} y \Delta A = \frac{E}{\rho} \sum y^2 \Delta A$$

The summation $\sum y^2 \Delta A$ is, by definition, the moment of inertia of the cross-sectional area with reference to the neutral plane; therefore,

$$M = \frac{EI}{\rho} \tag{f}$$

Equations (5-1) and (f), when combined, give the formula for bending stress in terms of M, y, and I.

$$\sigma = \frac{My}{I} \tag{g}$$

For maximum values of tensile or compressive stress, y is set equal to either c_t or c_c:

$$\sigma_t = \frac{Mc_t}{I} \qquad \sigma_c = \frac{Mc_c}{I} \tag{5-2}$$

In symmetrical beams, c_t and c_c are equal, and the flexure formula is simply written as

$$\sigma = \frac{Mc}{I} \tag{5-3}$$

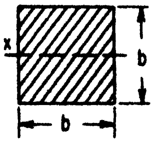
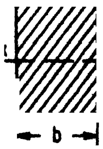
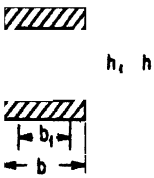
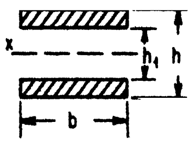
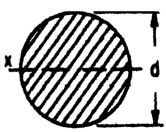
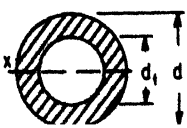
A common variation is obtained by writing Eq. (5-3) in terms of the *section modulus* Z, defined as the ratio I/c.

$$\sigma = \frac{M}{I/c} = \frac{M}{Z} \tag{5-4}$$

Table 5-1 lists the geometric properties, including the section moduli, of some common beam sections; the properties of standard structural shapes are tabulated in Appendix B.

Table 5-1

Properties of Areas

	Section	*Moment of Inertia*	*Section Modulus*
Square		$I_x = \frac{b^4}{12}$	$Z = \frac{b^3}{6}$
Rectangle		$I_x = \frac{bh^3}{12}$	$Z = \frac{bh^2}{6}$
Hollow Rectangle		$I_x = \frac{bh^3 - b_1h_1^3}{12}$	$Z = \frac{bh^3 - b_1h_1^3}{6h}$
Equal Rectangles		$I_x = \frac{b(h^3 - h_1^3)}{12}$	$Z = \frac{b(h^3 - h_1^3)}{6h}$
Circle		$I_x = \frac{\pi d^4}{64}$	$Z = \frac{\pi d^3}{32}$
Hollow Circle		$I_x = \frac{\pi(d^4 - d_1^4)}{64}$	$Z = \frac{\pi(d^4 - d_1^4)}{32d}$

Example 1. A simply supported timber beam 20 ft long carries a concentrated load P, as shown in Fig. 5-3(a). The beam has a 4 in. by 12 in. rectangular cross section. Determine the maximum value of P if the fiber stress is not to exceed 1200 psi, and (a) the 12 in. side is horizontal, (b) the 12 in. side is vertical.

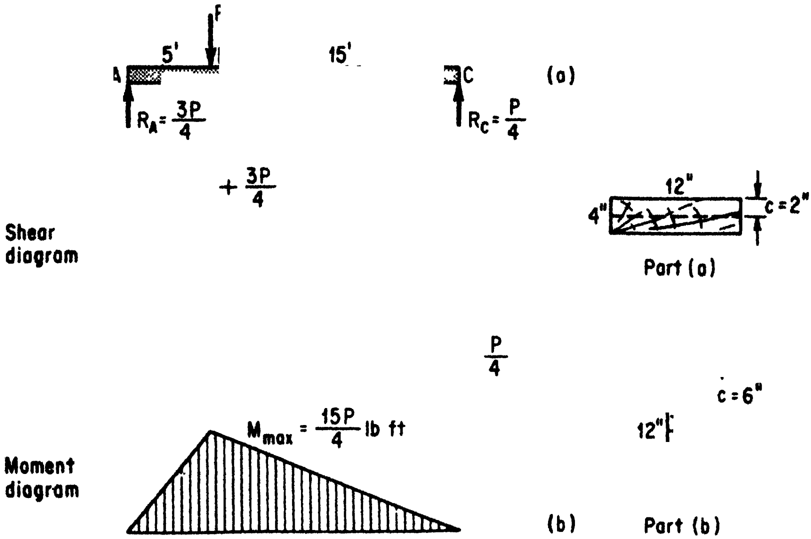


Fig. 5-3

Solution: The reactions $R_A = 3P/4$ and $R_C = P/4$ are first computed, and the shear and moment diagrams are drawn as shown in Fig. 5-3(b). The maximum moment, which occurs at B, is equal to the area of the shear diagram between points A and B.

$$M_{max} = 5\left(\frac{3P}{4}\right) = \frac{15P}{4} \text{ lb ft} = 45P \text{ lb in.}$$

Part (a). The moment of inertia about the neutral plane is

$$I_{NA} = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(12)(4)^3 = 64 \text{ in.}^4$$

By Eq. (5-3),

$$\sigma = \frac{Mc}{I}$$

$$1200 \quad \frac{45P(2)}{64}$$

$$P = 853 \text{ lb}$$

Part (b). With the 12 in. side vertical, the moment of inertia increases appreciably; thus

$$I_{NA} = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(4)(12)^3 = 576 \text{ in.}^4$$

By Eq. (5-3),

$$\sigma = \frac{Mc}{I}$$

$$1200 = \frac{45P(6)}{576}$$

$$P = 2560 \text{ lb}$$

Thus, three times the load can be supported when the beam is oriented with its long dimension vertical. This greater strength is readily apparent by comparing the section moduli.

With the short dimension vertical:

$$Z_a = \frac{I}{c} = \frac{1}{12}\frac{bh^3}{(h/2)} = \frac{bh^2}{6} \quad 12(4)^2 = 32 \text{ in.}^3$$

With the long dimension vertical:

$$Z = \frac{bh^2}{} = \frac{4(12)^2}{} = 96 \text{ in.}^3$$

Example 2. A steel band-saw blade $\frac{1}{2}$ in. wide and $\frac{1}{32}$ in. thick is driven by two 4-ft-diameter pulleys, as shown in Fig. 5-4. Determine the maximum stress developed in the blade and the magnitude of the internal moment.

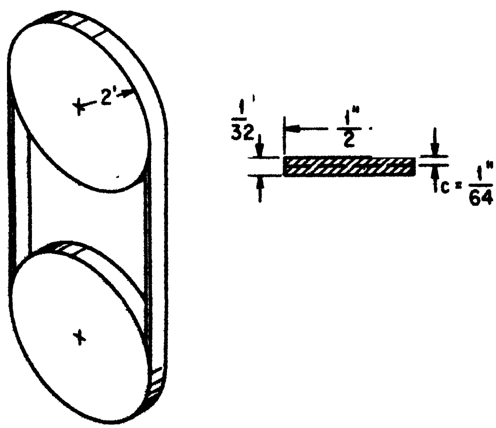


Fig. 5-4

Solution: By Eq. (5-1),

$$\sigma = \frac{Ec}{\rho} = \frac{30 \times 10^6}{2(12)} \left(\frac{1}{64}\right) = 19{,}500 \text{ psi}$$

The moment of inertia about the neutral plane is

$$I_{NA} = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(\tfrac{1}{2})(\tfrac{1}{32})^3 = 1.27 \times 10^{-6} \text{ in.}^4$$

By Eq. (5-3),

$$M = \frac{\sigma I}{c} = \frac{19{,}500(1.27 \times 10^{-6})}{\tfrac{1}{64}} = 1.58 \text{ lb in.}$$

Since the radius of curvature is constant, the internal moment is constant in the curved section of the blade.

Example 3. The 10-ft tee-beam illustrated in Fig. 5-5 supports a distributed load of 1000 lb per ft. Determine the maximum tensile and compressive bending stresses in the beam.

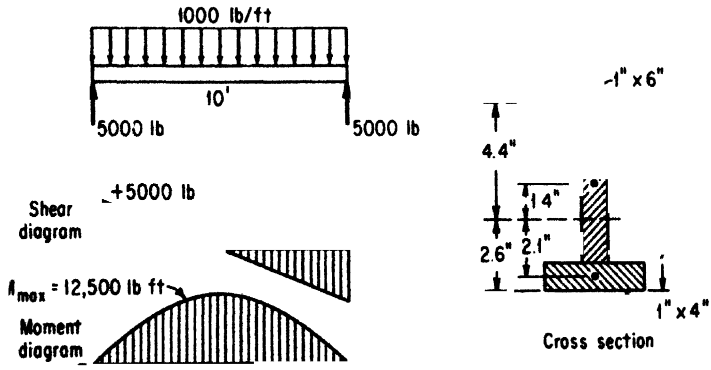


Fig. 5-5

Solution: Before we attempt to find the stress, the centroidal axis must be located and the moment of inertia with respect to this axis computed. The centroid is computed by taking the first moment of the area with reference to the base.

$$A\bar{y} = A_1y_1 + A_2y_2$$

$$y = \frac{4(1)(\frac{1}{2}) + 6(1)4}{4(1) + 6(1)} = 2.6 \text{ in. above the base}$$

The moment of inertia is computed by assuming the section to consist of two rectangles. The *parallel axis theorem* must be employed to transpose the inertia of each rectangle to the neutral, or centroidal, axis.

$$
\begin{aligned}
I_{NA} &= I_1 + I_2 \\
&= (\bar{I} + Ad^2)_1 + (\bar{I} + Ad^2)_2 \\
&= [\tfrac{1}{12}(4)(1)^3 + 4(1)(2.1)^2] + [\tfrac{1}{12}(1)6^3 + 6(1)(1.4)^2] \\
&= 0.33 + 17.64 + 18.00 + 11.76 \\
&= 47.7 \text{ in.}^4
\end{aligned}
$$

The bottom fibers of the beam are in tension, and the top fibers are in compression; hence,

$$\sigma_t = \frac{Mc_t}{I} = \frac{12{,}500(12)(2.6)}{47.7} = 8180 \text{ psi}$$

and

$$\sigma_c = \frac{Mc_c}{I} = \frac{12{,}500(12)(4.4)}{47.7} = 13{,}800 \text{ psi}$$

5-2 Longitudinal Shear in Beams

A second important factor to be considered in the determination of the strength of beams is *horizontal*, or *longitudinal*, shear. Many materials, wood, for example, are primarily weak in shear, and for this reason the load that can be supported may depend upon the ability of the beam to resist horizontal shearing forces.

The shearing tendency can be best described in terms of a beam, Fig. 5-6(a), that is composed of a number of thin layers placed with no adhesion one on top of the other. With a force P applied, as shown in Fig. 5-6(b), the laminations slide relative to one another as the beam sags. If the beam is solid, Fig. 5-6(c), this slipping tendency would be prevented by internal forces acting parallel to the axis of the beam. The intensity of these forces, the shearing stresses, act on every horizontal plane, since every such plane is a potential sliding surface.

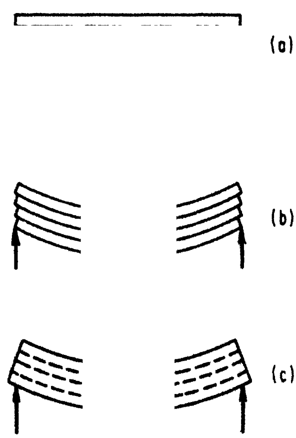


Fig. 5-6

To arrive at a relationship for the shearing stress, consider the beam shown in Fig. 5-7(a). At section a–d the moment is M_1, and at section b–c, an incremental

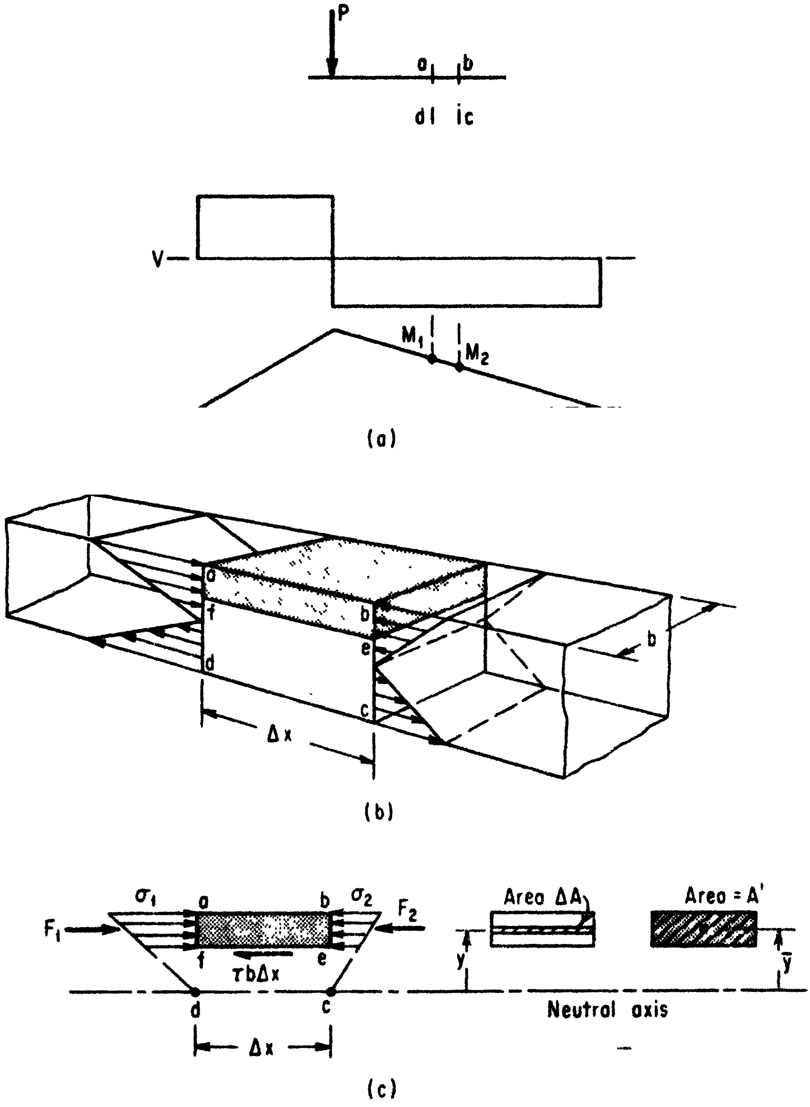


Fig. 5-7

distance Δx to the right, the moment is M_2. Next, consider an enlarged view of this section, Fig. 5-7(b), drawn to show the distribution of tensile and compressive bending stresses. In the element *abef*, the forces F_1 and F_2 are the resultants of the bending stress forces that act on the transverse planes *af* and *be*. The shear force $\tau b \Delta x$ that acts on plane *ef* is required to maintain equilibrium.

$$\tau b\Delta x = F_1 - F_2 \tag{a}$$

Since

$$F_1 = \sum \sigma_{y1}\Delta A \quad \text{and} \quad F_2 = \sum \sigma_{y2}\Delta A$$

it follows that

$$\tau b\Delta x = \sum \sigma_{y1}\Delta A - \sum \sigma_{y2}\Delta A \tag{b}$$

Next, substituting $\sigma_{y1} = M_1 y/I$ and $\sigma_{y2} = M_2 y/I$ gives

$$\tau b\Delta x = \frac{M_1}{I}\sum y\Delta A - \frac{M_2}{I}\sum y\Delta A$$

$$(M_1 - M_2) \sum y\Delta A \tag{c}$$

Since the difference $(M_1 - M_2)$ is small, Eq. (c) can be written

$$\tau b\Delta x = \frac{\Delta M}{I}\sum y\Delta A$$

Rearranging terms gives

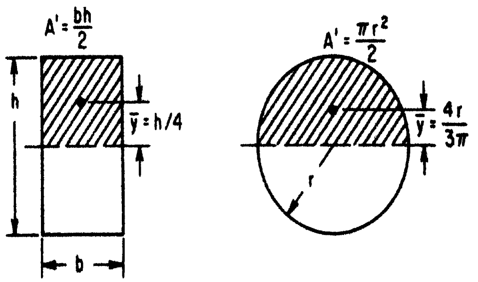
$$\frac{\Delta M}{\Delta x} \sum \frac{y\Delta A}{Ib} \tag{d}$$

Noting that $\Delta M/\Delta x = V$ and $\sum y\Delta A = \bar{y}A'$, the shear-stress equation can be written in its final form.

$$\tau = \frac{V}{Ib}A'\bar{y} \tag{5-5}$$

It is important to understand the meaning of $A'\bar{y}$; the term represents the first moment of the cross-sectional area lying above the plane at which shear is to be computed. The term b represents the width of the material comprising the beam at the shear plane under investigation.

In Eq. (5-5) V and I are constants for any given section; it follows, therefore, that the shearing stress is a maximum when the quantity $A'\bar{y}/b$ is a



(a) (b)

Fig. 5-8

maximum. In a rectangular beam, Fig. 5-8(a), the maximum shear stress occurs at the neutral axis and is equal to

$$= \frac{V}{Ib} A\bar{y} = \frac{V}{b^2h^3/12}\left(\frac{bh}{2}\right)\left(\frac{h}{4}\right) = \frac{3V}{2A} \tag{5-6}$$

where A is the area of the entire cross section. In a circular section, Fig. 5-8(b), the maximum shearing stress is again at the neutral axis and is equal to

$$\tau_{\max} = \frac{4V}{3A} \tag{5-7}$$

When the beam is one of the standard shapes, an I-beam or a wide flange beam, it is generally safe to assume that the flanges contribute very little in the way of resistance to shear; the beam is considered simply as a rectangular web section. It is common practice in structural design to approximate the maximum shearing stress from the equation

$$\tau_{\max} = \frac{3V}{2td} \tag{5-8}$$

where t and d are the web thickness and web depth respectively.

Example 4. The timber cantilever beam shown in Fig. 5-9 carries a load P at its free end. Find the safe value of P if the working stresses in tension or compression and in shear are $\sigma = 1200$ psi and $\tau = 100$ psi.

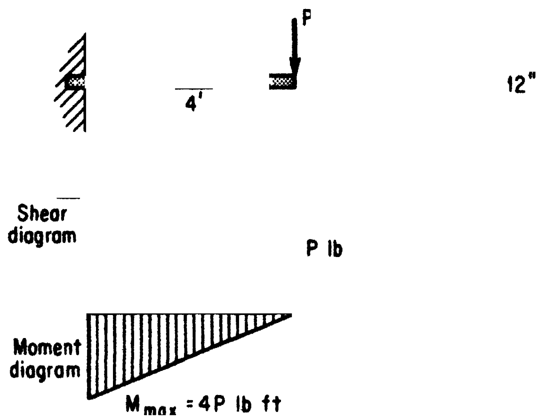


Fig. 5-9

Solution: The moment of inertia with reference to the neutral axis is first computed:

$$I = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(4)(12)^3 = 576 \text{ in.}^4$$

Next, a value of the load P is computed on the basis of the allowable flexural stress

$$\sigma = \frac{Mc}{I}$$

$$1200 = \frac{4(12)P(6)}{576}$$

$$P = 2400 \text{ lb}$$

A value of P is next determined on the basis of the permissible shearing stress; the vertical shear V in this instance is constant throughout the beam.

$$\tau = \frac{3}{2}\frac{V}{A}$$

$$100 = \frac{3}{2}\frac{P}{(4)12}$$

$$P = 3200 \text{ lb}$$

The lesser of the two values, $P = 2400$ lb, is the safe load.

Example 5. Determine the maximum value of the shearing stress in the box beam shown in Fig. 5-10.

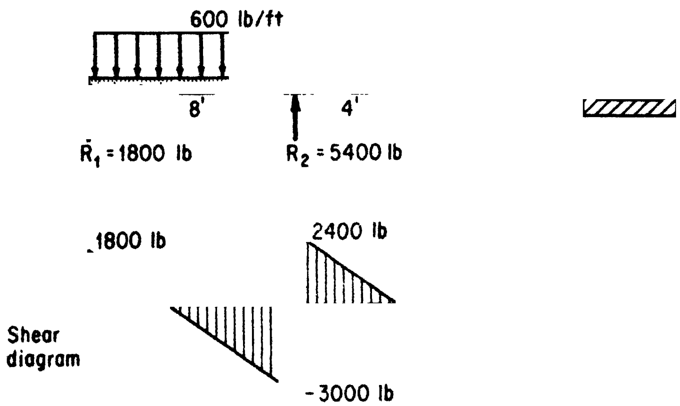


Fig. 5-10

Solution: The reactions R_1 and R_2 are computed by the usual methods:

$$R_1 = \frac{600(12)2}{8} = 1800 \text{ lb}$$

$$R_2 = \frac{600(12)6}{8} = 5400 \text{ lb}$$

When plotted, the shear diagram indicates the magnitude of maximum vertical shear to be 3000 lb just to the left of R_2. The terms I, $A'\bar{y}$, and b are computed prior to substitution into Eq. (5-5); thus,

$$I = \tfrac{1}{12}(6)(8)^3 - \tfrac{1}{12}(4)(6)^3 = 184 \text{ in.}^4$$

$$A'\bar{y} = A_1 y_1 - A_2 y_2 = 6(4)2 - 4(3)(1.5) = 30 \text{ in.}^4$$

$$b = 1 + 1 = 2 \text{ in.}$$

Therefore,

$$= \frac{V}{Ib} A'\bar{y} = \frac{3000(30)}{184(2)} = 245 \text{ psi}$$

5-3 Stresses in Built-up Beams

It is common construction practice to fabricate beams by joining various lightweight elements together to form a single strong section The sections illustrated in Fig 5-11 are typical examples of various built-up beams. The

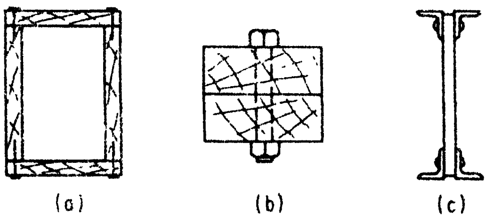


Fig. 5-11

first, Fig. 5-11(a), is a box beam fabricated by nailing or lagging planks together as shown. Another type of built-up beam, Fig. 5-11(b), consists simply of rectangular timbers bolted together to form a solid beam. A third type, the plate and angle girder shown in Fig. 5-11(c), consists of angle sections bolted or riveted to a web plate.

Of primary concern in built-up sections is the *spacing*, or *pitch*, between bolts or rivets that are used to secure the assembly. Consider the beam of Fig. 5-12(a), which is composed of three timbers bolted together to form a composite section. If the beam were truly solid. the shearing stress across a-a would be given by

$$\tau = \frac{V}{Ib} A'\bar{y}$$

where $A'\bar{y}$ represents the first moment of the cross-sectional area of the top plank with reference to the neutral axis. Since the beam is not solid, a typical

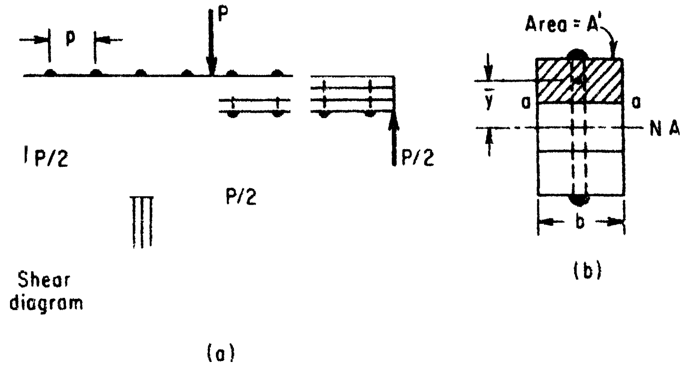


Fig. 5–12

bolt in the assembly must support a force in shear equal to the product of the shearing stress and the area bp, where p represents the distance or *pitch* between adjacent bolts

$$F = \tau b p = p\frac{VA'\bar{y}}{I} \tag{5-9}$$

In built-up beams the usual practice is to maintain a constant pitch, even though the vertical shear varies from point to point, and to base this pitch on the greatest value of vertical shear.

It is interesting to note that Eq. (5-9) can be used to determine pitch for all types of built-up sections; in every case the term $A'\bar{y}$ represents the first moment of the area that tends to slide relative to the beam itself. The example that follows will illustrate this concept.

Example 6. The built-up beam of Fig. 5-13(a) consists of four 6 in. by 4 in. by $\frac{7}{16}$ in. angles connected to a web plate by equally spaced $\frac{3}{4}$ in. rivets. The

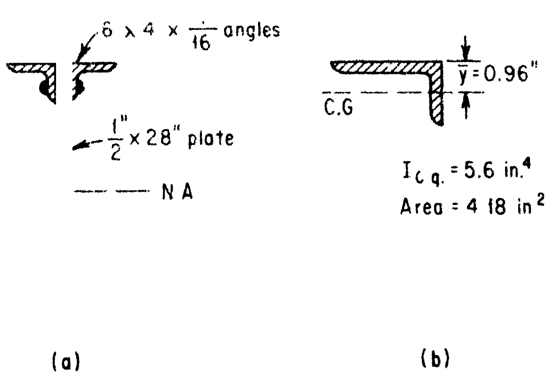


Fig. 5–13

web plate is ½ in. thick and has a depth of 28 in. Determine the rivet spacing if the allowable shearing stress in the rivets is 8000 psi. The beam carries a uniformly distributed load of 4000 lb per ft on a span of 20 ft.

Solution: The maximum vertical shear occurs at the supports and is equal to

$$V = \frac{wl}{2} = \frac{4000(20)}{2} = 40{,}000 \text{ lb}$$

If it were not for the rivets, the angles at the top (or bottom) would move relative to the web. The force necessary to prevent this motion is represented by Eq. (5-9), where $A'\bar{y}$ is the first moment of the shaded area about the neutral axis, and I is the moment of inertia of the entire cross section.

$$A'\bar{y} = 2(4.18)\,(14 - 0.96) = 109 \text{ in.}^3$$

$$\begin{aligned} I &= I_{\text{web}} + 4I_{\text{angle}} \\ &= \tfrac{1}{12}bh^3 + 4(\bar{I} + Ad^2) \\ &= \tfrac{1}{12}(\tfrac{1}{2})\,(28)^3 + 4[5.6 + 4.18(14 - 0.96)^2] \\ &= 3780 \text{ in.}^4 \end{aligned}$$

By Eq. (5-9),

$$F = p\frac{VA'\bar{y}}{I} = p\frac{(40{,}000)\,(109)}{3780}$$

$$F = 1150p \qquad \text{(a)}$$

Each rivet is in *double shear* and is capable of supporting a load of

$$F = \tau A = (2)8000\,\frac{\pi}{4}\left(\frac{3}{4}\right)^2 = 7070 \text{ lb} \qquad \text{(b)}$$

Equating Eqs. (a) and (b) gives the required pitch; thus,

$$1150p = 7070$$

$$p = 6.15 \text{ in.}$$

5-4 *Beams of Two or More Materials*

Certain advantages are often derived by combining two or more different materials to form a single beam; appearance, weight reduction, and strength are a few of the factors that often lead to the design of composite beams.

To simplify the computations, it is convenient to *transform* the composite section into a single material; to imagine, in other words, that the beam is composed of one homogeneous substance. The transformed beam and the composite beam must be of *equivalent sections*; they must deform equally

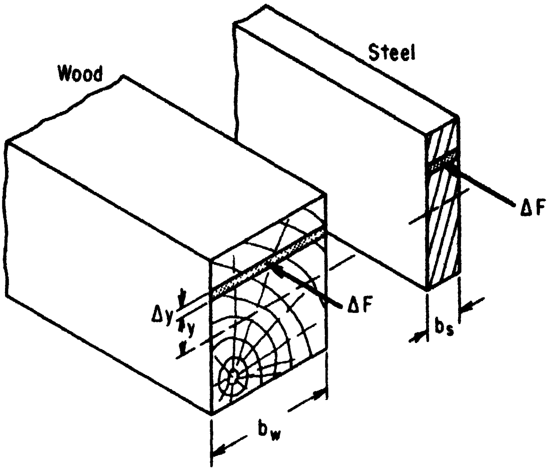


Fig. 5-14

under the action of the transverse loads. Consider the two beams, one wood and the other steel, shown in Fig. 5-14. Both sections have the same depth but vary in width. To be equivalent, the strains at corresponding distances from the neutral axis of both beams must be equal.

$$\epsilon_w = \epsilon_s \tag{a}$$

Since stress and strain are proportional, and since it is assumed that the stress does not exceed the proportional limit, it follows that

$$\frac{\sigma_w}{E_w} \quad \frac{\sigma_s}{E_s} \tag{5-10}$$

The beams have equal strengths, and both can sustain a similar force ΔF applied at equivalent distances from their neutral axes. This force, in terms of stress and area, is

$$\Delta F = \sigma_s b_s \Delta y = \sigma_w b_w \Delta y \tag{b}$$

Eqs. (5-10) and (b) can be combined to give

$$\sigma_s b_s = \frac{E_w}{E_s} \sigma_s b_w$$

$$b_s = \frac{E_w}{E_s} b_w \tag{c}$$

If the moduli of wood and steel are 1.5×10^6 psi and 30×10^6 psi respectively, the widths of the beams differ by a factor of 20.

$$b_s = \frac{1.5(10)^6}{30(10)^6} b_w = \frac{b_w}{20}$$

To be more general, the equivalent width of material 1 in terms of a second material 2 in a composite beam is

$$b_1 = \frac{E_2}{E_1} b_2 \tag{5-11}$$

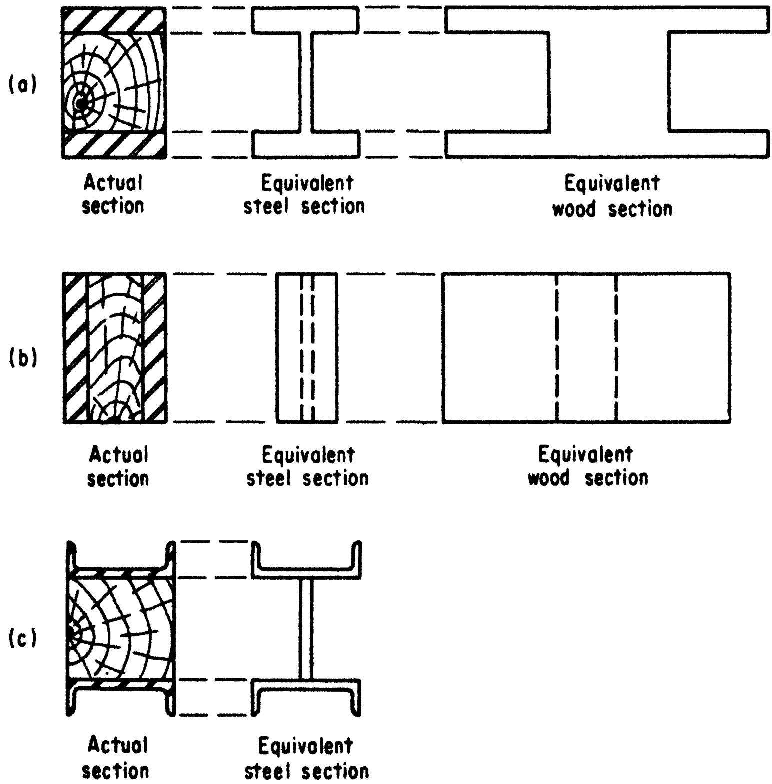


Fig. 5-15

The various beam sections of Fig. 5-15 illustrate the types of transformations possible. Wood and steel are used as examples, although any two or more materials can be combined to form a composite beam. In Fig. 5-15(a), a wood block is laminated with steel plates at the top and bottom. For analysis, this section can be transformed into an all-steel section or an all-wood section, as shown. In the second example, Fig. 5-15(b), a wood core is faced with steel plates; as before, the depth of the equivalent section remains the same; as an imaginary solid steel beam, the section is much narrower than as an imaginary solid wooden beam. This is realistic, since much less steel would be required to match the strength of the wood. It would be mathematically difficult to transform the steel channel, Fig. 5-15(c), into an equivalent wood section; for this case the wood is exchanged for an equivalent amount of steel.

Once the transformed section is obtained, the analysis of bending proceeds as though the beam were either steel or wood. The stresses computed, however, are those sustained by the imaginary section. One value of stress will be the actual value, whereas the other must be corrected through use of the proportion given in Eq. (5-11):

$$\frac{\sigma_w}{E_w} = \frac{\sigma_s}{E_s}$$

Example 7. A composite beam, Fig. 5-16(a), consists of a timber core reinforced by steel plates. The beam supports a distributed load of 500 lb per ft on a 12 ft span. Determine the bending stress in both materials. Steel and wood have moduli of elasticity of 30×10^6 psi and 1.5×10^6 psi, respectively.

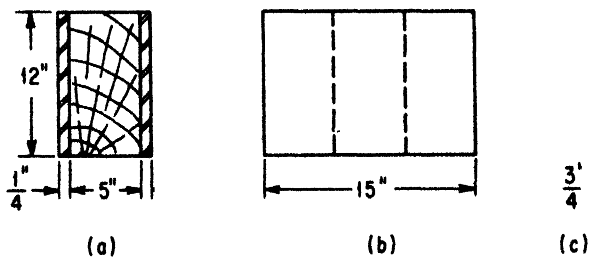


Fig. 5-16

Solution: The problem can be solved by transforming the composite section either to solid timber or to solid steel. Both methods will be applied to illustrate the approach.

Transformation to timber. An equivalent width of each steel plate in terms of the wood is given by Eq. (5-11):

$$b_w = \frac{E_s b_s}{E_w} = \frac{30(10)^6}{1.5(10)^6}\,\frac{1}{4} = 5 \text{ in.}$$

The composite section is 15 in. wide, as shown in Fig. 5-16(b), and has a moment of inertia of

$$I_{NA} = \frac{1}{12}bh^3 = \frac{1}{12}(15)(12)^3 = 2160 \text{ in.}^4$$

The maximum bending stress is

$$\sigma = \frac{Mc}{I}$$

where the moment M is

$$M = \frac{3000(6)}{2} \times 12 = 108{,}000 \text{ lb in.}$$

Therefore,

$$\sigma = \frac{108{,}000(6)}{2160} \quad 300 \text{ psi}$$

Numerical values are substituted into Eq. (5-10) to find the true stress in the steel.

$$\sigma_s = \frac{E_s}{E_w}\sigma_w = \frac{30(10)^6}{1.5(10)^6}(300) = 6000 \text{ psi}$$

Transformation to steel. An equivalent width of steel to replace the wood is

$$b_s = \frac{E_w}{E_b}b_w = \frac{1.5(10)^6}{30(10)^6}(5) = \frac{1}{4} \text{ in.}$$

The transformed steel section is, therefore, $\frac{3}{4}$ in. wide and has a moment of inertia of

$$I_{NA} = \frac{1}{12}bh^3 = \frac{1}{12}(\tfrac{3}{4})(12)^3 = 108 \text{ in.}^4$$

The flexure equation gives the true bending stress in the steel, whereas the stress in the wood must be corrected.

$$\sigma_s = \frac{Mc}{I} = \frac{108{,}000(6)}{108} = 6000 \text{ psi}$$

$$\sigma_w = \frac{E_w}{E_s}\sigma_s = \frac{1.5(10)^6}{30(10)^6}(6000) = 300 \text{ psi}$$

Example 8. Steel and aluminum are combined to form the composite beam shown in Fig. 5-17(a). Determine the greatest uniformly distributed load which can be supported on a span of 16 ft. Working stresses for the steel and aluminum are $\sigma_s = 15{,}000$ psi and $\sigma_a = 6000$ psi. $E_s = 30 \times 10^6$ psi; $E_a = 10 \times 10^6$ psi.

Solution: The midspan moment, by the usual methods, is

$$M = \tfrac{1}{8}(8w)8(12) = 384w \text{ lb in.}$$

A choice as to whether to convert the steel to aluminum or the aluminum to steel is arbitrary; however, if the latter transformation is employed, computations will involve smaller numbers.

The width of steel equivalent to the aluminum is

$$b_s = \frac{E_a}{E_s}b_a = \frac{10(10)^6}{30(10)^6}(3) = 1 \text{ in.}$$

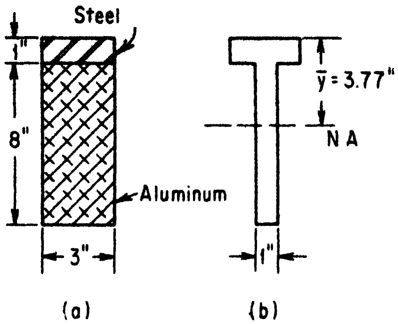


Fig. 5–17

First moments taken with respect to the top of the beam will locate the neutral axis.

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2}$$

$$= \frac{2(1)(0.5) + 9(1)(4.5)}{3(1) + 8(1)} = 3.77 \text{ in.}$$

The parallel axis theorem is employed to find the moment of inertia of the transformed section with respect to the neutral axis.

$$I = (\tfrac{1}{12}bh^3 + Ad^2) + (\tfrac{1}{12}bh^3 + Ad^2)$$

$$I_{NA} = [\tfrac{1}{12}(3)(1)^3 + 3(1)(3.77 - 0.5)^2] + [\tfrac{1}{12}(1)(8)^3 + 8(1)(5.23 - 4)^2]$$

$$= 0.25 + 32.08 + 42.67 + 12.10 = 87.1 \text{ in.}^4$$

The maximum compressive bending stress occurs at the top of the beam, and since both the original and transformed sections are of the same material in this region, the allowable stress is found by direct substitution.

$$\sigma_s = \frac{Mc}{I}$$

$$15{,}000 = \frac{384w(3.77)}{87.1}$$

$$w = \frac{15{,}000(87.1)}{384(3.77)} \quad 900 \text{ lb per ft}$$

The bottom portion of the original section is aluminum, and its corrected allowable stress in terms of the transformed beam is

$$\frac{E_s}{E_a} \quad \frac{30(10)^6}{10(10)^6}(6000) = 18{,}000 \text{ psi}$$

A permissible load in terms of the fiber stress at the bottom becomes

$$\sigma_s = \frac{Mc}{I}$$

$$18{,}000 = \frac{384w(5.23)}{87.1}$$

$$w = \frac{18{,}000(87.1)}{384(5.23)} = 780 \text{ lb per ft}$$

Aluminum governs the design, and the safe load is 780 lb per ft.

5-5 *Reinforced Concrete Beams*

Transformation concepts, developed in the previous article, can be applied in a rather interesting manner to reinforced concrete beams. Theory must be adjusted, however, to account for the fact that concrete has virtually no strength in tension. The steel reinforcing bars shown in Fig. 5-18(a) must assume a tensile load equal and opposite to the compressive load in the

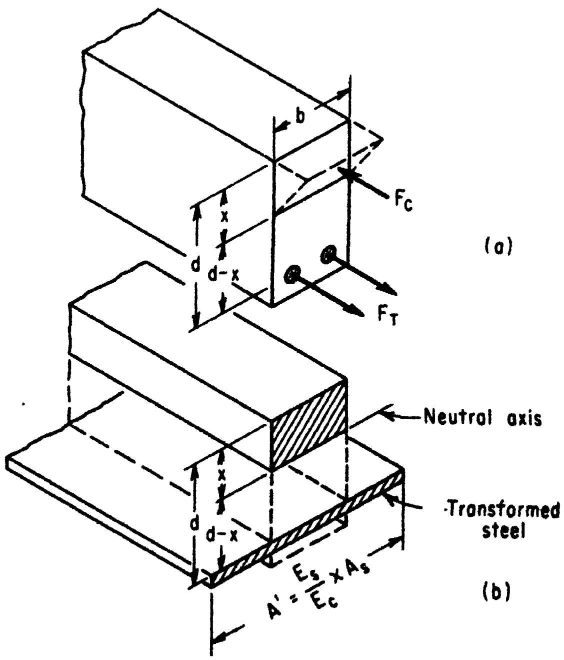


Fig. 5-18

concrete. Furthermore, the moment of the transformed area of the steel bars about the neutral axis must just equal the moment of the area of the *active* concrete that lies above the neutral axis; this computation locates the neutral axis.

Using standard notation, d is the distance from the top of the beam to the center line of the steel bars, and x the height of the concrete above the neutral axis. The width of the beam is designated as b and the transformed steel area as A', where

$$A' = \frac{E_{\text{steel}}}{E_{\text{concrete}}} \times A_{\text{steel}} \tag{5-12}$$

To simplify computations, the height of the transformed steel area is assumed to be negligible, and the single dimension $(d - x)$ locates the area below the neutral axis. Equating the first moments of the two areas will locate the neutral axis, and the moment of inertia can be computed. Stresses can then be determined by the methods outlined in the previous section. The examples that follow illustrate the complete approach.

Example 9. A reinforced-concrete beam, Fig. 5-19, is subjected to a bending moment of 500,000 lb in. Determine the maximum bending stress in the concrete and in the steel. Assume the ratio of the moduli of elasticity of steel to concrete to be 15.

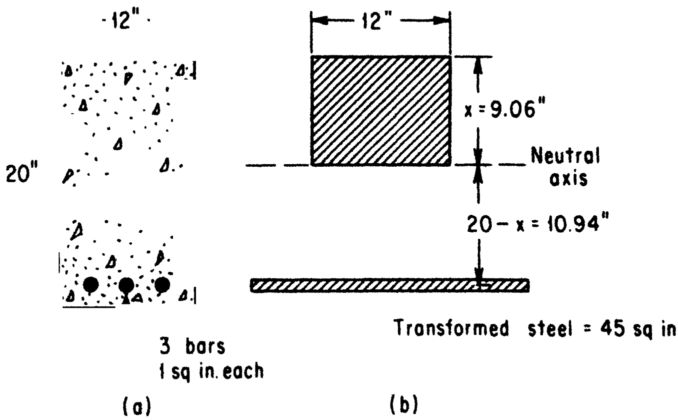


Fig. 5-19

Solution: The transformed area of the steel is

$$A' = \frac{E_s}{E_c}A_s = 15(3) = 45 \text{ sq in.}$$

The moments of the areas representing the *active* concrete and the steel must equate.

$$12(x)\left(\frac{x}{2}\right) = 45(20 - x)$$

$$x^2 + 7.5x - 150 = 0$$

Solving for x by the quadratic formula (only the positive root is of interest) gives

$$x = \frac{-7.5 \pm \sqrt{(7.5)^2 + 4(1)150}}{2(1)} = 9.06 \text{ in.}$$

Thus, $$x = 9.06 \quad \text{and} \quad 20 - x = 10.94$$

The moment of inertia, with respect to the neutral axis, and the bending stresses are then computed.

$$I_{NA} \quad \tfrac{1}{3}bh^3 + Ad^2$$

$$= \tfrac{1}{3}(12)(9.06)^3 + 45(10.94)^2 = 8360 \text{ in.}^4$$

$$= \frac{Mc}{I} = \frac{500{,}000(9.06)}{8360} = 542 \text{ psi}$$

$$(\sigma_s)_{\max} = 15\frac{Mc}{I} = \frac{15(500{,}000)10.94}{8360} \quad 9810 \text{ psi}$$

Example 10. The ideal design of a reinforced-concrete beam requires both the steel and the concrete to be stressed to their allowable limits; this is called *balanced reinforcement*. Determine the number of ⅝-in.-diameter steel reinforcing bars required for balanced reinforcement and the safe moment for the beam section shown in Fig. 5-20(a). The allowable stresses are $\sigma_c = 800$ psi and $\sigma_s = 18{,}000$ psi. Use $E_s/E_c = 15$.

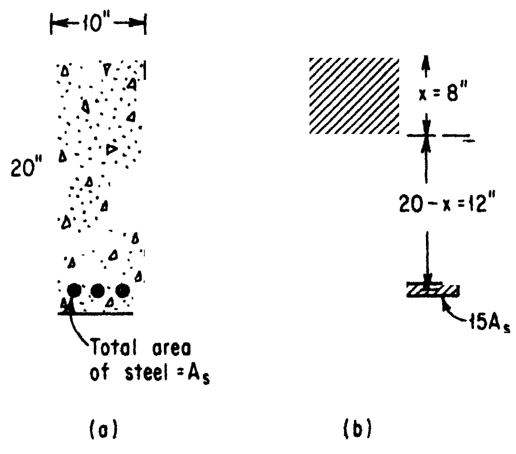


Fig. 5-20

Solution: Substitution of the values of the allowable stresses in the flexure formula provides a relationship for the distance x.

For the concrete: Mc

$$800 = \frac{M}{I}x \tag{a}$$

For the transformed steel: $15Mc$

$$18{,}000 = \frac{15M}{I}(20 - x) \tag{b}$$

Dividing Eq. (a) by Eq. (b) eliminates the common ratio M/I.

$$\frac{800}{18{,}000} = \frac{x}{15(20 - x)}$$

$$8(15)(20 - x) = 180x$$

$$20 - x = \tfrac{3}{2}x$$

$$x = 8 \text{ in.}$$

The required area of steel can now be found by equating first moments.

$$12(15)A_s = 10(8)(\tfrac{8}{2})$$

$$A_s = 1.78 \text{ sq in.}$$

The moment of inertia, based on the transformed area, and the safe bending moment are then computed.

$$I_{NA} = \tfrac{1}{3}bh^3 + Ad^2 = \tfrac{1}{3}(10)(8)^3 + (15 \times 1.78)(12)^2 = 5550 \text{ in.}^4$$

$$M = \frac{\sigma I}{c} = \frac{800}{8}(5550) = 555{,}000 \text{ lb in. or } 555 \text{ kip in.}$$

The required number of $\tfrac{5}{8}$-in.-diameter bars is

$$N = \frac{1.78}{\pi(\tfrac{5}{16})^2} = 5.8; \text{ use 6 bars}$$

PROBLEMS

5-1. A $\frac{1}{16}$-in.-diameter steel wire is coiled around a 5-ft-diameter mandril. Determine the moment and the bending stress in the wire.

5-2. A flat steel spring 1 in. wide and $\frac{1}{8}$ in. thick is subjected to end moments, as shown in Fig. P5-2. Determine the minimum radius of curvature to which the spring may be bent and the internal moment at this radius if the flexural stress is not to exceed 45,000 psi.

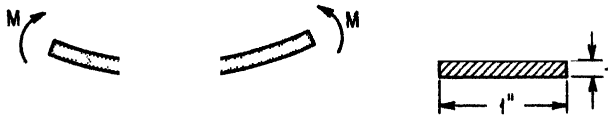


Fig. P5–2

5-3. Strain-gage measurements made on a beam that lacks symmetry about the neutral axis indicate the maximum compressive stress (at the top) to be 20,000 psi and the maximum tensile stress (at the bottom) to be 5000 psi. If the beam has a depth of 10 in., how far from the top is the neutral plane?

5-4. Determine the diameter of a circular bar that has a section modulus, with respect to a diameter, of 10 in.^{3}

5-5. Determine the section modulus of a rectangular beam 4 in. wide and 12 in. deep with respect to a line passing through the center and perpendicular to the 12 in. side.

5-6. A rectangular bar is simply supported as a beam, as shown in Fig. P5-6. Determine the maximum permissible value of P if the bending stress is not to exceed 20,000 psi.

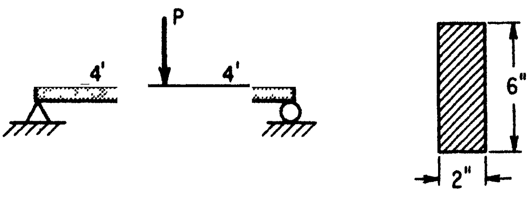


Fig. P5–6

5-7. Solve Prob. 5-6, assuming the beam to be a hollow tube with a wall thickness of $\frac{1}{8}$ in. The outside dimensions are the same as those of the solid bar.

5-8. The maximum fiber stress at a certain section in a rectangular beam 4 in. wide by 6 in. deep is 2400 psi. Determine the internal moment in the beam at this section.

5-9. Select the most economical (lightest weight) wide-flanged beam that can be safely subjected to a moment of 80,000 lb ft. The maximum bending stress in the beam is not to exceed 20,000 psi.

5-10. A wooden beam having a cross section of 4 in. wide by 12 in. deep carries two equal loads, as shown in Fig. P5-10. Find the magnitudes of these loads if the flexural stress is not to exceed 6000 psi and, (a) the beam bends about an axis parallel to the 4 in. face; (b) the beam bends about an axis parallel to the 12 in. face.

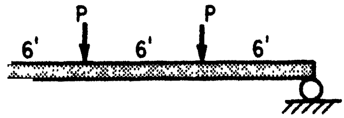


Fig. P5–10

5-11. Select the most economical beams, one an I-beam and the other a wide-flanged beam, to be used as a simply supported girder with a span of 20 ft. The beam carries a uniformly distributed load of 800 lb per ft. The flexural stress in either beam is not to exceed 15,000 psi.

5-12. A rectangular wooden beam supports the uniformly distributed load as shown in Fig. P5-12. Determine the flexural stress in the beam at all points where the shear is zero.

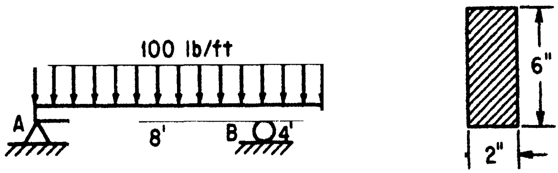


Fig. P5–12

5-13. The beam shown in Fig. P5-13 is constructed by welding $\frac{1}{2}$ in. by 10 in. cover plates to two 10 in., 20 lb per ft channels, as shown. What maximum uniformly distributed load can this beam support on a 20 ft span if the maximum fiber stress is not to exceed 20,000 psi?

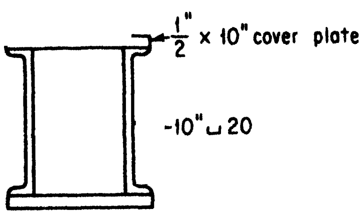


Fig. P5–13

5-14. The tee-beam carries two concentrated loads, as shown in Fig. P5-14. Find the maximum tensile and compressive stresses in the beam.

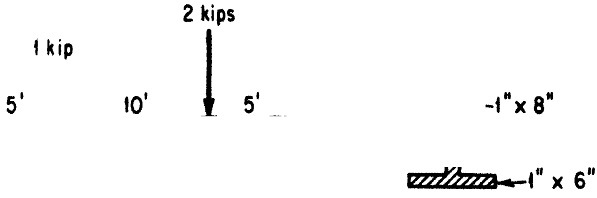


Fig. P5–14

5-15. Determine the greatest permissible distributed load w that the beam of Fig. P5-15 is capable of supporting if the allowable flexural tensile and compressive stresses are 10,000 psi and 6000 psi respectively.

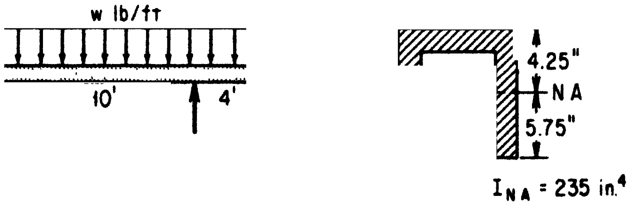


Fig. P5–15

5-16. A 10 WF 60 steel beam is loaded and supported as shown in Fig. P5-16. Determine the maximum bending stress on a section of the beam at the wall.

5-17. Compare the strengths of two beams, identical in all respects, except that one is hollow and the other solid, as shown in Fig. P5-17.

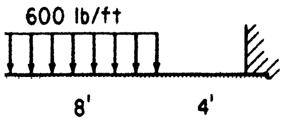


Fig. P5–16

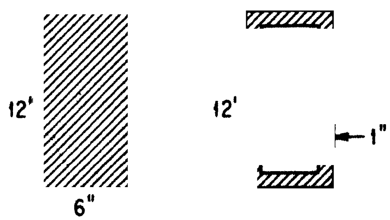


Fig. P5–17

5-18. A log is available from which a mill may cut either a 14 in. by 15 in. or a 10 in by 18 in. section. Which will make the stronger beam and by what factor?

5-19. Select the most economical wide-flanged steel beam 21 ft long to carry the following total load: a uniform load of 500 lb per ft throughout the entire span, a concentrated load of 18 kips at a point 6 ft from the left end, and an additional uniform load of 600 lb per ft on the right-hand third of the beam. Base the selection on an allowable bending stress of 20,000 psi and neglect the weight of the beam.

5-20. Two steel angles, each 6 in. by 4 in. by ½ in., are combined and used with their 4 in. legs at the top to form a tee section for a cantilever beam 10 ft long. If the beam supports a uniform load of 400 lb per ft over its entire length, find the maximum tensile and compressive stresses in the beam.

5-21. A 200 lb man stands in the middle of a 100 lb plank that is floating in water. The plank is 10 ft long, 2 in. thick, and 12 in. wide. Find the maximum bending stress in the plank.

5-22. A rectangular timber beam 10 ft long carries a uniformly distributed load of 500 lb per ft, as shown in Fig. P5-22. Determine the maximum shearing stress acting within the beam.

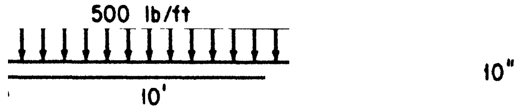


Fig. P5-22

5-23. A circular log 10 in. in diameter is used as a simply supported beam with a 12 ft span. The beam supports two equal loads P at its third points. Find the safe value of P if the allowable working stresses in tension or compression and in shear parallel to the grain are $\sigma = 1200$ psi and $\tau = 120$ psi respectively.

5-24. A rectangular section 4 in. wide and 12 in. deep is subjected to a vertical shear force of $V = 20$ kips. Determine the shearing stress on a horizontal plane (a) 2 in. below the top of the beam, and (b) 6 in. below the top of the beam.

5-25. A simply supported beam with an 8 ft span carries a uniformly distributed load of w lb per ft. The cross section of the beam is a rectangle 3 in. wide by 8 in. deep. Determine the maximum shearing stress in the beam if the maximum bending stress is 1200 psi.

5-26. Find the maximum shearing stress in the beam shown in Fig. P5-26. *Hint:* The maximum shearing stress occurs at the neutral plane.

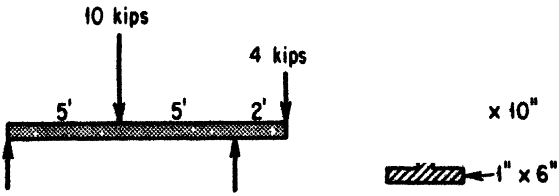


Fig. P5-26

5-27. The cross section of the timber beam, Fig. P5-27, is a rectangle 6 in. wide by 12 in. deep. Determine the maximum value of w if the allowable stresses in bending and shear are $\sigma = 1200$ psi and $\tau = 72$ psi respectively.

5-28. A timber beam has the cross section shown in Fig. P5-28. What is the maximum shearing stress in this section if the vertical shear V is 5000 lb?

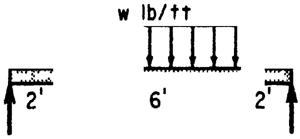


Fig. P5-27

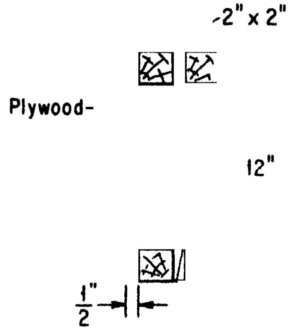


Fig. P5-28

5-29. The cross section of an aluminum wing channel is shown in Fig. P5-29. Determine the safe distributed load w lb per ft that can act on this section supported as a simple beam with 4 ft span. The permissible stresses in bending and shear are $\sigma = 2000$ psi and $\tau = 800$ psi respectively.

5-30. Two 6 in. by 6 in. timbers are bolted together to form a single solid beam, as shown in Fig. P5-30. The bolts each have a cross-sectional area of 0.625 in.^{2}, and the beam supports a uniformly distributed load of 500 lb per ft on a span of 20 ft. Determine the required pitch of the bolts if their shearing stress is not to exceed 8800 psi.

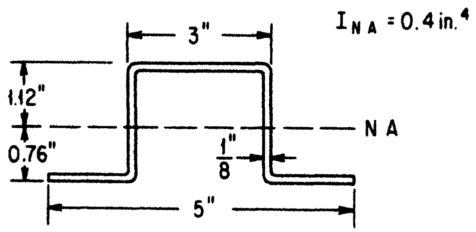


Fig. P5-29

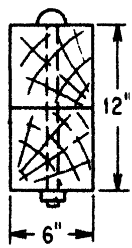


Fig. P5-30

5-31. Three 4 in. by 4 in. full-dimension timbers, Fig. P5-31, are fastened together by carriage bolts having cross-sectional areas of 0.4 in.^{2} each. What concentrated load P can the beam support if the following restrictions are placed on the design? The maximum shearing stress in the bolts is not to exceed 8800 psi, and the maximum flexural stress in the wood is not to exceed 2000 psi.

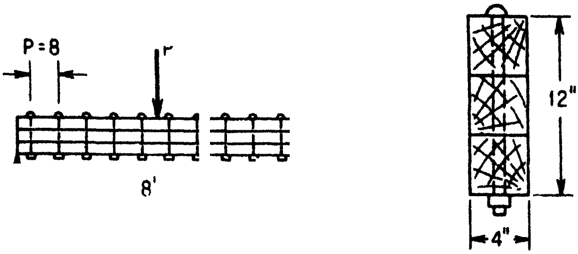


Fig. P5–31

5-32. Plywood and full-dimension 2 in. by 4 in. blocking are lagged together to form the beam section shown in Fig. P5-32. Find the required pitch of the screws if the beam supports a uniform load of 100 lb per ft on a 16 ft span. The screws have cross-sectional areas of 0.01 in.^{2} and a permissible working stress in shear of 8800 psi.

5-33. Assume the beam of Prob. 5-32 to carry a distributed load of w lb per ft sufficiently great to cause a shearing stress of 80 psi at the neutral axis of the beam. Determine the required pitch of the screws.

5-34. A timber beam is made of three 4 in. by 4 in. members glued together, as shown in Fig. P5-34. The beam is simply supported and has a span of 6 ft. Determine the safe load P that can be placed at midspan if the permissible shearing stresses in the wood and in the glue are 320 psi and 200 psi respectively.

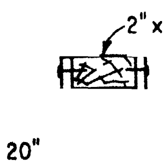


Fig. P5–32*

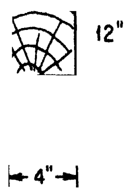


Fig. P5–34

5-35. A built-up steel girder has the cross section shown in Fig. P5-35. The moment of inertia of the cross section with reference to the neutral axis is $I_{NA} = 4470$ in.^{4} Find the proper rivet spacing if the girder is to support a maximum shear of $V = 90$ kips. The rivets have diameters of $\frac{3}{4}$ in. (area = 0.442 in.^{2}), and a permissible shearing stress of 8800 psi.

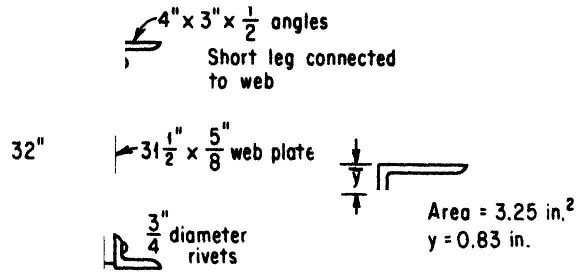


Fig. P5-35

5-36. A wide-flange beam and a channel are riveted together, as shown in Fig. P5-36. Find the maximum allowable vertical shear V the section is capable of supporting. The $\frac{3}{4}$-in.-diameter (area = 0.442 in.^{2}) rivets have a pitch of 4 in. and a permissible stress in shear of 8800 psi.

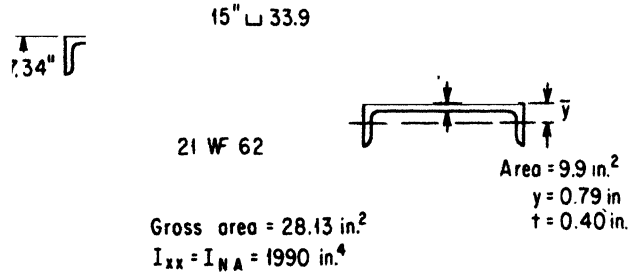


Fig. P5-36

5-37. A 20 ft timber beam 6 in. wide by 10 in. deep is reinforced with steel plates, 6 in. wide by 1 in. thick, top and bottom. Determine the maximum fiber stress in each material if the beam is simply supported and carries a uniformly distributed load of 400 lb per ft. $E_w = 1.5 \times 10^6$ psi.

5-38. An 8-ft-timber cantilever beam 10 in. wide by 18 in. deep is reinforced with steel plates, 18 in. wide by $\frac{1}{4}$ in. thick on either side. Determine the maximum fiber stress in each material if the beam supports a concentrated load of 10,000 lb at its free end. $E_w = 1.5 \times 10^6$ psi.

5-39. A 10 in. by 12 in. aluminum wide-flanged beam is blocked with oak timbers, as shown in Fig. P5-39. Calculate the bending stress in the timber and in the aluminum if the beam is subjected to a moment of 20 kip ft. $E_w = 2 \times 10^6$ psi; $E_a = 10 \times 10^6$ psi.

5-40. Steel and aluminum are combined to form the beam section shown in Fig. P5-40. Determine the maximum moment that can act on the section if the permissible bending stresses are $\sigma_a = 5000$ psi and $\sigma_s = 15{,}000$ psi. $E_a = 10 \times 10^6$ psi; $E_s = 30 \times 10^6$ psi.

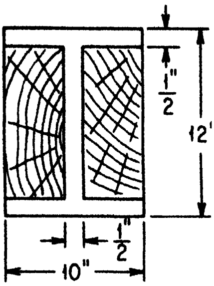


Fig. P5–39

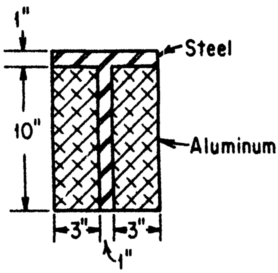


Fig. P5–40

5-41. A laminated-timber beam has the cross section shown in Fig. P5-41. Alternate layers of wood with the grain running at right angles to one another comprise the section. If the moduli parallel to the grain and across the grain are 2×10^6 psi and 0.5×10^6 psi respectively, determine the safe distributed load w lb per ft that the beam can carry on a 10 ft span. The permissible bending stresses in the strong and weak directions are $\sigma = 1200$ psi and $\sigma = 300$ psi respectively.

5-42. Two 10 in., 30 lb channels are combined with timber to form the composite section shown in Fig. P5-42. Determine the safe bending moment for this section if the maximum permissible bending stresses are $\sigma_w = 1200$ psi and $\sigma_s = 18{,}000$ psi. $E_w = 1.5 \times 10^6$ psi; $E_s = 30 \times 10^6$ psi.

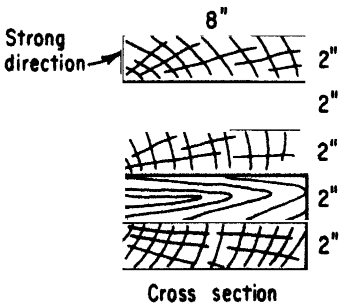


Fig. P5–41

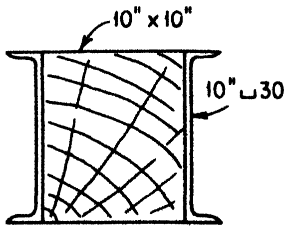


Fig. P5–42

5-43. A small cantilevered leaf spring is made of two pieces of brass and one piece of steel, each measuring 1 in. wide by 0.1 in. thick, brazed to form a solid section. Determine the stress in each material if a concentrated force $P = 10$ lb acts at the free end, as shown in Fig. P5-43. $E_b = 15 \times 10^6$ psi; $E_s = 30 \times 10^6$ psi.

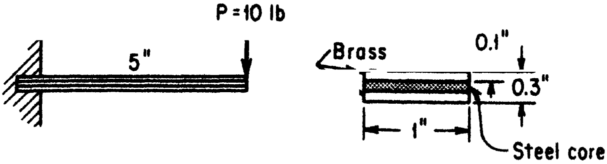


Fig. P5–43

5-44. Compare the strengths of the two laminated beam sections shown in Fig. P5-44. One has a soft wood core and steel flange plates and the other a steel core with soft wood flange plates. The permissible stresses for wood and steel are $\sigma_w = 800$ psi and $\sigma_s = 20{,}000$ psi. $E_s = 30 \times 10^6$ psi; $E_w = 1.5 \times 10^6$ psi.

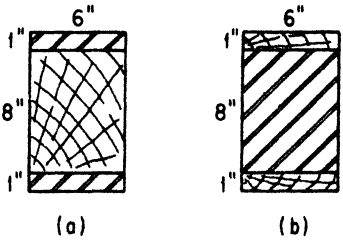


Fig. P5–44

5-45. A rectangular concrete beam is reinforced with four steel bars, each having an area of $\frac{1}{2}$ sq in., as shown in Fig. P5-45. Locate the neutral axis of the section. Assume the ratio $E_s/E_c = 15$.

5-46. A reinforced concrete beam, Fig. P5-46, is subjected to a moment of 400,000 lb in. Determine the maximum stress in the concrete and in the steel. The area of each of the four steel rods is 1 sq in. Assume the ratio $E_s/E_c = 15$.

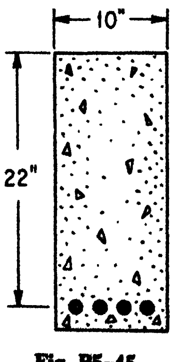


Fig. P5–45

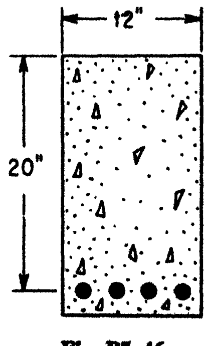


Fig. P5–46

5-47. A reinforced concrete tee-beam has the dimensions shown in Fig. P5-47. A moment of 1000 kip in. acts on the section. Determine: (a) the required area of the steel if the neutral axis is to be at section *a–a*; (b) the maximum stress in the concrete and in the steel. Use $E_s/E_c = 10$.

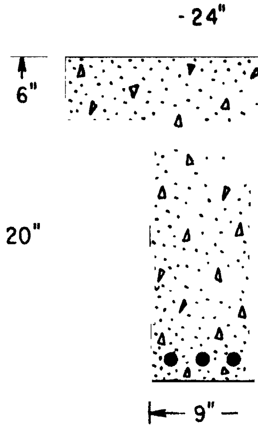


Fig. P5–47

5-48. A reinforced concrete floor slab has the dimensions shown in Fig. P5-48. Determine the allowable moment in this slab if the permissible stresses are $\sigma_s = 15{,}000$ psi and $\sigma_c = 900$ psi. Assume the ratio $E_s/E_c = 10$.

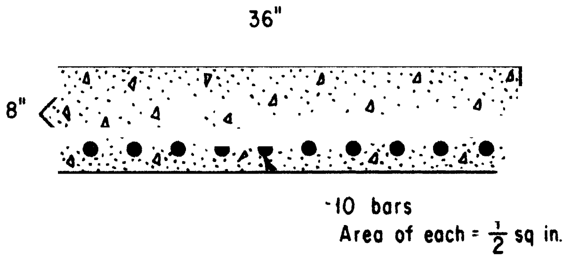


Fig. P5–48

5-49. Determine the required total area of reinforcing steel for the floor slab of Prob. 5-48 if both the concrete and steel are to be stressed to their maximum values.

5-50. A simply supported reinforced concrete beam 12 ft long has the cross section shown in Fig. P5-50. The beam supports a uniformly distributed load of *w* lb per ft (including its own weight) over the entire span. Determine *w* if both the concrete and the steel are stressed to their maximum values of $\sigma_c = 800$ psi and $\sigma_s = 16{,}000$ psi. Assume that $E_s/E_c = 10$. *Hint:* First find the required area of steel.

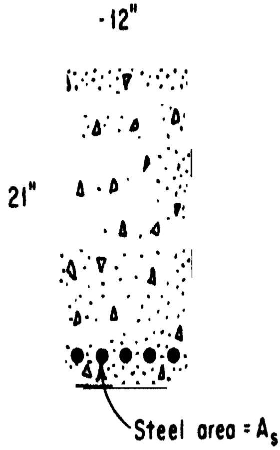


Fig. P5–50

CHAPTER 6

Deflection of Beams

In design, deformation often shares an equal importance with strength. A girder, for example, may have sufficient strength to withstand a particular floor load, but it may be too "soft" or too "bouncy" underfoot. Its deflection under load may also far surpass the elasticity of architectural materials, such as mortar and plaster, that might be fastened to it. It is necessary, therefore, to understand how beam deflections, as well as stresses, are computed.

There are many ways of approaching the problem of beam deflections; two of these, the *moment-area* method and the *superposition* method, will be considered in this chapter. Either singly or in combination, these two concepts will handle all of the more common beam-deflection problems.

6-1 Moment-Area Method

The reciprocal of radius of curvature ρ of a beam in terms of bending moment, modulus of elasticity, and moment of inertia is defined as follows:

$$\frac{1}{\rho} = \frac{M}{EI} \tag{6-1}$$

It is this equation that forms the basis of the moment-area method. Consider a portion AB of a beam, Fig. 6-1, distorted by some manner of loading (not shown). An incremental length Δx of the *elastic curve* can be defined by the radius ρ and the angle $\Delta\theta$ as

$$\Delta x = \rho\Delta\theta \tag{6-2}$$

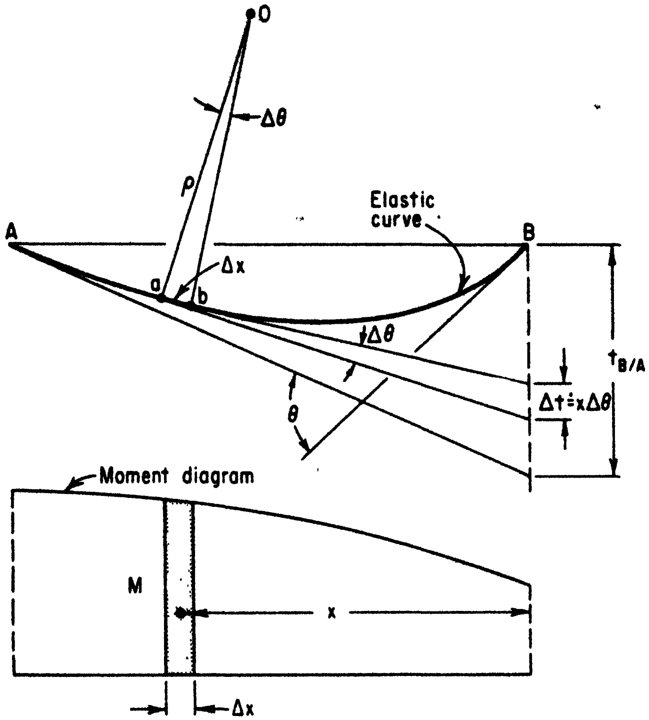


Fig. 6-1

Equations (6-1) and (6-2) are combined to give

$$\varDelta\theta = \frac{M\varDelta x}{EI}$$

Tangents drawn to the *elastic curve* at *a* and *b* are separated by the same angle as are the normals *oa* and *ob*. By imagining the beam to be composed of an infinite number of segments $\varDelta x$, the slope at any point on the elastic curve relative to a second point would be the sum $\sum \varDelta\theta$. Thus

$$\theta_{A/B} = \sum \varDelta\theta = \frac{1}{EI} \sum M\varDelta x \qquad (6\text{-}3)$$

where the term $\sum M\varDelta x$ is the area of the moment diagram between corresponding limits A and B on the elastic curve. Theorem I, a formal statement of Eq. (6-3), is as follows:

Theorem I. *The angle between tangents drawn at* A *and* B *on the elastic curve is equal to the area of the corresponding portion of the bending moment diagram, divided by* EI.

Consider, next, the deflection or *vertical deviation* of point B on the elastic curve relative to a tangent drawn at A. Returning to the small element ab, the distance Δt is very nearly equal to $x\Delta\theta$. This approximation is valid, since $\Delta\theta$, and for that matter θ, is very small. The vertical deviation of point B relative to a tangent drawn at A is the sum of all vertical deviations Δt between A and B.

$$t_{B/A} = \sum_A^B \Delta t = \sum_A^B x\Delta\theta = \sum_A^B x\left(\frac{M\Delta x}{EI}\right)$$

$$= \frac{1}{EI}\sum_A^B x(M\Delta x)$$

The summation $\sum_A^B x(M\Delta x)$ is physically equal to *the moment of the area of the moment diagram between* A *and* B *with respect to a vertical line drawn from* B. It is essential to understand the significance of the subscripts in terms representing tangential deviations.

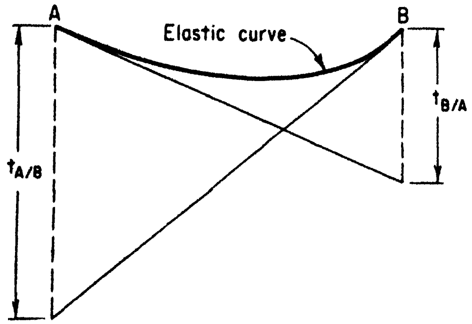


Fig. 6–2

Thus, $t_{A/B}$ and $t_{B/A}$ are not necessarily equal. The first represents the vertical distance between point A and a tangent drawn to the elastic curve at point B, whereas the second is the vertical distance between point B and a tangent drawn at point A. This notation is illustrated in Fig. 6-2. A memory aid: moments of the areas of the moment diagram are always taken relative to the deviation line. Hence,

$$t_{B/A} = \frac{1}{EI}\,[\text{area}]_{AB}\bar{x}_B$$

and (6-4)

$$t_{A/B} = \frac{1}{EI}\,[\text{area}]_{AB}\bar{x}_A$$

Equations (6-4) can be summarized as follows:

Theorem II. *The vertical deviation of point* B *on the elastic curve away from a tangent drawn at* A *is equal to the moment of the area of the bending moment diagram with respect to* B, *divided by* EI.

To simplify the calculations, moment diagrams are best drawn by the *method of parts* described in Chapter 4. For convenience, a table listing the properties of areas is presented again.

6–2 Statically Determinate Cantilever Beams

Deflections of fixed-end beams are easily computed by the moment-area method. Consider the beam shown in Fig. 6-3. Since the tangent drawn on the elastic curve at B is horizontal, the deflection δ_A at point A and the tangential deviation $t_{A/B}$ are exactly equal.

$$\delta_A = t_{A/B} = \frac{1}{EI}[\text{area}]_{AB}\bar{x}_A$$

$$= \frac{1}{EI}\left[\frac{1}{2}(l)(-Pl)\left(\frac{2}{3}l\right)\right]$$

$$= -\frac{Pl^3}{3EI}$$

The negative sign indicates the deflection to be below the unloaded, or free, position of point A.

The slope of the beam at the free end, by Theorem I, is

$$\theta = \frac{1}{EI}[\text{area}]_{AB} = \frac{1}{EI}\left[\frac{1}{2}(l)(-Pl)\right]$$

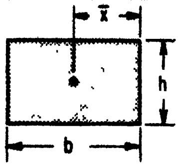
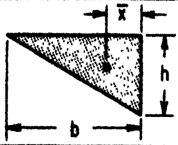
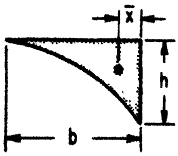
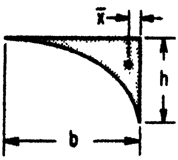
$$\frac{Pl^2}{2EI}\ \text{rad}$$

The minus sign merely indicates the slope of a tangent at A to be negative when measured relative to a tangent at B.

Example 1. The timber cantilever beam ($E = 1.5 \times 10^6$ psi) shown in Fig. 6-4 supports two loads, as indicated. Determine the maximum deflection of the beam.

Solution: A sketch is drawn to represent the probable elastic curve. From this sketch it can be seen that the maximum deflection occurs at the free end and is, therefore, equal to the tangential deviation $t_{A/B}$. The moment

Table 6-1

		Area	*Centroid* $\bar{x}$
	Rectangle	bh	$b/2$
	Triangle	$\frac{1}{2}bh$	$b/3$
	Parabola	$\frac{1}{3}bh$	$b/4$
	Cubic parabola	$\frac{1}{4}bh$	$b/5$

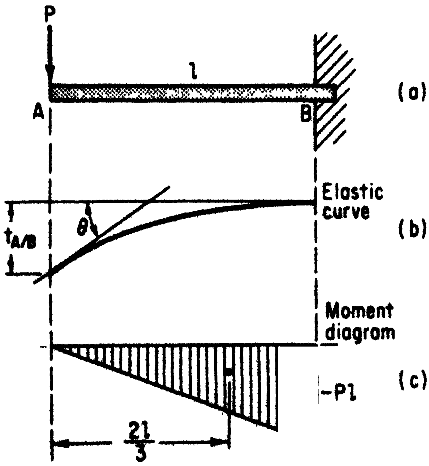


Fig. 6-3

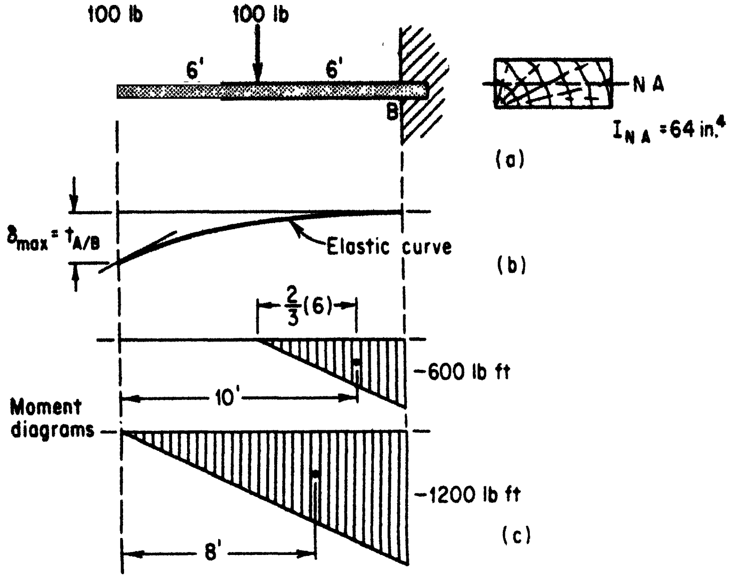


Fig. 6-4

diagram, drawn by *parts*, and Theorem II are used to compute the required deflection.

$$\delta_{max} = t_{A/B} = \frac{1}{EI}\,[\text{area}]_{AB}\bar{x}_A$$

$$= \frac{1}{EI}\left[\frac{1}{2}(6)(-600)(10) + \frac{1}{2}(12)(-1200)(8)\right]$$

$$= -\frac{75{,}600}{EI}\ \text{lb ft}^3 = -\frac{75{,}600(1728)}{EI}\ \text{lb in.}^3$$

Numerical values are substituted to obtain the desired answer:

$$\delta_{max} = \frac{-75{,}600(1728)}{1.5(10)^6(64)} = -1.36\ \text{in.}$$

Example 2. Find the distributed weight w lb per ft, Fig. 6-5, that will limit the maximum deflection to 1 in. $E = 1.5 \times 10^6$ psi; $I = 50$ in.^{4}

Solution: The moment diagram, which is a parabola, has an area and centroid of

$$A = \tfrac{1}{3}bh, \qquad \bar{x} = \tfrac{3}{4}b$$

The maximum deflection and the deviation $t_{A/B}$ are equal; hence,

$$\delta_{max} = t_{A/B} = \frac{1}{EI}\,[\text{area}]_{AB}\bar{x}_A$$

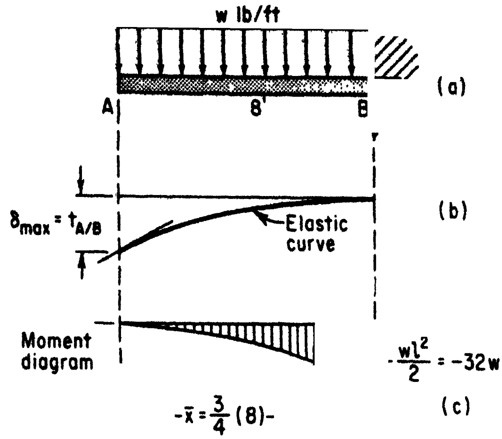


Fig. 6-5

When symbolic terms are used, δ_{max} is equal to

$$\delta_{max} = \frac{1}{EI}\left[\frac{1}{3}(l)\left(-\frac{wl^2}{2}\right)\left(\frac{3}{4}l\right)\right]$$

$$\frac{wl^4}{8EI} \text{ lb ft}^3$$

Substitution of numerical values gives

$$1 = -\frac{w(8)^4(1728)}{8(1.5)(10^6)50}$$

$$w = 84.8 \text{ lb per ft}$$

Example 3. Find the maximum value of $EI\delta$ for the cantilever beam shown in Fig. 6-6(a).

Solution: To simplify computations, the moment diagram is drawn in parts, Fig. 6-6(b); the left end of the beam is used as a reference. The diagram, then, consists of a triangle, a rectangle, and a parabola. To accomplish this step, the reactions are first computed.

$$\sum M = 0$$

$$M = 50(5)(4.5) = 1125 \text{ lb ft}$$

$$\sum F_y = 0$$

$$V = 50(5) = 250 \text{ lb}$$

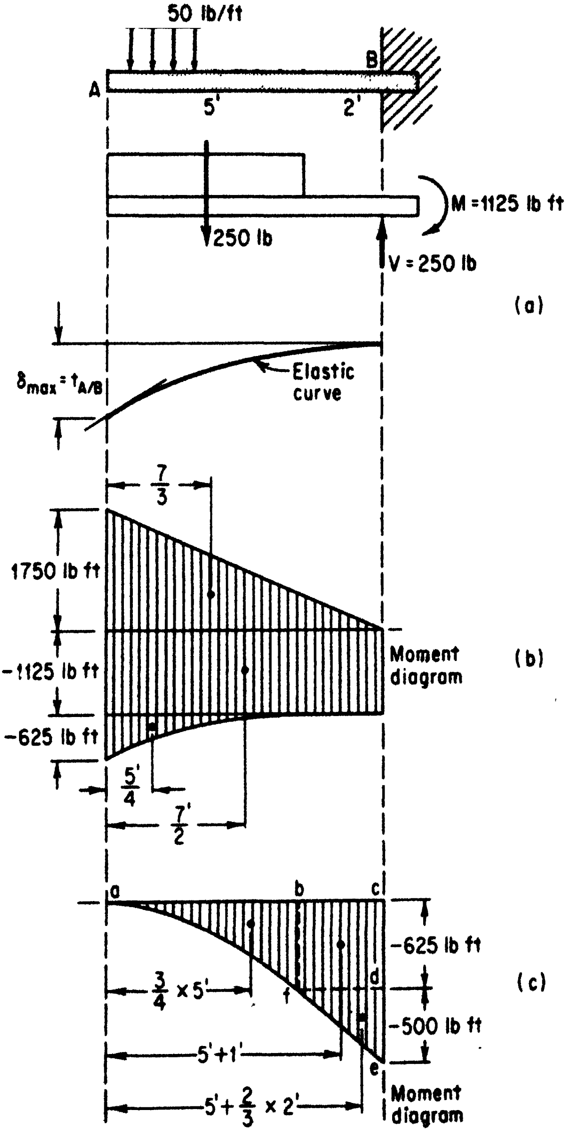


Fig. 6–6

The heights of the triangular, rectangular, and parabolic areas are, respectively,

$$M_1 = 250(7) = 1750 \text{ lb ft}$$

$$M_2 = -1125 \text{ lb ft}$$

$$M_3 = -50(5)(5/2) = -625 \text{ lb ft}$$

By Theorem II,

$$t_{A/B} = \frac{1}{EI}[\text{area}]_{AB}\bar{x}_A$$

$$EI\delta_{max} = [\tfrac{1}{2}(7)(1750)(\tfrac{7}{3}) + 7(-1125)(\tfrac{7}{2}) + \tfrac{1}{3}(5)(-625)(\tfrac{5}{4})]$$

$$= -14{,}600 \text{ lb ft}^3$$

Alternate Solution: The moment diagram can be drawn in total form, as shown in Fig. 6-6(c). To simplify computations, the diagram is considered to be the sum of three areas: parabola *abf*, rectangle *bcdf*, and triangle *def*. Ordinate *bf* represents the moment of the distributed load about a point 5 ft from the extreme left.

$$bf = -50(5)\tfrac{5}{2} = -625 \text{ lb ft}$$

Ordinate *de* is simply the numerical difference:

$$de = ce - cd$$

$$= -1125 + 625 = -500 \text{ lb ft}$$

By Theorem II,

$$\delta_{max} = \frac{1}{EI}[\text{area}]_{AB}\bar{x}_A$$

$$EI\delta_{max} = [\tfrac{1}{3}(5)(-625)(\tfrac{3}{4} \times 5) + 2(-625)(5+1) + \tfrac{1}{2}(2)(-500)(5+\tfrac{2}{3} \times 2)]$$

$$= -14{,}600 \text{ lb ft}^3$$

6-3 Simply Supported Beams

Determination of deflections in simply supported beams by the moment-area method is not quite as direct as with cantilever beams. The existence of a horizontal tangent at the support, in the latter, simplified computations, since the tangential deviation and the maximum deflection were equal.

A series of steps are necessary, for example, to find the deflection δ_C on the simply supported beam of Fig. 6-7(a). First, a diagram of the elastic curve and its associated geometry must be carefully drawn, Fig. 6-7(b). In this example δ_C is the desired deflection, $t_{C/A}$ is the vertical deviation of point C relative to a tangent at A, and $t_{B/A}$ is the vertical deviation of point B relative also to a tangent at A. By similar triangles,

$$\frac{\delta_C + t_{C/A}}{6} = \frac{t_{B/A}}{9}$$

Solving for the deflection gives

$$\delta_C = \tfrac{2}{3}t_{B/A} - t_{C/A}$$

By Theorem II:

$$EI\delta_C = \tfrac{2}{3}[\text{area}]_{AB}\bar{x}_B - [\text{area}]_{AC}\bar{x}_C \qquad \text{(a)}$$

Thought given to the construction of the moment diagram can greatly simplify computations; in this example the right end of the beam is selected as a reference line, and the diagram appears as shown in Fig. 6-7(c).

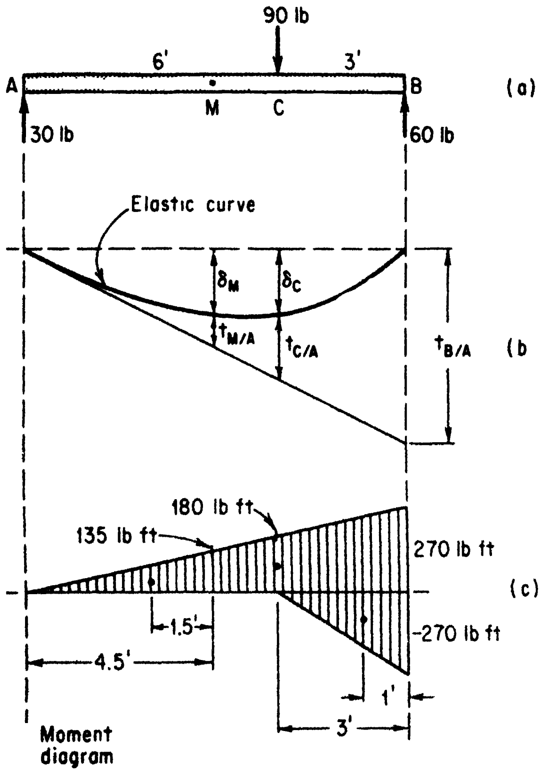


Fig. 6-7

Numerical data are substituted into Eq. (a):

$$EI\delta_C = \tfrac{2}{3}[\tfrac{1}{2}(9)(270)(\tfrac{9}{3}) + \tfrac{1}{2}(3)(-270)(\tfrac{3}{3})] - [\tfrac{1}{2}(6)(180)(\tfrac{6}{3})]$$
$$= 1080 \text{ lb ft}^3 \text{ directed as shown}$$

The deflection at midspan is computed in a similar manner.

$$\frac{\delta_M + t_{M/A}}{4.5} = \frac{t_{B/A}}{9}$$

$$\delta_M = \tfrac{1}{2}t_{B/A} - t_{M/A}$$

$$EI\delta_M = \tfrac{1}{2}[\text{area}]_{AB}\bar{x}_B - [\text{area}]_{AM}\bar{x}_M$$
$$= \tfrac{1}{2}[\tfrac{1}{2}(9)(270)(\tfrac{9}{3}) + \tfrac{1}{2}(3)(-270)(\tfrac{3}{3})] - [\tfrac{1}{2}(4.5)(135)(\tfrac{4.5}{3})]$$
$$= 1164 \text{ lb ft}^3 \text{ directed as shown}$$

It will be shown in later discussion that the exact value of $EI\delta_{max}$ is 1176 lb ft.^{3} The answer obtained for midspan deflection is, therefore, approximately equal to the maximum deflection.

Example 4. A uniformly distributed load w lb per ft acts across the entire span of a simply supported beam, as shown in Fig. 6-8. Determine the maximum value of $EI\delta$.

Solution: The moment diagram is constructed by the method of parts, as shown in Fig. 6-8(c). Since the loading is symmetrical, the maximum deflection occurs at the midspan M, a point at which the elastic curve has a zero slope. Tangential deviation of A relative to M, therefore, equals the maximum deflection.

Theorem II is applied to the area of the bending moment diagram between points A and M.

$$\delta_{max} = t_{A/M} = \frac{1}{EI}[\text{area}]_{AM}\bar{x}_A$$

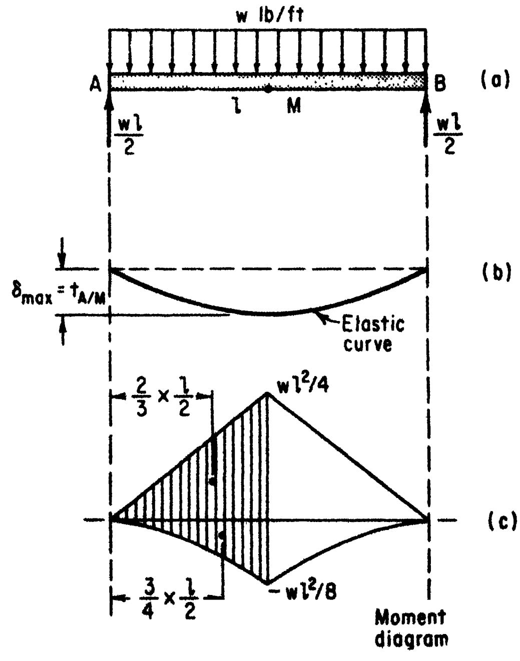


Fig. 6-8

Numerical values are substituted, and δ_{max} is computed.

$$EI\delta_m \quad \left[\left(\frac{1}{2}\right)\left(\frac{l}{2}\right)\left(\frac{wl^2}{4}\right)\left(\frac{2}{3}\times\frac{l}{2}\right) - \frac{1}{3}\left(\frac{l}{2}\right)\left(\frac{wl^2}{8}\right)\left(\frac{3}{4}\times\frac{l}{2}\right)\right]$$
$$= wl^4\,(1/48 - 1/128)$$
$$= 5/384\; wl^4 \text{ directed as shown}$$

6–4 Overhanging Beams

The treatment of beams which extend beyond their supports can be best explained by considering the example that follows.

The beam of Fig. 6-9(a) supports a concentrated load of 120 lb midway between supports and a second load of 60 lb extending beyond the right support, as shown. It is conceivable that point C could be either below or above the original axis of the beam. These two possibilities are illustrated in Figs. 6-9(b) and 6-9(c). If C is assumed to be below the axis, geometry indicates δ_C to be related by similar triangles to tangential deviations $t_{C/A}$ and $t_{B/A}$ in the following manner:

$$\frac{\delta_C + t_{C/A}}{14} = \frac{t_{B/A}}{12}$$

$$\delta_C = 7/6\, t_{B/A} - t_{C/A} \tag{a}$$

If the second manner of bending is assumed, the geometry indicates that

$$\frac{t_{C/A}}{14}\;\frac{\delta_C}{} = \frac{t_{B/A}}{12}$$

$$\delta_C = -\,7/6\, t_{B/A} + t_{C/A} \tag{b}$$

Numerically, Eqs. (a) and (b) are equal; they differ, however, in algebraic sign. One or the other of the two possibilities must be assumed in writing the geometric relationship; an incorrect guess will be acknowledged by a minus sign.

The moment diagram is constructed in parts with reference to reaction B, as shown in Fig. 6-9(d). Here again, there are many ways of drawing the moment diagram, and the method selected should be one which will keep the computations simple and direct.

It is assumed that δ_C will be below the axis, and numerical data are substituted into Eq. (a).

$$\delta_C = 7/6\, t_{B/A} - t_{C/A}$$

$$\delta_C = \left(\frac{7}{6}\right)\frac{1}{EI}\,[\text{area}]_{AB}\bar{x}_B - \frac{1}{EI}\,[\text{area}]_{AC}\bar{x}_C$$

$$EI\delta_C = \tfrac{7}{6}[\tfrac{1}{2}(12)(600)(\tfrac{12}{3}) + \tfrac{1}{2}(6)(-720)(\tfrac{6}{3})] - [\tfrac{1}{2}(12)(600)(\tfrac{12}{3} + 2) + \tfrac{1}{2}(6)(-720)(\tfrac{6}{3} + 2) + \tfrac{1}{2}(2)(-120)(\tfrac{2}{3} \times 2)]$$
$$= -1040 \text{ lb ft}^3$$

Since the answer is negative, point C is above the horizontal axis of the beam, opposite to that assumed.

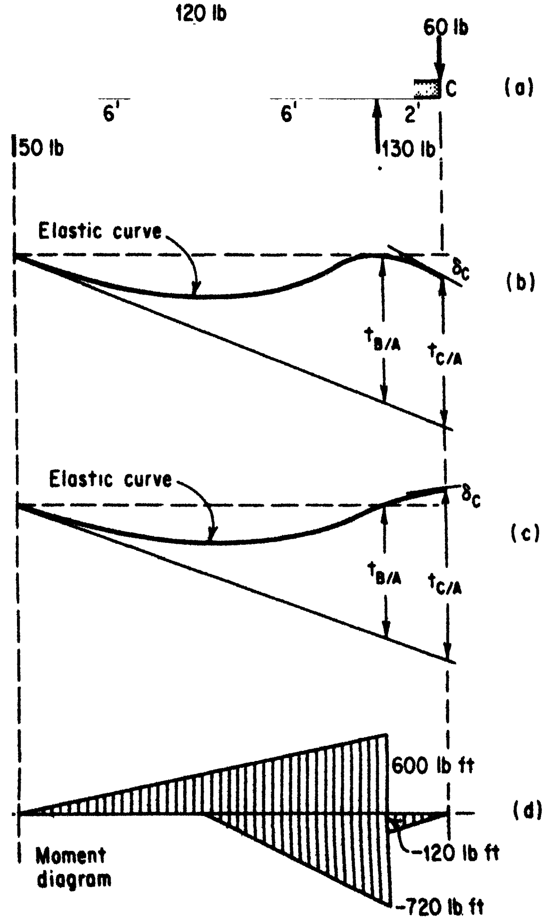


Fig. 6-9

6-5 Propped Beams

The propped beam is the first encounter in this text with statically indeterminate beam loading. It is unfortunate that the phrase *statically* *indeterminate* carries a feeling of something mysterious and incalculable. In truth, however, it simply means that, although the equations of statics apply, they are not sufficient; there are more unknowns than there are equations of equilibrium. The missing "link" in the computations can be supplied by the moment-area concept.

The propped beam, Fig. 6-10(a), is an example of an indeterminate structure. Three unknown reactions R, M, and V are shown in the free-body diagram, Fig. 6-10(b). If points A and B remain on the same horizontal line, the tangential deviation $t_{A/B}$ is zero. Thus, the reaction R can be obtained by applying Theorem II to the moment diagram of Fig. 6-10(c).

$$t_{A/B} = \frac{1}{EI}[\text{area}]_{AB}\bar{x}_A = 0$$

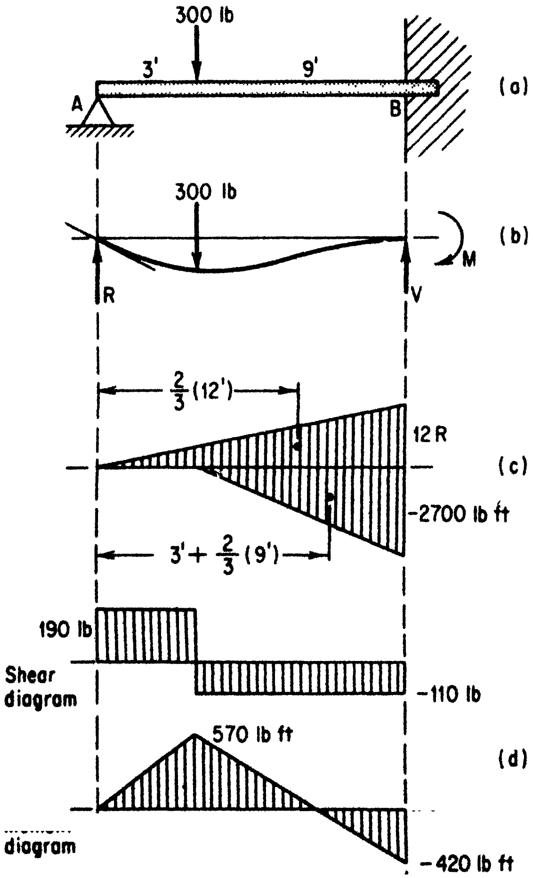


Fig. 6-10

Hence,

$$[\text{area}]_{AB}\bar{x}_A = 0$$

$$\tfrac{1}{2}(12)(12R)(\tfrac{2}{3} \times 12) + \tfrac{1}{2}(9)(-2700)(3 + \tfrac{2}{3} \times 9) = 0$$

$$576R = 109{,}400$$

$$R = 190 \text{ lb}$$

Two unknown reactions remain, and these can be found through the equations of statics.

$$\sum M = 0$$

$$M = 12(190) - 2700 = -420 \text{ lb ft}$$

$$\sum F_y = 0$$

$$V = 300 - 190 = 110 \text{ lb}$$

To obtain maximum values of M and V, the shear and the moment diagrams can be drawn in their total forms, Fig. 6-10(d). Although the moment-area method can be used to find deflections in indeterminate beams, the computation becomes rather tedious. For this reason, the superposition method, which is described in Sec. 6-7, is the more practical approach.

6-6 Restrained Beams

The restrained beam, Fig. 6-11(a), is *doubly indeterminate.* Four unknown reactions, consisting of two end moments and two vertical forces, as shown in Fig. 6-11(b), must be found before a moment diagram can be constructed. Here again, the moment-area method proves to be an efficient approach.

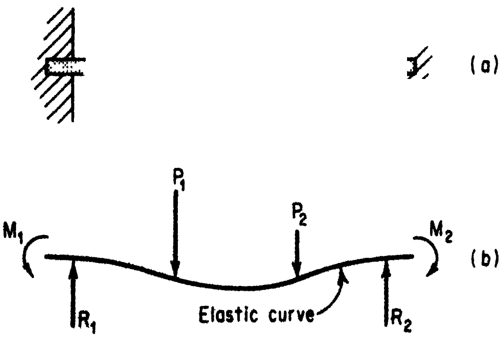


Fig. 6-11

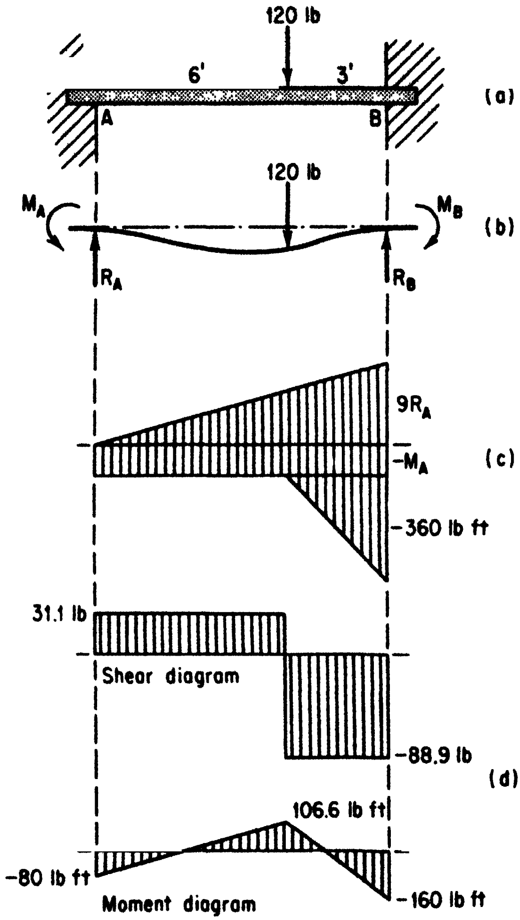


Fig. 6-12

Consider the restrained beam of Fig. 6-12(a). Two moment-area equations can be written for this beam:

$$t_{A/B} = \frac{1}{EI}[\text{area}]_{AB}\bar{x}_A = 0 \tag{a}$$

and

$$t_{B/A} = \frac{1}{EI}[\text{area}]_{AB}\bar{x}_B = 0 \tag{b}$$

The walls act as perfect restraints; the slope of the beam at A and B is zero, and the two points are on a common horizontal tangent. For this reason

Theorem I also applies and can be used to replace either Eq. (a) or Eq. (b).

$$\theta_{A/B} = \frac{1}{EI}[\text{area}]_{AB} = 0 \qquad \text{(c)}$$

For the sake of illustration, Eqs. (a) and (b) will be used in this example.

Either wall may be selected as a reference for the moment diagram, and in this example the right wall is used. Three areas comprise the diagram, as shown in Fig. 6-12(c); one is completely dimensioned, whereas the other two contain unknowns R_A and M_A.

Numerical data are substituted into Eq. (a):

$$t_{A/B} = 0$$

$$\frac{1}{EI}[\text{area}]_{AB}\bar{x}_A \qquad 0$$

$$\tfrac{1}{2}(9)(9R_A)(\tfrac{2}{3} \times 9) - 9(M_A)(\tfrac{9}{2}) - \tfrac{1}{2}(3)(360)(6 + \tfrac{2}{3} \times 3) = 0$$

$$18R_A - 3M_A = 320 \qquad \text{(a)}$$

A second moment-area equation is written, and data are substituted in a similar manner.

$$t_{B/A} = 0$$

$$\frac{1}{EI}[\text{area}]_{AB}\bar{x}_B = 0$$

$$\tfrac{1}{2}(9)(9R_A)(\tfrac{9}{3}) - 9(M_A)(\tfrac{9}{2}) - \tfrac{1}{2}(3)(360)(\tfrac{3}{3}) = 0$$

$$9R_A - 3M_A = 40 \qquad \text{(b)}$$

The unknowns R_A and M_A can be computed by solving Eqs. (a) and (b) simultaneously.

$$\begin{aligned} 18R_A - 3M_A &= 320 \\ -9R_A + 3M_A &= -40 \\ \hline 9R_A \qquad &= 280 \\ R_A = {}^{280}\!/_9 &= 31.1 \text{ lb} \end{aligned}$$

and

$$3M_A = 9R_A - 40 = 9({}^{280}\!/_9) - 40$$

$$M_A = 80 \text{ lb ft (directed as shown)}$$

Two unknowns remain, R_B and M_B, and these are found through equations of static equilibrium.

$$\sum F_y = 0$$

$$R_A + R_B = 120$$

$$R_B = 120 - 31.1 = 88.9 \text{ lb}$$

$$\sum M = 0$$

$$M_B = 9R_A \quad M_A - 360$$

$$= 9(31.1) \quad 80 - 360 = -160 \text{ lb ft}$$

The shear and moment diagrams can now be drawn, Fig. 6-12(d), and the critical values calculated.

When the loading is symmetrical, computations are greatly simplified, since forces R_A and R_B are equal, as are moments M_A and M_B. The reactions can easily be found, since they share the external load equally.

The example that follows illustrates a complete analysis of a symmetrically loaded, restrained beam.

Example 4. Find the reactions and maximum deflection of the restrained beam shown in Fig. 6-13(a). Sketch the shear and moment diagrams and indicate critical values.

Solution: The free-body diagram is drawn as shown in Fig. 6-13(b). Since the loading is symmetrical, reactions R_A and R_B are each equal to one-half the external load.

$$R_A = R_B = {}^{10}\!/_2 = 5 \text{ kips}$$

Next, Theorem I is applied to the moment diagram constructed as shown in Fig. 6-13(c). Tangents to the elastic curve at points A and B are horizontal; thus $\theta_{A/B} = 0$.

$$\theta_{A/B} \quad \frac{1}{EI}[\text{area}]_{AB} = 0$$

$$\tfrac{1}{2}(12)(60) - 12M_A - \tfrac{1}{2}(6)60 = 0$$

$$M_A = M_B = 15 \text{ kip ft (directed as shown)}$$

Shear and moment diagrams are constructed as shown in Fig. 6-13(d). The moment at midspan, computed in terms of the area of the shear diagram. is

$$\Delta M = \text{area under shear diagram}$$

$$M_C - M_A = 5(6) = 30$$

$$M_C = 30 + M_A = 30 - 15 = 15 \text{ kip ft}$$

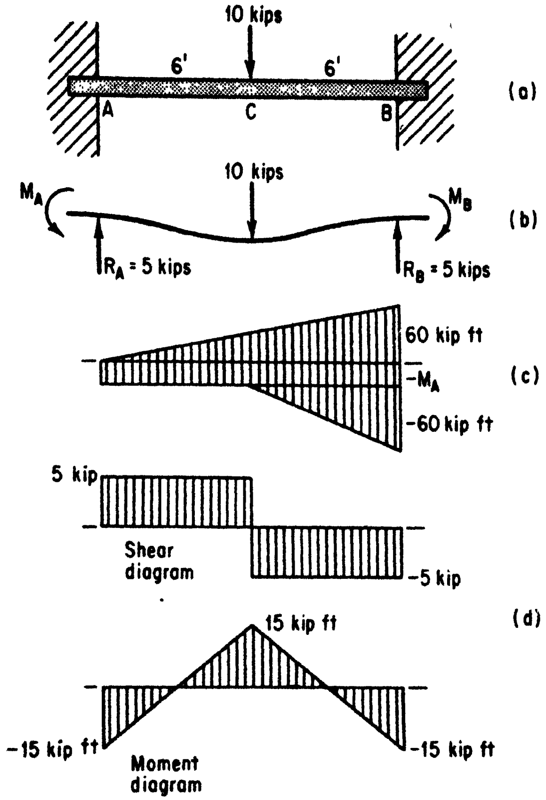


Fig. 6-13

Because of symmetry, a tangent drawn to the elastic curve at midspan is horizontal. Theorem II, therefore, can be applied directly to find δ_{max}.

$$\delta_{max} = \frac{1}{EI}[\text{area}]_{AC}\bar{x}_C$$

$$\delta_{max} = \frac{1}{EI}\left[\frac{1}{2}(6)(30)\left(\frac{6}{3}\right) + 6(-15)3\right]$$

$$= -\frac{90}{EI}\text{ ft}$$

6-7 Deflections by Superposition

Superposition, which simply means "to place one upon the other," is a powerful concept with many applications in engineering. It can be used to find deflections, moments, and stresses in both simple and complex beams.

Consider two identical beams, Fig. 6-14(a), made of the same material and with the same physical dimensions. One beam supports a concentrated load at midspan, and the other, a uniformly distributed load over its entire length. Each member deflects through a known distance, as indicated. By superposing the two loads, deflections and reactions become the algebraic sum of those of the individual beams. This is illustrated in Fig. 6-14(b).

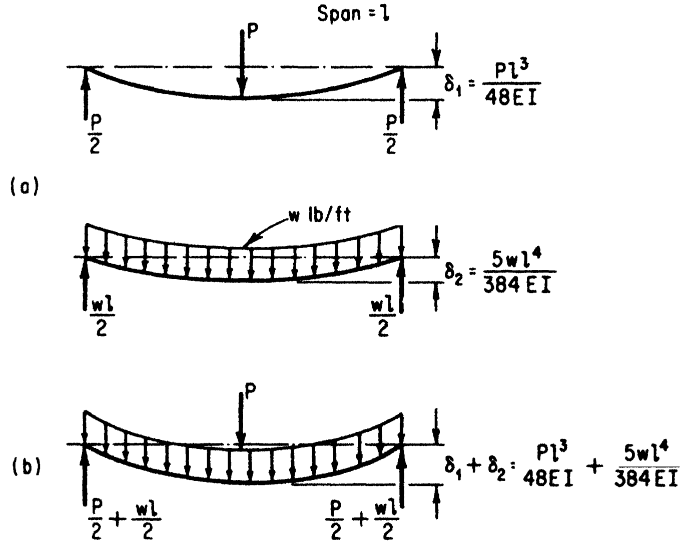
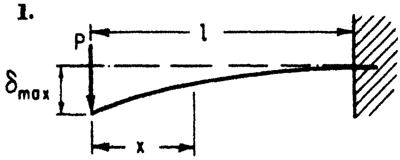


Fig. 6-14

Tabulated values of deflection, Table 6-2, are generally employed with the superposition method. Ingenuity often plays an important part in the use of these tables, as the examples which follow illustrate.

Table 6-2 Beam Deflection Equations

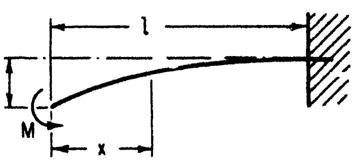
1.



$$\delta_x = \frac{P}{6EI}(2l^3 - 3l^2x + x^3)$$

$$\delta_{max} = \frac{Pl^3}{3EI}$$

2.

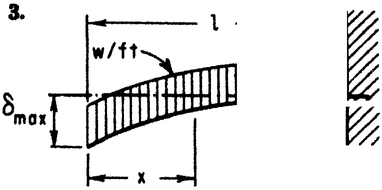


$$\delta_x = \frac{M(l-x)^2}{2EI}$$

$$\delta_{max} = \frac{Ml^2}{2EI}$$

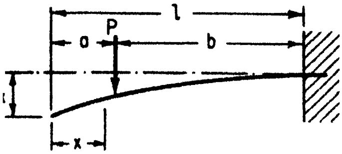
Table 6-2 (cont.)

3.



$$\delta_x = \frac{w}{24EI}(x^4 - 4l^3x + 3l^4)$$

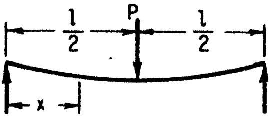
$$\delta_{max} = \frac{wl^4}{8EI}$$



$$\delta_x = \frac{Pb^2}{6EI}(3l - 3x - b) \quad \text{for } x<a$$

$$\delta_x = \frac{P(l-x)^2}{6EI}(3b - l + x) \quad \text{for } x>a$$

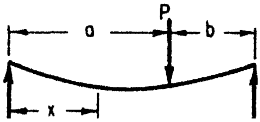
$$\delta_{max} = \frac{Pb^2}{6EI}(3l - b)$$



$$= \frac{Px}{48EI}(3l^2 - 4x^2) \quad \text{for } x < \frac{l}{2}$$

$$= \frac{Pl^3}{48EI} \quad \text{at } x = \frac{l}{2}$$

6.



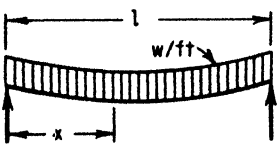
$$\delta_x = \frac{Pbx}{6lEI}(l^2 - x^2 - b^2) \quad \text{for } x<a$$

$$\delta_x = \frac{Pb}{6lEI}\left[\frac{l}{b}(x-a)^3 + (l^2 - b^2)x - x^3\right] \quad \text{for } x>a$$

$$\delta = \frac{Pb}{48EI}(3l^2 - 4b^2) \quad \text{at center if } a>b$$

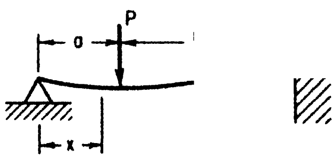
$$\delta_{max} = \frac{Pb(l^2-b^2)^{3/2}}{9\sqrt{3}\,lEI} \quad \text{at } x = \sqrt{\frac{l^2-b^2}{3}}$$

7.



$$\delta_x = \frac{wx}{24EI}(l^3 - 2lx^2 + x^3)$$

$$\delta_{max} = \frac{5wl^4}{384EI} \quad \text{at center}$$

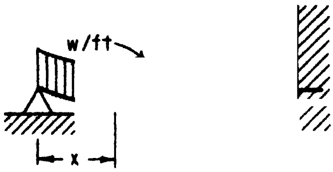


$$\delta_x = \frac{Pb^2x}{12EIl^3}(3al^2 - 2lx^2 - ax^2) \quad \text{for } x<a$$

$$\delta_x = \frac{Pa(l-x)^2}{12EIl^3}(3l^2x - a^2x - 2a^2l) \quad \text{for } x>a$$

$$\delta = \frac{Pa^2b^3}{12EIl^3}(3l + a) \quad \text{at point of load}$$

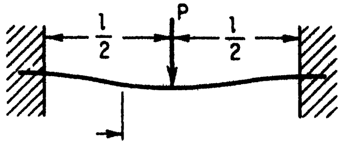
Table 6-2 (cont.)



$$\delta_x = \frac{wx}{48EI}(l^3 - 3lx^2 + 2x^3)$$

$$\delta_{max} = \frac{wl^4}{185EI} \text{ at } x = 0{\cdot}422l$$

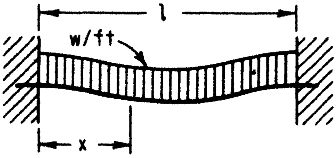
10.



$$\delta_x = \frac{Px^2}{48EI}(3l - 4x)$$

$$\delta_{max} = \frac{Pl^3}{192EI} \text{ at center}$$

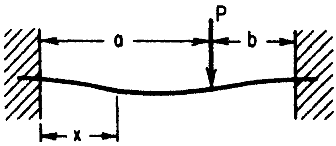
11.



$$\delta_x = \frac{wx^2}{24EI}(l - x)^2$$

$$\delta_{max} = \frac{wl^4}{384EI} \text{ at center}$$

12.



$$\delta_x = \frac{Pb^2x^2}{6EIl^3}(3al - 3ax - bx) \text{ for } x<a$$

$$\delta_{max} = \frac{2Pa^3b^2}{3EI(3a+b)^2} \text{ at } x = \frac{2al}{3a+b},\ a>b$$

$$\delta = \frac{Pa^3b^3}{3EIl^3} \text{ at point of load}$$

Example 5. Use the method of superposition to find the deflection at the free end of the cantilever beam shown in Fig. 6-15.

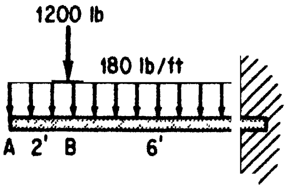


Fig. 6-15

Solution: Two basic loadings, Case 3 and Case 4 of Table 6-2, are superposed. The deflection at the free end is the sum:

$$\delta_A = \frac{wl^4}{8EI} + \frac{Pb^2}{6EI}(3l - b)$$

Numerical data are substituted to find δ_A:

$$\frac{180(8)^4}{8EI} + \frac{1200(6)^2}{6EI}[3(8) - 6]$$

$$\frac{222{,}000}{EI} \text{ ft}$$

Example 6. Find the midspan value of the deflection for the restrained beam shown in Fig. 6-16(a).

Solution: Case 10 of Table 6-2 represents the concentrated load at midspan; the deflection caused by the distributed load, which does not appear in the table, lies hidden in Case 11. This can be explained as follows: consider two identical beams, Fig. 6-16(b), each supporting a distributed load over half a span. Superposed, the beams are equivalent to Case 11. The "half beams" deflect equally; therefore,

$$\delta + \delta = \frac{wl^4}{384EI}$$

and

$$\frac{wl^4}{768EI}$$

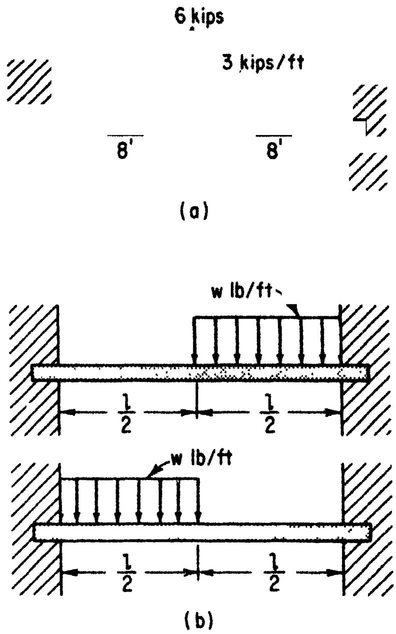


Fig. 6–16

The midspan deflection of the original beam can now be calculated.

$$\delta = \frac{Pl^3}{192EI} + \frac{wl^4}{768EI}$$

$$\frac{6(16)^3}{192EI} + \frac{3(16)^4}{768EI}$$

$$EI\delta = 384 \text{ kip ft}^3$$

6–8 Continuous Beams

The superposition method can be applied to analyze *continuous beams*—beams that have more supports than necessary to maintain equilibrium. While this is not the only method of attack,[1] it is one that can be easily used.

Consider the continuous beam of Fig. 6-17(a). The reactions are to be determined, and both a shear and a moment diagram are to be drawn.

The member can be viewed in the following way: imagine first, Fig. 6-17(b), that the distributed load is simply supported and is typical of Case 7

[1] For a classical method of approach, the reader is referred to the *three moment method* described in advanced texts on the subject of mechanics of materials.

given in Table 6-2. The midspan deflection is $5wl^4/384EI$ and the end reactions are $wl/2$, each. Now imagine that a load P acts upward on a similar beam with sufficient force to cause the midspan deflection to equal the former value, Fig. 6-17(c). This second loading is typical of Case 5, where $\delta = Pl^3/48EI$.

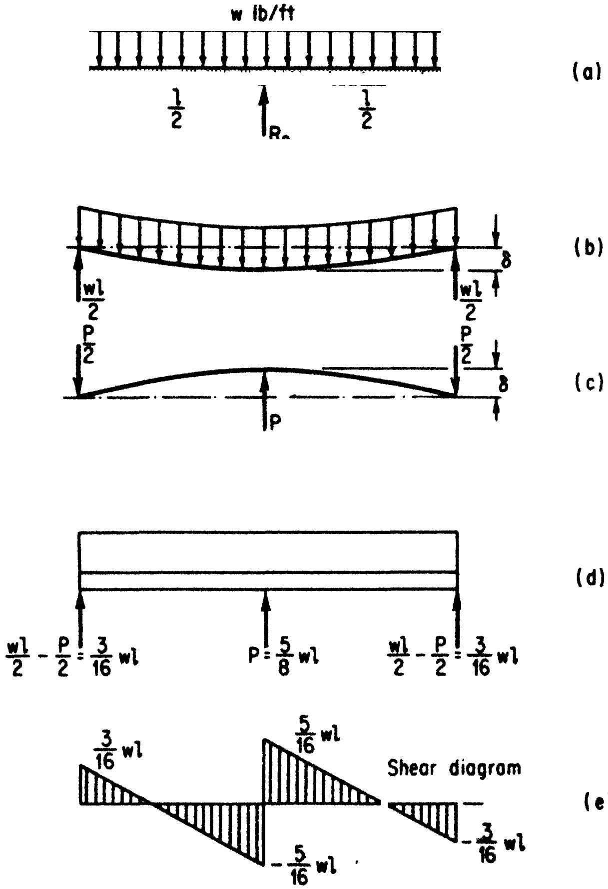


Fig. 6-17

Equating the deflections will give a value of P which, in reality, is the center reaction R_2.

$$\frac{Pl^3}{48EI} = \frac{5l^4}{384EI}$$

$$P = \tfrac{5}{8}wl = R_2$$

The end reactions, Fig. 6-17(d), are the sums of those for the two cases; of course, in the first instance the reaction is upward and in the second, downward.

$$R_1 = R_3 = \frac{wl}{2} - \frac{1}{2}\left(\frac{5}{8}\right)wl = \frac{3}{16}wl$$

The shear diagram, from which the moment diagram could be constructed, is shown in Fig. 6-17(e).

Example 7. Find the moment over the center support for the continuous beam shown in Fig. 6-18(a).

Solution: The deflections δ_1 and δ_2 must add to give δ_3, since all three supports lie on the same horizontal line. Case 5 and Case 6 of Table 6-2 are used to find these deflections.

For the beam of Fig. 6-18(b):

$$\delta_1 = \frac{Pb}{6lEI}\left[\frac{l}{b}(x-a)^3 + (l^2 - b^2)x - x^3\right] \quad \text{for } x > a$$

$$= \frac{4(9)}{6(12)EI}\left[\frac{12}{9}(6-3)^3 + (12^2 - 9^2)6 - 6^3\right] = \frac{99}{EI}$$

For the beam of Fig. 6-18(c):

$$\delta_2 = \frac{Pbx}{6lEI}(l^2 - x^2 - b^2) \quad \text{for } x < a$$

$$= \frac{8(3)6}{6(12)EI}(12^2 - 6^2 - 3^2) = \frac{198}{EI}$$

For the beam of Fig. 6-18(d):

$$\delta_3 = \frac{Pl^3}{48EI}$$

$$= \frac{P(12)^3}{48EI} = \frac{36P}{EI}$$

Thus

$$\delta_3 = \delta_1 + \delta_2$$

$$\frac{36P}{EI} = \frac{99}{EI} + \frac{198}{EI}$$

$$P = \frac{99 + 198}{36} = 8.25 \text{ kips}$$

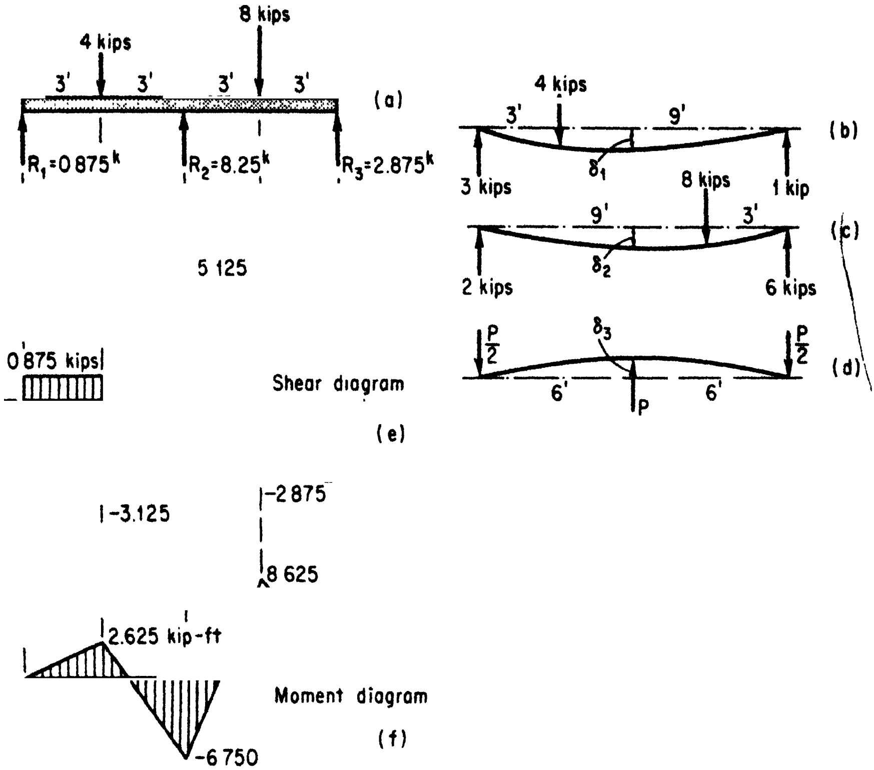


Fig. 6-18

With reference to the individual free-body diagrams, the reactions are found to be

$$R_1 = 3 + 2 - \frac{8.25}{2} = 0.875 \text{ kips}$$

$$R_2 = P = 8.25 \text{ kips}$$

$$R_3 = 1 + 6 - \frac{8.25}{2} = 2.875 \text{ kips}$$

The shear and moment diagrams are constructed in the usual manner and appear as shown in Figs. 6-18(e) and (f); the maximum moment is found to be under the 8 kip load.

6–9 Special Techniques

Interesting situations arise when loads act on beams which, in turn, are supported by deformable bodies. These bodies may be cables, springs, or columns. In general, equations can be written which relate the deformations of the various components, and these equations are then combined with those of statics to complete the solution. Examples that follow illustrate typical problems involving this type of loading.

Example 8. The steel cantilever beam, Fig. 6-19, is partially supported at its free end by a coil spring. Find the force in the spring. The moment of inertia of the beam is 1.728 in.[4] and the constant of the spring is 1000 lb per in.

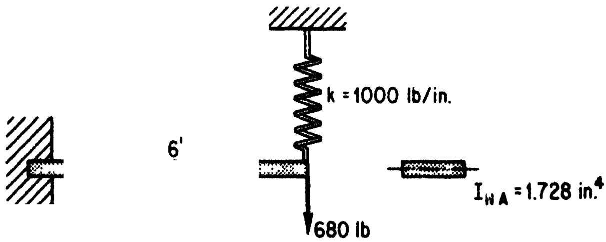


Fig. 6-19

Solution: The beam and spring deflect equally as they share the load.

$$\delta_b = \delta_s$$

$$\left(\frac{Pl^3}{3EI}\right)_b = \left(\frac{P}{k}\right)_s$$

$$\frac{P_b(6 \times 12)^3}{3(30 \times 10^6)(1.728)} = \frac{P_s}{10^3}$$

$$2.4P_b = P_s \tag{a}$$

and

$$P_b + P_s = 680 \tag{b}$$

The forces P_s and P_b are obtained by solving Eqs. (a) and (b) simultaneously.

$$P_b + 2.4P_b = 680$$

$$3.4P_b = 680$$

$$P_b = 200 \text{ lb}$$

$$P_s = 2.4(200) = 480 \text{ lb}$$

Example 9. Two steel cantilever beams help support load F as shown in Fig. 6-20(a). Find F if the deflection at A is to be $\frac{1}{4}$ in. The moment of inertia of each member is 1.5 in.[4]

Solution: Two conditions at point A provide the necessary equations for

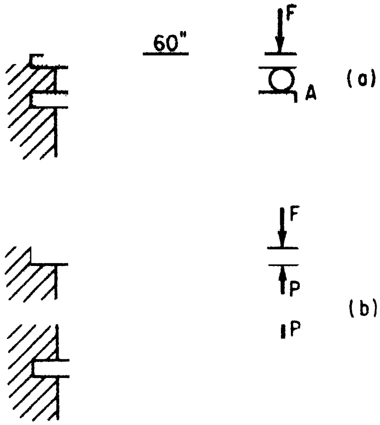


Fig. 6-20

this problem. First, a mutual force P acts on both beams, Fig. 6-20(b), and second, the deflections of both beams are equal.

With reference to Table 6-2, the deflection of the upper beam is

$$\delta_A = \frac{Fl^3}{3EI} \quad \frac{Pl^3}{3EI} \tag{a}$$

and for the lower beam

$$\delta_A = \frac{Pl^3}{3EI} \tag{b}$$

Equations (a) and (b) are combined to eliminate the mutual force P.

$$2\delta_A = \frac{Fl^3}{3EI}$$

$$= \frac{6\delta_A EI}{l^3}$$

$$= \frac{6(0.5)(30 \times 10^6)1.5}{(60)^3}$$

$$= 625 \text{ lb}$$

PROBLEMS

6-1. A cantilever beam of length l is acted upon by a couple M at the free end. Determine the maximum deflection in terms of M, l, E, and I.

6-2. Determine the deflection at the free end if the couple in Prob. 6-1 is applied at the midpoint of the beam.

6-3. The steel beam shown in Fig. P6-3 has a moment of inertia of 172.8 in.^{4} Determine the magnitude of the load P if the deflection at the free end is 0.25 in.

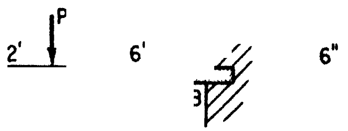


Fig. P6-3

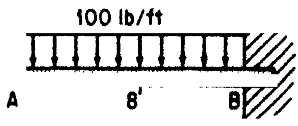


Fig. P6-4

6-4. The cantilever beam shown in Fig. P6-4 is subjected to a distributed load of 100 lb per ft over the entire span. Find the slope and deflection at the free end. $E = 1.5 \times 10^6$ psi.

6-5. Three equal concentrated loads are supported by the cantilever beam of Fig. P6-5. Find the product $EI\delta_{max}$.

6-6. Find the depth d of the beam shown in Fig. P6-6 if the maximum deflection is to equal 1 in. $E = 1.5 \times 10^6$ psi.

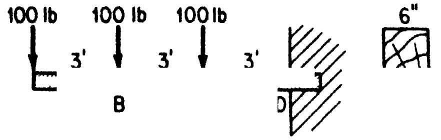


Fig. P6-5

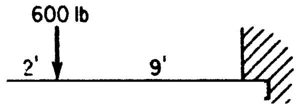


Fig. P6-6

6-7. Determine the maximum value of $EI\delta$ for the cantilever beam shown in Fig. P6-7.

6-8. Find the safe load P for the beam shown in Fig. P6-8 if the deflection is not to exceed 1 in., the bending stress is not to exceed 1800 psi, and the permissible shearing stress is limited to 800 psi. $E = 1.5 \times 10^6$ psi.

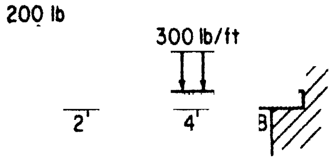


Fig. P6-7

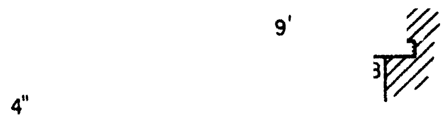


Fig. P6-8

6-9. Determine the value of the moment M in terms of P and l if the deflection of the beam shown in Fig. P6-9 is zero at the free end.

6-10. Determine the deflection at the free end of the cantilever beam in Fig. P6-10. E is 10×10^6 psi and $I = 17.28$ in.^{4}

Fig. P6-9

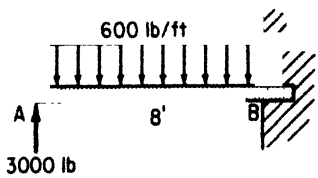


Fig. P6-10

6-11. The maximum bending stress in the steel 10 WF 33 cantilever beam of Fig. P6-11 is 15,000 psi. Find the deflection at the free end.

6-12. The cantilever beam of Fig. P6-12 supports a distributed load over a portion of its span, as shown. Find the value of $EI\delta$ at the free end.

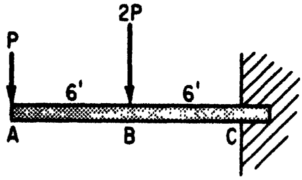


Fig. P6-11

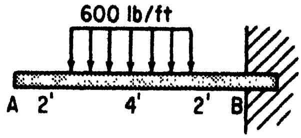


Fig. P6-12

6-13. A simply supported beam of length l supports a concentrated load P at the midspan. Determine the maximum deflection by the moment-area method.

6-14. Use the moment-area method to find the midspan value of $EI\delta$ for the beam shown in Fig. P6-14.

6-15. The permissible midspan deflection for the oak beam shown in Fig. P6-15 is $\frac{1}{2}$ in. The beam has a cross section 6 in. wide by 12 in. deep and a modulus of elasticity $E = 1.5 \times 10^6$ psi. Find the safe value of P.

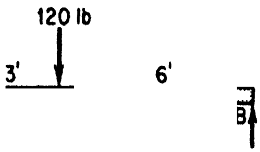


Fig. P6-14

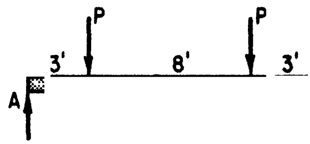


Fig. P6-15

6-16. A 3 ton flywheel is supported at the center of a 4-in.-diameter steel shaft, as shown in Fig. P6-16. Find (a) the maximum deflection by the moment-area method, (b) the maximum bending stress in the shaft.

6-17. An 8 WF 40 steel beam supports a distributed load, as shown in Fig. P6-17. Use the moment-area method to find the maximum deflection.

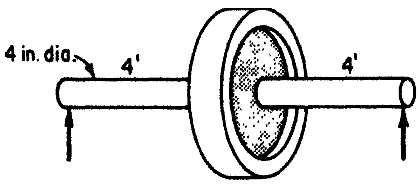


Fig. P6-16

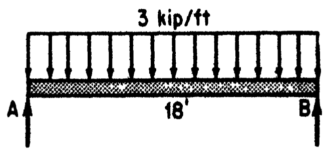


Fig. P6-17

6-18. Calculate the midspan value of $EI\delta$ for the simply supported beam of Fig. P6-18. Use the moment-area method.

6-19. A 12 WF 65 steel beam supports a distributed load symmetrically arranged, as shown in Fig. P6-19. Determine the midspan value of δ by the moment-area method.

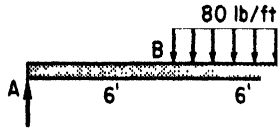


Fig. P6–18

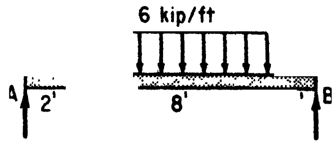


Fig. P6–19

6-20. A 4-in.-diameter steel line-shaft is subjected to belt pulls, as shown in Fig. P6-20. Use the moment-area method to determine the deflection at midspan.

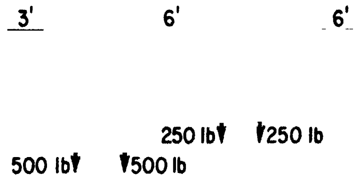


Fig. P6–20

6-21. Three steel coils, each weighing 10,000 lb, are lashed to a trailer bed, as shown in Fig. P6-21. Find the deflection of the bed at the center coil. Assume the trailer bed to consist of two 12 in. 35 lb per ft standard I beams, as shown in the section. Neglect the weights of the beams and the effects of the flooring, blocking, and guy wires. Use the moment-area method.

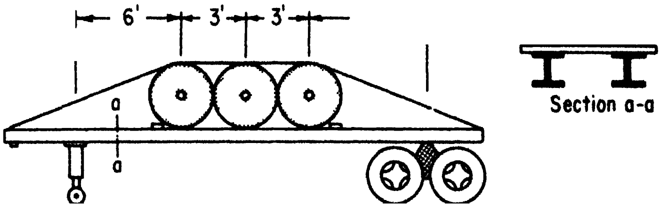


Fig. P6–21

6-22. Use the moment-area method to determine the maximum deflection of a timber beam which supports a distributed load of 200 lb per ft over a span of 12 ft plus a concentrated load of 1000 lb at midspan. The beam is 4 in. wide by 12 in. deep. $E = 1.5 \times 10^6$ psi.

6-23. Compare the deflections of two identical beams; one has a total load of W lb distributed uniformly over its entire span, and the second has the load W concentrated at the midspan.

6-24. Prove that the maximum bending stress in a rectangular beam subjected to a concentrated load at midspan is $\sigma = 6Eh\delta/l^2$, where h is the depth of the beam, δ the maximum deflection, and l the span.

6-25 through **6-28.** For the beams shown in the respective figures, find tb value of $EI\delta$ at the free end by the moment-area method.

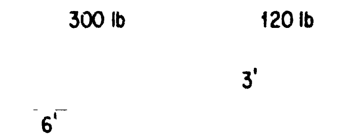


Fig. P6-25

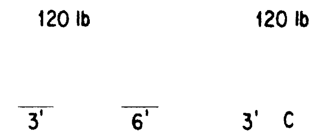


Fig. P6-26

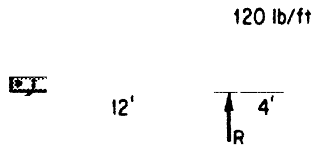


Fig. P6-27

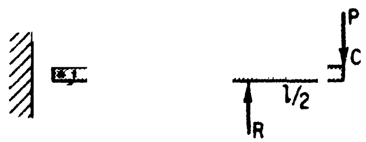


Fig. P6-28

6-29. Find the value of P, Fig. P6-29, that will make the deflection at C be zero.

6-30 through **6-34.** For each of the beams shown in the respective figures, find the reaction at the prop and then draw the shear and moment diagrams. Indicate the maximum values of V and M.

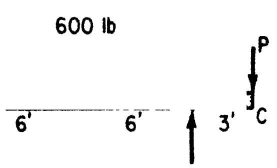


Fig. P6-29

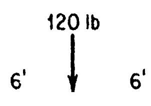


Fig. P6-30

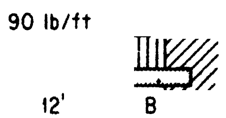


Fig. P6-31

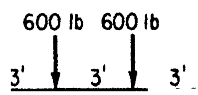


Fig. P6-32

Fig. P6-33

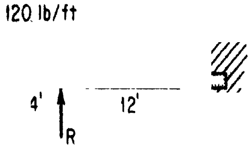


Fig. P6-34

6-35. Prove that the reaction at the prop for the beam shown in Fig. P6-35 is equal to the sum of the prop reactions for the beam of Probs. 6-30 and 6-31.

6-36. Two props are employed to help support the beam of Fig. P6-36 Find the reactions R_1 and R_2.

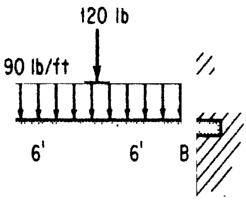


Fig. P6-35

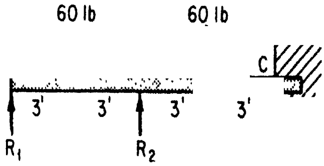


Fig. P6-36

6-37 through **6-41.** For each of the restrained beams shown in the respective figures, find, by the moment-area method, the forces and moments that act at the supports. Draw the shear and moment diagrams for each and indicate maximum values.

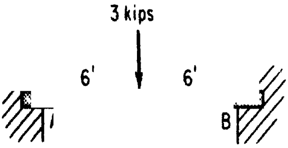


Fig. P6-37

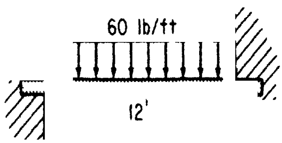


Fig. P6-38

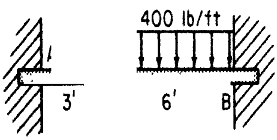


Fig. P6-39

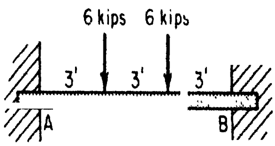


Fig. P6-40

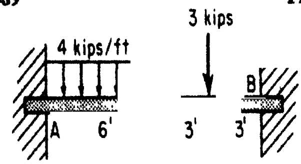


Fig. P6-41

6-42. Determine the midspan deflection of the beam shown in Fig. P6-37 in terms of E and I. Use the moment-area method and apply the results of Prob. 6-37.

6-43. The beam of Fig. P6-38 is a 5 in. by 3 in. steel I-beam girder. Determine the midspan deflection by applying the moment-area method to the results of Prob. 6-38.

6-44. Use the moment-area method to find the deflection at point C for the beam of Fig. P6-40. The beam is timber ($E = 1.5 \times 10^6$ psi) and has a square cross section measuring 6 in. on edge. Apply the results of Prob. 6-40.

6-45. The beam of Fig. P6-41 is an 8 in., 23 lb standard steel I-beam. Use the results of Prob. 6-41 and the moment-area method to determine the midspan deflection.

6-46. Use the superposition method to determine the deflection at the free end of the cantilever beam shown in Fig. P6-46.

6-47. Find the maximum deflection of the cantilever beam shown in Fig. P6-47. The beam is wood ($E = 1.5 \times 10^6$ psi) and has a square cross section 6 in. on edge. Use the method of superposition.

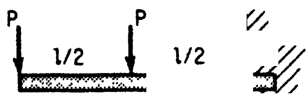


Fig. P6–46

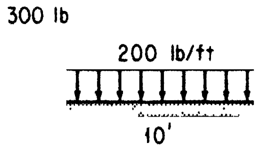


Fig. P6–47

6-48. Use the method of superposition to find the midspan deflection of the timber beam shown in Fig. P6-48. The beam is 6 in. wide by 12 in. deep and has a modulus of elasticity of 1.5×10^6 psi.

6-49. Use the superposition method and find the value of P, Fig. P6-49, if the steel beam is to deflect 0.2 in. The moment of inertia of the beam is 17.28 in.^{4}

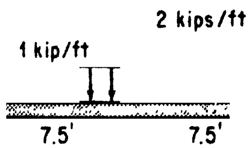


Fig. P6–48

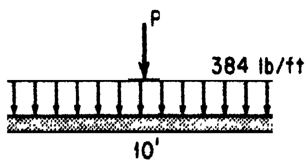


Fig. P6–49

6-50. Two loads of W lb act on the beam shown in Fig. P6-50. One of the loads is concentrated at the center of the span and the other is distributed uniformly over the entire length. Find the deflection, by the superposition method, in terms of E, I, l, and W.

6-51. Find the least moment of inertia of the steel beam of Fig. P6-51 if the midspan deflection is not to exceed $\frac{1}{400}$ of the span.

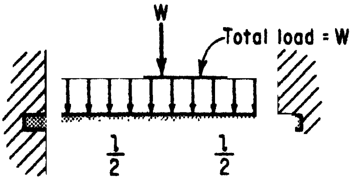


Fig. P6-50

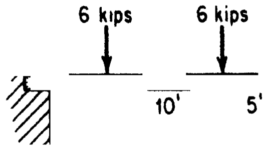


Fig. P6-51

6-52. A simple beam having a span of 12 ft supports a concentrated load P at the midspan. The beam has a rectangular cross section, 6 in. wide and 12 in. deep, and a modulus $E = 2 \times 10^6$ psi. Determine the safe value of P under the following conditions:

Maximum allowable bending stress	1000 psi
Maximum allowable longitudinal stress	100 psi
Maximum allowable deflection	Span/400

6-53. A cantilever beam having an 8 ft span supports a uniformly distributed load of w lb per ft. The beam is a steel I-beam having a depth of 10 in. and a moment of inertia of 172.8 in.^{4} Find the safe value of w under the following limitations:

Maximum bending stress	20,000 psi
Maximum deflection	Span/500

6-54. A timber cantilever beam ($E = 2 \times 10^6$ psi) having a depth of 12 in. supports a concentrated load P at its free end. Find the length of the beam if the following conditions are to be satisfied simultaneously:

Deflection	Span/400
Bending stress	1000 psi

6-55. A steel tie-bar having a cross-sectional area of 0.2 in.^{2} helps to support a timber cantilever beam, as shown in Fig. P6-55. The beam has a moment of inertia of 600 in.^{4} and a modulus of elasticity of 2×10^6 psi. Determine the axial force in the tie-bar.

6-56. A steel post having a cross-sectional area of 0.5 in.^{2} is placed under a restrained steel beam, as shown in Fig. P6-56. The beam has a moment of inertia of 100 in.^{4} Find the stress in the post if its temperature is raised 100°F. $\alpha = 6.5 \times 10^{-6}$ in./in./°F.

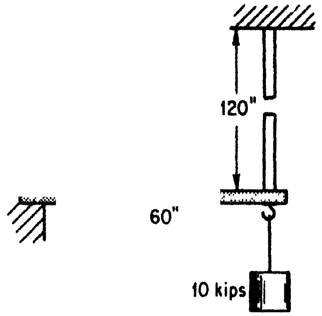


Fig. P6–55

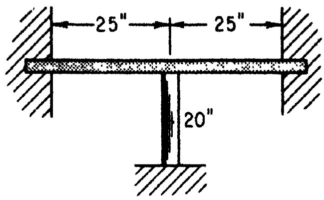


Fig. P6–56

6-57. A brass tube having a cross-sectional area of $\frac{1}{2}$ in.^{2} is placed between two steel cantilever beams, as shown in Fig. P6-57. The moments of inertia of the beams are 100 in.^{4} and 200 in.^{4} respectively. Find the stress in the tube if its temperature is increased 100°F. $E_b = 12 \times 10^6$ psi; $E_s = 30 \times 10^6$ psi; $\alpha = 10 \times 10^{-6}$ in./in./°F.

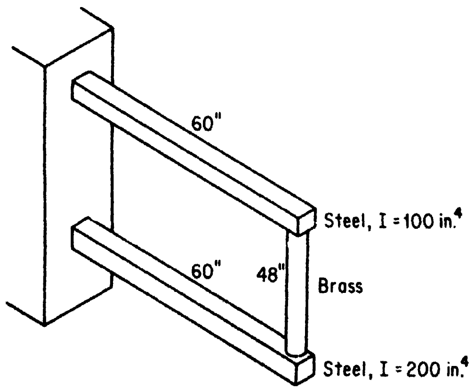


Fig. P6–57

6-58. A steel beam having a moment of inertia of 24 in.^{4} is supported by three columns, as shown in Fig. P6-58. The columns have a cross-sectional area of 1 in.^{2} and are supported to prevent buckling. Find the force in the center column, if the temperature of all three increases 100°F. $E_b = 12 \times 10^6$ psi; $E_s = 30 \times 10^6$ psi; $\alpha_b = 10 \times 10^{-6}$ in./in./°F; $\alpha_s = 6.5 \times 10^{-6}$ in./in./°F.

6-59. Find the force F necessary to cause point A, Fig. P6-59, to move 0.2 in. downward. The beams are aluminum and have a centroidal moment of inertia of 0.27 in.^{4} each. $E_a = 10 \times 10^6$ psi.

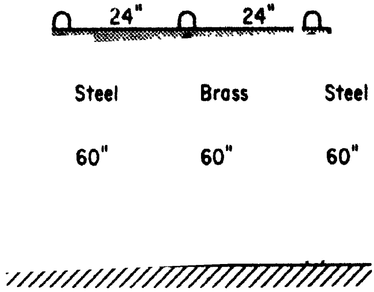


Fig. P6–58

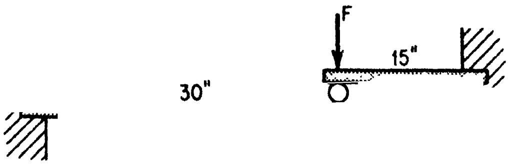


Fig. P6–59

6-60. A spring placed under a cantilever beam, as shown in Fig. P6-60, limits the deflection to one-half the value obtained if the beam could deflect freely. The beam is made of steel and is 12 in. wide by 1 in. deep. Find the constant of the spring.

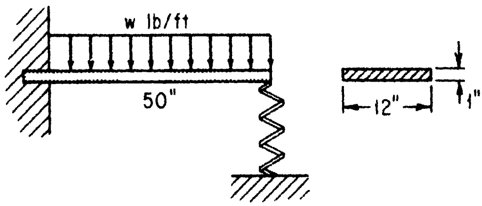


Fig. P6–60

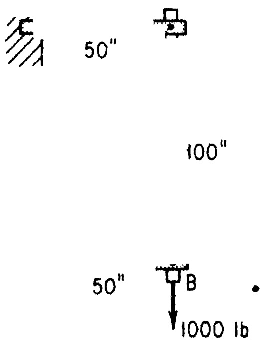


Fig. P6–61

6-61. A steel tie-rod AB is pin-connected to identical steel cantilever beams as shown in Fig. P6-61. The cross-sectional area of the tie-rod is 0.1in.^{2}, and each beam has a moment of inertia $I_{NA} = 20$ in.^{4} Find the stress in the tie-rod

CHAPTER 7

Combined Loading

In the development of basic concepts, loads have been applied to members in four ways: *along the axis of the member, in direct shear, in torsion, and in bending*. This chapter will consider the analysis and effects of combined loading.

7-1 General Considerations

More often than not, in an actual structure, the fundamental loading arrangements act in combination. Figure 7-1 illustrates how a single force can produce conditions equivalent to all four basic loads. On a section within the bar, the component F_z acts in two ways: it both pushes and bends, whereas the component F_y produces three distinct actions within the bar: direct shear, torsion, and bending.

If considered individually, the stresses in this example could be calculated by formulas developed in previous chapters. The problem of combined stress is approached by using these basic equations and the concepts of superposition.

7-2 Combined Axial and Bending Loads

Probably the most common case of combined loading occurs when bending moments and axial forces act together. Although the combination is usually not one of choice, there are instances when properties can be enhanced by superimposing compressive and bending loads; prestressed concrete, which will be discussed in detail, is an example.

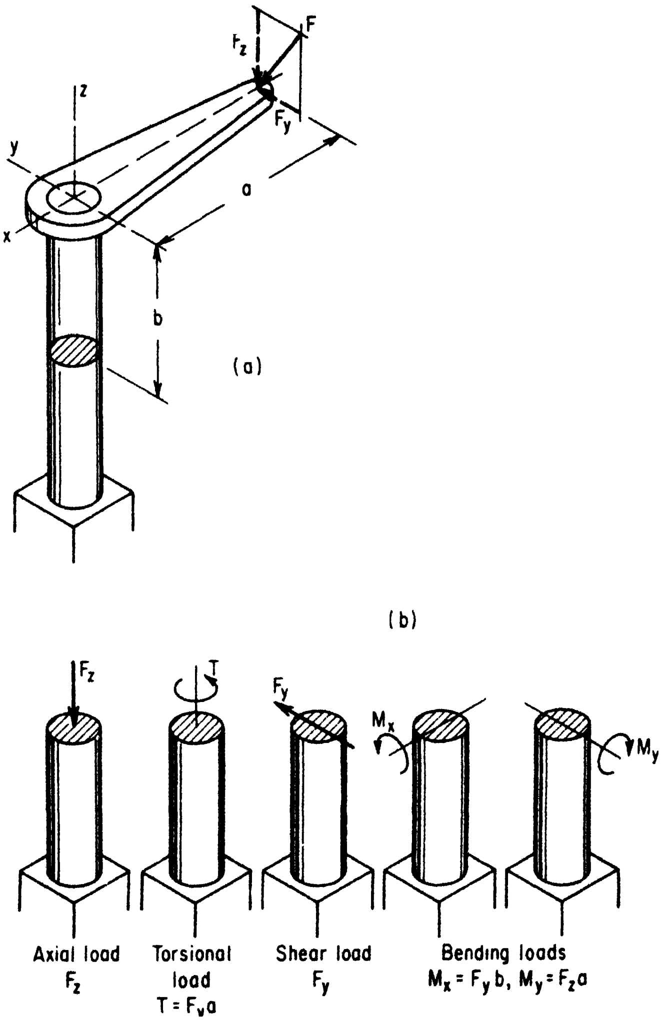


Fig. 7-1

All cases of combined axial loading and bending can be analyzed by superposition methods. Figure 7-2(a) shows a beam subjected to bending loads. For the sake of simplicity, the beam is assumed to have a rectangular cross section. Stresses, then, vary directly with distance from the neutral axis, and a stress pattern exists within the member typical to that shown. If this same beam were subjected to an axial load, Fig. 7-2(b), the stresses acting on a transverse section would be uniformly distributed across the entire area. When the loads act together, Fig. 7-2(c), the resultant stress is the algebraic, or superposed, sum of the two previous patterns. In this

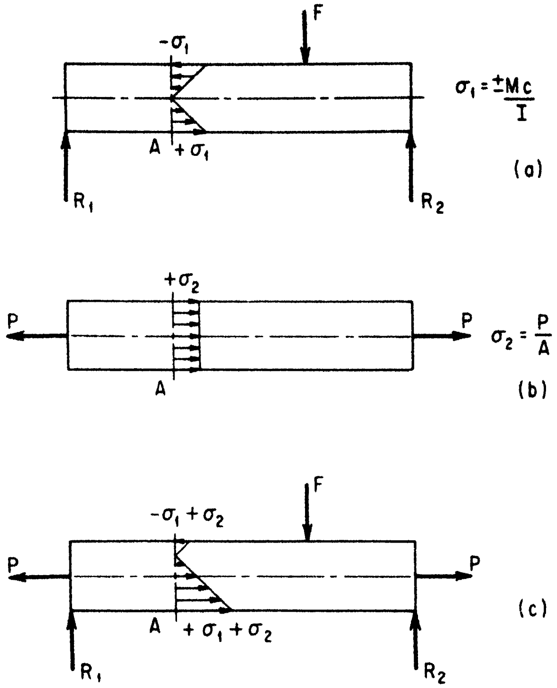


Fig. 7-2

illustration, stresses in the top fibers diminish, while those in the bottom fibers increase; in effect, the neutral axis is shifted upward. The combined stress at a point within the section is found by adding the axial stress and the flexural stress at that point:

$$\sigma = \pm \frac{P}{A} \pm \frac{My}{I} \tag{7-1}$$

In terms of maximum values, the stress is given by

$$\sigma = \pm \frac{P}{A} \pm \frac{Mc}{I} \tag{7-2}$$

It must be remembered that algebraic signs are as much a part of Eqs. (7-1) and (7-2) as are magnitudes and that tensile stress and compressive stress assume opposite signs.

Example 1. A concentrated load of 600 lb is supported by a pin-connected truss, as shown in Fig. 7-3(a). Find the maximum values of the tensile and compressive stresses that act on a transverse section at D. The horizontal member is 2 in. wide and 6 in. deep.

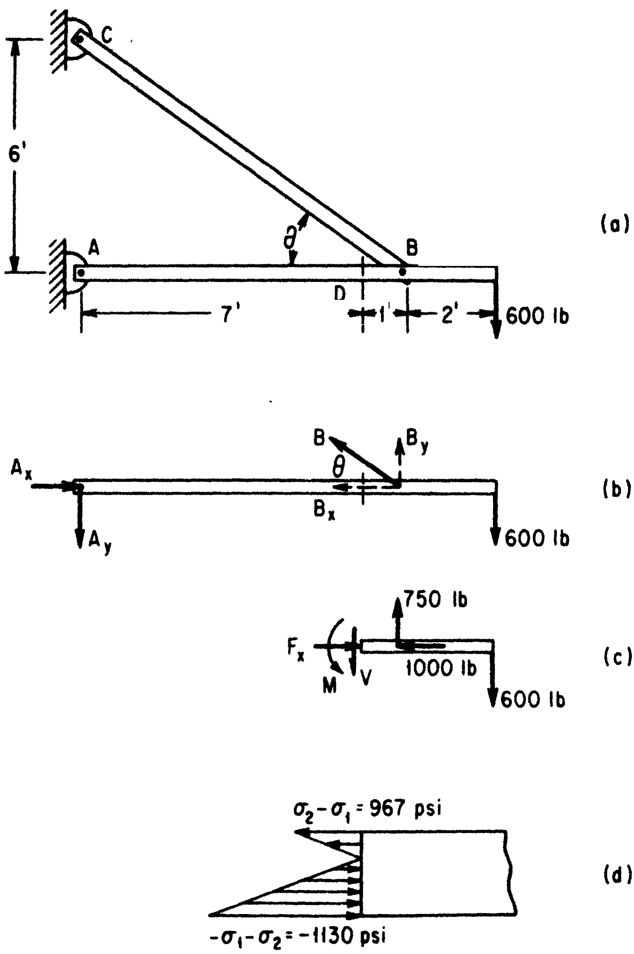


Fig. 7-3

Solution: Figure 7-3(b) shows a free-body diagram of the horizontal member. The vertical component of the pin reaction at B is computed by summing moments about A.

$$\sum M_A = 0$$
$$8B_y = 600(10)$$
$$B_y = 750 \text{ lb}$$

The ratio B_y/B_x is equal to the tangent of the angle θ; hence,

$$\frac{B_y}{B_x} = \tan\ \theta = \frac{3}{4}$$

$$B_x = \tfrac{4}{3}B_y = \tfrac{4}{3}(750) = 1000 \text{ lb}$$

The internal reactions at D become apparent when the horizontal member is sectioned as shown in Fig. 7-3(c). The reactions F_x, V, and M are

$$F_x = 1000 \text{ lb}$$

$$V = 750 - 600 = 150 \text{ lb}$$

$$M = -600(3) + 750(1) = -1050 \text{ lb ft}$$

Since only maximum axial stresses are to be computed, the direct shear V does not enter into the computations. The direct stress and flexure stress can be computed as follows:

$$\sigma_1 = -\frac{P}{A} = -\frac{1000}{(2 \times 6)} = -83.3 \text{ psi}$$

$$\sigma_2 = \pm\frac{Mc}{I} = \pm\frac{(1050 \times 12)3}{2(6)^3/12} = \pm 1050 \text{ psi}$$

The maximum compressive stress occurs at the bottom fibers, where σ_1 and σ_2 add algebraically:

$$\sigma_{c_{\max}} = -\sigma_1 - \sigma_2 = -83.3 - 1050 = -1133.3 \text{ psi}$$

Fibers at the top of the section are in tension; the difference between direct stress and bending stress is equal to the tensile stress.

$$\sigma_{t_{\max}} = -\sigma_1 + \sigma_2 = -83.3 + 1050 = +966.7 \text{ psi}$$

Superposition of the direct and flexural stresses is shown in Fig. 7-3(d).

Example 2. The vertical member of the jib, Fig. 7-4(a), is a 12 WF 40 beam. Find the axial stresses σ_t and σ_c in this member if $x = 5$ ft. Assume the column to be adequately braced to prevent buckling.

Solution: A force and a moment act at section a–a, as shown in the free-body diagram, Fig. 7-4(b). The axial compressive stress σ_1 is constant and equal to

$$\sigma_1 = -\frac{P}{A} \qquad \frac{10{,}000}{11.77} = -850 \text{ psi}$$

The bending stress σ_2 is a function of the moment M, and at $x = 5$ ft is

$$\sigma_2 = \pm\frac{M}{Z} = \pm\frac{10{,}000(5 \times 12)}{51.9} = \pm 11{,}600 \text{ psi}$$

The bending stress σ_2 acts in compression at the inside face of the beam and in tension at the outside face; thus we have, at the inside face:

$$\sigma_c = -\sigma_1 - \sigma_2 = -850 - 11{,}600 = -12{,}450 \text{ psi}$$

at the outside face:

$$\sigma_t = -\sigma_1 + \sigma_2 = -850 + 11{,}600 = 10{,}750 \text{ psi}$$

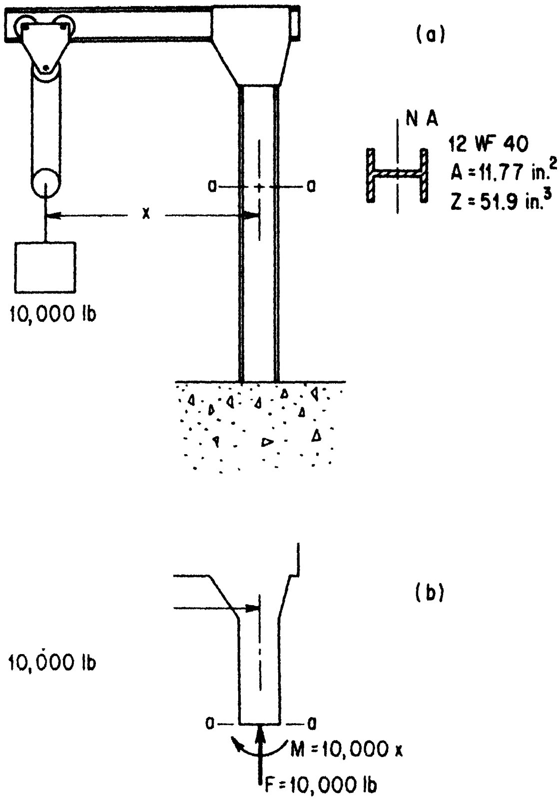


Fig. 7-4

7-3 Prestressed Concrete

In certain circumstances the effects of combined loading are a desirable feature rather than a condition one would wish to avoid. *Prestressed concrete*—or perhaps the more descriptive phrase, precompressed concrete—is an example.

Concrete is weak in tension, and for this reason steel reinforcing bars are used whenever concrete members must sustain tensile loads. For many cases, however, where the spans are great or the loads are heavy, the use of reinforced concrete becomes impractical. In such an instance, prestressed concrete is an economical solution.

The desire in prestressing is to eliminate tensile stress; this is done by stretching reinforcing steel so as to superimpose on the concrete compressive stresses equal or greater than the expected tensile stress. The strengthening effect is somewhat like the squeeze put on a horizontal row of boxes when they are to be moved from place to place; with sufficient pressure they may be lifted and carried, even though those in the center are unsupported.

The actual fabrication of prestressed beams consists of casting concrete in a form containing high-strength steel bars under tension. This tension is maintained, Fig. 7-5(a), until the concrete has hardened and cured. The

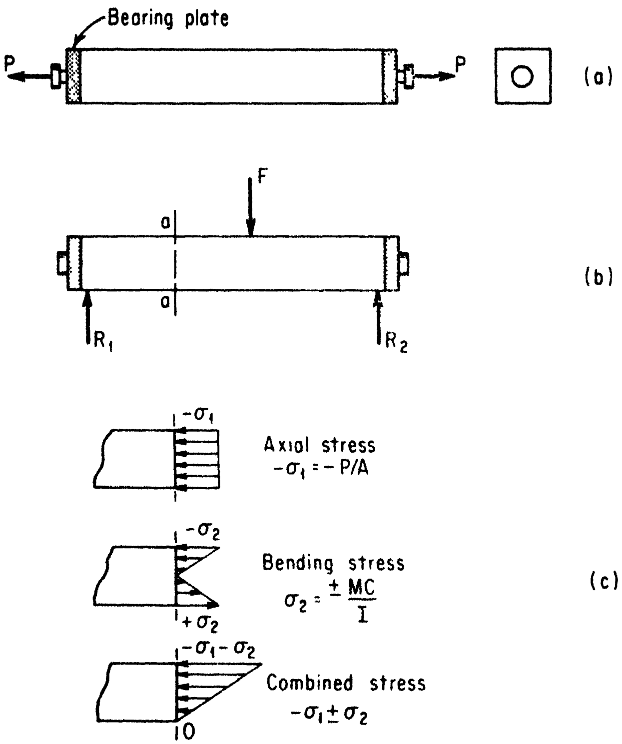


Fig. 7-5

tensile load P is then released, Fig. 7-5(b), and the beam is *precompressed.* When the reinforcing bars are centrally located,[1] the compressive stresses and bending stresses add or subtract to give the superimposed stress pattern, Fig. 7-5(c).

[1] Usual practice is to locate reinforcing bars closer to the tension side of the beam. This permits an even greater amount of precompression for a given amount of steel because of the eccentricity of the load. The reader is referred to the many fine texts on concrete design for a more detailed discussion of prestressed concrete.

Example 3. A concrete beam, Fig. 7-6, having a width of 6 in. and a depth of 12 in. is precompressed by a force of 100,000 lb acting through centrally located reinforcing bars. Find the safe distributed load w lb per ft that can be supported by the beam.

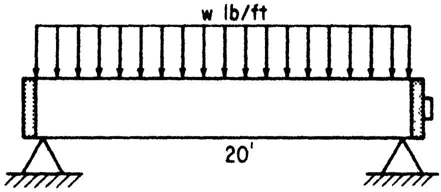


Fig. 7-6

Solution: The precompressive stress is equal to the applied load divided by the cross-sectional area of the beam.

$$\sigma_1 = -\frac{P}{A} = -\frac{100{,}000}{6(12)} = -1390 \text{ psi}$$

The maximum bending stress occurs at midspan, where the moment has its greatest value, $wl^2/8$ lb ft.

$$\sigma_2 \quad \pm\frac{Mc}{I} = \pm\frac{M}{Z}$$

$$= \pm\left(\frac{wl^2}{8} \times \frac{1}{bh^2/6}\right) = \pm\left[\frac{w(20)^2(12)}{8}\right]\left[\frac{6}{6(12)^2}\right]$$

$$= \pm 4.17w \text{ psi}$$

The safe load is obtained by equating the tensile bending stress to the compressive prestress.

$$\sigma_2 = \sigma_1$$

$$4.17w = 1390$$

$$w = 333 \text{ lb per ft}$$

7-4 Bending in Two Directions

Still another form of combined stress occurs when members are acted upon by loads, Fig. 7-7(a), which induce bending in two directions, as illustrated in Fig. 7-7(b). At a distance a from the free end, bending moments M_x and M_y act to produce the stress distribution patterns shown in Fig. 7-7(c), where

$$\sigma_1 = \pm \frac{M_y}{Z_y} = \pm \frac{F_x a}{Z_y}$$

and (7-3)

$$\sigma_2 = \pm \frac{M_x}{Z_x} = \pm \frac{F_y a}{Z_x}$$

where Z_x and Z_y are the section moduli with respect to the x-axis and the y-axis. The *superposition* of bending stresses σ_1 and σ_2 is illustrated in Fig. 7-7(d). Stress magnitudes at each of the four corners are as follows:

$$\sigma_b = \sigma_1 + \sigma_2$$

$$\sigma_c = -\sigma_1 + \sigma_2$$

$$\sigma_d = -\sigma_1 - \sigma_2$$

$$\sigma_e = \sigma_1 - \sigma_2$$

In this illustration the maximum tensile stress occurs at b, and the maximum compressive stress at d.

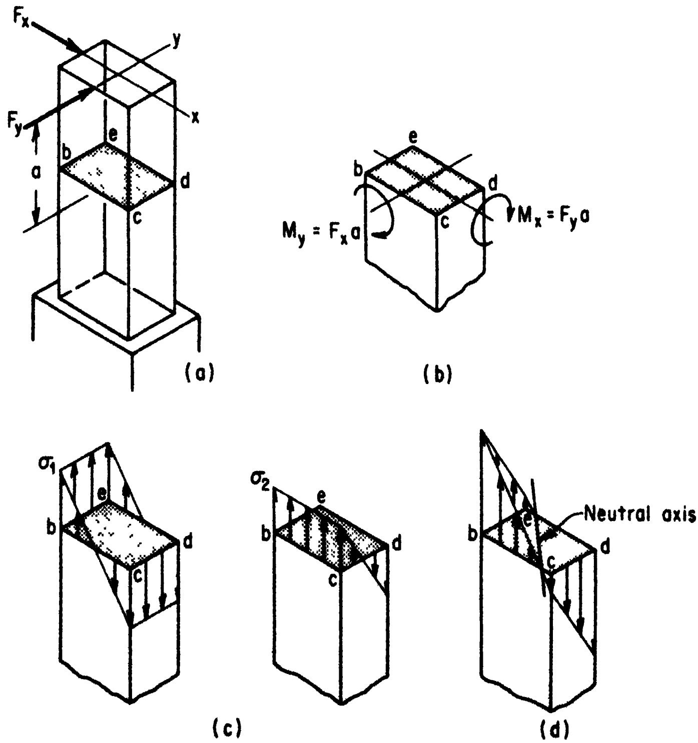


Fig. 7-7

There is always a line of zero stress associated with two-directional bending; in effect, this line is the neutral axis of the beam. For this example, the axis of zero stress is a line joining the points of zero stress on faces *bc* and *ed*, Fig. 7-7(d).

Example 4. Find the maximum stress in the cantilever beam of Fig. 7-8. The beam has a width of 4 in. and a depth of 9 in.

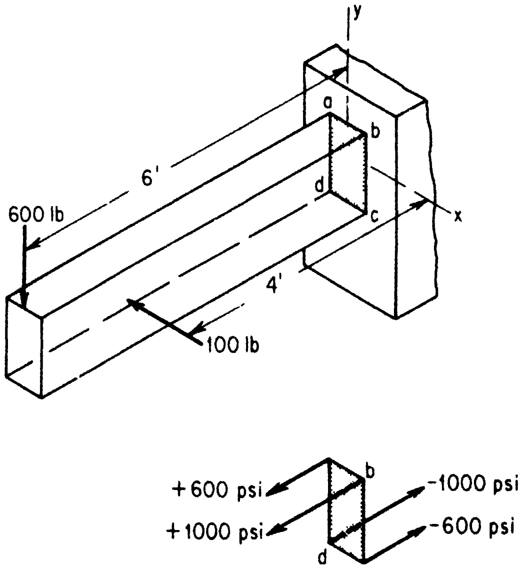


Fig. 7-8

Solution: The section moduli are computed as follows:

$$Z_x = \frac{I}{c} = \frac{bh^3}{12(h/2)} = \frac{bh^2}{6} = \frac{4(9)^2}{6} = 54 \text{ in.}^3$$

$$Z_y = \frac{b^2h}{6} = \frac{(4)^2\, 9}{6} = 24 \text{ in.}^3$$

Stresses, produced by the individual bending loads, are

$$\sigma_1 = \pm\frac{M}{Z_x} = \pm\frac{600(6 \times 12)}{54}$$

$= \pm$ 800 psi (tension on the top face and compression on the bottom face)

$$\sigma_2 = \pm\frac{M}{Z_y} = \pm\frac{100(4 \times 12)}{24}$$

$= \pm$ 200 psi (tension on the near side and compression on the far side)

The maximum stresses occur along edges b and d at the supports.

$$\sigma_b = \sigma_1 + \sigma_2 = 800 + 200 = 1000 \text{ psi (tension)}$$

$$\sigma_d = -\sigma_1 - \sigma_2 = -800 - 200 = -1000 \text{ psi (compression)}$$

Minimum values of stress occur at corners a and c and are, respectively,

$$\sigma_a = \sigma_1 - \sigma_2 = 800 - 200 = 600 \text{ psi (tension)}$$

$$\sigma_c = -\sigma_1 + \sigma_2 = -800 + 200 = -600 \text{ psi (compression)}$$

Example 5. Two loads are applied to a 2-in.-diameter circular shaft simply supported as shown in Fig. 7-9(a). Find the maximum normal stress in the shaft.

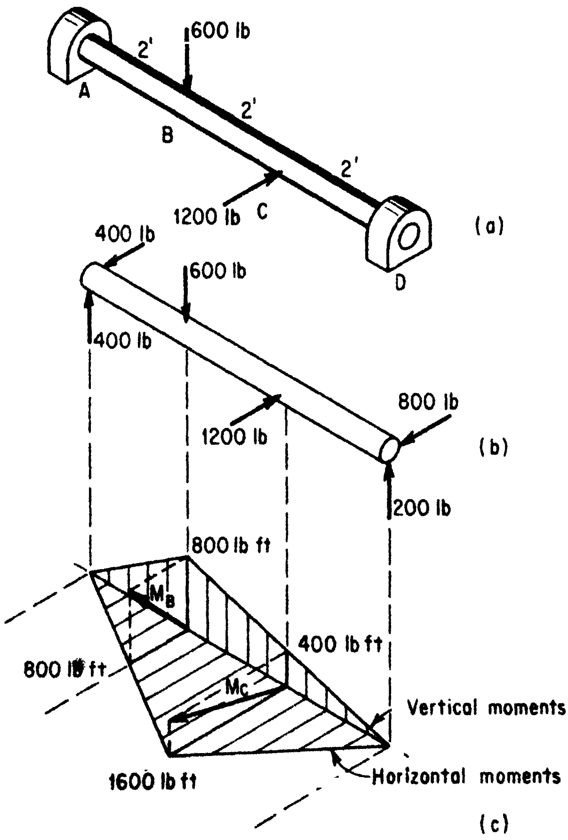


Fig. 7-9

Solution: The bearing reactions are found in the usual way, and the free-body diagram is drawn as shown in Fig. 7-9(b). Vertical and horizontal bending-moment diagrams are next constructed, as illustrated in Fig. 7-9(c).

Since moments are directed magnitudes, they can be added vectorially, just as force vectors are added. The maximum moment, which will occur either at B or C, is the vector sum M_B or M_C; thus

$$M_B = \sqrt{(800)^2 + (800)^2} = 1130 \text{ lb ft}$$

$$M_C = \sqrt{(400)^2 + (1600)^2} = 1650 \text{ lb ft}$$

The critical section, therefore, is at point C.

$$\sigma_{\max} = \frac{Mc}{I}$$

where

$$c = 1 \text{ in.}$$

and

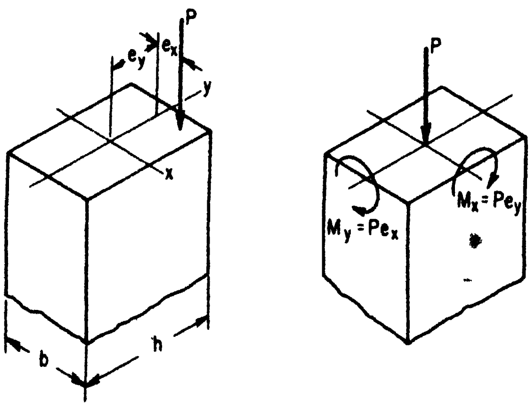
$$I = \frac{\pi D^4}{64} = \frac{\pi (2)^4}{64} = \frac{\pi}{4}$$

Thus

$$\frac{(1650 \times 12)(1)}{\pi/4} \qquad 25{,}200 \text{ psi}$$

7-5 *Axial Loading Applied Eccentrically*

Bending stresses and axial stresses are induced when loads are applied *eccentrically*, as shown in Fig. 7-10(a). The equivalent loading, Fig. 7-10(b),



(a) (b)

Fig. 7-10

consists of bending moments M_x and M_y and a direct load of P. In a prismatic bar, like that shown in the figure, the stress at any corner would be

$$\sigma = \pm \frac{P}{A} \pm \frac{M_x}{Z_x} \pm \frac{M_y}{Z_y} \tag{7-4}$$

where Z_x and Z_y represent the section moduli with respect to the x-axis and the y-axis. The positive and negative signs indicate the stress to be either tension or compression.

Equation (7-4) is a particularly important relationship which governs the design of short columns constructed of materials weak in tension. A permissible amount of eccentricity can be found by equating the direct compressive stress to the tensile stresses induced by bending.

$$\frac{P}{A} = \frac{M_x}{Z_x} + \frac{M_y}{Z_y} = \frac{Pe_y}{bh^2/6} + \frac{Pe_x}{b^2h/6}$$

The section moduli of a prismatic bar, Fig. 7-11(a), in terms of area is

$$Z_x = \frac{bh^2}{6} = \frac{Ah}{6}$$

and

$$Z_y = \frac{b^2h}{6} = \frac{Ab}{6}$$

Hence,

$$\frac{P}{A} = \frac{6Pe_y}{Ah} + \frac{6Pe_x}{Ab}$$

$$1 = \frac{6e_y}{h} + \frac{6e_x}{b}$$

By setting $e_x = 0$, the permissible eccentricity e_y is

$$e_y = \frac{h}{6}$$

and similarly, if $e_y = 0$,

$$e_x = \frac{b}{6}$$

A neutral axis can be drawn by joining the points which represent the greatest permissible eccentricity in each coordinate direction, Fig. 7-11(b). If the compressive force acts anywhere to the right of this axis, tensile stresses will be created within the member. By symmetry, four neutral axes exist in the section; these lines form the boundaries of a *core* or *kern*, as shown in Fig. 7-11(c). A compressive force acting within this *core* will not produce tensile stresses within the member. Every geometric section has its characteristic *kern*. In a solid circular section, the *kern* is the central circular area shown in Fig. 7-12.

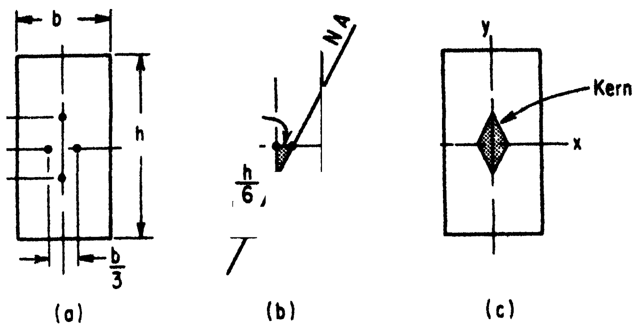


Fig. 7-11

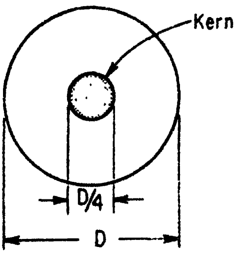


Fig. 7-12

Example 6. Determine the maximum and minimum stresses in the eccentrically loaded post of Fig. 7-13(a). The member is a 16 WF 50 section with the following geometric properties:

Area	14.7 in.^{2}
Section modulus	80.7 in.^{3}
Depth	16.25 in.

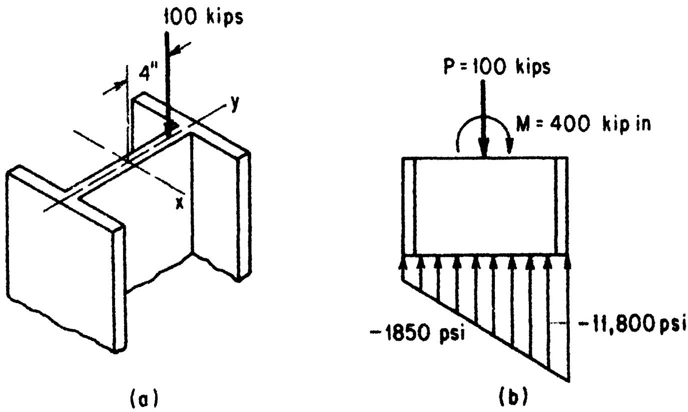


Fig. 7-13

Solution: The eccentric load is equivalent to a direct load of 100 kips superimposed on a bending load of 400 kip in., as shown in Fig. 7-13(b). On the face of the beam nearest the load, stresses add, and on the opposite face, they subtract.

$$\frac{P}{A} - \frac{M}{Z} = -\frac{100{,}000}{14.7} - \frac{400{,}000}{80.7}$$

$$= -11{,}800 \text{ psi}$$

$$\sigma_{min} = -\frac{P}{A} + \frac{M}{Z} = -\frac{100{,}000}{14.7} + \frac{400{,}000}{80.7}$$

$$= -1850 \text{ psi}$$

Both σ_{max} and σ_{min} are negative; the 4 in. eccentricity, therefore, is within the *kern* of the section. While it is not required, the limit of eccentricity in this example can be easily found by equating the direct stress to the bending stress.

$$\frac{P}{A} = \frac{M}{Z}$$

$$\frac{100}{14.7} \quad \frac{100e_y}{80.7}$$

$$\frac{80.7}{14.7} \quad 5.49 \text{ in.}$$

Hence, if e_y exceeds 5.49 in., tensile stresses will be developed in the post.

Example 7. Two square members, one solid and the other hollow, are being considered for the eccentrically loaded strut of Fig. 7-14. The members have equal weights, and, therefore, equal cross-sectional areas and the eccentricity in each is the same. Compare the load-carrying capacity of the two sections.

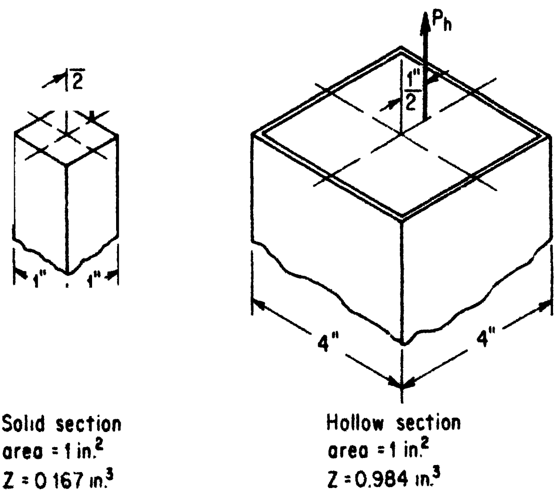


Fig. 7-14

Solution: For a given limiting stress, the allowable loads in each strut can be computed as follows:

Solid strut:

$$\frac{P_s}{A} + \frac{P_s e}{Z}$$

$$\frac{P_s}{1 \times 1} + \frac{P_s(0.5)}{0.167} \quad 4P_s \qquad \text{(a)}$$

Hollow strut:

$$\frac{P_h}{A} + \frac{P_h e}{Z}$$

$$= \frac{P_h}{1 \times 1} \cdot \frac{P_h(0.5)}{0.984} \quad 1.51P_h \qquad \text{(b)}$$

The ratio P_h/P_s can be found by equating (a) and (b):

$$1.51P_h = 4P_s$$

$$\frac{P_h}{P_s} = \frac{4}{1.51} = 2.65$$

For the same weight of material, the hollow strut will carry almost three times the load. This is why hollow members are favored in the design of eccentrically loaded struts.

7-6 Biaxial Stress

A familiar example of a member subjected to *biaxial stress,* or *normal stress in two directions,* is the pressure vessel described in Chapter 1. The walls of the vessel, Fig. 7-15, are stressed in both the longitudinal and circumferential directions.

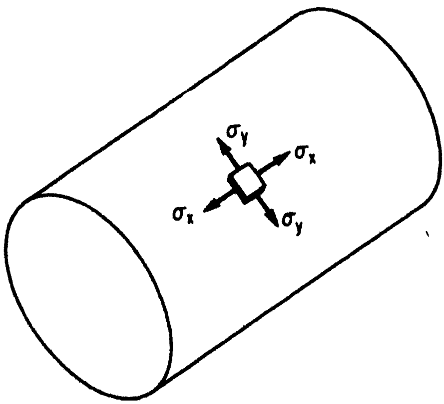


Fig. 7-15

The effects of biaxial stress can be best analyzed by considering a member of uniform thickness, Fig. 7-16(a), acted upon by the *orthogonal forces* F_x and F_y. *The state of stress within the member* can be computed by isolating a small triangular element, as shown in Fig. 7-16(b). Let the diagonal plane of this element have an area A; the vertical and horizontal planes will then

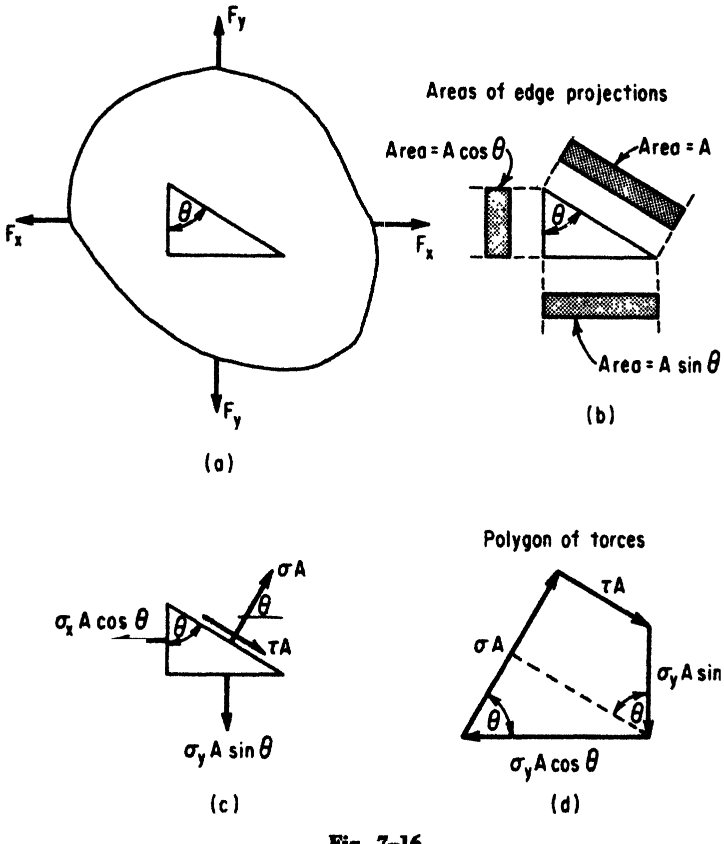


Fig. 7-16

have areas $A \cos \theta$ and $A \sin \theta$, respectively. The element is subjected to normal and shear forces on the diagonal plane, Fig. 7-16(c), and these forces must statically balance the x- and y-components of force imposed by the loading. This is the same as saying that the vector addition of the forces must form a closed polygon, as shown in Fig. 7-16(d). The purpose of this analysis is, of course, to find the normal and shear forces that accompany biaxial stress, and this can be accomplished by applying trigonometry to the polygon of Fig. 7-16(d); thus

$$\sigma A = (\sigma_x A \cos\theta)\cos\theta + (\sigma_y A \sin\theta)\sin\theta$$

$$\sigma = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta \tag{a}$$

and

$$\tau A = (\sigma_x A \cos\theta)\sin\theta - (\sigma_y A \sin\theta)\cos\theta$$

$$\tau = (\sigma_x - \sigma_y)\sin\theta\cos\theta \tag{b}$$

In Eqs. (a) and (b), *tensile stresses are assumed to be positive and compressive stresses negative: a shear stress that tends to impose clockwise rotation on the element is considered to be positive shear.*

By making the following substitutions, Eqs. (a) and (b) can be expressed in terms of the double angle 2θ.

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)$$

$$\sin\theta\cos\theta = \tfrac{1}{2}\sin 2\theta$$

Equations (a) and (b), written in terms of the double angle, are

$$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta \tag{7-5}$$

$$\tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta \tag{7-6}$$

The maximum and minimum values of stress, obtained through these equations, are called *principal stresses*; the principal normal stresses occur when

$$\cos 2\theta = \pm 1$$

$$2\theta = 0°,\ 180°$$

$$\theta = 0°,\ 90°$$

In other words, there are no planes within the element subjected to simple biaxial loading that will have normal stresses greater than σ_x or σ_y. This can be demonstrated by first allowing $\cos 2\theta$ to equal $(+1)$ and then (-1):

$\cos 2\theta = 1$: $$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}(1) = \sigma_x$$

$\cos 2\theta = -1$: $$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}(-1) = \sigma_y$$

The shear stress is maximum or minimum at $\sin 2\theta = \pm 1$; hence,

$$\begin{aligned} \sin 2\theta &= \pm 1 \\ 2\theta &= 90°, 270° \\ \theta &= 45°, 135° \end{aligned}$$

Thus, *for biaxial stress the maximum (or minimum) shear stress occurs on planes inclined at* 45 *deg to the principal normal planes.*

$$\frac{\sigma_x - \sigma_y}{2} \quad \text{at} \quad \theta = 45°$$

$$\tau_{\min} = -\frac{\sigma_x - \sigma_y}{2} \quad \text{at} \quad \theta = 135°$$

When the biaxial stresses are equal (in other words, when $\sigma_x = \sigma_y$), all planes within the member will be free of shearing stresses and will have a normal stress equal to the biaxial stress. This is observed when σ_x and σ_y are set equal to one another in Eqs. (7-5) and (7-6).

$$\sigma_x = \sigma_y$$

$$\sigma = \frac{\sigma_x + \sigma_x}{2} + \frac{\sigma_x - \sigma_x}{2} \cos 2\theta = \sigma_x$$

$$\tau = \frac{\sigma_x - \sigma_x}{2} \sin 2\theta = 0$$

Although substitution of numerical values into Eqs. (7-5) and (7-6) is not difficult, the equations are very rarely used as such. The reason is that a much simpler approach exists in their graphical interpretation.

A set of coordinated axes are drawn as shown in Fig. 7-17(a). These axes represent normal and shearing stresses, and must not be confused with an x-, y-coordinate system. The coordinates σ_x and σ_y are located on the horizontal axis, the axis that represents the coordinates of *all* normal stresses. A circle is then drawn having a diameter equal to the difference $(\sigma_x - \sigma_y)$. This is shown in Fig. 7-17(b). The radius of this circle is, therefore, $(\sigma_x - \sigma_y)/2$, and its center lies at a distance $(\sigma_x + \sigma_y)/2$ from the shear axis. All possible inclined planes within a biaxially stressed element are represented by this circle, as are the values of normal and shear stress that act on these planes. This is evident in Fig. 7-17(c), where point A represents the coordinates of stress σ and τ that act on a plane within an element inclined at an angle θ, as shown in Fig. 7-17(d). The angle is measured relative to the x-axis of the element and a normal to the plane under investigation. In the circle, the angle is doubled and is measured relative to a line drawn between

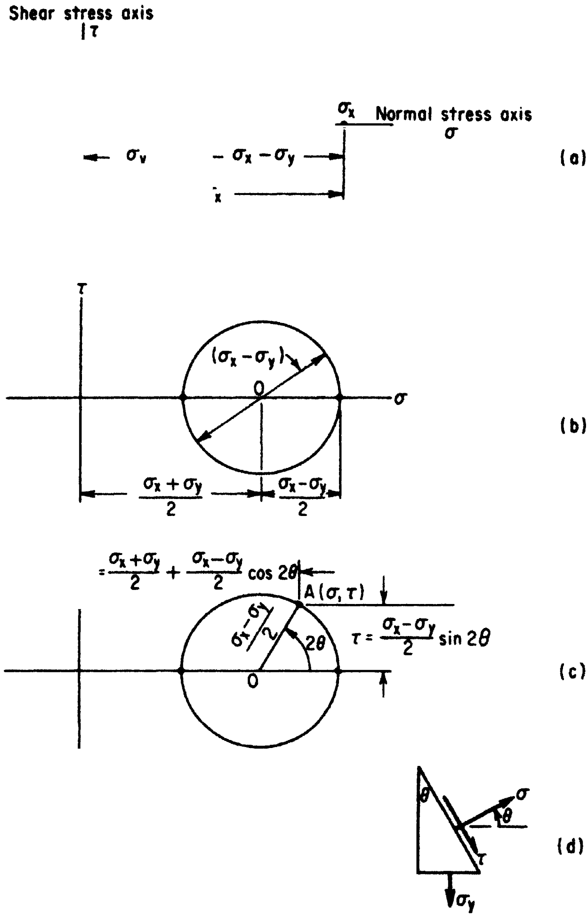


Fig. 7-17

the center of the circle O and σ_x, and the radius line OA. If the angle is measured counterclockwise in the element, it must be measured counterclockwise in the circle.

The circle of stress was devised by Otto Mohr, a German engineer, in 1882, and is called *Mohr's circle.*

In its symbolic form, Mohr's circle might appear to be the more cumbersome approach to the problem of combined biaxial stress. The examples that follow, however, will illustrate the simplicity of the graphical method.

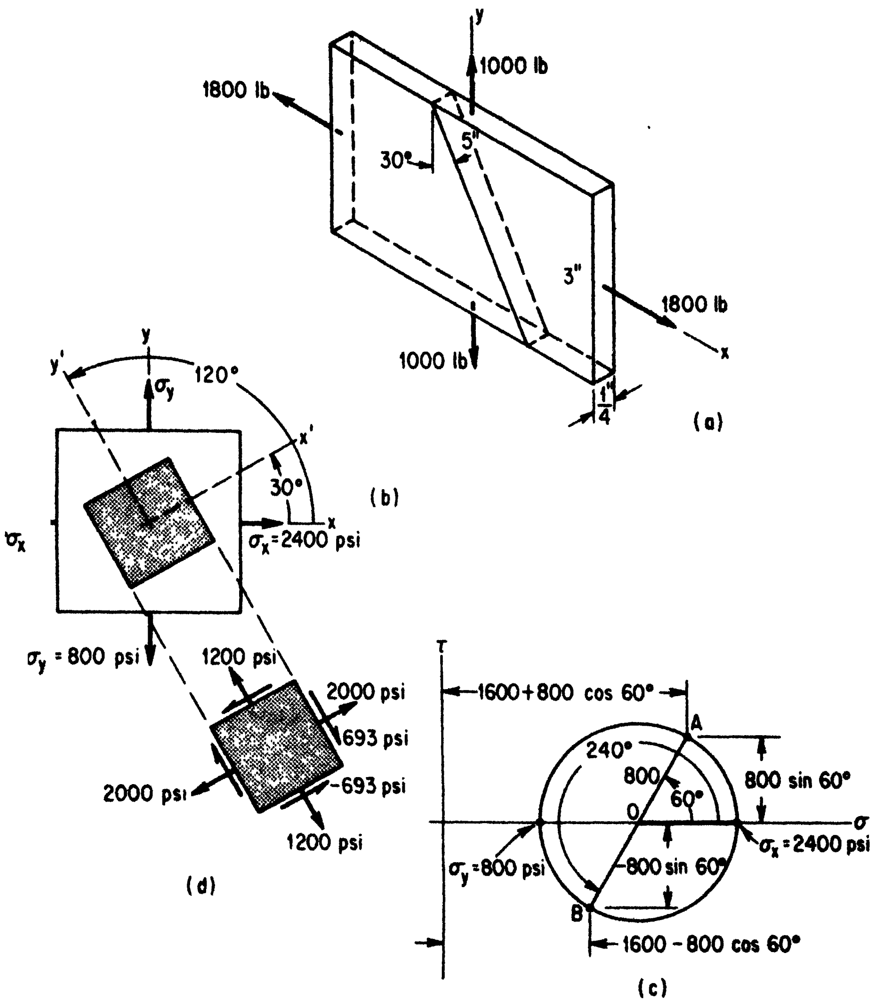


Fig. 7-18

Example 8. A rectangular plate, Fig. 7-18(a), is subjected to the forces shown. Determine the normal and shearing stresses that act on a plane within the member, inclined at 30 deg, as shown.

Solution: The biaxial stresses σ_x and σ_y are first computed:

$$\sigma_x = \frac{P}{A} = \frac{1800}{3(\frac{1}{4})} = 2400 \text{ psi}$$

$$\sigma_y = \frac{}{A} \qquad \frac{1000}{5(\frac{1}{4})} \qquad 800 \text{ psi}$$

An element is isolated, Fig. 7-18(b), to represent the state of stress under consideration, and Mohr's circle is constructed, as shown in Fig. 7-18(c). The intersection of the radius line with the circumference of the circle represents all possible states of stress on all possible planes within the element. At $\theta = 30$ deg ($2\theta = 60$ deg), the state of stress, Fig. 7-18(d), is represented by the coordinates σ and τ of point A, where

$$\sigma = 1600 + 800 \cos 60^\circ$$
$$= 1600 + 800(0.5) = 2000 \text{ psi}$$
$$\tau = 800 \sin 60^\circ$$
$$= 800(0.866) = 693 \text{ psi}$$

Point B on the circle gives the coordinates of the stress which act on a face at right angles to that described by point A. In the element these faces are 90 deg apart; in Mohr's circle they are 180 deg apart, since all angles are doubled. The shear-stress coordinate at B is negative and is, therefore, drawn as a counterclockwise couple on the element.

To understand better the importance of algebraic signs, the previous problem is repeated with the 1800 lb force acting in compression.

Example 9. A rectangular plate, Fig. 7-19(a), is subjected to the forces shown. Determine the normal and shearing stresses that act on a plane within the member, inclined at 30 deg, as shown.

Solution: The plate is subjected to stresses that have the same magnitude as those in Example 8; they differ in sign, however, since σ_x acts in compression. The state of stress drawn on the original body is shown in Fig. 7-19(b). Mohr's circle, which represents an infinite variety of possible states of stress, is drawn as shown in Fig. 7-19(c). Since σ_x is negative, it appears to the left of the shear axis, and, for that matter, to the left of σ_y. This is important to note, because, for the case of biaxial stress, angles in Mohr's circle are measured relative to the radius line $O\sigma_x$.

The coordinates of point A represent the state of stress on the face of the element inclined at 30 deg, as shown in Fig. 7-19(d); thus,

$$\sigma = -1600 \cos 60^\circ - 800 = -1600 \text{ psi}$$
$$\tau = -1600 \sin 60^\circ = -1390 \text{ psi}$$

Point B on the circumference of the circle represents the state of stress on the adjacent face of the element.

$$\sigma = 1600 \cos 60^\circ - 800 = 0$$
$$\tau = 1600 \sin 60^\circ = 1390 \text{ psi}$$

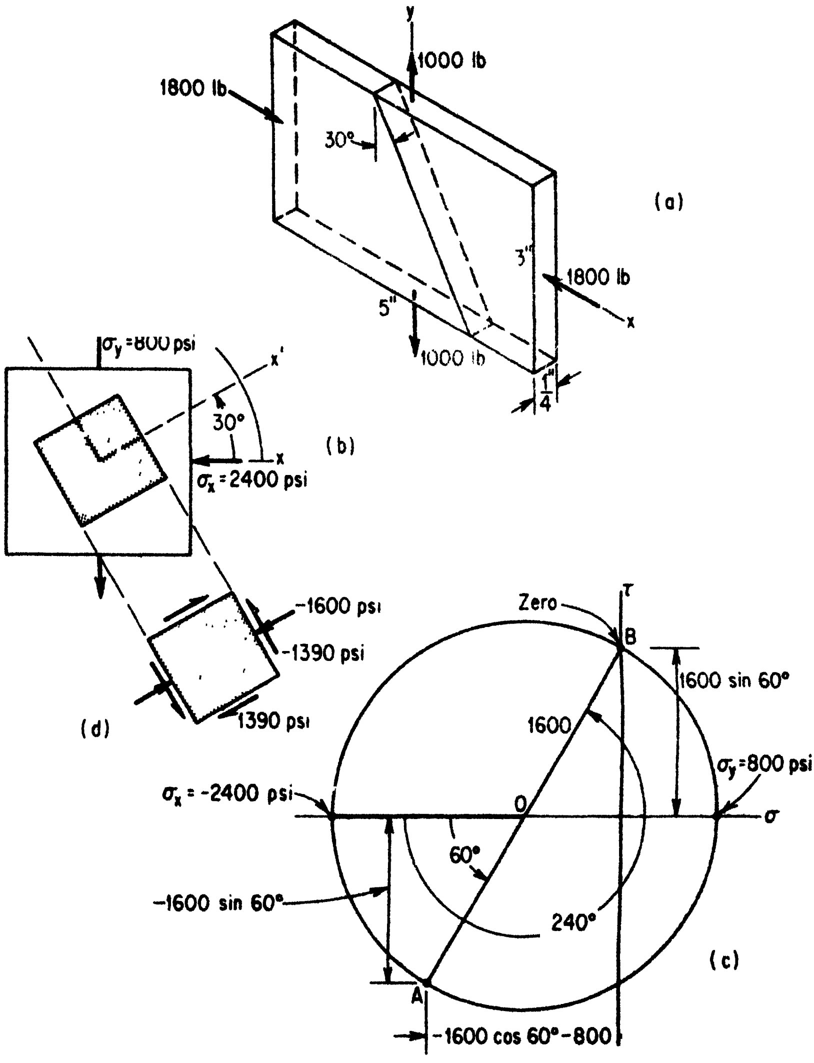


Fig. 7-19

7-7 Pure Shear

Although pure shear is not considered as a combined load in itself, it does produce normal and shearing stresses within members on which it acts. By way of illustration, consider an element of area on the surface of a torsion bar, Fig. 7-20(a). The state of stress on this element, Fig. 7-20(b), consists of shearing stresses τ_{xy} and τ_{yx}. The subscripts used imply that τ_{xy} *acts in the*

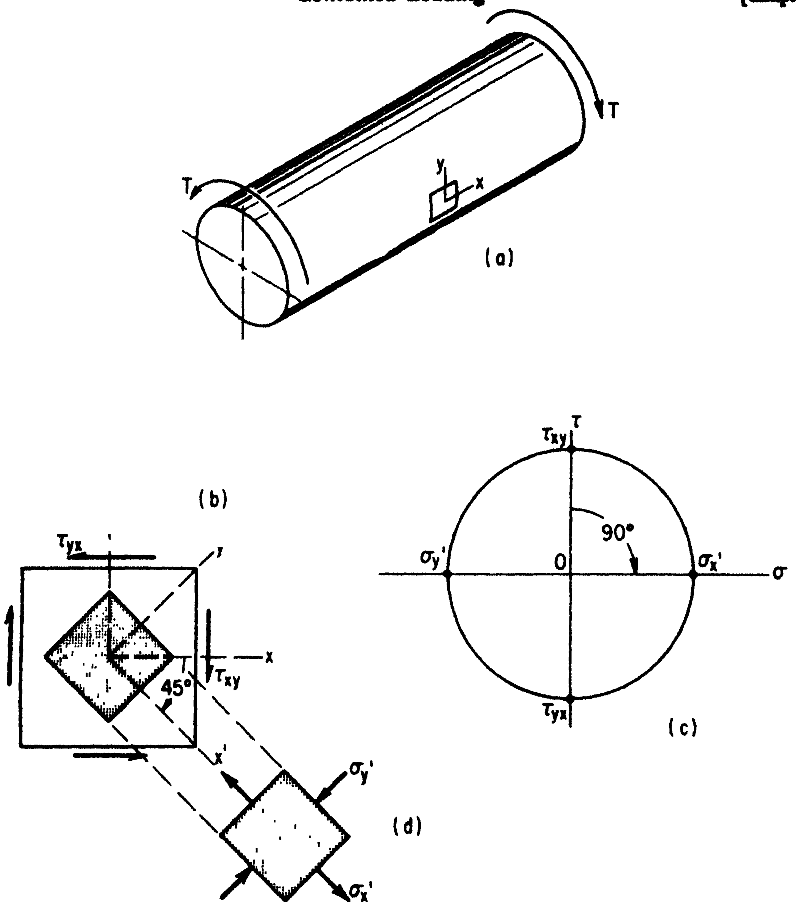


Fig. 7-20

y-*direction on a face normal to the* x-*axis.* Since the element must be in static equilibrium, shearing stresses always act in pairs to form *stress-couples* of equal magnitudes. In other words, all four faces of the element are in shear, and the stresses are such that

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \text{and} \quad \sum M = 0$$

Mohr's circle for pure shear is constructed, Fig. 7-20(c), by locating points τ_{xy} and τ_{yx} on the shear axis and then drawing a circle having a diameter of $2\tau_{xy}$. In pure shear, Mohr's circle is always symmetrical about the normal and shear axis.

The radius of the circle, the line drawn from O to τ_{xy}, represents the x-axis on the element under investigation. As before, all angles measured in the circle are double those of the element and must be similarly directed. Thus, the principal stresses in this example are at $2\theta = 90$ deg and $2\theta = 270$ deg measured clockwise in the circle; these occur on the faces of the element oriented at $\theta = 45$ deg and $\theta = 135$ deg measured clockwise from the positive x-axis. This is illustrated in Fig. 7-20(d), where the x'- and the y'-axis are the principal axes, or the axes perpendicular to faces acted on by normal stress alone.

Example 10. A state of stress on an element is shown in Fig. 7-21(a). (a) Determine the stress components that act on faces of the element rotated 15 deg clockwise. (b) Find the values of the principal stresses.

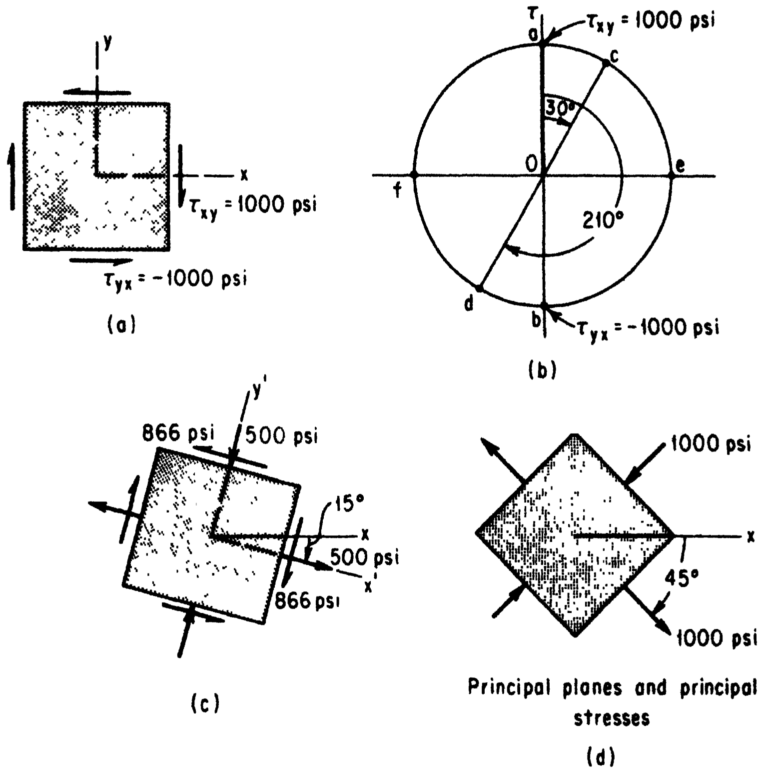


Fig. 7-21

Solution:

Part (a). Mohr's circle is drawn as shown in Fig. 7-21(b). Radius line Oa represents the state of stress on a face of the element normal to the x-axis, and Ob, the state of stress on a face normal to the y-axis. By rotating the radius line, first 30 deg and then 210 deg, stress components that act on faces of the element normal to the x'- and the y'-axis can be found.

$$\sigma_{x'} = 1000 \sin 30° = 500 \text{ psi}$$

$$\tau_{x'} = 1000 \cos 30° = 866 \text{ psi}$$

$$\sigma_{y'} = -1000 \sin 30° = -500 \text{ psi}$$

$$\tau_{y'} = -1000 \cos 30° = -866 \text{ psi}$$

Part (b). Points e and f on Mohr's circle are the coordinates of the principal stresses. These stresses, which are shown in Fig. 7-21(c), act on planes inclined at

$$2\theta = 90° \text{ (clockwise from } Oa)$$

$$\theta = 45°$$

and

$$2\theta = 270° \text{ (clockwise from } Oa)$$

$$\theta = 135°$$

Hence:

$$\sigma_{x'} = 1000 \text{ psi}$$

$$\sigma_{y'} = -1000 \text{ psi}$$

$$\tau_{x'y'} = \tau_{y'x'} = 0$$

Example 11. Determine the horsepower rating of a 4-in.-diameter shaft rotating at a speed of 315 rpm. The normal stress in the shaft is not to exceed 12,000 psi, and the shear stress 8000 psi.

Solution: In the case of pure shear, maximum normal stresses, those that act on the principal planes, have magnitudes equal to the maximum shearing stress, a fact illustrated by the previous example. In this problem, therefore, the limiting shear stress governs the design, since τ_{max} is less than σ_{max}.

$$\tau = \frac{Tc}{J}$$

$$T = \frac{\tau J}{c} = \frac{8000\pi(4)^4}{32(2)} = 32{,}000\pi \text{ lb in.}$$

$$\text{hp} = \frac{Tn}{63{,}000} = \frac{32{,}000\pi(315)}{63{,}000} = 500 \text{ hp}$$

7–8 *Generalized Plane Stress*

The simplicity of Mohr's circle as a method of determining the stress at a point can be fully appreciated when it is applied to ***generalized plane stress***: the case of biaxial and shearing stresses acting simultaneously.

To draw Mohr's circle for generalized plane stress, Fig. 7-22(a), locate the coordinates (σ_x, τ_{xy}) and (σ_y, τ_{yx}) on a set of normal and shear axes, as shown in Fig. 7-22(b). The straight line that joins these points is the diameter of

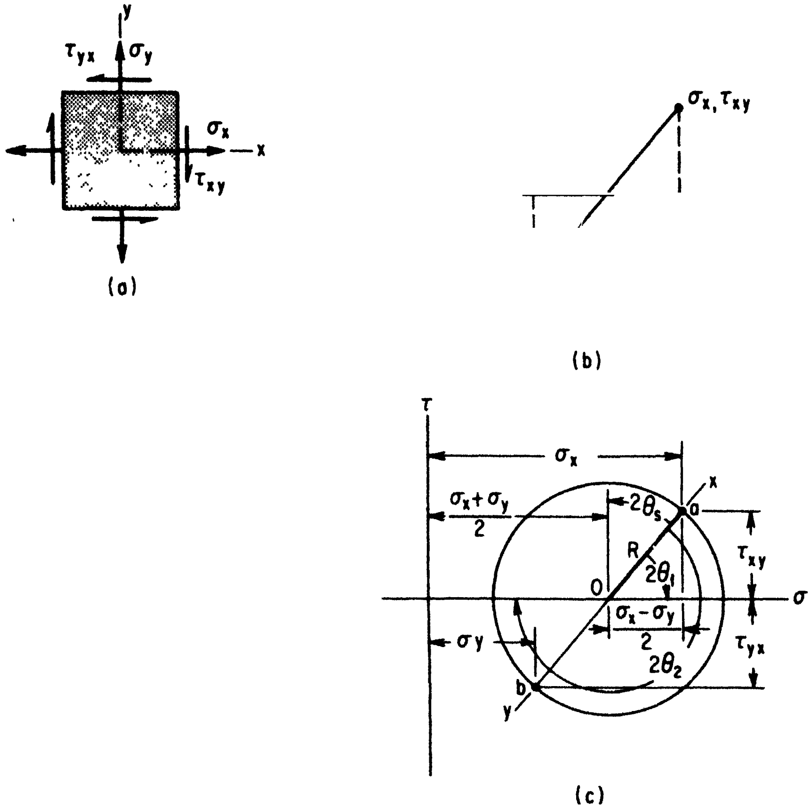


Fig. 7-22

the circle. The radius line Oa, Fig. 7-22(c), represents the x-axis of the element, and line Ob the y-axis. As before, all angles are doubled in the circle, and the x- and the y-axis appear to be 180 deg apart. The radius R has the value

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

and the principal stresses, therefore, are

$$\sigma = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \tag{7-7}$$

The principal planes on which these stresses act are defined in terms of the double angles $2\theta_1$ and $2\theta_2$, where

$$\tan 2\theta_1 = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\theta_1 = \frac{1}{2} \arctan \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

and

$$\theta_2 = \theta_1 + 90°$$

As before, maximum and minimum shearing stresses are equal to the radius of Mohr's circle:

$$\tau_{\substack{\max \\ \min}} = \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

and the planes on which these act are defined by the angles θ_s and $(\theta_s + 90°)$.

Example 12. A plane element in a body is subjected to the following stresses: $\sigma_x = 9$ ksi, $\sigma_y = -5$ ksi, $\tau_{xy} = 4$ ksi. Use Mohr's circle to determine: (a) the principal stresses and the principal planes, (b) the principal shearing stresses and the principal shearing planes.

Solution: The element and its accompanying state of stress is shown in Fig. 7-23(a). Mohr's circle is drawn as shown in Fig. 7-23(b). The radius of the circle is computed as follows:

$$R = \sqrt{7^2 + 4^2} = 8.06 \text{ ksi}$$

To determine the principal stresses, the radius line R must be rotated clockwise through an angle $2\theta_{p1}$ and then through an angle $(2\theta_{p1} + 180 \text{ deg})$. Stresses evaluated at these two positions are

$$\sigma_{max} = 2 + 8.06 = 10.06 \text{ ksi}$$

$$\sigma_{min} = 2 - 8.06 = -6.06 \text{ ksi}$$

The direction of the principal planes are next computed:

$$\tan 2\theta_{p1} = 4/7 = 0.5714$$

$$2\theta_{p1} = 29.7°$$

$$\theta_{p1} = 14.85° \text{ (clockwise from the positive } x\text{-axis)}$$

and

$$\theta_{p2} = \theta_{p1} + 90° = 104.85° \text{ (clockwise from the positive } x\text{-axis)}$$

Fig. 7-23(c) shows the principal stresses and the planes on which they act.

Points c and d on Mohr's circle represent the coordinates of the principal shearing stresses; thus

Point c: $\sigma = 2$ ksi, $\tau = 8.06$ ksi

Point d: $\sigma = 2$ ksi, $\tau = -8.06$ ksi

The planes of principal shear are located at

$$2\theta_{s1} = 90 - 2\theta_{p1} \quad 90 - 29.7 = 60.3^\circ$$

$$\theta_{s1} = 30.15^\circ \text{ (counterclockwise from the positive } x\text{-axis)}$$

and

$$\theta_{s2} = \theta_{s1} + 90^\circ = 30.15 + 90$$

$$= 120.15^\circ \text{ (counterclockwise from the positive } x\text{-axis)}$$

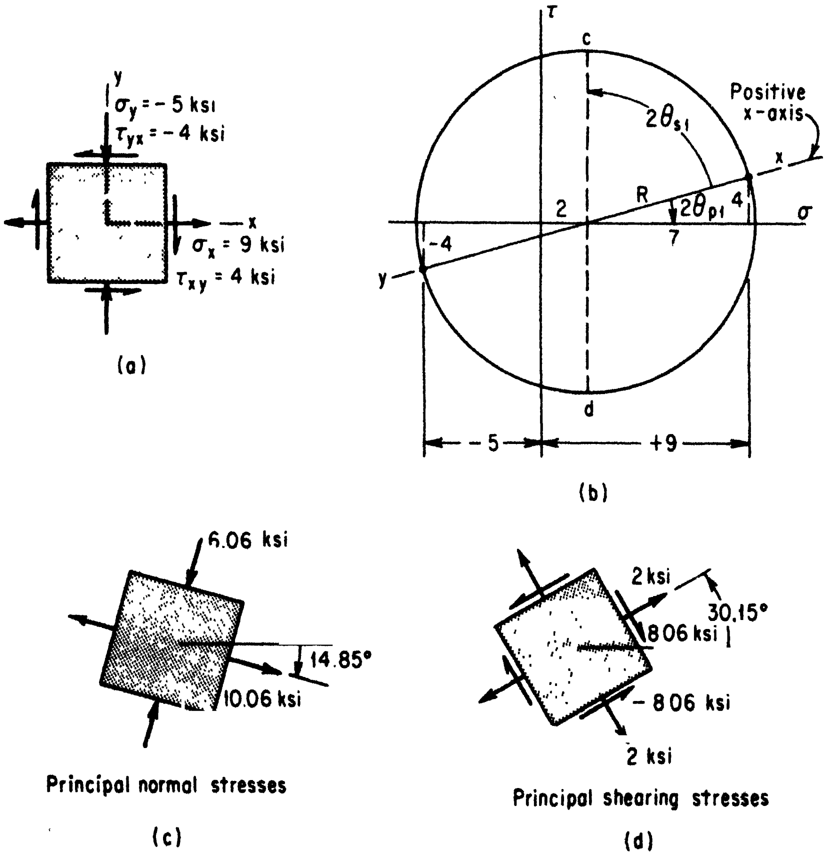


Fig. 7-23

It is important to note that the angle to any plane is always measured relative to the positive x-axis as defined by the original state of stress. Thus, if the planes of principal shear are directed counterclockwise in the circle, they must be directed counterclockwise on the element. This is illustrated in Fig. 7-23(d).

Example 13. A propeller mounted on a 4-in.-diameter solid steel shaft produces a torque of 40,000 lb in. and a thrust of 20,000 lb on the shaft. Determine the maximum normal stress and the maximum shearing stress developed in the member.

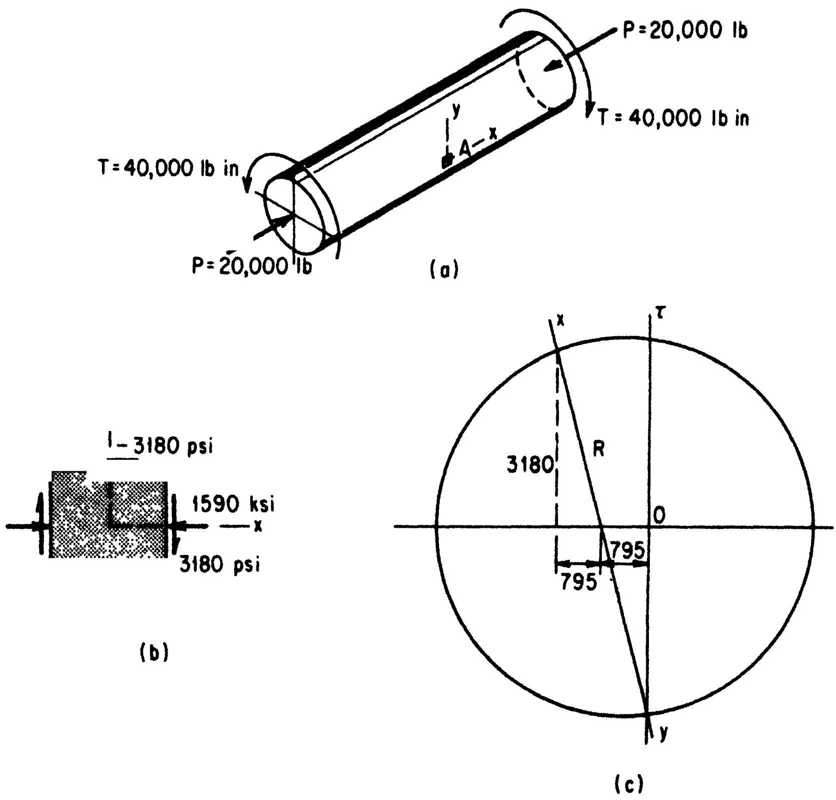


Fig. 7-24

Solution: The combined loading produces shearing stresses and compressive stress within the member.

$$\tau = \frac{Tc}{J} = \frac{40{,}000(2)}{\pi(4)^4/32} = 3180 \text{ psi}$$

$$\sigma = \frac{P}{A} = \frac{20{,}000}{\pi(2)^2} = 1590 \text{ psi}$$

The state of stress at A in Fig. 7-24(a) can be represented as shown in

Fig. 7-24(b). Mohr's circle for the loading, Fig. 7-24(c), is constructed; note that since σ_x is negative and τ_{xy} positive, the x-axis is up and to the left.

The maximum normal stress is compressive and equal to

$$\sigma_{\max} = -795 - R = -795 - \sqrt{(3180)^2 + (795)^2}$$

$$= -795 - 3278 = -4073 \text{ psi}$$

The maximum shearing stress in the shaft is equal to the radius R.

$$\tau_{\max} = 3278 \text{ psi}$$

PROBLEMS

7-1. A timber beam supports a load $P = 3600$ lb at its free end, as shown in Fig. P7-1. The beam has a cross section 3 in. wide by 12 in. deep. Determine the maximum tensile and compressive stresses acting in the beam.

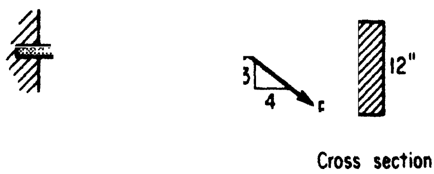


Fig. P7-1

7-2. Find the safe value of the force P in Prob. 7-1 if the following limitations are placed on the design:

Maximum tensile stress	1400 psi
Maximum compressive stress	1200 psi

7-3. Determine the moment M, Fig. P7-3, necessary to cause the stress at A, the top fibers, to be zero.

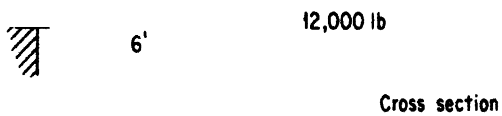


Fig. P7-3

7-4. An 8 WF 17 beam, Fig. P7-4, is supported by cables as shown. Determine the maximum tensile and compressive stresses in the beam at a section just to the right of midspan.

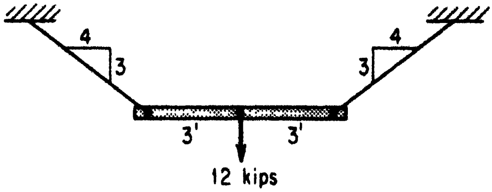


Fig. P7–4

7-5. Compare the load-carrying capacity of the two 1 in. square bars shown in Fig. P7-5. The upper bar is slightly curved, and the lower bar is perfectly straight. *Hint:* Assume the maximum stress in each member to be the same.

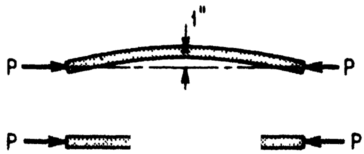


Fig. P7–5

7-6. Two 500 lb sign panels are supported by two 2 in. by 4 in. short columns, as shown in Fig. P7-6. Determine the maximum tensile and compressive stresses in the vertical supports if a wind load of 20 lb per sq ft acts on one face of the sign as indicated.

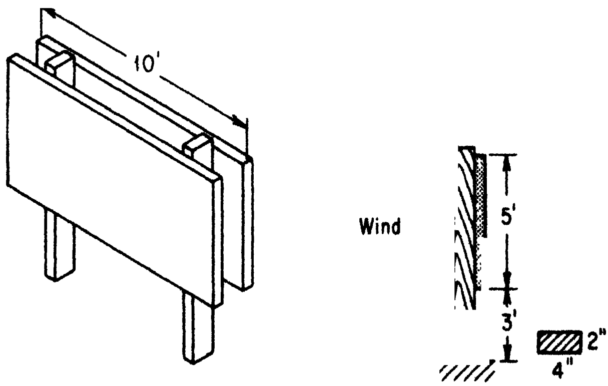


Fig. P7–6

7-7. A 24-in.-diameter steel smokestack, weighing 100 lb per lin ft, is securely supported at its base, as shown in Fig. P7-7. Determine the maximum compressive stress in the stack if a wind load of 50 lb per lineal foot acts as shown.

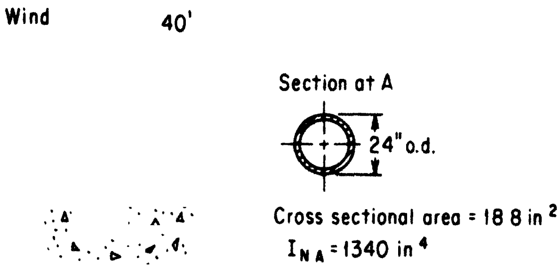


Fig. P7-7

7-8. Determine the maximum fiber stress on section A–A of the machine frame shown in Fig. P7-8 if $P = 100{,}000$ lb.

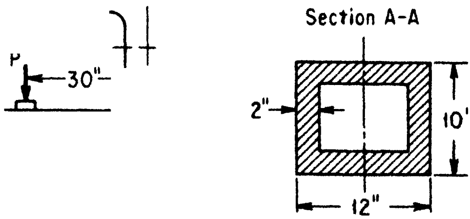


Fig. P7-8

7-9. Determine the clamping force P that can be exerted by the C-clamp, Fig. P7-9, if the allowable axial stress is not to exceed 15,000 psi.

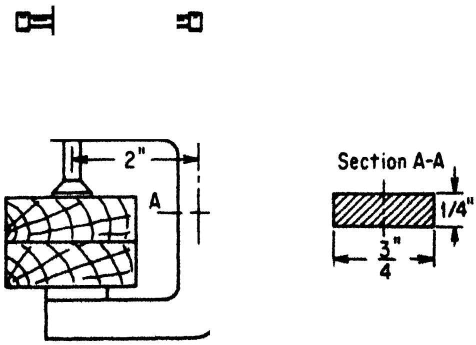


Fig. P7-9

7-10. A concrete retaining wall is to sustain a force of 1000 lb, as shown in Fig. P7-10. Determine the required height h if the stress along a–a is to be zero. Concrete weighs 150 lb per cu ft.

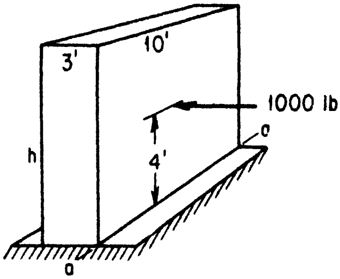


Fig. P7-10

7-11. If the retaining wall of Prob. 7-10 is 15 ft high, find the greatest force F that can be applied at a distance of 5 ft up from the base. The stress along a–a is to be zero.

7-12. Ten cubical blocks measuring 6 in. on edge are held together by a 100 lb force to form a beam, as shown in Fig. P7-12. What is the greatest force P that could act at midspan? Neglect the weight of the blocks. *Hint:* The bottom edge of the beam cannot be in tension.

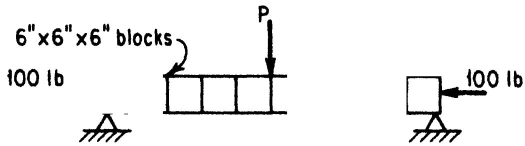


Fig. P7-12

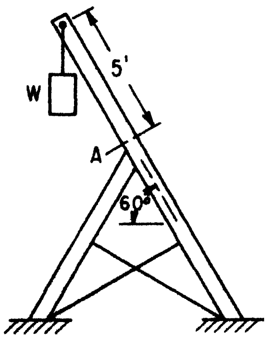


Fig. P7-13

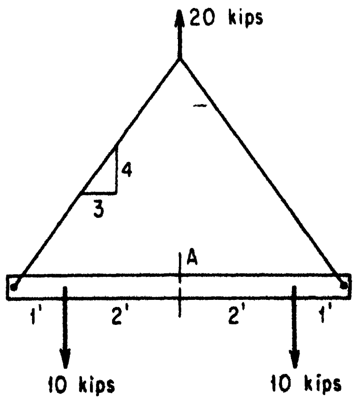


Fig. P7-14

7-13. Find the greatest force W that can be supported by the structure shown in Fig. P7-13, if the stress at A is not to exceed 15,000 psi. The members have cross-sectional areas of 10 in.^{2} and section moduli of 40 in.^{3}

7-14. A beam, suspended by cables as shown in Fig. P7-14, carries two 10 kip loads. Find the maximum normal stress that acts at section A. The beam has a cross-sectional area of 15 in.^{2} and a section modulus of 60 in.^{3}

7-15. A steel bar is supported at A and B, as shown in Fig. P7-15. Find the load F that can act at midspan if the temperature of the bar increases 50°F. The normal stress in the bar is not to exceed 20,000 psi. $E = 30 \times 10^6$ psi; $\alpha = 6.5 \times 10^{-6}$ in./in./°F.

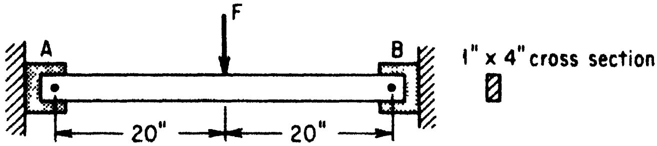


Fig. P7-15

7-16. A simply supported concrete beam 40 ft long is to support a uniformly distributed load of 1000 lb per ft, which includes its own weight. The beam has a depth of 30 in. and a width of 12 in., and is prestressed by high-strength steel bars located at the neutral axis. Find the necessary precompressive force.

7-17. Use the results of Prob. 7-16 to determine the required number of high-strength steel bars required to produce the precompression load. The bars have a cross-sectional area of 0.06 in.^{2} and a working stress of 140,000 psi.

7-18. A slab bridge of 50 ft span is to be made of prestressed concrete. The bridge is 10 ft wide and has a depth of 2 ft and is to carry a load of 800 lb per sq ft, which includes its own weight. Determine the required number of high-strength (140,000 psi) steel bars necessary in the structure. The bars have a cross-sectional area of 0.20 in.^{2} and are located so that the neutral axis remains at the centroid of the beam.

7-19. A concrete beam having the cross section shown in Fig. P7-19 is placed in a state of precompression by 20 high-strength steel bars, each having a diameter of $\frac{1}{4}$ in. and each initially stressed to 160,000 psi. On curing, the concrete shrinks, and the stress in the steel is reduced 10 per cent. Find the permissible bending moment that may be applied to the beam.

7-20. Two equal loads P act at right angles to one another at the free end of a cantilever beam, as shown in Fig. P7-20. The beam has a square cross section. Show that the maximum bending stress in the beam is equal to that obtained by applying the resultant R of these two forces along a diagonal. *Hint:* The moment of inertia of a square about the diagonal is $I = d^4/12$.

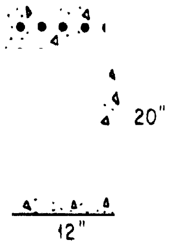


Fig. P7-19

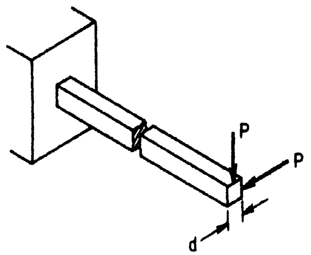


Fig. P7-20

7-21. A force of 100 lb acts as shown on the cantilever beam of Fig. P7-21. Find the maximum normal stress in the beam if θ is the angle whose tangent is $\frac{3}{4}$.

7-22. Find the required inclination θ in Prob. 7-21 if the neutral axis is to lie on the diagonal of the section.

7-23. Compute the magnitudes of the maximum and minimum normal stresses that act in the beam of Fig. P7-23. The beam is a standard 10 WF 33 girder.

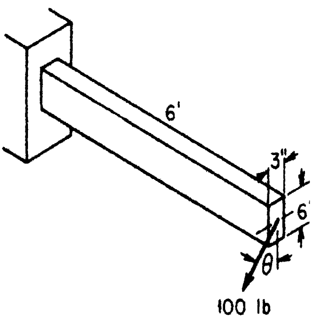


Fig. P7-21

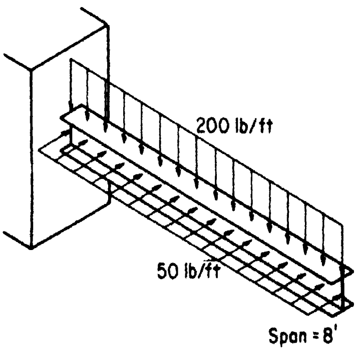


Fig. P7-23

7-24. Find the width b of the beam shown in Fig. P7-24, if the maximum normal stress is to equal 1200 psi. Assume the force to pass through the geometric center of the cross section.

7-25. For the beam of Fig. P7-25, find the normal stress at each of the four edges, a, b, c, and d at a section 4 ft from the free end.

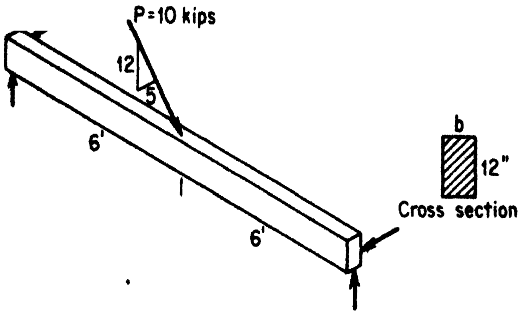


Fig. P7-24

7-26. The column of Fig. P7-26 is subjected to two axial loads and one bending load, as shown. Determine the maximum tensile stress in the member if the column has a section modulus of $Z = 40$ in.^{3} and a cross-sectional area $A = 10$ in.^{2}

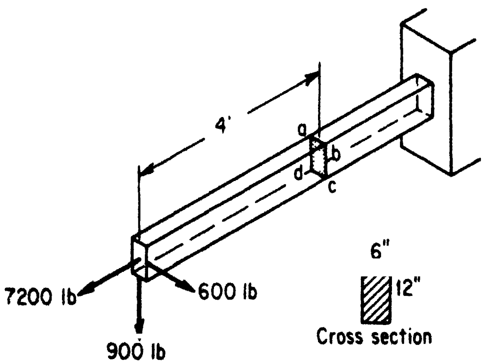


Fig. P7-25

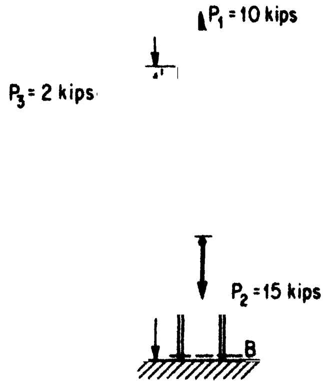


Fig. P7-26

7-27. Find the force P_2 for the column of Prob. 7-26, if the tensile stresses at sections A and B are to be equal. Assume the bending load P_3 and the axial load P_1 to have the values indicated in the figure.

7-28. Find the maximum bending stress in the tubular shaft shown in Fig. P7-28. The section modulus of the shaft is $Z = 15$ in.^{3}, and the bearings are self-aligning and can offer no resistance to bending.

7-29. Determine the maximum bending stress in the circular shaft of Fig. P7-29. Assume the belt "pulls" act at right angles to one another and that the section modulus of the shaft is $Z = 3$ in.^{3}

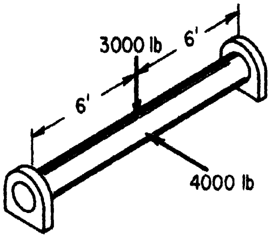


Fig. P7-28

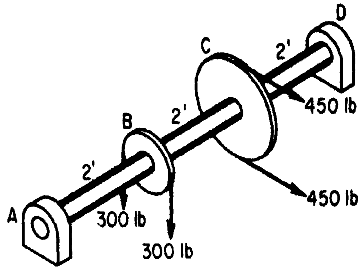


Fig. P7-29

7-30. Three equal loads act on a 2-in.-diameter steel shaft, as shown in Fig. P7-30. Find the maximum value of P if the bending stress is not to exceed 20 ksi.

7-31. An eccentric load of $P = 3600$ lb acts on a short column as shown in Fig. P7-31. Find the stresses that act on the column faces which are parallel to the x-axis.

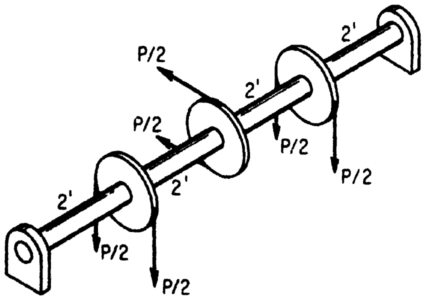


Fig. P7-30

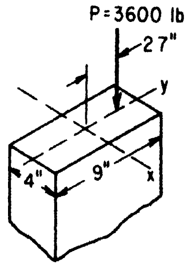


Fig. P7-31

7-32. Find the permissible load P, in Prob. 7-31, if the following limitations are placed on the design:

Maximum compressive stress	600 psi
Maximum tensile stress	400 psi

7-33. Determine the permissible load P that can act on the short column shown in Fig. P7-33. The column is a 14 WF 68 section, and the maximum stress is not to exceed 18,000 psi.

7-34. Find the force P, Fig. P7-34, if the tensile stress in the short column is to be zero.

7-35. The short column of Fig. P7-35 supports an eccentric load, as shown. Determine the values of axial stress within the member at each of the four corners.

Fig. P7–33

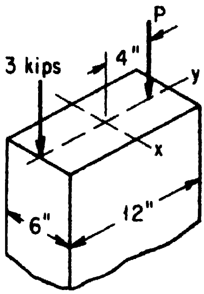


Fig. P7–34

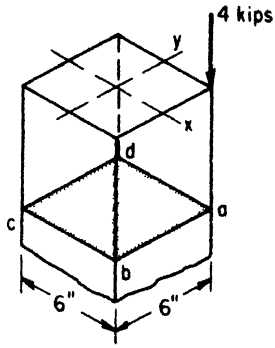


Fig. P7–35

7-36. Prove that the *kern* of a circular section of diameter D is a concentric circular area having a diameter of $D/4$.

7-37. Determine the permissible eccentricity, Fig. P7-37, of a load P that acts on a 24-in.-diameter post if the maximum tensile stress is not to exceed 10 per cent of the permissible compressive stress.

7-38. Two loads act on a short column as shown in Fig. P7-38. Determine the maximum normal stress developed in the section.

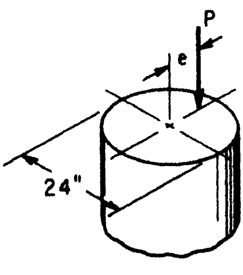


Fig. P7–37

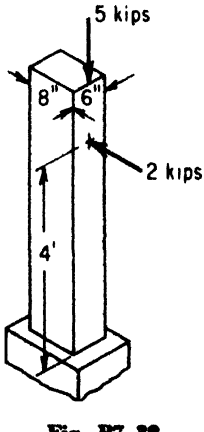


Fig. P7–38

7-39. The cantilever beam of Fig. P7-39 supports an eccentric load P applied at an angle, as shown. The beam is a 14 WF 61 section. Find the safe value of P if the normal stress is not to exceed 15,000 psi.

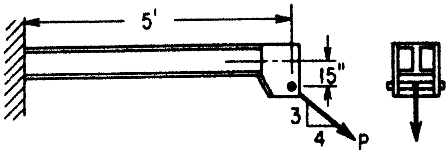


Fig. P7–39

7-40 through **7-45.** A state of biaxial stress is indicated in each of the respective figures. Determine, by means of Mohr's circle, components of stress that act on the faces of an element oriented as indicated.

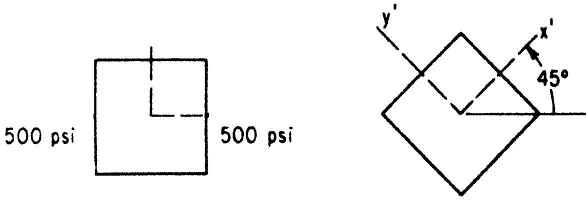


Fig. P7–40

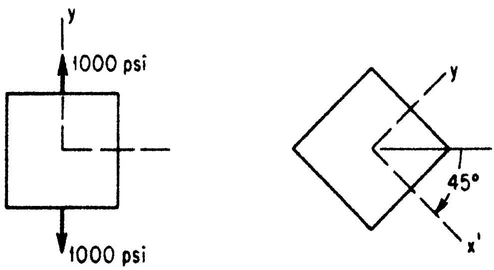


Fig. P7–41

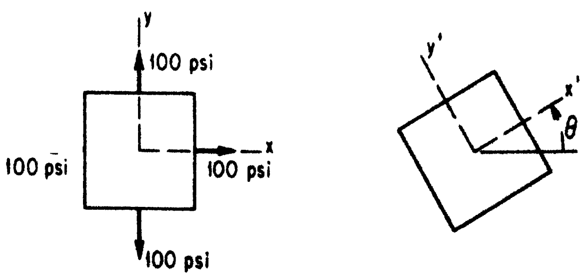


Fig. P7–42

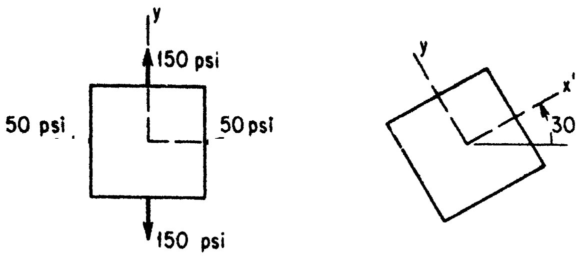


Fig. P7-43

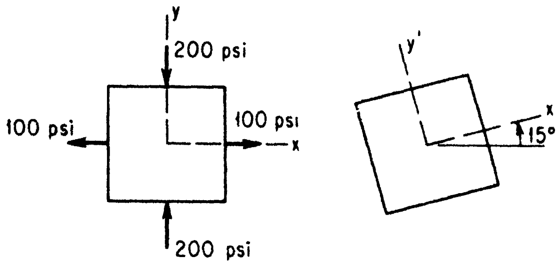


Fig. P7-44

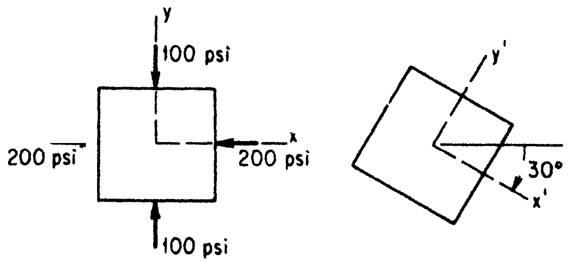


Fig. P7-45

7-46. Two pieces of 1 in. by 6 in. steel are welded along a diagonal, as shown in Fig. P7-46. Determine the safe value of P, using Mohr's circle, if the permissible stresses in the weld are $\sigma = 5000$ psi and $\tau = 1000$ psi.

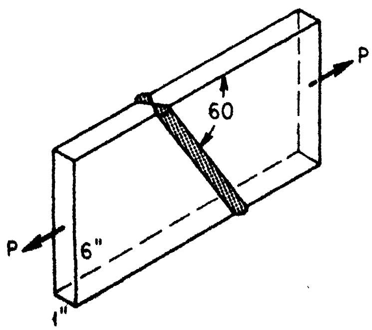


Fig. P7-46

7-47. Two pieces of 2 in. by 6 in. Douglas fir are scarfed and glued as shown in Fig. P7-47. Determine the safe value of P if the compressive stress in the wood is not to exceed 1000 psi, nor the shear stress in the joint, 200 psi.

Fig. P7-47

7-48. A closed pressure vessel is spirally welded as shown in Fig. P7-48. Determine the components of stress that act in the weld if the pressure within the vessel is 20 psi. The vessel has a diameter of 5 ft and a wall thickness of $\frac{1}{4}$ in.

7-49. Three pieces of steel are welded to form a tee-beam, as shown in Fig. P7-49. Determine the maximum shear stress developed at A if the beam supports two equal loads, as indicated. Each beam has a moment of inertia of $I_{NA} = 60$ in.^{4} and a depth of 4 in. *Hint:* Assume the stress at A to be *biaxial* and neglect the effect of stress concentration at the joints.

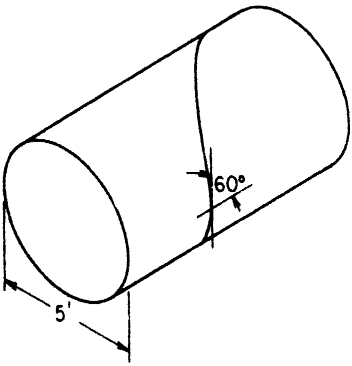


Fig. P7-48

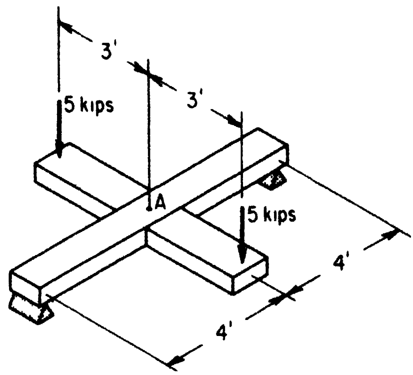


Fig. P7-49

7-50. The tee-beam of Fig. P7-50 is used as a shock absorber in a towing rig. This device protects the cables from the shock loading. The steel beam is fabricated by welding three 1 in. by 6 in. steel plates together, as shown. Determine the state of the stress at A in terms of P and the permissible value of P if the shearing stress at A is not to exceed 12,000 psi. Neglect the effects of stress concentration and assume the weld to be as strong as the parent metal.

7-51. A plane element in a body is subjected to pure shearing stresses $\tau_{xy} = 5000$ psi and $\tau_{yx} = -5000$ psi. Determine the components of stress that act on a plane inclined at (a) 30 deg counterclockwise from the positive x-axis; (b) 30 deg clockwise from the positive x-axis.

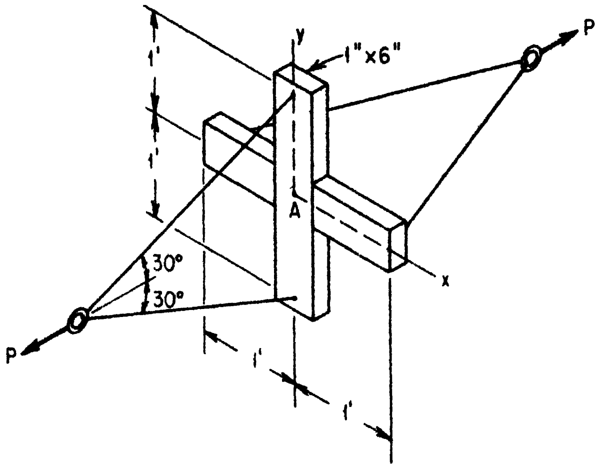


Fig. P7-50

7-52. Determine the principal stresses associated with a state of pure shear $\tau_{xy} =$ 10 ksi, $\tau_{yz} = -10$ ksi.

7-53. A plane element on a body is subjected to the two states of stress superimposed upon one another, as shown in Fig. P7-53. Find the components of stress that act on a face of the element inclined at 15 deg with the x-axis, as indicated. *Hint:* Draw Mohr's circle for each case and add the results.

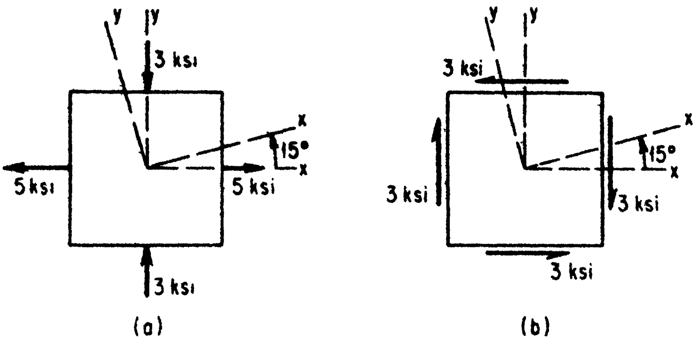


Fig. P7-53

7-54 through **7-57.** A plane element on a body is subjected to the state of stress shown in each of the respective figures. Determine: (a) the principal stresses and their directions; (b) the maximum shearing stress and the direction of the planes on which they occur.

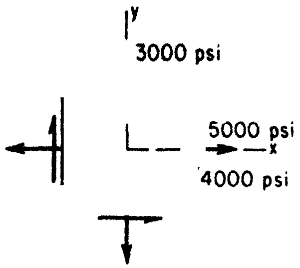


Fig. P7–54

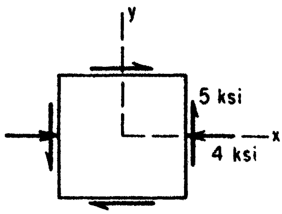


Fig. P7–55

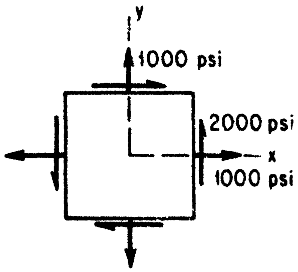


Fig. P7–56

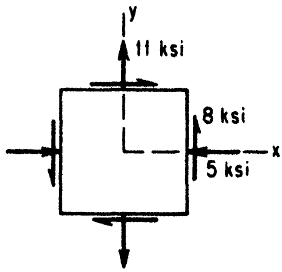


Fig. P7–57

7-58. A solid circular shaft 4 in. in diameter is subjected to an axial compressive force of 100 kips and a twisting moment of 25 kip in. Find the maximum normal stress in the shaft.

7-59. A tensile force P lb and twisting moment of $P/8$ lb in. act together on a 2-in.-diameter circular shaft. Find P if the maximum tensile stress is not to exceed 20,000 psi.

7-60. A thin-walled closed tube having a diameter of 2 in. and a wall thickness of 0.10 in. is subjected to an internal pressure of 200 psi and a twisting moment of 314 lb in. Determine the maximum normal and shearing stresses that act in the tube.

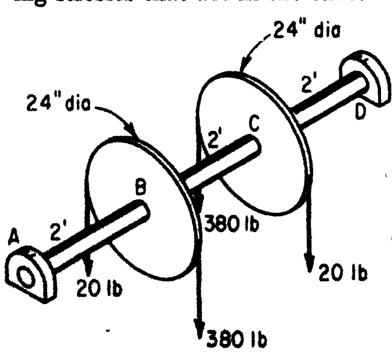


Fig. P7–61

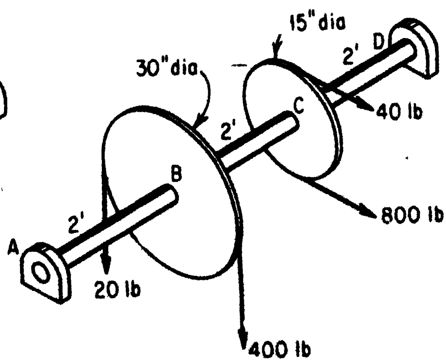


Fig. P7–62

7-61. Determine the principal stresses and the maximum shearing stress in the 2-in.-diameter line-shaft of Fig. P7-61. *Hint:* Bending and torsion act together on the shaft.

7-62. The 2-in.-diameter line-shaft of Fig. P7-62 is subjected to two-dimensional bending, as well as torsion. Find the maximum shearing stress in the shaft.

7-63. The bracket shown in Fig. P7-63 is subjected to two forces, as indicated. Determine the maximum normal and maximum shearing stresses that act in the circular section *a-a*.

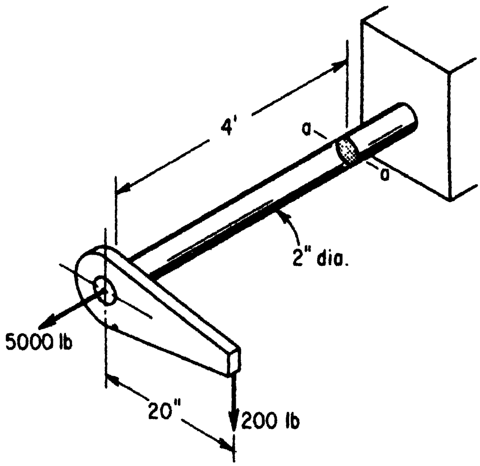


Fig. P7-63

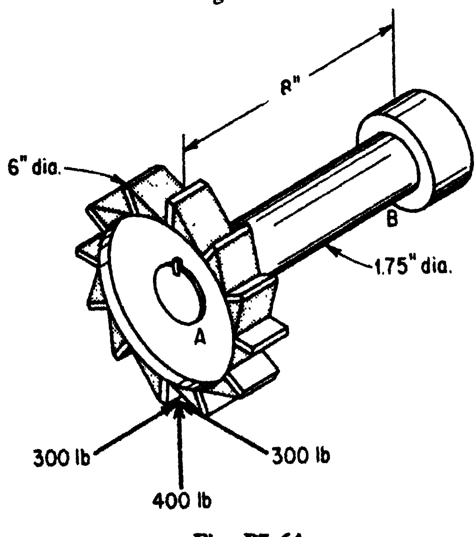


Fig. P7-64

7-64. A 6-in.-diameter milling cutter is acted upon by three components of force, as shown in Fig. P7-64. Determine the maximum normal stress and the maximum shearing stress in the cutter shaft.

CHAPTER 8

Welded, Bolted, and Riveted Connections

There are many well-established methods of connecting components together to form structures and machines; these include fusion welding, bolting, riveting, brazing, soldering, and gluing. Size, shape, and service requirements of the structure or machine are among the many factors which influence the selection of connectors. By far, however, the most important methods are fusion welding, bolting, and riveting.

8–1 Welded Connections

Gas and arc welding techniques have improved to the point that welding is now the most important single method of joining metallic components. There is virtually no size limit to welding, and the process can be used with equal dependability in both shop and field.

Some of the more common types of welded joints are shown in the composite drawing of Fig. 8-1. With the exception of the seam weld, all require an addition of *weld metal* to the *parent metal*. The strength of a welded connection depends, therefore, on the properties of both the parent metal and the weld metal, and the geometry of the weld. Involved in the latter is the minimum cross-sectional area through which failure can occur, and specified values for the allowable stress. In a butt weld, Fig. 8-2(a), the minimum cross-sectional area is assumed to be the length of the joint times the thickness of the thinner plate; thus

$$P = \sigma_t bt \tag{8-1}$$

where σ_t is the permissible tensile stress in the weld.

In a fillet weld, Fig. 8-2(b), the minimum cross-sectional area depends upon the width of the *throat* (roughly the length of the *fillet leg* times the sine

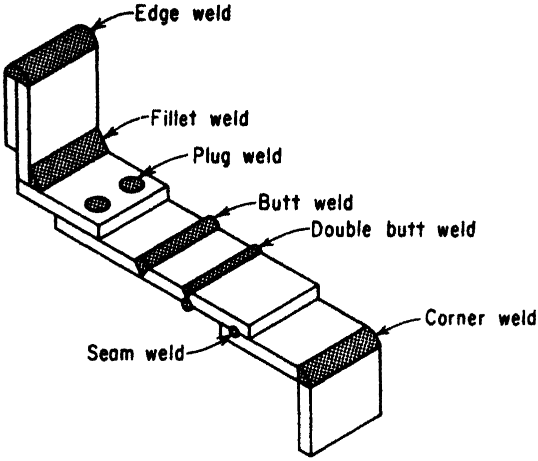


Fig. 8-1

of 45 deg), and the length of the weld.

$$\text{minimum area} = 0.707(\text{fillet size})L$$

Hence,

$$P = \tau(0.707)\,(\text{fillet size})L \tag{8-2}$$

where τ is the permissible shearing stress.

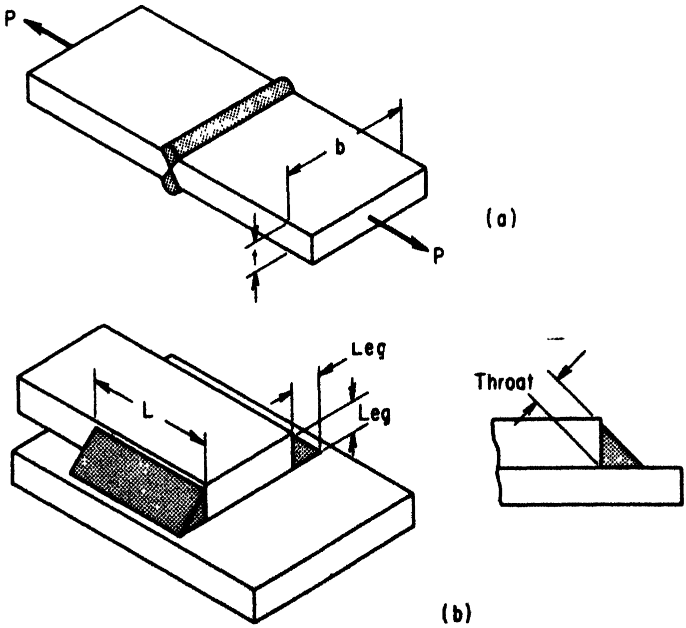


Fig. 8-2

Working stresses for welded joints are governed largely by codes, established either by industries involved or by local building regulations. Typical values of working stress for steel weldments are

Shear stress	14,000 psi
Tensile stress	16,000 psi
Compressive stress	18,500 psi

Since practically all fillet welds are subject to shear, the strength can be given in terms of fillet size. Thus, for an $\frac{1}{8}$ in. fillet weld.

$$P = (14{,}000)(0.707)(0.125) = 1250 \text{ lb/in.}$$

Based on this computation, the allowable shearing loads per inch of weld for various fillet sizes are as follows:

Size of fillet (in.)	*Safe load* (lb per in.)
$\frac{1}{8}$	1250
$\frac{3}{16}$	1875
$\frac{1}{4}$	2500
$\frac{5}{16}$	3125
	3750
	5000
	6250
$\frac{3}{4}$	7500

These are easy numbers to remember, since each contains the decimal equivalent of the fillet size.

Example 1. Two plates are joined by three sections of $\frac{3}{8}$ in. fillet weld, as shown in Fig. 8-3(a). What length L is required if the joint is to be 100 per cent efficient, i.e., if the weld is to be as strong as the parent metal? Assume an allowable tensile strength of the plate to be 20,000 psi.

Solution: The load P is determined as follows

$$P = \sigma A = 20{,}000(6 \times \tfrac{1}{2}) = 60{,}000 \text{ lb}$$

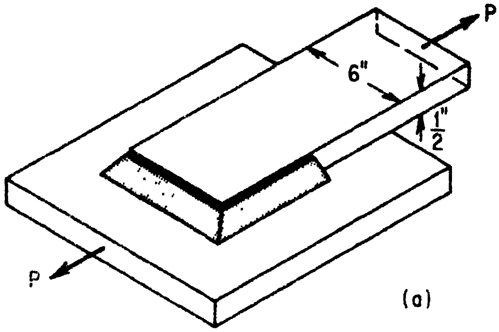
In terms of the free-body diagram, Fig. 8-3(b), the load P must be supported by two longitudinal welds and one transverse weld.

$$2P_1 + P_2 = 60{,}000 \text{ lb}$$

Substitution of numerical values from the table of permissible loads gives

$$2(3750)L + 3750(6) = 60{,}000$$

$$L = 5.0 \text{ in.}$$



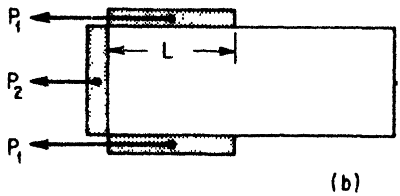


Fig. 8–3

Example 2. A 4 in. by 3 in. by $\frac{3}{8}$ in. standard steel angle is attached to the face of a gusset plate by $\frac{1}{2}$ in. fillet welds, as shown in Fig. 8-4(a). Determine the lengths L_1 and L_2 if the angle is to support a tensile load of 75 kips applied at its centroid.

Solution: Forces and moments must be balanced.

$$\sum F = 0$$

$$P_1 + P_2 = 75{,}000$$

and

$$\sum M = 0$$

$$2.63P_1 = 1.37P_2$$

The permissible load on a $\frac{1}{2}$ in. fillet is 5000 lb per in.; therefore,

$$5000L_1 + 5000L_2 = 75{,}000$$

$$L_1 + L_2 = 15 \qquad \text{(a)}$$

and

$$2.63(5000)L_1 = 1.37(5000)L_2$$

$$L_2 = 1.92L_1 \qquad \text{(b)}$$

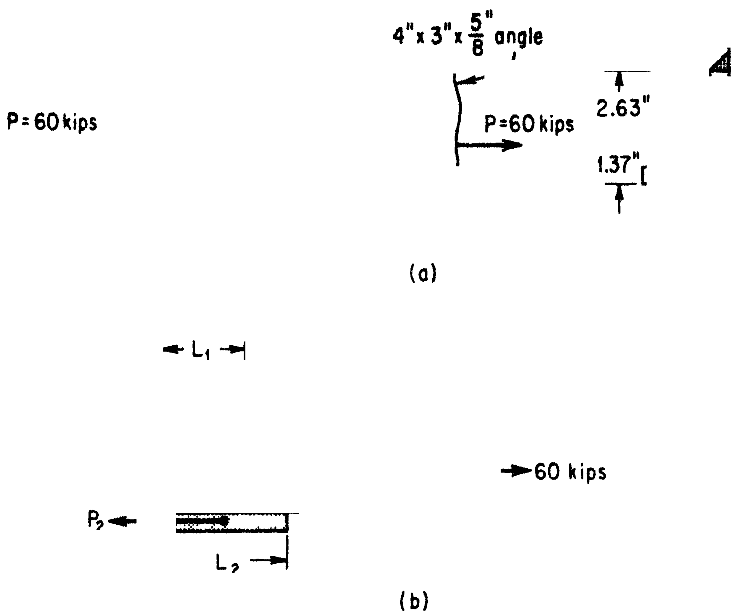


Fig. 8-4

Equations (a) and (b) are solved simultaneously for L_1 and L_2.

$$L_1 + 1.92L_1 = 15$$

$$L_1 = \frac{15}{2.92} = 5.14 \text{ in.}$$

$$L_2 = 15 - 5.14 = 9.86 \text{ in.}$$

8-2 Riveted and Bolted Joints: Single Connector

It is customary to consider the strength of a riveted or bolted connection to depend on three quantities: the shearing strength of the rivets or bolts, the crushing or bearing strength of the connected plates, and the tensile strength of the connected plates. It is presumed that the joint will ultimately fail in one of the three ways shown in Fig. 8-5.

The allowable load P_s based on permissible shearing stress is

$$P_s = \tau A_s \tag{8-3}$$

where τ and A_s are the working stress in shear and the area being sheared, respectively.

In bearing, failure is caused by the pressure force between the cylindrical surfaces; the bearing strength P_b, therefore, depends upon the ultimate compressive stress of the connector or the plate and the projected area *td* of the rivet or bolt hole:

$$P_b = \sigma_b A_b = \sigma_b td \tag{8-4}$$

where σ_b is the permissible bearing stress.

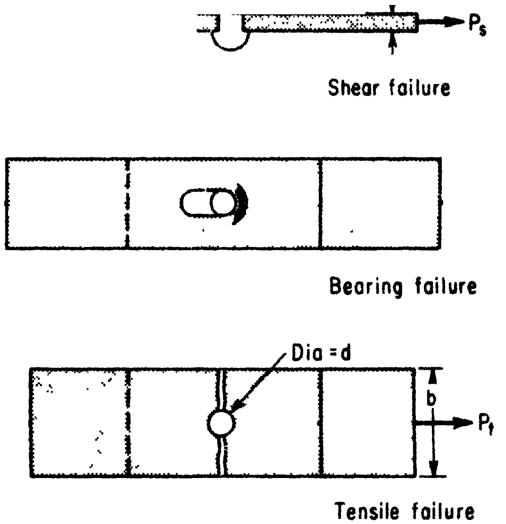


Fig. 8-5

In tension, the strength of the joint depends upon the tensile strength of the plates and the minimum area that must sustain the force. If a joint has a single connector, the permissible force P_t is

$$P_t = \sigma_t A_t = \sigma_t (b - d)t \tag{8-5}$$

where

σ_t = the working stress in tension.
b = the width of the plate.
d = the diameter of the rivet or bolt hole.
t = the thickness of the plate.

Efficiency of a riveted or bolted joint is defined as the *ratio of the permissible load to the strength of the plate itself.*

As with welded joints, suitable working stresses for riveted and bolted connections are governed by either industrial practice or local building codes.

Example 3. A single $\frac{7}{8}$-in.-diameter bolt is used to connect two plates, as shown in Fig. 8-6. Determine the strength and the efficiency of the joint if the following working stresses apply: $\tau = 15{,}000$ psi, $\sigma_b = 32{,}000$ psi, $\sigma_t = 18{,}000$ psi.

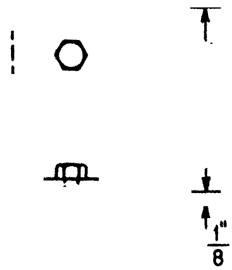


Fig. 8-6

Solution: The safe loads, based on the shearing strength of the bolt, the strength of the plate in the vicinity of the hole, and the crushing strength of the plate, are each investigated.

Shear: $P_s = \tau A_s = 15{,}000\pi(\tfrac{3}{16})^2 = 1660$ lb

Tension: $P_t = \sigma_t A_t = 18{,}000(\tfrac{1}{8})(1 - \tfrac{3}{8}) = 1410$ lb

Bearing: $P_b = \sigma_b A_b = 32{,}000(\tfrac{1}{8})\tfrac{3}{8} = 1500$ lb

The smallest value, 1410 lb, is the design load. Efficiency is the ratio of the design load to the strength of the plate expressed in per cent; thus

$$e = \frac{1410}{18{,}000(\frac{1}{8})(1)} \times 100 = 62.7 \text{ per cent}$$

8-3 Riveted and Bolted Joints: Multiple Connectors

When multiple connectors are used in a fitting or joint, and the line of action of the load passes through the centroid of these connectors, as in Fig. 8-7(a), the rivets (or bolts) are assumed to deform equally. Each connector, therefore, carries an equal share of the load. The shear and bearing forces are distributed so that

$$\frac{\quad}{nA_s} \tag{8-6}$$

and

$$\sigma_b \quad \frac{P}{nA_b} \tag{8-7}$$

where τ and σ_b are the shear and bearing stresses, respectively, and n is the number of connectors. As before, A_s and A_b are the areas in shear and in bearing, respectively.

The tensile stress in the plate varies with the rivet or bolt geometry. The full load is carried by the plate across section 1–1, in Fig. 8-7, while two-thirds of the load is carried by the plate at section 2–2. This is the case since one-third of the load is partially supported by the rivet in row 1–1, as illustrated in the free-body diagram, Fig. 8-7(b).

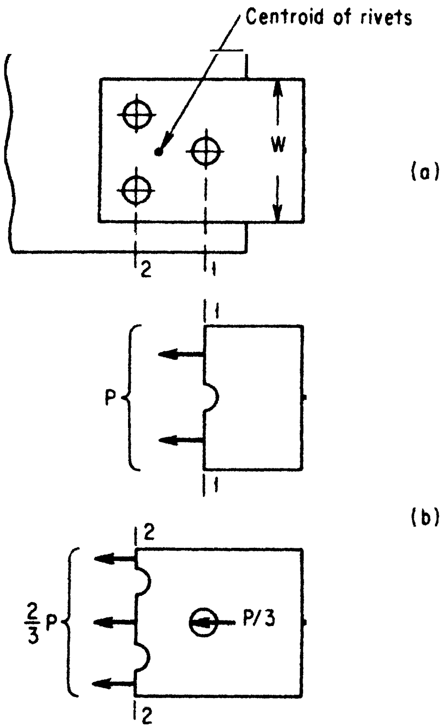


Fig. 8-7

In the analysis of concentrically loaded joints fastened by more than one connector, shear and bearing are generally considered first. Free-body diagrams are then drawn and tension is investigated along each transverse row of rivets or bolts. The least value of the load P is the design load.

Example 4. A *triple-riveted lap joint* is shown in Fig. 8-8. Determine the safe load P and the efficiency of the joint. The rivets are $\frac{3}{4}$ in. in diameter, and the following permissible stresses apply: $\tau = 15{,}000$ psi, $\sigma_b = 32{,}000$ psi, $\sigma_t = 20{,}000$ psi.

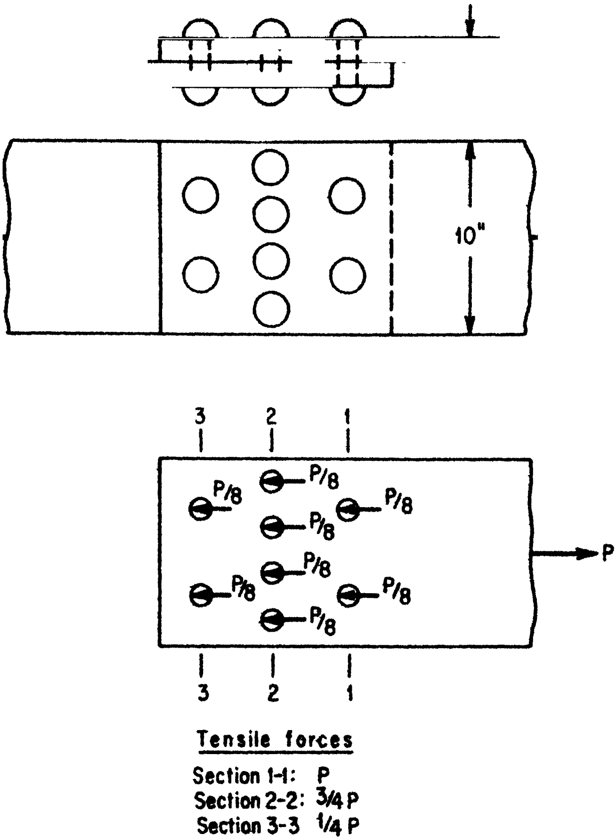


Fig. 8-8

Solution: **Since eight rivets are acting to support the load, each rivet carries a force of** $P/8$.

Permissible load in shear:

$$P_s = \tau A_s = 15{,}000(8)\pi(\tfrac{7}{16})^2 = 72{,}100 \text{ lb}$$

Permissible load in bearing:

$$P_b = \sigma_b A_b = 32{,}000(8)\,(\tfrac{7}{8})\,(\tfrac{3}{8}) = 84{,}000 \text{ lb}$$

Permissible load in tension:

At section 1–1

$$P_t = \sigma_t A_t = 20{,}000(10 - \tfrac{7}{8} \times 2)\,(\tfrac{3}{8}) = 61{,}900 \text{ lb}$$

At section 2–2

$$\tfrac{3}{4}P_t = \sigma_t A_t = 20{,}000(10 - \tfrac{7}{8} \times 4)\,(\tfrac{3}{8})$$

$$P_t = 65{,}000 \text{ lb}$$

At section 3–3

$$\frac{P_t}{4} = \sigma_t A_t = 20{,}000(10 - \tfrac{7}{8} \times 2)(\tfrac{3}{8})$$

$$P_t = 247{,}500 \text{ lb}$$

Thus, tension at section 1–1 governs the design, and the allowable load is 61,900 lb. The efficiency of this connection is the ratio of the safe load to the strength of the plate.

$$e = \frac{61{,}900}{20{,}000(10)(\tfrac{3}{8})} \times 100 = 82.5 \text{ per cent}$$

Example 5. Figure 8-9 shows a standard AISC (American Institute of Steel Construction) connection for joining a 16 WF 88 beam to a column. The fastening consists of two 4 in. by $3\frac{1}{2}$ in. by $\frac{3}{8}$ in. angles each $11\frac{1}{2}$ in. long joined to the web of the beam by four $\frac{7}{8}$-in.-diameter rivets. The AISC code provides the following working stresses: 15 ksi in shear, 20 ksi in tension, 32 ksi in bearing for single shear, and 40 ksi in bearing for double shear. Find the load capacity of the connection.

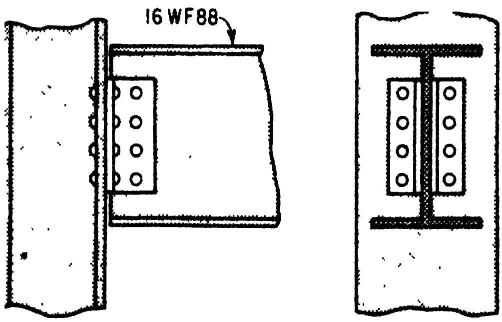


Fig. 8–9

Solution: Since a tensile failure cannot occur, only shear and bearing will be investigated.

Shear. Four rivets are in double shear; hence,

$$P_s = \tau A_s = 15(8)\pi(\tfrac{7}{16})^2 = 72.2 \text{ kips}$$

Bearing on web. The four rivets that bear on the web of the beam are in double shear; the allowable bearing stress, therefore, is 40 ksi. The web thickness, found in Appendix B, is 0.504 in.; hence,

$$P_b = \sigma_b A_b = 40(4)(0.504)\tfrac{7}{8} = 70.6 \text{ kips}$$

Bearing on angles. Eight bearing surfaces are involved, each supporting a rivet that is in single shear; thus

$$P_b = \sigma_b A_b = 32(8)\left(\tfrac{3}{8}\right)\left(\tfrac{7}{8}\right) = 84 \text{ kips}$$

The capacity of the connection is, therefore, 70.6 kips.

In an actual design, the thickness of the plate to which the connection is made must also be investigated. If this plate is thin, the permissible load may be reduced.

8-4 Eccentrically Loaded Connections

When the line of action of the applied load does not pass through the centroid of the connectors, *torsion* as well as a *direct shear* acts on the fastening. This can be shown by considering the eccentrically loaded rivet joint of Fig. 8-10(a). The external load P acts at a distance e measured from the

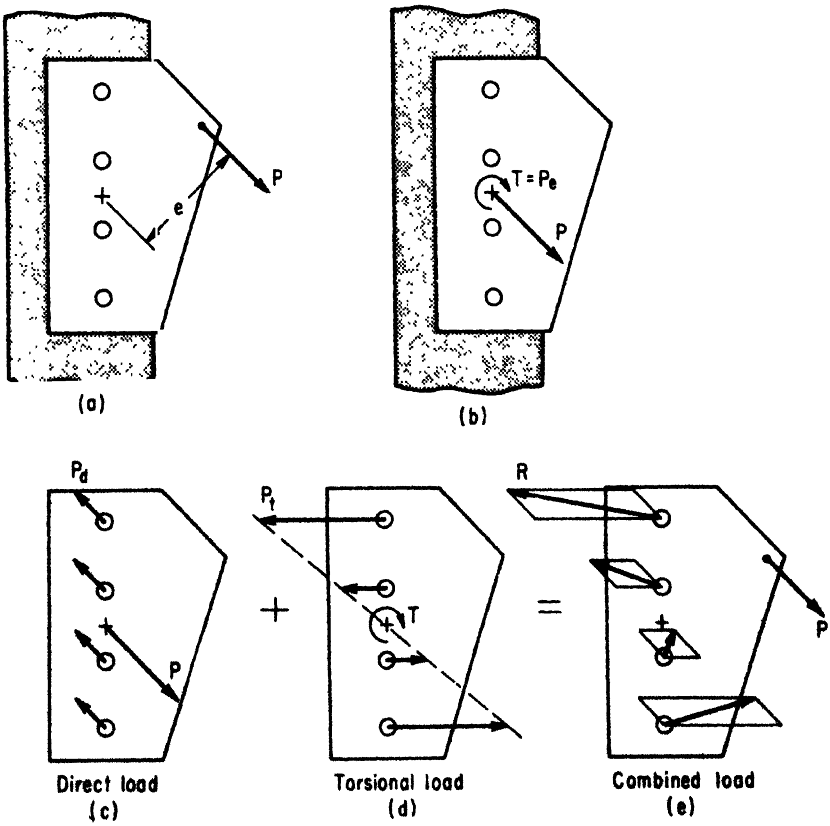


Fig. 8-10

centroid of the rivet array. This is *statically equivalent* to a direct load P and a moment Pe, as shown in Fig. 8-10(b). The resisting force of the rivets acts to oppose the direct force P by sharing the load equally, Fig. 8-10(c). In resisting the torque, however, the rivets act as the pins of a coupling, and the reactive forces on each are proportional to their respective centroidal distances. The sum of the moments of these reactive forces must balance the couple Pe. This is shown in Fig. 8-10(d). The load carried by any one rivet is the *vector sum of the direct force and the coupling force*, as shown in Fig. 8-10(e).

Eccentrically applied loads are not a desirable design feature, since the effective strength of the joint is appreciably reduced. The coupling effect can be minimized, however, by keeping the eccentricity small and by careful consideration of rivet geometry.

Example 6. Compare the load-carrying capacity of the two rivet arrangements shown in Fig. 8-11. Half-inch-diameter rivets are used in each, and the working stress in shear is $\tau = 15{,}000$ psi.

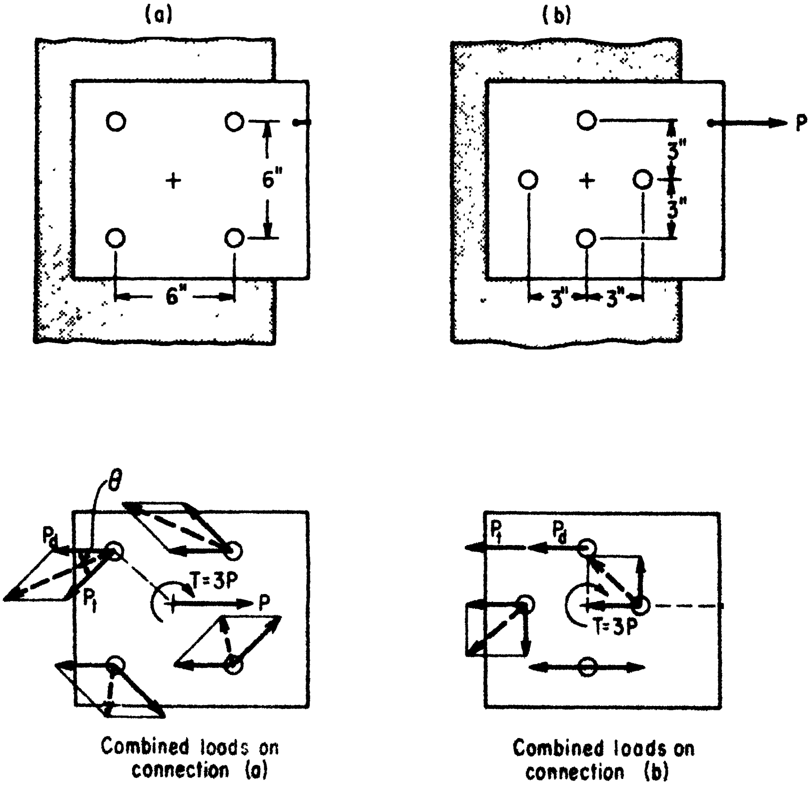


Fig. 8-11

Solution: Direct loads and coupling loads are the same for both connections; the difference in load-carrying capacity will depend, therefore, on geometry alone.

Connection (a). The direct load P_d on each of the rivets acts horizontally to the left and is equal to $P/4$. The torsional load P_t on the rivets is the same for all, since the rivets are equidistant from the centroid. P_t can be found by a moment summation:

$$\sum M = 0$$

$$4P_t\sqrt{3^2 + 3^2} = 3P$$

$$P_t \quad \frac{P}{4\sqrt{2}}$$

By observing the vectors, the two top rivets seem to be the most severely loaded in the group. The resultant force, by the cosine law, for either of these rivets is

$$\begin{aligned} R &= \sqrt{(P_d)^2 + (P_t)^2 + 2P_dP_t \cos\theta} \\ &= \sqrt{\left(\frac{P}{4}\right)^2 + \left(\frac{P}{4\sqrt{2}}\right)^2 + 2\left(\frac{P}{4}\right)\left(\frac{P}{4\sqrt{2}}\right)\cos 45^\circ} \\ &= 0.395P \end{aligned}$$

The permissible load, based on the safe working stress, is

$$\begin{aligned} 0.395P &= 15{,}000\left(\frac{\pi}{4}\right)\left(\frac{1}{2}\right)^2 \\ P &= 7460 \text{ lb} \end{aligned}$$

Joint (b). The direct force on each of the rivets is $P/4$, and the torsional load is

$$4(3)P_t = 3P$$

$$P_t \quad \frac{P}{4}$$

The uppermost rivet is the most severely loaded in this arrangement; the forces, in this case, are colinear and add directly:

$$R = P_d + P_t = \frac{P}{4} + \frac{P}{4} = \frac{P}{2}$$

$$\frac{P}{2} = 15{,}000\left(\frac{\pi}{4}\right)\left(\frac{1}{2}\right)^2$$

$$P = 5890 \text{ lb}$$

Rivet arrangement (a) is the stronger of the two.

Example 7. A steel gusset plate is bolted to a machine frame, as shown in Fig. 8-12. Find the greatest load that acts in any one bolt.

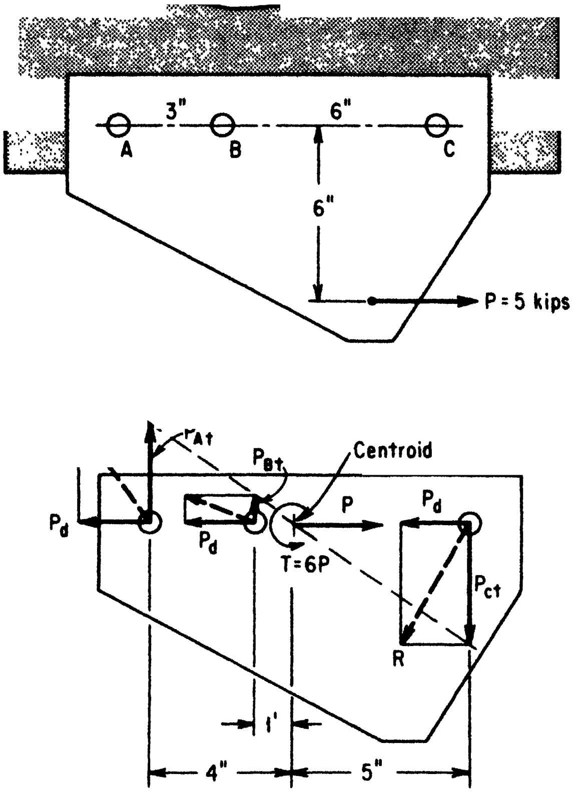


Fig. 8-12

Solution: The centroid of the bolt array is found by taking first moments of the bolt areas about the left connector.

$$3A\bar{x} = A(0) + A(3) + A(9)$$

$$\bar{x} = 4 \text{ in.}$$

By observation, bolt C is seen to be the most severely loaded connector. The direct load on C is $P/3 = 1.67$ kips, and the torsional load can be found by summing moments of the torsional loads about the centroid.

$$4P_{At} + P_{Bt} + 5P_{Ct} = 6P = 30$$

By similar triangles,

$$P_{A_t} = \tfrac{4}{5}P_{C_t}$$

and

$$P_{B_t} = \tfrac{1}{5}P_{C_t}$$

Therefore,

$$\tfrac{16}{5}P_{C_t} + \tfrac{1}{5}P_{C_t} + 5P_{C_t} = 30$$

$$P_{C_t}(3.2 + 0.2 + 5) = 30$$

$$P_{C_t} = \frac{30}{8.4} = 3.57 \text{ kips}$$

The resultant force on bolt C is the vector sum of the direct load and the torsional load.

$$R = \sqrt{(1.67)^2 + (3.57)^2} = 3.94 \text{ kips}$$

In this example the torsion load is far greater than the direct load; this clearly indicates the severe effects of eccentricity.

PROBLEMS

8-1. Two ½ in. by 6 in. steel plates are double-butt welded, as shown in Fig. P8-1. Determine the safe load P and the efficiency of the joint if the permissible stress in the steel plate is 18,000 psi.

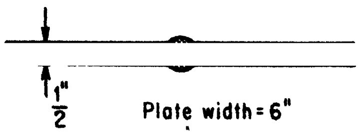


Fig. P8-1

8-2. Determine the strength and efficiency of the welded connection shown in Fig. P8-2. The working stress of the steel plate is 20,000 psi.

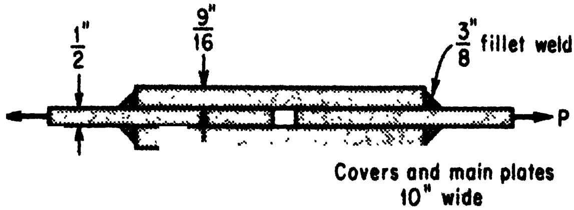


Fig. P8-2

8-3. Determine the minimum size of the fillet welds in Prob. 8-2 if the joint is to be 100 per cent efficient.

8-4. Tanks and boilers operating under high pressure are fabricated by welding "dished" heads to cylindrical shells by the various methods shown in Fig. P8-4. Determine the safe working pressures based on the strength of the weld for each of the attachments. Assume the vessels, which are capable of sustaining these pressures, to have equal diameters of 60 in. and to have shell and head thicknesses of $\frac{3}{8}$ in.

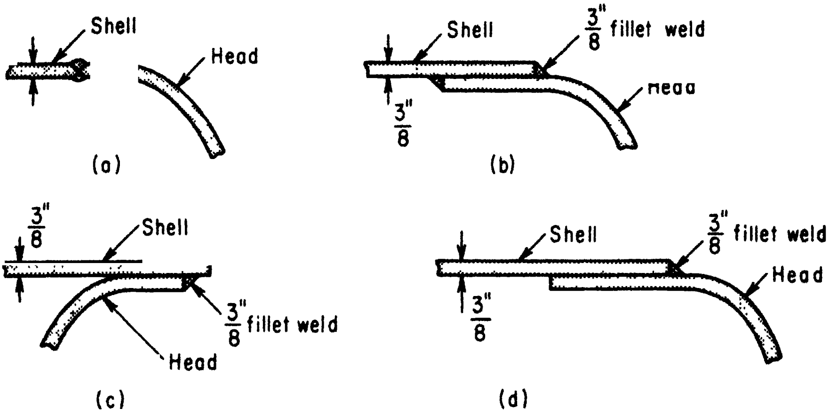


Fig. P8-4

8-5. A 30-ft-diameter spherical gas tank is fabricated by welding together two $\frac{1}{2}$-in.-thick steel hemispheres. Find the safe working pressure for the tank and the efficiency of the joint based on a working stress of 20,000 psi for the steel plate.

8-6. The structural joint of Fig. P8-6 is fillet-welded and has strength equal to that of the angle. Determine the proper length L_1 and L_2 if the permissible stress in the steel members is 18,000 psi. Assume the load to be applied at the centroid of the angle.

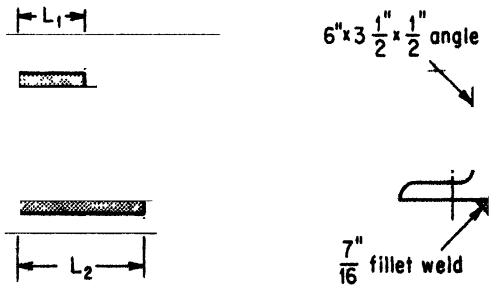


Fig. P8-6

8-7. An additional transverse fillet weld 6 in. long is added along *AB* to the structural joint of Prob. 8-6. Determine the appropriate lengths L_1 and L_2.

8-8. Cover plates are used to join two sections of 14 in. wide-flange beam, as shown in Fig. P8-8. Determine the required length L of $\frac{1}{2}$ in. fillet weld required if the bending moment in the beam at section A–A is 150,000 lb ft. Assume the vertical shear to be negligible in this section.

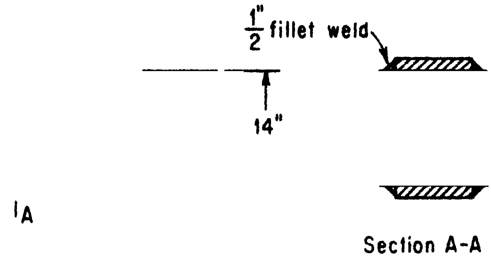


Fig. P8–8

8-9. The telescoping tubes are welded to form the joint shown in Fig. P8-9. Determine the safe load P and the efficiency of the connection based on a working stress of 25,000 psi in the tubing.

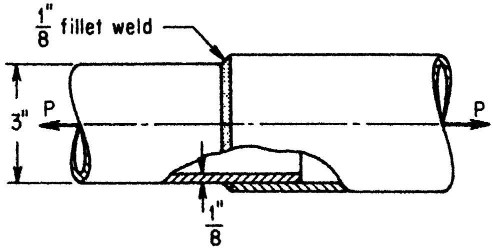


Fig. P8–9

8-10. To increase the strength in the welded joint, the larger of the square tubes is "fish-tailed," as shown in Fig. P8-10. Determine the efficiency of the joint if the permissible stress in the base metal is 20,000 psi.

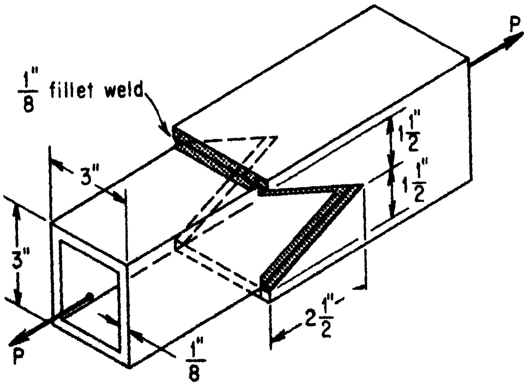


Fig. P8–10

8-11. Determine the permissible torque that can be applied to the plug-welded steel shaft of Fig. P8-11. Assume a working stress in shear of 14,000 psi in the weld.

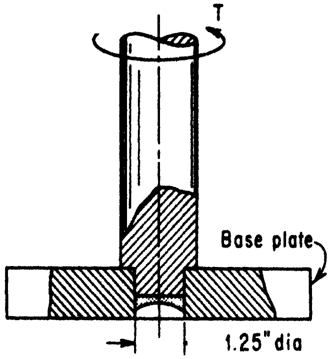


Fig. P8-11

8-12. Four legs are used to support the water storage tank shown in Fig. P8-12. The tank is 8 ft in diameter and has a capacity of 20,000 gallons. Determine the appropriate length L of $\frac{5}{16}$ in. fillet weld required on each side of the four legs. Water weighs 62.4 lb per cu ft.

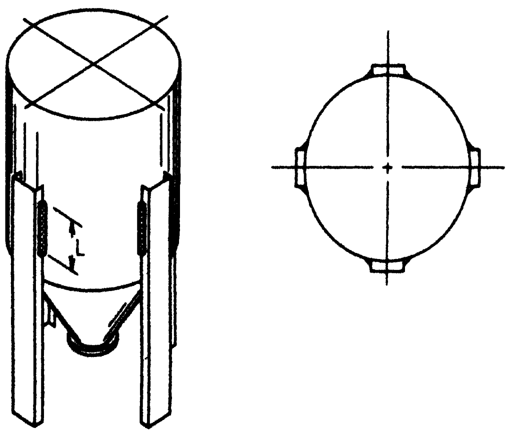


Fig. P8-12

8-13. Two $\frac{1}{4}$ in. plates are connected by a single $\frac{1}{2}$-in.-diameter rivet, as shown in Fig. P8-13. Determine the strength and efficiency of the joint if the following working stresses apply: $\tau = 15{,}000$ psi; $\sigma_b = 32{,}000$ psi; $\sigma_t = 18{,}000$ psi.

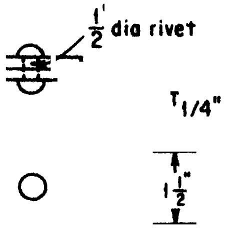


Fig. P8-13

8-14. What is the safe load and the efficiency of the connection described in Prob. 8-13 if the rivet diameter is increased to $\frac{5}{8}$ in.?

8-15. Two cover plates are employed in the connection shown in Fig. P8-15. The $\frac{5}{8}$ in. bolts have a cross-sectional area of 0.3068 sq. in. Determine the strength and efficiency of the joint if the following working stresses apply: $\tau = 10{,}000$ psi; $\sigma_b = 25{,}000$ psi; $\sigma_t = 16{,}000$ psi. *Hint:* The bolts are in double shear.

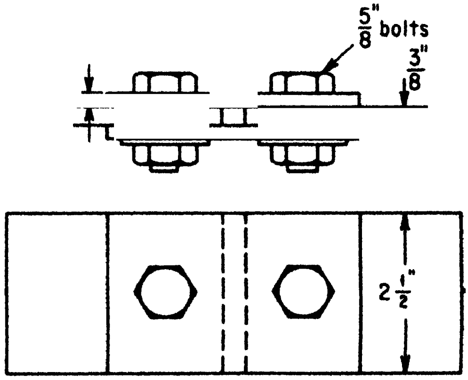


Fig. P8-15

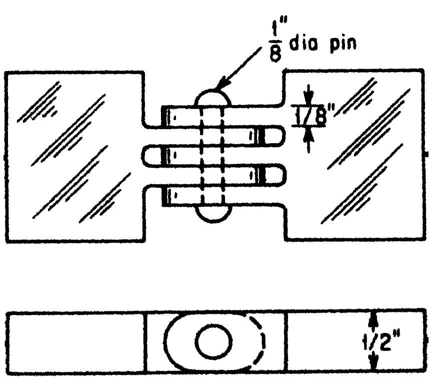


Fig. P8-16

8-16. A stainless-steel pin having a diameter of $\frac{1}{8}$ in. is used to secure the two halves of the brass hinge shown in Fig. -16. Determine the permissible load hat can be supported as indicated psi, σ_b 12,000 psi, and $\sigma_t = 6000$ psi.

8-17. Determine the proper bolt diameter d (nearest standard size by $\frac{1}{8}$ in. increments), width w, and thickness t of the connection shown in Fig. P8-17, if it is to withstand a pull of $P = 10{,}000$ lb. The working stresses are $\tau = 15{,}000$ psi, $\sigma_b = 32{,}000$ psi, $\sigma_t = 18{,}000$ psi.

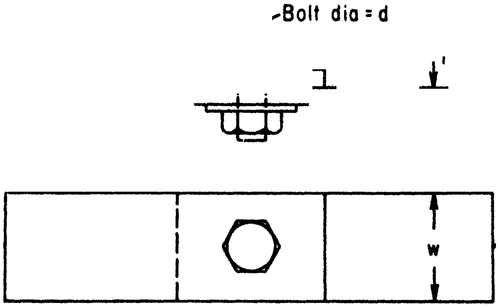


Fig. P8-17

8-18. Find the critical dimensions d, w, t_1, and t_2 for the coupling shown in Fig. P8-18. Base the computation on the following working stresses: $\tau = 15{,}000$ psi, $\sigma_b = 30{,}000$ psi, $\sigma_t = 20{,}000$ psi.

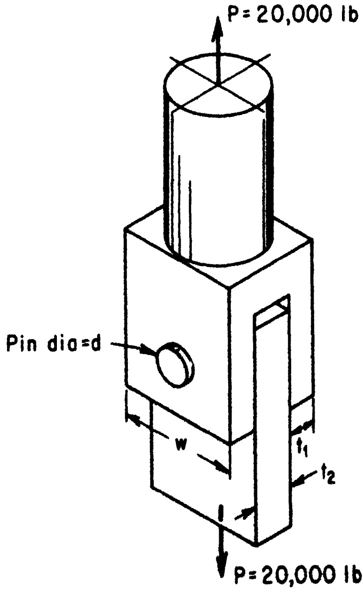


Fig. P8-18

8-19. Five $\frac{1}{4}$-in.-diameter rivets are used to secure the two plates shown in Fig. P8-19. Determine the tensile stress in the top plate at each of the five sections indicated.

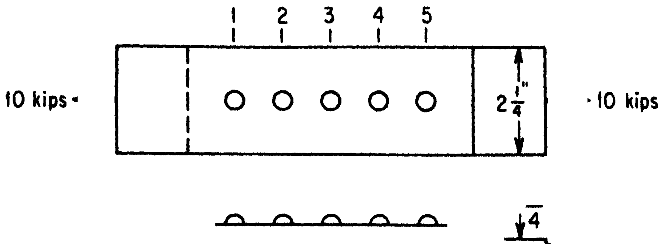


Fig. P8-19

8-20. Determine the safe load P that can be supported by the riveted lap joint shown in Fig. P8-20. The rivets have $\frac{5}{8}$ in. diameters, and the permissible stresses are $\tau = 15{,}000$ psi, $\sigma_b = 32{,}000$ psi, and $\sigma_t = 20{,}000$ psi. What is the efficiency of the joint?

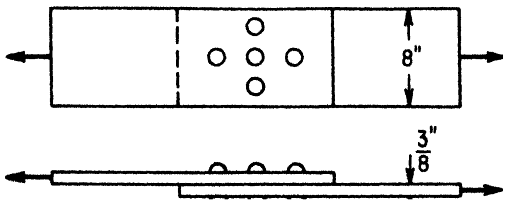


Fig. P8-20

8-21. Nine $\frac{7}{8}$-in.-diameter rivets are used to secure the three plates of Fig. P8-21. Find the design load P and the efficiency of the joint if the following stresses apply: $\tau = 15{,}000$ psi, $\sigma_b = 32{,}000$ psi for single shear, $\sigma_b = 40{,}000$ psi for double shear, $\sigma_t = 18{,}000$ psi.

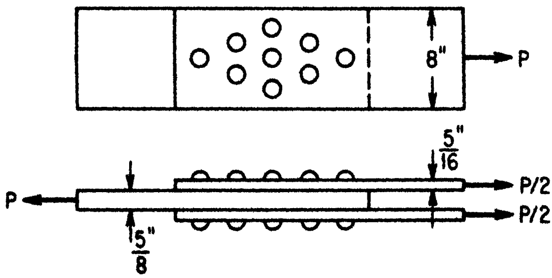


Fig. P8-21

8-22. Two steel plates 10 in. wide by $\frac{3}{8}$ in. thick are to be lapped and connected by using $\frac{3}{4}$-in.-diameter rivets. Determine the number of rivets required and their arrangement if the joint is to have maximum efficiency. The following stresses apply: τ = 15,000 psi, σ_b = 32,000 psi, σ_t = 20,000 psi.

8-23. Determine the safe load *P* and the efficiency for the connection shown in Fig. P8-23. The allowable stresses are τ = 15,000 psi, σ_b = 32,000 psi for single shear, σ_b = 40,000 psi for double shear, σ_t = 20,000 psi.

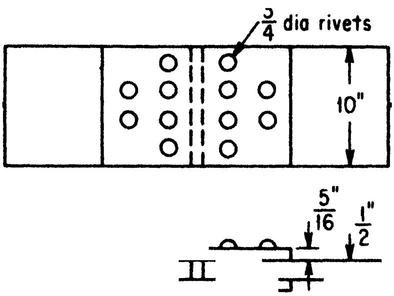


Fig. P8-23

8-24. Tension member *B*, Fig. P8-24, consists of two 4 in. by 3 in. by $\frac{3}{8}$ in. angles bolted back to back to a $\frac{1}{2}$ in. gusset plate. The member is to carry a maximum load based on a tensile stress of 20,000 psi. Determine the required number of $\frac{7}{8}$-in.-diameter *in-line* bolts. See Prob. 8-23 for the permissible loading stresses.

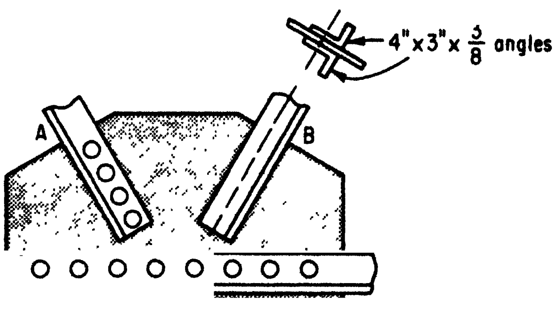


Fig. P8-24

8-25. A 21 WF 62 beam is attached to two 14 WF 68 columns by means of the standard beam connection shown in Fig. P8-25. Two 4 in. by $3\frac{1}{2}$ in. by $\frac{7}{16}$ in. angles and ten $\frac{7}{8}$-in.-diameter rivets are used at each column. Determine the safe load P as governed by bearing and shear. The permissible stresses are those of Prob. 8-23.

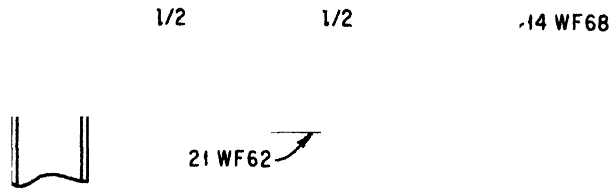


Fig. P8-25

8-26. Determine the safe load P that may be carried by the eccentrically loaded bolted connection shown in Fig. P8-26. The working stress in shear is 15,000 psi, and the bolt diameters are $\frac{7}{8}$ in.

8-27. The riveted connection shown in Fig. P8-27 can carry either a centric load P' or an eccentric load of P. Find the ratio of P/P'. Assume the stress in the most severely loaded rivet to be the same for both methods of loading.

8-28. Five $\frac{7}{8}$-in.-diameter rivets are used to connect the gusset plate to a column, as shown in Fig. P8-28. Find the safe load P if the permissible shearing stress is 15,000 psi.

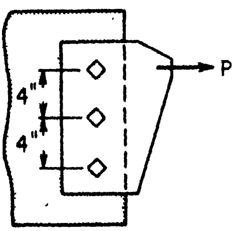


Fig. P8-26

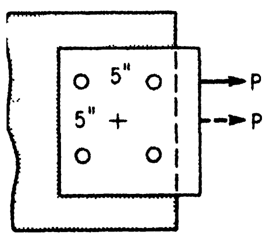


Fig. P8-27

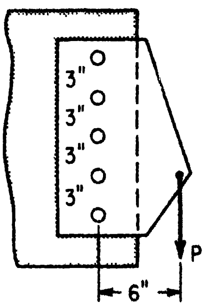


Fig. P8-28

8-29. Find the permissible eccentricity e in the connection of Fig. P8-29 if the maximum load any one rivet can support is 5 kips.

8-30. A load of 20 kips acts on a riveted connection shown in Fig. P8-30. Find the maximum and minimum rivet loads.

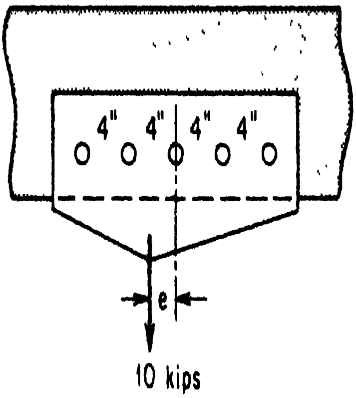


Fig. P8-29

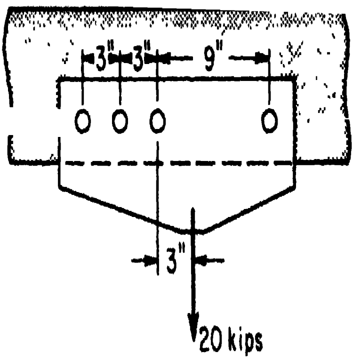


Fig. P8-30

8-31. Find P for the rivet arrangement shown in Fig. P8-31. The maximum load for any one rivet is 5000 lb.

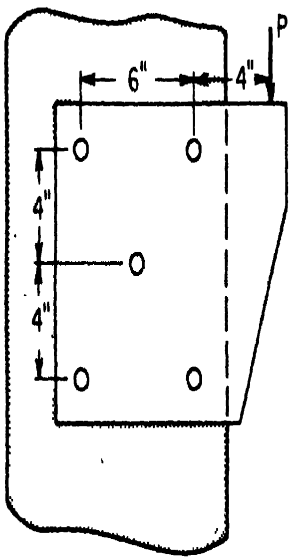


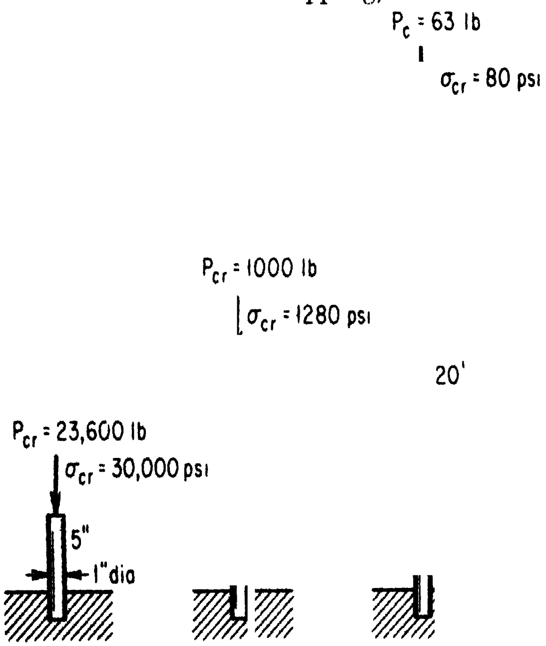
Fig. P8-31

CHAPTER 9

Columns

A member that fails in compression by *buckling*, or *collapsing*, is defined as a column. The term buckling refers to an *unstable state*, and the force causing this instability is called the *critical force*. Little concern is given to the degree of buckling; once instability is reached, the member is presumed to have failed. Stress that accompanies buckling failure is always less than that required for a direct compressive failure.

A rather forcible illustration of the effects of buckling appears in Fig. 9-1. Critical (a better word would be crippling) loads on the three flagpole



Critical loads and critical stresses on flag-pole columns

Fig. 9-1

columns, each a 1-in.-diameter steel bar, are indicated, together with the stress at a point of instability. The shortest member, in the true sense of the word, is not a column, but rather a *post*; failure here is due to direct compression. The illustration shows the relationship between length and instability.

9–1 Long Columns: Euler's Equation

Fig. 9–2

In 1744 Leonard Euler[1] published a lengthy paper covering a variety of cases of elastic bending. Created by the studies is the now-famous "Euler formula," an expression for the buckling strength of hinge-ended columns like that of Fig. 9-2. The equation is a statement of the critical buckling load P_{cr} in terms of the elastic and geometric properties of the column.

$$P_{cr} = \frac{\pi^2 EI}{l^2} \qquad (9\text{-}1)$$

where

E = modulus of elasticity.
I = least moment of inertia.
l = length.

It is rather interesting to observe that the critical buckling load does not depend upon the strength of the column material, but rather on elasticity and geometry. High-strength alloy steels and common structural steels used as similar columns both fail under the same critical load.

The stress at buckling can be found by dividing both sides of the Euler equation by the cross-sectional area A of the column.

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{l^2 A}$$

Since the square of the radius of gyration[2] r is equal to the ratio I/A, the critical stress becomes

$$\pi^2 E r^2 \qquad \frac{\pi^2 E}{(l/r)^2} \qquad (9\text{-}2)$$

When expressed in this form, the term l/r is called the *slenderness ratio.*

In terms of pure mathematics, the slenderness ratio can assume any value other than zero; from the practical point of view, however, severe restrictions

[1] Leonard Euler (1707-1783), has been called the most prolific mathematician in history; he is probably the greatest man of science that Switzerland has produced.

[2] In mechanics of materials the r is generally used to denote radius of gyration, whereas in engineering mechanics k is the usual symbol.

must be placed on the term. Imagine, for example, that a steel column has a slenderness ratio equal to 1. This would mean that the critical stress at buckling would be roughly

$$\frac{\pi^2 E}{(l/r)^2} = \frac{\pi^2(30)10^6}{1} = 300{,}000{,}000 \text{ psi}$$

a ridiculous number.

For any given material, then, there must be a limit to the least value of l/r: a limit that can be found by allowing the critical stress to equal the yield stress. For steel, the slenderness ratio must be greater than 100, if a yield stress of approximately 30,000 psi is assumed.

$$\left(\frac{l}{r}\right)^2_{\min} = \frac{\pi^2 E}{\sigma_{yp}} = \frac{100(30)10^6}{30{,}000}$$

$$\left(\frac{l}{r}\right)_{\min} = 100$$

For high-strength steels, the slenderness ratio can be considerably smaller; for low-strength steels, higher. Good design practice generally governs the upper limit of the slenderness ratio. For structural-steel design, a rule of thumb places the greatest permissible l/r at 200.

Another way of determining the validity of Euler's equation is to divide the critical load, as found in the equation, by the cross-sectional area. If the compressive stress is less than the yield stress, the column is *long* and will fail by buckling. If the stress is greater than the yield stress, the member will fail by fracture or by plastic deformation.

An important point that must be remembered when using Euler's equation is that the moment of inertia (or radius of gyration) must be the least value for the section.

Example 1. An aluminum alloy has a modulus of elasticity of 10×10^6 psi and a yield strength of 6000 psi. Determine the least value of the slenderness ratio for which Euler's equation applies.

Solution: The least value of the slenderness ratio is found by substituting numerical values into Eq. 9-2.

$$\sigma_{cr} = \frac{\pi^2 E}{(l/r)^2}$$

$$\left(\frac{l}{r}\right)_{\min} = \sqrt{\frac{\pi^2 E}{\sigma_{yp}}} = \sqrt{\frac{(\pi)^2 10(10)^6}{6000}} \quad 128$$

Example 2. A structural-steel column 20 ft long is to support an axial load of 50 kips. Use a factor of safety of 3 and find the lightest wide-flanged section that can support this load.

Solution: To satisfy the required factor of safety, the critical design load is

$$P_{cr} = 3(50) = 150 \text{ kips}$$

Numerical data are substituted into Euler's equation, and the least moment of inertia is computed:

$$I = \frac{P_{cr}l^2}{\pi^2 E} = \frac{150(10)^3(20 \times 12)^2}{\pi^2 30(10)^6}$$
$$= 29.2 \text{ in.}^4$$

A suitable column must, therefore, have a moment of inertia in its weak direction of at least 29.2 in.^{4} Tables in the Appendix give the lightest beam as an 8 WF 31. The least radius of gyration of this section is 2.01; therefore,

$$\frac{l}{r} = \frac{20(12)}{2.01} = 119$$

This section is satisfactory, since its slenderness ratio is greater than 100. If the ratio l/r were less than 100, the next lightest beam would be checked, and so on, until a satisfactory section could be found.

9-2 *Effects of Bracing and End Restraints*

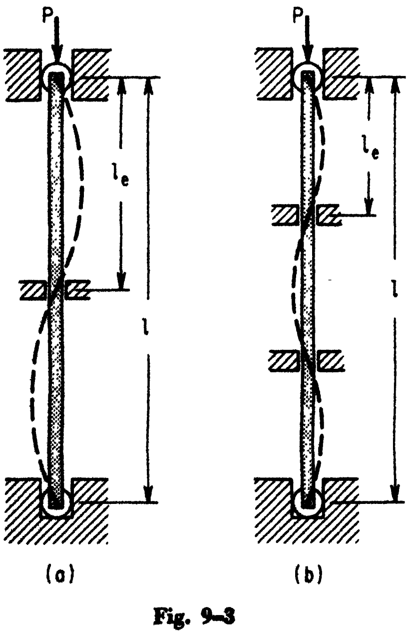


Fig. 9-3

The braced column of Fig. 9-3(a) is, in effect, two columns, one on top of the other; to fail by buckling each half must collapse. The *effective length* of this member is $l/2$, thereby making the column four times stronger than when unbraced; thus

$$P_{cr} = \frac{\pi^2 EI}{(l/2)^2} = \frac{4\pi^2 EI}{l^2} \qquad (9\text{-}3)$$

By similar reasoning, bracing at the third points, Fig. 9-3(b) increases the column strength nine times.

$$P_{cr} = \frac{\pi^2 EI}{(l/3)^2} = \frac{9\pi^2 EI}{l^2} \qquad (9\text{-}4)$$

Experience has shown that Euler's formula can be corrected to apply to the variety of end conditions illustrated in Fig. 9-4. In the second and third cases, the end restraints tend to strengthen the column, whereas the lack of restraints in the fourth case weakens the column. It is usual practice to consider columns "hinge-ended" when in doubt as to the degree of restraint offered by the supports. This simply increases the factor of safety in the design.

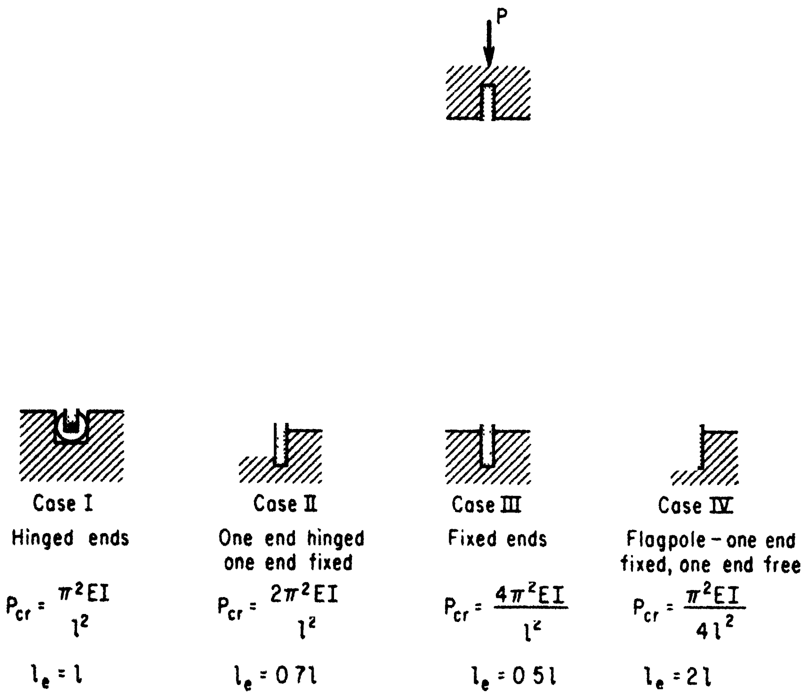


Fig. 9-4

Example 3. A 5 in. I-beam weighing 10 lb per ft is to be used as a hinge-ended column 18 ft long. The beam is braced in its weakest direction at the midpoint. Determine the safe load P based on a factor of safety of 2.5.

Solution: Two possibilities exist: the column can fail in its strongest direction, where its length is considered to be the full 18 ft, or it can fail in its weakest direction, where its effective length is 9 ft. Both possibilities must be investigated, as well as the slenderness ratio for each case.

As an 18 ft column in the strong direction:

The slenderness ratio must be checked to insure that Euler's equation applies. Data are obtained from Appendix B.

$$\frac{l}{r} = \frac{18(12)}{2.05} = 105$$

Since the slenderness ratio is greater than 100, Euler's equation is valid:

$$P_{cr} = \frac{\pi^2 EI}{l^2} = \frac{\pi^2(30)10^6(12.1)}{(18 \times 12)^2} = 76{,}700 \text{ lb}$$

The design load is

$$P = \frac{76{,}700}{2.5} = 30{,}700 \text{ lb}$$

As a 9 ft column in the weak direction:

$$\frac{l}{r} = \frac{9(12)}{0.65} = 166$$

Euler's equation is valid; hence,

$$P_{cr} = \frac{\pi^2 EI}{l^2} = \frac{\pi^2(30)10^6(1.2)}{(9 \times 12)^2} = 30{,}500 \text{ lb}$$

$$P = \frac{30{,}500}{2.5} = 12{,}200 \text{ lb}$$

The weak direction governs the design. This conclusion could have been reached by merely comparing the two slenderness ratios; the configuration which gives the greatest slenderness ratio governs the design.

Example 4. Two 12 in., 25 lb channels are latticed together to form a column. The distance between the channels is designed so that the moments of inertia about the principal axes are equal. Find the critical (minimum) length of this column, assuming one end to be fixed and the other hinged. $E = 30 \times 10^6$ psi; $\sigma_{yp} = 30{,}000$ psi.

Solution: The critical load for this column is given in Fig. 9-4:

$$P_{cr} = \frac{2\pi^2 EI}{l^2}$$

where $P_{cr} = \sigma_{yp} A$

$$I = 2(143.5) = 287 \text{ in.}^4$$

$$A = 2(7.32) = 14.64 \text{ in.}^2$$

Substitution of numerical data gives

$$l^2 = \frac{2\pi^2 EI}{\sigma_{yp} A} = \frac{2\pi^2(30 \times 10)^6\, 287}{30{,}000(14.64)} = 38.7 \times 10^4$$

$$l = 622 \text{ in. or } 51.8 \text{ ft}$$

Alternate Solution: If the column were hinge-ended, the critical slenderness ratio would be approximately 100.

$$l/r \approx 100$$

$$l = 100\sqrt{\frac{I}{A}} = 100\sqrt{\frac{287}{14.64}} = 443 \text{ in.}$$

With one end fixed and the other hinged, the effective length, given in Fig. 9-4, is

$$l_e = 0.7l$$

Therefore

$$l = \frac{l_e}{0.7} = \frac{443}{0.7} = 634 \text{ in. or } 52.8 \text{ ft}$$

which is a fair approximation to the exact answer.

9-3 *Long Columns Loaded Between Supports*

When long columns sustain concentric loads applied at points other than the ends, Fig. 9-5, Euler's formula becomes

$$P_{cr} = \frac{\pi^2 EI}{l_r^2} \qquad (9\text{-}5)$$

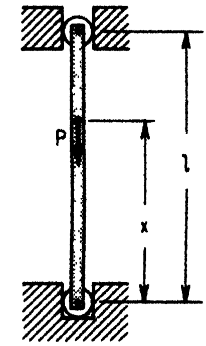


Fig. 9-5

where l_r is called the *reduced length* and is similar to the *effective length* employed in correcting for various end conditions. In other words, l_r is the length of a hinged column which has the same critical buckling load as the column loaded between supports.

Values of l_r may be obtained from the graph[1] of Fig. 9-6, which relates the ratios x/l and l_r/l. The former is obtained from known data and the latter from the graph.

For validity, Euler's equation must give a critical stress less than the yield stress, which means that a failure will occur through buckling rather than compression. This point is easily checked by simply computing the value of the critical stress at the buckling load:

$$\sigma_a = \frac{P}{A}$$

[1] James Dow, "Columns Loaded Between Supports," *Machine Design*, February, 1961, pp. 167–168.

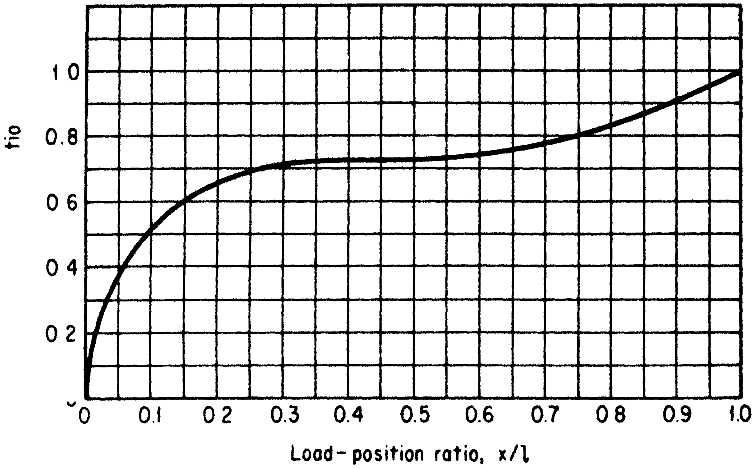


Fig. 9-6

Example 5. The control linkage shown in Fig. 9-7 consists of a high-strength steel rod and a pivot arm. The load is transmitted to the rod through pin A. Determine (a) the critical value of F based on the buckling strength of the rod, (b) the stress in the rod at the critical load. $E = 30 \times 10^6$ psi; $\sigma_{yp} = 60{,}000$ psi.

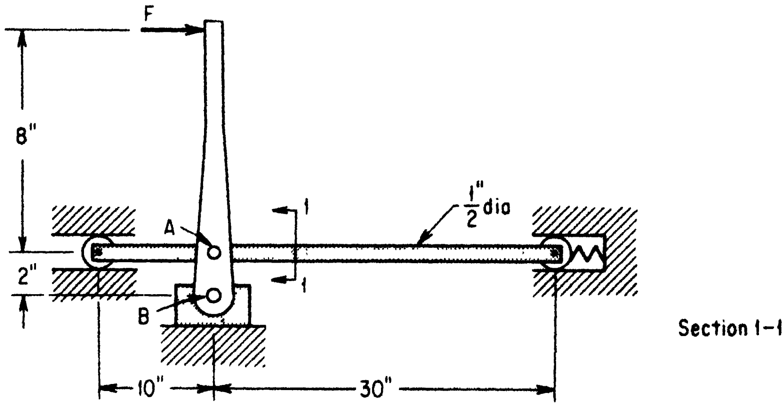


Fig. 9-7

Solution: The ratio of $x/l = 30/40 = 0.75$; l_r/l from the graph of Fig. 9-6 is 0.8; hence,

$$l_r = 0.8(40) = 32 \text{ in.}$$

Pin A can tolerate a critical force of

$$P_{cr} = \frac{\pi^2 EI}{l_r^2}$$

where

$$I = \frac{\pi d^4}{64} = \frac{\pi(0.5)^4}{64} = 0.00307 \text{ in.}^4$$

Thus

$$P_A = P_{cr} = \frac{\pi^2(30 \times 10^6)(0.00307)}{(32)^2} = 887 \text{ lb}$$

Stress, at the critical load, is

$$\frac{887}{\pi(0.25)^2} = 4520 \text{ psi}$$

which is well below the yield stress of 60,000 psi.

Moments taken about the pin B will give the critical force F:

$$10F = 2(887)$$

$$F = 177 \text{ lb}$$

9-4 Intermediate Columns: Empirical Formulas

A graph of Euler's equation, Fig. 9-8, shows the critical stress to decrease as the slenderness ratio increases, the useful portion of the curve being below the proportional limit. The range of columns whose slenderness ratios are less than Euler's critical value are called *short columns* and *intermediate*

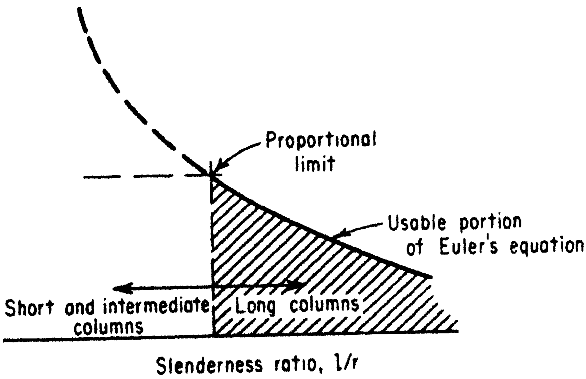


Fig. 9-8

columns. Limiting values of compressive stress govern the design of the former class, and empirical formulas based on observation and experience govern the latter. The newest and most practical of the intermediate column formulas[1] is based on studies made by the American Institute of Steel Construction. The formula defines the permissible stress as

$$\sigma = \frac{\left[1 - \frac{(l/r)^2}{2C_c^2}\right]\sigma_{yp}}{\text{f.s.}} \tag{9-6}$$

where

$$\text{f.s.} = \text{factor of safety} = \frac{5}{3} + \frac{3(l/r)}{8C_c} - \frac{(l/r)^3}{8C_c^3}$$

and

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_{yp}}}$$

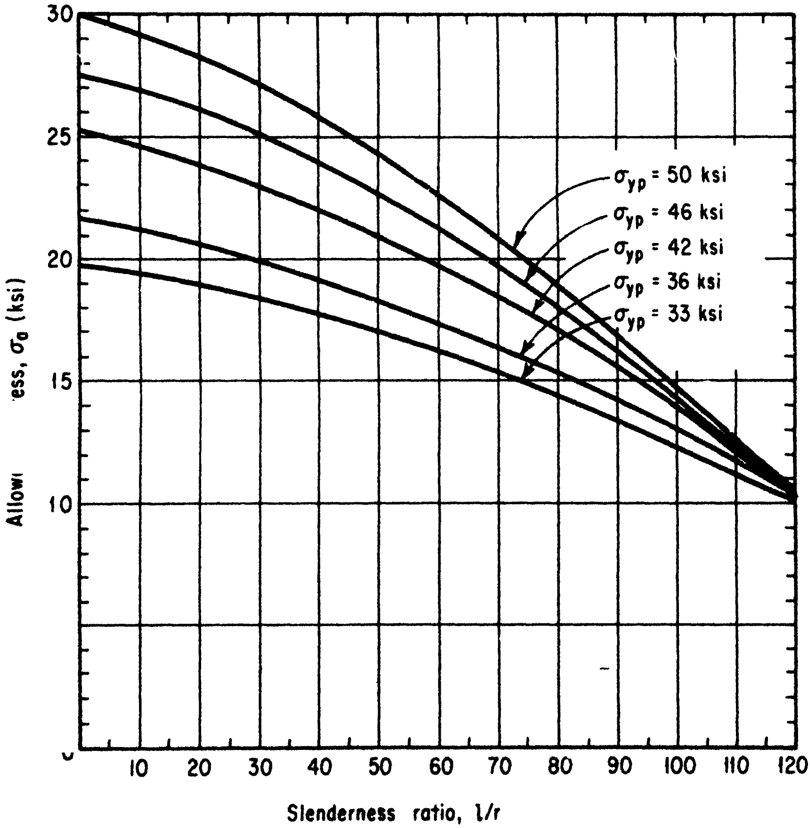


Fig. 9–9

[1] Adopted November 30, 1961 by The American Institute of Steel Construction and reprinted by permission.

The term σ_{yp} is the stress at the yield point for a given grade of steel; the term σ_a is the permissible axial stress. Equation (9-6) is valid for any axial-loaded hinge-ended column whose slenderness ratio is less than the constant C_c.

The AISC equation is interesting from a very practical point of view in that it provides a *variable factor of safety*—high, where the column is most vulnerable. Similar provisions have been included in the British and German design standards for some time and can be justified by the insensitivity of columns to accidental eccentricities.

Allowable stresses, as calculated by Eq. (9-6), for five grades of steel are presented in the graph of Fig. 9-9. It is interesting to see how the values of σ_a converge to those permitted by Euler's equation for high values of l/r.

Example 6. Determine the permissible load that can be carried by a 10 ft 10 WF 45 steel column. The stress at the yield point of this particular steel is 33,000 psi.

Solution: Appendix B gives the least radius of gyration of this section as $r = 2$ in. The slenderness ratio, therefore, is

$$\frac{10(12)}{} = 60$$

The constant C_c and the factor of safety are next computed:

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_{yp}}} = \sqrt{\frac{2\pi^2(30 \times 1(}{33{,}000}} = 134$$

$$\text{f.s.} = \frac{5}{3} + \frac{3(l/r)}{8C_c} - \frac{(l/r)^3}{8C_c^3}$$

$$= \frac{5}{3} + \frac{3(60)}{8(134)} - \frac{(60)^3}{8(134)^3} \quad 1.83$$

Numerical values are substituted into Eq. (9-6).

$$\sigma_{-} = \frac{\left[1 - \frac{(l/r)^2}{2C_c^2}\right]\sigma_{yp}}{\text{f.s.}}$$

$$= \frac{\left[1 - \frac{(60)^2}{2(134)^2}\right]33{,}000}{1.83} \cdot \quad 16{,}200 \text{ psi}$$

Since this column has a cross-sectional area of 13.24 in.^{2}, the allowable load is

$$P_a = \sigma_a A = 16{,}200(13.24) = 214{,}000 \text{ lb or } 214 \text{ kips}$$

Alternate Solution:

The graph of Fig. 9-9 gives an allowable stress of 16.2 ksi at $l/r = 60$ for 33,000 psi grade steel; therefore,

$$P_a = 16.2(13.24) = 214 \text{ kips}$$

9-5 Eccentrically Loaded Columns: Conservative Approach

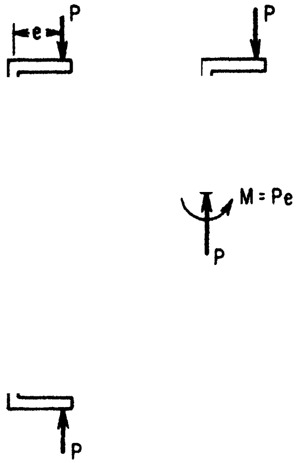


Fig. 9-10

When the line of action of the axial force does not coincide with the centroidal axis of the column, bending stresses as well as compressive stresses are induced in the member. This is illustrated in Fig. 9-10, where the moment M is the product of the force P and the eccentricity e; the combined stress has a maximum magnitude of

$$= \frac{P}{A} + \frac{Pe}{Z} \qquad (9\text{-}7)$$

where A and Z are the area and section modulus respectively.

For a column acted upon by an axial load P_o and an eccentric load P, a more realistic situation, the maximum stress is

$$\frac{P_o + P}{A} + \frac{Pe}{Z} \qquad (9\text{-}8)$$

In these equations the maximum stress is actually the working stress σ_a obtained through Eq. (9-6) or by the graph of Fig. 9-9, and is always based on the section's least radius of gyration.

Example 7. A 12 WF 40 steel column 12 ft long supports an axial load of 100 kips. Determine the additional load that may be applied in the weak direction at an eccentricity $e = 2$ ft. The steel has a yield point stress of 42,000 psi.

Solution: The properties of the section are

$$A = 11.77 \text{ in.}^2$$

$$r_{min} = 1.94 \text{ in.}$$

$$Z_{min} \quad 11.0 \text{ in.}^3$$

$$l \quad \frac{12(12)}{1.94} = 74.2$$

The working stress is estimated from the graph of Fig. 9-9.

$$\sigma_a = 18 \text{ ksi}$$

Numerical values are substituted into Eq. (9-8):

$$\sigma_{\max} = \frac{P_o + P}{A} + \frac{Pe}{Z}$$

$$18 \quad \frac{100 + P}{11.77} + \frac{P(2 \times 12)}{11.0}$$

$$P = 4.19 \text{ kips}$$

Example 8. Select a wide-flanged section to act as a column 15 ft long. The axial compressive load is 80 kips, and the eccentric load is 60 kips applied in the strong direction, as shown in Fig. 9-11. The steel has a yield point stress of 33 ksi.

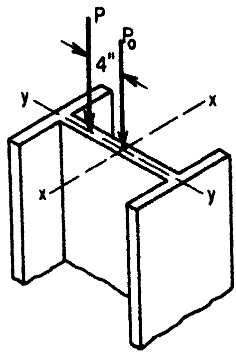


Fig. 9-11

Solution: This problem is more difficult than the previous example, since the properties of the section are not known; trial-and-error methods must be used.

If the total load were applied concentrically and if the slenderness ratio were zero, the maximum stress from the graph of Fig. 9-9 would be approximately 20 ksi. The column would require an area of at least

$$A = \frac{P}{A} = \frac{80 + 60}{20} = 7 \text{ in.}^2$$

Column sections having less cross-sectional area need not be considered.

Several members must be tried until an agreement between the left- and right-hand portions of Eq. (9-8) can be found.

Trial 1. 12 WF 40

$$A = 11.77 \text{ in.}^2$$

$$r_{\min} = 1.94 \text{ in.}$$

$$Z_{xx} = 51.9 \text{ in.}^3$$

$$\frac{l}{r} = \frac{15(12)}{1.94} = 92.8$$

$$\sigma_{\max} = \frac{P + P_o}{A} + \frac{Pe}{Z}$$

$$\sigma_{\max} = \frac{80 + 60}{11.77} + \frac{60(4)}{51.9} = 16.5 \text{ ksi}$$

The graph of Fig. 9-9 gives a maximum stress of 13.1 ksi for this section. A heavier member is, therefore, required.

Trial 2. 10 WF 49

$$A = 14.4 \text{ in.}^2$$

$$r_{\min} = 2.54 \text{ in.}$$

$$Z_{xx} = 54.6 \text{ in.}^3$$

$$\frac{l}{r} = \frac{15(12)}{2.54} = 70.9$$

$$\sigma_{\max} = \frac{80 + 60}{14.4} + \frac{60(4)}{54.6} = 14.1 \text{ ksi}$$

This section is satisfactory, since the allowable stress from the graph of Fig. 9-9 is 15.3 ksi, a greater value.

Further trials may result in finding a lighter, and, therefore, more economical section.

PROBLEMS

9-1. Find the greatest value of the slenderness ratio for each of the following vertical columns: (a) a square cross-section section, 1 in. on edge and 10 ft long; (b) a plate 5 ft wide, $\frac{1}{4}$ in. thick, and 3 ft high; (c) a solid circular section, 2 in. in diameter and 5 ft long; (d) an 8 WF 31 section 20 ft long; (e) two 9 in., 20 lb channels, 6 ft long placed back to back; (f) four 3 in. by 3 in. by $\frac{1}{4}$ in. angles welded together to form a box-section 10 ft long; (g) a 6 in. by 4 in. by $\frac{1}{2}$ in. angle 3 ft long.

9-2. Two 12 in., 30 lb channels are latticed together, Fig. P9-2, to form a column section having equal moments of inertia I_x and I_y. Find the length of this column if it has a slenderness ratio of 110.

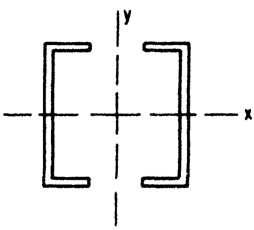


Fig. P9-2

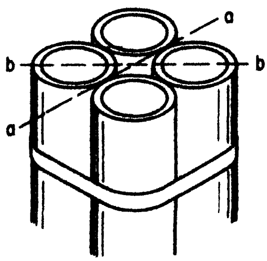


Fig. P9-3

9-3. Four sections of $3\frac{1}{2}$-in.-diameter standard pipe are strapped together to form the column section shown in Fig. P9-3. Each pipe has an outside diameter of 4 in., a cross-sectional area of 2.68 sq in., and a diametral moment of inertia of 4.79 in.^{4} Find the radius of gyration of the section about axis a–a and about axis b–b.

9-4. A certain high-tensile-strength steel has a modulus of elasticity of 30×10^6 psi and a yield point stress of 75,000 psi. Find the limiting value of the slenderness ratio for which Euler's equation is valid. Is this a minimum or maximum value?

9-5. Douglas fir has a modulus of elasticity of 1.76×10^6 psi and a yield stress parallel to the grain of 5000 psi. Find the limiting value of the slenderness ratio for which Euler's equation is valid.

9-6. Show that the buckling stress in a long rectangular column having cross-sectional dimensions of b and d is $\pi^2E/12(l/d)^2$, providing $d > b$.

9-7. An axially-loaded aluminum column having a rectangular cross section 2 in. by 3 in. is pinned at each end. Determine the range of lengths for which Euler's equation is valid. $\sigma_{yp} = 18{,}000$ psi; $E = 10 \times 10^6$ psi.

9-8. Determine the safe axial load that may act on a 15 ft hinge-ended 8 WF 24 steel column. Use a factor of safety of 2.5.

9-9. Determine the critical load F that can be applied as indicated to the steel column shown in Fig. P9-9. The member has rounded ends and is braced at the midpoint in the weak direction. Use Euler's equation and the following data: $\sigma_{yp} = 45{,}000$ psi; $E = 30 \times 10^6$ psi.

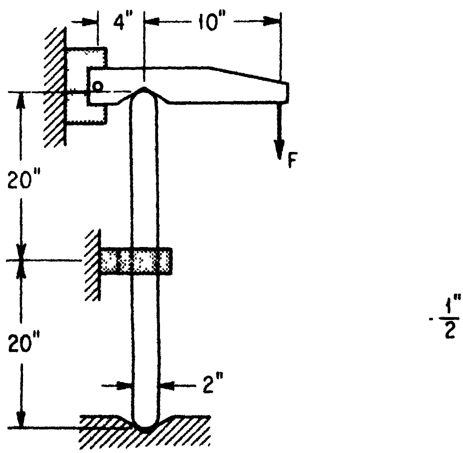


Fig. P9-9

9-10. What column width d would be required in Prob. 9-9 if the critical loads in the x- and y-directions are to be the same? Assume the $\frac{1}{2}$-in. dimension of the column to be unchanged.

9-11. Two steel columns with square cross sections support a rigid beam of negligible weight, as shown in Fig. P9-11. Column A is fixed at one end and pinned at the other, and column B is pinned at both ends. Find the magnitude and location of the critical load P that can be supported by the assembly. Use Euler's equation.

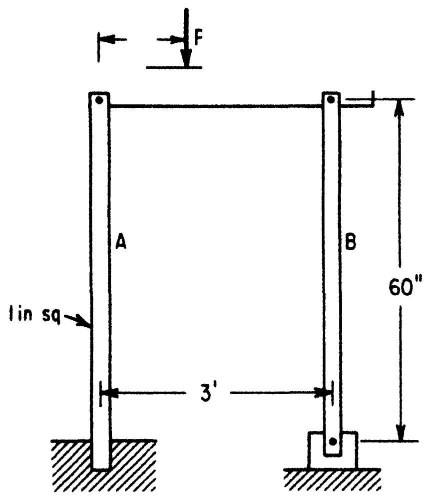


Fig. P9–11

9-12. Four aluminum tubes are latticed together to form the 200 ft antenna shown in Fig. P9-12. Guy wires with equal tensions T support the member. Determine (a) the permissible cable tension, using Euler's equation with a factor of safety of 2, and (b) the stress in the tubes. Each section has a diametral moment of inertia of 1.9 in.^{4} and a cross-sectional area of 1.33 sq in. $E = 10 \times 10^6$ psi.

9-13. A $\frac{1}{4}$-in.-diameter steel rod is rigidly supported at its base, as shown in Fig. P9-13. The rod is free to expand 0.05 in. before it makes contact with the rigid support. Determine the minimum temperature change necessary to cause buckling. $E = 30 \times 10^6$ psi; $a = 6.5 \times 10^{-6}$ in./in./°F.

9-14. Use the graph of Fig. 9-6 to find the reduced length for each of the following conditions for the column shown in Fig. P9-14: (a) $l = 10$ ft, $x = 6$ ft; (b) $l = 12$ ft, $x = 4$ ft; (c) $l = 20$ ft, $x = 18$ ft; (d) $l = 60$ in., $x = 10$ in.

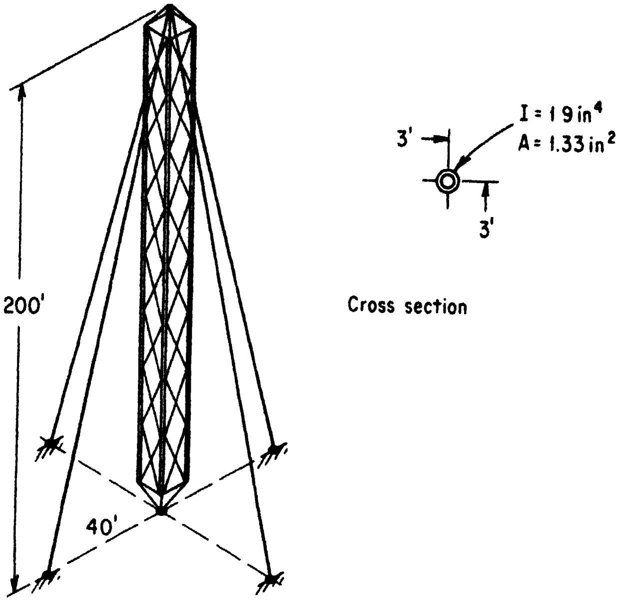


Fig. P9-12

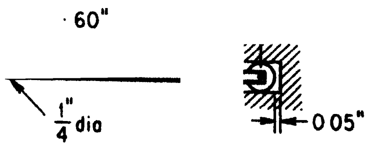


Fig. P9-13

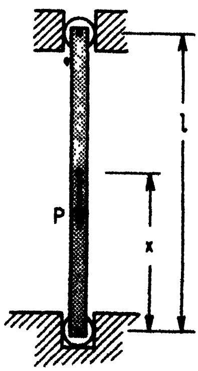


Fig. P9-14

9-15. A 3 in., 5 lb per ft steel channel is used as a hinge-ended column 10 ft long. Find (a) the critical axial load that may be applied at a point 6 ft from the base, and (b) the critical stress for this loading arrangement.

9-16. Through what range of values of x, in Prob. 9-15, is Euler's equation valid? The steel has a yield-point stress of 30,000 psi.

9-17. Pressure applied at C is used to operate the push-rod switch mechanism shown in Fig. P9-17. Find the safe value of F, based on a factor of safety of 2, and the resulting contact force at A. The rod is a 1-in.-diameter steel tube and has the properties shown in the figure.

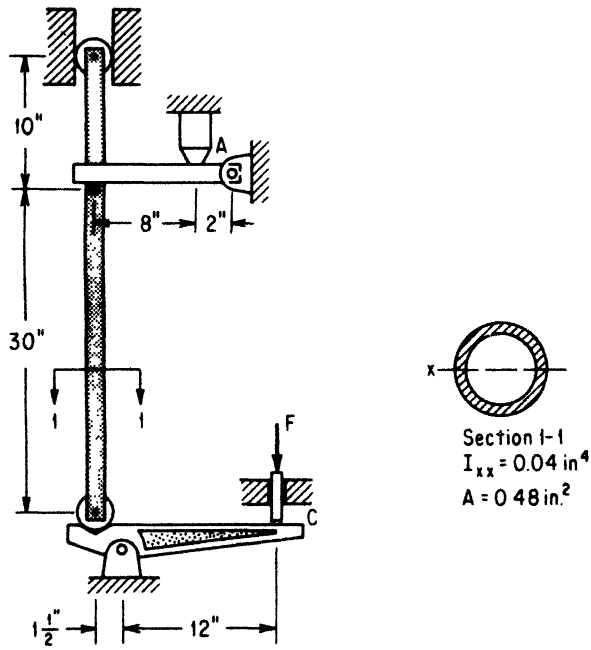


Fig. P9-17

9-18. A 10 WF 77 steel section 12 ft long is used as a hinge-end column. Use the AISC formulas to determine the load capacity of the column. The steel has a yield-point stress of 36 ksi.

9-19. A column has a slenderness ratio of 70 and is made of steel that has a yield-point stress of 40,000 psi. Use the AISC formulas to determine the permissible stress and the appropriate factor of safety.

9-20. What are the maximum and minimum safety factors given by the AISC formulas for the permissible range of l/r?

9-21. What load can be carried by two 12 in., 25 lb steel channels 8 ft long

(a) if they are fastened back to back; (b) if they are latticed together to form a section having equal moments of inertia in the two principal directions? The steel has a yield-point stress of 42 ksi.

9-22. Find a suitable wide-flanged steel beam (33 ksi stress grade) to be used as a 20 ft column capable of supporting an axial load of 150 kips. Use the AISC formulas.

9-23. A steel pipe 8 ft long with outside and inside diameters of 6 in. and 4 in., respectively, is used as a column. Determine the safe axial load based on the AISC formulas. The steel has a yield-point stress of 36 ksi.

9-24. A 12 ft boxed column is fabricated by welding four 3 in. by 3 in. by $\frac{1}{2}$ in. steel angles together. Determine, by means of the AISC formula, the safe concentric load that may be carried. The steel has a yield-point stress of 50 ksi.

9-25. Determine the permissible column load that may be supported by a 6 in. long, $\frac{1}{2}$-in.-diameter alloy steel rod. The yield-point stress is 90,000 psi, and the modulus of elasticity is 30×10^6 psi. Use either the AISC formulas or the Euler equation, whichever applies.

9-26. A 12 WF 65 steel beam is used as a column 20 ft long. Use the AISC formulas and Eq. (9-7) to determine the maximum load at an eccentricity of 12 in. in (a) the weak direction; (b) the strong direction. Assume the steel to have a yield-point stress of 33 ksi.

9-27. What is the maximum beam reaction that may be carried by a 14 WF 34 steel column 12 ft long? Use the AISC formulas and Eq. (9-7). The beam reaction occurs at the outside flange of the column, as shown in Fig. P9-27. Assume this steel to have a yield-point stress of 33 ksi.

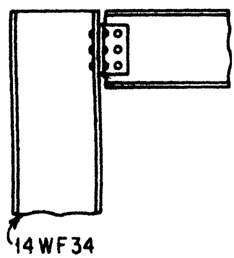


Fig. P9-27

9-28. Determine the maximum permissible eccentricity *e* for the column of Fig. P9-28. The section, a 14 WF 74 steel beam, has a yield-point stress of 36 ksi. Use Eq. (9-8) and the AISC formulas and assume the ends of the column to be hinge-supported.

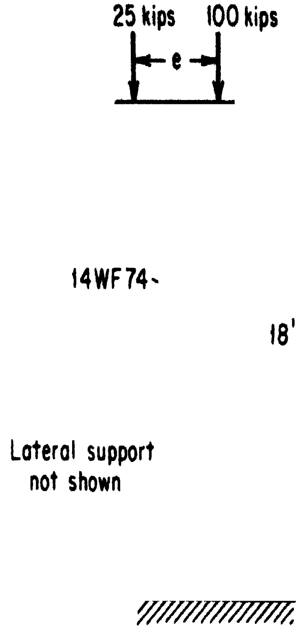


Fig. P9–28

9-29. A cam-operated plunger bar is shown in Fig. P9-29. The bar, a "flag-pole" column, is made of steel having a yield-point stress of 70,000 psi. Determine the maximum and minimum critical values of P for the limits of cam rotation shown. The greatest eccentricity is 2 in.

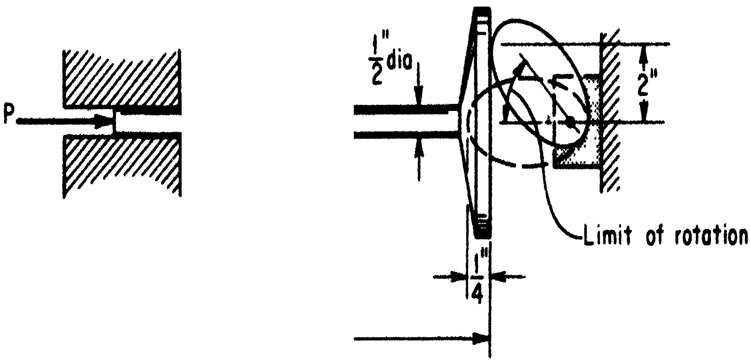


Fig. P9–29

CHAPTER 10

Experimental Stress Analysis

> Mathematics is the most exact science, and its conclusions are capable of absolute proof. But this is so only because mathematics does not attempt to draw absolute conclusions. All mathematical truths are relative, conditional—Charles Proteus Steinmetz (1923)[1]

This chapter is concerned with the more significant experimental methods used in the study of mechanics of materials. Although the pure theorist will cringe at the importance about to be attributed to the experimental method, he cannot deny its many accomplishments. In truth, a theory or a formula is only useful when it can predict with reasonable accuracy what will happen physically. To a degree, the equations of torsion, bending, and axial loading —theoretical equations all—give reasonable and accurate answers; they fail, however, to account for the variance in the physical properties of materials, the stress and strain in the region of the applied load, the abrupt change in geometry, and the effects of dynamic loading.

The designer is, more often than not, in these regions of the unknown, and he frequently relies on intuition based on past experience. A difficulty that accompanies this approach is the high cost of both success and failure. The high cost of success is measured by the waste of raw materials caused by overdesigning; the high cost of failure is usually represented in newspaper headlines.

Experimental stress analysis has one great claim to fame: it provides a *non-destructive* and *non-disruptive* method of examining a part in actual service.

[1] From *Men of Mathematics*, Copyright 1937 by E. T. Bell, by permission of Simon and Schuster, Inc.

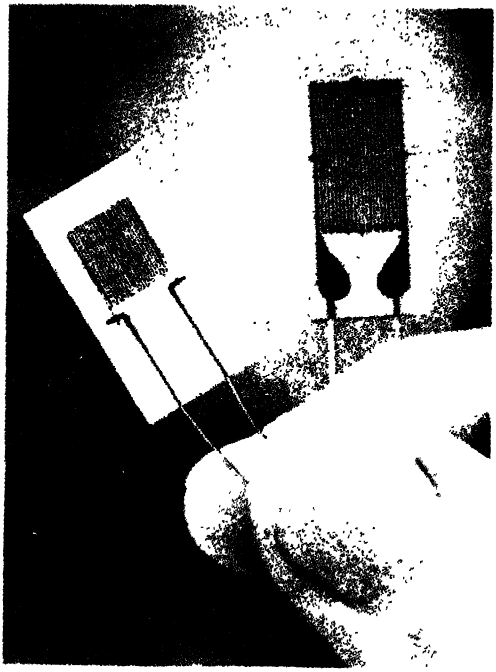


Fig. 10-1. Wire filament and foil filament strain gages. *Courtesy Baldwin-Lima-Hamilton Corporation.*

10-1 The Bonded Resistance Strain Gage

One of the most versatile tools available for the measurement of strain is the bonded resistance strain gage, Fig. 10-1, a device which undergoes a minute change in electrical resistance as its length changes. The physical bases of the gage were observed by Lord Kelvin over 100 years ago; it took some 70 years before ingenious thinking put the concept to work in the form of the strain gage.

There are two essential and inseparable parts to a bonded resistance strain gage: a resistive filament, which may take the form of a fine wire or thin foil, and a backing or carrier to support and electrically insulate the filament.

Gages are available in a variety of shapes and sizes designed to meet specific requirements. There are *general purpose gages*, Fig. 10-2, with grid lengths from $\frac{3}{8}$ to 6 in., and *rosettes*, which consist of two or more gages mounted on a single backing. Etched foil gages, a rather recent development, are also available in a variety of shapes and sizes, some as small as $\frac{1}{64}$ in. gage length. These appear as shown in Fig. 10-3.

A number of different materials are used as carriers of the resistive filaments, and the selection of a backing depends upon the temperature at

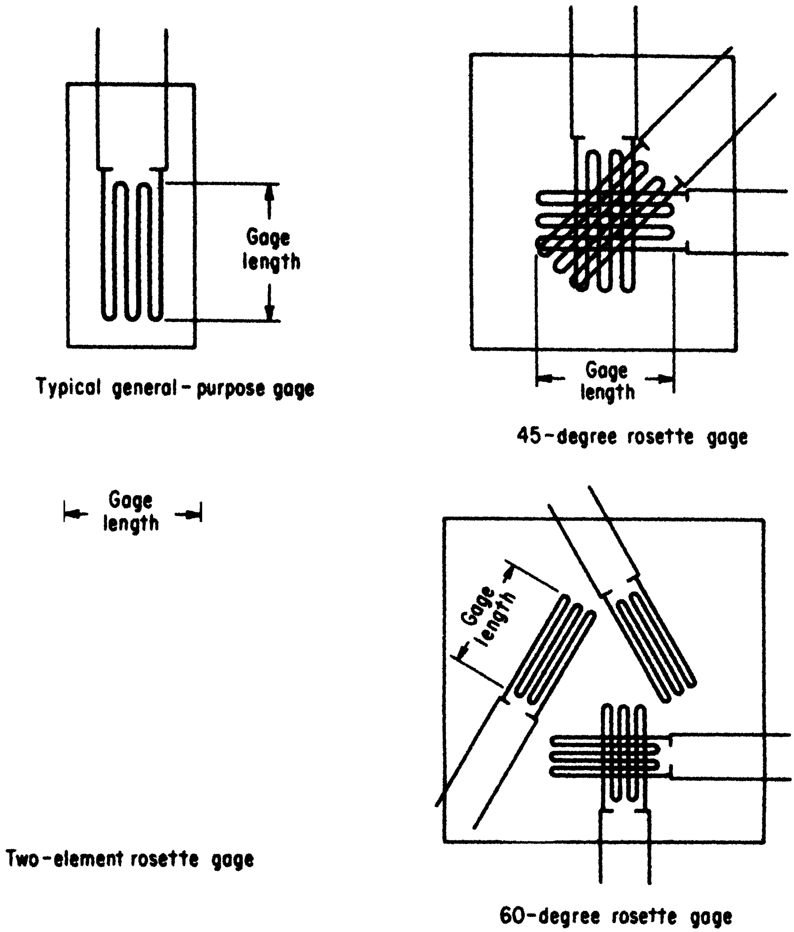


Fig. 10–2. Typical general purpose bonded wire strain gages. *Courtesy Baldwin-Lima-Hamilton Corporation.*

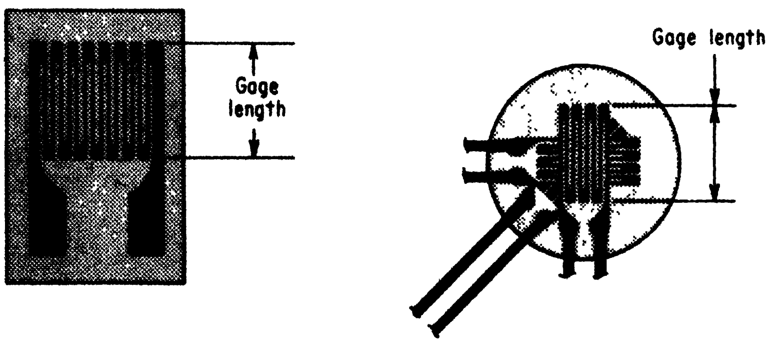
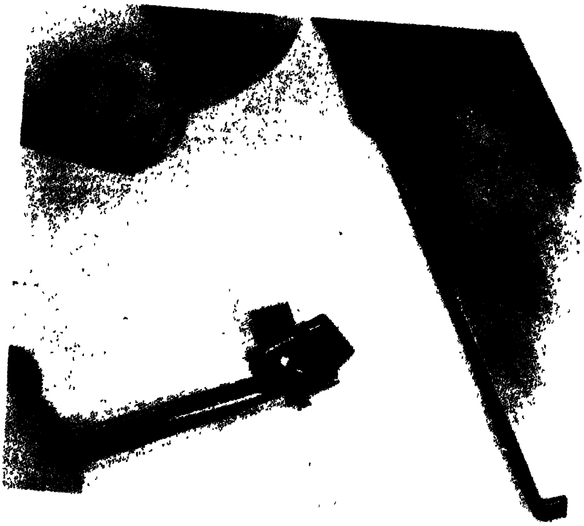
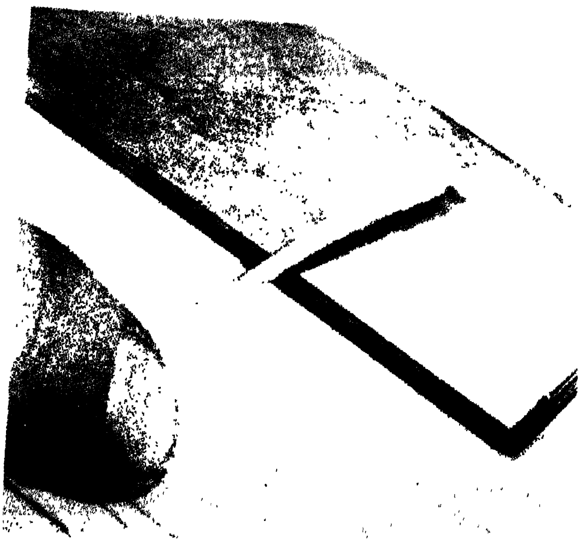


Fig. 10–3. Etched foil gages. *Courtesy Baldwin-Lima-Hamilton Corporation.*



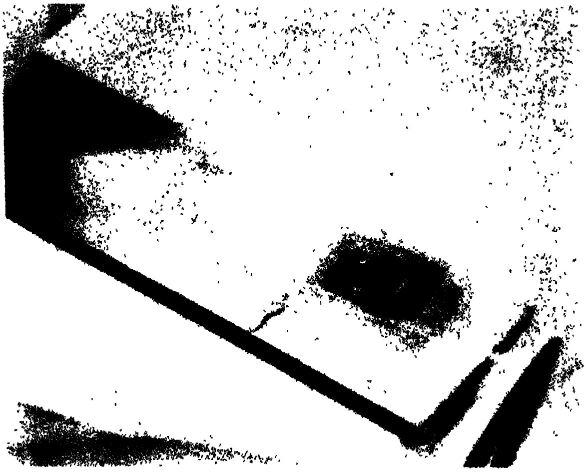
a. Gage has been picked up with transparent tape, plastic protective backing removed, and cement catalyst applied to gage.



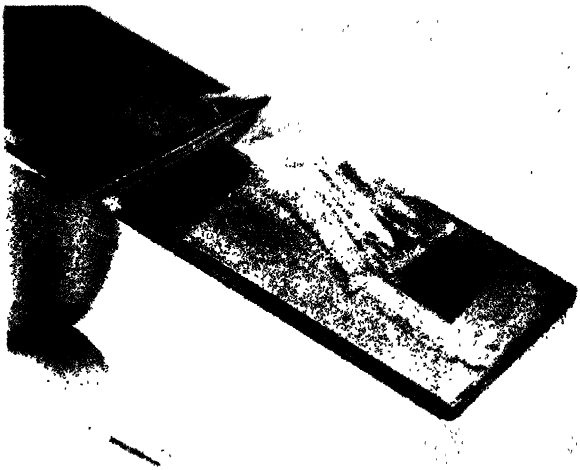
b. Mounting surface has been cleaned and neutralized. Meanwhile catalyst has dried.

Fig. 10-4. Rapid method of installing Budd Metal Film strain gages with catalytic adhesives. *Courtesy Instruments Division of the Budd Company.*

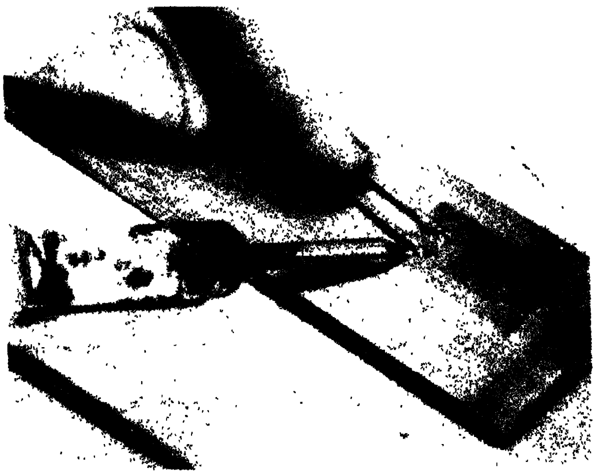
c. Gage has been oriented on surface applied with cement and smoothed out.

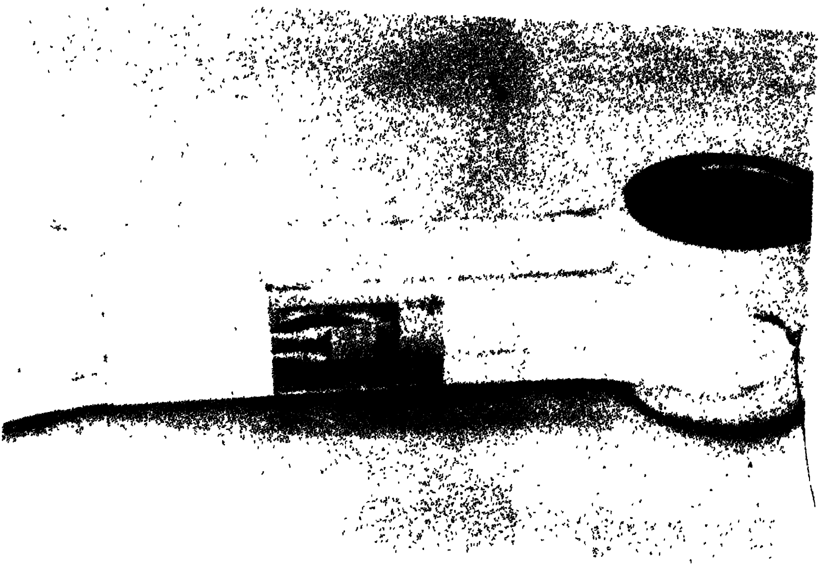


d. While cement sets, lead wires have been tinned. Transparent tape has been peeled from gage.



e. Lead wires have been attached and supported and water-proofing compound applied.





(a)

(b)

Fig. 10–5. Application of etched foil gages for the analysis of strain in (a) an automotive connecting rod, (b) a critical section of a crankshaft journal. *Courtesy General Motors Engineering Staff.*

which the gage is to be used. Paper- or epoxy-backed gages are limited to strain measurements below 180°F, whereas bakelite-based gages can be used successfully up to 450°F. At elevated temperatures ceramics are used to hold and insulate the filaments. Strain gages are sensitive devices which can give true measures of strain only if properly bonded to a clean, scale-free surface. The backing material used in the gage generally dictates the appropriate cement and cementing procedure to use. For paper-backed gages, Duco Household Cement is universally used, and Bakelite cements (Phenol-Resins) are used with Bakelite-backed gages.

There are remarkable adhesives now available which transform, with the addition of catalysts and moderate pressure, from free-flowing liquids to rigid plastics. These adhesives, which are particularly compatible with epoxy-back strain gages, provide a simple and rapid method of gage installation. The five simple steps outlined in Fig. 10-4 illustrate the bonding procedure generally used with these adhesives; Fig. 10-5 shows two typical applications of the epoxy-backed foil gage in the automotive field.

10–2 Strain Gage Instrumentation

A relationship between two physical quantities, resistance and length, forms the basis of strain-gage analysis. The sensitivity of a strain gage is dependent upon its *gage factor* G, a constant defined by the ratio

$$G = \frac{\Delta R/R}{\Delta L/L} \tag{10-1}$$

where R and L are the initial values of resistance and length; ΔR and ΔL are the change in electrical resistance and the accompanying change in length, respectively. Gage factor, initial resistance, and initial length are fixed quantities, established in the manufacture of the gage; the variables in Eq. (10-1) are the change in resistance ΔR and the change in length ΔL. Many American-made gages have nominal resistances of $R = 120$ ohms and gage factors of $G = 2$. The magnitude of ΔR in terms of strain can be found by substituting these two numerical values into Eq. (10-1):

$$\Delta R = G \cdot \frac{\Delta L}{L} \cdot R$$

$$\Delta R = GR\epsilon$$

$$= 2(120)\epsilon$$

For these particular constants, a strain ϵ of 1 microinch (one-millionth of an inch) would be equivalent to a change in resistance of

$$\Delta R = 240 \times 10^{-6} = 0.000240 \text{ ohms}$$

hardly a very large value, yet a measurable one.

A basic circuit capable of detecting and measuring these small changes in resistance is the *Wheatstone bridge*; for the most part, instrumentation, both simple and elaborate, makes use of this circuit. A diagram of a typical arrangement is shown in Fig. 10-6. The strain gage, suitably mounted on a

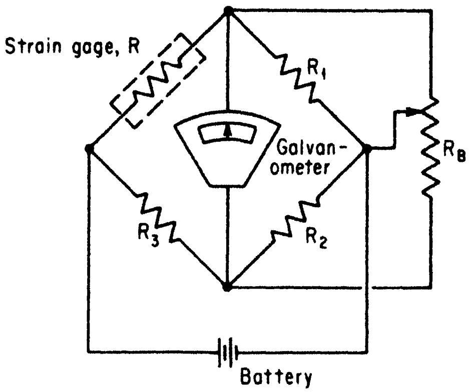


Fig. 10-6

structure to be tested, is used as one leg of the bridge. Fixed resistors R_1, R_2, and R_3, which might be non-acting strain gages, are used as the remaining three legs. A galvanometer, power supply, and balancing resistor R_B complete the circuit. R_B is a very high resistance, usually 50,000 or 100,000 ohms, used to establish a *zero* or *null* balance of the galvanometer. Changes in resistance of the strain gage appear in the circuit as a galvanometer deflection. If the balancing resistor is calibrated, a measured dial rotation can restore the galvanometer to null balance, and strain can be read directly.

Since most materials expand when heated, strain gages are extremely sensitive to temperature variation. A change of one degree Fahrenheit in steel, for example, is equivalent to 6.5 microinches of strain, which, in turn, could be misinterpreted as a stress of 195 psi:

$$\sigma = \epsilon E = (6.5 \times 10^{-6})(30 \times 10^{6}) = 195 \text{ psi}$$

To eliminate temperature errors, a *dummy*, or *compensating gage*, alike in all respects to the measuring gage, is mounted on a material similar to that to be tested. This gage replaces R_3 in the circuit of Fig. 10-6, the hope being that the temperature of the measuring gage and the dummy gage will undergo equal changes and, therefore, will balance out expansion and contraction strains.

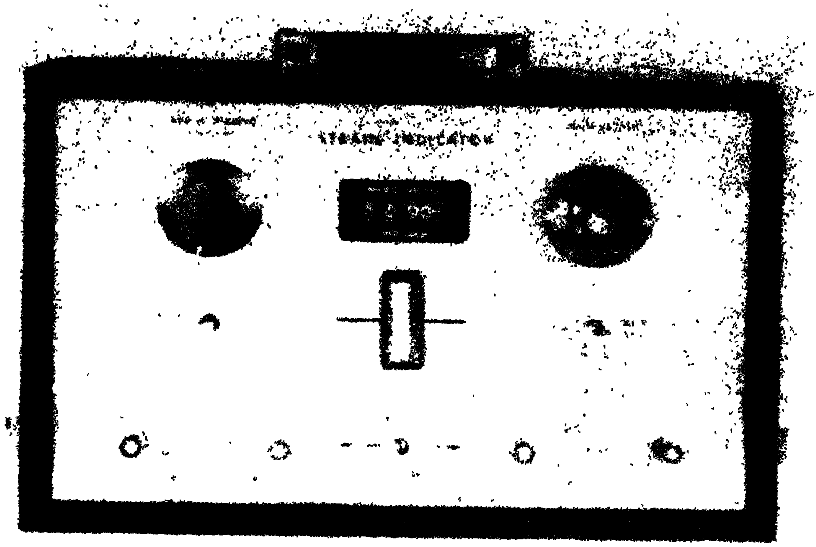


Fig. 10–7. Baldwin Model 120 Strain Indicator. This instrument, which features digital readout, may be used with strain gages in one, two, or four arm networks. Dynamic strains up to 50 cps and 5000 microinches per inch may be seen by coupling the indicator to a standard cathode ray oscilloscope. *Courtesy Baldwin-Lima-Hamilton Corporation.*

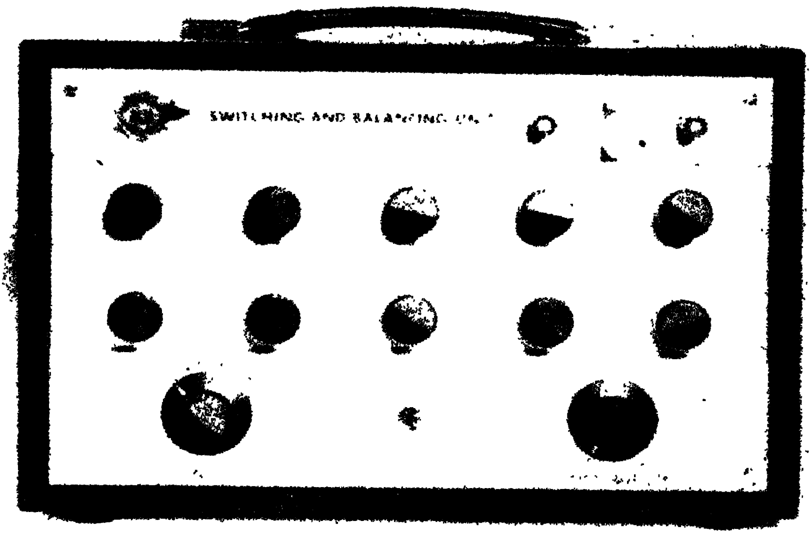


Fig. 10–8. Baldwin Model 225 Switching and Balancing Unit. The unit provides a means for initially balancing each of several bridges to zero and quickly switching active and compensating gages into a strain indicator. *Courtesy Baldwin-Lima-Hamilton Corporation.*

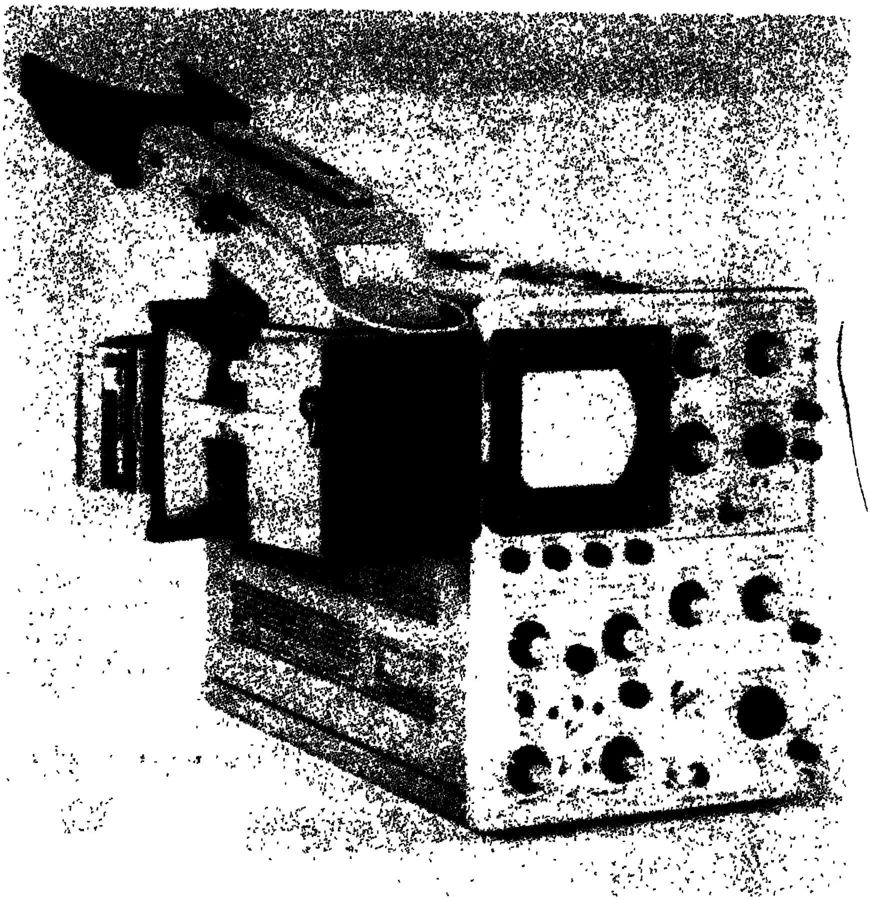


Fig. 10-9. Tektronix oscilloscope, strain gage plug-in unit, and Polaroid Land Camera. The unit is self-contained and requires no external equipment other than strain gages. *Courtesy Tektronix, Inc.*

Instruments are available, like the strain indicator shown in Fig. 10-7, which feature *digital readout* of strain in microinches per inch. This instrument can also be used with a standard cathode ray oscilloscope, and dynamic strains up to 50 cps and 5000 microinches per inch can be seen and measured. A switching and balancing network, like that shown in Fig. 10-8, allows as many as ten gages to be read with a single strain indicator.

A variety of strain gage instrumentation is used for amplifying and recording strain measurements; oscilloscopes with camera attachments, like that shown in Fig. 10-9, are capable of recording high-frequency dynamic strain. Oscillographs, Fig. 10-10, are available which will simultaneously record the output of several gages.

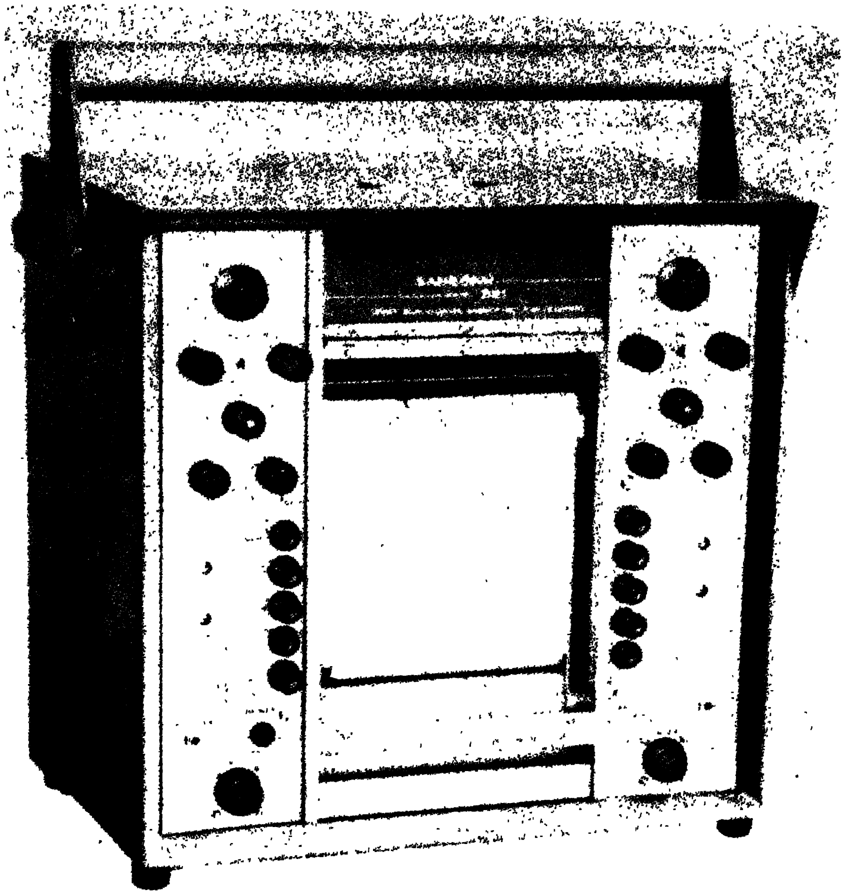


Fig. 10–10. Sanborn 2 Channel Carrier Recorder. This instrument provides the power for two separate strain gage bridges and records the outputs of these bridges. *Courtesy Sanborn Company.*

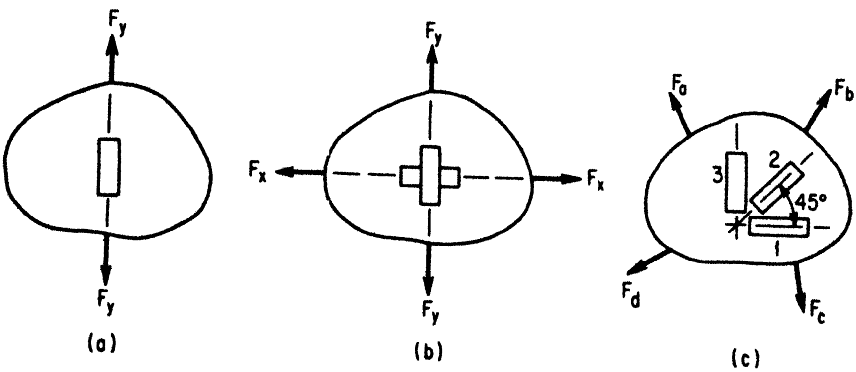


Fig. 10–11

10-3 Interpretation of Strain-Gage Data

For simple stress, like that in Fig. 10-11(a), a single gage mounted in the direction of the load is used. Strain readings are readily converted to stress by multiplying the indicated strain by the modulus of elasticity.

$$\sigma = \epsilon E \tag{10-2}$$

For biaxial stress, Fig. 10-11(b), principle strains in terms of principal stresses are

$$\epsilon_x = \frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E}$$

and (10-3)

$$\epsilon_y = \frac{\sigma_y}{E} - \mu\frac{\sigma_x}{E}$$

where μ is Poisson's ratio.

Since the measured quantity is strain, it is convenient to express stress in terms of strain in Eq. (10-3). By solving simultaneously for σ_x and σ_y, Eq. (10-3) can be rewritten as

$$\sigma_x = \frac{E}{1-\mu^2}(\epsilon_x + \mu\epsilon_y)$$

(10-4)

$$\sigma_y = \frac{E}{1-\mu^2}(\epsilon_y + \mu\epsilon_x)$$

Numerical values of measured strain ϵ_x and ϵ_y can be substituted into Eq. (10-4), and the values of stress can be computed.

When directions of principal stresses are not known, as is frequently the case, the problem becomes somewhat more complicated. A minimum of three strain readings in known directions is required for the analysis. A variety of rosette gages are available for this purpose, and each has a characteristic set of equations for stress based on indicated strain. Principal stresses in terms of indicated strains ϵ_1, ϵ_2, and ϵ_3 for a rectangular rosette, Fig. 10-11(c), are

$$\sigma_{\substack{max\\min}} = \frac{E}{2}\left\{\frac{\epsilon_1 + \epsilon_3}{1-\mu} \pm \frac{1}{1+\mu}\sqrt{(\epsilon_1 - \epsilon_3)^2 + [2\epsilon_2 - (\epsilon_1 + \epsilon_3)]^2}\right\}$$

(10-5)

$$\tau_{max} = \frac{E}{2(1+\mu)}\sqrt{(\epsilon_1 - \epsilon_3)^2 + [2\epsilon_2 - (\epsilon_1 + \epsilon_3)]^2}$$

θ_p (the angle from gage 1 to σ_{max} axis) is equal to

$$\frac{1}{2}\tan^{-1}\left[\frac{2\epsilon_2 - (\epsilon_1 + \epsilon_3)}{\epsilon_1 - \epsilon_3}\right] \tag{10-6}$$

The slide rule, scratch paper, and eraser approach to these equations can be rather tedious; fortunately, however, electronic computers can be programmed to handle hundreds of these and similar calculations in a matter of seconds.

10–4 Brittle Coatings

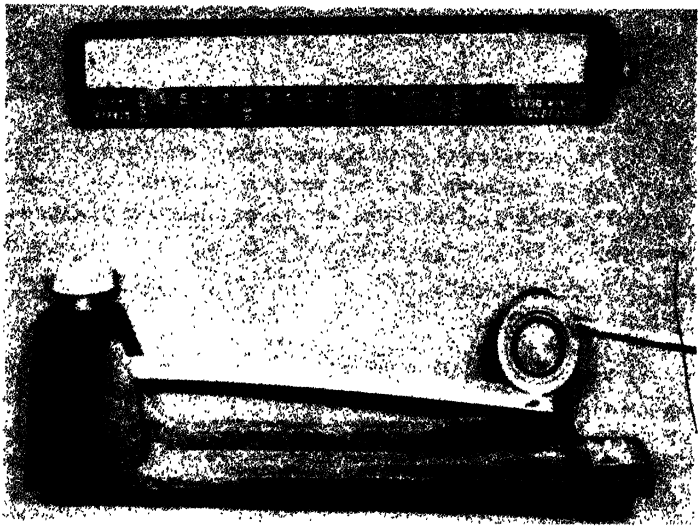
The crazing and cracking in the glazes on antique pottery is caused by unequal expansion rates. The glaze, which is a thin, brittle, glasslike substance with little tensile strength, cracks as the body of the pottery expands with temperature. A simple observation like this led to the brittle coating techniques used in stress analysis. These methods involve coating a part to be studied with a brittle lacquerlike substance.[1] Calibration strips are always processed along with the part to be tested, and when dry, these strips are subjected to a known strain in a test fixture, like that shown in Fig. 10-12(a). The brittle coating fractures under the strain, and note is made of the minimum value of stress required to produce a failure in the coating.

Brittle coatings respond to both static and dynamic loading, and the usual procedure in static testing is to observe the start and the spread of the cracks as measured loads are applied. Since it is difficult to apply measured dynamic loads, several coatings, each with a different strain sensitivity, are used. This technique gives a quantitative picture of the magnitude as well as the distribution of strain.

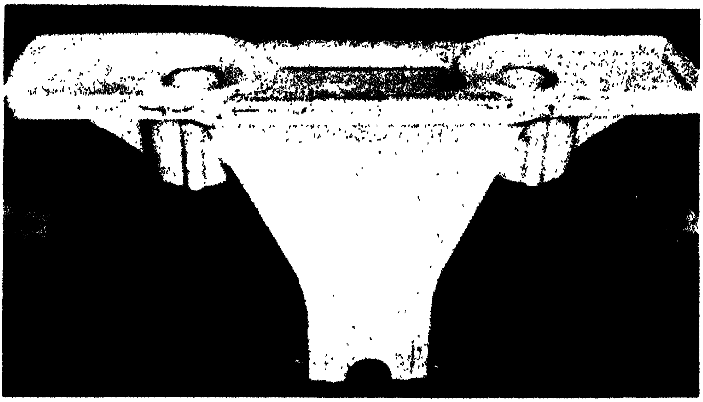
With careful handling, brittle coatings are capable of a quantitative accuracy of 10 per cent. Where more precise results are desired, strain gages can be applied at critical regions indicated by the coating.

Ceramic-base coatings, a rather recent development, are available for the analysis of strain at temperatures up to 700°F. These coatings are, in reality, glazes suspended in a vehicle that is sprayed on the part, dried, and then fired at 1000°F, a process similar to porcelain enameling. Success of the ceramic coating depends upon a controlled difference between the coefficient of expansion of the coating and that of the base metal. If the coefficient of expansion of the coating is greater than that of the base metal, residual tension will be *locked in* the coating when it cools from the firing

[1] These coatings are sold under the trade name of Stresscoat and are manufactured by Magnaflux Corporation, Chicago, Illinois.



(a)



(b)

Fig. 10–12. (a) View of Stresscoat Calibrator and strain scale, with test strip in place. (b) Stresscoat crack patterns on a heavily loaded aluminum beam saddle for the double rear axle tandem suspension of a truck. *Courtesy Magnaflux Corporation.*

temperature. This develops a coating which cracks at low strain levels. If, on the other hand, the coefficient of expansion of the coating is less than that of the base metal, residual compression will be locked in the coating, which then requires higher strain levels to initiate crack patterns.

10–5 Photoelastic Stress Analysis

The photoelastic approach requires exacting techniques and is the most mathematical of the methods discussed. Basis of the method is the fact that certain transparent materials, like glass and Bakelite, become *birefringent*, or *doubly refracting*, when stressed. This simply means that these substances will divide an incident ray of light into two beams, which travel at different velocities through the material. In addition, the two beams transmitted through the stressed body are *polarized* at right angles to one another, as shown in Fig. 10-13.

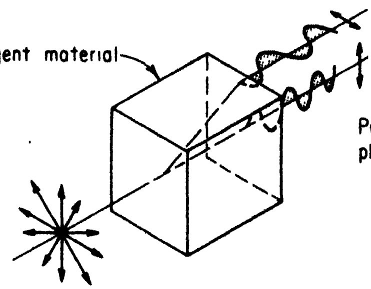


Fig. 10–13

In photoelastic stress analysis, polarized light is used to reveal the presence of strains in a transparent *model* of a part to be studied. The light is split into two components, which vibrate in the directions of the principal stresses. A second polarizing screen, called *an analyzer*, permits the passage of only the component of light in its plane of polarization. A pattern of alternate bright and dark lines indicates the variation in refraction produced in the strained model. Figure 10-14 is a photoelastic stress pattern of an "eye-bar" in tension. The dark bands, called *stress fringes*, or *isochromatics*, pass through points of constant difference in principal stress. A variety of techniques, some borrowed from the field of physical optics, some developed especially for photoelasticity, are used to translate fringe patterns into stress magnitudes. In general, optical observations, mechanical measurements, and mathematical methods are involved in the interpretation of data. Also involved is the mechanical ability to machine accurate, stress-free models and calibration bars.

The photoelastic method has made an enormous contribution to the theories of mechanics of materials, and in particular to the development of

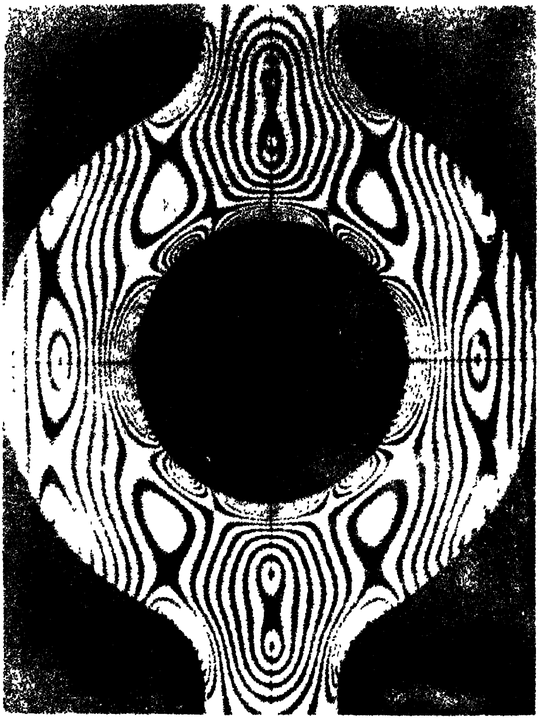


Fig. 10-14. Photoelactic stress pattern of an "eyebar" in tension. *Courtesy Prof. W. M. Murray, MIT.*

stress-concentration factors. It does not, however, easily lend itself as a method to be used by the untrained or novice in the field of stress analysis.[1]

10-6 Photoelastic Coatings

New techniques have recently been developed in which the *actual structure* to be stress-analyzed is covered with a photoelastic coating. When loads are applied to the structure, strains are transmitted to the plastic coating, which then becomes birefringent. The strain distribution, which is visible when the plastic is illuminated with suitable polarizing instruments, appears as a colored fringe pattern. These fringes, called *isochromatics*, are the loci of points where the difference in principal strains is constant; an area of uniform color denotes an area of uniform shear strain. Since the *birefringence* (or color) is directly proportional to the difference in principal strains, the method can produce quantitative results. A typical fringe pattern on an actual part is shown in Fig. 10-15.

The photoelastic coating, in effect, acts as an infinite number of strain gages uniformly distributed over the surface being studied. In many

[1] For a more detailed, non-technical discussion of the photoelastic method, see *Photoelastic Stress Analysis*, Eastman Kodak Co.

Fig. 10-15. Photostress pattern on a portion of the main landing gear of a supersonic bomber. High stress concentrations, particularly in the fillet area, are indicated by the close fringe pattern. *Courtesy Instruments Division of the Budd Company.*

respects, the technique combines the functions of photoelasticity, brittle lacquers, and electrical strain gages. The process does have its limitations, however, since stresses cannot be measured in areas of an assembly which are inaccessible to light. Correction factors must also be introduced for measurement at temperatures above 100°F, and errors in measurement are inherent when the thickness of the plastic is comparable to the thickness of the part under study.

The photoelastic coating, sold under the trade name of Photostress,[1] can be applied in either of two ways, depending upon the contour of the structure. If flat surfaces are to be studied, the plastic in sheet form can be bonded directly to the part. When the surfaces to be analyzed are curved, the plastic can be brushed on and allowed to polymerize directly on the part. An alternate method involves the pouring of a flat sheet of plastic and allowing it to polymerize partially; the flexible sheet can then be formed about the complex surface and allowed to cure.

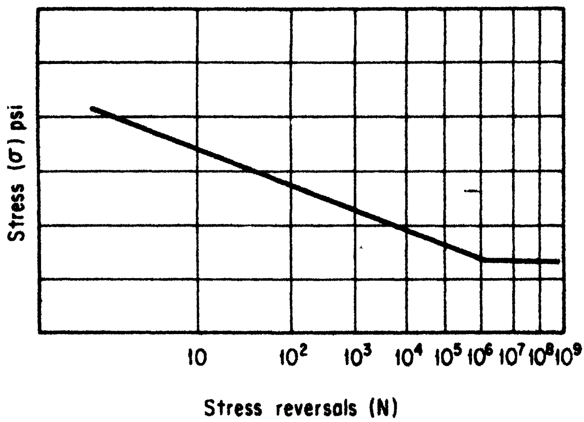
10-7 The Measurement of Fatigue Strength

At one time or another everyone has bent a paper clip, a wire, or a nail back and forth until it broke. The mechanism of failure, called *fatigue*, is

[1] Photostress is produced by Instruments Division of The Budd Company, Phoenixville, Pa.

the result of a cyclic reversal of stress from tension to compression. The exact mechanism of fatigue failure is not completely understood; it seems reasonable, however, to assume the failure to progress in three stages: (1) the start of a surface crack; (2) the propagation of the crack under repeated stress; (3) final rupture of the weakened section. There is little argument concerning the second and third stages; the question debated is how a crack can start under cyclic stresses lower than the static rupture stress. Attempts have been made to correlate fatigue characteristics of materials with other engineering properties, such as tensile strength, yield strength, impact strength, and so on; for the most part, these attempts have been unsuccessful. Some general observations, based on a mountainous amount of experimental data, indicate that (a) most structural materials, like metals, wood, and some plastics, will fail by cracking under cyclic loads at stresses lower than the ultimate tensile stress; (b) fatigue in metals generally depends upon the number of cycles of load change in a given stress range rather than the rate of loading; (c) most metals have a safe range of stress, called the *fatigue limit*, or *endurance limit*, below which failure will not occur even after a large number of cycles; (d) notches, grooves, stamp marks, and machining marks, appreciably decrease the fatigue strength.

The knowledge of fatigue characteristics of materials is largely dependent upon experimental data obtained by repeated loading in either bending, tension, or torsion. Data are usually presented in graphic form in which fatigue life in terms of cycles N of load reversal is plotted against a stress σ to cause failure. Because of the large number of cycles involved, N is plotted on a logarithmic scale. The curves, referred to as S–N diagrams,



Typical S–N Diagram

Fig. 10–16

appear as shown in Fig. 10-16. For most metals, a limiting stress exists, below which a material will endure what appears to be an infinite number of stress cycles. This stress is called the *fatigue limit*, or *endurance limit*.

The *S–N* diagram can only be of value in designing against fatigue if the part in question is always subjected to a constant fatigue cycle. This condition is rarely the case, since most structural members, for instance, the automobile spring, are stressed at a variety of levels under normal operations; this variety of levels has a decided effect on fatigue life. Experience and experimentation have disclosed that the two most important factors affecting fatigue life are surface roughness and geometric discontinuity. Carefully finished surfaces and liberal fillets are common features of a good design when dynamic loads are involved.

APPENDIX A

Review Problems

Each problem in this section uses one or more of the basic concepts discussed in the text. No attempt has been made to present this review in a preferred order of topic sequence or difficulty, but rather to represent the subject of mechanics of materials in its most general form.

More than twenty state engineering registration boards have generously supplied the problems in this section. Therefore, the problems represent the types of questions which will probably be encountered in the portion of the examination devoted to mechanics of materials.

A-1. A wood-stave tank, Fig. A-1, filled with water is bound together with threaded steel rods $\frac{3}{4}$ in. in diameter. If the allowable tensile stress for the steel is 20 ksi, how many rods are required to resist the water pressure?

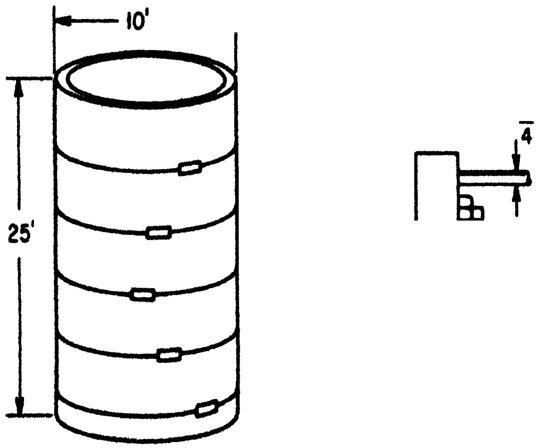


Fig. A-1

A-2. Two plates are welded together, Fig. A-2, to carry a concrete block wall over a doorway. If the simple effective span length is 6 ft and the load per foot is 400 lb, find: (a) the maximum tensile stress; (b) the maximum compressive stress; (c) the shearing stress between the plates.

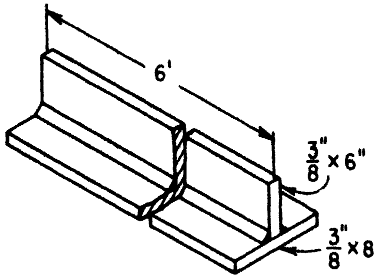


Fig. A-2

A-3. A perfectly straight column, rigidly fixed at the bottom and free at the top, must support a concentric load of 35 kips. The column is a 4 in. o.d. steel pipe with $\frac{1}{4}$-in.-thick walls. The steel has a modulus of elasticity of 30×10^6 psi and an elastic limit of 40,000 psi. How long can the column be and still have a factor of safety against buckling of 1.5?

A-4. Two 4-in.-diameter solid shafts are connected through a coupling with six $\frac{3}{4}$ in. bolts located on a 10 in. bolt circle. If the shafts rotate at 150 rpm, what horsepower can be transmitted without exceeding a shearing stress of 10,000 psi in the bolts or in the shafts?

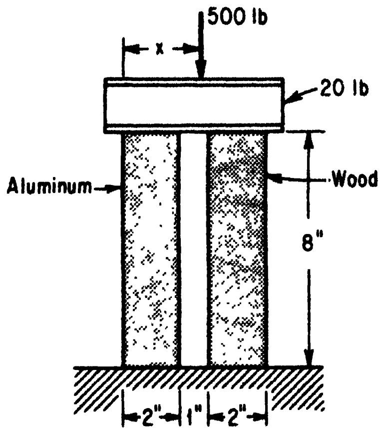


Fig. A-5

A-5. A block of wood and a block of aluminum, each 2 in. by 4 in. by 8 in., support a 20 lb beam and a concentrated load of 500 lb, as shown in Fig.

A-5. If the beam is assumed to be rigid and level before the 500 lb load is applied, where should the load be placed if the beam is to remain horizontal? $E_a = 10 \times 10^6$ psi; $E_w = 2 \times 10^6$ psi.

A-6. A timber cantilever beam 10 in. wide by 2 in. deep and 10 ft long supports a load of 150 lb at a point 3 ft from the free end. Find: (a) the slope at the free end; (b) the deflection at the free end; (c) the maximum bending stress. $E = 1.5 \times 10^6$ psi.

A-7. A short timber compression member has a cross section of 6 in. by 8 in. It is reinforced by the addition of two $\frac{1}{2}$ in. by 6 in. steel plates placed to form a composite post. What is the maximum load this member can carry? The limiting stresses for timber and steel are $\sigma_t = 1000$ psi and $\sigma_s = 18{,}000$ psi. $E_t = 1.5 \times 10^6$ psi; $E_s = 30 \times 10^6$ psi.

A-8. A circular open-link chain is made of $\frac{3}{8}$-in.-round bar, and the links have an outside diameter of $1\frac{3}{4}$ in. If the maximum tensile stress is limited to 30,000 psi, what is the greatest load that can be safely supported by the chain?

A-9. Determine the size of a circular steel shaft to transmit 100 hp at 1000 rpm with an allowable extreme fiber stress in shear of 10,000 psi. The angle of twist must not be greater than 1 degree per foot of length. $G = 12 \times 10^6$ psi.

A-10. A square steel bar of 1 in. by 1 in. cross section and 6 ft long is to be used as a column. The ends are perfectly free to rotate, but may not be displaced. Find the maximum load this column may support if the allowable stress is 30,000 psi. $E = 30 \times 10^6$ psi.

A-11. A steel punch has a diameter of 0.750 in. In punching a hole in a plate, a total compressive force of 35,000 lb acts on the punch. What is the actual diameter when the load is applied? $E = 30 \times 10^6$ psi; $\mu = 0.25$.

A-12. A cantilever beam 8 ft long is fixed at the right end. It carries a uniformly distributed load of 100 lb per ft, including its own weight, and a concentrated load of 1000 lb at the free end. Write the bending-moment equation for any point along the beam as a function of the length x measured from the free end.

A-13. An 8 in., 40 lb wide-flange beam, used as a column, is 30 ft long. It is supported at the middle in a direction normal to the web but is unsupported in the direction parallel to the web. Find the safe load for this column by means of Euler's equation. Assume a factor of safety of 3 and "round-end" conditions. $E = 30 \times 10^6$ psi; $\sigma_{pl} = 30{,}000$ psi.

A-14. A vertical, steel, cylindrical standpipe is 20 ft in diameter and 50 ft high. Compute the maximum circumferential stress exerted within the walls when the tank is filled with water. The walls are $\frac{3}{8}$ in. thick.

A-15. An aluminum bar 10 in. long is placed on a steel bar 12 in. long, as shown in Fig. A-15. Both bars have the same cross section. A gap of 0.01 in. exists between the top of the aluminum bar and the rigid support. What stress is produced in the aluminum when the temperature is raised 100°F? $E_s = 30 \times 10^6$ psi; $E_a = 10 \times 10^6$ psi; $a_s = 6.5 \times 10^{-6}$ in./in./°F; $a_a = 12.5 \times 10^{-6}$ in./in./°F.

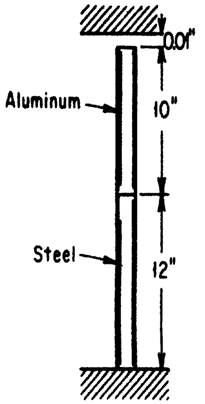


Fig. A-15

A-16. An I-beam 14 in. deep and 20 ft long is simply supported at its ends. The beam has a moment of inertia of 440 in.^{4} (a) What load may be placed at midspan if the deflection is limited to $\frac{1}{4}$ in.? (b) What is the maximum bending stress for this value of load? (c) What distributed load would produce the same deflection? (d) What is the maximum bending stress associated with the distributed load of part (c)? Neglect the weight of the beam.

A-17. A 1-in.-diameter steel pipe 4 ft long acts as a spreader bar, Fig. A-17. What pull P may be applied through the cables and connectors? Use Euler's formula for pinned-end conditions and assume a factor of safety of 3. $I = 0.087$ in.^{4}; $A = 0.494$ in.^{2}

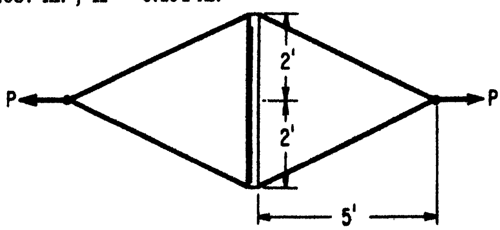


Fig. A-17

A-18. A short length of 7-in.-diameter steel propeller shaft, turning at 160 rpm, is subjected to an axial compressive load of 231 tons. Find the greatest horsepower that may be transmitted if the maximum shearing stress must not exceed 10,000 psi nor the maximum normal stress exceed 13,500 psi.

A-19. A steel beam 16 ft long is fixed against rotation at one end and simply supported at the other. The load is uniformly distributed over the entire length, and its magnitude is 1000 lb per ft. The moment of inertia of the beam section is 100 in.^{4} Determine the reaction at the simply supported end if the end settles $\frac{1}{2}$ in. when the load is applied.

A-20. Calculate the spacing of $\frac{7}{8}$-in.-diameter rivets required to resist a girder shear of 70 kips in the composite section shown in Fig. A-20. The permissible shearing stress is 8000 psi.

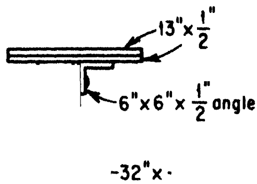


Fig. A-20

A-21. Select the most economical wide-flange steel beam capable of supporting the loads shown in Fig. A-21. The limiting bending stress is 20 ksi. The beam is braced, and it is not necessary to check for deflection or shear. Neglect the weight of the beam.

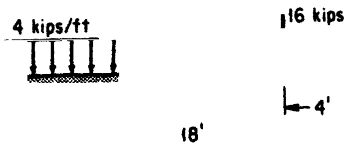


Fig. A-21

A-22. A steel column section consists of a 12 in. by $\frac{1}{2}$ in. web plate, four 6 in.

by 4 in. by ½ in. angles, with the 4 in. leg attached to the web, and two 14 in. by ½ in. cover plates. What is the least radius of gyration of the section?

A-23. A 10 in., 25.4 lb standard I-beam is supported as shown in Fig. A-23. Determine the deflection at C. Neglect the weight of the beam.

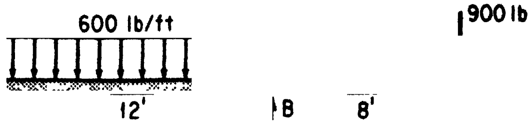


Fig. A-23

A-24. If the allowable bending stress is 15,000 psi, what is the required section modulus for the beam of Fig. A-24?

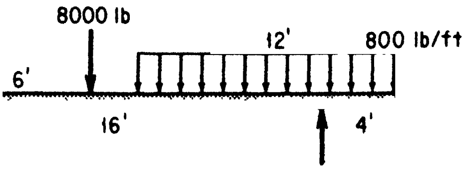


Fig. A-24

A-25. A flanged and bolted coupling is to be used to connect two solid shafts, each 3 in. in diameter. How many ¾ in. bolts must be used in a 6 in. bolt circle, if the shearing stresses in the shaft and in the bolts are not to exceed 12,000 psi and 8000 psi respectively?

A-26. The frame of a riveting machine, Fig. A-26, is acted upon by the forces shown. Find the maximum tensile and compressive stresses at section *a–a*.

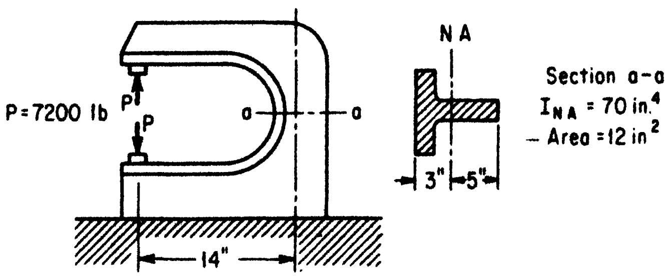


Fig. A-26

A-27. The composite compression member shown in Fig. A-27 consists of a block, 4 in. by 4 in. by 6 in., of each of the following materials: steel, copper, bronze, and aluminum. A load of 800,000 lb is applied in a way to cause each block to decrease in height the same amount. Find the stress in each block $E_s = 30 \times 10^6$ psi; $E_c = 15 \times 10^6$ psi; $E_b = 12 \times 10^6$ psi; $E_a = 10 \times 10^6$ psi.

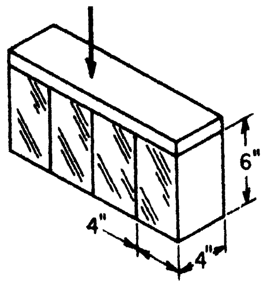


Fig. A-27

A-28. A Douglas fir column $4\frac{1}{2}$ in. by $9\frac{1}{2}$ in. with a modulus of $E = 1.6 \times 10^6$ psi and a spruce column of the same size with a modulus of $E = 1.2 \times 10^6$ psi are bolted together to form a short column. If lateral bending is prevented, what portion of a load of 70,000 lb will each carry?

A-29. A load of 12 kips is supported on a bracket, which, in turn, is fastened to a column by a single vertical row of 5 rivets spaced on 3 in. centers. The load is 9 in. out from a vertical center line that passes through the rivets. What is the maximum shearing force developed in the rivets?

A-30. The rungs of a ladder are $1\frac{1}{8}$ in. in diameter and are carried by side rails spaced a maximum of 12 in. apart. The limiting bending stress in the rungs is 2400 psi. What is the maximum centrally placed concentrated load which may be applied?

A-31. A 6 in. by 12 in. Douglas fir beam (1600 psi stress grade) is reinforced by the addition of two 12 in., 20.7 lb channels properly bolted to the wood beam. What is the relative strength of the reinforced beam to the original wood beam? The allowable stress in the steel is 20,000 psi, and the modulus of the wood is $E = 1.5 \times 10^6$ psi.

A-32. A series of Douglas fir joists 16 ft long carry a uniformly distributed load of 45 lb per sq ft plus a cross-partition 4 ft from one end which carries a load of 300 lb per running ft. If the joists are spaced 16 in. on center, what size joists ($1\frac{5}{8}$ in. wide) are needed? The stresses are limited to 1600 psi in bending and 100 psi in shear.

A-33. A horizontal beam is 22 ft long. It is supported at a point 4 ft from the left end and 6 ft from the right. The beam supports a concentrated load of 6000 lb at its center and a distributed load of 1000 lb per ft from the left support to the extreme right end. Draw the shear and moment diagrams and determine maximum values in each.

APPENDIX B

Tables

1. Typical Physical Properties of Metals
2. Typical Physical Properties of Timber
3. American Standard Steel I-Beams, Properties for Designing
4. Steel Wide-Flange Beams, Properties for Designing
5. American Standard Steel Channels, Properties for Designing
6. Steel Angles with Equal Legs, Properties for Designing
7. Steel Angles with Unequal Legs, Properties for Designing
8. Properties of Areas
9. Deflections of Variously Loaded Beams

Tables 3 through 7 are taken from the *AISC Manual of Steel Construction* and are reproduced by permission of The American Institute of Steel Construction.

Table 1

Typical Physical Properties of Metals

Material	*Nominal composition (essential elements) per cent*	*Form and condition*	*Typical mechanical properties*			*Typical physical constants*					
			Yield strength (0.2% offset) 1000 psi	Tensile strength 1000 psi	Elongation in 2 in., per cent	Hardness brinell	Density lb/cu in.	Specific gravity	Thermal expansion coefficient (32° – 212°F) $x10^{-6}$ in./in./°F	Tensile modulus of elasticity $x\ 10^6$ psi	Torsional modulus of elasticity $x\ 10^6$ psi
Carbon steel AISI–SAE 1020	Fe —Bal. Mn—0.45 Si —0.25 C —0.20	Annealed Hot-rolled Hardened	38 42 62	65 68 90	30 32 25	130 135 179	0.284	7.86	6.7	30	11.6
Cast gray iron	C —3.4 Si —1.8 Mn—0.5 Fe —Bal.	Cast		25 min	0.5 max	180	0.260	7.20	6.7	13	
Ingot iron	Fe —99.9 plus	Hot-rolled Annealed	29 19	45 38	26 45	90 67	0.284	7.86	6.8	30.1	11.8
Malleable iron	C —2.5 Si —1.0 Mn—0.55 max	Cast	33	52	12	130	0.264	7.32	6.6	25	
Stainless steel type 431	Fe —Bal. Cr —16 Ni — 2	Annealed Heat-treated	85 150	120 195	25 20	250 400	0.280	7.75	6.5	29	10.5

Wrought iron	Fe —Bal. Slag—2.5	Hot-rolled	30	48	30	100	0.278	7.70	6.35	29	
Aluminum alloy	Al —99 plus	Annealed–0 Cold-rolled–H14 Cold-rolled–H18	5 17 22	13 18 24	45 20 15	23 32 44	0.098	2.71	13.1	10	3.8
Copper	Cu —99.9 plus	Annealed Cold-drawn Cold-rolled	10 40 40	32 45 46	45 15 5	42 90 100	0.322	8.91	9.3	17	6.4
Magnesium alloy	Mg —Bal. Al —9.0 Zn —2.0 Mn—0.10 min	Sand-cast	14	24	6	50	0.066	1.83	14.5	6.5	2.4
Manganese bronze	Cu —58.5 Zn —39.2 Fe —1.0 Sn —1.0 Mn—0.3	Annealed Cold-drawn	30 50	60 80	30 20	95 180	0.302	8.36	11.2	15	5.6
Nickel (pure)	Ni —99.99	Annealed	8.5	46	30		0.322	8.91	7.4	30	11
Red brass (wrought)	Cu —85 Zn —15	Annealed Cold-drawn Cold-rolled	15 55 60	40 70 75	50 15 7	50 120 135	0.316	8.75	9.8	17	6.4
Titanium (commercially pure)	Ti —Bal. Fe —0.2 max N_2 —0.05 max C —0.08 max H_2 —0.015 max	Annealed	70	90	23	200	0.163	4.54	5.0	16.5	6.6

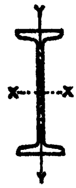
Table 2

Typical Properties of Structural Timber

Species	*Allowable unit stresses* (psi)				*Modulus of elasticity* (psi)
	Tension	Horizontal shear	Compression (perpendicular to grain)	Compression (parallel to grain)	
Douglas fir	2000	120	450	1500	1.8×10^6
Hemlock	1300	80	360	850	1.2×10^6
Pine (southern)	3000	160	470	2300	1.8×10^6
Pine (Norway)	1200	75	360	900	1.3×10^6
Redwood	1700	110	320	1500	1.3×10^6

Table 3

I AMERICAN STANDARD STEEL I BEAMS PROPERTIES FOR DESIGNING



Nominal Size*	Weight per Foot	Area	Depth	Flange Width	Flange Thickness	Web Thickness	Axis X-X I	Axis X-X I/c	Axis X-X r	Axis Y-Y I	Axis Y-Y I/c	Axis Y-Y r
In.	Lb.	In.²	In.	In.	In.	In.	In.⁴	In.³	In.	In.⁴	In.³	In.
24 x 7⅞	120.0	35.13	24.00	8.048	1.102	.798	3010.8	250.9	9.26	84.9	21.1	1.56
	105.9	30.98	24.00	7.875	1.102	.625	2811.5	234.3	9.53	78.9	20.0	1.60
24 x 7	100.0	29.25	24.00	7.247	.871	.747	2371.8	197.6	9.05	48.4	13.4	1.29
	90.0	26.30	24.00	7.124	.871	.624	2230.1	185.8	9.21	45.5	12.8	1.32
	79.9	23.33	24.00	7.000	.871	.500	2087.2	173.9	9.46	42.9	12.2	1.36
20 x 7	95.0	27.74	20.00	7.200	.916	.800	1599.7	160.0	7.59	50.5	14.0	1.35
	85.0	24.80	20.00	7.053	.916	.653	1501.7	150.2	7.78	47.0	13.3	1.38
20 x 6¼	75.0	21.90	20.00	6.391	.789	.641	1263.5	126.3	7.60	30.1	9.4	1.17
	65.4	19.08	20.00	6.250	.789	.500	1169.5	116.9	7.83	27.9	8.9	1.21
18 x 6	70.0	20.46	18.00	6.251	.691	.711	917.5	101.9	6.70	24.5	7.8	1.09
	54.7	15.94	18.00	6.000	.691	.460	795.5	88.4	7.07	21.2	7.1	1.15
15 x 5½	50.0	14.59	15.00	5.640	.622	.550	481.1	64.2	5.74	16.0	5.7	1.05
	42.9	12.49	15.00	5.500	.622	.410	441.8	58.9	5.95	14.6	5.3	1.08
12 x 5¼	50.0	14.57	12.00	5.477	.659	.687	301.6	50.3	4.55	16.0	5.8	1.05
	40.8	11.84	12.00	5.250	.659	.460	268.9	44.8	4.77	13.8	5.3	1.08
12 x 5	35.0	10.20	12.00	5.078	.544	.428	227.0	37.8	4.72	10.0	3.9	.99
	31.8	9.26	12.00	5.000	.544	.350	215.8	36.0	4.83	9.5	3.8	1.01
10 x 4⅝	35.0	10.22	10.00	4.944	.491	.594	145.8	29.2	3.78	8.5	3.4	.91
	25.4	7.38	10.00	4.660	.491	.310	122.1	24.4	4.07	6.9	3.0	.97
8 x 4	23.0	6.71	8.00	4.171	.425	.441	64.2	16.0	3.09	4.4	2.1	.81
	18.4	5.34	8.00	4.000	.425	.270	56.9	14.2	3.26	3.8	1.9	.84
7 x 3⅝	20.0	5.83	7.00	3.860	.392	.450	41.9	12.0	2.68	3.1	1.6	.74
	15.3	4.43	7.00	3.660	.392	.250	36.2	10.4	2.86	2.7	1.5	.78
6 x 3⅜	17.25	5.02	6.00	3.565	.359	.465	26.0	8.7	2.28	2.3	1.3	.68
	12.5	3.61	6.00	3.330	.359	.230	21.8	7.3	2.46	1.8	1.1	.72
5 x 3	14.75	4.29	5.00	3.284	.326	.494	15.0	6.0	1.87	1.7	1.0	.63
	10.0	2.87	5.00	3.000	.326	.210	12.1	4.8	2.05	1.2	.82	.65
4 x 2⅝	9.5	2.76	4.00	2.796	.293	.326	6.7	3.3	1.56	.91	.65	.58
	7.7	2.21	4.00	2.660	.293	.190	6.0	3.0	1.64	.77	.58	.59
3 x 2⅜	7.5	2.17	3.00	2.509	.260	.349	2.9	1.9	1.15	.59	.47	.52
	5.7	1.64	3.00	2.330	.260	.170	2.5	1.7	1.23	.46	.40	.53

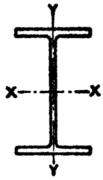
* Steel I-beams are designated by giving their depth in inches first; then the letter I to designate an I-beam; then the weight in pounds per linear foot. For example, 24 I 120.0.

Table 4

STEEL WIDE FLANGE BEAMS

PROPERTIES FOR DESIGNING

(ABRIDGED LIST)



Nominal Size*	Weight per Foot	Area	Depth	Flange Width	Flange Thickness	Web Thickness	Axis X-X I	Axis X-X I/c	Axis X-X r	Axis Y-Y I	Axis Y-Y I/c	Axis Y-Y r
In.	Lb.	In.²	In.	In.	In.	In.	In.⁴	In.³	In.	In.⁴	In.³	In.
36 x 16½	230	67.73	35.88	16.475	1.260	.765	14988.4	835.5	14.88	870.9	105.7	3.59
36 x 12	150	44.16	35.84	11.972	.940	.625	9012.1	502.9	14.29	250.4	41.8	2.38
33 x 15¾	200	58.79	33.00	15.750	1.150	.715	11048.2	669.6	13.71	691.7	87.8	3.43
33 x 11½	130	38.26	33.10	11.510	.855	.580	6699.0	404.8	13.23	201.4	35.0	2.29
30 x 15	172	50.65	29.88	14.985	1.065	.655	7891.5	528.2	12.48	550.1	73.4	3.30
30 x 10½	108	31.77	29.82	10.484	.760	.548	4461.0	299.2	11.85	135.1	25.8	2.06
27 x 14	145	42.68	26.88	13.965	.975	.600	5414.3	402.9	11.26	406.9	58.3	3.09
27 x 10	94	27.65	26.91	9.990	.747	.490	3266.7	242.8	10.87	115.1	23.0	2.04
24 x 14	130	38.21	24.25	14.000	.900	.565	4009.5	330.7	10.24	375.2	53.6	3.13
24 x 12	100	29.43	24.00	12.000	.775	.468	2987.3	248.9	10.08	203.5	33.9	2.63
24 x 9	76	22.37	23.91	8.985	.682	.440	2096.4	175.4	9.68	76.5	17.0	1.85
21 x 13	112	32.93	21.00	13.000	.865	.527	2620.6	249.6	8.92	289.7	44.6	2.96
21 x 9	82	24.10	20.86	8.962	.795	.499	1752.4	168.0	8.53	89.6	20.0	1.93
21 x 8¼	62	18.23	20.99	8.240	.615	.400	1326.8	126.4	8.53	53.1	12.9	1.71
18 x 11¾	96	28.22	18.16	11.750	.831	.512	1674.7	184.4	7.70	206.8	35.2	2.71
18 x 8¾	64	18.80	17.87	8.715	.686	.403	1045.8	117.0	7.46	70.3	16.1	1.93
18 x 7½	50	14.71	18.00	7.500	.570	.358	800.6	89.0	7.38	37.2	9.9	1.59
16 x 11½	88	25.87	16.16	11.502	.795	.504	1222.6	151.3	6.87	185.2	32.2	2.67
16 x 8½	58	17.04	15.86	8.464	.645	.407	746.4	94.1	6.62	60.5	14.3	1.88
16 x 7	50	14.70	16.25	7.073	.628	.380	655.4	80.7	6.68	34.8	9.8	1.54
	36	10.59	15.85	6.992	.428	.299	446.3	56.3	6.49	22.1	6.3	1.45
14 x 16	142	41.85	14.75	15.500	1.063	.680	1672.2	226.7	6.32	660.1	85.2	3.97
	†320	94.12	16.81	16.710	2.093	1.890	4141.7	492.8	6.63	1635.1	195.7	4.17
14 x 14½	87	25.56	14.00	14.500	.688	.420	966.9	138.1	6.15	349.7	48.2	3.70
14 x 12	84	24.71	14.18	12.023	.778	.451	928.4	130.9	6.13	225.5	37.5	3.02
	78	22.94	14.06	12.000	.718	.428	851.2	121.1	6.09	206.9	34.5	3.00

* Steel WF beams are designated by giving their nominal depth in inches first; then the letters WF to designate a wide-flange beam; then the weight in pounds per linear foot. For example, 36 WF 230.

† Column core section.

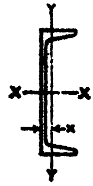
Table 4 (cont.)

Nominal Size	Weight per Foot	Area	Depth	Flange Width	Flange Thickness	Web Thickness	Axis X-X I	Axis X-X $\frac{I}{c}$	Axis X-X r	Axis Y-Y I	Axis Y-Y $\frac{I}{c}$	Axis Y-Y r
In.	Lb.	In.^{2}	In.	In.	In.	In.	In.^{4}	In.^{3}	In.	In.^{4}	In.^{3}	In.
14 x 10	74	21.76	14.19	10.072	.783	.450	796.8	112.3	6.05	133.5	26.5	2.48
	68	20.00	14.06	10.040	.718	.418	724.1	103.0	6.02	121.2	24.1	2.46
	61	17.94	13.91	10.000	.643	.378	641.5	92.2	5.98	107.3	21.5	2.45
14 x 8	53	15.59	13.94	8.062	.658	.370	542.1	77.8	5.90	57.5	14.3	1.92
	43	12.65	13.68	8.000	.528	.308	429.0	62.7	5.82	45.1	11.3	1.89
14 x 6¾	38	11.17	14.12	6.776	.513	.313	385.3	54.6	5.87	24.6	7.3	1.49
	34	10.00	14.00	6.750	.453	.287	339.2	48.5	5.83	21.3	6.3	1.46
	30	8.81	13.86	6.733	.383	.270	289.6	41.8	5.73	17.5	5.2	1.41
12 x 12	85	24.98	12.50	12.105	.796	.495	723.3	115.7	5.38	235.5	38.9	3.07
	65	19.11	12.12	12.000	.606	.390	533.4	88.0	5.28	174.6	29.1	3.02
12 x 10	53	15.59	12.06	10.000	.576	.345	426.2	70.7	5.23	96.1	19.2	2.48
12 x 8	40	11.77	11.94	8.000	.516	.294	310.1	51.9	5.13	44.1	11.0	1.94
12 x 6½	36	10.59	12.24	6.565	.540	.305	280.8	45.9	5.15	23.7	7.2	1.50
	31	9.12	12.09	6.525	.465	.265	238.4	39.4	5.11	19.8	6.1	1.47
	27	7.97	11.95	6.500	.400	.240	204.1	34.1	5.06	16.6	5.1	1.44
10 x 10	112	32.92	11.38	10.415	1.248	.755	718.7	126.3	4.67	235.4	45.2	2.67
	100	29.43	11.12	10.345	1.118	.685	625.0	112.4	4.61	206.6	39.9	2.65
	89	26.19	10.88	10.275	.998	.615	542.4	99.7	4.55	180.6	35.2	2.63
	77	22.67	10.62	10.195	.868	.535	457.2	86.1	4.49	153.4	30.1	2.60
	49	14.40	10.00	10.000	.558	.340	272.9	54.6	4.35	93.0	18.6	2.54
10 x 8	45	13.24	10.12	8.022	.618	.350	248.6	49.1	4.33	53.2	13.3	2.00
	39	11.48	9.94	7.990	.528	.318	209.7	42.2	4.27	44.9	11.2	1.98
	33	9.71	9.75	7.964	.433	.292	170.9	35.0	4.20	36.5	9.2	1.94
10 x 5¾	29	8.53	10.22	5.799	.500	.289	157.3	30.8	4.29	15.2	5.2	1.34
	21	6.19	9.90	5.750	.340	.240	106.3	21.5	4.14	9.7	3.4	1.25
8 x 8	67	19.70	9.00	8.287	.933	.575	271.8	60.4	3.71	88.6	21.4	2.12
	58	17.06	8.75	8.222	.808	.510	227.3	52.0	3.65	74.9	18.2	2.10
	48	14.11	8.50	8.117	.683	.405	183.7	43.2	3.61	60.9	15.0	2.08
	40	11.76	8.25	8.077	.558	.365	146.3	35.5	3.53	49.0	12.1	2.04
	35	10.30	8.12	8.027	.493	.315	126.5	31.1	3.50	42.5	10.6	2.03
	31	9.12	8.00	8.000	.433	.288	109.7	27.4	3.47	37.0	9.2	2.01
8 x 6½	28	8.23	8.06	6.540	.463	.285	97.8	24.3	3.45	21.6	6.6	1.62
	24	7.06	7.93	6.500	.398	.245	82.5	20.8	3.42	18.2	5.6	1.61
8 x 5¼	20	5.88	8.14	5.268	.378	.248	69.2	17.0	3.43	8.5	3.2	1.20
	17	5.00	8.00	5.250	.308	.230	56.4	14.1	3.36	6.7	2.6	1.16

Table 5

AMERICAN STANDARD STEEL CHANNELS

PROPERTIES FOR DESIGNING



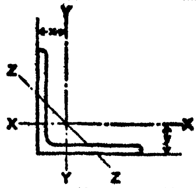
Nominal* Size	Weight per Foot	Area	Depth	Flange Width	Flange Average Thickness	Web Thickness	Axis X-X I	Axis X-X I/c	Axis X-X r	Axis Y-Y I	Axis Y-Y I/c	Axis Y-Y r	Axis Y-Y x
In.	Lb.	In.^{2}	In.	In.	In.	In.	In.^{4}	In.^{3}	In.	In.^{4}	In.^{3}	In.	In.
†18 x 4	58.0	16.98	18.00	4.200	.625	.700	670.7	74.5	6.29	18.5	5.6	1.04	.88
	51.9	15.18	18.00	4.100	.625	.600	622.1	69.1	6.40	17.1	5.3	1.06	.87
	45.8	13.38	18.00	4.000	.625	.500	573.5	63.7	6.55	15.8	5.1	1.09	.89
	42.7	12.48	18.00	3.950	.625	.450	549.2	61.0	6.64	15.0	4.9	1.10	.90
15 x 3⅜	50.0	14.64	15.00	3.716	.650	.716	401.4	53.6	5.24	11.2	3.8	.87	.80
	40.0	11.70	15.00	3.520	.650	.520	346.3	46.2	5.44	9.3	3.4	.89	.78
	33.9	9.90	15.00	3.400	.650	.400	312.6	41.7	5.62	8.2	3.2	.91	.79
12 x 3	30.0	8.79	12.00	3.170	.501	.510	161.2	26.9	4.28	5.2	2.1	.77	.68
	25.0	7.32	12.00	3.047	.501	.387	143.5	23.9	4.43	4.5	1.9	.79	.68
	20.7	6.03	12.00	2.940	.501	.280	128.1	21.4	4.61	3.9	1.7	.81	.70
10 x 2⅝	30.0	8.80	10.00	3.033	.436	.673	103.0	20.6	3.42	4.0	1.7	.67	.65
	25.0	7.33	10.00	2.886	.436	.526	90.7	18.1	3.52	3.4	1.5	.68	.62
	20.0	5.86	10.00	2.739	.436	.379	78.5	15.7	3.66	2.8	1.3	.70	.61
	15.3	4.47	10.00	2.600	.436	.240	66.9	13.4	3.87	2.3	1.2	.72	.64
9 x 2½	20.0	5.86	9.00	2.648	.413	.448	60.6	13.5	3.22	2.4	1.2	.65	.59
	15.0	4.39	9.00	2.485	.413	.285	50.7	11.3	3.40	1.9	1.0	.67	.59
	13.4	3.89	9.00	2.430	.413	.230	47.3	10.5	3.49	1.8	.97	.67	.61
8 x 2¼	18.75	5.49	8.00	2.527	.390	.487	43.7	10.9	2.82	2.0	1.0	.60	.57
	13.75	4.02	8.00	2.343	.390	.303	35.8	9.0	2.99	1.5	.86	.62	.56
	11.5	3.36	8.00	2.260	.390	.220	32.3	8.1	3.10	1.3	.79	.63	.58
7 x 2⅛	14.75	4.32	7.00	2.299	.366	.419	27.1	7.7	2.51	1.4	.79	.57	.53
	12.25	3.58	7.00	2.194	.366	.314	24.1	6.9	2.59	1.2	.71	.58	.53
	9.8	2.85	7.00	2.090	.366	.210	21.1	6.0	2.72	.98	.63	.59	.55
6 x 2	13.0	3.81	6.00	2.157	.343	.437	17.3	5.8	2.13	1.1	.65	.53	.52
	10.5	3.07	6.00	2.034	.343	.314	15.1	5.0	2.22	.87	.57	.53	.50
	8.2	2.39	6.00	1.920	.343	.200	13.0	4.3	2.34	.70	.50	.54	.52
5 x 1¾	9.0	2.63	5.00	1.885	.320	.325	8.8	3.5	1.83	.64	.45	.49	.48
	6.7	1.95	5.00	1.750	.320	.190	7.4	3.0	1.95	.48	.38	.50	.49
4 x 1⅝	7.25	2.12	4.00	1.720	.296	.320	4.5	2.3	1.47	.44	.35	.46	.46
	5.4	1.56	4.00	1.580	.296	.180	3.8	1.9	1.56	.32	.29	.45	.46
3 x 1½	6.0	1.75	3.00	1.596	.273	.356	2.1	1.4	1.08	.31	.27	.42	.46
	5.0	1.46	3.00	1.498	.273	.258	1.8	1.2	1.12	.25	.24	.41	.44
	4.1	1.19	3.00	1.410	.273	.170	1.6	1.1	1.17	.20	.21	.41	.44

* Steel channels are designated by giving their depth in inches first; then the symbol ⊔ to designate a channel; then the weight in pounds per linear foot. For example 15 ⊔ 50.0.

† Car and shipbuilding channel; not an American Standard.

Table 6

STEEL ANGLES
EQUAL LEGS
PROPERTIES FOR DESIGNING



Size	Thickness	Weight per Foot	Area	AXIS X-X AND AXIS Y-Y				AXIS Z-Z
				I	I/c	r	x or y	r
In.	In.	Lb.	In.^{2}	In.^{4}	In.^{3}	In.	In.	In.
8 x 8	1 1/8	56.9	16.73	98.0	17.5	2.42	2.41	1.56
	1	51.0	15.00	89.0	15.8	2.44	2.37	1.56
	7/8	45.0	13.23	79.6	14.0	2.45	2.32	1.57
	3/4	38.9	11.44	69.7	12.2	2.47	2.28	1.57
	5/8	32.7	9.61	59.4	10.3	2.49	2.23	1.58
	9/16	29.6	8.68	54.1	9.3	2.50	2.21	1.58
	1/2	26.4	7.75	48.6	8.4	2.50	2.19	1.59
6 x 6	1	37.4	11.00	35.5	8.6	1.80	1.86	1.17
	7/8	33.1	9.73	31.9	7.6	1.81	1.82	1.17
	3/4	28.7	8.44	28.2	6.7	1.83	1.78	1.17
	5/8	24.2	7.11	24.2	5.7	1.84	1.73	1.18
	9/16	21.9	6.43	22.1	5.1	1.85	1.71	1.18
	1/2	19.6	5.75	19.9	4.6	1.86	1.68	1.18
	7/16	17.2	5.06	17.7	4.1	1.87	1.66	1.19
	3/8	14.9	4.36	15.4	3.5	1.88	1.64	1.19
	5/16	12.5	3.66	13.0	3.0	1.89	1.61	1.19
5 x 5	7/8	27.2	7.98	17.8	5.2	1.49	1.57	.97
	3/4	23.6	6.94	15.7	4.5	1.51	1.52	.97
	5/8	20.0	5.86	13.6	3.9	1.52	1.46	.98
	1/2	16.2	4.75	11.3	3.2	1.54	1.43	.98
	7/16	14.3	4.18	10.0	2.8	1.55	1.41	.98
	3/8	12.3	3.61	8.7	2.4	1.56	1.39	.99
	5/16	10.3	3.03	7.4	2.0	1.57	1.37	.99
4 x 4	3/4	18.5	5.44	7.7	2.8	1.19	1.27	.78
	5/8	15.7	4.61	6.7	2.4	1.20	1.23	.78
	1/2	12.8	3.75	5.6	2.0	1.22	1.18	.78
	7/16	11.3	3.31	5.0	1.8	1.23	1.16	.78
	3/8	9.8	2.86	4.4	1.5	1.23	1.14	.79
	5/16	8.2	2.40	3.7	1.3	1.24	1.12	.79
	1/4	6.6	1.94	3.0	1.1	1.25	1.09	.80
3 1/2 x 3 1/2	1/2	11.1	3.25	3.6	1.5	1.06	1.06	.68
	7/16	9.8	2.87	3.3	1.3	1.07	1.04	.68
	3/8	8.5	2.48	2.9	1.2	1.07	1.01	.69
	5/16	7.2	2.09	2.5	.98	1.08	.99	.69
	1/4	5.8	1.69	2.0	.79	1.09	.97	.69
3 x 3	1/2	9.4	2.75	2.2	1.1	.90	.93	.58
	7/16	8.3	2.43	2.0	.95	.91	.91	.58
	3/8	7.2	2.11	1.8	.83	.91	.89	.58
	5/16	6.1	1.78	1.5	.71	.92	.87	.59
	1/4	4.9	1.44	1.2	.58	.93	.84	.59
	3/16	3.71	1.09	.96	.44	.94	.82	.59
2 1/2 x 2 1/2	1/2	7.7	2.25	1.2	.72	.74	.81	.49
	3/8	5.9	1.73	.98	.57	.75	.76	.49
	5/16	5.0	1.47	.85	.48	.76	.74	.49
	1/4	4.1	1.19	.70	.39	.77	.72	.49
	3/16	3.07	.90	.55	.30	.78	.69	.49

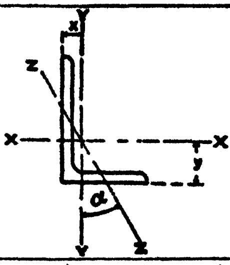


Table 7

STEEL ANGLES UNEQUAL LEGS

PROPERTIES FOR DESIGNING

Size	Thickness	Weight per Foot	Area	Axis X-X				Axis Y-Y				Axis Z-Z	
				I	I/c	r	y	I	I/c	r	x	r	Tan α
In.	In.	Lb.	In.²	In.⁴	In.³	In.	In.	In.⁴	In.³	In.	In.	In.	
8 x 6	1	44.2	13.00	80.8	15.1	2.49	2.65	38.8	8.9	1.73	1.65	1.28	.543
	7/8	39.1	11.48	72.3	13.4	2.51	2.61	34.9	7.9	1.74	1.61	1.28	.547
	3/4	33.8	9.94	63.4	11.7	2.53	2.56	30.7	6.9	1.76	1.56	1.29	.551
	5/8	28.5	8.36	54.1	9.9	2.54	2.52	26.3	5.9	1.77	1.52	1.29	.554
	9/16	25.7	7.56	49.3	9.0	2.55	2.50	24.0	5.3	1.78	1.50	1.30	.556
	1/2	23.0	6.75	44.3	8.0	2.56	2.47	21.7	4.8	1.79	1.47	1.30	.558
	7/16	20.2	5.93	39.2	7.1	2.57	2.45	19.3	4.2	1.80	1.45	1.31	.560
8 x 4	1	37.4	11.00	69.6	14.1	2.52	3.05	11.6	3.9	1.03	1.05	.85	.247
	7/8	33.1	9.73	62.5	12.5	2.53	3.00	10.5	3.5	1.04	1.00	.85	.253
	3/4	28.7	8.44	54.9	10.9	2.55	2.95	9.4	3.1	1.05	.95	.85	.258
	5/8	24.2	7.11	46.9	9.2	2.57	2.91	8.1	2.6	1.07	.91	.86	.262
	9/16	21.9	6.43	42.8	8.4	2.58	2.88	7.4	2.4	1.07	.88	.86	.265
	1/2	19.6	5.75	38.5	7.5	2.59	2.86	6.7	2.2	1.08	.86	.86	.267
	7/16	17.2	5.06	34.1	6.6	2.60	2.83	6.0	1.9	1.09	.83	.87	.269
7 x 4	7/8	30.2	8.86	42.9	9.7	2.20	2.55	10.2	3.5	1.07	1.05	.86	.318
	3/4	26.2	7.69	37.8	8.4	2.22	2.51	9.1	3.0	1.09	1.01	.86	.324
	5/8	22.1	6.48	32.4	7.1	2.24	2.46	7.8	2.6	1.10	.96	.86	.329
	9/16	20.0	5.87	29.6	6.5	2.24	2.44	7.2	2.4	1.11	.94	.87	.332
	1/2	17.9	5.25	26.7	5.8	2.25	2.42	6.5	2.1	1.11	.92	.87	.335
	7/16	15.8	4.62	23.7	5.1	2.26	2.39	5.8	1.9	1.12	.89	.88	.337
	3/8	13.6	3.98	20.6	4.4	2.27	2.37	5.1	1.6	1.13	.87	.88	.339
6 x 4	7/8	27.2	7.98	27.7	7.2	1.86	2.12	9.8	3.4	1.11	1.12	.86	.421
	3/4	23.6	6.94	24.5	6.3	1.88	2.08	8.7	3.0	1.12	1.08	.86	.428
	5/8	20.0	5.86	21.1	5.3	1.90	2.03	7.5	2.5	1.13	1.03	.86	.435
	9/16	18.1	5.31	19.3	4.8	1.90	2.01	6.9	2.3	1.14	1.01	.87	.438
	1/2	16.2	4.75	17.4	4.3	1.91	1.99	6.3	2.1	1.15	.99	.87	.440
	7/16	14.3	4.18	15.5	3.8	1.92	1.96	5.6	1.9	1.16	.96	.87	.443
	3/8	12.3	3.61	13.5	3.3	1.93	1.94	4.9	1.6	1.17	.94	.88	.446
	5/16	10.3	3.03	11.4	2.8	1.94	1.92	4.2	1.4	1.17	.92	.88	.449
6 x 3½	1/2	15.3	4.50	16.6	4.2	1.92	2.08	4.3	1.6	.97	.83	.76	.344
	3/8	11.7	3.42	12.9	3.2	1.94	2.04	3.3	1.2	.99	.79	.77	.350
	5/16	9.8	2.87	10.9	2.7	1.95	2.01	2.9	1.0	1.00	.76	.77	.352
	1/4	7.9	2.31	8.9	2.2	1.96	1.99	2.3	0.85	1.01	.74	.78	.355
5 x 3½	3/4	19.8	5.81	13.9	4.3	1.55	1.75	5.6	2.2	.98	1.00	.75	.464
	5/8	16.8	4.92	12.0	3.7	1.56	1.70	4.8	1.9	.99	.95	.75	.472
	1/2	13.6	4.00	10.0	3.0	1.58	1.66	4.1	1.6	1.01	.91	.75	.479
	7/16	12.0	3.53	8.9	2.6	1.59	1.63	3.6	1.4	1.01	.88	.76	.482
	3/8	10.4	3.05	7.8	2.3	1.60	1.61	3.2	1.2	1.02	.86	.76	.486
	5/16	8.7	2.56	6.6	1.9	1.61	1.59	2.7	1.0	1.03	.84	.76	.489
	1/4	7.0	2.06	5.4	1.6	1.61	1.56	2.2	.83	1.04	.81	.76	.492

Table 7 (cont.)

Size	Thick-ness	Weight per Foot	Area	AXIS X-X				AXIS Y-Y				AXIS Z-Z	
				I	I/c	r	y	I	I/c	r	x	r	Tan α
In.	In.	Lb.	In.²	In.⁴	In.³	In.	In.	In.⁴	In.³	In.	In.	In.	
5 x 3	½	12.8	3.75	9.5	2.9	1.59	1.75	2.6	1.1	.83	.75	.65	.357
	7/16	11.3	3.31	8.4	2.6	1.60	1.73	2.3	1.0	.84	.73	.65	.361
	⅜	9.8	2.86	7.4	2.2	1.61	1.70	2.0	.89	.84	.70	.65	.364
	5/16	8.2	2.40	6.3	1.9	1.61	1.68	1.8	.75	.85	.68	.66	.368
	¼	6.6	1.94	5.1	1.5	1.62	1.66	1.4	.61	.86	.66	.66	.371
4 x 3½	⅝	14.7	4.30	6.4	2.4	1.22	1.29	4.5	1.8	1.03	1.04	.72	.745
	½	11.9	3.50	5.3	1.9	1.23	1.25	3.8	1.5	1.04	1.00	.72	.750
	7/16	10.6	3.09	4.8	1.7	1.24	1.23	3.4	1.4	1.05	.98	.72	.753
	⅜	9.1	2.67	4.2	1.5	1.25	1.21	3.0	1.2	1.06	.96	.73	.755
	5/16	7.7	2.25	3.6	1.3	1.26	1.18	2.6	1.0	1.07	.93	.73	.757
	¼	6.2	1.81	2.9	1.0	1.27	1.16	2.1	.81	1.07	.91	.73	.759
4 x 3	⅝	13.6	3.98	6.0	2.3	1.23	1.37	2.9	1.4	.85	.87	.64	.534
	½	11.1	3.25	5.1	1.9	1.25	1.33	2.4	1.1	.86	.83	.64	.543
	7/16	9.8	2.87	4.5	1.7	1.25	1.30	2.2	1.0	.87	.80	.64	.547
	⅜	8.5	2.48	4.0	1.5	1.26	1.28	1.9	.87	.88	.78	.64	.551
	5/16	7.2	2.09	3.4	1.2	1.27	1.26	1.7	.73	.89	.76	.65	.554
	¼	5.8	1.69	2.8	1.0	1.28	1.24	1.4	.60	.90	.74	.65	.558
3½ x 3	½	10.2	3.00	3.5	1.5	1.07	1.13	2.3	1.1	.88	.88	.62	.714
	7/16	9.1	2.65	3.1	1.3	1.08	1.10	2.1	.98	.89	.85	.62	.718
	⅜	7.9	2.30	2.7	1.1	1.09	1.08	1.9	.85	.90	.83	.62	.721
	5/16	6.6	1.93	2.3	.95	1.10	1.06	1.6	.72	.90	.81	.63	.724
	¼	5.4	1.56	1.9	.78	1.11	1.04	1.3	.59	.91	.79	.63	.727
3½x2½	½	9.4	2.75	3.2	1.4	1.09	1.20	1.4	.76	.70	.70	.53	.486
	7/16	8.3	2.43	2.9	1.3	1.09	1.18	1.2	.68	.71	.68	.54	.491
	⅜	7.2	2.11	2.6	1.1	1.10	1.16	1.1	.59	.72	.66	.54	.496
	5/16	6.1	1.78	2.2	.93	1.11	1.14	.94	.50	.73	.64	.54	.501
	¼	4.9	1.44	1.8	.75	1.12	1.11	.78	.41	.74	.61	.54	.506
3 x 2½	½	8.5	2.50	2.1	1.0	.91	1.00	1.3	.74	.72	.75	.52	.667
	7/16	7.6	2.21	1.9	.93	.92	.98	1.2	.66	.73	.73	.52	.672
	⅜	6.6	1.92	1.7	.81	.93	.96	1.0	.58	.74	.71	.52	.676
	5/16	5.6	1.62	1.4	.69	.94	.93	.90	.49	.74	.68	.53	.680
	¼	4.5	1.31	1.2	.56	.95	.91	.74	.40	.75	.66	.53	.684
3 x 2	½	7.7	2.25	1.9	1.0	.92	1.08	.67	.47	.55	.58	.43	.414
	7/16	6.8	2.00	1.7	.89	.93	1.06	.61	.42	.55	.56	.43	.421
	⅜	5.9	1.73	1.5	.78	.94	1.04	.54	.37	.56	.54	.43	.428
	5/16	5.0	1.47	1.3	.66	.95	1.02	.47	.32	.57	.52	.43	.435
	¼	4.1	1.19	1.1	.54	.95	.99	.39	.26	.57	.49	.43	.440
	3/16	3.07	.90	.84	.41	.97	.97	.31	.20	.58	.47	.44	.446
2½ x 2	⅜	5.3	1.55	.91	.55	.77	.83	.51	.36	.58	.58	.42	.614
	5/16	4.5	1.31	.79	.47	.78	.81	.45	.31	.58	.56	.42	.620
	¼	3.62	1.06	.65	.38	.78	.79	.37	.25	.59	.54	.42	.626
	3/16	2.75	.81	.51	.29	.79	.76	.29	.20	.60	.51	.43	.631

Table 8 Properties of Areas

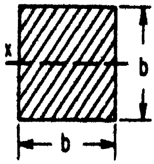
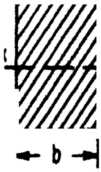
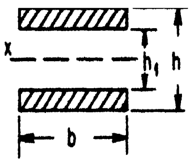
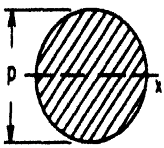
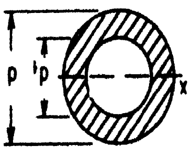
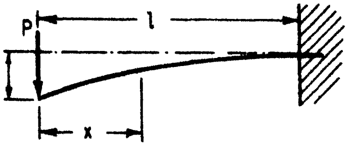
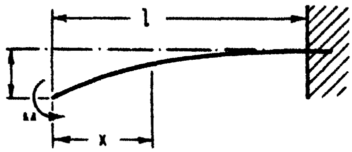
Square		$I_x = \frac{b^4}{12}$	$Z = b^3$
Rectangle		$I_x = \frac{bh^3}{12}$	$Z = bh^2$
Hollow rectangle		$I_x = \frac{bh^3 - b_1 h_1^3}{12}$	$Z = \frac{bh^3 - b_1 h_1^3}{6h}$
Equal rectangles		$I_x = \frac{b(h^3 - h_1^3)}{12}$	$Z = \frac{b(h^3 - h_1^3)}{6h}$
Circle		$I_x = \frac{d^4}{64}$	$Z = \frac{d^3}{32}$
Hollow circle		$I_x = \frac{(d^4 - d_1^4)}{64}$	$Z = (d^4 - d_1^4)$

Table 9 Beam Deflection Equations



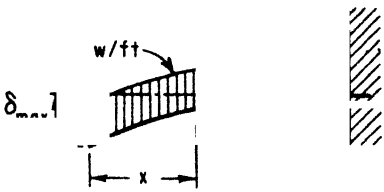
$$\delta_x = \frac{P}{6EI}(2l^3 - 3l^2x + x^3)$$

$$\delta_{max} = \frac{Pl^3}{3EI}$$



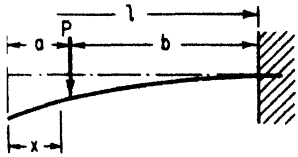
$$\delta_x = \frac{M(l-x)^2}{2EI}$$

$$\delta_{max} = \frac{Ml^2}{2EI}$$



$$\delta_x = \frac{w}{24EI}(x^4 - 4l^3x + 3l^4)$$

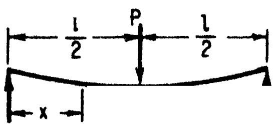
$$\delta_{max} = \frac{wl^4}{8EI}$$



$$\delta_x = \frac{Pb^2}{6EI}(3l - 3x - b) \quad \text{for } x<a$$

$$\delta_x = \frac{P(l-x)^2}{6EI}(3b - l + x) \quad \text{for } x>a$$

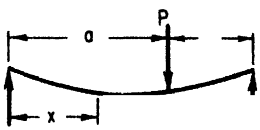
$$= \frac{Pb^2}{6EI}$$



$$\delta_x \quad \frac{Px}{48EI}(3l^2 - 4x^2) \quad \text{for } x < \frac{l}{2}$$

$$\delta_{max} \quad \frac{Pl^3}{48EI} \quad \text{at } x =$$

6.



$$\delta_x = \frac{Pbx}{6lEI}(l^2 - x^2 - b^2) \quad \text{for } x<a$$

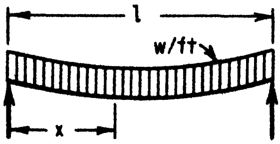
$$\delta_x = \frac{Pb}{6lEI}\left[\frac{l}{b}(x-a)^3 + (l^2-b^2)x - x^3\right] \quad \text{for } x>a$$

$$\frac{Pb}{48EI}(3l^2 - 4b^2) \quad \text{at center if } a>b$$

$$\delta_{max} = \frac{Pb(l^2-b^2)^{3/2}}{9\sqrt{3}lEI} \quad \text{at } x \quad \sqrt{\frac{l^2-b^2}{3}}$$

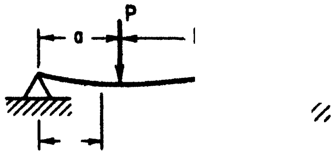
Table 9 (cont.)

7.



$$\delta_x = \frac{wx}{24EI}(l^3 - 2lx^2 + x^3)$$

$$\delta_{max} \quad \frac{5wl^4}{384EI} \text{ at center}$$

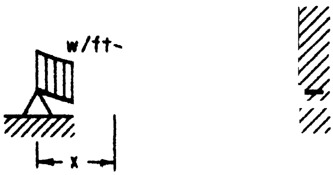


$$\delta_x = \frac{Pb^2x}{12EIl^3}(3al^2 - 2lx^2 - ax^2) \text{ for } x<a$$

$$\delta_x = \frac{Pa(l-x)^2}{12EIl^3}(3l^2x - a^2x - 2a^2l) \text{ for } x>a$$

$$\delta = \frac{Pa^2b^3}{12EIl^3}(3l+a) \text{ at point of load}$$

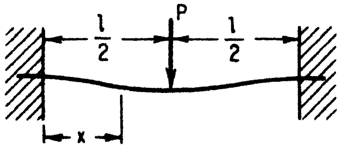
9.



$$\delta_x = \frac{wx}{48EI}(l^3 - 3lx^2 + 2x^3)$$

$$\delta_{max} = \frac{wl^4}{185EI} \text{ at } x = 0.422l$$

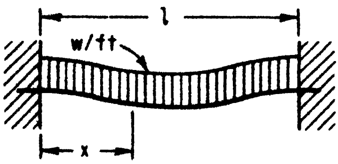
10.



$$\delta_x = \frac{Px^2}{48EI}(3l - 4x)$$

$$\delta_{max} = \frac{Pl^3}{192EI} \text{ at center}$$

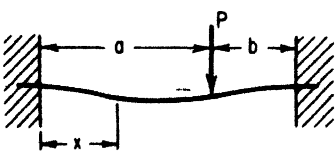
11.



$$\delta_x = \frac{wx^2}{24EI}(l-x)^2$$

$$\delta_{max} = \frac{wl^4}{384EI} \text{ at center}$$

12.



$$\frac{Pb^2x^2}{6EIl^3}(3al - 3ax - bx) \text{ for } x<a$$

$$\delta_{max} = \frac{2Pa^3b^2}{3EI(3a+b)^2} \text{ at } x = \frac{2al}{3a+b}, a>b$$

$$\delta = \frac{Pa^3b^3}{3EIl^3} \text{ at point of load}$$

APPENDIX C

Answers to Problems

CHAPTER 1

1-1. $F_{\text{axial}} = 100$ lb; $F_{\text{shear}} = 0$; $M = 400$ lb in.

1-2. $F_{\text{axial}} = 2800$ lb; $M = 12{,}000$ lb ft

1-3. 150 lb downward

1-4. $F_{\text{axial}} = 96$ lb; $F_{\text{shear}} = 72$ lb; $M = 144$ lb in.

1-5. $M_B = 5000$ lb ft; $M_C = 3750$ lb ft

1-6. $M_B = 5000$ lb ft; $M_C = 5000$ lb ft

1-7. $F_{\text{axial}} = 200$ lb; $F_{\text{shear}} = 66.7$ lb; $M = 200$ lb ft

1-8. $M = Wx(l - x)/2$

1-9. $F_{\text{axial}} = 867$ lb; $F_{\text{shear}} = 0$; $M = 0$

1-10. $F_{\text{axial}} = 0$; $F_{\text{shear}} = 1500$ lb; $M = 7500$ lb ft

1-11. $F_{\text{axial}} = 13$ kips; $F_{\text{shear}} = 2.5$ kips; $M = 12.5$ kip ft

1-12. $F_{\text{shear}} = 500$ lb; $M_z = 2000$ lb ft; $T_x = 1500$ lb ft

1-13. $F_{\text{axial}} = 0$; $F_{\text{shear}} = 5000$ lb; $T_x = 1000$ lb ft; $M_y = 6000$ lb ft; $M_z = 16{,}000$ lb ft

1-14. $F_y = 5000 - 150y$

1-15. $V = W(l - d)/l$; $M = Wx(l - d)/l$

1-16. $F_y = 18.1$ lb; $F_x = 8$ lb; $T = 320$ lb in.; $M_x = 218$ lb in.; $M_y = 144$ lb in.

1-17. $F_N = F_S = 707$ lb

1-18. $F_S = 4330$ lb; $F_N = 2500$ lb

1-19. $F_N = 260$ lb; $F_S = 180$ lb

1-20. $F_N = 650$ lb; $F_S = 0$

1-21. $F_N = 5200$ lb; $F_S = 0$

1-22. $F_x = 212$ lb; $F_y = 354$ lb

1-23. 9550 psi

1-24. $AB = 2.78$ in.^{2}; $AD = 1.67$ in.^{2}; $BD = 2.50$ in.^{2}

1-25. $AB = 1.56$ in.^{2}; $BC = 2.08$ in.^{2}; $CD = 1.25$ in.^{2}; $BD = 0$

1-26. $\sigma_{AB} = 2000$ psi tension; $\sigma_{BC} = 1000$ psi compression; $\sigma_{CD} = 5000$ psi tension

1-27. 22,400 lb; 28,500 psi

1-28. 13.1 in.^{2}

1-29. 10,000 psi; 1000 psi

1-30. 4000 psi; 5330 psi

1-31. 5330 psi and 7110 psi (maximum values)

1-32. 20,600 lb in.

1-33. (a) 64.8 kips; (b) 91.2 kips

1-34. $W_t = 0.248$ lb; $W_s = 1.425$ lb

1-35. 1100 lb

1-36. 1.65 in.; 2.3 in; 0.796 in.

1-37. 5025 lb; 203 psi

1-38. $A_A = 2.4$ in.^{2}; $A_B = 0.8$ in.^{2}; $A_C = 1.2$ in.^{2}

1-39. A: 10 WF 33; B: 16 WF 58

1-40. $\sigma_n = 833$ psi; $\tau = 1440$ psi

1-41. 70.7 kips

1-42. 46.2 kips; 2.89 ksi

1-43. A shear failure; the maximum load in shear is $\frac{5}{8}$ of the maximum load in tension.

1-44. 1000 lb

1-45. 33.9 kips

1-46. 62.1 kips; 2.49 ft to right of A

1-47. 2680 lb

1-48. 3260 lb

1-49. 5000 lb

1-50. 3300 lb

1-51. 29,400 lb

1-52. 0.2 in.

1-53. 0.1 in.

1-54. 43.6 psi

1-55. 0.4 in.

1-57. 24,900 psi

1-58. 22.9 ft

1-59. 0.25 in.

1-60. 28 bolts

CHAPTER 2

2-1. 0.0025 in./in.

2-2. 1.2 in.

2-3. 0.0012 in./in.; − 0.0003 in./in.; − 0.0003 in./in.

2-4. 0.075 in.^{3} (decrease)

2-5. 0.00024 in./in.

2-6. 8.17 in.

2-7. 0.24 in.

2-8. 0.161 deg

2-9. 2:1

2-10. (a) 15×10^6 psi; (b) 30,000 psi; (c) 35,000 psi; (d) 43.8 per cent; (e) 34.9 per cent; (f) 49,000 psi; (g) 75,300 psi

2-12. (a) 14.7×10^6 psi; (b) 23,600 psi; (c) 70,000 psi; (d) 22.5 per cent; (e) 28.6 per cent; (f) 53,500 psi; (g) 75,000 psi

2-14. 0.0608 in.

2-15. 0.111 in.

2-16. 50 kips; 0.160 in.

2-17. − 0.0192 in.

2-18. 14,700 lb

2-19. No

2-20. 50,000 lb

2-21. 28,300 lb

2-22. 3.53 in.^{2}

2-23. 10.6 in.^{2}

2-24. $\sigma_{AB} = 16$ ksi tension; $\sigma_{BC} = 24$ ksi compression

2-25. $\sigma_{AB} = 14.8$ ksi tension; $\sigma_{BC} = 25.3$ ksi compression

2-26. $\sigma_{AB} = 17$ ksi tension; $\sigma_{BC} = 30$ ksi compression

2-27. $\sigma_a = 5410$ psi compression; $\sigma_b = 9240$ psi compression

2-28. $A_c = 146$ in.^{2}; $A_s = 30.6$ in.^{2}; $\sigma_c = 167$ psi; $\sigma_s = 2500$ psi

2-29. 72,900 lb

2-30. $\sigma_s = 26{,}100$ psi tension; $\sigma_a = 1320$ psi compression

2-31. 17,500 lb

2-32. 6.82 ft to right of A

2-33. 0.6 in. to right of center line

2-34. 0.000222 in.

2-35. 0.25

2-36. 0.145; 2×10^6 psi

2-37. $P.L. = 40{,}000$ psi; $E = 32 \times 10^6$ psi; $\mu = 0.25$

2-38. 0.00042 in.

2-39. $\Delta L = 0.04$ in.; $\Delta A = -0.00066$ in.^{2}; $\Delta V = 0.014$ in.^{3} (increase)

2-40. 18.7 in.

2-41. 0.196 in.^{2} (increase)

2-42. 0.281 in.

2-43. 25,400 psi (tension)

2-44. 7500 psi

2-45. 16.5°F

2-46. $\sigma_s = 27{,}300$ psi (tension); $\sigma_b = 13{,}650$ psi (tension)

2-47. $\sigma_s = 18{,}100$ psi; $\sigma_b = 9050$ psi; $\sigma_a = 6030$ psi

2-48. ± 89.5°F

2-49. 0.15 in.

2-50. 86.8°F

2-51. 11.7×10^{-6} in./in./°F

2-52. 85.5°F (decrease)

2-53. 114°F (increase)

2-54. 2.67 in. to left of center line

2-55. 0.008 in./in.

2-56. 2400 lb

2-57. 0.00078 rad

2-58. 36,000 lb

2-59. 14,000 lb

2-60. 0.339

2-61. 4×10^6 psi

2-62. 0.4

CHAPTER 3

3-1. 1570 lb in.

3-2. 28.7 deg

3-3. 18,600 psi

3-4. The hollow shaft is $\frac{15}{16}$ as strong as the solid shaft.

3-5. 5000π lb in.; 5.73 deg
3-6. 6550 psi
3-7. 2.33 in.
3-8. 8.82 deg
3-9. $\tau_b = 7570$ psi; $\tau_s = 6310$ psi
3-10. $a/b = 0.842$
3-11. 6.82 deg
3-12. 944 lb in.
3-13. 6000 lb in.; 46.4 deg
3-14. − 37.7 deg
3-15. $T_A = 1000$ lb in.; $T_B = 24{,}000$ lb in.
3-16. 50 hp
3-17. 1.48 in.
3-18. 3.89 hp
3-19. 114 rpm; a minimum value
3-20. 3180 psi; 1.28 deg
3-21. $\tau_{AB} = 3820$; $\tau_{BC} = 11{,}500$ psi; $\theta_{A/C} = 8.78$ deg
3-22. $d_{AB} = 1.37$ in.; $d_{BC} = 2.07$ in.; $d_{CD} = 1.56$ in.
3-23. 2.56 deg
3-24. 0.157×10^6 lb in./rad
3-25. 25.4 in.
3-26. 1 lb in./rad = 1.45×10^{-6} kip ft/deg
3-27. 2.34 ft
3-28. 0.8×10^6 lb in./rad; 0.716 deg
3-29. 0.179×10^6 lb in./rad; 3.20 deg
3-30. 2×10^6 lb in./rad; 0.286 deg
3-31. 0.583×10^6 lb in./rad; 0.982 deg
3-32. 2.57×10^4 lb in. /rad; 22.3 deg
3-33. 1.9×10^4 lb in./rad; 30.2 deg
3-34. $3k$
3-35. 8.82 deg
3-36. 6.82 deg
3-37. 46.4 deg
3-38. $T_A = 1000$ lb in.; $T_B = 24{,}000$ lb in.
3-39. $\tau_{max} = 1660$ psi; $\delta = 0.144$ in.; $k = 694$ lb/in.
3-40. $R = 5$ in.; $k = 5$ lb/in.; $\tau_{max} = 2090$ psi
3-41. $P = 649$ lb; $n = 22$ coils; $\delta = 6.49$ in.

3-42. $n = 24.4$ coils; $\tau_{max} = 69{,}300$ psi; $\delta = 5$ in.

3-43. $d = 0.2$ in.; $n = 19.1$ coils; $k = 15.7$ lb/in.

3-44. $R = 0.7$ in.; $d = 0.22$ in.; $\delta = 0.785$ in.

3-45. $\tau_{max} = 55{,}700$ psi; $P = 656$ lb

3-46. 1600 lb/in.

3-47. 6 ft

3-48. 6.4 ft to the right of A

3-49. $R_A = 200$ lb; $R_B = 400$ lb

3-50. 130 lb/in.

3-51. 236 hp

3-52. 8 bolts

3-53. 58.9 hp

3-54. 8910 psi

3-55. 10 bolts

CHAPTER 4

4-1. $V = -100$ lb; $M = -100x$; $M_{max} = -800$ lb ft

4-2. $V = 0$; $M = -300$ lb ft

4-3. $V = 0$; $M = 500$ lb ft

4-4. $V = -100x$; $M = -50x^2$; $V_{max} = -600$ lb; $M_{max} = -1800$ lb ft

4-5. $0 < x < 4$: $V = -100$ lb; $M = -100x$
$4 < x < 10$: $V = -100$ lb; $M = -100x + 600$

4-6. $0 < x < 2$: $V = -100$ lb; $M = -100x$
$2 < x < 10$: $V = -100 - 50(x - 2)$; $M = -100x - 25(x - 2)^2$
$M_{max} = -2600$ lb ft at $x = 10$ ft

4-7. $0 < x < 3$: $V = 350$ lb; $M = 350x$
$3 < x < 10$: $V = -150$ lb; $M = 350x - 500(x - 3)$
$V_{max} = 350$ lb; $M_{max} = 1050$ lb ft, 3 ft to right of A

4-8. $M_{max} = -15$ kip ft, 5 ft to right of A

4-9. $M_{max} = 1200$ lb ft, 6 ft to right of A

4-10. $0 < x < 5$: $V = -1250$ lb; $M = -1250x$
$5 < x < 15$: $V = 0$; $M = -6250$ lb ft
$15 < x < 20$: $V = 1250$ lb; $M = 25{,}000 - 1250x$

4-11. $V = 1000 - 200x$; $M = 1000x - 100x^2$

4-12. $0 < x < 4$: $V = 400$ lb; $M = 400x$
$4 < x < 12$: $V = 400 - 150(x - 4)$; $M = 400x - 75(x - 4)^2$

4-13. $0 < x < 3$: $V = 900$ lb; $M = 900x$

$3 < x < 9$: $V = 900 - 300(x - 3)$; $M = 900x - 150(x - 3)^2$

$9 < x < 12$: $V = -900$ lb; $M = 900x - 1800(x - 6)$

4-14. $M_{max} = 2880$ lb ft, 4.8 ft to right of A

4-15. $M_{max} = -2000$ lb ft

4-16. $M_{max} = -1400$ lb ft

4-17. $M_{max} = -3600$ lb ft

4-18. $M_{max} = 882$ lb ft, 4.2 ft to right of A; $M_B = 720$ lb ft

4-19. $M_{max} = 1910$ lb ft, 4.38 ft to right of A; $M_B = 1825$ lb ft

4-20. $M_B = 1040$ lb ft; $M_C = 2720$ lb ft; $M_D = 1560$ lb ft

4-21. $V_B = 1000$ lb; $V_C = -1000$ lb; $M_B = M_C = 2500$ lb ft

$M_{max} = 5000$ lb ft at midspan

4-22. $M_B = 500$ lb ft; $M_C = 0$; $M_D = -500$ lb ft; $M_E = 2100$ lb ft

4-23. Moment at hinge is zero; $M_{1_{max}} = 200$ lb ft, 2 ft to right of A;

$M_{2_{max}} = -600$ lb ft at C; $M_{3_{max}} = 528$ lb ft, 3.25 ft to left of D

4-24. $M_{max} = 1390$ lb ft, 2.54 ft to right of A

4-25. $M_{max} = 10{,}900$ lb ft, at midspan

4-26. 22,750 lb ft^3

4-27. 54,000 lb ft^3

4-28. 83,300 lb ft^3

4-29. 27,000 lb ft^3

4-30. 102,400 lb ft^3

4-31. 57,700 lb ft^3

4-32. $-$ 18,600 lb ft^3

4-33. 10,880 lb ft^3

4-34. $M_{max} = 2500$ lb ft

4-35. $M_{max} = 1250$ lb ft

4-36. $M_{max} = -1600$ lb ft

4-37. $M_{max} = -9000$ lb ft

4-38. $M_{max} = -960$ lb ft

CHAPTER 5

5-1. 31,250 psi; 0.749 lb in.

5-2. 41.7 in.; 117 lb in.

5-3. 8 in.

5-4. 4.67 in.

5-5. 96 in.^{3}
5-6. 10,000 lb
5-7. 2300 lb
5-8. 4800 lb ft
5-9. 14 WF 34
5-10. (a) 8000 lb; (b) 2670 lb
5-11. 12 I 31.8; 12 WF 27
5-12. 450 psi, 3 ft to right of A; 800 psi at B
5-13. 2620 lb/ft
5-14. σ_t = 2860 psi; σ_c = 5530 psi
5-15. 2554 lb/ft
5-16. 6870 psi
5-17. Solid section is 1.62 times as strong
5-18. The 10 in. by 18 in. section is 1.03 times as strong
5-19. 16 WF 40
5-20. σ_t = 13,700 psi; σ_c = 27,600 psi
5-21. 375 psi
5-22. 125 psi
5-23. 2450 lb
5-24. (a) 347 psi; (b) 625 psi
5-25. 100 psi
5-26. 733 psi
5-27. 1150 lb/ft
5-28. 360 psi
5-29. 29.8 lb/ft
5-30. 8.8 in.
5-31. 7920 lb
5-32. 6 in.
5-33. 3.73 in.
5-34. 14,400 lb
5-35. 3.92 in.
5-36. 60.2 kips
5-37. σ_s = 3700 psi; σ_w = 154 psi
5-38. σ_s = 17,800 psi; σ_w = 890 psi
5-39. σ_a = 2660 psi; σ_w = 487 psi
5-40. 1050 kip in.
5-41. 900 lb/ft

5-42. 74.3 kip ft

5-43. $\sigma_b = 3210$ psi; $\sigma_s = 2140$ psi

5-44. Beam (a) is 1.2 times stronger than beam (b)

5-45. 8.88 in. below top

5-46. $\sigma_c = 400$ psi; $\sigma_s = 6000$ psi

5-47. (a) 2.16 in.^{2}; (b) $\sigma_c = 579$ psi; $\sigma_s = 19{,}300$ psi

5-48. 390 kip in.

5-49. 3.24 in.^{2}

5-50. 2910 lb/ft

CHAPTER 6

6-1. $\pm \dfrac{Ml^2}{2EI}$

6-2. $\pm \dfrac{3Ml^2}{8EI}$

6-3. 6940 lb

6-4. $\theta = 0.0228$ rad; $\delta = 1.64$ in.

6-5. $-$ 40,500 lb ft^3

6-6. 7.65 in.

6-7. $-$ 30,400 lb ft^3

6-8. 868 lb; deflection governs.

6-9. $M = \dfrac{2Pl}{3}$

6-10. 2.048 in. upward

6-11. 0.575 in.

6-12. $-$ 134,400 lb ft^3

6-13. $\dfrac{Pl^3}{48EI}$

6-14. 1550 lb ft^3

6-15. 5430 lb

6-16. 0.293 in.; 2290 psi

6-17. 1.61 in.

6-18. 10,800 lb ft^3

6-19. 0.152 in.

6-20. 0.335 in.

6-21. 0.417 in.

6-22. 0.18 in.

6-23. $\delta_{conc} = 1.6\delta_{dist}$

6-25. 2700 lb ft^3 (point C is above B)

6-26. 2880 lb ft^3 (point C is below B)

6-27. 19,200 lb ft^3

6-28. $\dfrac{Pl^3}{8}$

6-29. 360 lb

6-30. $R = 37.5$ lb; $V_{max} = -82.5$ lb; $M_{max} = -270$ lb ft

6-31. $R = 405$ lb; $V_{max} = -675$ lb; $M_{max} = -1620$ lb ft

6-32. $R = 400$ lb; $V_{max} = -800$ lb; $M_{max} = -1800$ lb ft

6-33. $R = 900$ lb; $V_{max} = -600$ lb; $M_{max} = -1800$ lb ft

6-34. $R = 587$ lb; $V_{max} = -480$ lb; $M_{max} = -960$ lb ft

6-35. $R = 443$ lb

6-36. $R_1 = 20.4$ lb; $R_2 = 72.8$ lb

6-37. $R_A = R_B = 1.5$ kips; $M_A = M_B = -4.5$ kip ft

6-38. $R_A = R_B = 360$ lb; $M_A = M_B = -720$ lb ft

6-39. $R_A = 711$ lb; $R_B = 1690$ lb; $M_A = -1600$ lb ft; $M_B =$ 2400 lb ft

6-40. $R_A = R_B = 6$ kips; $M_A = M_B = -12$ kip ft

6-41. $R_A = 20$ kips; $R_B = 7$ kips; $M_A = -34.6$ kip ft; M_B · 20 kip ft

6-42. $EI\delta = 27$ kip ft^3

6-43. 0.012 in.

6-44. 0.288 in.

6-45. 0.11 in.

6-46. $\dfrac{7Pl^3}{16EI}$

6-47. 3.73 in.

6-48. 1.32 in.

6-49. 480 lb

6-50. $\dfrac{3Wl^3}{384EI}$

6-51. 24 in.^{4}

6-52. 4000 lb

6-53. 1125 lb/ft

6-54. 7.5 ft

6-55. 7500 lb

6-56. 16,800 psi

6-57. 2182 psi

6-58. 2280 lb

6-59. 540 lb

6-60. 720 lb/in.

6-61. 4030 psi

CHAPTER 7

7-1. $\sigma_t = 1520$ psi; $\sigma_c = 1360$ psi

7-2. 3180 lb

7-3. 12,000 lb in.

7-4. $\sigma_t = 16{,}900$ psi; $\sigma_c = 13{,}700$ psi

7-5. The straight bar will carry 7 times as much load as the curved bar

7-6. $\sigma_t = 6120$ psi; $\sigma_c = 6240$ psi

7-7. 4510 psi

7-8. 16,600 psi

7-9. 165 lb

7-10. 17.8 ft

7-11. 6750 lb

7-12. 6.67 lb

7-13. 17,900 lb

7-14. 2500 psi

7-15. 2730 lb

7-16. 480,000 lb

7-17. 58 bars

7-18. 268 bars

7-19. 39,300 lb ft

7-21. 800 psi

7-22. $\theta = \text{arc tan } \frac{1}{2}$

7-23. $\sigma_{max} = \pm\ 4280$ psi; $\sigma_{min} = \pm\ 107$ psi

7-24. 15.3 in.

7-25. $\sigma_a = 800$ psi, $\sigma_b = 0$; $\sigma_c = -\ 600$ psi; $\sigma_d = -\ 200$ psi

7-26. 4000 psi; the critical section is at A

7-27. 24 kips directed downward

7-28. 12,000 psi

7-29. 5060 psi

7-30. 463 lb

7-31. $\sigma_{max} = -\ 280$ psi; $\sigma_{min} =$

7-32. 1800 lb; tension governs

7-33. 92 kips

7-34. 2 kips

7-35. $\sigma_a = -0.778$ ksi; $\sigma_b = -0.111$ ksi; $\sigma_c = 0.556$ ksi; $\sigma_d = -0.111$ ksi

7-37. 3.67 in.

7-38. 1.69 ksi

7-39. 49,300 lb

7-40. $\sigma_{x'} = 250$ psi, $\tau_{x'} = 250$ psi; $\sigma_{y'} = 250$ psi, $\tau_{y'} = -250$ psi

7-41. $\sigma_{x'} = 500$ psi, $\tau_{x'} = 500$ psi; $\sigma_{y'} = 500$ psi, $\tau_{y'} = -500$ psi

7-42. Mohr's circle has a radius of zero length; hence, for any value of θ, $\sigma = 100$ psi and $\tau = 0$

7-43. $\sigma_{x'} = 0$, $\tau_{x'} = -86.6$ psi; $\sigma_{y'} = 100$ psi, $\tau_{y'} = 86.6$ psi

7-44. $\sigma_{x'} = 80$ psi, $\tau_{x'} = 75$ psi, $\sigma_{y'} = -180$ psi, $\tau_{y'} = -75$ psi

7-45. $\sigma_{x'} = -175$ psi, $\tau_{x'} = 43.3$ psi; $\sigma_{y'} = -125$ psi, $\tau_{y'} - 43.3$ psi

7-46. Shear governs; $P = 13{,}900$ lb

7-47. Shear governs; $P = 9600$ lb

7-48. $\sigma = 2100$ psi; $\tau = -520$ psi

7-49. $\tau_{max} = 7000$ psi

7-50. $\sigma_x = 5.95P$; $\sigma_y = -6.048P$; $P = 2000$ lb

7-51. (a) $\sigma = -4330$ psi, $\tau = 2500$ psi; (b) $\sigma = 4330$ psi, $\tau = 2500$ psi

7-52. $\sigma = \pm 10$ ksi

7-53. $\sigma = 2.96$ ksi; $\tau = 4.6$ ksi

7-54. (a) $\sigma_{max} = 8120$ psi at 37.8° CW from x-axis; $\sigma_{min} = -120$ psi at 127.8° CW from x-axis

(b) $\tau_{max} = 4120$ psi at 7.25° CCW from x-axis; $\tau_{min} = -4120$ psi at 97.25° CCW from x-axis

7-55. (a) $\sigma_{min} = -7.39$ ksi at 34.1° CW from x-axis; $\sigma_{max} = 3.39$ ksi at 124.1° CW from x-axis

(b) $\tau_{max} = 5.39$ ksi at 100.9° CCW from x-axis; $\tau_{min} = -5.39$ ksi at 10.9° CCW from x-axis

7-56. (a) $\sigma_{max} = 3000$ psi at 45° CCW from x-axis; $\sigma_{min} = -1000$ psi at 135° CCW from x-axis

(b) Planes of maximum and minimum shear are those given in problem

7-57. (a) $\sigma_{max} = 14.3$ ksi at 112.5° CW from x-axis; $\sigma_{min} = -8.3$ ksi at 22.5° CW from x-axis

(b) $\tau_{max} = 11.3$ ksi at 67.5° CW from x-axis; $\tau_{min} = -11.3$ ksi at 22.5° CCW from x-axis

7-58. $-$ 8430 psi

7-59. 59,300 lb

7-60. $\sigma_{max} = 2620$ psi; $\tau_{max} = 1120$ psi

7-61. $\sigma_{max} = 12{,}790$ psi; $\sigma_{min} = -590$ psi; $\tau_{max} = 6690$ psi

7-62. $\tau_{max} = 9540$ psi

7-63. $\sigma_{max} = 14{,}240$ psi; $\tau_{max} = 7330$ psi

7-64. $\sigma_{max} = -6560$ psi; $\tau_{max} = 3340$ psi

CHAPTER 8

8-1. $P = 48{,}000$ lb; $e = 88.9$ per cent

8-2. $P = 75{,}000$ lb; $e = 75$ per cent

8-3. 0.505 in. leg

8-4. (a) 400 psi; (b) 500 psi; (c) 250 psi; (d) 250 psi

8-5. $P = 88.8$ psi; $e = 80$ per cent

8-6. $L_1 = 6.41$ in.; $L_2 = 12.1$ in.

8-7. $L_1 = 4.33$ in; $L_2 = 8.17$ in.

8-8. $L = 12.9$ in.

8-9. $P = 11{,}800$ lb; $e = 41.7$ per cent

8-10. 76.9 per cent

8-11. 5370 lb in.

8-12. 6.67 in.

8-13. $P = 2940$ lb; $e = 43.6$ per cent

8-14. $P = 3940$ lb; $e = 58.3$ per cent

8-15. $P = 5860$ lb; $e = 39.1$ per cent

8-16. $P = 375$ lb

8-17. $d = 1$ in.; $t = 0.3125$ in.; $w = 2.78$ in.

8-18. $d = 0.922$ in.; $t_1 = 0.362$ in.; $t_2 = 0.724$ in.; $w = 2.3$ in.

8-19. $\sigma_1 = 20{,}000$ psi; $\sigma_2 = 16{,}000$ psi; $\sigma_3 = 12{,}000$ psi; $\sigma_4 = 8000$ psi; $\sigma_5 = 4000$ psi

8-20. $P = 23$ kips; $e = 38.3$ per cent

8-21. $P = 79.1$ kips; $e = 87.9$ per cent

8-22. $P = 69.4$ kips; $e = 92.5$ per cent; 12 rivets arranged in rows of 1:2:3:3:2:1

8-23. $P = 79.5$ kips; $e = 79.5$ per cent

8-24. Use 5 bolts.

8-25. 140 kips; bearing on web governs

8-26. 10.8 kips

8-27. $\frac{P}{P'} = 0.633$

8-28. 20.2 kips

8-29. 6 in.

8-30. $F_D = 9.29$ kips; $F_A = 2.14$ kips

8-31. 10,000 lb

CHAPTER 9

9-1. (a) 416; (b) 499; (c) 120; (d) 119; (e) 82.7; (f) 51.2; (g) 41.4

9-2. 39.3 ft

9-3. $r_{a-a} = r_{b-b} = 2.41$ in.

9-4. $\frac{l}{r} \geqq 62.8$

9-5. $\frac{l}{r} \geqq 59$

9-7. $l \geqq 42.7$ in.

9-8. 66.5 kips

9-9. 4400 lb

9-10. 1 in.

9-11. $P = 20.6$ kips; $x = 1$ ft

9-12. $T = 3780$ lb; $\sigma = 2790$ psi

9-13. 132° F

9-14. (a) 7.4 ft; (b) 8.6 ft; (c) 18.2 ft; (d) 37.5 in.

9-15. $P_{cr} = 9380$ lb; $\sigma_{cr} = 6420$ psi

9-16. 4.8 in. $< x <$ 120 (approximate answer)

9-17. $F_C = 724$ lb; $F_A = 28{,}900$ lb

9-18. 405 kips

9-19. $\sigma_a = 18$ ksi; f.s. = 1.86

9-20. f.s. = 1.67 for $\frac{l}{r} = 0$; f.s. = 1.92 for $\frac{l}{r} = C_c$

9-21. (a) 222 kips; (b) 348 kips

9-22. 8 WF 48

9-23. 283 kips

9-24. 242 kips

9-25. 8000 lb

9-26. 31.3 kips; 77.1 kips

9-27. 51.2 kips

9-28. 39.8 in.

9-29. $P_{max} = 8190$ lb; $P_{min} = 275$ lb

APPENDIX A

A-1. 12 rods

A-2. $\sigma_t = 1700$ psi; $\sigma_c = 5270$ psi; $\tau = 664$ psi

A-3. 85 in.

A-4. 299 hp

A-5. 1.46 ft

A-6. (a) 0.053 rad; (b) 4.86 in.; (c) 1890 psi

A-7. 151,200 lb

A-8. 211 lb

A-9. 1.48 in.

A-10. 4770 lb

A-11. 0.7505

A-12. $M = -1000x - 50x^2$

A-13. 111,500 lb

A-14. 6930 psi

A-15. 7360 psi

A-16. (a) 11,500 lb; (b) 10,900 psi; (c) 917 lb/ft; (d) 8750 psi

A-17. 3720 lb

A-18. 770 hp

A-19. 5680 lb

A-20. 4.36 in.

A-21. 14 WF 30

A-22. 3.17 in.

A-23. 0.181 in.

A-24. 29.8 in.^{3}

A-25. 6 bolts

A-26. $\sigma_t = 4920$ psi; $\sigma_c = 6600$ psi

A-27. $\sigma_s = 22{,}400$ psi; $\sigma_c = 11{,}200$ psi; $\sigma_b = 8960$ psi; $\sigma_a = 7470$ psi

A-28. $F_{fir} = 40{,}000$ lb; $F_{spruce} = 30{,}000$ lb

A-29. 7.59 kips

A-30. 224 lb

A-31. $M_{reinforced}/M_{wood} = 4.34$

A-32. $1\frac{5}{8}$ in. by 8.82 in. (use standard 2 in. by 10 in. lumber)

A-33. $V_{max} = -11{,}000$ lb; $M_{max} = 24{,}500$ lb ft

Index