THEORY
OF
MACHINES

B. B. LOW

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THEORY OF MACHINES

B. B. LOW, M.A.(Cantab.), M.I.Mech.E.

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PREFACE TO THIRD EDITION

THE ground covered on the subject of Vibrations has been increased by the addition of two chapters. An article on Hooke's Joint is now included in Chapter III and part of Article 202 has been rewritten and illustrated to demonstrate that when a suspended mass vibrates the sum of the three types of energy is constant. The latter change was made because the procedure often adopted of regarding $\frac{1}{2}kx^2$ as the strain energy due to vibration seems to require investigation since it is incorrect and yet leads to the right result.

The author renews his thanks, as expressed on p. vi, to the various authorities who have allowed their examination questions to be used.

B. B. LOW.

May 1958.

EXTRACT FROM PREFACE TO SECOND EDITION

THIS book deals with the subject of *Theory of Machines* to about the standard required in most Universities and in the leading Engineering Institutions. It consists of the greater part of an earlier work, *Engineering Mechanics*, first published in 1942, together with eight additional chapters.

The author has included several matters which are not in the usual text-books, as for example the analytical treatment of brakes, and in particular the theory given for brakes with external-contracting pivoted shoes which has not previously appeared in print. Consequently it is hoped that the book may be of use to engineers and designers, as well as to students.

The student is recommended to study at least part of the chapter on Dimensions and Dynamical Similarity, as a dimensional check on an equation can prevent mistakes being made.

The author is greatly indebted to Professor W. Steeds, O.B.E., B.Sc., to Mr J. E. Taylor, M.Sc.Tech., and to his son Mr E. D. Low, M.A., for reading the manuscript and giving valuable criticism. He also thanks the following firms for the information they kindly supplied:—

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Some of the examples and exercises have been selected from examination papers set by the Universities of Cambridge and London, by the Board of Education (now Ministry of Education), and by the Institutions of Civil and Mechanical Engineers; these are designated [C.U.], [U.L.], [B.E.], [Inst. C.E.] and [I.Mech.E.] respectively. The author acknowledges with thanks the permission—granted by the Syndics of the Cambridge University Press, the Senate of the University of London, the Controller of H.M. Stationery Office, the Council of the Institution of Civil Engineers, and the Council of the Institution of Mechanical Engineers—to use these questions.

B. B. LOW.

October 1953.

THE author is entirely responsible for the solution of the worked examples and for the answers to the exercises. Where examination questions are used the authorities concerned are in no way committed to approval of the solutions and answers given.

CONTENTS

THEORY OF MACHINES

CHAPTER I

VELOCITY—ACCELERATION—VECTORS

1. Greek Alphabet.—A few of the Greek letters are used in this volume, and the alphabet is given here for reference.

Α	α	alpha	Ι	ι	iōta	Ρ	ρ	rho
Β	β	bēta	Κ	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	Τ	τ	tau
Δ	δ	delta	Μ	μ	mu	Υ	υ	upsīlon
Ε	ϵ	epsīlon	Ν	ν	nu	Φ	ϕ	phi
Ζ	ζ	zeta	Ξ	ξ	xi	Χ	χ	chi
Η	η	ēta	Ο	o	omīcron	Ψ	ψ	psi
Θ	θ	thēta	Π	π	pi	Ω	ω	ōmega

2. Velocity and Speed.—All motion is relative motion, and when a body is said to be fixed or at rest it is only at rest relative to another body. In engineering work, motion is generally measured relative to the frame of a machine or to the earth. The rate of change of position of a point is called the *velocity* of the point and it involves direction and sense as well as rate. The rate without consideration of direction is called *speed.* Direction is defined by saying the motion is along a certain straight line, say AB, then the sense of the direction is either from A towards B or from B towards A. If the motion of a point is along a curve, then at any instant the direction of motion is along the tangent to the curve at the point.

Speed is *uniform* or *variable* according as equal or unequal distances are traversed in equal intervals of time, however short these intervals may be. A point may have uniform speed along any path, either straight or curved, but if the

A

velocity is to be uniform the motion must be in a straight line since velocity involves direction as well as rate. The velocity of a point is variable when its speed is changing or when its direction of motion is changing or when both speed and direction are changing.

A moving point has *linear velocity* and the magnitude of this velocity is measured in units of length per unit time—for instance, feet per second or miles per hour. Suppose the speed, or magnitude of the velocity, is constant and is denoted by v, then if a distance s is travelled in time t,

$$s = vt \quad \text{or} \quad v = s/t.$$

If s is in feet and t is in seconds, then v is in feet per second.

A line turning about a point has *angular velocity*. If the angular velocity is uniform and its magnitude is denoted by ω, and if θ is the angle turned through in time t, then

$$\theta = \omega t \quad \text{or} \quad \omega = \theta/t.$$

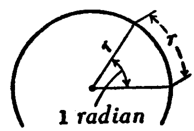

FIG. 1.

If θ is in radians and t is in seconds, then ω is in radians per second. It may be remarked here that a radian is the angle subtended at the centre of a circle by an arc whose length is equal to the radius (Fig. 1) and that $360° = 2\pi$ radians.

3. **Distance-Time Graphs.**—Let the distance s travelled by a point be plotted against time t, using axes OS and

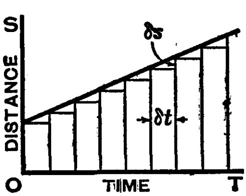

FIG. 2.

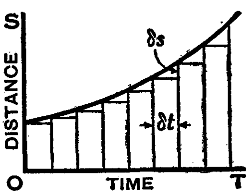

FIG. 3.

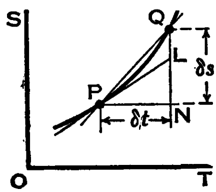

FIG. 4.

OT (Fig. 2), then if equal distances δs are traversed in equal times δt the speed is $\frac{\delta s}{\delta t}$, which is constant, and the graph is a straight line. If the distances δs are not equal in all the equal intervals δt, the graph is a curve (Fig. 3) and the speed is variable since $\frac{\delta s}{\delta t}$ is different for each interval of time. For any particular interval of time the value of $\frac{\delta s}{\delta t}$ is called the *average speed* during that interval.

To find the instantaneous value of the speed at any moment, consider points P and Q near together on the curve, a short length of which is shown enlarged in Fig. 4. Draw PN and QN parallel to OT and OS, respectively, and intersecting at N. The average speed between P and Q is $\frac{QN}{PN}$ or $\frac{\delta s}{\delta t}$, where $\delta s = QN$ and $\delta t = PN$, and the nearer Q is to P the more nearly does $\frac{\delta s}{\delta t}$ represent the speed at P. Now let Q approach P, then ultimately the chord PQ becomes the tangent PL to the curve at the point P. The value to which $\frac{\delta s}{\delta t}$ approaches, as δt approaches zero, is written $\frac{ds}{dt}$ and this is the slope of the tangent and the speed at P. Therefore the speed at any point P may be found graphically by drawing a tangent to the curve and finding the slope $\frac{LN}{PN}$, measuring LN with the distance scale and measuring PN with the time scale. Although it is often difficult to draw a tangent with great accuracy, the resultant error due to inaccuracy in measurement should be made as small as possible by drawing LN at a greater distance from P than is shown in the Fig.

Sometimes a short length PQ of the curve is so nearly straight that $\frac{\delta s}{\delta t}$ gives a close approximation to the speed at the point P. When the curvature is more noticeable it will be more nearly correct to take $\frac{\delta s}{\delta t}$ as the speed at the middle of the interval. When the speed is constant, then the graph is a straight line and at any point the value of $\frac{ds}{dt}$ is the same as the ratio $\frac{\delta s}{\delta t}$.

If the equation of the curve is known, the value of $\frac{ds}{dt}$ should be found by differentiation. This method is discussed in Chap. III.

Although the distance-time graph does not show the direction of motion of a point at any instant, this direction is usually known and the speed $\frac{ds}{dt}$ is often called the velocity.

4. Acceleration.—The rate of change of a velocity is called *acceleration* and it may be uniform or variable. Since velocity involves speed and direction, acceleration involves change of speed or change of direction or change of both speed and direction. Generally, in continuous motion, acceleration is regarded as positive or negative according as the velocity is increasing or decreasing. Negative acceleration is also called *deceleration* or *retardation*. When the direction of motion is reversed at intervals, as, for instance, during vibrating motion, the acceleration is usually taken as positive or negative according as it is in the direction of positive or negative displacement. Acceleration is *linear acceleration* when the variable velocity is linear and it is *angular acceleration* when the variable velocity is angular.

If a point starting from rest moves along a straight line with a uniform acceleration f, then the velocity v after time t is

$$v = ft \quad \text{and} \quad f = v/t.$$

If v is in feet per second and t is in seconds, then f is in feet per second per second. Similarly, for uniform angular acceleration where motion is from rest, the relation between angular velocity ω, angular acceleration a, and time t is

$$\omega = at \quad \text{or} \quad a = \omega/t.$$

If ω is in radians per second and t is in seconds, then α is in radians per second per second.

The linear acceleration of a falling body whose motion is unresisted is denoted by g, and the value of g is approximately 32·2 feet per second per second. Actually this acceleration, due to gravity, varies slightly at different places on the earth and depends on the altitude, but for engineering purposes it is sufficiently accurate to take it as a constant.

5. Velocity - Time Graphs.—Velocity-time graphs are shown in Figs. 5 and 6. When the velocity increases at a uniform rate (Fig. 5) equal increments δv occur in equal intervals of time δt, the graph is a straight line and the acceleration is $\frac{\delta v}{\delta t}$, which is constant.

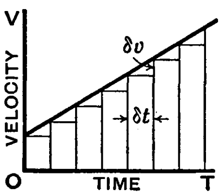

FIG. 5.

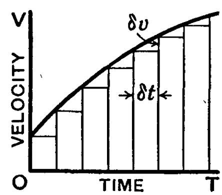

FIG. 6.

When the velocity increases at a varying rate (Fig. 6) the increment δv has different values in the equal intervals of time δt. The average acceleration over one of these intervals is $\frac{\delta v}{\delta t}$, and the value to which this ratio approaches if δt is made to approach zero is written $\frac{dv}{dt}$, which is the instantaneous value of the acceleration. When the acceleration is constant, $\frac{dv}{dt}$ has the same value as $\frac{\delta v}{\delta t}$.

The acceleration at any instant may be found graphically by drawing a tangent to the velocity-time graph and measuring the slope, which is the graphical interpretation of $\frac{dv}{dt}$ provided the lengths are measured with the appropriate velocity and time scales. Owing to inaccuracies in drawing and measurement, this graphical method only gives approximate results.

The use of the notation $\frac{dv}{dt}$ for the instantaneous value of the acceleration is exactly the same as the use of $\frac{ds}{dt}$ for the instantaneous value of the velocity which was dealt with in Art. 3. When the equation of the velocity-time graph is known, the value of $\frac{dv}{dt}$ should be found by differentiation. Since $v=\frac{ds}{dt}$, $\frac{dv}{dt}$ may be written $\frac{d\left(\frac{ds}{dt}\right)}{dt}$ or more briefly $\frac{d^2s}{dt^2}$, but it should be understood that $\frac{d^2s}{dt^2}$ is to be taken as one symbol denoting acceleration or rate of change of velocity.

It will be shown later (Arts. 28 and 29) that when motion is along a curve there is an acceleration along the normal at each point, even when the velocity is of uniform magnitude. Therefore, in general, the slope of the velocity-time graph will not give the total acceleration unless the motion is along a straight line.

6. Fluxional Notation for Velocity and Acceleration.— Instead of writing velocity as $\frac{ds}{dt}$ and acceleration as $\frac{dv}{dt}$ or $\frac{d^2s}{dt^2}$, the notation $\dot{s}$ for velocity and $\dot{v}$ or $\ddot{s}$ for acceleration is sometimes very convenient because it is written more quickly and saves space. $\dot{s}$ is read as *s dot* and $\ddot{s}$ is read as *s double dot* or *s two dot.* Similarly, $\dot{\theta}$ denotes angular velocity and $\ddot{\theta}$ or $\dot{\omega}$ denotes angular acceleration.

7. Relations between Linear Motion and Circular Motion. —Let P be any point on a disc which is rotating in its own plane about its centre O and let OP$=r$ (Fig. 7). When OP has turned through θ radians the distance s which P has travelled is given by $s=\theta r$ and at any instant the direction of motion of P is perpendicular to the radius OP.

Since $s=\theta r$, the rate of change of s must be r times the rate of change of θ, therefore, if at any instant v is the linear velocity of P and ω is the angular velocity of OP,

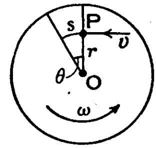

FIG. 7.

$$v=\omega r \quad \text{or} \quad \omega=v/r.$$

For a given value of ω the value of v is proportional to r, for instance if r is doubled then v is also doubled.

At any moment let the acceleration of P tangential to its path be f and let the angular acceleration of OP be α. Since $v=\omega r$, the rate of change of v must be r times the rate of change of ω, therefore

$$f=\alpha r \quad \text{or} \quad \alpha=f/r,$$

and for a given value of α, f is proportional to r.

The units will now be considered. If distance is measured in feet and time in seconds, then since an angle in radians is an arc divided by a radius, or feet divided by feet, or a ratio which cannot have dimensions, therefore

$$\omega=\frac{v}{r}=\frac{\text{feet}}{\text{sec.}}\cdot\frac{1}{\text{feet}}=\frac{\text{radians}}{\text{sec.}},$$

and
$$\alpha=\frac{f}{r}=\frac{\text{feet}}{\text{sec.}^2}\cdot\frac{1}{\text{feet}}=\frac{\text{radians}}{\text{sec.}^2}.$$

8. **Vectors.**—Quantities such as displacement, velocity, and acceleration which involve direction as well as magnitude are called *vector quantities* and may be represented by straight lines called *vectors*. For instance, a displacement from A to B (Fig. 8) may be represented by a line *ab* drawn parallel to AB and to some definite scale so that the length *ab* is proportional to the length AB. The directions of the two

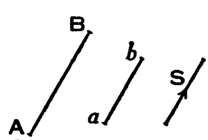

FIG. 8.

lines must be the same and it is for this reason that they are made parallel. The *sense* of the direction, that is whether the displacement is from A to B or from B to A, is given by the order in which the letters *a* and *b* are mentioned. A displacement from A to B is represented by the vector *ab*, and a displacement from B to A is represented by the vector *ba*. The sense may also be shown by putting an arrowhead on the vector, which may then be labelled with one letter, such as S in the Fig., but with this method the vector can only represent one sense.

Suppose a point P (Fig. 9) is moved in the directions PA and PB simultaneously and it is required to find the resultant displacement. Draw a line *pa* parallel to the direction PA and of such length that it represents to some scale the displacement in the direction PA. Similarly, draw *pb* parallel to PB to represent the displacement in the direction PB. Complete the parallelogram *pacb* and join *pc*, then *pc* represents the resultant displacement of the point P to the same scale that *pa* and *pb* represent the separate displacements. The parallelogram *pacb* is called a *vector parallelogram* or *parallelogram of vectors.* The resultant *pc* is the vector sum of the vectors *pa* and *pb*, and this vector addition may be written as

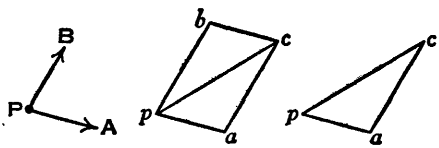

FIG. 9.

$$pa + pb = pc.$$

Various notations are used to ensure that it is understood that such an equation represents the addition of vectors and not an algebraic sum, for instance,

$$\overline{pa} + \overline{pb} = \overline{pc},$$

or

$$\overrightarrow{pa} + \overrightarrow{pb} = \overrightarrow{pc},$$

but in most cases these notations are not essential.

Since *ac* is equal and parallel to *pb*, the vector sum of *pa* and *pb* may also be found by drawing the triangle *pac*, called a *vector triangle*, then

$$pa + pb = pa + ac = pc.$$

The sum of any number of vectors may be found by drawing a polygon. Suppose it is required to find the resultant of the vectors R, S, T, and U (Fig. 10). Draw *ab* parallel and equal to R, *bc* parallel and equal to S, *cd* parallel and equal to T, and *de* parallel and equal to U. (Of course *ab*, *bc*, etc. may be made proportional to the corresponding vectors R, S, etc.) Join *ae*, then *ae* is the resultant of the given vectors and *abcde* is called a *vector polygon*. Join *ac* and *ad*, then *ac* is the resultant of *ab* and *bc*, *ad* is the resultant of *ac* and *cd*, and *ae* is the resultant of *ad* and *de*, therefore *ae* is the resultant of the given vectors.

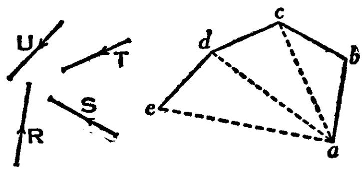

FIG. 10.

So far only displacement vectors have been considered, but velocity is displacement divided by time and therefore a vector triangle or a vector polygon may be drawn in which each vector represents a velocity. Similarly, since acceleration is change of velocity divided by time, a vector triangle or a vector polygon may be drawn in which each vector represents an acceleration.

9. Resolution of Vectors.—Since the vector sum or resultant of two vectors *ab* and *bc* (Fig. 11) may be obtained by drawing a triangle *abc*, conversely the resultant *ac* may be resolved or split up into the two vectors *ab* and *bc*. These two vectors are called *component vectors* or *components* and each may be drawn in any direction as indicated in Fig. 12. It is often desirable to have the components in mutually perpendicular directions (Fig. 13); in this case the fixing of the direction of one component also fixes the direction of the other component.

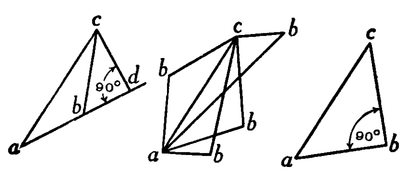

FIG. 11. FIG. 12. FIG. 13.

It will be noticed in Fig. 11 that by drawing *cd* perpendicular to *ab* and meeting *ab* produced at *d*, the component *bc* is resolved into components *bd* and *dc*, *bd* being in the same direction as *ab*; therefore, unless two components are mutually perpendicular, each one may be resolved so as to have a component in the direction of the other.

10. Analytical Determination of Resultants and Components.—Suppose it is required to find by calculation the resultant R of vectors P and Q (Fig. 14) which are inclined to one another at an angle β.

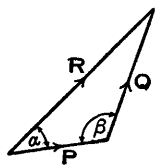

FIG. 14.

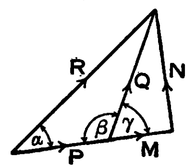

FIG. 15.

The resultant may be calculated from the cosine formula

$$R = \sqrt{P^2 + Q^2 - 2PQ \cos \beta}.$$

Let α be the angle between P and R, then

$$\frac{\sin \alpha}{\sin \beta} = \frac{Q}{R} \quad \text{or} \quad \alpha = \sin^{-1}\left(\frac{Q}{R} \sin \beta\right).$$

The same results could also be obtained without using the cosine formula. Let Q be resolved in two directions, along and perpendicular to the direction of P. Denoting these components by M and N (Fig. 15) and writing γ for $180° - \beta$, then $M = Q \cos \gamma$ and $N = Q \sin \gamma$.

The resultant R is the hypotenuse of a right-angled triangle and is given by

$$R = \sqrt{(P+M)^2 + N^2},$$

also $$\sin \alpha = \frac{N}{R} \quad \text{and} \quad \alpha = \sin^{-1} \frac{N}{R},$$

or $$\tan \alpha = \frac{N}{P+M} \quad \text{and} \quad \alpha = \tan^{-1} \frac{N}{P+M}.$$

The second method is better illustrated by the example shown in Fig. 16. Suppose it is required to find the resultant R of vectors P_1, P_2, and P_3, inclined at angles θ_1, θ_2, and θ_3, respectively, to an axis OX.

Draw the axis OY at right angles to OX, then project P_1, P_2, and P_3 on to the axes by drawing the perpendiculars

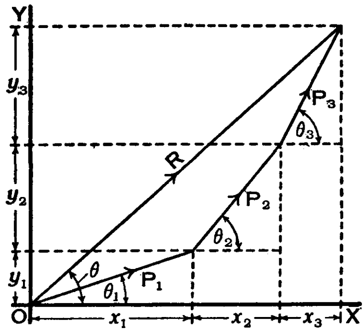

FIG. 16.

as shown by the dotted lines and let the projections be denoted by x_1, y_1, etc.

Then

$$x_1 = P_1 \cos \theta_1, \quad x_2 = P_2 \cos \theta_2, \quad x_3 = P_3 \cos \theta_3,$$
$$y_1 = P_1 \sin \theta_1, \quad y_2 = P_2 \sin \theta_2, \quad y_3 = P_3 \sin \theta_3,$$

and $$R = \sqrt{(x_1 + x_2 + x_3)^2 + (y_1 + y_2 + y_3)^2}.$$

If R is inclined at an angle θ to OX, then

$$\theta = \sin^{-1} \frac{y_1 + y_2 + y_3}{R} = \tan^{-1} \frac{y_1 + y_2 + y_3}{x_1 + x_2 + x_3}.$$

CHAPTER II

VELOCITY DIAGRAMS—INSTANTANEOUS CENTRES

11. Relative Velocity.—Consider points A and B (Fig. 17) moving in the plane of the paper and having velocities v_a and v_b, respectively, relative to the paper. It is required to find the velocity of B relative to A—that is, the velocity with which B appears to be moving when viewed from A.

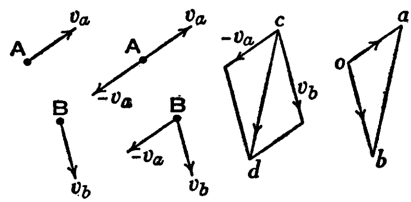

FIG. 17.

Give A and B velocities equal to $-v_a$, then A will be at rest relative to the paper but the velocity of B relative to A will be unaltered. Draw the parallelogram of velocities for B, then the diagonal cd represents the velocity of B relative to the paper and therefore relative to A because A is now at rest.

The same result may be obtained more quickly from the vector triangle oab, drawing oa and ob to represent v_a and v_b, respectively, and then joining ab. The velocity of B relative to A is represented by ab, for it can be seen that $ab = cd$. The arrowheads shown on the triangle oab are unnecessary in practice, but they have been put on to emphasize the fact that the vectors oa and ob represent the *actual velocities* v_a and v_b, neither of these being reversed.

In a similar way, by bringing the point B to rest, it can be shown that the velocity of A relative to B is ba.

The point o, the starting-point when drawing the triangle, is usually called a *pole*.

In the triangle oab there are two ways of getting from

o to b, either direct or via a, and the vector equation representing these two routes is

$$ob = oa + ab.$$

In words—

Velocity of B = *Velocity of* A + *Velocity of* B *relative to* A.

This vector equation may also be written in the form

$$oa = ob - ab,$$

but $$-ab = ba,$$

therefore $$oa = ob + ba.$$

In words—

Velocity of A = *Velocity of* B + *Velocity of* A *relative to* B.

This result also follows immediately from the triangle oab, for there are two ways of getting from o to a, either direct or via b.

12. Special Cases of Relative Velocity.—(1) Let A and B be two points on a rigid link moving in the plane of the paper (Fig. 18). The distance from A to B cannot change and so there can be no relative motion between A and B along the line AB. Therefore relative motion must be perpendicular to AB. If rotary motion of the link is clockwise, then relative to B the velocity of A is in the direction Aa perpendicular to AB, and relative to A the velocity of B is in the direction Bb perpendicular to AB.

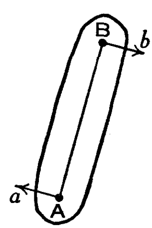

FIG. 18.

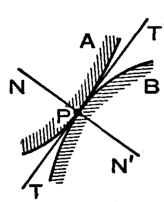

FIG. 19.

(2) Let A and B (Fig. 19) be two rigid links having relative motion in the plane of the paper and being continuously in contact. Let P be the point of contact for an instant, let TPT′ be the common tangent, and let NPN′ be the common normal. Since the links are rigid and remain in contact, there can be no relative motion along the common normal NPN′. Therefore if one link slides on the other, the relative motion at P must be along the common

tangent TPT′. If the motion is pure rolling, then there is no relative motion at P.

13. Angular Velocity Ratios.—Two examples of the determination of angular velocity ratios, in which the results of the preceding Art. are used, will now be given.

Example 1.—Let a link AB turning about a fixed point A with an angular velocity ω_1 (Fig. 20) turn a link CD about a fixed point D with an angular velocity ω_2, the points B and C being coupled by a link BC. The three links all move in the plane of the paper. It is required to find the value of the velocity ratio ω_2/ω_1 when the links are in the positions shown.

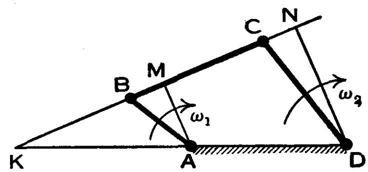

FIG. 20.

Produce CB and DA to intersect at K, and draw AM and DN perpendicular to BC, meeting BC at M and N, respectively.

Assuming that the link BC does not alter in length, the points B and C cannot have any relative velocity along BC. The component of the velocity of B in the direction BC is $\omega_1 . \text{AM}$, and the component of the velocity of C in the direction BC is $\omega_2 . \text{DN}$. Therefore

$$\omega_1 . \text{AM} = \omega_2 . \text{DN} \quad \text{or} \quad \frac{\omega_2}{\omega_1} = \frac{\text{AM}}{\text{DN}}.$$

Since the triangles KAM and KDN are similar,

$$\frac{\text{AM}}{\text{DN}} = \frac{\text{KA}}{\text{KD}}.$$

Therefore
$$\frac{\omega_2}{\omega_1} = \frac{\text{KA}}{\text{KD}}.$$

Example 2.—A cam AB rotating about a fixed point A with an angular velocity ω_1, turns a link DC about a fixed point D with an angular velocity ω_2 (Fig. 21). Assuming that the cam and the link are always in contact, it is required to find the value of the ratio ω_2/ω_1 for the position shown.

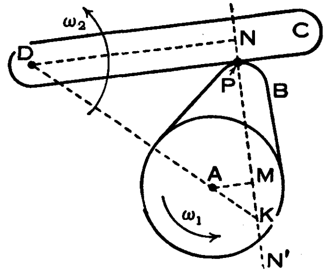

FIG. 21.

Let P be the point of contact. Draw the common normal NPN′ and draw AM and DN perpendicular to it. Draw a line through D and A to intersect NN′ at K.

There cannot be any relative velocity, between the cam and the link, along the common normal NPN′, therefore, as in the preceding example, $\omega_1 . \mathrm{AM} = \omega_2 . \mathrm{DN}$, or

$$\frac{\omega_2}{\omega_1} = \frac{\mathrm{AM}}{\mathrm{DN}} = \frac{\mathrm{KA}}{\mathrm{KD}}.$$

14. Velocities of Points on a Rigid Body—Velocity Image.—Let A and B (Fig. 22) be points on a rigid body having plane motion and let the velocities of the points A and B be v_a and v_b, respectively. Assume that v_a is known in direction and magnitude, but that v_b is known in direction only. It is required to find the magnitude of v_b and then to show how the velocity of any other point on the body may be determined. All points in any line perpendicular to the plane of the paper, which is a plane of motion, have the same velocity, and therefore it is sufficient to consider A and B and other points as being in the plane of the paper.

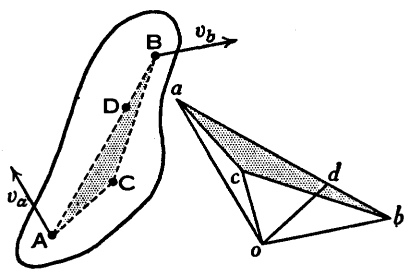

FIG. 22.

The velocity diagram *oab* may now be constructed. To some convenient scale draw *oa* equal and parallel to v_a and draw *ob* parallel to v_b. The length of *ob* is as yet unknown. Since A and B are points on a rigid body, there cannot be relative motion between them along the line AB, therefore the relative motion between A and B must be in the direction perpendicular to AB. Therefore draw *ab* perpendicular to AB, intersecting *ob* at *b*, then the length *ob* represents the magnitude of v_b, the velocity of the point B. The vector equation may be written—

Velocity of B = Velocity of A + Velocity of B relative to A,

or $$ob = oa + ab.$$

Similarly, $$oa = ob + ba.$$

Now consider any other point C on the body and in the plane of the paper. Join AC and BC. The velocity of C relative to A is perpendicular to AC and the velocity of C relative to B is perpendicular to BC. Therefore, in the velocity diagram, draw *ac* and *bc* perpendicular to AC and BC, respectively, and let *c* be the point of intersection, then *ac* represents the velocity of C relative to A and *bc* represents the velocity of C relative to B. Join *oc*, then *oc* represents the velocity of the point C, as can be seen from either of the vector equations

$$oc = oa + ac \quad \text{or} \quad oc = ob + bc.$$

The triangles *abc* and ABC are similar, since each side of one is perpendicular to a side of the other; the triangle *abc* is called the *velocity image* of the triangle ABC. The velocity of any point D in the line AB is given by *od*, where *d* is found by dividing *ab* so that $\frac{ad}{ab} = \frac{\mathrm{AD}}{\mathrm{AB}}$. The angular velocity of the body AB may be found by dividing the velocity of B relative to A (or of A relative to B) by the length AB—that is,

$$\text{Angular velocity of AB} = \frac{ab}{\mathrm{AB}} = \frac{ba}{\mathrm{AB}},$$

and the direction in this particular case is clockwise.

15. Instantaneous Centre.—Another way of determining velocities is by means of the instantaneous centre. Suppose A and B (Fig. 23) are points on a rigid body having plane motion and that A and B are moving in the plane of the paper. Let v_a and v_b be the velocities of A and B, respectively, when the body is in the given position, and let the directions of these velocities be defined by the angles α and β as indicated. Usually one velocity is known in both magnitude and direction, but the other is known in direction only.

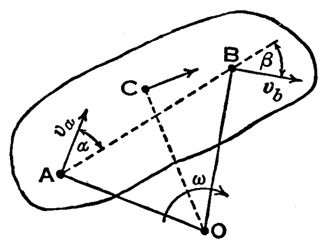

FIG. 23.

Draw AO and BO perpendicular to the directions of the velocities of A and B, respectively, and let AO and BO intersect at O, then O is called the *instantaneous centre* or *virtual centre* of the body relative to the paper.

Since the direction of motion of the point A is perpendicular to AO, the body can turn for an instant about any point in AO without affecting the direction of motion of A; similarly, since the direction of motion of the point B is perpendicular to BO, the body can turn for an instant about any point in BO without affecting the direction of motion of B. Therefore the body can turn about the point O for an instant without affecting the directions of the motions of A and B.

Let ω be the angular velocity of the body, then

$$v_a=\omega . \mathrm{OA}, \quad v_b=\omega . \mathrm{OB}, \quad \text{and by division} \quad \frac{v_a}{v_b}=\frac{\mathrm{OA}}{\mathrm{OB}}.$$

Therefore, if the magnitude of one velocity is known, the magnitude of the other can be determined.

The same result will now be obtained in another way to show that the use of the instantaneous centre is justifiable. Since A and B are points on a rigid body, there cannot be any relative motion between them in the line AB, therefore, resolving along AB,

$$v_a \cos \alpha=v_b \cos \beta, \quad \text{or} \quad \frac{v_a}{v_b}=\frac{\cos \beta}{\cos \alpha}.$$

Now $\quad \cos \alpha=\sin \mathrm{OAB}$ and $\cos \beta=\sin \mathrm{OBA}$, therefore $$\frac{v_a}{v_b}=\frac{\sin \text{OBA}}{\sin \text{OAB}}=\frac{\text{OA}}{\text{OB}},$$

and this is the relation obtained before, therefore the instantaneous centre method gave the correct result. The velocity of any other point C on the body and in the plane of the paper is ω.OC and its direction is perpendicular to OC.

It will be noticed that OAB is a triangle of velocities turned through a right angle, for OA and OB are perpendicular to the directions of motion of A and B, respectively, and AB is perpendicular to the direction of the relative motion between A and B. The magnitudes of the velocities are represented by the lengths of the sides of the triangle, to the scale on which OA represents the magnitude of the velocity of A.

The method of instantaneous centres is convenient and easy to apply in simple mechanisms, but the velocity diagram method is to be preferred in complex cases. Examples in which both methods are illustrated are given in Arts. 17 and 18.

The instantaneous centre of a moving body is continually changing its position unless the body has rotary motion only, and the locus of the instantaneous centre is called a *centrode.* A line drawn through an instantaneous centre perpendicular to the plane of motion is called an *instantaneous axis,* and the locus of this axis is a surface called an *axode.*

16. Permanent and Fixed Centres.—Two particular cases of instantaneous centres are illustrated in Fig. 24, which shows the slider-crank mechanism. Relative to the crank CB, the connecting-rod AB turns about the point B, and B is the instantaneous centre of AB relative to CB or of CB relative to AB. The point B is also called a *permanent centre* since it always connects AB and CB, although its position is continually changing relative to the paper or the fixed link AC. The point A, connecting the slider to the rod AB, is another permanent centre.

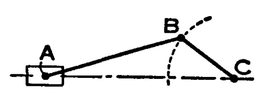

FIG. 24.

The crank CB turns about the fixed point C, and C is the instantaneous centre of CB relative to the paper or the fixed link AC. The point C is also called a *fixed centre* since its position does not change.

17. Velocities in the Slider-Crank Mechanism.—In this Art. a simple example illustrates the use of a velocity diagram and of an instantaneous centre.

A slider at A (Figs. 25 and 26) is connected by a rod AB to the end B of a crank CB of length r. The crank rotates in a clockwise direction about the point C with uniform angular velocity ω, and the slider reciprocates along the line AC. It is required to find the velocity v_a of the slider A and the angular velocity Ω of the rod AB when the crank is in the given position. (The analytical determination of the velocity of the slider is given in Art. 27.)

Let v_b be the velocity of the point B, then $v_b = \omega r$ and its direction is perpendicular to CB.

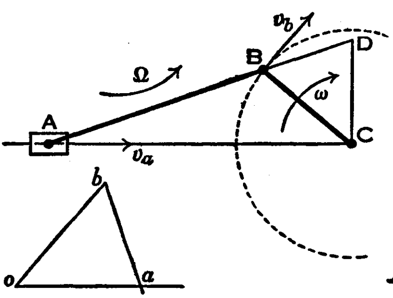

FIG. 25.

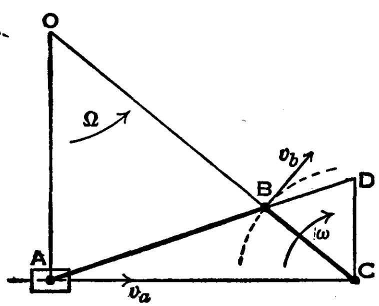

FIG. 26.

Method 1.—*Velocity Diagram.*—From a convenient pole o (Fig. 25) draw ob perpendicular to CB and oa parallel to AC, then, using a suitable scale, make $ob = v_b = \omega r$. Since AB is a rigid link, the velocity of A relative to B is perpendicular to AB, therefore draw ba perpendicular to AB and intersecting oa at a.

Velocity of A = Velocity of B + Velocity of A relative to B,

or
$$oa = ob + ba.$$

Therefore oa represents v_a, the velocity of A.

The angular velocity of AB is $\Omega=\frac{ba}{\text{AB}}$, where ba is measured with the velocity scale and AB is measured with the scale to which the mechanism is drawn. The direction of this angular velocity is anticlockwise.

In this particular example the required velocities may also be obtained as follows. Produce AB to intersect at D a line drawn from C perpendicular to AC.

The triangles CBD and *oba* are similar, since CB, CD, and BD are perpendicular to *ob*, *oa*, and *ba*, respectively,

therefore $$\frac{v_a}{v_b}=\frac{oa}{ob}=\frac{\text{CD}}{\text{CB}},$$

and CD represents the velocity of A to the same scale that CB represents the velocity of B. Also, measuring BD with the same scale and measuring AB with the scale to which the mechanism is drawn, $\Omega=\frac{\text{BD}}{\text{AB}}$.

Since $v_b=\omega.\text{CB}$, using the mechanism scale, therefore $v_a=\omega.\text{CD}$ and $\Omega=\omega\frac{\text{BD}}{\text{AB}}$. The velocities can be found for various positions of the crank by repeating the construction.

Method 2.—*Instantaneous Centre.*—Draw AO (Fig. 26) perpendicular to AC, the direction of motion of A, and produce CB, which is perpendicular to the direction of motion of B, to intersect AO at O. Since A and B are moving perpendicular to OA and OB, respectively, therefore O is the instantaneous centre of AB relative to the paper—that is, relative to AC, for AC is not moving.

Since B is a point in BC which is turning about C, $v_b=\omega.\text{CB}$; also since B is a point in AB which is turning for an instant about O, $v_b=\Omega.\text{OB}$.

$$\text{Therefore}\quad \Omega.\text{OB}=\omega.\text{CB},\quad \text{or}\quad \Omega=\omega\frac{\text{CB}}{\text{OB}}.$$

In some cases the point O may be off the paper, therefore produce AB to intersect at D a line drawn from C perpendicular to AC, then OAB and CDB are similar triangles

and
$$\frac{CB}{OB}=\frac{BD}{AB}.$$

Therefore
$$\Omega=\omega\frac{BD}{AB}.$$

Also $v_a=\Omega.OA=\omega\dfrac{BD.OA}{AB}$, but $\dfrac{OA}{AB}=\dfrac{CD}{BD}$,

therefore
$$v_a=\omega.CD,$$

and since $v_b=\omega.CB$, v_a is represented by CD to the same scale that CB represents v_b.

18. Valve Velocity in a Hackworth Valve Gear.—The Hackworth valve gear is shown diagrammatically in Fig. 27. A link CM, rotating anticlockwise with uniform velocity about the fixed centre C, drives a link MN which is pivoted to a slider at N, the slider being constrained by guides to move in a straight line NL. The link MN drives a slider V through a link PV, the slider being constrained by guides to move in a straight line HV. The valve, which is not shown, has the same motion as the slider V.

Given that the linear velocity of the point M is 5 feet per second, it is required to find the velocity of the slider V when the configuration of the mechanism is as shown.

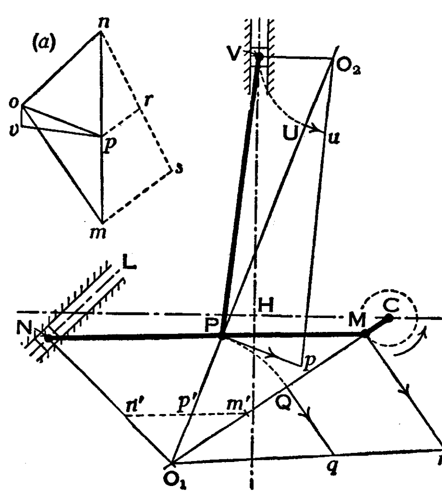

FIG 27.

Method 1.—*Velocity Diagram.* — The velocity diagram, shown at (*a*), is constructed as follows. From any pole *o* draw *om* parallel to the direction of motion of the point M, that is perpendicular to CM, making the length *om* represent 5 feet per second to some convenient scale. The actual length of *om* is 0·75 inch, but on the original drawing from which Fig. 27 was prepared it was twice as long, and a much larger scale would give greater accuracy. Draw *on* parallel to NL, the direction of motion of N, and draw *mn* perpendicular to MN, meeting *on* at *n*, then *mn* is the velocity image of MN. Divide *mn* at *p* so that $\frac{np}{nm}=\frac{NP}{NM}$. This has been done by drawing a line *ns* at a convenient angle to *nm*, making $ns=\frac{1}{2}NM$ and $nr=\frac{1}{2}NP$, joining *sm* and drawing *rp* parallel to *sm*. Join *op*, then *op* represents the velocity of P.

Finally, draw *ov* parallel to VH and *pv* perpendicular to PV, meeting *ov* at *v*, then *ov* represents the velocity of the point V in direction and magnitude. By measurement $ov=0{\cdot}11$ inch, therefore the velocity of V is $\frac{5}{0{\cdot}75}\times 0{\cdot}11=0{\cdot}73$ foot per second, approximately.

Method 2.—*Instantaneous Centres.*—Produce CM to intersect at O_1 a line drawn from N perpendicular to NL, then O_1 is the instantaneous centre of the link MN. Join O_1P, then P is moving perpendicular to O_1P. Produce O_1P to intersect at O_2 a line drawn from V perpendicular to HV, then O_2 is the instantaneous centre of the link PV. To some scale, O_1M, O_1P, and O_1N represent the velocities of M, P, and N, respectively, and to some other scale, O_2P and O_2V represent the velocities of P and V, respectively, but it is unnecessary to know these scales to find the velocity of the slider V.

Velocity of $V=5\times\frac{O_1P}{O_1M}\times\frac{O_2V}{O_2P}$ feet per second, and if the lengths O_1P, etc., are measured, then the numerical value of the velocity of V may be calculated, but a graphical solution is given in the Fig.

Draw M*m* perpendicular to O_1M and equal to 0·75 inch to represent the velocity of M—that is, 5 feet per second. As previously stated, a larger scale would give greater accuracy. Join O_1m. With centre O_1 and radius O_1P

draw an arc PQ meeting O_1M at Q, then draw Qq parallel to Mm and meeting O_1m at q. Draw Pp perpendicular to O_1P and equal to Qq, then Pp represents the velocity of P in direction and magnitude.

Join O_2p. With centre O_2 and radius O_2V draw an arc VU meeting O_2P at U, then draw Uu parallel to Pp and meeting O_2p at u. The magnitude of the velocity of V is represented by Uu and its true direction is along VH. The proof will be left to the student. By measurement $Uu = 0{\cdot}11$ inch, therefore the velocity of V is $\frac{5}{0{\cdot}75} \times 0{\cdot}11 = 0{\cdot}73$ foot per second, approximately, as before. If it happens that an instantaneous centre is off the paper and cannot be used, the difficulty may be overcome graphically. Suppose, for instance, the point O_1 is inaccessible and the velocity of the point P is wanted. Along MO_1 mark off Mm' equal to Mm. Draw $m'n'$ parallel to MN and meeting NO_1 at n'. Divide $m'n'$ at p' so that $\frac{n'p'}{n'm'} = \frac{NP}{NM}$ and join $p'P$. The construction for obtaining the point p' would be similar to that shown by the dotted lines in the velocity diagram at (a).

The velocities of P and M are proportional to their distances from O_1, also, since $m'n'$ is parallel to MN,

$$\frac{O_1P}{O_1M} = \frac{Pp'}{Mm'},$$

therefore $$\frac{\text{Velocity of P}}{\text{Velocity of M}} = \frac{O_1P}{O_1M} = \frac{Pp'}{Mm'}.$$

Therefore Pp' is the magnitude of the velocity of P to the same scale that Mm' is the magnitude of the velocity of M. The direction of the velocity of P is perpendicular to Pp', clockwise about O_1, since M is moving clockwise about O_1. Similarly, to the same scale, Nn' is the magnitude of the velocity of N and the direction is perpendicular to Nn', clockwise about O_1.

19. Relative Instantaneous Centres of any Three Links in a Plane.—Let the thick lines A, B, and C (Fig. 28) represent *any* three links which are in the same plane, and let O_{AB}, O_{BC}, and O_{AC} be the three instantaneous centres for relative motion between the links. For instance, when there is relative motion between A and B, then, for an instant, either link turns relative to the other about the point O_{AB}. Similarly for B and C and for A and C. It will now be proved that the three instantaneous centres lie in one straight line.

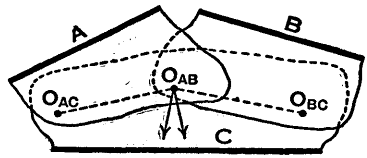

FIG. 28.

Since the links as drawn do not overlap, it may be helpful to imagine a separate piece of paper attached to each in order to overlap at the instantaneous centres as indicated in the Fig.

As B moves relative to C, it turns for an instant about O_{BC}, and the point O_{AB}, considered as a point on B, moves perpendicular to the straight line $O_{AB}O_{BC}$. Also, as A moves relative to C it turns for an instant about O_{AC}, and the point O_{AB}, considered as a point on A, moves perpendicular to the straight line $O_{AC}O_{AB}$. Now the relative motion between A and B is one of rotation about O_{AB}, therefore this point can move only in one direction relative to C, and since this direction is perpendicular to $O_{AB}O_{BC}$ and to $O_{AC}O_{AB}$, it follows that $O_{AB}O_{BC}$ and $O_{AC}O_{AB}$ must be in one straight line. Therefore the instantaneous centres O_{AC}, O_{AB}, and O_{BC} lie in one straight line.

Example.—Consider the mechanism known as a *four-bar chain*, consisting of four connected links and labelled, 1, 2, 3, and 4 in Fig. 29. Any one of these links may be fixed, leaving the other three links free to move.

The various instantaneous centres will now be determined. Relative motion between links 1 and 2 can only occur by rotation about the point labelled O_{12}, and this is the

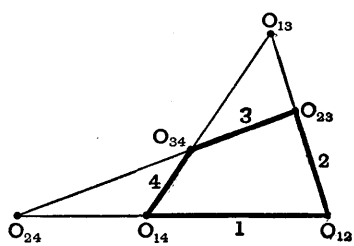

FIG. 29.

instantaneous centre for link 1 relative to link 2 or *vice versa*. Similarly, the instantaneous centres O_{23}, O_{34}, and O_{14} are the points so indicated. Now, suppose link 1 is fixed, then the directions of motion of the end points of link 3 are known—that is, the point O_{23} may move at right angles to link 2 and the point O_{34} may move at right angles to link 4, therefore O_{13} is at the intersection of links 2 and 4 produced, and the link 3 may turn for an instant about the instantaneous centre O_{13}. The same point would be arrived at if link 3 were supposed fixed and the motion of link 1 were considered. The point O_{24} is obtained in a similar manner.

Now consider any three of the four links and it will be seen that their instantaneous centres lie in a straight line. For instance, the instantaneous centres of the links 1, 2, and 3 lie in the straight line $O_{12}O_{23}O_{13}$.

20. Centrodes.—As defined in Art. 15, a centrode is the locus of an instantaneous centre. It will now be demonstrated that if any two links of a mechanism move in one plane then their relative motion may be produced by rolling together two centrodes fixed to the links.

Let A_1B_1, A_2B_2, A_3B_3, and A_4B_4 be four positions of a

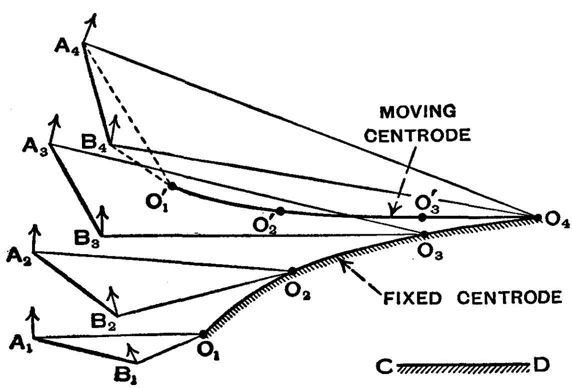

Fig. 30.

moving link AB (Fig. 30) relative to a link CD which will be taken as fixed, and let the points A and B be moving in the directions shown by the arrows when AB is in each of its four positions.

Draw A_1O_1 and B_1O_1 perpendicular to the directions of motion of A_1 and B_1, respectively, then the point O_1 at the intersection of A_1O_1 and B_1O_1 is the instantaneous centre for the position A_1B_1. The instantaneous centres O_2, O_3, and O_4 are found in a similar way. Now draw a smooth curve through these points. If a great many positions of the moving link were considered, then the curve could be drawn fairly accurately. This curve is the locus of the instantaneous centre of the moving link, therefore it is a centrode and since it is fixed relative to the fixed link CD it may be called the *fixed centrode.*

Suppose now that the moving link AB had a piece of tracing paper attached to it when in its first position. The instantaneous centres could be marked on the tracing paper as it moved with the link and a centrode could be drawn through these points. This centrode may be called the *moving centrode.*

The moving centrode can be drawn in one or more positions without using the tracing paper. For instance, for the position when AB is at A_4B_4, begin by drawing the triangle $A_4B_4O_1'$ equal to the triangle $A_1B_1O_1$. It is clear that if the moving link returns to its first position, taking the triangle $A_4B_4O_1'$ with it, then the point O_1' would coincide with the point O_1. By drawing two more triangles (not shown in the Fig.) the points O_2' and O_3' can be found, then a smooth curve through the points O_1', O_2', O_3', and O_4 gives one position of the moving centrode.

Every point on the moving centrode is in turn coincident with a point on the fixed centrode, and the moving link turns for an instant about each of these coincident points. Therefore the moving centrode rolls on the fixed centrode, and the relative motion between AB and CD may be obtained by rolling the one curve on the other.

The same relative motion between the two links is obtained if the link AB and the centrode $O_1'O_2'O_3'O_4$ are fixed and the centrode $O_1O_2O_3O_4$ is allowed to roll taking with it the link CD.

The student is advised to draw his own figure to a large scale, then to obtain the moving centrode on a sheet of tracing paper and roll one centrode on the other. It is worth doing.

21. Centrodes for Double Slider Mechanism.—The link AB (Fig. 31) is pivoted at its ends to sliders which are constrained so that they can move along fixed straight lines DD′ and EE′ intersecting at C. It is required to find the centrodes for the relative motion between the link AB and the straight lines DD′ and EE′ which will be designated simply as the link DCE.

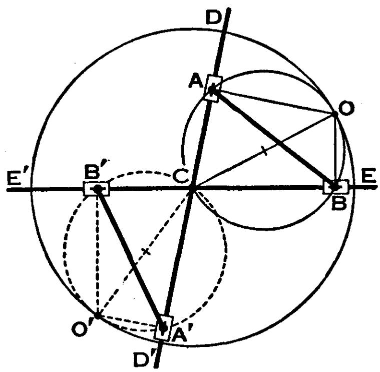

FIG. 31.

Draw AO perpendicular to CD, and BO perpendicular to CE, letting AO and BO intersect at O, then O is the instantaneous centre for the link AB when it is in the position shown.

Now suppose the link AB to be fixed and the link DCE to be free, in order to find how the point O moves relative to AB. Since the angles CAO and CBO are right angles and since the angle ACB is constant and the length AB is constant, therefore the points C, A, O, and B lie on a circle and the locus of O is a circle drawn with CO as a diameter.

To find how the point O moves relative to the link DCE, fix DCE and let AB be free to move. Since CO is of constant length, the locus of O is a circle drawn with its centre at C and with radius equal to CO.

The length CO may be expressed in terms of AB and the angle ACB. Since CAOB is a cyclic quadrilateral, the angle COB is equal to the angle CAB, therefore

$$CO = \frac{CB}{\sin COB} = \frac{CB}{\sin CAB} = \frac{AB}{\sin ACB}.$$

The motion of the link AB relative to the link DCE may be obtained by fixing it to the circle CAOB and rolling this circle, without slipping, on the inside of the fixed circle DOEO′. The moving link AB is shown in another position at A′B′.

Exercises II

1. The points A and B (Fig. 32) are moving in the directions shown. The speed of A is 20 feet per second and the speed of B is 40 feet per second. Find the velocity of B relative to A.

2. If in the preceding exercise the given velocity of B is the velocity of B relative to A, find the true velocity of B.

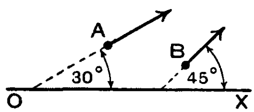

FIG. 32.

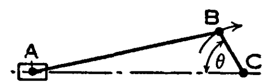

FIG. 33.

3. In the slider-crank mechanism (Fig. 33) AB = 4 feet, BC = 1 foot, and the velocity of the crank pin B is 10 feet per second. Find the velocity of the slider A when the crank angle θ has the values 30°, 45°, 60°, and 90°.

4. State the position of the instantaneous centre of the connecting-rod AB (Fig. 33), (*a*) when $\theta = 0°$, (*b*) when $\theta = 90°$.

5. The line of stroke of a piston A (Fig. 34) is offset a perpendicular distance of $\frac{1}{2}$ inch from the centre C of the crankshaft. The crank CB is rotating clockwise at 1500 revolutions per minute, CB = $2\frac{1}{2}$ inches, and AB = 11 inches. Find the velocity of the piston A in feet per second when the angle DCB = 105°, DC being parallel to the line of stroke.

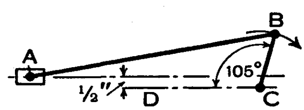

FIG. 34.

6. In the preceding exercise, assume that the outside diameters of the crankshaft journal and the crank pin B are 2 inches and $1\frac{3}{4}$ inches, respectively. Find the rubbing speed, in feet per second, of each journal when the crank is in the given position.

7. (*a*) A locomotive driving wheel rolls along a rail without slipping. Where is the instantaneous centre and what is the velocity of the top point of the wheel if the middle point is moving at 80 feet per second?

(*b*) The wheel turns and slips on the rail when the locomotive is at rest. Where is the instantaneous centre of the wheel?

(*c*) The wheel is 6 feet in diameter and a point on the rim has a velocity of 12 feet per second relative to the locomotive, which is moving at 5 feet per second. Where is the instantaneous centre of the wheel, what is the sliding velocity of the bottom point, and what is the velocity of the top point?

(*d*) Using the instantaneous centre, find the velocity of the foremost point of the wheel when the latter is moving as described in (*c*).

8. A water turbine blade is shown in Fig. 35. The edge A is moving at 50 feet per second in the direction A*a*, and the edge B is moving at 58 feet per second in the direction B*b*. The tangent to the blade at A makes an angle ϕ with A*a*, and the tangent at B makes an angle of 15° with B*b*. Water enters at A without shock and with a velocity of 100 feet per second, inclined at 20° to A*a*, as shown, and leaves at B with a velocity v inclined at an angle θ to B*b*. The velocity of the water relative to the blade is the same at B as at A.

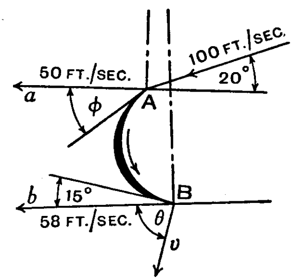

FIG. 35.

Find (*a*) the angle ϕ, (*b*) the velocity of the water relative to the blade, (*c*) the velocity v, and (*d*) the angle θ.

9. Fig. 36 shows the mechanism of the Robinson Air Engine. The crank OC is 2$\frac{3}{4}$ inches long. Connecting-rod PC is 12 inches long. BC = 5 inches, O′B = 12 inches, BE = 4 inches, EF = 6$\frac{1}{2}$ inches.

Draw the velocity diagram for the mechanism when the crank angle $\theta = 30°$ and the crank OC rotates at 180 revolutions per minute. State the velocities of the two pistons in feet per second. [U.L.]

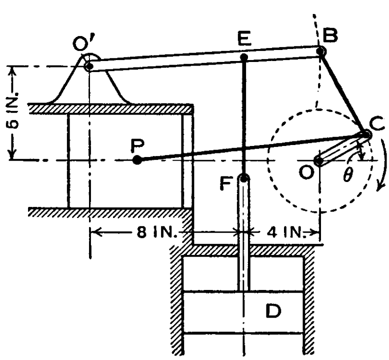

FIG. 36.

10. The link AD of the four-bar chain ABCD (Fig. 37) is fixed and the link AB is rotating clockwise at 20 revolutions per minute. The lengths of the links are as indicated. For the configuration shown, where the angle BAD = 135°, find the angular velocity in radians per second of BC and of CD, the

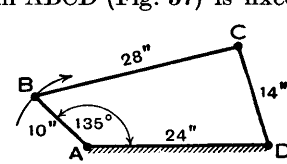

FIG. 37.

velocity of the point C, and the velocity of the middle point of BC.

11. In the mechanism shown in Fig. 38 the crank BC is connected by links to three crossheads A, F, and H. The line of stroke of A passes through the centre C, and F and H move parallel to AC. If BC is rotating clockwise at 150 revolutions per minute, find the velocities of A, F, and H when the angle ACB is 120° as shown. The dimensions are as follows: BC = 8 inches, AB = 28 inches, BE = 31 inches, AE = 13 inches, DE = 26 inches and is one straight link, DF = 8·5 inches, and EH = 8·1 inches. The lines of stroke of F and H are at perpendicular distances of 13·6 inches from the line AC.

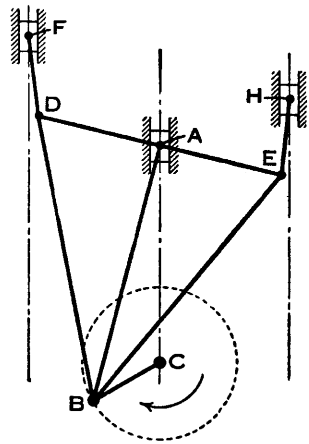

FIG. 38.

12. Referring to the diagram of the Hackworth valve gear (Fig. 27), find the velocity of the slider V when the angle HCM is 90° and the point M has a velocity of 4 feet per second. The other particulars are to be taken as follows: VH is perpendicular to CH and CH = 14 inches, NL is inclined at 45° to CH and intersects CH produced at 33 inches from C, CM = 3 inches, MN = $33\frac{1}{2}$ inches, MP = 15 inches, and PV = 30 inches.

13. Fig. 39 is a diagram of the mechanism of a valve gear for a certain setting, the lengths of the links which are not given in

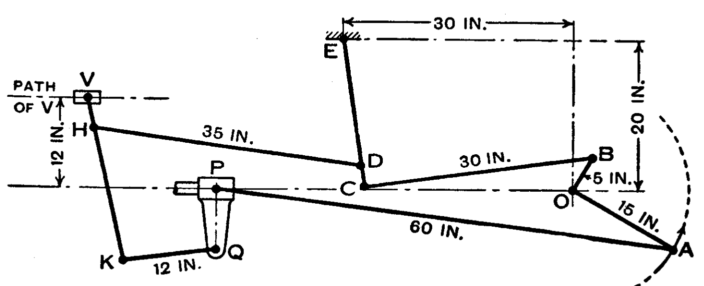

FIG. 39.

the figure being as follows: VH = 4 inches, VK = 22 inches, PQ = 8 inches, ED = 17 inches, EC = 20 inches.

The cranks OA and OB rotate together, being 90° apart, and the velocity of the main crank-pin A is 30 feet per second. Draw a velocity diagram and determine the velocity of the valve spindle when the crank angle AOP is 150°. (Scales—Draw the mechanism $\frac{1}{8}$th full size; 1 inch = 5 feet per second.) [C.U.]

14. Four links AB, AC, BD, and DC are such that AB = DC and AC = BD. They are jointed at A, B, C, and D in such a way that the links AC and BD cross each other. Prove that the relative motion of AB and CD is the same as that obtained by the rolling of two equal ellipses on one another with foci A and B and C and D, respectively.

Further, show that if the ellipses are free to rotate about A and C, respectively, and one is given a uniform angular velocity, the fractional fluctuation of speed of the other is given by $\frac{4a}{1-a^2}$ where $a = \frac{AB}{AC}$. [C.U.]

CHAPTER III

ANALYSIS OF VELOCITY AND ACCELERATION

22. An Application of Differentiation and Integration.—Denoting displacement by s and time by t, then velocity, or the rate of change of displacement, is given in magnitude by

$$v=\frac{ds}{dt},$$

the direction being the same as the direction of motion.

It is understood that the direction of motion is known whenever the term velocity is used. If the direction is unknown, then, strictly speaking, the term speed should be used, because velocity involves direction as well as magnitude.

Acceleration or the rate of change of velocity, *in the direction of motion*, is given by

$$f=\frac{dv}{dt}=\frac{d^2s}{dt^2}.$$

It will be seen later (Arts. 28 and 29) that when the motion is along a curve there is an acceleration perpendicular to the direction of motion.

If displacement is completely defined by one or more expressions, then the velocity and acceleration may be obtained by differentiation. Conversely, the velocity and displacement may be found from the acceleration by the process of integration.

Example 1.—The displacement s of a body along a straight line is given by the relation $s=4t^2-9$. It is required to find expressions for the velocity and the acceleration.

Since $$s=4t^2-9,$$

$$\text{velocity } v=\frac{ds}{dt}=8t,$$

and $$\text{acceleration } f=\frac{dv}{dt}=\frac{d^2s}{dt^2}=8.$$

If s is measured in feet and t in seconds, then the velocity is in feet per second and the acceleration is in feet per second per second.

Example 2.—The acceleration of a body having rectilinear motion is 8 ft./sec.^{2}. It is required to find the velocity and the displacement in terms of time t, given that the velocity is 16 ft./sec. when $t=2$ sec. and that the displacement is zero when $t=1{\cdot}5$ sec.

Since $$\frac{d^2s}{dt^2}=8,$$

integrating, $$v=\frac{ds}{dt}=8t+\text{A},$$

where A is an arbitrary constant or constant of integration.

Now $v=16$ when $t=2$ and substitution in the velocity equation gives

$$16=8\times 2+\text{A}, \quad \text{from which} \quad \text{A}=0,$$

therefore

$$v=\frac{ds}{dt}=8t,$$

where t is in sec. and v is in ft./sec.

Integrating again, $$s=4t^2+\text{B},$$

where B is a constant of integration.

Now $s=0$ when $t=1{\cdot}5$ and substitution in the displacement equation gives

$$0=4\times 1{\cdot}5^2+\text{B}, \quad \text{from which} \quad \text{B}=-9,$$

therefore $$s=4t^2-9,$$

where t is in sec. and s is in ft.

It should be noticed that the two integrations produce two arbitrary constants and that these constants cannot be evaluated unless two conditions are known. In this example the known conditions are the velocity and the displacement at certain times.

23. Alternative Expressions for Acceleration.—In the preceding Art. acceleration has been written as

$$f=\frac{d^2s}{dt^2}=\frac{dv}{dt} \quad . \quad . \quad . \quad (1).$$

Now
$$\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}=v\frac{dv}{ds},$$

so that
$$f=v\frac{dv}{ds} \quad . \quad . \quad . \quad . \quad (2),$$

is another way of expressing acceleration.

In (1) the independent variable is time t, and in (2) it is displacement s.

Separating the variables and integrating between appropriate limits, say v_1, v_2, t_1, t_2, s_1, and s_2, then from (1)

$$\int_{v_1}^{v_2} dv=\int_{t_1}^{t_2} fdt,$$

and from (2),
$$\int_{v_1}^{v_2} vdv=\int_{s_1}^{s_2} fds.$$

Both these forms are used in the next Art.

24. Uniform Acceleration—Formulæ.—Suppose that a body moving in a straight line with uniform acceleration f has an initial velocity u and that at time t the displacement is s and the velocity is v, both t and s being zero when $v=u$. It is required to find the relations between f, v, u, t, and s.

Since
$$f=\frac{dv}{dt},$$

therefore
$$\int_u^v dv=\int_0^t fdt.$$

Integrating, $v - u = ft,$

or $v = u + ft$ . . . . (1).

Since $f = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds},$

therefore $\int_u^v vdv = \int_0^s fds.$

Integrating, $\left[\frac{1}{2}v^2\right]_u^v = fs,$

$\frac{1}{2}(v^2 - u^2) = fs,$

or $v^2 = u^2 + 2fs$ . . . . (2).

From (1) and (2), eliminating f,

$s = \frac{1}{2}(u + v)t$ . . . . (3).

From (1) and (3), or from (1) and (2), eliminating v,

$s = ut + \frac{1}{2}ft^2$ . . . . (4).

If the initial velocity u is zero, then equations (1), (2), (3), and (4) become

$v = ft,\quad v^2 = 2fs,\quad s = \frac{1}{2}vt,$ and $s = \frac{1}{2}ft^2$. (5),

respectively.

Similar formulæ may be obtained for uniformly accelerated rotary motion.

Example.—A vehicle travels from A to B, a distance of 400 yards. Starting from rest at A, the vehicle is uniformly accelerated for 16 seconds to a speed of 24 miles per hour. It travels at this speed until it is 64 yards from B, then it is uniformly retarded and brought to rest at B. It is required to find the time taken to travel from A to B, the acceleration and the retardation.

Foot and second units will be used. The uniform speed is $v = 24 \times \frac{88}{60} = 35{\cdot}2$ ft./sec. Let t_1, t_2, and t_3 be the times of acceleration, uniform speed, and retardation respectively, and let s_1, s_2, and s_3 be the corresponding distances. Also let f_1 be the acceleration and f_3 the retardation.

The velocity-time graph for the journey is shown in

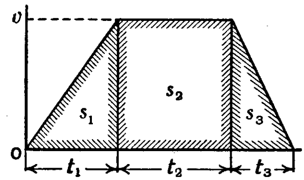

FIG. 40.

Fig. 40, the shaded areas representing the distances as indicated.

$$t_1 = 16 \text{ sec.} \quad s_1 = \tfrac{1}{2}vt_1 = \tfrac{1}{2} \times 35{\cdot}2 \times 16 = 281{\cdot}6 \text{ feet.}$$

Since $\qquad s_1 + s_2 + s_3 = 400 \times 3 = 1200$ feet

and $s_3 = 64 \times 3 = 192$ feet, therefore

$$s_2 = 1200 - s_1 - s_3 = 1200 - 281{\cdot}6 - 192 = 726{\cdot}4 \text{ feet.}$$

Now $s_2 = vt_2$, therefore $t_2 = \dfrac{s_2}{v} = \dfrac{726{\cdot}4}{35{\cdot}2} = 20{\cdot}64$ sec.

Also $s_3 = \frac{1}{2}vt_3$, therefore $t_3 = \dfrac{2s_3}{v} = \dfrac{2 \times 192}{35{\cdot}2} = 10{\cdot}91$ sec.

Total time is $t_1 + t_2 + t_3 = 16 + 20{\cdot}64 + 10{\cdot}91 = 47{\cdot}55$ sec.

$$\text{Acceleration } f_1 = \frac{v}{t_1} = \frac{35{\cdot}2}{16} = 2{\cdot}20 \text{ ft./sec.}^2.$$

$$\text{Retardation } f_3 = \frac{v}{t_3} = \frac{35{\cdot}2}{10{\cdot}91} = 3{\cdot}23 \text{ ft./sec.}^2.$$

25. Resultant Velocity and Resultant Acceleration.—If at time t a particle P has displacements x and y (Fig. 41) measured parallel to mutually perpendicular axes OX and OY respectively, then its velocities are

$$\dot{x} \text{ and } \dot{y},$$

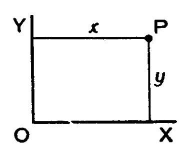

FIG. 41.

using the fluxional notation to save space (see Art. 6, p. 6).

The accelerations of P are

$$\ddot{x} \text{ and } \ddot{y}.$$

The resultant velocity at time t is (Fig. 42)

$$V=(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}},$$

in a direction making an angle ϕ with the x axis, and

$$\tan\phi=\dot{y}/\dot{x}.$$

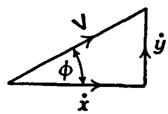

FIG. 42.

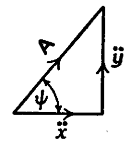

FIG. 43.

The resultant acceleration at time t is (Fig. 43)

$$A=(\ddot{x}^2+\ddot{y}^2)^{\frac{1}{2}}$$

in a direction making an angle ψ with the x axis, and

$$\tan\psi=\ddot{y}/\ddot{x}.$$

It is to be noted that the resultant acceleration *cannot*, in general, be obtained by differentiating the resultant velocity.

Example 1.—A particle P (Fig. 44) is moving in a circular path of radius $OP=r$ with uniform speed v. It is required to investigate the motion and to find the acceleration of the particle by considering the x and y components of the displacement.

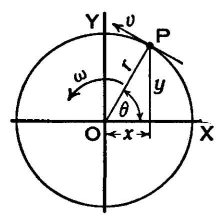

FIG. 44.

In Art. 28, p. 47, it is shown by using vectors that the acceleration is v^2/r or $\omega^2 r$, where ω is the angular velocity of the radius OP, and the direction of the acceleration is along the radius OP from P towards O. This result is now to be found analytically. The vector method is very much simpler in this case, but the analytical method is instructive.

Let OP make an angle θ with OX at time t, then from the Fig. it can be seen that

$$x=r\cos\theta,\quad y=r\sin\theta.$$

Now $\dfrac{dx}{dt}=\dfrac{dx}{d\theta}\dfrac{d\theta}{dt}$ or $\dot{x}=\dfrac{dx}{d\theta}\dot{\theta}$ and similarly $\dot{y}=\dfrac{dy}{d\theta}\dot{\theta}$, therefore,

differentiating with respect to t, the velocities are

$$\dot{x}=-r\sin\theta\,\dot{\theta},\quad \dot{y}=r\cos\theta\,\dot{\theta}.$$

Differentiating again with respect to t, the accelerations are

$$\ddot{x} = -r \cos\theta\, \dot{\theta}^2, \quad \ddot{y} = -r \sin\theta\, \dot{\theta}^2.$$

The resultant velocity is

$$(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}} = (r^2 \sin^2\theta\, \dot{\theta}^2 + r^2 \cos^2\theta\, \dot{\theta}^2)^{\frac{1}{2}} = r\dot{\theta} = r\omega = v,$$

which is obviously correct.

If the direction of this velocity is inclined at an angle ϕ to the axis OX, then

$$\tan\phi = \frac{\dot{y}}{\dot{x}} = \frac{r\cos\theta\,\dot{\theta}}{-r\sin\theta\,\dot{\theta}} = -\cot\theta,$$

and a little consideration will show that this direction is perpendicular to the radius OP.

The resultant acceleration is

$$(\ddot{x}^2 + \ddot{y}^2)^{\frac{1}{2}} = (r^2 \cos^2\theta\, \dot{\theta}^4 + r^2 \sin^2\theta\, \dot{\theta}^4)^{\frac{1}{2}} = r\dot{\theta}^2 = r\omega^2 = v^2/r.$$

If the direction of this acceleration is inclined at an angle ψ to the axis OX, then

$$\tan\psi = \frac{\ddot{y}}{\ddot{x}} = \frac{-r\sin\theta\,\dot{\theta}^2}{-r\cos\theta\,\dot{\theta}^2} = \tan\theta,$$

from which $\psi = \theta$ or $\theta + 180°$. Since the two components of the acceleration are negative, the second of these angles is the one required. Therefore the resultant acceleration of P is along the radius OP and is directed from P towards O.

It is evident that differentiating the resultant velocity v would not produce the resultant acceleration; it would merely show that the component of the acceleration in the direction of the resultant velocity is zero.

Example 2.—The x and y components of the displacement of a body at time t are given by

$$x = 2t^2 + 3 \quad \text{and} \quad y = t^3 - 4t^2 - 2,$$

using foot and second units. It is required to find the resultant velocity and resultant acceleration when $t = 5$ seconds.

Since $x=2t^2+3$ and $y=t^3-4t^2-2$,

therefore $\dot{x}=4t$, $\dot{y}=3t^2-8t$,

and $\ddot{x}=4$, $\ddot{y}=6t-8$.

When $t=5$, $\dot{x}=4\times 5=20$ and $\dot{y}=3\times 5^2-8\times 5=35$.

Resultant velocity is $(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}=(20^2+35^2)^{\frac{1}{2}}=40{\cdot}31$ ft./sec. If ϕ is the angle the direction of this velocity makes with the x axis,

$$\tan\phi=\tfrac{35}{20}=1{\cdot}75 \quad\text{and}\quad \phi=60^\circ\ 15'.$$

When $t=5$, $\ddot{x}=4$ and $\ddot{y}=6\times 5-8=22$.

Resultant acceleration is $(\ddot{x}^2+\ddot{y}^2)^{\frac{1}{2}}=(4^2+22^2)^{\frac{1}{2}}=22{\cdot}36$ ft./sec.^{2}. If ψ is the angle the direction of this acceleration makes with the x axis,

$$\tan\psi=\tfrac{22}{4}=5{\cdot}5 \quad\text{and}\quad \psi=79^\circ\ 42'.$$

26. Expressions for the Approximate Velocity and Acceleration in the Slider-Crank Mechanism.—Let r be the length of the crank CB (Fig. 45), l the length of the connecting-rod AB, and ω the uniform angular velocity of the crank. At time t let x be the displacement of the slider A from the beginning of its stroke, and let θ and ϕ be the angles turned through by the crank and connecting-rod respectively.

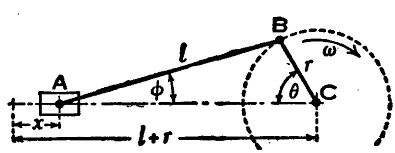

FIG. 45.

Displacement $x=l+r-l\cos\phi-r\cos\theta$.

Now $l\sin\phi=r\sin\theta$ or $\sin\phi=\dfrac{r}{l}\sin\theta$

and $$\cos\phi=(1-\sin^2\phi)^{\frac{1}{2}}=\left(1-\frac{r^2}{l^2}\sin^2\theta\right)^{\frac{1}{2}}$$

$$=1-\frac{r^2}{2l^2}\sin^2\theta, \quad\text{approximately,}$$

expanding by the binomial theorem and neglecting all terms after the second. This approximation gives sufficient accuracy for the usual values of r/l which occur in practice.

Therefore
$$x = l + r - l\left(1 - \frac{r^2}{2l^2}\sin^2\theta\right) - r\cos\theta$$
$$= r(1 - \cos\theta) + \frac{r^2}{2l}\sin^2\theta.$$

Velocity
$$v = \frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt}$$
$$= \left\{r\sin\theta + \frac{r^2}{2l}2\sin\theta\cos\theta\right\}\frac{d\theta}{dt}$$
$$= \omega r\left\{\sin\theta + \frac{r}{2l}\sin 2\theta\right\},$$

where
$$\omega = \frac{d\theta}{dt}.$$

Acceleration
$$f = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{d\theta}\frac{d\theta}{dt}$$
$$= \omega r\left\{\cos\theta + \frac{r}{2l}2\cos 2\theta\right\}\frac{d\theta}{dt}$$
$$= \omega^2 r\left\{\cos\theta + \frac{r}{l}\cos 2\theta\right\}.$$

Expressions for the exact acceleration are given in Ex. 17 and Ex. 18, p. 45.

Since the acceleration is the rate of change of the velocity, the former is zero when the latter is a maximum, for the slope of the velocity-time curve is then zero. The student is asked to plot a velocity-time curve in Ex. 13, p. 44, and an acceleration-time curve in Ex. 14, p. 44.

The method of determining the velocity of the slider by a velocity diagram was explained in Art. 17, p. 19, and the acceleration diagram is dealt with in Art. 32, p. 50.

27. Hooke's Joint.—This joint, shown diagrammatically in Fig. 46 at (*a*) enables power to be transmitted through two shafts whose axes intersect. The shafts A and B have forked ends connected by a crosspiece, shown separately at (*b*), which has journals with mutually perpendicular axes COC_1 and DOD_1. The relation between ω_b and ω_a, the angular velocities of B and A respectively, will be obtained.

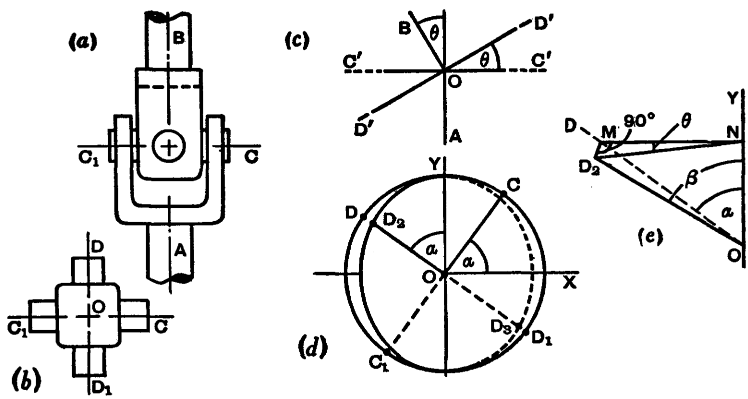

FIG. 46.

At (*c*) the angle between the centre lines of the shafts A and B is θ; $C'OC'$ and $D'OD'$ are the planes, perpendicular to the paper, in which the axes COC_1 and DOD_1 move. Imagine these planes to be represented by circular discs fixed to the ends of the shafts, then in the elevation at (*d*) the disc on B appears as an ellipse. Suppose A and COC_1 turn through an angle α from the axis OX. If A and B were in line, DOD_1 would then make an angle α with OY, but its actual position is D_2OD_3 on the ellipse, arrived at by turning DOD_1 about COC_1.

Let the true value of the angle D_2OY be β, as shown in the enlarged pictorial view at (*e*), where D_2M and MN are at right angles to OD and OY respectively and D_2 and N are joined; the angle D_2NM is θ and the angle D_2MN is 90°.

$$\tan \alpha = \frac{MN}{ON}, \quad \tan \beta = \frac{D_2N}{ON}, \quad \text{and} \quad \cos \theta = \frac{MN}{D_2N},$$

from which $$\tan \alpha = \tan \beta \cos \theta.$$

Differentiating with respect to time assuming θ is constant,

$$\sec^2 \alpha \frac{d\alpha}{dt} = \sec^2 \beta \frac{d\beta}{dt} \cos \theta,$$

from which $$\frac{\omega_b}{\omega_a} = \frac{\sec^2 \alpha}{\sec^2 \beta \cos \theta} = \frac{\cos \theta}{1 - \sin^2 \theta \cos^2 \alpha}.$$

This ratio is a maximum when $\cos \alpha = \pm 1$, or $\alpha = 0$ or $180°$, then $\omega_b/\omega_a = 1/\cos \theta$. It is a minimum when $\cos \alpha = 0$, or $\alpha = 90°$ or $270°$, then $\omega_b/\omega_a = \cos \theta$.

$\omega_b = \omega_a$ when $1 - \sin^2 \theta \cos^2 \alpha = \cos \theta$.

Exercises III

Take $g = 32{\cdot}2$ *ft./sec.*^{2}

1. If a body moves along a straight line according to the law $x = t^3 - 2t^2 + 7$, using foot and second units, find the velocity and the acceleration when $t = 5$ sec.

2. Given that the acceleration of a body moving along a straight line is 0·8 ft./sec.^{2}, that the velocity is 15 ft./sec. when $t = 9$ sec., and that the displacement is 14 ft. when $t = 0$, find the velocity and displacement when $t = 16$ sec.

3. The equation $x = \sin 3t - 4 \cos 3t$ gives displacement x in feet at time t in seconds. Find the lowest positive value of the time when the velocity is zero and then find the value of the acceleration at that time.

4. Two trains whose lengths are 106 and 120 feet, moving in opposite directions along parallel lines, meet when their velocities are 30 and 45 miles per hour respectively. At this instant the former has an acceleration of 2 feet per sec. per sec. and the latter of 1 foot per sec. per sec. Prove that they will pass each other in two seconds. [C.U.]

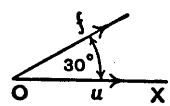

FIG. 47.

5. A particle has an initial velocity $u = 50$ ft./sec. along an axis OX and an acceleration $f = 4$ ft./sec.^{2} in a direction inclined at 30° to OX (Fig. 47). Find the direction of motion and the distance of the particle from O when 10 seconds has elapsed.

6. Two motor cyclists A and B are travelling towards towns C and D along straight roads OC and OD, the angle COD being 60°. At noon, A passes O at 10 miles per hour and has a constant acceleration of 4 miles per hour per minute, and B passes O at noon (just escaping a collision) at 20 miles per hour, and has a constant acceleration of 2 miles per hour per minute. Find the time when their distances from O are equal, and show (1) that the velocity of A relative to B is then 45·8 . . . miles per hour, and (2) that the distance between A and B is then increasing at the rate of 45 miles per hour. [C.U.]

7. A tramcar, starting from rest, is uniformly accelerated for 16 seconds to a speed of 30 miles per hour which is maintained for a distance of 280 yards, then it is uniformly retarded until it is brought to rest at a distance of 500 yards from the starting-point. Find the values of the acceleration, the retardation, and the total time for the journey.

8. A train starting from rest has an acceleration f feet per second per second, where $f = \frac{1}{1000}(t - 60)^2$, and t is the time in seconds. Find, graphically or analytically, the velocity attained and the distance passed over in the first minute. [C.U.]

9. The acceleration of a vehicle, starting from rest, is given by $f = 1{\cdot}65 - 0{\cdot}0052s$ in foot and second units. Find the velocity when $s = 125$ feet and when $s = 250$ feet.

10. A train passes three consecutive mile posts A, B, and C. The time taken to travel from A to B is 80 seconds and from B to C is 56 seconds. Assuming that the acceleration is uniform, find the speeds at A and C in miles per hour and the acceleration in feet per second per second.

11. If an aeroplane travels a distance r in a steady wind along a course making an acute angle θ with the direction from which the wind blows, and returns to its starting-point, the speed of the plane in still air being u miles per hour, and the wind-velocity being v miles per hour, show that it must be *steered* in directions making equal angles α on either side of its course, where $u \sin \alpha = v \sin \theta$.

Show also that if t_1, t_2 are the times of going and returning respectively,

(i) $r(t_1 + t_2) = 2ut_1t_2 \cos \alpha$;
(ii) $r(t_1 - t_2) = 2vt_1t_2 \cos \theta$;
(iii) $r^2 = t_1t_2(u^2 - v^2)$. [U.L.]

12. A train moving with constant acceleration is found to cover two consecutive distances of 200 yards in 10 and 9 seconds respectively. Determine the acceleration of the train, and find its velocity at the end of the first of these distances. [C.U.]

13. The displacement of a piston from the beginning of its stroke is given approximately by

$$x = 0{\cdot}230(1 - \cos 50\pi t) + 0{\cdot}026 \sin^2 50\pi t,$$

where x is in feet and the time t is in seconds.

Show that this equation may be written as

$$x = 0{\cdot}243 - 0{\cdot}230 \cos 50\pi t - 0{\cdot}013 \cos 100\pi t,$$

then, by differentiating either form, obtain an equation for the velocity of the piston.

Find the velocity when t has the values 0, $\frac{1}{200}$, $\frac{1}{100}$, $\frac{3}{200}$, and $\frac{1}{50}$ second, then, plotting these results, sketch the velocity-time graph.

14. If the velocity of the piston in the preceding exercise is

$$\frac{dx}{dt} = 11{\cdot}50\pi \sin 50\pi t + 1{\cdot}3 \sin 100\pi t,$$

find the acceleration when t has the values 0, $\frac{1}{200}$, $\frac{1}{100}$, $\frac{3}{200}$, and $\frac{1}{50}$ second, and plot the acceleration-time curve.

15. Given that the velocity of a piston is approximately

$$v = \omega r\left\{\sin \theta + \frac{r}{2l} \sin 2\theta\right\},$$

where ω is the uniform angular velocity of the crank, θ is the crank angle, and r and l are respectively the lengths of the crank and connecting-rod, find to the nearest degree the first value of θ when v is a maximum, taking $l=5r$.

16. In the slider-crank mechanism, Fig. 45, p. 39, $l \sin \phi = r \sin \theta$. Assuming that $\frac{d\theta}{dt}$ or ω, the angular velocity of the crank, is uniform, show by differentiation that the angular velocity and the angular acceleration of the connecting-rod are respectively

$$\frac{d\phi}{dt}=\frac{\omega r \cos \theta}{(l^2-r^2 \sin^2 \theta)^{\frac{1}{2}}},$$

and

$$\frac{d^2\phi}{dt^2}=\frac{\omega^2 r(r^2-l^2) \sin \theta}{(l^2-r^2 \sin^2 \theta)^{\frac{3}{2}}}.$$

17. Given that the displacement of a piston from the beginning of its stroke is

$$x=l+r-l\left\{1-\frac{1}{n^2} \sin^2 \theta\right\}^{\frac{1}{2}}-r \cos \theta,$$

where $n=l/r$, show that an exact expression for the acceleration is

$$\frac{d^2x}{dt^2}=\omega^2 r\left\{\frac{\sin^2 2\theta}{4(n^2-\sin^2 \theta)^{\frac{3}{2}}}+\frac{\cos 2\theta}{(n^2-\sin^2 \theta)^{\frac{1}{2}}}+\cos \theta\right\},$$

where the angular velocity $\omega=\frac{d\theta}{dt}$ is uniform.

Then show that when the acceleration is zero

$$\sin^6 \theta-n^2 \sin^4 \theta-n^4 \sin^2 \theta+n^4=0.*$$

18. For a direct acting engine CP is the crank, PD the connecting-rod, and the angles PCD and PDC are θ and ϕ respectively. If the crank rotates with a uniform angular velocity ω, prove that the acceleration of D is given by the expression

$$\frac{\omega^2 r}{k}\left\{\cos \phi+k \cos \theta-\frac{1-k^2}{\cos^3 \phi}\right\},$$

where k is the ratio of CP to PD and r is the length CP. Show that a close approximation to the acceleration of D is given by

$$\omega^2 r\left\{\cos \theta+\left(k+\frac{k^3}{4}\right) \cos 2\theta-\frac{k^3}{4} \cos 4\theta\right\}. \qquad \text{[C.U.]}$$

19. Fig. 48 shows a mechanism used for giving an intermittent motion to the film in a projection apparatus. A pin C, fixed to

* See D. A. Low's *Applied Mechanics*, p. 304, for roots of this equation for various values of n.

a wheel, rotating about an axis A with uniform angular velocity ω, engages with a slotted piece which rotates about an axis B. The slots are at right angles to one another and the ratio of AB to AC is $\sqrt{2}$. Show that during the motion of the slotted piece its angular velocity is given by $\dfrac{\sqrt{2}\cos\theta - 1}{3 - 2\sqrt{2}\cos\theta}\omega$, where θ is the angle CAB.

Find similar expressions for the velocity of sliding of the pin in a slot, and the angular acceleration of the slotted piece. [C.U.]

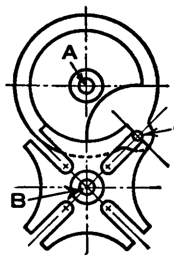

FIG. 48.

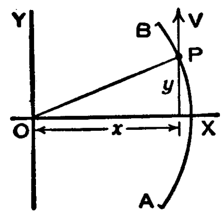

FIG. 49.

20. During the operation of a cam mechanism a point P moves along a short circular arc AB (Fig. 49) whose equation is $x^2 + y^2 = r^2$. It is arranged that the velocity in the direction parallel to the axis OY is constant and equal to V. Show that acceleration in the direction parallel to the axis OX is $-V^2r^2/x^3$.

CHAPTER IV

ACCELERATION DIAGRAMS

28. Radial Acceleration of a Point Moving in a Circular Path.—Let P and Q (Fig. 50) be two positions near together of a point moving with a velocity v of constant magnitude, in a circular path with centre C and of radius r, and let δt be the time taken to move from P to Q.

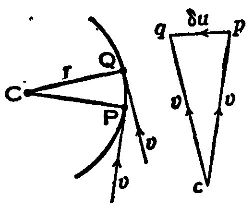

FIG. 50.

Since at any instant the direction of the velocity is tangential to the circle, by drawing the velocity triangle *cpq* it can be seen that in passing from P to Q the moving point has its velocity changed by an amount *pq*, due to the change in the direction from *cp* to *cq*. Let this change of velocity be denoted by δu.

If Q is very near to P, the arc PQ may be regarded as a straight line and then CPQ and *cpq* are similar triangles with *cp*, *cq*, and *pq* perpendicular to CP, CQ, and PQ, respectively.

Therefore $\dfrac{pq}{cp}=\dfrac{\mathrm{PQ}}{\mathrm{CP}}$ or $\dfrac{\delta u}{v}=\dfrac{\mathrm{PQ}}{r}$, approximately, but $\mathrm{PQ}=v\delta t$, therefore $\dfrac{\delta u}{v}=\dfrac{v\delta t}{r}$ or $\dfrac{\delta u}{\delta t}=\dfrac{v^2}{r}$, approximately.

The nearer Q is to P the more nearly exact does this approximation become. If δt is made to approach zero, Q becomes indefinitely near to P and $\dfrac{\delta u}{\delta t}$ approaches the value $\dfrac{du}{dt}$ which will be the acceleration of the point P. This acceleration acts along the radius towards C the centre of the circle and is called the *centripetal acceleration* of the point P.

Therefore the centripetal acceleration is

$$\frac{du}{dt}=\frac{v^2}{r}=\omega^2 r,$$

where $\omega=v/r$ is the angular velocity of the radius CP.

29. Radial Acceleration of a Point Moving in any Curved Path.—The arguments of the preceding Art. may be applied to a point P moving with a velocity v of constant magnitude along any curved path, by taking the point C (Fig. 50) as the centre of curvature of the curve at the point P. Then $CP=r$ is the radius of curvature of the curve at the point P. The length CQ is not equal to r, but approaches this value as Q approaches P, therefore CPQ and *cpq* may be regarded as similar triangles and the proof continued as before.

The centripetal acceleration of the point P is v^2/r, where v is its velocity along the curve and r is the radius of curvature of the curve at the point P. The formula is also true when the magnitude of the velocity is varying.

30. Resultant Acceleration of a Point on a Body Rotating with Varying Velocity.—Let A and B (Fig. 51) be points on a rigid body moving with plane motion so that the line AB rotates in the plane of the paper with varying angular velocity ω about the point A, which is fixed, and let the length AB be r.

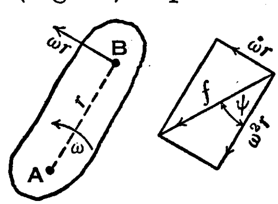

FIG. 51.

At any instant the velocity of the point B is ωr along the tangent to the path of B—that is, perpendicular to AB.

The angular acceleration of AB is $\frac{d\omega}{dt}$ which will be written as $\dot{\omega}$, and the acceleration of B perpendicular to AB is $\dot{\omega} r$. The sense of the acceleration $\dot{\omega} r$ is the same as the sense of the velocity ωr if this velocity is increasing.

The acceleration of B along BA is $\omega^2 r$. Drawing the acceleration diagram for the point B and denoting the resultant acceleration of B by the symbol f, then it can be seen that

$$f = \sqrt{\dot{\omega}^2 r^2 + \omega^4 r^2} = r\sqrt{\dot{\omega}^2 + \omega^4}$$

and this resultant acts in a direction inclined to BA at an angle ψ, where $\psi = \tan^{-1}(\dot{\omega}/\omega^2)$.

A velocity is always in the direction of motion, but a resultant acceleration cannot be continually in the direction of motion unless the motion is in a straight line.

31. Accelerations of Points on a Rigid Body—Acceleration Image.—Suppose the accelerations of two points A and B (Fig. 52) on a rigid body having plane motion are known, and let them be f_a and f_b, respectively, the directions being as shown. The points A and B are assumed to be in the plane of the paper, which is a plane of motion.

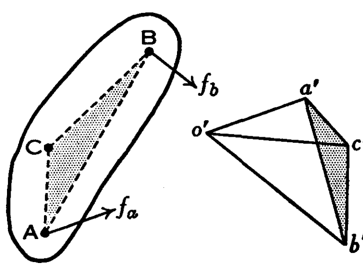

FIG. 52.

In constructing a velocity diagram a pole o was used, and a vector oa represented the velocity of a point A. In an acceleration diagram the pole will be designated o', and a vector $o'a'$ will be used to represent the acceleration of a point A.

Therefore, from a convenient pole o' mark off $o'a'$ and $o'b'$ to represent the given accelerations in direction and magnitude. Join $a'b'$, then $a'b'$ represents the acceleration of B relative to A, and $b'a'$ represents the acceleration of A relative to B.

The vector equation is

$$o'a' = o'b' + b'a'.$$

In words—

Acceleration of A = *Acceleration of* B
+ *Acceleration of* A *relative to* B.

To find the acceleration of any other point C on the body and in the plane of the paper, join AC and BC and draw the triangle $a'b'c'$ similar to the triangle ABC. Join $o'c'$, then $o'c'$ represents the acceleration of C. The triangle $a'b'c'$ is called the *acceleration image* of the triangle ABC, or the line $a'b'$ is the acceleration image of the line AB.

In practice the work is not so easy as in the foregoing description, since it is unusual for both f_a and f_b to be known completely. Suppose f_b is known in magnitude and direction, but f_a is known only in direction and it is required to find its magnitude. The length and direction of $o'b'$ is known as before and the direction $o'a'$ is known, but the length of $o'a'$ is unknown. The point a' is found by considering the acceleration of A relative to B. An example is given in the next Art.

32. Accelerations in the Slider-Crank Mechanism.—The crank CB (Fig. 53) rotates with constant angular velocity ω and it is required to find the acceleration of the slider A and the angular acceleration of the connecting-rod AB when the mechanism is in the given position. (The analytical determination of the acceleration of the slider has been given in Art. 27.)

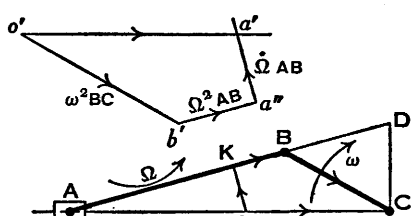

FIG. 53.

Let Ω be the angular velocity of the connecting-rod AB. Produce AB to intersect at D a line drawn from C perpendicular to AC, then, as shown in Art. 17, $\Omega = \omega \frac{BD}{AB}$.

To find the acceleration of the slider A, the acceleration diagram is drawn with the help of the vector equation—

Acceleration of A = Acceleration of B
+ Acceleration of A relative to B.

From a convenient pole o' draw $o'b'$ parallel to BC to represent, to a suitable scale, ω^2BC the acceleration of B.

The acceleration of A relative to B consists of two components, one along AB towards B and equal to Ω^2AB and another perpendicular to AB and equal to $\dot{\Omega}$AB. Now Ω^2AB can be calculated, therefore draw $b'a''$ parallel to AB to represent Ω^2AB, then draw $a''a'$ perpendicular to AB. Since $\dot{\Omega}$ is unknown, the magnitude of $\dot{\Omega}$AB cannot be calculated and therefore the length of $a''a'$ is as yet unknown. However, it is known that the acceleration of A is along AC, therefore the closing line $o'a'$ of the acceleration diagram can be drawn parallel to AC to intersect $a''a'$ at a'.

Therefore the acceleration of A is $o'a'$ measured with the acceleration scale. The angular acceleration of AB is $\dot{\Omega}=\dfrac{a''a'}{\text{AB}}$, where $a''a'$ is measured with the acceleration scale. The arrowhead on the vector $a''a'$ shows the direction of this component of the acceleration of A relative to B, and therefore it follows that the direction of $\dot{\Omega}$ is clockwise. Since the angular velocity of AB is anti-clockwise, the clockwise acceleration indicates a retardation.

The acceleration diagram may also be drawn to a particular scale on the mechanism diagram ABC.

Since $\Omega=\omega\dfrac{\text{BD}}{\text{AB}}$, therefore $\Omega^2\text{AB}=\omega^2\dfrac{(\text{BD})^2}{\text{AB}}$. Along BA mark off a point K so that $\text{KB}=\dfrac{(\text{BD})^2}{\text{AB}}$ and draw KL at right angles to AB to intersect AC at L. Then ω^2LC is the acceleration of A, or LC represents the acceleration of A to the same scale that BC represents the acceleration of B.

The proof, put briefly, is as follows. The figures CBKL and $o'b'a''a'$ are similar since each side in one is parallel to a side in the other, taken in order, and since BC is proportional to ω^2BC or to $o'b'$ and KB is proportional to ω^2KB or Ω^2AB or proportional to $b'a''$. Therefore the acceleration of A is ω^2LC. Also, $\omega^2\text{LK}=\dot{\Omega}\text{AB}$ and therefore $\dot{\Omega}=\omega^2\dfrac{\text{LK}}{\text{AB}}$. Constructions for drawing the line KL are given in the next two Arts.

33. Klein's Construction.—The slider-crank mechanism is shown again in Fig. 54. Produce AB to intersect at D a line drawn from C perpendicular to AC. Draw a circle with AB as diameter. With centre B and radius BD draw another circle cutting the first circle at M and N. Join MN, intersecting AB at K and AC at L.

Since MN is a chord common to both circles and since both circles have their centres in AB, therefore MN is perpendicular to AB. Join AM and BM. The right-angled triangles KBM and MBA are similar, therefore $\frac{KB}{BM}=\frac{BM}{AB}$ or $KB=\frac{(BM)^2}{AB}$. But BM = BD, therefore $KB=\frac{(BD)^2}{AB}$. Therefore, as shown in the preceding Art., the acceleration of the slider A is $\omega^2 LC$. The student's attention is drawn to Ex. 6, p. 61, where the angular velocity of the crank is not uniform.

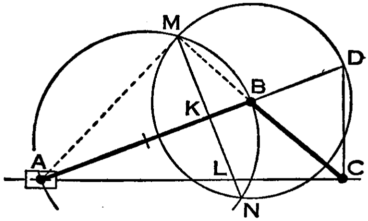

FIG. 54.

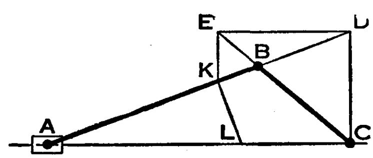

FIG. 55.

34. Mohr's Construction.*—As in Klein's construction, produce AB to intersect at D a line drawn from C perpendicular to AC (Fig. 55). Draw DE parallel to CA to intersect CB produced at E. Draw EK parallel to DC, intersecting AB at K, then draw KL perpendicular to AB, intersecting AC at L.

The triangles EBK and CBD are similar, therefore $\frac{KB}{BD}=\frac{EB}{BC}$. Also the triangles EBD and CBA are similar, therefore $\frac{EB}{BC}=\frac{BD}{AB}$.

* Also attributed to *Rittershaus* and others.

Therefore $\frac{KB}{BD}=\frac{BD}{AB}$ or $KB=\frac{(BD)^2}{AB}$, therefore K is the required point in AB and, since KL is perpendicular to AB, L is the required point in AC. Therefore the acceleration of the slider A is ω^2LC.

35. Centre of Zero Acceleration.—Let the accelerations of points A and B on a rigid body having plane motion (Fig. 56) be f_a and f_b in the directions Aa and Bb, respectively, and let $o'a'b'$ be the acceleration diagram. It is assumed that the points A and B are in the plane of the paper, which is a plane of motion. The acceleration of any point C on the body, and in the plane of the paper, is represented on the acceleration diagram by $o'c'$ if the triangle $a'b'c'$ is the image of the triangle ABC (Art. 31).

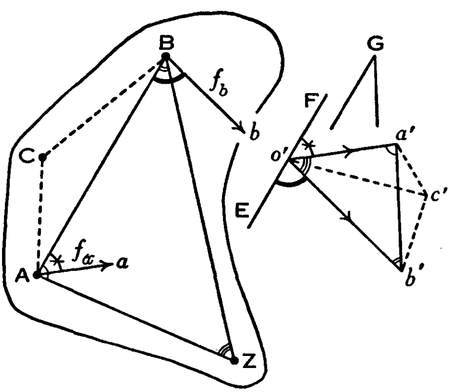

FIG. 56.

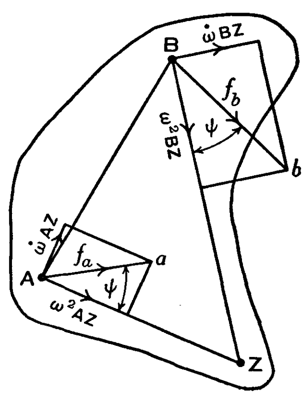

FIG. 57.

If the point c' coincided with the pole o', then the acceleration of C would be zero. Therefore draw the triangle ABZ so that the triangle $a'b'o'$ is its image, then the pole o' is the image of the point Z. The point Z has no acceleration and it is called the *centre of zero acceleration* or the *acceleration centre.*

It will now be shown that the angle ZAa is equal to the angle ZBb. Draw EF through o' and parallel to AB. Produce EF and $b'a'$ to intersect at G. (In the figure the lines intersecting at G have been drawn parallel to EF and

$b'a'$, respectively.) The angles which are known to be equal are indicated by similar marks.

$$\text{Angle ZA}a = \text{angle ZAB} - \text{angle } a\text{AB}$$
$$= \text{angle } o'a'b' - \text{angle F}o'a' = \text{angle at G.}$$
$$\text{Angle ZB}b = \text{angle AB}b - \text{angle ABZ}$$
$$= \text{angle E}o'b' - \text{angle } o'b'a' = \text{angle at G.}$$

Therefore $\quad$ angle Z$\text{A}a$ = angle Z$\text{B}b$.

The position of the point Z may be arrived at in another way. Suppose the body has an angular velocity ω and an angular acceleration $\dot{\omega}$. Draw AZ (Fig. 57) so that the angle $\text{ZA}a = \psi = \tan^{-1}\dfrac{\dot{\omega}}{\omega^2}$, then the length of AZ may be determined as follows:—

$$\text{Acceleration of A} = \text{Acceleration of Z} + \text{Acceleration of A relative to Z.}$$

Therefore $f_a = \text{Acceleration of Z} + \text{AZ}\sqrt{\omega^4 + \dot{\omega}^2}$.

Make the length AZ such that $f_a = \text{AZ}\sqrt{\omega^4 + \dot{\omega}^2}$, then the acceleration of the point Z will be zero, and Z will be the centre of zero acceleration.

Now join BZ. Since the acceleration of Z is zero, the acceleration of B is $f_b = \text{BZ}\sqrt{\omega^4 + \dot{\omega}^2}$ and the angle $\text{ZB}b = \psi = \tan^{-1}\dfrac{\dot{\omega}}{\omega^2}$. The acceleration of any other point may be found in a similar way.

The point Z may be obtained by drawing AZ and BZ inclined at the angle ψ to Aa and Bb, respectively, and intersecting at Z. If, as shown, f_b is greater than f_a, then ZB is greater than ZA and therefore the angles ψ are measured in clockwise directions from the lines Bb and Aa, for it can be seen that ZB would be less than ZA if the angles ψ were measured in anticlockwise directions. If only one acceleration is given, or if the directions of two accelerations are given without their magnitudes, then the direction of $\dot{\omega}$ must be known in order to determine the direction of angle ψ. The angle ψ is measured clockwise or anticlockwise from the direction of a given acceleration according as $\dot{\omega}$ is clockwise or anticlockwise.

The point Z may be inside or outside a body, but in the latter case it must still be regarded as rigidly attached to the body.

36. Acceleration of a Point Moving along a Rotating Straight Line.—Let P be a point moving along a straight line OP which is rotating about O (Fig. 58). It is required to find the acceleration of the point P along and perpendicular to OP.

Suppose that at a given instant OP $=r$, OP is inclined at an angle θ to a fixed line OX, the angular velocity of OP is $\frac{d\theta}{dt}$ or ω, and the velocity of P along OP is $\frac{dr}{dt}$ or u. Also, suppose that during a short time δt, θ increases to

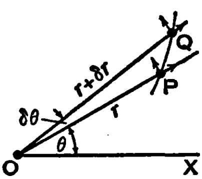

FIG. 58.

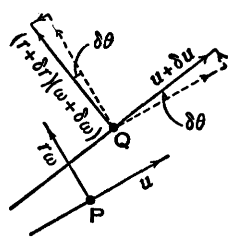

FIG. 59.

$\theta+\delta\theta$, P moves to Q, so that r increases to $r+\delta r$, ω becomes $\omega+\delta\omega$, and u becomes $u+\delta u$.

At P the velocity perpendicular to OP is $r\omega$, and at Q the velocity perpendicular to OQ is $(r+\delta r)(\omega+\delta\omega)$.

If the velocities at Q are resolved in directions parallel and perpendicular to OP, as indicated in the enlarged view (Fig. 59), then the changes in the velocities of P in these directions, and consequently the accelerations of P, may be determined.

Parallel to OP—

$$\frac{\text{Change of vel.}}{\text{Time}}=\frac{(u+\delta u)\cos\delta\theta-(r+\delta r)(\omega+\delta\omega)\sin\delta\theta-u}{\delta t}.$$

Let Q approach P, that is let δt approach zero, then $\delta\theta$ approaches zero, $\cos\delta\theta$ approaches 1, $\sin\delta\theta$ approaches $\delta\theta$, the products $r\delta\omega\frac{\delta\theta}{\delta t}$, $\omega\delta r\frac{\delta\theta}{\delta t}$, $\delta r\,\delta\omega\frac{\delta\theta}{\delta t}$ ultimately vanish, $\frac{\delta u}{\delta t}$ is written $\frac{du}{dt}$ and $\frac{\delta\theta}{\delta t}$ is written $\frac{d\theta}{dt}$ or ω, and the expression becomes the acceleration of P along OP, therefore

$$\text{Acceleration of P along OP} = \frac{du}{dt} - \omega^2 r \quad . \quad (1).$$

Perpendicular to OP—

$$\frac{\text{Change of vel.}}{\text{Time}} = \frac{(u+\delta u)\sin\delta\theta + (r+\delta r)(\omega+\delta\omega)\cos\delta\theta - r\omega}{\delta t}.$$

Let δt approach zero, then the expression becomes the acceleration of P perpendicular to OP. Therefore, in a similar manner to the previous simplification,

Acceleration of P perpendicular to OP

$$= u\frac{d\theta}{dt} + \frac{dr}{dt}\omega + r\frac{d\omega}{dt} = 2u\omega + r\frac{d\omega}{dt} \qquad (2),$$

since $$\frac{d\theta}{dt} = \omega \quad \text{and} \quad \frac{dr}{dt} = u.$$

Expression (1) might have been written down straightaway, for it consists of $\frac{du}{dt}$, the rate of change of the velocity of P along OP, and the centripetal acceleration $\omega^2 r$ due to the rotation of OP. In expression (2) the term $r\frac{d\omega}{dt}$ was also expected; the term $2u\omega$ may astonish the student, but it is accounted for by the fact that OP is rotating whilst changing in length. The acceleration $2u\omega$ is sometimes called the *compound supplementary acceleration* or the *Coriolis acceleration.*

The importance of these results is illustrated by the example worked out in the next Art., and another method of obtaining them is suggested in Ex. 17, p. 65.

37. Shaping Machine Mechanism.—A mechanism for a shaping machine is shown diagrammatically in Fig. 60. A crank CB turning about a fixed centre C drives a slider through a pin B. The slider slides on a link AE turning about a fixed centre A and causes AE to oscillate and, through a link EF, drive a slider at F to and fro along the line HF. The various dimensions in inches are: AC$=5$, CH$=4\frac{1}{2}$, CB$=2\frac{1}{2}$, AE$=11$, and EF$=8$. The angle CHF $=90°$ and the angle HCB$=30°$. The crank CB is turning uniformly at 30 r.p.m., and it is required to find the velocity and acceleration of the slider at F when the mechanism has the given configuration.

Let the angular velocities of CB, AE, and EF, in radians per second, be denoted by ω, Ω_1, and Ω_2, respectively. Also let D be the point on the link AE immediately under the pin B. For an instant B and D are coincident, but B slides away from D as the crank rotates. Let the velocity of B relative to D be u.

The angular velocity of CB is $\omega = 30 \times \dfrac{2\pi}{60} = \pi$ rad./sec.

The velocity of B, perpendicular to CB, is $\pi \times \dfrac{2 \cdot 5}{12} = 0 \cdot 654$ ft./sec.

The velocity diagram may now be constructed. The scale used in Fig. 60 is 1 inch$=0\cdot5$ ft./sec., but the student should work the problem on drawing-paper, using a scale of say 1 inch$=0\cdot1$ ft./sec. Starting with a convenient pole o, draw ob perpendicular to CB to represent to scale the velocity of B. Since the point D moves in the direction perpendicular to AE and the point B moves along AE, draw od perpendicular to AE and draw bd parallel to AE, meeting od at d, then od represents the velocity of D and db represents u, the velocity of B relative to D.

Produce od to e, making $\dfrac{oe}{od} = \dfrac{\text{AE}}{\text{AD}}$, then oe represents the velocity of E. The construction is shown by the dotted lines; os makes any convenient angle with od, $os =$ AE and $or =$ AD, rd is joined and se is drawn parallel to rd to meet oe at e.

Since the point F moves along HF and its motion relative to E is perpendicular to FE, draw *of* parallel to HF and draw *ef* perpendicular to EF, meeting *of* at *f*, then *ef* represents the velocity of F relative to E and *of* represents the velocity of F. Measuring *of* with the velocity scale it is found that the velocity of F is 0·89

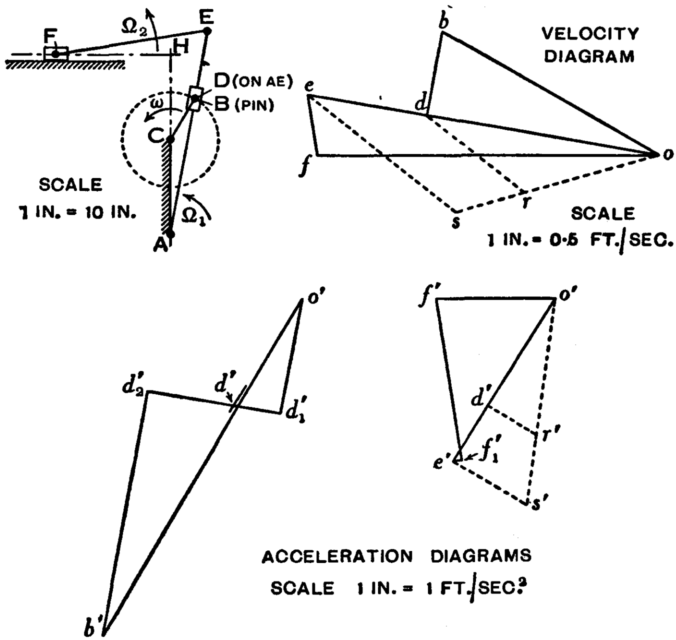

FIG. 60.

ft./sec. Also the velocity of B relative to D is *db* or $u = 0{\cdot}225$ ft./sec.

Two acceleration diagrams are shown, but this is only for clearness; they should be combined and drawn as one diagram. The acceleration of the point B may be expressed in two ways: as a point on the crank CB it is ω^2CB acting towards C, and as a point moving along the link AE it is $\dot{u} - \Omega_1^2\text{AD}$ along AE in the sense A to E and $\dot{\Omega}_1\text{AD} + 2u\Omega_1$ perpendicular to AE in the sense in which the point D is moving, provided in each case the numerical result is positive. It will be found later that $\dot{u}$ is negative. It should be noted that $-\Omega_1^2 AD$ and $\dot{\Omega}_1 AD$ are the accelerations of the point D, and although Ω_1 is easily calculated, $\dot{\Omega}_1$ is more troublesome to find. Once the resultant acceleration of D is obtained, the accelerations of E and F may be found without difficulty.

The acceleration scale is 1 inch = 1 ft./sec.², but the student should use a much larger scale, say 1 inch = 0·2 ft./sec.².

The acceleration of B is $\omega^2 CB = \pi^2 \times \frac{2\cdot5}{12} = 2\cdot06$ ft./sec.², therefore, from a convenient pole o' draw $o'b'$ parallel to BC and equal to 2·06 inches.

Now $\Omega_1 = \frac{od}{AD}$. By measurement $od = \frac{1\cdot23}{2}$ ft./sec., and AD = 7·3 inches, therefore $\Omega_1 = \frac{1\cdot23}{2} \times \frac{12}{7\cdot3} = 1\cdot01$ rad./sec.

Also $\Omega_1^2 AD = 1\cdot01^2 \times \frac{7\cdot3}{12} = 0\cdot62$ ft./sec.² and the direction is towards A, therefore draw $o'd_1'$ parallel to DA and equal to 0·62 inch.

Next draw $d_1'd_2'$ perpendicular to AD, then this is the direction of the acceleration $\dot{\Omega}_1 AD$, the magnitude and sense of which are unknown, and it is the direction of the acceleration $2u\Omega_1$ the magnitude of which may be calculated. The sense of $2u\Omega_1$ is from d_1' to d_2' because, as seen from the velocity diagram, $u = db$ is positive or in the sense A to E.

The acceleration of B relative to D along AE is $\dot{u}$, therefore draw $b'd_2'$ parallel to AE, intersecting $d_1'd_2'$ at d_2', then $d_2'b'$ represents $\dot{u}$, which is evidently a retardation since the sense is opposite to the sense of the velocity u.

As already found, $u = 0\cdot225$ ft./sec. and $\Omega_1 = 1\cdot01$ rad./sec., therefore $2u\Omega_1 = 2 \times 0\cdot225 \times 1\cdot01 = 0\cdot455$ ft./sec.². From d_2' mark off $d_2'd' = 0\cdot455$ inch along $d_2'd_1'$, then $d'd_2'$ represents $2u\Omega_1$ and, by subtraction, $d_1'd'$ represents $\dot{\Omega}_1 AD$.

Join $o'd'$, then this is the total acceleration of the point D. To prevent confusion the line $o'd'$ has been drawn in the other diagram on the right, but as previously mentioned the two diagrams should be drawn as one.

Produce $o'd'$ to e' making $\frac{o'e'}{o'd'}=\frac{\text{AE}}{\text{AD}}$. The construction is as shown by the dotted lines where $o'r'=\text{AD}$, $o's'=\text{AE}$, $r'd'$ is joined and $s'e'$ is drawn parallel to $r'd'$ to meet $o'd'$ produced at e'. The acceleration of E is represented by $o'e'$.

The acceleration of F may now be obtained.

Acceleration of F = Acceleration of E
+ Acceleration of F relative to E.

The acceleration of F is along the line HF, therefore draw $o'f'$ parallel to HF, the length of $o'f'$ being unknown. The acceleration of F relative to E is $\Omega_2^2\text{FE}$ along FE towards E and $\dot{\Omega}_2\text{FE}$ perpendicular to FE, where Ω_2 and $\dot{\Omega}_2$ are respectively the angular velocity and angular acceleration of FE.

$$\text{Now } \Omega_2=\frac{fe}{\text{FE}} \text{ and } \Omega_2^2\text{FE}=\frac{(fe)^2}{\text{FE}}=0{\cdot}16^2\times\frac{12}{8}=0{\cdot}038 \text{ ft./sec.}^2,$$

the value of fe being obtained from the velocity diagram. Therefore draw $e'f_1'$ parallel to FE and equal to 0·038 inch, then draw $f_1'f'$ perpendicular to FE and intersecting $o'f'$ at f'. The length $f_1'f'$ represents $\dot{\Omega}_2\text{FE}$. The acceleration of F is represented by $o'f'$, and by measurement with the acceleration scale it is found to be 0·62 ft./sec.^{2}, approximately.

Exercises IV

1. A vertical single cylinder engine running at 1800 revolutions per minute has a crank $2\frac{1}{4}$ inches long and a connecting-rod $10\frac{1}{2}$ inches long, and the line of stroke of the piston passes through the centre of the crankshaft. Draw the acceleration diagram when the crank is at an angle of 30° from the top dead centre, and state the magnitude of the acceleration of the piston in feet per second per second.

2. A crank BC of length r feet is turning about the point C at ω radians per second and is connected to a slider A by a rod AB

of length l feet, the line of stroke of A being along AC. Draw acceleration diagrams to find the acceleration of the slider, (*a*) when the angle ACB is 0°, (*b*) when the angle ACB is 180°.

3. The slider-crank mechanism as shown in Fig. 61 has the line of stroke of the slider A offset a perpendicular distance of 2 inches from the centre C. AB = 30 inches, BC = 8 inches, and BC is rotating clockwise at 200 r.p.m. Draw an acceleration diagram to find the acceleration of the slider when the angle DCB is 60°, the datum line DC being parallel to the line of stroke.

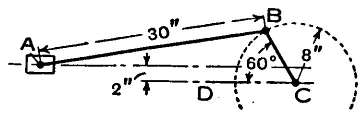

FIG. 61.

4. Referring to the preceding exercise and Fig. 61, find the acceleration of the slider A for each of the positions when AB and BC are in line, (*a*) not overlapping, (*b*) overlapping.

5. In the slider-crank mechanism ABC (Fig. 62) the line KL is obtained as explained in Art. 33 and BL is joined. From any point E in AB, E*e* is drawn parallel to AC to intersect BL at *e* and *e*C is joined. If BC rotates with a uniform angular velocity ω, show that the acceleration of the point E is $\omega^2 . eC$.

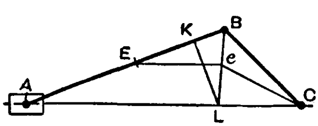

FIG. 62.

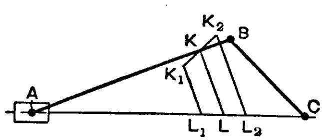

FIG. 63.

6. Let ω and $\dot{\omega}$ be the angular velocity and angular acceleration, respectively, of the crank BC in the slider-crank mechanism (Fig. 63). The line KL is drawn as explained in Art. 33, then if $\dot{\omega} = 0$ the acceleration of the slider A is ω^2LC. Draw K_1K_2 through K and perpendicular to BC, making $KK_1 = KK_2 = \frac{\dot{\omega}BC}{\omega^2}$, then draw K_1L_1 and K_2L_2 parallel to KL to meet AC at L_1 and L_2.

Draw acceleration diagrams and show that the acceleration ot A is $\omega^2 . L_1C$ when the angular acceleration $\dot{\omega}$ is clockwise and $\omega^2 . L_2C$ when $\dot{\omega}$ is anticlockwise.

7. Referring to Fig. 63 and the preceding exercise, show that the angular acceleration of the rod AB is zero for the given configuration if $\dot{\omega} = \frac{AK}{AB}\omega^2 \tan\theta$ and is clockwise, θ being the angle ACB.

8. When a link AB, which is 6 inches long, is in the given position (Fig. 64) its angular motion is clockwise and the velocity and acceleration of the point A and the acceleration of the point B are as indicated. Find the angular velocity and angular acceleration of the link and the velocity of the point B.

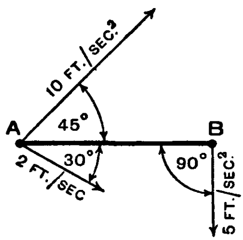

FIG. 64.

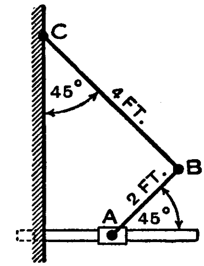

FIG. 65.

9. In the mechanism illustrated in Fig. 65 the slider A is moving to the right with a velocity v and an acceleration a, prove that the horizontal and vertical components of the acceleration of B are $\frac{a}{2} - \frac{3v^2}{8\sqrt{2}}$ and $\frac{a}{2} - \frac{v^2}{8\sqrt{2}}$ respectively. [C.U.]

10. Fig. 66 illustrates the arrangement of the crank and connecting-rods of each pair of cylinders of a 12-cylinder 60° Vee engine. Draw velocity and acceleration diagrams for the mechanism, and find the velocity and acceleration of the pistons and the angular velocity and angular acceleration of the

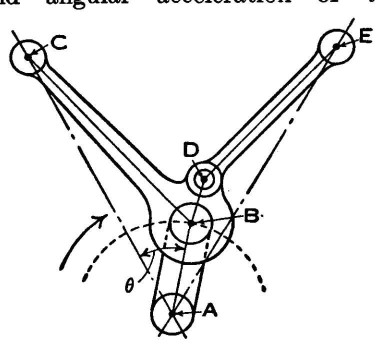

FIG. 66.

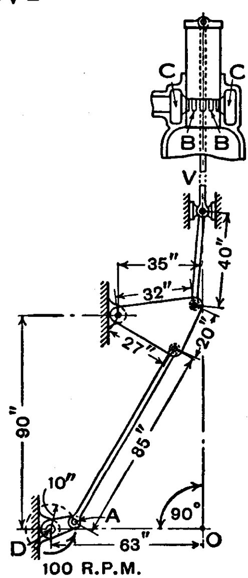

FIG. 67.

connecting-rods when the crank angle θ is 40°. Engine speed, 2400 r.p.m.

AB = 2·5 inches, BC = 6·25 inches, BD = 1·25 inches, DE = 5 inches, angle CBD = 60°. (Scales—In vector diagrams represent velocity and acceleration of B by 2·5 inches.) [C.U.]

11. Fig. 67 is a diagram of a cylindrical scavenge valve and the gear for operating it in the cylinder head of a large two-stroke Diesel engine. The valve-rod V is operated from the auxiliary crank DA of radius 10 inches which rotates at a constant speed of 100 r.p.m., and the scavenge air enters the cylinder from the receiver C through the ring of ports BB.

Draw velocity and acceleration diagrams for the mechanism for the position in which the crank pin A is above the line DO and the angle ADO is 15°, and find the velocity and the acceleration of the valve-rod V. (Scales for vector diagrams—1 inch = 2 feet per second; 1 inch = 15 feet per second per second.) [C.U.]

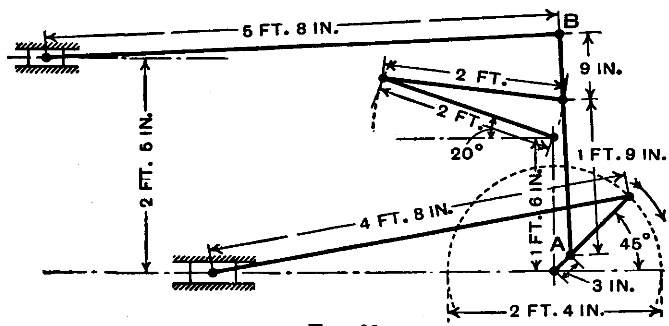

FIG. 68.

12. Draw the diagram of Marshall's valve gear as shown in Fig. 68. If the crank is 14 inches long and rotates at 100 r.p.m., draw the velocity diagram and the acceleration diagram; find the angular velocity and angular acceleration of the link AB. [U.L.]

13. Fig. 69 is a diagrammatic representation of the crank and slotted lever mechanism as applied to a shaping machine. Neglecting the obliquity of EF and assuming in consequence that the acceleration of F is the same as the acceleration of E in the direction at right angles to CD, show how to find this acceleration for any position of the crank CP when it is rotating uniformly.

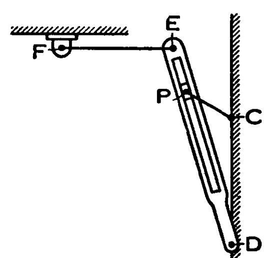

FIG. 69.

For the position in which PCD is

a right angle prove that the acceleration of F is

$$\frac{n^6+1}{(n^2+1)^{\frac{7}{2}}}\omega^2 . \mathrm{DE},$$

where n is the ratio of $\frac{\mathrm{DC}}{\mathrm{CP}}$ and ω is the angular velocity of the crank. [C.U.]

14. Draw velocity and acceleration diagrams for the quick-return mechanism (Fig. 70) in the position shown, when the crank BC is rotating clockwise at a uniform speed of 60 revolutions per minute.

The slotted lever AD and the crank BC turn about fixed centres A and B respectively. BC = 4 inches. DE = 6 inches. The line of stroke of E is at right angles to AB and at a perpendicular distance of 12 inches from A.

(Scales—1 inch = 5 inches per second; 1 inch = 25 inches per second per second.) [C.U.]

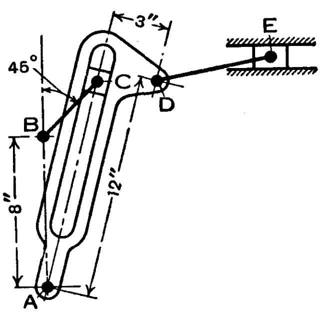

Fig. 70.

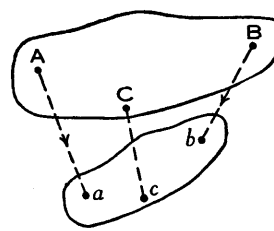

Fig. 71.

15. ABC (Fig. 71) is a link of a mechanism moving in a plane. Aa and Bb represent in magnitude, direction, and sense the accelerations of the points A and B respectively. On ab an image of the link ABC is constructed. Prove that if the triangle acb is similar to the triangle ACB, then the acceleration of C is completely represented by the vector Cc.

Prove, also, that a similar construction will give the velocity of any point of the link. [C.U.]

16. In a four-bar chain ABCD, in which AD is the fixed link, AB and DC intersect at I and CB and DA intersect at K. (See Fig. 72 which was not in the original examination question.) AN and AM are drawn parallel to DC and BC respectively and cut these lines in N and M. F is the point of intersection of KM and AN, and E is the point of intersection of IN and AM. FS is drawn perpendicular to AN, and ES is drawn perpendicular

to AM. FS and ES intersect in S. Prove that the acceleration of C is given by ω^2NS, where ω is the uniform angular velocity of AB. [C.U.]

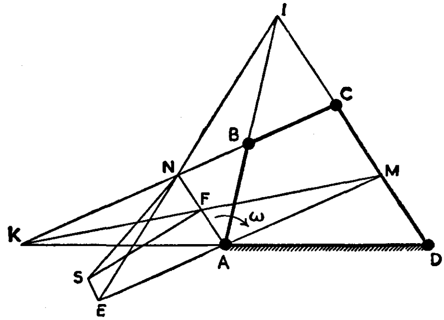

FIG. 72.

17. A point P is moving along a line OP which is turning in the plane of the paper about O (Fig. 73). Referred to fixed rectangular axes OX and OY, the co-ordinates of the point P are x and y. Denoting the length OP by r and the angle POX by θ, then

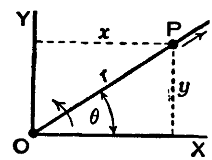

FIG. 73.

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Show by differentiation that

$$\frac{d^2x}{dt^2} = -r\left\{\frac{d^2\theta}{dt^2}\sin\theta + \left(\frac{d\theta}{dt}\right)^2\cos\theta\right\} + \frac{d^2r}{dt^2}\cos\theta - 2\frac{dr}{dt}\frac{d\theta}{dt}\sin\theta,$$

$$\frac{d^2y}{dt^2} = r\left\{\frac{d^2\theta}{dt^2}\cos\theta - \left(\frac{d\theta}{dt}\right)^2\sin\theta\right\} + \frac{d^2r}{dt^2}\sin\theta + 2\frac{dr}{dt}\frac{d\theta}{dt}\cos\theta,$$

then, by resolving these accelerations along and perpendicular to the line OP, show that

$$\text{Acceleration of P along OP is } \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2,$$

$$\text{Acceleration of P perpendicular to OP is } r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}.$$

18. A body moving with plane motion has its instantaneous centre at O and its centre of zero acceleration at Z (Fig. 74).

If the direction of the acceleration of a point P is perpendicular to OP, that is, if the acceleration is in the same direction as the velocity, show that all points such as P lie on a circle passing through O, P, and Z; also show that the acceleration of O is along the diameter OD_1.

If the direction of the acceleration of a point Q is along QO, that is, if the acceleration is in the direction perpendicular to the velocity, show that all points such as Q lie on a circle passing through O, Q, and Z.

Show that the diameters OD_1 and OD_2 of the two circles are mutually perpendicular and that the points D_1, Z, and D_2 lie in a straight line.

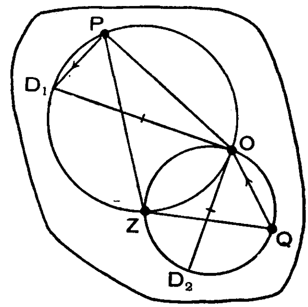

FIG. 74.

19. The crank CB of the slider-crank mechanism ABC (Fig. 75) is rotating about C with uniform angular velocity ω. From the instantaneous centre O of the rod AB, the line OK is drawn parallel to AC to intersect AB produced at K, then KP is drawn parallel to OA to intersect BC produced at P. A circle is drawn through the points O, A, and

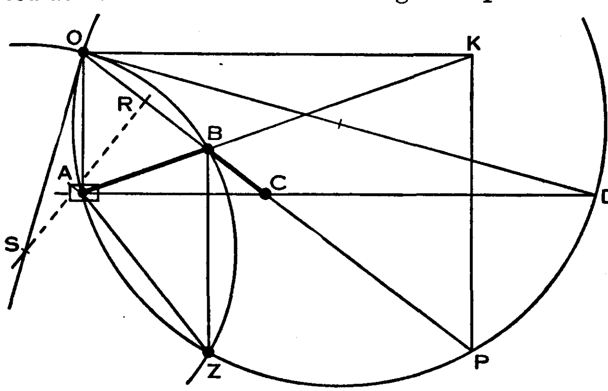

FIG. 75.

P, and then the diameter OD is drawn. Next, OS is drawn perpendicular to OD, and OB is bisected at right angles by the line RS. The lines OS and RS intersect at S. With centre S and radius SO a circle is drawn through the points O and B, intersecting the circle OAP again at Z.

Prove that $BC.BP = OB^2$ and then show that the point P, considered as a point attached to the rod AB, has no acceleration in the direction PC. (*Hint.*—Considering the points P, B, and C, write down the total acceleration of the point P and then find the condition which makes the component in the direction PC zero.)

Show that the point Z is the centre of zero acceleration of the rod AB, and that the acceleration of the slider A may be expressed as $\omega^2 \frac{BC}{ZB} \times ZA$.

Finally, show that the point Z could also be obtained by drawing the circle OAP and a circle passing through the points A, B, and C.

CHAPTER V

FORCE, TORQUE, WORK, AND ENERGY

38. Newton's Laws of Motion.—*First Law.*—Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state.

Second Law.—Rate of change of momentum is proportional to the impressed force and takes place in the direction of the straight line in which the force acts.

Third Law.—To every action there is always an equal and opposite reaction.

39. Force and Acceleration.—Denoting force, mass, velocity, and time by P, M, v, and t, respectively, then *momentum*, the product of mass and velocity, is Mv and from Newton's second law of motion

$$\mathrm{P} \propto \frac{d}{dt}(\mathrm{M}v)$$

or

$$\mathrm{P} = \mathrm{C}\frac{d}{dt}(\mathrm{M}v),$$

where C is a constant, or, using suitable units,

$$\mathrm{P} = \frac{d}{dt}(\mathrm{M}v).$$

When M is constant, $\mathrm{P} = \mathrm{M}\dfrac{dv}{dt} = \mathrm{M}f,$

where dv/dt or f is the acceleration,

or $\qquad$ Force = Mass × Acceleration.

If s is displacement, $v = ds/dt$ and acceleration may be written as

$$\frac{d^2s}{dt^2} = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}.$$

Also it is sometimes convenient to write mass M as W/g, where W is the weight of the mass and g is the acceleration due to gravity.

Therefore $$P=Mf=\frac{W}{g}\frac{dv}{dt}=\frac{W}{g}v\frac{dv}{ds}$$

are some of the different ways of expressing the same equation.

The dimensions of force in terms of mass M, length L, and time T are ML/T^2 or MLT^{-2}.

Units.—A force of 1 *pound-weight* is the attraction which the earth exerts *in the latitude of London* on a certain standard piece of platinum whose mass is described as being 1 *pound.* Another unit of force is the *poundal,* and 32·2 poundals are approximately equal to 1 pound-weight.

A force of 1 poundal will give a mass of 1 pound an acceleration of 1 foot per second per second.

Therefore if a mass of M pounds is acted on by a force of P poundals and the acceleration is f feet per second per second, the equation of motion is

$$P=Mf.$$

A force of 1 pound-weight will give a mass of 1 pound an acceleration of 32·2 feet per second per second, correct to three figures, and this acceleration is the value of g, the acceleration due to gravity, *in the latitude of London.* It follows that a force of 1 pound-weight will give a mass of 32·2 pounds an acceleration of 1 foot per second per second.

Therefore if a mass of M pounds is acted on by a force of P pounds-weight and the acceleration is f feet per second per second, the equation of motion is

$$P=\frac{M}{32{\cdot}2}f \quad \text{or} \quad 32{\cdot}2\ P=Mf.$$

This is often written as $Pg=Mf$, which, strictly, is incorrect. The value of g varies slightly from place to place, whereas the units of force (as defined above), mass, and acceleration do not depend on position.

If a new unit of mass equal to 32·2 pounds is chosen and if M_s is the number of these units in the mass, then the equation becomes

$$P = M_s f.$$

This larger unit of mass is sometimes called a *slug*.

Suppose now a mass of M units, of W pounds-weight, is acted on by a force equal to its own weight, then the acceleration will be g feet per second per second and the equation may be written as $W = Mg$ if the units of M are suitably selected. Solving for M gives $M = W/g$ and it is evident that mass may be replaced by the ratio of its weight to the acceleration due to gravity. Therefore if the equation $P = Mf$ is written as

$$P = \frac{W}{g} f,$$

the force P will be in pounds-weight, provided the weight W is in the same units and the units of f are the same as those of g.

Much controversy has ranged over the question of writing W/g for M and probably will continue for all time, but the method is correct and very convenient. Most of the American books appear to use W/g, although it is not always clear whether force/acceleration or mass/number is intended. This discussion may seem unimportant to the student, but it is essential that he should know what he is doing.

The advantages of using W/g and regarding it as force/acceleration are that a unit check may be used in an equation and that the use of different units is facilitated.

For example, considering $P = \frac{W}{g} f$, which is the simplest equation in which mass appears:—

(1) With lb.-weight, foot and second units,

$$\text{lb.-wt.} = \frac{\text{lb.-wt.}}{\text{ft./sec.}^2} \times \text{ft./sec.}^2,$$

or $$\text{lb.-wt.} = \text{lb.-wt}$$

(2) With ton-weight, inch and second units,

$$\text{tons-wt.} = \frac{\text{tons-wt.}}{\text{inches/sec.}^2} \times \text{inches/sec.}^2,$$

or

$$\text{tons-wt.} = \text{tons-wt.}$$

Usually an engineer talks of a force of so many pounds when he means pounds-weight, but it is doubtful whether this ever leads to confusion.

40. Torque and Angular Acceleration.—Consider a small mass m travelling, with velocity v at time t, in a circular path of radius $OA = r$ and acted on by a tangential accelerating force P (Fig. 76).

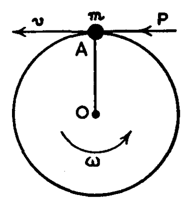

FIG. 76.

Since $P = \frac{d}{dt}(mv)$, $Pr = \frac{d}{dt}(mrv)$.

But $v = \omega r$ where ω is the angular velocity of the radius OA, therefore

$$Pr = \frac{d}{dt}(mr^2\omega).$$

If a body consisting of a large number of particles of mass m is rotating with angular velocity ω at time t, each particle being acted on by a tangential force P, then, using the symbol Σ to denote the words "the sum of,"

$$\Sigma Pr = \frac{d}{dt}\left\{\Sigma(mr^2)\omega\right\},$$

or

$$T = \frac{d}{dt}(I\omega),$$

where $T = \Sigma Pr$ denotes the *turning moment* or *torque* on the body and $I = \Sigma mr^2$ is known as the *moment of inertia* of the body (see Art. 44).

The product $I\omega$ is the *angular momentum* of the body, therefore the rate of change of angular momentum of the body is equal to the applied torque. Of course, for this equality to hold good, suitable units must be used.

Since $I\omega = \Sigma(mr^2)\omega = \Sigma(mr^2v/r) = \Sigma mvr$ and mv is a momentum, Σmvr, or $I\omega$, is the moment of a momentum. Therefore angular momentum is also called *moment of momentum*.

If the moment of inertia I is constant,

$$T = I\frac{d\omega}{dt} = Ia,$$

where $d\omega/dt$ or a is the angular acceleration.

Units.—If $I = \Sigma mr^2$ is in pound-feet² and a is in rad./sec.², then T will be in poundal-feet. If mass m is written as w/g and w is in pounds-weight and g is in feet/sec.², then T will be in pound-weight-feet or, more briefly although strictly incorrectly, pound-feet.

41. Impulse.—If a force P acts on a mass M for a time t and changes the velocity of the mass from u to v, then, since

$$P = M\frac{dv}{dt}$$

$$\int_0^t Pdt = M\int_u^v dv = M(v - u).$$

If P is a variable and has a mean value P′, then

$$\int_0^t Pdt = P't \quad \text{and} \quad P't = M(v - u).$$

The product $P't$ is called an *impulse* and is equal to the change of momentum it produces. When t is indefinitely small, as in the case of a blow or collision, P′ is very large and is called an *impulsive force*. Generally in such a case it is not possible to measure t accurately and therefore P′ cannot be determined.

In a similar way it can be shown that

$$\int_0^t Tdt = T't = I(\omega_2 - \omega_1),$$

where T′ is the mean value of a variable torque T acting for a time t, $\omega_2 - \omega_1$ is the change of angular velocity, and $I(\omega_2 - \omega_1)$ is the change of angular momentum. The product $T't$ is the impulse of the torque and is equal to the change of angular momentum it produces. When t is indefinitely small, T' is very large and is called an *impulsive torque* or *impulsive couple.*

42. Conservation of Momentum—Linear Momentum.—*If, in any direction, the sum of the components of the external forces acting on a system of bodies is zero, the total momentum of the system is constant in that direction.* For example, if two bodies act on one another with a force P for a time t, the impulse is Pt, and if there is no external force acting in the direction of the force P, the momentum gained by one is equal to that lost by the other, therefore the total momentum of the system is unchanged.

Angular Momentum or Moment of Momentum.—*If, in a system of bodies, the sum of the moments of the external forces about any fixed axis is zero, the angular momentum of the system about that axis is constant.* For example, if two discs rotating about a common axis act on one another with a torque T for a time t, and there is no external torque acting on the discs, the angular momentum gained by one is equal to that lost by the other and the total angular momentum of the system is unchanged.

43. Centrifugal Force.—When a body of mass m moves in a circular path, an inward radial acceleration has to be provided and its value is v^2/r or $\omega^2 r$ (Art. 28, p. 47).

Since $P=mf$, the inward radial force which is required in order to produce the inward radial acceleration is $P=mv^2/r=m\omega^2 r$ and this is called the *centripetal force.* The mass resists this inward force with an equal and opposite outward force called the *centrifugal force,* but this force does *not* act on the mass. If centripetal force ceases to act, for example if the mass is travelling in a circle at the end of a radial string on a horizontal surface and the string is suddenly cut, the centrifugal force disappears too and the mass goes off in a tangential direction.

It is common practice, however, *to regard centrifugal force as an outward radial force acting on the mass* and then to treat the forces acting on the mass as being in equilibrium.

Centrifugal force regarded in this way is merely a particular case of a reversed effective force or reversed accelerating force (Art. 74, p. 132). This is an artifice which produces the desired result, but it is important to understand that it is only an artifice.

The following example on a conical pendulum shows that either centripetal force or the artificial centrifugal force may be used to produce identical results.

Consider the conical pendulum shown in Fig. 77 where a string OA supports a body of mass m and weight W, which is moving in a horizontal circle of radius r about a vertical axis OY. Let the angular velocity be ω and let the angle YOA be θ. It is required to find ω in terms of W, θ, m, and r.

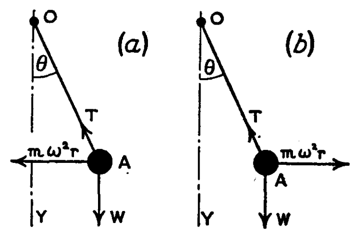

FIG. 77.

The horizontal component of the tension T in the string is T sin θ and this component is equal to and provides the centripetal force $m\omega^2r$ shown at (a), that is

$$T \sin \theta = m\omega^2 r.$$

Also, resolving vertically, in which direction there is no accelerating force,

$$T \cos \theta = W.$$

From these equations, $\omega = \sqrt{\dfrac{W \tan \theta}{mr}}$.

Using centrifugal force and referring to the Fig. at (b), $m\omega^2r$ is regarded as a radial force acting outwards on the mass, and all the forces acting on the mass are treated as being in statical equilibrium.

Resolving horizontally,

$$T \sin \theta = m\omega^2 r,$$

and resolving vertically,

$$T \cos \theta = W.$$

These two equations are identical with those arrived at when centripetal force was used, thus the two methods produce the same results.

44. Moment of Inertia.—If m is the mass of a particle of a body at a perpendicular distance r from an axis through O perpendicular to the paper (Fig. 78), then the *moment of inertia of the body* about the axis through O is defined as $I = \Sigma mr^2$, where Σ denotes "the sum of" and the summation includes all the particles of the body.

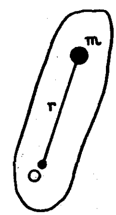

FIG. 78.

If M is the total mass of the body and k is such that

$$I = Mk^2 = \Sigma mr^2 \quad \text{or} \quad k = \sqrt{(I/M)},$$

then k is defined as the *radius of gyration* of the body about the axis through O.

The *moment of inertia of an area* is defined in a similar manner by substituting area for mass. If, in Fig. 78, a is an element of area at a distance r from O and the whole figure has an area A,

$$I = Ak^2 = \Sigma ar^2 \quad \text{and} \quad k = \sqrt{(I/A)}.$$

Sometimes the moment of inertia of an area is called the *second moment of an area*, which is a more descriptive phrase since an area, having no mass, cannot have inertia. However, the term moment of inertia is more usual in practice.

Units.—For a mass, since $I = Mk^2$, the units of I will depend on those of M and k. If M is in pounds and k is in feet, then I will be in pounds $\times$ feet^2. If M is expressed as W/g, force/acceleration, with say W in pounds-weight and g in feet per second per second, and if k is in feet, then I will be in $\dfrac{\text{lb.-wt.}}{\text{ft./sec.}^2} \times \text{ft.}^2$ or lb.-wt. $\times$ ft. $\times$ sec.^{2}.

For an area, since $I = Ak^2$, the units of I will be those of length^4; if length is in inches then I will be in inches^4.

45. Moment of Inertia—Theorems.—Two theorems and their corollaries are given below.

Theorem I.—If I_x and I_y are the moments of inertia of a plane figure (Fig. 79) about axes OX and OY in its plane and perpendicular to one another, and if I_z is the moment of inertia of the figure about an axis OZ perpendicular to the plane XOY, then $I_z = I_x + I_y$. I_z is called a *polar moment of inertia.*

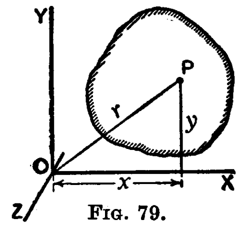

FIG. 79.

Consider a small element P, of area a, whose distances from OY, OX, and O are x, y, and r, respectively, then $r^2 = x^2 + y^2$, $ar^2 = ax^2 + ay^2$, and $\Sigma ar^2 = \Sigma ax^2 + \Sigma ay^2$, therefore $I_z = I_y + I_x$.

Corollary 1.—If OZ is a fixed axis perpendicular to the plane of the figure and if OX and OY are any two mutually perpendicular axes in the plane, then $I_x + I_y$, being equal to I_z, is constant.

Corollary 2.—Since $I_x + I_y$ is constant, it follows that if I_x is a maximum, I_y is a minimum.

Theorem II.—Let I be the moment of inertia of a surface or body about an axis XX passing through its centre of gravity G (Fig. 80), and let I_1 be the moment of inertia of the surface or body about an axis X_1X_1 parallel to XX and at a perpendicular distance r from it. Let A be the area of the surface and M the mass of the body. It can be shown (see D. A. Low's *Applied Mechanics*, Art. 68, p. 51) that—

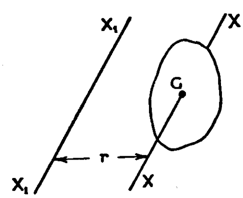

FIG. 80.

For the surface $\qquad I_1 = I + Ar^2.$

For the body $\qquad I_1 = I + Mr^2.$

Corollary 1.—If k and k_1 are the radii of gyration about the axes XX and X_1X_1, respectively, $I = Ak^2$ or Mk^2 and $I_1 = Ak_1^2$ or Mk_1^2. Hence $k_1^2 = k^2 + r^2$.

Corollary 2.—The radius of gyration about a given axis passing through the centre of gravity is less than the radius of gyration about an axis parallel to the given axis, and the axis about which the radius of gyration is least must pass through the centre of gravity.

46. Moment of Inertia—Routh's Rule.—

Moment of inertia about an axis of symmetry

$$= mass \times \frac{sum\ of\ squares\ of\ perpendicular\ semi\text{-}axes}{3,\ 4,\ or\ 5}.$$

The denominator is to be 3, 4, *or* 5, *according as the body is rectangular, elliptical, or ellipsoidal.*

In this rule the word mass may be interpreted as either mass or area. Also it is to be understood that the semi-axes are perpendicular to the axis about which the moment of inertia is required. For instance, for a circle (this is the particular case of an ellipse where the major and minor axes are equal) there is one semi-axis if the moment of inertia is about a diameter, and there are two semi-axes if it is about an axis through the centre perpendicular to the plane of the figure.

In the case of right solids, all sections perpendicular to the axis about which the moment of inertia is required must be either rectangular, circular, or elliptical and must all be equal.

The rule cannot be used to find the moment of inertia of a square about a diagonal, but this is the same as the moment of inertia about an axis through the centre and parallel to one side. The proof follows at once from Art. 45, Theorem I, Corollary 1.

47. Moment of Inertia.—*Example.*—To find the moment of inertia and radius of gyration of a right circular cylinder, of radius R, length l, and mass M, about the longitudinal axis.

Using Routh's rule, there are two perpendicular semi-axes, each equal to R, therefore

$$I = M\frac{R^2 + R^2}{4} = M\frac{R^2}{2}$$

and $$k = \sqrt{(I/M)} = R/\sqrt{2}.$$

Working from first principles, let m be the mass per unit volume, then $m = M/\pi R^2 l$.

For a thin cylinder of radius r and thickness δr (Fig. 81) the mass is $2\pi r\delta rlm$ or $2r\delta r\mathrm{M}/\mathrm{R}^2$ and the moment of inertia is $2r^3\delta r\mathrm{M}/\mathrm{R}^2$.

The total moment of inertia is

$$\mathrm{I}=\frac{2\mathrm{M}}{\mathrm{R}^2}\int_0^{\mathrm{R}} r^3dr=\frac{2\mathrm{M}}{\mathrm{R}^2}\frac{\mathrm{R}^4}{4}=\mathrm{M}\frac{\mathrm{R}^2}{2}$$

and, as before, $k=\mathrm{R}/\sqrt{2}$.

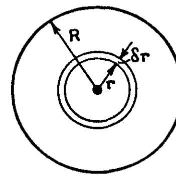

FIG. 81.

Other examples of moments of inertia are given in the exercises at the end of this Chapter, in D. A. Low's *Applied Mechanics*, p. 52, and in the author's *Mathematics*, p. 252.

48. **Work.**—Work is measured by the product of force and the distance through which it acts. In Fig. 82 force P is plotted against displacement s. The area of the shaded strip of width δs represents $\mathrm{P}\delta s$, the work done during the small displacement δs. The area under the curve between the ordinates at A and B represents the work done whilst the displacement increases from $s=s_1$ to $s=s_2$, therefore

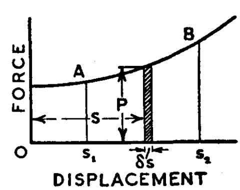

FIG. 82.

$$\text{Work between these limits}=\int_{s_1}^{s_2}\mathrm{P}ds.$$

If the equation to the curve AB is known, the value of this integral may usually be obtained. If the equation is unknown, the actual area under the curve may be found graphically, and then, taking into account the force and displacement scales, the work done may be evaluated. Suppose 1 inch horizontally represents a feet and 1 inch vertically represents p pounds, then 1 square inch represents ap foot-pounds. Hence, if the whole area considered is A square inches, the work done will be apA foot-pounds.

From what has been said about units of force, it follows that p should be in pounds-weight instead of pounds, and then, if displacement is in feet, the work done would be in foot-pounds-weight. However, in practice " weight " is omitted and the work is said to be in foot-pounds. Other commonly used units are inch-pounds, foot-tons, and inch-tons.

49. Work in Raising a System of Masses.—When a number of masses are raised through different heights, or when all the parts of one mass are not raised through the same height, the amount of work done is obtained by multiplying the total weight lifted by the distance through which the centre of gravity of the system is raised (Proof is given in D. A. Low's *Applied Mechanics*, p. 24).

50. Work done by a Torque.—Suppose a uniform torque T pound-feet acts through an angle of θ radians (Fig. 83). Let T be replaced by a force P pounds acting at a radius r feet, the product Pr being equal to T. The force P is displaced a distance $s = r\theta$ and the work done is

$$Ps = Pr\theta = T\theta \text{ foot-pounds.}$$

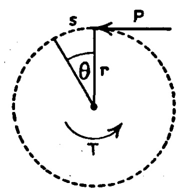

FIG. 83.

If the torque starts from zero and increases uniformly to the value T, then the mean value is $\frac{1}{2}$T and the work done is $\frac{1}{2}T\theta$. In general, if a varying torque T acts through an angle θ,

$$\text{Work done} = \int_0^\theta T d\theta.$$

If the equation between T and θ is known, the value of this integral may usually be obtained, otherwise graphical integration must be used as explained in Art. 48.

51. Energy.—Energy is the capacity for doing work.

Potential Energy is energy due to position or configuration. If a body of weight W is lifted a height h from the ground, the work done is Wh and this is stored up as potential energy due to the position of the body. If the body is then allowed to fall a distance h it can be made to do work equal to Wh. A deflected spring has potential energy stored in it, for it can be made to do work in returning to its unstrained position.

Kinetic Energy is energy due to motion. If a body of weight W which has been lifted through a height h is allowed to fall, it will lose potential energy and gain kinetic energy. For instance, when it has fallen a distance $\frac{1}{4}h$ its potential energy will be $\frac{3}{4}Wh$ and its kinetic energy will be $\frac{1}{4}Wh$. Just as it reaches the ground its potential energy will be zero and its kinetic energy will be Wh.

52. Conservation of Energy.—Energy is indestructible but may change its form. For example, work done against friction will generate heat. If there is no friction and no external work is done, then the total energy in any mechanical system is constant.

In Mechanics, the conservation of energy is stated by the relation

Potential Energy + Kinetic Energy = Constant.

53. Kinetic Energy of Translation and Rotation.—Suppose a body starts from rest and acquires a velocity v in time t. It is required to find the kinetic energy.

If the mass is M, the displacement is s and the accelerating force is P, then

$$P = M\frac{dv}{dt} = Mv\frac{dv}{ds}.$$

Therefore $$\int_0^s P\,ds = M\int_0^v v\,dv = \tfrac{1}{2}Mv^2,$$

and this is the kinetic energy. It follows that if the velocity of the body changes from v_1 to v_2, the change of kinetic energy is $\frac{1}{2}M(v_2^2 - v_1^2)$.

The kinetic energy of a rotating body is found in a similar way. Suppose the moment of inertia about the axis of rotation is I and that at time t the torque is T, the angle turned through from rest is θ and the acquired angular velocity is ω.

Since $$T = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\,\frac{d\theta}{dt} = I\omega\frac{d\omega}{d\theta},$$

therefore $$\int_0^\theta T\,d\theta = I\int_0^\omega \omega\,d\omega = \tfrac{1}{2}I\omega^2$$

and this is the kinetic energy due to rotation. If the angular velocity changes from ω_1 to ω_2, the change of kinetic energy is $\frac{1}{2}I(\omega_2^2 - \omega_1^2)$.

If a body has motions of translation and rotation at the same time, the linear velocity of the axis of rotation being v and the angular velocity about the axis of rotation being ω, then the total kinetic energy is

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2.$$

54. Power.—Power is the rate of doing work. If a force P is working at a speed V, then the work done per unit time is PV.

Horse-power is the name given to the particular rate of 33,000 foot-pounds per minute or 550 foot-pounds per second. If a force of P pounds is doing work at the rate of V feet per minute, then

$$\text{Horse-power} = \frac{PV}{33{,}000}.$$

If a torque of T pound-feet is turning a shaft at N revolutions per minute, the angle turned through per minute is $\theta = 2\pi N$ radians, the work done per minute is $T\theta = 2\pi NT$,

and $$\text{Horse-power} = \frac{2\pi NT}{33{,}000}.$$

55. Vector Representation of Angular Velocity and Torque.—Consider a disc C or any other body rotating about an axis AB (Fig. 84) with an angular velocity ω. If the direction of rotation is clockwise when viewed from A, the angular velocity may be represented in magnitude and sense by a line *oc* drawn parallel to the axis AB, to a suitable scale, the sense being determined, by using the right-handed screw convention, as from *o* to *c*, the direction in which a right-handed screw would advance when turned clockwise in a fixed nut. Viewed from B, the direction of

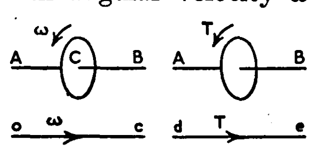

FIG. 84. FIG. 85.

rotation is anticlockwise, but a right-handed screw would still move in the direction AB if turned anticlockwise, therefore *oc* still represents ω. If the direction of ω is reversed, then ω is represented by *co*.

What has been said about angular velocity may also be applied to angular momentum, because the latter is a constant times velocity.

In the same way a torque T (Fig. 85) may be represented by a line *de* drawn parallel to the axis AB, to a suitable scale.

56. Gyrostatic Motion.—In Fig. 86, OX, OY, and OZ are mutually perpendicular axes, and a disc C is rotating about OX with a constant angular velocity ω, the direction being clockwise when viewed from O. Suppose now that a torque T is applied to the disc, in a clockwise direction when looking from Z to O. If the diameter MN is parallel to OZ, the torque is clockwise about this diameter when viewed from N. It is required to investigate the motion of the disc due to the application of the torque T.

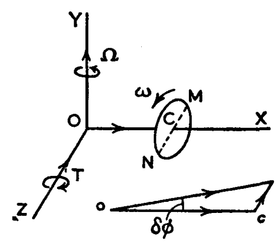

FIG. 86.

Let I be the moment of inertia of the disc about OX, then the angular momentum about OX is $I\omega$. Draw the vector *oc* parallel to OX, to a suitable scale, to represent $I\omega$. Now a torque T applied for a time δt will produce a change of angular momentum equal to $T\delta t$, therefore draw *cc'* parallel to ZO to represent $T\delta t$ and join *oc'*. Let the angle *coc'* be $\delta\phi$. The triangle *coc'* is in the ZOX plane and the angle *occ'* represents 90°.

Since ω is a constant angular velocity, *oc'* must also represent $I\omega$. As $\delta\phi$ is very small, *cc'* is approximately equal to $oc\delta\phi$, therefore

$$T\delta t = I\omega\delta\phi \quad \text{or} \quad T = I\omega\frac{\delta\phi}{\delta t}, \quad \text{approximately.}$$

When δt is made indefinitely small and approaches zero,

$$T = I\omega\frac{d\phi}{dt} = I\omega\Omega,$$

where $\Omega = \frac{d\phi}{dt}$ is the angular velocity in the ZOX plane.

Therefore it is seen that instead of the axis OX turning about OZ, due to the torque T, it turns about the axis OY in an anticlockwise direction when viewed from Y. This motion of the axis of spin OX, about the axis OY, is called *precession* and the rotating disc *precesses*.

It is to be noted that the precession *tends* to cause the axis of spin to move round so that it coincides with the axis of the applied torque and so that the spin and the torque would then have the same sense. Actually, of course, when the axis of spin moves round, the axis of torque also moves round because the angle between these axes remains a right angle.

When a torque is applied in such a way as to cause a spinning body to precess, the motion is called *gyrostatic*.

Exercises V

1. A body, weighing 100 lb. and supported on a horizontal plane, is moved by a horizontal force of 20 lb. against a constant frictional force of 15 lb. If the body starts from rest, find the value of its acceleration and also its speed at the end of 15 seconds. If the 20-lb. force is then removed, what further time will elapse before the body comes to rest?

2. A pile-driver hammer weighing 1500 lb. falls a distance of 2·5 feet on to an inelastic pile weighing 1000 lb. and drives it 1·5 inches into the ground against a uniform resistance. Find the velocity with which the pile begins to move and the time during which it is moving. Find also the value of the ground resistance (*a*) neglecting the weights of the hammer and the pile, (*b*) taking these weights into account.

3. Two masses M and m, moving along a straight line in the same direction with velocities U and u respectively, collide and continue their motion with a common velocity v. Show that the loss of kinetic energy is

$$\frac{Mm(U-u)^2}{2(M+m)}.$$

4. A body, initially at rest, is acted on by a force which varies as the square of the time and is 50 lb. after 20 sec. Find the time-average and the space-average of the force during the first 20 sec.

5. What would be the answers to the preceding exercise if the force varies as the time and is 100 lb. after 20 sec.?

6. A car weighing 1 ton starts from rest on a level road. The difference between the tractive effort and the resistance to motion is initially 500 lb. and falls, the decrease being proportional to the distance travelled, until it is zero at the end of 250 yards. Find the maximum speed in miles per hour.

7. If a car weighing 1 ton starts from rest on a level road and the accelerating force in pounds is $500 - 0{\cdot}045v^2$, where v is the speed in miles per hour, find the speed at the moment when the car has travelled 250 yards.

8. A shell weighing 2000 lb. is fired horizontally with a velocity of 2100 ft. per sec. from a gun mounted on a truck, the combined weight of gun and truck being 170 tons. If the truck is standing on an incline of 1 in 5, measured as a sine, find the distance it will travel up the slope.

9. If in the preceding exercise a constant force of F tons is acting down the incline during the motion, find its value so that the truck may come to rest after travelling 5 feet.

10. Two men exerting together a force of 90 lb. weight put a railway wagon into motion. The wagon weighs 6 tons and the resistance to motion is 10 lb. per ton. How far does the wagon advance in 1 minute; and at what rate, in horse-power, are the men working at the end of the minute?

If the men can at most do work at the rate of 0·8 horse-power, at what constant speed can they keep the wagon moving? [C.U.]

11. A flywheel is secured to a shaft which is supported horizontally in bearings, and a weight W is carried by a taut string attached to and wrapped round the shaft (Fig. 87). Starting from rest, the weight W descends a distance of 4 feet, then the string is detached from the shaft and the latter does a further 150 turns before coming to rest. Assuming that a constant resisting couple C acts on the shaft during the whole experiment, find its value in lb.-inches.

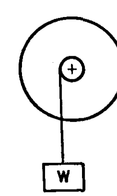

Fig. 87.

Weight of flywheel and shaft is 80 lb. and their radius of gyration is 10 inches. Diameter of shaft is $1\frac{1}{2}$ inches and W = 15 lb.

12. A flywheel is fixed on a shaft, 1·5 inches in diameter, which is supported on two parallel rails having a slope of 1 in 10 (measured as a sine). If the flywheel and shaft start from rest and in 10 seconds roll 6·5 feet down the slope without

slipping, find their radius of gyration about the centre line of the shaft.

13. For starting a small diesel engine by hand, all compression is relieved while the handle is swung until a speed of 100 r.p.m. is attained. Full compression is then brought in, and the energy stored in the flywheel is relied upon to move the piston past top dead centre at an instantaneous crankshaft speed of 50 r.p.m. If full compression starts at 40° after bottom dead centre, and the work done during compression is given by $\int \frac{C \sin \theta}{1 - \cos \theta} d\theta$, where $C = 125$ lb.-ft., calculate the radius of gyration of the flywheel whose weight is 2 cwt.

14. A body weighing 2000 lb., starting from rest, is propelled in a straight line against a constant resistance of 40 lb. by a constant horse-power which is equal to 1·2. Find the time taken for the body to reach a speed of 10 ft. per sec.

15. A car has a speed of 60 m.p.h. round a curve on a banked track of 100 yards radius. If there is no lateral force between the car and the track, find the angle of banking.

If a car weighing 2000 lb. travelled round the track at 70 m.p.h., what would be the lateral force between the car and the track?

16. A railway track of 4 ft. $8\frac{1}{2}$ in. gauge is laid in a curve of 3000 ft. mean radius. Find the super-elevation necessary if a train travelling at 70 m.p.h. is to exert the same pressure on the outer rail as it will on the inner rail when travelling at 30 m.p.h.

17. A liner of 50,000 tons displacement is driven by engines which are capable of maintaining a propulsive force of 250 tons at all speeds. Resistance to motion varies as the square of the speed and the vessel reaches a steady speed at 30 m.p.h. Find the horse-power developed at this speed, and if the engines are then shut off, find the time which elapses before the speed has fallen to 15 m.p.h. [C.U.]

18. Water is pumped at the rate of 150 tons per hour through a pipe to a height of 40 feet where it is delivered through a nozzle of 3 inches diameter. Find the horse-power required to drive the pump if 35 per cent. of the energy supplied is wasted in frictional resistances. Take the weight of a cubic foot of water as 62·3 lb.

19. A railway truck weighing 8 tons, travelling at 4 m.p.h. on a level track, collides with a stationary truck weighing 12 tons. When the spring buffers are in contact and the distance between the trucks is reduced by x inches the reaction is $0{\cdot}85x$ tons. Find the common velocity which the trucks have for an instant, the loss of kinetic energy, and the compression in inches of the buffer springs.

20. A motor-car weighing 2000 lb. exerts 18 horse-power when travelling at a uniform speed of 20 m.p.h. up a slope of 1 in 7 (measured as a sine). Find (*a*) the horse-power exerted in overcoming gravity, and (*b*) the resistance due to friction, air pressure, etc.

21. On the level the maximum speed of a train of total weight W tons is V miles per hour, the resistance to the motion when the speed is v miles per hour being $(a+bv)$ tons weight, where a and b are constants. The pull of the engine is constant and equal to P tons weight. Show that the maximum horse-power of the engine when the train is on an up-gradient of angle α, under the same resistance, is

$$\frac{448}{75}\mathrm{PV}\left(1-\frac{\mathrm{W}\sin\alpha}{b\mathrm{V}}\right).$$

Calculate the value of V, given that the resistances corresponding to speeds of 5 and 25 m.p.h. are respectively 1 and $3\frac{1}{2}$ tons weight, that $\mathrm{W}=450$, that $\sin\alpha=1/200$, and that the values of the maximum horse-power developed on the level and on the incline are respectively 360 and 252. [C.U.]

22. A horizontal shaft in fixed bearings carries an eccentric of radius a, the distance of whose centre from that of the shaft is c. The eccentric bears against the horizontal surface of a bar carrying a weight. The bar is so constrained that it is only free to move vertically, remaining parallel to itself. The shaft is slowly turned so that the bar rises and falls. Find an expression for the couple which must be applied to the shaft in terms of the angle through which it has been turned. The weight of the eccentric may be neglected, and the only friction to be taken account of is that between the bar and the eccentric, the coefficient for which is μ.

If $\mu=\frac{1}{4}$ and $a=2c$, prove that the efficiency of the arrangement, when used as a machine for raising the bar and weight from the lowest to the highest position by a half-turn of the shaft is approximately 0·56. [C.U.]

23. An aeroplane travelling at a speed V relative to the air experiences a resistance $\mathrm{R}=a\mathrm{V}^2+b/\mathrm{V}^2$, where a and b are constants within certain limits of V. Show that, within these limits of V, the power absorbed in air resistance has a minimum value H_0, at a speed V_0, where

$$\mathrm{H}_0=4\left(\frac{ab^3}{27}\right)^{\frac{1}{4}},\qquad \mathrm{V}_0^4=b/3a.$$

Assuming that the effective thrust power Z of the propeller is independent of V, find the greatest rate of gain of height, and show that the aeroplane is then climbing at an angle $\sin^{-1}\dfrac{Z-H_0}{WV_0}$ to the horizontal, where W is the weight of the aeroplane. [C.U.]

24. Let it be assumed that the effective rate of working of the engine of a motor-car is represented by the expression $hn(n_1-n)$ where n is the number of revolutions per second and h, n_1 are certain constants; also that the velocity v is equal to kn where k is a constant. If there is a fixed resistance to the motion of the car equal to λ times its weight (w), find the speed with which the car will ascend an incline of angle α. If the value of k is capable of adjustment, show that to secure the best speed k must be equal to

$$hn_1/2w(\lambda+\sin\alpha),$$

the best speed being

$$hn_1^2/4w(\lambda+\sin\alpha).$$

If the maximum power of the engine is 20 H.P., the weight of the car is one ton, and the best speed up a slope of one in ten is 20 miles per hour, show that the best speed on the level is nearly 50 miles per hour and find the best speed up a slope of one in five. [C.U.]

25. A particle of mass m moves in a vertical plane in a medium whose resistance is km multiplied by the velocity. Show that the component velocities referred to horizontal and vertical axes are

$$\dot{x}=Ae^{-kt},\qquad \dot{y}=Be^{-kt}-\frac{g}{k},$$

and that the resultant acceleration is in a fixed direction. Prove also that the vertical distance of the particle from a line through the point of projection parallel to the direction of the resultant acceleration is gt/k. [C.U.]

26. A motor-car is travelling round a left-hand bend of 20 feet mean radius at a speed of 15 m.p.h. The flywheel and crankshaft rotate at 1500 r.p.m. in a clockwise direction when viewed from the front; their weight is 100 lb. and their radius of gyration is 7·5 inches. The distance between the front and back axles is 8 feet. Find the change in the pressures on the wheels due to gyrostatic action of the flywheel and crankshaft.

27. A four-wheeled vehicle is rounding a curve of 200 feet mean radius at a speed of 30 m.p.h. The wheel track is 4 feet 6 inches and each wheel has a diameter of 2·5 feet, a radius of gyration of 1 foot, and weighs 50 lb. Find the alteration of pressure between each wheel and the track, due to gyrostatic action.

Moments of Inertia

In the exercises which follow, show that the values of the moments of inertia are as given.

28. A straight and uniform slender rod, of length l and mass M, about an axis perpendicular to it and passing through one end, $I = \frac{1}{3}Ml^2$.

If the axis passes through the centre of the rod instead of through one end, $I = \frac{1}{12}Ml^2$.

29. Rectangle or parallelogram, base of length b and height h, about an axis coinciding with the base, $I = \frac{1}{3}bh^3$.

If the axis passes through the centre of gravity and is parallel to the base, $I = \frac{1}{12}bh^3$.

If the axis passes through the centre of gravity and is perpendicular to the plane of the rectangle, $I = \frac{1}{12}bh(b^2 + h^2)$.

30. A square, length of side a, about a diagonal, $I = \frac{1}{12}a^4$.

31. Triangle, base of length b and altitude h, about an axis coinciding with the base, $I = \frac{1}{12}bh^3$.

If the axis passes through the centre of gravity of the triangle and is parallel to the base, $I = \frac{1}{36}bh^3$.

If the axis passes through the vertex of the triangle and is parallel to the base, $I = \frac{1}{4}bh^3$.

32. Circle of radius R, about an axis passing through its centre and perpendicular to its plane, $I = \frac{1}{2}\pi R^4$.

About a diameter, $I = \frac{1}{4}\pi R^4$.

33. A solid cylinder of length l, radius R and mass M, about a diameter at one end, $I = M(\frac{1}{4}R^2 + \frac{1}{3}l^2)$.

34. A solid sphere of radius R and mass M, about a diameter, $I = \frac{2}{5}MR^2$. (It is known that the volume of a sphere is $\frac{4}{3}\pi R^3$.)

35. A solid right circular cone, about a diameter in the base, taking R as the base radius, H as the altitude, and M as the mass, $I = \frac{3}{20}M(R^2 + \frac{2}{3}H^2)$.

CHAPTER VI

DIMENSIONS—DYNAMICAL SIMILARITY

57. Dimensions.—In Mechanics most quantities may be expressed in terms of one or more of the fundamental quantities, mass M, length L, and time T, although sometimes it may be convenient to use force F instead of mass M.

A mass is of one dimension in M, a length is of one dimension in L, and a period of time is of one dimension in T. An area being length multiplied by length, or L^2, is of two dimensions in L, and a volume, or L^3, is of three dimensions in L.

Velocity, being length divided by time, or L/T or LT^{-1}, is of one dimension in L and minus one in T. Acceleration, or L/T^2 or LT^{-2}, is of one dimension in L and minus two in T. An angle measured in radians is an arc divided by a radius, or L/L, which is a ratio or mere number and is dimensionless. Therefore angular velocity has the dimension 1/T or T^{-1}, and angular acceleration has the dimensions $1/T^2$ or T^{-2}.

Since force = mass × acceleration, the dimensions of force F in terms of M, L, and T are ML/T^2 or MLT^{-2}.

In terms of F and L, stress or force/area has the dimensions F/L^2 or FL^{-2}, and in terms of M, L, and T this becomes MLT^{-2}/L^2 or $ML^{-1}T^{-2}$.

The dimensions of various quantities are tabulated on p. 90 in terms of M, L, and T and of F, L, and T.

In any equation having a physical meaning and which is true in any system of units, the dimensions of all terms must be the same. This is shown in the examples which follow.

Example 1.—Consider the relation between velocity, acceleration, and time, that is $v = ft$.

Quantity.	Dimensions.	
	In Terms of M, L, T.	In Terms of F, L, T.
Mass	M	$FL^{-1}T^2$
Force	MLT^{-2}	F
Moment of Force	ML^2T^{-2}	FL
Momentum or Impulse	MLT^{-1}	FT
Angular Momentum	ML^2T^{-1}	FLT
Work or Energy	ML^2T^{-2}	FL
Power	ML^2T^{-3}	FLT^{-1}
Moment of Inertia	ML^2	FLT^2
Stress	$ML^{-1}T^{-2}$	FL^{-2}
Density	ML^{-3}	$FL^{-4}T^2$
Viscosity	$ML^{-1}T^{-1}$	$FL^{-2}T$
Kinematic Viscosity	L^2T^{-1}	L^2T^{-1}

The dimensional equation is

$$L/T = (L/T^2)T$$

or

$$LT^{-1} = LT^{-1}.$$

Example 2.—Consider the equation $s = ut + \frac{1}{2}ft^2$.

The dimensional equation is

$$L = (L/T)T + (L/T^2)T^2$$

or

$$L = L + L.$$

Since numerical coefficients are dimensionless, they do not appear in dimensional equations and therefore a dimensional check is not complete evidence that an equation is correct.

Example 3.—It is required to examine the dimensions of the terms in the equation connecting the loss of potential energy with the gain of kinetic energy when a body rolls down an incline.

The equation is $\quad Wh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2,$

where the symbols have the usual meanings.

The dimensional equation is

$$\left(\frac{ML}{T^2}\right)L = M\left(\frac{L}{T}\right)^2 + ML^2\left(\frac{1}{T}\right)^2$$

or

$$ML^2T^{-2} = ML^2T^{-2} + ML^2T^{-2}.$$

Example 4.—The relation between the torque T_q and the shear stress f in a shaft of diameter d is $T_q = \pi d^3 f/16$. It is required to check the dimensions of this formula.

This is an example where it is simpler to use F than M.

The dimensional equation is

$$FL = L^3F/L^2$$

or

$$FL = FL.$$

In terms of M, L, and T the dimensional equation is

$$\left(\frac{ML}{T^2}\right)L = L^3\left(\frac{ML}{T^2}\right)\bigg/L^2$$

or

$$ML^2T^{-2} = ML^2T^{-2}.$$

Example 5.—A list of formulæ gave the stress f in a leaf spring as

$$f = \frac{3}{2}\frac{Wl^a}{nbt^2}$$

where W is load, l is length, n is the number of leaves, and b and t are respectively the breadth and thickness of each leaf. The index a was a numerical value which had been altered and made illegible. It is required to find the value of a. [This simple problem actually occurred and the method of dimensions produced the answer in far less time than it could have been obtained in any other way.]

In terms of F and L the dimensional equation is

$$F/L^2 = FL^a/L^3$$

or

$$FL^{-2} = FL^{a-3}.$$

Therefore $\quad a - 3 = -2 \quad$ and $\quad a = 1.$

58. Dimensional Method of Determining Indices in Formulæ.—Sometimes it is convenient to use the method of dimensions to find how one quantity varies in relation to other quantities. The method is shown in the following examples:—

Example 1.—A torque T_q is transmitted through a shaft of diameter d and it is required to use the dimensional method to find how the torque varies with d and with the shear stress f.

Assume that $$T_q = Cd^a f^b,$$

where C is a constant and a and b are unknown indices. Since a constant is dimensionless, this method will not enable the value of C to be determined. In general, it is incorrect to assume only one term on the right-hand side as it might be necessary to have a number of terms, but this point will be explained further in the next Art. In the present example, and in each of the two which follow, it happens that only one term is required on the right-hand side.

Since $$T_q = Cd^a f^b,$$

the dimensional equation, in terms of F and L, is

$$FL = L^a (F/L^2)^b$$

or $$FL = L^{a-2b} F^b.$$

Equating indices of F, which must be the same on each side of the equation,

$$b = 1.$$

Similarly, equating indices of L,

$$a - 2b = 1,$$

therefore $$a - 2 = 1 \quad \text{or} \quad a = 3.$$

The dimensional equation becomes

$$FL = L^3 (F/L^2)$$

and the required equation is

$$T_q = Cd^3 f,$$

showing that the torque is proportional to the stress and the cube of the diameter.

Example 2.—To find how the periodic time t of a pendulum varies with the mass m, the length l, and with g.

Assume $$t = Cm^a l^b g^c,$$

where C is a constant and a, b, and c are unknown indices.

The dimensional equation is

$$T = M^a L^b (L/T^2)^c$$

or
$$T = M^a L^{b+c} T^{-2c}.$$

Equating indices

of M, $\qquad a = 0,$

of L, $\qquad b + c = 0,$

of T, $\qquad -2c = 1.$

From these equations,

$$c = -\tfrac{1}{2} \quad \text{and} \quad b = -c = \tfrac{1}{2},$$

therefore
$$T = L^{\frac{1}{2}}(L/T^2)^{-\frac{1}{2}}$$

and the required equation is

$$t = C(l/g)^{\frac{1}{2}}.$$

Example 3.—To find how the deflection y of a beam varies with the linear dimensions and the modulus of elasticity E, given that y is directly proportional to the applied load W.

Assume
$$y = CWd^aE^b,$$

where C is a constant and d may be any linear dimension of the beam, such as span, depth, breadth, thickness of a flange, etc. Since E = stress/strain, its dimensions are those of stress.

The dimensional equation is

$$L = FL^a(F/L^2)^b$$

or
$$L = F^{1+b}L^{a-2b}.$$

Equating indices

of F, $\qquad 1 + b = 0,$

of L, $\qquad a - 2b = 1.$

From these equations,

$$b = -1 \quad \text{and} \quad a = 1 + 2b = -1.$$

Therefore
$$y = \frac{CW}{dE}.$$

59. Dimensional Method continued—Dynamical Similarity.—In the examples in the preceding Art. values were obtained for all the indices, but in many cases this is not possible.

Consider the relation

$$S = Cx^a y^b z^c,$$

where S, x, y, and z are physical quantities, C is a constant, and a, b, and c are indices to be determined. Now the right-hand side might have to be represented by a series of terms instead of by one term. If the values of a, b, and c can be obtained when the right-hand side is taken as being one term, it is evident that this one term could not be expanded as a series because the separate terms would all be identical except for their coefficients and could be added together to make one term. But it is not until the indices have been found that it is known that there is only one term.

If S is a function of x, y, and z, or in symbols

$$S = \phi(x, y, z),$$

where ϕ denotes " a function of " (f or F are often used instead of ϕ), it should be assumed that

$$S = \Sigma C x^a y^b z^c,$$

where Σ denotes that the sum of a number of terms is intended and C is a constant in each term.

Physical considerations require that all the terms of the series must have the same dimensions as the quantity which the series represents, hence the dimensions of the function itself ($\Sigma C x^a y^b z^c$) may be equated to those of the quantity it represents (S). In cases where all the indices can be found, the function can be expressed in one term.

When equating indices it saves space to use the notation [M] to denote " equating indices of M," and similarly for [L] and [T] and other dimensions which may occur.

It has been stated above that in many cases it is not possible to obtain the values of all the assumed indices; it should be noted, however, that it may be possible to determine enough indices to provide some useful information, and this is the basis of the solution of problems in dynamical similarity. In considering dynamical similarity, the idea in general is to study the behaviour of one body and to predict what will be the behaviour of a similar body. For instance, information about the design of full-size ships can be obtained from experiments on models.

The examples which follow will help to make the subject clear. The first is one in which all the indices can be found.

Example 1.—In a grinding wheel of outside radius r it is required to find how the permissible angular velocity ω varies with r, with the stress f in the material of the wheel and with its density ρ. (See *Grinding Machinery* by J. J. Guest, p. 29. Arnold.)

Assume $$\omega = \Sigma a f^l r^m \rho^n,$$

where a is a constant and l, m, and n are indices to be determined.

The dimensional equation is

$$\frac{1}{T} = \left(\frac{ML}{T^2L^2}\right)^l L^m \left(\frac{M}{L^3}\right)^n$$

or $$T^{-1} = M^{l+n} L^{-l+m-3n} T^{-2l}.$$

[T] $$2l = 1$$

[M] $$l + n = 0$$

[L] $$-l + m - 3n = 0.$$

From these equations,

$$l = \tfrac{1}{2}, \quad n = -\tfrac{1}{2}, \quad \text{and} \quad m = -1.$$

Therefore $$\omega = a f^{\frac{1}{2}} r^{-1} \rho^{-\frac{1}{2}}$$

or $$\omega r = a\sqrt{\frac{f}{\rho}}.$$

Regarding the density as constant, the limiting value of the circumferential speed ωr is proportional to the square root of the permissible stress f.

Example 2.—To find how the deflections y of geometrically similar beams vary with the linear dimensions d, the modulus of elasticity E, and the applied load W.

This problem is similar to Example 3 in the preceding Art., but there it was given that y is directly proportional to W.

Since y is a function of W, d, and E, or

$$y = \phi_1(\mathrm{W}, d, \mathrm{E}),$$

assume
$$y = \Sigma \mathrm{A}\mathrm{W}^a d^b \mathrm{E}^c,$$

where A is a constant in each term and a, b, and c are unknown indices.

The dimensional equation is

$$\mathrm{L} = \mathrm{F}^a \mathrm{L}^b (\mathrm{F}/\mathrm{L}^2)^c$$

or
$$\mathrm{L} = \mathrm{F}^{a+c} \mathrm{L}^{b-2c}.$$

[L]
$$b - 2c = 1$$

[F]
$$a + c = 0.$$

From these equations, obtaining b and c in terms of a,

$$c = -a \quad \text{and} \quad b = 1 - 2a.$$

As there are three unknown quantities and only two equations, numerical values cannot be obtained for a, b, and c.

Therefore
$$y = \Sigma \mathrm{A}\mathrm{W}^a d^{1-2a} \mathrm{E}^{-a}$$

or
$$y = d\Sigma \mathrm{A}\left(\frac{\mathrm{W}}{d^2\mathrm{E}}\right)^a$$

or
$$\frac{y}{d} = \phi_2\left(\frac{\mathrm{W}}{d^2\mathrm{E}}\right).$$

Both y/d and $\mathrm{W}/d^2\mathrm{E}$ are dimensionless. As the index a is unknown, the function of $\mathrm{W}/d^2\mathrm{E}$ may or may not be a series, but if it is a series each term will be dimensionless and the equation will still be dimensionally correct.

It follows that, in geometrically similar beams, if the values of $\mathrm{W}/d^2\mathrm{E}$ are equal, then corresponding values of y/d will be equal.

For instance, suppose the linear dimensions (d_1) of one beam are ten times those (d_2) of another beam, and let W_1 and W_2 respectively be similarly situated loads on the two beams.

If $$\frac{W_1}{d_1^2E}=\frac{W_2}{d_2^2E} \quad \text{or} \quad \frac{W_1}{(10d_2)^2E}=\frac{W_2}{d_2^2E},$$

then, if E is the same for each beam—that is, if the beams are both made of the same material—W_1 must be 100 times W_2.

Now $$\frac{y_1}{d_1}=\frac{y_2}{d_2} \quad \text{or} \quad \frac{y_1}{10d_2}=\frac{y_2}{d_2},$$

therefore $$y_1=10y_2.$$

Example 3.—The resistance to the motion of a ship is

$$R=\phi_1(l, v, \rho, \nu, g),$$

where l is any linear dimension of the ship, v is the speed, ρ is the density of the water and ν is its kinematic viscosity (dimensions L^2/T), and g is the acceleration due to gravity. It is required to compare the resistance of a full-size ship with that of a model.

Assume $$R=\Sigma A l^a v^b \rho^c \nu^d g^e,$$

where A is a constant in each term, and a, b, c, d, and e are unknown indices.

The dimensional equation is

$$\frac{ML}{T^2}=L^a\left(\frac{L}{T}\right)^b\left(\frac{M}{L^3}\right)^c\left(\frac{L^2}{T}\right)^d\left(\frac{L}{T^2}\right)^e$$

or $$MLT^{-2}=L^{a+b-3c+2d+e}M^cT^{-b-d-2e}.$$

[M] $$c=1$$

[L] $$a+b-3c+2d+e=1$$

[T] $$-b-d-2e=-2,$$

from which

$$c=1, \quad b=2-d-2e, \quad \text{and} \quad a=2-d+e.$$

D

Therefore
$$\begin{aligned} R &= \Sigma A l^{2-d+e} v^{2-d-2e} \rho \nu^d g^e \\ &= \Sigma A \rho l^2 v^2 (l^{-d+e} v^{-d-2e} \nu^d g^e) \\ &= \rho l^2 v^2 \Sigma A \left(\frac{\nu}{lv}\right)^d \left(\frac{lg}{v^2}\right)^e \\ &= \rho l^2 v^2 \phi_2\left(\frac{\nu}{lv}, \frac{lg}{v^2}\right). \end{aligned}$$

The ratio lv/ν is known as *Reynolds' Number* and is dimensionless, as is lg/v^2. Now function $1/x$ is also a function of x and therefore, writing Reynolds' Number instead of its reciprocal,

$$R = \rho l^2 v^2 \phi_3\left(\frac{lv}{\nu}, \frac{lg}{v^2}\right)$$

or
$$\frac{R}{\rho l^2 v^2} = \phi_3\left(\frac{lv}{\nu}, \frac{lg}{v^2}\right).$$

It should be noted that both sides of the equation are now dimensionless.

If it is arranged that lv/ν and lg/v^2 are each the same for a model as for a ship, then $R/\rho l^2 v^2$ will be the same for a model as for a ship.

Example 4.—The amplitude of vibration of a rotating body is

$$\delta = \Sigma A l^r \rho^s E^t C^u I^v J^x \omega^y g^z,$$

where A is a constant in each term, l is a linear dimension of the body, ρ is the density, E is the modulus of elasticity, C is the modulus of rigidity, I is the moment of inertia of a cross-section about a transverse axis through the centre of gravity of the cross-section, J is the polar moment of inertia of a cross-section, ω is the angular velocity, g is the acceleration due to gravity, and r, s, t, u, v, x, y, and z are unknown indices. It is required to investigate the motions of geometrically similar bodies.

The dimensional equation is

$$L = L^r \left(\frac{M}{L^3}\right)^s \left(\frac{ML}{T^2L^2}\right)^t \left(\frac{ML}{T^2L^2}\right)^u L^{4v} L^{4x} \left(\frac{1}{T}\right)^y \left(\frac{L}{T^2}\right)^z$$

or $\quad L = L^{r-3s-t-u+4v+4x+z}M^{s+t+u}T^{-2t-2u-y-2z}.$

$$[L] \qquad r - 3s - t - u + 4v + 4x + z = 1$$

$$[M] \qquad s + t + u = 0$$

$$[T] \qquad -2t - 2u - y - 2z = 0.$$

From these equations

$$y = -2t - 2u - 2z,$$

$$s = -t - u,$$

and

$$r = 1 - 2t - 2u - 4v - 4x - z.$$

Therefore

$$\begin{aligned}\delta &= \Sigma A l^{1-2t-2u-4v-4x-z}\rho^{-t-u}E^tC^uI^vJ^x\omega^{-2t-2u-2z}g^z\\ &= l\Sigma A(l^{-2t}\rho^{-t}E^t\omega^{-2t})(l^{-2u}\rho^{-u}C^u\omega^{-2u})(l^{-4v}I^v)(l^{-4x}J^x)(l^{-z}\omega^{-2z}g^z)\\ &= l\Sigma A\left(\frac{E}{l^2\rho\omega^2}\right)^t\left(\frac{C}{l^2\rho\omega^2}\right)^u\left(\frac{I}{l^4}\right)^v\left(\frac{J}{l^4}\right)^x\left(\frac{g}{l\omega^2}\right)^z\\ &= l\phi\left(\frac{E}{l^2\rho\omega^2}, \frac{C}{l^2\rho\omega^2}, \frac{I}{l^4}, \frac{J}{l^4}, \frac{g}{l\omega^2}\right).\end{aligned}$$

Dividing by l, the left-hand side of the equation becomes δ/l which is dimensionless, and it will of course be found that the terms on the right-hand side are also dimensionless.

Suppose transverse vibrations are being considered, then $C/l^2\rho\omega^2$ and J/l^4 will not enter into the problem and I/l^4 will be the same for geometrically similar bodies.

The relationship is now reduced to

$$\frac{\delta}{l} = \phi\left(\frac{E}{l^2\rho\omega^2}, \frac{g}{l\omega^2}\right).$$

If $g/l\omega^2$ can be neglected and if $E/l^2\rho\omega^2$ is the same for a model as for the actual body, then δ/l will be the same for each. It is not practicable to vary $g/l\omega^2$ and $E/l^2\rho\omega^2$ simultaneously so that each is the same for a model as for the actual body.

The inclusion of the moment of inertia I as one of the variables in the initial equation allows the model to have a differently shaped cross-section from that of the actual body. Taking l to represent longitudinal measurements only, then provided I/l^4 is the same for the model as for the actual body, the cross-sections need not be similar.

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Exercises VI

Check the dimensions of the equations given in Exercises 1 to 9.

1. The resistance to the penetration of a pile is given by

$$R = W + w + \frac{W^2 h}{(W + w)d}$$

where W and w are weights and h and d are lengths.

2. Power transmitted by a wire rope is given by

$$H = \left(f - \frac{Ed}{D} - \frac{wv^2}{g}\right)\frac{(1 - n)v}{550}$$

where H is horse-power transmitted per unit net section of the rope, f is maximum stress, E is modulus of elasticity (a stress), d is diameter of each wire of rope, D is diameter of pulley, w is weight per unit length per unit net section of rope, v is velocity of rope, g is acceleration due to gravity, and $n = e^{-\mu\theta}$ where μ is a constant and θ is an angle.

3. The maximum horse-power transmitted by a belt is

$$H = \frac{bt(1 - n)}{550} \times \tfrac{2}{3} f \sqrt{\frac{fg}{3w}}$$

where b and t are the breadth and thickness, n is a numerical fraction, f is stress, g is acceleration due to gravity, and w is the weight per unit length per unit area of cross-section.

4. $$Qa = \frac{1}{a}\frac{dv}{dt}\{I_1 + n^2 I_2\}$$

where Q is force, a is length, v is velocity, t is time, I_1 and I_2 are moments of inertia of masses, and n is a constant.

5. Horse-power transmitted by a water turbine is

$$\text{H.P.} = \frac{Q\omega}{550g}\{v_1 r_1 \cos \alpha_1 + v_2 r_2 \cos \alpha_2\}$$

where Q is weight of water per unit time, ω is angular velocity, g is acceleration due to gravity, v_1 and v_2 are linear velocities, r_1 and r_2 are radii, and α_1 and α_2 are angles.

6. $$\theta - \phi = \omega\sqrt{\frac{I}{n}}\sin\left(\sqrt{\frac{n}{I}}t\right) - \frac{f}{n}.$$

θ and ϕ are angles, ω is an angular velocity, I is the moment of inertia of a mass, n is a twisting moment per radian, f is a frictional moment, and t is time.

7. Two of the formulæ for the critical load P on a strut are

$$P = \frac{\pi^2}{l^2}EI \quad \text{and} \quad P = fA\Big/\left\{1 + a\left(\frac{l}{k}\right)^2\right\}$$

where l is length, E is modulus of elasticity, I is moment of inertia of the area of a cross-section about a transverse axis, f is stress, A is area, k is radius of gyration, and a is a constant.

8. The periodic time t of the torsional vibration of a shaft fixed at one end and carrying a flywheel at the other end is

$$t = 2\pi\sqrt{\frac{32Il}{C\pi d^4}}$$

where I is moment of inertia of flywheel, l is length, C is modulus of rigidity (a stress), and d is diameter of shaft.

9. The following is a formula which occurred in some research work:

$$V^2 = \frac{\pi}{2}H\frac{d^2t^2}{m(t+l)}\Big/\left\{1 - \frac{\pi\rho d^2 t^2}{8m(t+l)}\right\}$$

where V is velocity, H is a Brinell Number (dimensions same as stress), m is mass, ρ is density, and d, t, and l are linear measurements.

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10. The frequency of vibration of a stretched wire is given by

$$f = \phi(l, P, m)$$

where l is length, P is stretching force, and m is mass per unit length.

Show that
$$f=\frac{A}{l}\sqrt{\frac{P}{m}}$$
where A is a constant.

11. If the periodic time t of a pendulum varies with mass m, length l, the acceleration g due to gravity, and with the arc of swing s on one side of the mean position, show that
$$t=\left(\frac{l}{g}\right)^{\frac{1}{2}}\phi(a)$$
where $a=s/l$ is the amplitude.

12. (*a*) Supposing that in a fluid flywheel the torque transmitted is given by
$$T_q=\phi_1(N, D, \rho)$$
where N is revolutions per unit time, D is a linear dimension, and ρ is the density of the fluid, show that
$$T_q=kN^2D^5\rho$$
where k is a constant.

(*b*) If the torque is a function of viscosity μ as well as of N, D, and ρ, show that
$$T_q=N^2D^5\rho\phi_2(\mu/ND^2\rho).$$

The torque also depends on the percentage slip, that is on $\frac{N-n}{N}\times 100$, where N and n are the speeds of the two parts of the flywheel respectively, but as this quantity is dimensionless it does not enter into the above calculations.

13. If
$$s=\phi_1(u, t, f),$$
where s is displacement, u is velocity, t is time, and f is acceleration, show that
$$s=ut\phi_2\left(\frac{ft}{u}\right).$$

Given $u=5$, $f=2$, and the tabulated corresponding values of

t	1	2	3	4	5
s	6	14	24	36	50

t and s, plot s/ut against ft/u and show that
$$s/ut=1+\tfrac{1}{2}ft/u \quad \text{or} \quad s=ut+\tfrac{1}{2}ft^2.$$

14. If any deflection y of a whirling shaft is a function of length l, mass m per unit length, modulus of elasticity E, moment of inertia I about a diameter of a cross-section, and angular velocity ω, show that

$$y = l\phi_1\left(\frac{E}{m\omega^2}, \frac{I}{l^4}\right)$$

and that this may also be written as

$$y = l\phi_2\left(\frac{EI}{m\omega^2 l^4}, \frac{I}{l^4}\right).$$

15. Given that $\qquad R = \phi_1(\rho, v, l, \nu)$

where R is the resistance to the motion of a body through a fluid of density ρ, v is the velocity, l is length, and ν is kinematic viscosity, show that

$$R = \rho v^2 l^2 \phi_2\left(\frac{vl}{\nu}\right).$$

16. Prove that the resistance of the air to a projectile can be written as

$$R = \rho v^2 d^2 \phi\left(\frac{v}{a}, \frac{vd}{\nu}\right)$$

where R is air resistance, ρ is air density, a is velocity of sound in air, v is velocity of projectile of diameter d. and ν is kinematic viscosity.

17. Given that

$$f = \phi_1(l, \rho, E, C, I, J, \omega, g)$$

where f is frequency and the other symbols are those used in Example 4, Art. 59, show that

$$f = \omega\phi_2(E/l^2\rho\omega^2, C/l^2\rho\omega^2, I/l^4, J/l^4, g/l\omega^2).$$

CHAPTER VII

SIMPLE HARMONIC MOTION

60. Simple Harmonic Motion.—If a point P (Fig. 88) moves with constant speed in a circular path, then a point p which is the projection of P on a fixed diameter XX′ moves with *simple harmonic motion.* Put in another way, it may be said that uniform circular motion looked at edgeways appears to be simple harmonic motion.

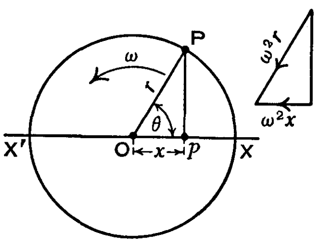


Fig. 88.

Let O be the centre of the circle of radius OP$=r$, let the constant angular velocity of OP be ω, and at time t let the angle XOP be θ and let O$p=x$.

Since P is moving with uniform speed in a circular path, the acceleration of P is $\omega^2 r$ along PO (see Art. 28, p. 47). The component of this acceleration, in the direction parallel to OX, is $-\omega^2 x$ and this is also the acceleration of the point p. The sign is negative because the sense of the acceleration is towards O and the positive direction of x is away from O to the right. When x is positive the acceleration is negative, and when x is negative the acceleration is positive.

The acceleration of the point p is

$$\frac{d^2x}{dt^2} = -\omega^2 x,$$

therefore

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad . \quad . \quad . \quad (1),$$

and this is the equation of simple harmonic motion.

Simple harmonic motion need not be in a straight line; for instance, a flywheel may oscillate with this type of motion which is also approximately that of a simple pendulum. Any body has simple harmonic motion if its acceleration is proportional to its displacement from the mean position and is directed towards the mean position. The equation of motion can always be reduced to the form given in (1).

From Fig. 88 it is seen that

$$x = r \cos \theta \quad . \quad . \quad . \quad (2),$$

and this is a solution of (1), for differentiating with respect to t,

$$\frac{dx}{dt} = -r \sin \theta \frac{d\theta}{dt} \quad \text{and} \quad \frac{d^2x}{dt^2} = -r \cos \theta \left(\frac{d\theta}{dt}\right)^2,$$

then substituting the values of $\frac{d^2x}{dt^2}$ and x in (1), writing ω for the angular velocity $\frac{d\theta}{dt}$,

$$-\omega^2 r \cos \theta + \omega^2 r \cos \theta = 0,$$

showing that (1) is satisfied when $x = r \cos \theta$.

Now suppose θ is measured from a fixed line OA, the angle XOA having a value ϕ (Fig. 89), then as before $\frac{d^2x}{dt^2} = -\omega^2 x$, but in this case

$$x = r \cos (\theta + \phi) \quad . \quad . \quad . \quad (3),$$

and this also is a solution of (1), as may be proved by differentiation and substitution.

Since $\cos (\theta + \phi) = \sin (\theta + \phi + 90°) = \sin (\theta + \phi')$, where $\phi' = \phi + 90°$, it follows that

$$x = r \sin (\theta + \phi') \quad . \quad (4)$$

is another way of expressing the solution given in (3).

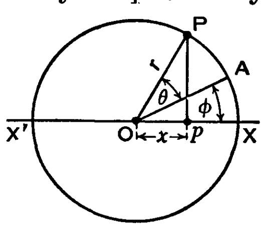


Fig. 89.

Again, since $\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$, therefore, substituting in (3),

$$x=r(\cos\theta\cos\phi-\sin\theta\sin\phi)$$

or

$$x=A\cos\theta+B\sin\theta \quad . \quad . \quad . \quad (5),$$

writing A for $r\cos\phi$ and B for $-r\sin\phi$. Since r, $\cos\phi$, and $\sin\phi$ are constants, therefore A and B are constants. The same result could be obtained by expanding (4).

Equations (3), (4), and (5) are merely different ways of writing the general solution of the differential equation (1), and whichever form of solution is used, the values of the constants are obtained from the conditions given in any particular problem.

Since $\theta=\omega t$, equations (3), (4), and (5) may also be written

$$x=r\cos(\omega t+\phi) \quad . \quad . \quad . \quad (6),$$

$$x=r\sin(\omega t+\phi') \quad . \quad . \quad . \quad (7),$$

$$x=A\cos\omega t+B\sin\omega t \quad . \quad . \quad (8).$$

The velocity is obtained by differentiation; for instance, from (8) the velocity is

$$\frac{dx}{dt}=-\omega A\sin\omega t+\omega B\cos\omega t \quad . \quad . \quad (9).$$

The maximum value of the displacement x is r and is called the *amplitude*. A journey from X to X′ and back to X is called a *cycle*; but a cycle may start at any point and is completed on arrival back at the same point, the motion then being in the same direction as at first. The time taken to complete one cycle is called the *periodic time* or the *period*. The number of cycles completed in unit time is called the *frequency*.

If T is the periodic time and f is the frequency, then, measuring ω in radians per unit time,

$$\omega T=2\pi, \quad T=2\pi/\omega, \quad \text{and} \quad f=1/T=\omega/2\pi.$$

Referring to Fig. 88, the angle XOP is called the *phase angle*. The ratio (*angle* XOP)/2π, which is the same thing as the ratio (*time taken to travel from* X *to* p)/(*periodic time*), is called the *phase*, but sometimes the term phase is used to denote phase angle. The two simple harmonic motions $x_1=a\cos(\omega t+\phi_1)$ and $x_2=b\cos(\omega t+\phi_2)$ differ in phase by the angle $\phi_2-\phi_1$ or by $(\phi_2-\phi_1)/2\pi$ of a period and either of these forms is the *phase difference*.

In some problems, instead of using equations (6), (7), or (8), it is convenient to refer back to the corresponding circular motion and work from first principles. The two methods are illustrated in the examples which follow.

Example 1.—A body moves with simple harmonic motion. The amplitude is 3 inches and the periodic time is 5 seconds. It is required to obtain expressions for the displacement, the velocity, and the acceleration in terms of time t.

Since $\omega T=2\pi$ and $T=5$ sec., therefore $\omega=2\pi/5$ rad./sec.

If x is the displacement from the mean position at time t, then the equation of motion is

$$\frac{d^2x}{dt^2}+\omega^2x=0$$

or

$$\frac{d^2x}{dt^2}+\frac{4\pi^2}{25}x=0.$$

The solution, using the form given in equation (8), is

$$x=A\cos\omega t+B\sin\omega t,$$

and the velocity is

$$\frac{dx}{dt}=-\omega A\sin\omega t+\omega B\cos\omega t.$$

Assuming that time is measured from the instant when the body is at one end of its travel, then $\dfrac{dx}{dt}=0$ when $t=0$, and substituting these values in the velocity equation,

$$0=0+\omega B, \quad \text{or} \quad B=0.$$

Hence the equation of motion reduces to

$$x=A\cos\omega t.$$

The amplitude is 3 inches, so $x=3$ when $t=0$, therefore $A=3$.

Therefore $x=3\cos\omega t=3\cos\frac{2\pi}{5}t$ inches,

velocity $\frac{dx}{dt}=-\frac{6\pi}{5}\sin\frac{2\pi}{5}t$ in./sec.,

and acceleration $\frac{d^2x}{dt^2}=-\frac{12\pi^2}{25}\cos\frac{2\pi}{5}t$ in./sec.^{2}.

The same results could have been obtained by using equations (6) or (7).

Example 2.—A ship is rolling with a period of 10 seconds. A man at the masthead 100 feet above the deck is swung to and fro 25 feet on either side of the vertical with a motion which is approximately horizontal and simple harmonic. The man weighs 200 lb., and his horizontal hold failing at 50 lb. he is thrown off the mast. The width of the deck being 80 feet, prove that he falls clear of the ship. [Assume $\pi^2=10$ and $g=32$ f.s. units.] [C.U.]

Since force = mass × acceleration, or $P=Mf$, the acceleration of the man when his hold fails is

$$f=\frac{P}{M}=\frac{50\times 32}{200}=8 \text{ ft./sec.}^2.$$

Now $T=10$ sec., $\omega T=2\pi$, therefore $\omega=\frac{2\pi}{10}=\frac{\pi}{5}$ rad./sec.

The amplitude is 25 ft., therefore the maximum horizontal acceleration of the masthead is

$$25\omega^2=25\left(\frac{\pi}{5}\right)^2=10 \text{ ft./sec.}^2.$$

Since acceleration is proportional to the displacement from the mean position, it follows that the man loses his hold when he is $\frac{8}{10}\times 25$ or 20 ft. from the mean position, and this is approximately $40-20$ or 20 ft. from a vertical line at the side of the ship (Fig. 90), ignoring the slopes of the mast and the deck.

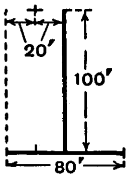


FIG. 90.

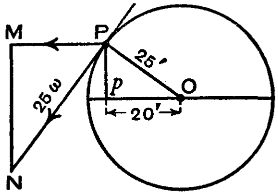


FIG. 91.

The linear velocity of the corresponding circular motion (Fig. 91) is 25ω and, resolving horizontally, the velocity of the man at the instant he loses his hold is $v = 25\omega\dfrac{PM}{PN}$.

From the similar triangles PMN and PpO, $\dfrac{PM}{PN} = \dfrac{Pp}{PO}$.

Therefore

$$v = 25\omega\frac{Pp}{PO} = 25\omega\frac{\sqrt{(25^2 - 20^2)}}{25} = 15\omega = 15 \times \frac{\pi}{5} = 3\pi \text{ ft./sec.}$$

The time taken to travel horizontally another 20 ft. at this velocity is $\dfrac{20}{3\pi}$ sec.

Let h be the distance fallen in time t, then $h = \frac{1}{2}gt^2$.

When $t = \dfrac{20}{3\pi}$, $h = \frac{1}{2} \times 32\left(\dfrac{20}{3\pi}\right)^2 = 71$ ft. approximately.

Therefore the man has fallen 71 ft. by the time he is over the side of the ship, and since he is then $100 - 71 = 29$ ft. above the deck it is evident that he will fall clear of the ship.

61. Simple Pendulum.—A particle A of mass m, suspended from a fixed point O by a light string OA of length l, swings in a vertical plane through a small angle α on each side of the vertical OY (Fig. 92). It is required to find the equation of motion and the periodic time. (The magnitude of the angle α is discussed at the end of Art. 63.)

Suppose that at time t the displacement of the particle A from its mean position Y is s, measured along the arc YA, and that the angle YOA is θ.

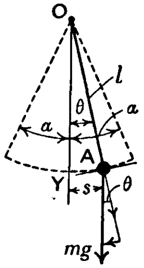


FIG. 92.

Since $s=l\theta$, differentiating twice gives $\frac{d^2s}{dt^2}=l\frac{d^2\theta}{dt^2}$, and this is the acceleration of the particle along the tangent at A, the direction in which s increases being regarded as positive.

The weight of the particle is $w=mg$ and its component along the tangent at A is $mg \sin \theta$ acting in the negative direction.

Now force = mass × acceleration,

therefore
$$-mg \sin \theta = ml\frac{d^2\theta}{dt^2}$$

or
$$\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin \theta=0.$$

When θ is small, $\sin \theta=\theta$ approximately, then

$$\frac{d^2\theta}{dt^2}+\frac{g}{l}\theta=0,$$

and this is the equation of simple harmonic motion.

By comparison with the standard form

$$\frac{d^2\theta}{dt^2}+\omega^2\theta=0,$$

it is seen that $\omega^2=g/l$ or $\omega=\sqrt{(g/l)}$.

Therefore, provided the amplitude a is small, the periodic time

$$T=2\pi/\omega=2\pi\sqrt{(l/g)}.$$

62. Bifilar Suspension.—A plate weighing 2W lb. is suspended by two vertical strings a distance $2a$ apart and each of length l (Fig. 93). The centre of gravity is midway between the points of attachment of the strings and in the same horizontal line. The plate is turned through a small angle about the vertical axis through the centre of gravity and is then released. It is required to investigate the subsequent motion.

Taking time and angular displacement as zero when the position of statical equilibrium is reached, suppose that at time t the plate has turned through an angle θ, the force in each string is F, and that each string then makes an angle ϕ with the vertical.

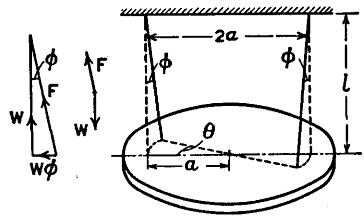


FIG. 93.

Since θ and ϕ are small, $l\phi = a\theta$ or $\phi = a\theta/l$ approximately. Neglecting vertical acceleration, the vertical component of the force F in each string is equal to W, half the weight of the plate, and the horizontal component is approximately $W\phi$, which produces a torque on the plate equal to $W\phi a$.

If T_q is the torque on the plate at time t, then

$$T_q = 2W\phi a = 2Wa^2\theta/l.$$

If $I = \frac{2W}{g}k^2$ is the moment of inertia of the plate about the axis of rotation, then since the torque is negative when θ is positive

$$T_q = -I\frac{d^2\theta}{dt^2} = -\frac{2W}{g}k^2\frac{d^2\theta}{dt^2}.$$

Therefore
$$\frac{2Wa^2\theta}{l} = -\frac{2W}{g}k^2\frac{d^2\theta}{dt^2}$$

or
$$\frac{d^2\theta}{dt^2} + \frac{ga^2}{lk^2}\theta = 0$$

which is the equation of simple harmonic motion.

By comparison with the standard form

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0,$$

it is seen that
$$\omega = \sqrt{\left(\frac{ga^2}{lk^2}\right)}$$

and periodic time

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\left(\frac{lk^2}{ga^2}\right)}=2\pi\frac{k}{a}\sqrt{\frac{l}{g}}.$$

63. Simple Pendulum—A Closer Approximation to the Value of the Periodic Time.—The equation of motion for a simple pendulum is, from Art. 61,

$$\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta=0,$$

which is true for all values of θ. The solution of this equation involves an elliptic integral which cannot be evaluated exactly and the answer is obtained in the form of a series, the degree of approximation depending on the number of terms used.

Only the main steps of the working are indicated, and those readers who are not interested in the mathematics may pass on to the final result.

Multiplying the equation by $2\frac{d\theta}{dt}$ and integrating, taking the amplitude as α, then

$$\frac{d\theta}{dt}=\sqrt{\frac{2g}{l}}(\cos\theta-\cos\alpha)^{\frac{1}{2}}.$$

Since $\cos\theta=1-2\sin^2\frac{\theta}{2}$ and $\cos\alpha=1-2\sin^2\frac{\alpha}{2}$,

$$\frac{d\theta}{dt}=\sqrt{\frac{2g}{l}}\left(2\sin^2\frac{\alpha}{2}-2\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}.$$

Integrating, $$\int_0^{\frac{1}{4}T}dt=\tfrac{1}{2}\sqrt{\frac{l}{g}}\int_0^{\alpha}\frac{d\theta}{\left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}}$$

where T is the periodic time.

Putting $\sin\frac{\theta}{2}=b\sin\phi$, where $b=\sin\frac{\alpha}{2}$, it can be shown that

$$T=4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}(1-b^2\sin^2\phi)^{-\frac{1}{2}}d\phi.$$

Expanding by the binomial theorem and integrating term by term gives

$$T=2\pi\sqrt{\frac{l}{g}}\left\{1+\frac{1}{2^2}b^2+\frac{3^2}{2^2.4^2}b^4+\frac{3^2.5^2}{2^2.4^2.6^2}b^6+\frac{3^2.5^2.7^2}{2^2.4^2.6^2.8^2}b^8+\ldots\right\}$$

where, as already stated, $b=\sin\frac{a}{2}$ and a is the amplitude. It can be seen now that the periodic time T is not independent of the amplitude a.

Denoting the expression in brackets by F, then

$$T=2\pi\sqrt{\frac{l}{g}}\times F,$$

and F, being a function of the amplitude a, may be called the *amplitude factor*.

When $a=10°$,

$$F=1+0{\cdot}001899+0{\cdot}000008+0{\cdot}000000=1{\cdot}001907,$$

therefore $$T=2\pi\sqrt{\frac{l}{g}}\times 1{\cdot}001907,$$

and the error is less than $\frac{1}{5}$ of 1 per cent. when the factor F is omitted. Evidently an amplitude of 10° might fairly be described as small when the periodic time is being calculated, and in this case the theory of Art. 61 is sufficiently accurate.

The graph of F plotted against a is shown in Fig. 94. When $F=1{\cdot}01$, the value of a is about 23°, so for this angle the error in the simple formula $T=2\pi\sqrt{(l/g)}$ is about 1 per cent. When $a=60°$, the error is about 7 per cent.

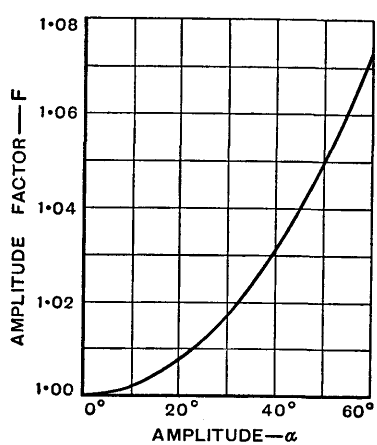


FIG. 94.

Exercises VII

Take $g = 32{\cdot}2$ ft./sec.²

1. In a certain problem the equation of motion is

$$1{\cdot}24\frac{d^2\theta}{dt^2} + 458\theta = 0,$$

the unit of time being the second. Show that the motion is simple harmonic, by comparing the given equation with the standard equation, then find the periodic time and the frequency.

2. A body moving with simple harmonic motion has a frequency of 141 cycles per minute. Using foot and second units, find expressions for the displacement x to satisfy the given conditions in the following cases:—

(i) When $t = 0$, $x = \frac{1}{8}$ ft., and $\frac{dx}{dt} = 0$.

(ii) When $t = 0$, $x = 0$; when $t = \frac{15}{141}$ sec., $x = \frac{1}{8}$ ft.

(iii) When $t = 0$, $x = \frac{1}{16}$ ft., and $\frac{dx}{dt} = -\frac{141\pi}{160\sqrt{3}}$ ft./sec.

3. If in simple harmonic motion the maximum velocity is V, the amplitude is a, and the velocity is v when the displacement from the mean position is x, show that $v = \pm V\sqrt{(1 - x^2/a^2)}$. If $v = 4$ ft./sec. when $x = 1$ ft., and $v = 2$ ft./sec. when $x = 1{\cdot}5$ ft., find the values of a, V, and the periodic time T.

4. A body moving with simple harmonic motion has a maximum velocity of 25 ft./sec. Find its velocity (i) when it is a distance equal to half the amplitude from its mean position, (ii) when it has travelled from its mean position for half the time it takes to reach an extreme position.

5. A body moving in a straight line with simple harmonic motion has a maximum acceleration of 10 ft./sec.² and performs 100 complete oscillations a minute. Find the amplitude and the maximum velocity.

6. The reciprocating parts of a single cylinder engine have a mass weighing 11·4 lb. and the length of the crank is 4·25 inches. Neglecting the obliquity of the connecting-rod and taking the engine speed as 2400 r.p.m., find the maximum velocity and the maximum acceleration of the piston; also find the maximum accelerating force.

7. A body weighing 10 pounds is suspended by a helical spring which is fixed at its upper end (Fig. 95). The spring is 6 inches long when unloaded and 8 inches long when carrying the load. The body is pulled down a distance of 0·75 inch and released. Find the periodic time of the vibrations and the maximum velocity of the body, assuming that there is no air resistance and neglecting the mass of the spring.

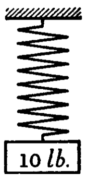


FIG. 95.

8. A body is moving harmonically and makes 20 oscillations a second. Find its greatest velocity if the distance between its extreme positions is 4 inches. Determine its velocity and acceleration when 1 inch from the mid-point of its motion.

A flat plate with bodies resting upon it begins to oscillate vertically through a distance 4 inches: determine within what limit the number of vibrations per minute must be kept if the bodies upon the plate are not to be thrown off by the vibration. [C.U.]

9. If a point moves with S.H.M., show that the graph connecting its velocity with distance along its path may be represented by a circle, if a particular scale is chosen for the velocity ordinate.

Show also that, during motion from one end of the travel to the other, the mean velocity with respect to the distance is $\frac{\pi}{4}$ times the maximum velocity, and with respect to the time is $\frac{2}{\pi}$ times the maximum velocity. [C.U.]

10. Two axes Ox and Oy are respectively horizontal and vertical. On Ox a point P moves with simple harmonic motion (S.H.M.), the mean position being O. A point Q similarly describes S.H.M. on the axis Oy. Each point makes a complete oscillation in one second, and the maximum displacement from O in each case is 1 foot. The motions are so timed that when Q is at O (moving downwards), P is at the maximum distance to the right from O, *i.e.* 1 foot. Show that, relative to Q, P describes a circle with uniform angular velocity. Show also that the acceleration of P with respect to Q is constant in magnitude (but not in direction), and give its numerical value. [C.U.]

11. For a simple pendulum having a small amplitude the periodic time is approximately $T=2\pi\sqrt{(l/g)}$. Taking logarithms and differentiating, show that

$$\frac{\delta T}{T}=\frac{\delta l}{2l}-\frac{\delta g}{2g} \text{ approximately,}$$

where δT, δl, and δg are small increments in T, l, and g respectively.

Find the periodic time of a simple pendulum whose length is

36 inches, and then find the increase in the periodic time due to an increase of $\frac{1}{2}$ inch in the length.

12. Calculate the length of a clock pendulum whose periodic time is 2 seconds. If the length is increased by 0·2 per cent., find the number of seconds a day the clock would lose.

13. A homogeneous disc of uniform thickness and 6 inches in diameter is suspended with its axis vertical by two vertical light threads 2 inches apart. The threads are 2 feet long and are symmetrically situated with regard to the centre of the disc so that each bears half the weight.

The disc is turned about its axis through a small angle and then released. Show that the resulting motion is very nearly simple harmonic, and find the time of a complete oscillation.

If the disc is replaced by one of twice its diameter, find the new time of oscillation. [C.U.]

14. Find the value of the amplitude factor F, correct to four places of decimals, in the formula $T = 2\pi(l/g)^{\frac{1}{2}}F$ when the amplitude is 30°.

CHAPTER VIII

ANALYSIS OF CAMS

64. Internal-Combustion Engine Cams.—The cams discussed in this chapter are of the types used for operating the valves in internal-combustion engines, where the cams rotate and give reciprocating motion to the tappets and valves. A small clearance between a valve and its tappet ensures that the former closes properly on to its seat when the engine is hot, but when the tappet has moved through a distance equal to the clearance and touches the valve stem, then the two pieces move together and continue to do so until the valve closes again. In general the tappet and valve will be briefly described as the *follower*.

Mathematical expressions are found for the displacement, velocity, and acceleration of the follower, the velocity and acceleration being obtained by differentiation. It is important to have a knowledge of the acceleration in order that a suitable valve spring may be designed to deal with the inertia forces.

Problems on cams which are used for various purposes are included in the exercises at the end of this chapter. A vector method is given in Ex. 8, p. 131, and this should be studied carefully.

65. Types of Cams.—The cams which will be analysed are shown in Figs. 96 to 99. Each cam rotates about a centre O with a constant angular velocity, and a follower F reciprocates along an axis passing through the centre O. In Fig. 96 the follower has a flat foot which bears on the cam, but in each of the other three types the contact is between the cam and a roller which is attached to the follower. The profiles of the cams are made up of circular arcs except in Fig. 97, where AB and A′B′ are straight lines. The cams will be called *straight, convex,* or *concave,* according as the lines AB and A′B′ are straight, convex, or concave.

66. The Cycle of Operations.—The motion of the follower F (Figs. 96 to 99) begins when the contact between it and the cam is at A. As the cam turns and the contact moves from A to B, the cam accelerates the follower, and the velocity of the latter reaches a maximum when the contact is at B. From B to D the follower is retarded by a spring,

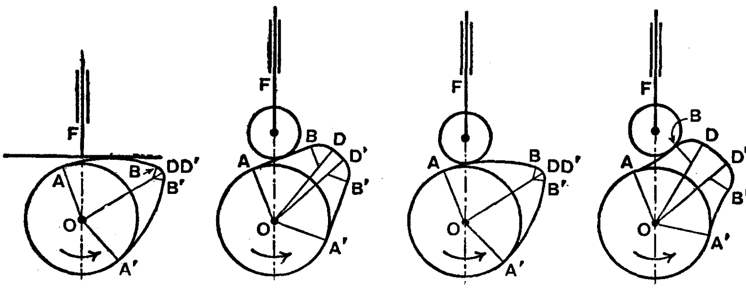


FIG. 96. FIG. 97. FIG. 98. FIG. 99.

not shown, so that the velocity is zero when the contact is at D and the follower has its full lift. The arc DD′ (Figs. 97 and 99) has its centre at O, the centre of rotation, therefore the follower is at rest whilst the contact is on this arc. The period of rest is sometimes called a *dwell.* In Figs. 96 and 98 the points D and D′ coincide and so there is no dwell at full lift.

From D′ to B′ the follower is accelerated downwards by the spring and has its maximum velocity at B′. Finally the cam retards the follower from B′ to A′, and at A′ the follower is again at rest. From A′ to A the follower remains at rest and then the cycle begins again at A.

The cams shown are symmetrical, and therefore in each case it will only be necessary to examine the motion of the follower whilst the valve is opening. Any modification of the dimensions on the closing side of the cam would, of course, require investigation in practice, but the same set of equations would be used with different constants.

In the Articles which follow, AB and BD will be described as the *first curve* and *second curve* respectively.

67. Convex Cam with Flat-Footed Follower.—The cam is shown in two positions in Figs. 100 and 101. The centres of the first curve AB and the second curve BD are at c_1 and c_2 respectively, and the various radii are as indicated. The length OD, although not a radius in this case, is marked r_5 because in two other examples the cams are drawn with a dwell at full lift. To ensure smooth

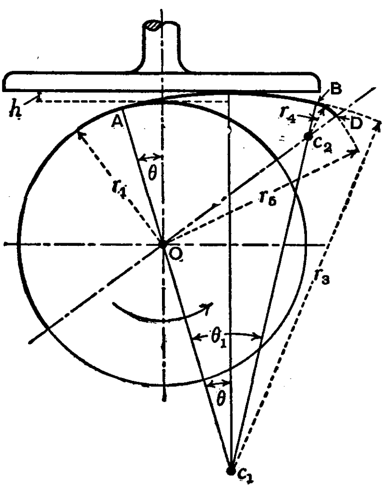


FIG. 100.

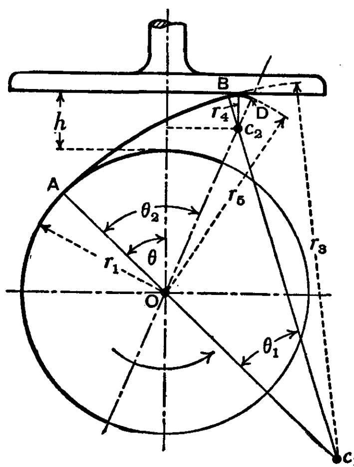


FIG. 101.

running, any two connected arcs should be drawn so as to have a common tangent at the junction.

First Curve AB (Fig. 100). Let h be the lift of the follower when the cam has turned through an angle θ, taking θ as zero when the contact is at A. On account of tappet clearance, the valve lift will be less than the tappet lift by an amount equal to the clearance. Let the angle Ac_1B be θ_1.

$$\text{Lift} \qquad h = r_3 - (r_3 - r_1)\cos\theta - r_1$$
$$= (r_3 - r_1)(1 - \cos\theta).$$

Velocity $$v=\frac{dh}{dt}=\frac{dh}{d\theta}\frac{d\theta}{dt}$$
$$=\frac{d\theta}{dt}(r_3-r_1)\sin\theta.$$

Acceleration $$\frac{dv}{dt}=\frac{dv}{d\theta}\frac{d\theta}{dt}$$
$$=\left(\frac{d\theta}{dt}\right)^2(r_3-r_1)\cos\theta.$$

These three equations hold from $\theta=0$ to $\theta=\theta_1$. The acceleration will have its greatest value when $\cos\theta$ is a maximum—that is, when $\theta=0$.

Second Curve BD (Fig. 101). Let the angle AOD be θ_2.

Lift $$h=(r_5-r_4)\cos(\theta_2-\theta)+r_4-r_1.$$

Velocity $$v=\frac{dh}{dt}=\frac{dh}{d\theta}\frac{d\theta}{dt}$$
$$=\frac{d\theta}{dt}(r_5-r_4)\sin(\theta_2-\theta).$$

Acceleration $$\frac{dv}{dt}=\frac{dv}{d\theta}\frac{d\theta}{dt}$$
$$=-\left(\frac{d\theta}{dt}\right)^2(r_5-r_4)\cos(\theta_2-\theta).$$

These three equations hold from $\theta=\theta_1$ to $\theta=\theta_2$. The acceleration will have its greatest value when $\cos(\theta_2-\theta)$ is a maximum—that is, when $\theta=\theta_2$.

To keep the follower in contact with the cam whilst the second curve is in operation, a spring is required which will give to the follower a downward acceleration which is not less than

$$\left(\frac{d\theta}{dt}\right)^2(r_5-r_4)\quad\text{when}\quad\theta=\theta_2,$$

and not less than

$$\left(\frac{d\theta}{dt}\right)^2(r_5-r_4)\cos(\theta_2-\theta_1)\quad\text{when}\quad\theta=\theta_1.$$

The total lift is $r_5 - r_1$, and the value of the lift when $\theta = \theta_1$ may be calculated from either of the lift equations.

Units.—If the uniform angular velocity of the cam, $\frac{d\theta}{dt}$, is in radians per second and the radii are in feet, then the lift h will be in feet, the velocity will be in feet per second, and the acceleration will be in feet per second per second.

68. Spring Force.—Given the acceleration and the mass, it is required to find the force. If the spring force is P when the acceleration is f, and if M is the mass of the follower, including the spring washer, etc., and one-third the mass of the spring, then the spring force may be calculated from the formula

$$P = Mf.$$

69. Straight Cam with Roller Follower.—In Figs. 102 and 103 the centre s of the roller moves along the straight line OF; also, relative to the cam, the point s moves along the straight line A_1B_1 and the circular arc B_1D_1. Taking the radius of the roller as r_2, then A_1B_1 is parallel to AB and at a perpendicular distance r_2 from AB, and the radius of the arc B_1D_1 is $r_2 + r_4$ where r_4 is the radius of the arc BD, both arcs being drawn with centre C_2. If the follower made point contact or knife-edge contact at s with the cam $A_1B_1D_1$, then $A_1B_1D_1$ is the equivalent cam which would give the same motion to the knife-edge follower as is given to the roller follower by the cam ABD.

First Curve AB or A_1B_1 (Fig. 102). Let h be the lift when the cam has turned through an angle θ and let the angle A_1OB_1 be θ_1.

Lift $$h = \frac{r_1 + r_2}{\cos\theta} - (r_1 + r_2).$$

Velocity $$v = \frac{dh}{dt} = \frac{dh}{d\theta}\frac{d\theta}{dt}$$

$$= \frac{d\theta}{dt}(r_1 + r_2)\frac{\tan\theta}{\cos\theta}.$$

$$\text{Acceleration } \frac{dv}{dt}=\frac{dv}{d\theta}\frac{d\theta}{dt}$$

$$=\left(\frac{d\theta}{dt}\right)^2(r_1+r_2)\frac{1+2\tan^2\theta}{\cos\theta}.$$

These equations hold from $\theta=0$ to $\theta=\theta_1$. As θ increases, $\tan\theta$ increases and $\cos\theta$ decreases, therefore the acceleration increases and its value will be greatest when $\theta=\theta_1$.

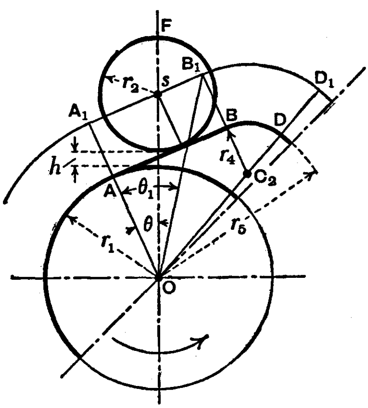


FIG. 102.

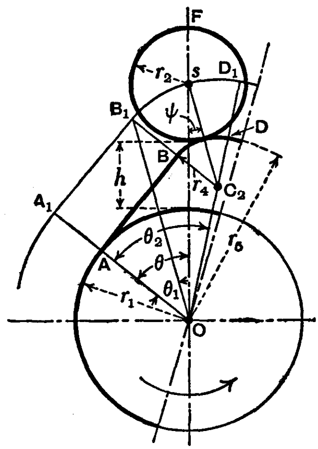


FIG. 103.

Second Curve BD or B_1D_1 (Fig. 103). The cam is shown with a dwell at full lift, but this does not affect the analysis. Let the angle A_1OD_1 be θ_2 and let the angle OsC_2 be ψ, a variable; also, to shorten the work which follows, let $r_5-r_4=d$ and $r_2+r_4=e$.

From the Figure, $(r_2+r_4)\sin\psi=(r_5-r_4)\sin(\theta_2-\theta)$

or $$e\sin\psi=d\sin(\theta_2-\theta),$$

therefore $$\sin\psi=\frac{d}{e}\sin(\theta_2-\theta)$$

and $$\cos\psi=(1-\sin^2\psi)^{\frac{1}{2}}$$

$$=\frac{1}{e}\{e^2-d^2\sin^2(\theta_2-\theta)\}^{\frac{1}{2}}.$$

Lift $\quad h = d\cos(\theta_2-\theta)+e\cos\psi-(r_1+r_2)$

$$= d\cos(\theta_2-\theta)+\{e^2-d^2\sin^2(\theta_2-\theta)\}^{\frac{1}{2}}-(r_1+r_2).$$

Velocity $v=\dfrac{dh}{dt}=\dfrac{dh}{d\theta}\dfrac{d\theta}{dt}$

$$=\frac{d\theta}{dt}\left[d\sin(\theta_2-\theta)+\frac{d^2\sin 2(\theta_2-\theta)}{2\{e^2-d^2\sin^2(\theta_2-\theta)\}^{\frac{1}{2}}}\right].$$

Acceleration $\dfrac{dv}{dt}=\dfrac{dv}{d\theta}\dfrac{d\theta}{dt}$

$$=\left(\frac{d\theta}{dt}\right)^2\left[-d\cos(\theta_2-\theta)-\frac{d^2\cos 2(\theta_2-\theta)}{\{e^2-d^2\sin^2(\theta_2-\theta)\}^{\frac{1}{2}}}\right.$$
$$\left.-\frac{d^4\sin^2 2(\theta_2-\theta)}{4\{e^2-d^2\sin^2(\theta_2-\theta)\}^{\frac{3}{2}}}\right]$$

which after simplification

$$=-\left(\frac{d\theta}{dt}\right)^2\left[d\cos(\theta_2-\theta)+\frac{d^2\{e^2\cos 2(\theta_2-\theta)+d^2\sin^4(\theta_2-\theta)\}}{\{e^2-d^2\sin^2(\theta_2-\theta)\}^{\frac{3}{2}}}\right].$$

These equations hold from $\theta=\theta_1$ to $\theta=\theta_2$. Whether the acceleration has its greatest value when $\theta=\theta_1$ or when $\theta=\theta_2$ depends on the values of d and e. The graphs in Fig. 108, Art. 72, illustrate this fact.

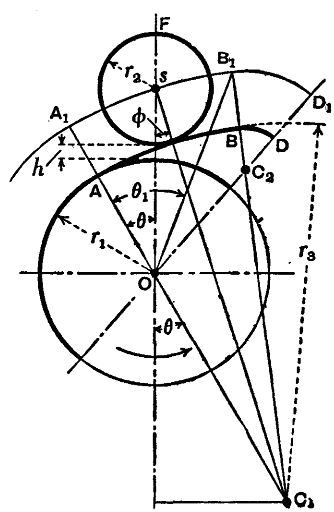


FIG. 104.

70. Convex Cam with Roller Follower.—In this case the first curve AB is a circular arc with centre C_1 and of radius r_3 (Fig. 104). The equivalent cam is $A_1B_1D_1$ as described in the preceding Art., except that here the part A_1B_1 is a circular arc with centre C_1 and of radius r_3+r_2.

First Curve AB or A_1B_1. Let h be the lift when the cam has turned through an angle θ, let the angle A_1OB_1 be θ_1, and let the angle OsC_1 be ϕ, a variable; also let $r_3 - r_1 = a$ and $r_3 + r_2 = b$.

From the Figure, $(r_3 + r_2) \sin \phi = (r_3 - r_1) \sin \theta$,

or $$b \sin \phi = a \sin \theta,$$

therefore $$\sin \phi = \frac{a}{b} \sin \theta$$

and $$\cos \phi = (1 - \sin^2 \phi)^{\frac{1}{2}} = \frac{1}{b}(b^2 - a^2 \sin^2 \theta)^{\frac{1}{2}}.$$

Lift $$h = b \cos \phi - a \cos \theta - (r_1 + r_2)$$
$$= -a \cos \theta + \{b^2 - a^2 \sin^2 \theta\}^{\frac{1}{2}} - (r_1 + r_2).$$

Velocity $$v = \frac{dh}{dt} = \frac{dh}{d\theta}\frac{d\theta}{dt}$$
$$= \frac{d\theta}{dt}\left[a \sin \theta - \frac{a^2 \sin 2\theta}{2(b^2 - a^2 \sin^2 \theta)^{\frac{1}{2}}}\right].$$

Acceleration $$\frac{dv}{dt} = \frac{dv}{d\theta}\frac{d\theta}{dt}$$
$$= \left(\frac{d\theta}{dt}\right)^2\left[a \cos \theta - \frac{a^2 \cos 2\theta}{(b^2 - a^2 \sin^2 \theta)^{\frac{1}{2}}} - \frac{a^4 \sin^2 2\theta}{4(b^2 - a^2 \sin^2 \theta)^{\frac{3}{2}}}\right]$$

which after simplification
$$= \left(\frac{d\theta}{dt}\right)^2\left[a \cos \theta - \frac{a^2(b^2 \cos 2\theta + a^2 \sin^4 \theta)}{(b^2 - a^2 \sin^2 \theta)^{\frac{3}{2}}}\right].$$

These equations hold from $\theta = 0$ to $\theta = \theta_1$.

Second Curve BD or B_1D_1. The equations are the same as those obtained for the second curve of the straight cam with the roller follower in the preceding Art.

71. Concave Cam with Roller Follower.—The concave cam (Fig. 105) is the same as the convex cam (Fig. 104) except for the change of position of C_1, the centre of the first curve AB. Therefore the formulæ derived for the first curve of the convex cam with roller follower (Art. 70) may be applied to the concave cam by changing the sign of the radius r_3. However, for the sake of clearness, the analysis will be outlined for the concave cam.

First Curve AB or A_1B_1. Let the angle A_1OB_1 be θ_1 and let the angle FsC_1 be ϕ, a variable; also let $r_3+r_1=a$ and $r_3-r_2=b$.

Now $$(r_3-r_2)\sin\phi=(r_3+r_1)\sin\theta$$

or $$b\sin\phi=a\sin\theta,$$

therefore $$\cos\phi=\frac{1}{b}\{b^2-a^2\sin^2\theta\}^{\frac{1}{2}}.$$

Lift $$\begin{aligned}h&=a\cos\theta-b\cos\phi-(r_1+r_2)\\&=a\cos\theta-\{b^2-a^2\sin^2\theta\}^{\frac{1}{2}}-(r_1+r_2).\end{aligned}$$

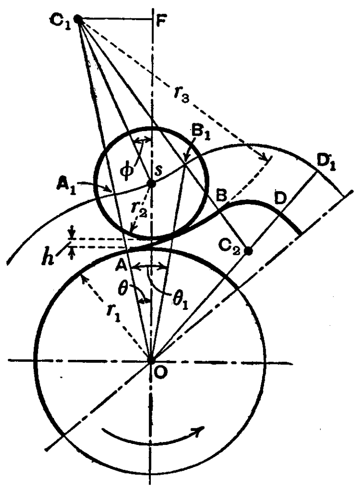


FIG. 105.

These equations should be compared with the corresponding equations for the convex cam with roller follower. (If r_3 had been made negative in the convex cam equations, then $-r_3-r_1=a$ or $r_3+r_1=-a$, and $-r_3+r_2=b$ or $r_3-r_2=-b$. With these changes, the convex cam lift equations give the concave cam lift.)

Since there are only two alterations in sign, the velocity and acceleration equations will be as follows:—

Velocity $$v=\frac{d\theta}{dt}\left[-a\sin\theta+\frac{a^2\sin 2\theta}{2(b^2-a^2\sin^2\theta)^{\frac{1}{2}}}\right].$$

$$\text{Acceleration } \frac{dv}{dt} = \left(\frac{d\theta}{dt}\right)^2 \left[-a \cos \theta + \frac{a^2(b^2 \cos 2\theta + a^2 \sin^4 \theta)}{(b^2 - a^2 \sin^2 \theta)^{\frac{3}{2}}} \right].$$

As before, the three equations hold from $\theta = 0$ to $\theta = \theta_1$.

Second Curve BD or B_1D_1. The equations are the same as those obtained for the second curve of the straight cam with roller follower (Art. 69).

72. Graphs of Lift, Velocity, and Acceleration.—The graphs * shown in Figs. 106 to 108 refer to two concave

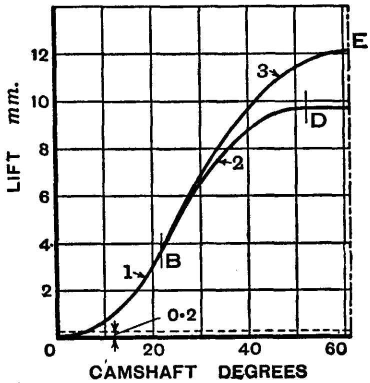


FIG. 106.

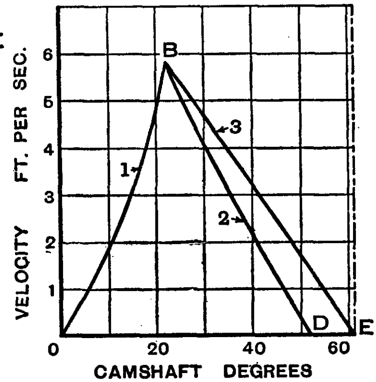


FIG. 107.

cams having roller followers. The curves labelled 1 and 2 in each of the three Figures concern a concave cam and roller having the following dimensions: $r_1 = 14{\cdot}75$ mm., $r_2 = 7{\cdot}5$ mm., $r_3 = 35{\cdot}0$ mm., $r_4 = 6{\cdot}0$ mm., and $r_5 = 24{\cdot}5$ mm. The crankshaft speed is 1400 r.p.m., and so the camshaft speed is 700 r.p.m. The maximum tappet lift is $r_5 - r_1 = 24{\cdot}5 - 14{\cdot}75 = 9{\cdot}75$ mm. The lift, velocity,

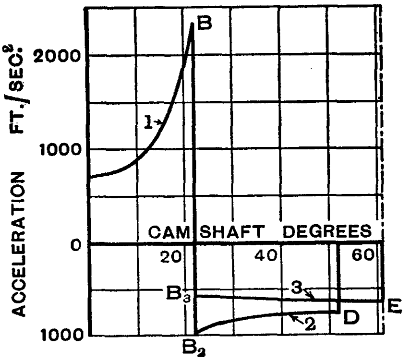


FIG. 108.

* From an article, "Internal-Combustion Engine Cams," by B. B. Low, *Engineering*, May 25, 1923.

and acceleration are given by the curves labelled 1 when the first curve of the cam is in operation, and by the curves labelled 2 when the second curve of the cam is in operation. The other cam and the curves 3 will be mentioned in the next Art.

The clearance between the tappet and the valve stem is 0·2 mm., or 0·0079 inch approximately. The clearance angle, or the angle through which the cam turns whilst the clearance is being taken up, is found by calculation from the lift equation to be 5° 41′. The question of clearance is important because the cam should be designed so that it begins to open the valve at the right moment.

At B the valve has its maximum velocity and there is a sudden change from positive to negative acceleration. At D the valve is fully open, its velocity is zero, and its acceleration is as shown. This particular cam was designed to turn through an angle of approximately 111° whilst the valve is in operation. Therefore half the cam angle is $5° 41' + 55° 30' = 61° 11'$. By calculation, using the given dimensions of the cam, $\theta_1 = 22° 6'$ (the angle at B on the curves) and $\theta_2 = 51° 53'$ (the angle at D on the curves).

Since θ_2 is less than half the cam angle, there is a dwell at full lift. The angle turned through by the cam during the dwell is $2(61° 11' - 51° 53') = 18° 36'$.

A valve spring is needed to provide the moving mass with the negative acceleration shown at the point B_2. The spring will provide more than enough acceleration at the point D, but this cannot be avoided. The spring force is referred to in Ex. 12, p. 131.

73. Elimination of Dwell by Increasing Lift.—It is shown in Fig. 109 how a dwell at full lift may be eliminated by increasing the lift. Before the design is altered, the second curve is BD with its centre at C_2 and its radius is C_2B. The lift is complete at D and the dwell begins. To make the alteration, let BC_2 be produced to intersect the centre line OE at C_3, then with centre C_3 and radius C_3B draw the arc BE, as shown dotted in the figure. With BE as the second curve of the cam, the lift is complete at E, on the centre line, and there is no dwell.

Before examining the effects of the alteration, the length C_3B must be known. For the cam having the dimensions given in the preceding Art., $C_2B = r_4 = 6$ mm. Half the cam angle is the angle AOE, which is 61° 11′. The length C_2C_3 is approximately 3 mm., which will be taken as the exact value, then $C_3B = 9$ mm., and by calculation the angle AOE is found to be 61° 5′, which is near enough to its given value.

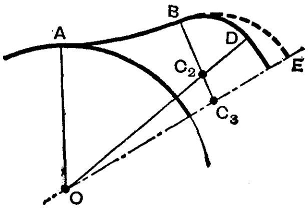


FIG. 109.

By calculation the maximum lift, OE − OA, is equal to 12·1 mm. With the original cam the maximum lift was 9·75 mm., therefore the increase in lift is 12·1 − 9·75 = 2·35 mm. Allowing for clearance, the increase in lift is

$$\frac{2{\cdot}35}{9{\cdot}75 - 0{\cdot}2} \times 100 = 24{\cdot}6 \text{ per cent.}$$

Now examine Figs. 106 to 108 and compare the curves labelled 2 and 3. The curves 3 refer to the cam with the increased lift. The point which the author wishes to emphasise is that the increased lift actually gives a reduction in the retardation. In foot and second units the value at B_2 is 990, but at B_3 it is 584 and at E it is 655. Therefore the spring force required at the beginning of the second curve (that is, at the point B on the actual cam in Fig. 109) is reduced to $\frac{584}{990}$, or 59 per cent., of its original value and this is certainly an advantage.

Increasing the lift of a valve might cause it to foul the cylinder-head or the piston; also the bearings of the camshaft would have to be slightly larger in diameter to clear the cams if the camshaft is assembled by being pushed through the bearings from one end. Whenever it is possible, however, it is evident that dwell should be eliminated by increasing the radius of the second curve of the cam, because the increased radius reduces the retardation to be provided by the spring and enables a weaker spring to be used. Another point is that an increase in lift should make it possible to get more mixture (burnt or unburnt) through the valve.

In conclusion it may be remarked that the reader who wishes to find further information concerning the relative advantages of the various cams may refer to the article already quoted in the footnote on p. 126.

Exercises VIII

1. A cam with a flat-footed follower has the dimensions shown in Fig. 110 which is not drawn to scale. The curves AB and BD are circular arcs. Find the radius r of the curve AB and also the angle θ_1 through which the cam turns whilst this curve is in operation. Assuming that the cam rotates at 1000 r.p.m., find the accelerations of the follower when contact is at the points A, B, and D.

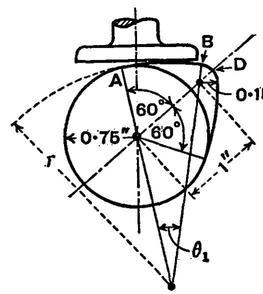


FIG. 110.

2. In Fig. 110 the contact between the cam and the follower begins at the point A and moves to the right, assuming that the cam rotates in an anticlockwise direction. Prove that the contact reaches its extreme right-hand position at the point B.

To reduce uneven wear on the under surface of the flat foot of the follower, this foot is often circular, and the follower is made to rotate about its vertical axis by being offset from the central plane of the cam.

If the vertical axis of the follower passes through the axis of the camshaft and is $\frac{3}{16}$ inch from the central plane of the cam, calculate the greatest distance from the vertical axis of the follower at which wear can occur on the foot, taking the thickness of the cam as $\frac{3}{8}$ inch, the radius $r=4$ inches, and the angle $\theta_1=13°$.

3. A valve of a four-stroke engine is open whilst the crankshaft turns through 220° and is fully open during 28°. The times of opening and closing are equal, and the crankshaft speed is 1800 r.p.m. The lift of the valve (neglecting clearance) is $\frac{3}{8}$ inch, and the acceleration and deceleration are uniform.

(*a*) Find the value of the acceleration if the valve has its maximum velocity at half lift.

(*b*) Calculate the lifts corresponding to the camshaft angles 6°, 12°, 18°, 24°, 30°, 36°, and 42°.

(*c*) Plot curves of acceleration and lift on camshaft angle bases, from 0° to 110°.

4. The profile of a cam is a circle of 3 inches diameter, and the centre about which it rotates is $\frac{1}{2}$ inch from the centre of the circular profile. The follower is provided with a flat palm, at right angles to the line of stroke, which bears directly on the cam surface. The line of stroke is horizontal and passes through the centre of rotation. The weight of the parts actuated by the cam is 5 lb., and a spring is fitted to maintain contact between the cam and the follower. The spring exerts a force of 8 lb. at the beginning and 20 lb. at the end of the out-stroke.

(*a*) Obtain an expression for the acceleration of the follower in terms of the cam angle.

(*b*) Find the greatest speed at which the arrangement will run satisfactorily. [U.L.]

5. Find the greatest speed at which the mechanism in the preceding exercise will run satisfactorily if the line of stroke is vertical: (*a*) when the follower is above the cam; (*b*) when the follower is below the cam.

6. A fuel injection pump of variable stroke is operated by a compound cam controlled by a governor. Reduced to its essentials, the mechanism is as shown in Fig. 111.

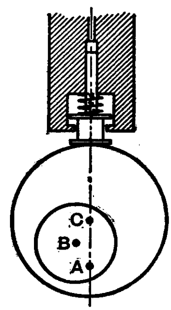


Fig. 111.

A, B, and C are the respective centres of the shaft which operates the pump, a disc fixed to the shaft, and an eccentric sheave mounted on the disc. The sheave can be turned round the disc by the governor mechanism and in this way the throw of the cam can be changed. The stroke of the pump is small compared with the total throw of the cam, and the return stroke, which is effected by means of a spring, is limited by the stops shown in the figure. AB and BC are each equal to 1 inch, and the stroke of the pump is $\frac{1}{4}$ inch when the angle ABC is 120°.

Through what angle must the sheave be turned in order to reduce the stroke to $\frac{1}{8}$ inch? By what fraction of a revolution of the shaft will the period of injection be retarded? [B.E.]

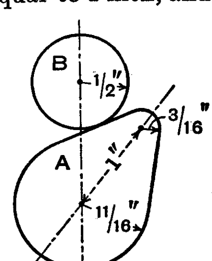


Fig. 112.

7. A petrol engine, working on the usual Otto cycle, has a straight-sided valve cam A of the form shown in Fig. 112: the roller B is constrained to move in a vertical path. The mass of the roller together with all the parts which move with it is 8 oz. Show that when the engine is running at 2400 revolutions per minute there is a sudden

change of acceleration equal to about 8000 feet per second per second, at the instant when the roller passes from the straight flank on to the curve of smaller radius, and that a valve spring giving a force of over 61 lb. weight is necessary if the roller is to remain in contact with the cam throughout the lift. [C.U.]

8. A cam rotates with uniform angular velocity ω about a fixed centre A and operates a roller whose centre C moves on a straight line through A, the point of contact being P. If B is the centre of curvature of the cam at P, show that the acceleration of C is given by ω^2AZ where Z is obtained as follows: CB meets the line through A perpendicular to CA in N. K is a point on CB such that $BN^2 = BC.BK$ and KZ is drawn perpendicular to CB to meet CA in Z. Show also that in the particular case where the cam surface at the point of contact is flat, KN = NC. [C.U.]

9. A cam rotating uniformly about A (Fig. 113), with an angular velocity ω, gives a reciprocating motion in a vertical line through A to a rod PF, the rod being kept in contact with the cam by a spring. The part PD of the profile of the cam is an involute of the circle with radius AC, the part EP is straight, and the angles EPC and PCA are right angles. Prove that in the position shown the change of acceleration of P due to the change of curvature of the cam is given by ω^2AP $\sec^4 \theta$. [C.U.]

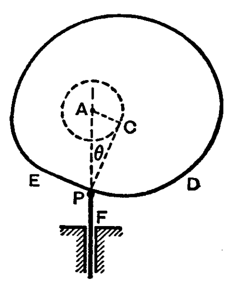


FIG. 113.

10. The fuel pump of a two-stroke oil engine is operated by a cam on the engine crankshaft. The tappet clearance is such that the cam makes contact during 60° of the revolution of the engine crank. The motion of the pump plunger during this time is simple harmonic. The plunger and its attached parts weigh $\frac{1}{2}$ lb., the stroke is $\frac{1}{2}$ inch, and the oil pressure produces a load of 80 lb. on the plunger at the end of its stroke. What is the greatest force that must be exerted by the spring in order to maintain contact between cam and roller when the engine runs at 900 r.p.m.? [U.L.]

11. As explained in Art. 72, the curves 1 and 2 in Figs. 106 to 108 concern a concave cam with roller follower, and the dimensions of the mechanism, in millimetres, are as follows: $r_1 = 14{\cdot}75$, $r_2 = 7{\cdot}5$, $r_3 = 35{\cdot}0$, $r_4 = 6{\cdot}0$, and $r_5 = 24{\cdot}5$. Show that $\theta = 22° \, 6'$ at the points B on the curves, and calculate the corresponding lift of the tappet.

12. Referring to the preceding exercise and Fig. 108, calculate the acceleration at the point B_2, taking the crankshaft speed to be 1400 r.p.m. and $\theta_2 = 51° \, 53'$; then assuming that the weight of the valve and tappet, etc., is $\frac{3}{4}$ lb., find the least spring force which is required at this point to prevent the tappet leaving the cam.

CHAPTER IX

MOTION OF RIGID BODIES IN TWO DIMENSIONS

74. Definitions—d'Alembert's Principle.—If a force P is applied to a particle of mass m and gives it an acceleration f, the direction of the force and the acceleration being the same, then $P = mf$ (Fig. 114). The force P is called the *applied force* and the product mf is called the *effective force*. If the effective force were reversed, then it would be in equilibrium with the applied force.

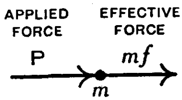


FIG. 114.

A *rigid body* may be defined as one in which the distances between the particles of which it is composed remain unchanged by the action of applied forces. Actually there is no such thing as a perfectly rigid body, but in many practical problems it is sufficiently accurate to regard a body as being rigid.

When applied forces accelerate a rigid body there are also internal forces and effective forces acting on every particle of the body. By what is known as *d'Alembert's principle* the internal forces are in equilibrium amongst themselves, and the applied forces are in equilibrium with the *reversed* effective forces. Alternatively, it may be said that the effective forces form a system which is equivalent to the system of applied forces.

In the figures in this chapter the effective forces are not shown reversed, but there will be no difficulty in distinguishing between an applied force and an effective force because the latter is the product of a mass and an acceleration.

To save space the fluxional notation is used for denoting velocity and acceleration (see Art. 6, p. 6).

75. Equations of Motion—Considering Applied and Effective Forces.—Suppose a body BC (Fig. 115), of mass M, moves due to the action of applied forces P_1, P_2, P_3, etc., then it will be shown that—

(1) The centre of gravity G moves as though the whole mass M were collected at G and all the applied forces were transferred to G with their lines of action parallel to their given lines.

(2) The body turns about its centre of gravity G, under the action of the given forces, as though G were fixed.

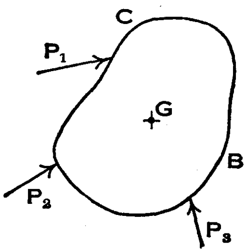


FIG. 115.

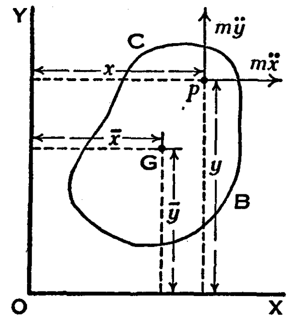


FIG. 116.

Let the co-ordinates of the centre of gravity G of the body BC (Fig. 116) be $\bar{x}$ and $\bar{y}$, at time t, with respect to fixed axes OX and OY, and let the co-ordinates of a particle p of mass m be x and y with respect to the same axes.

The components of the effective forces acting on the particle p, parallel to the axes OX and OY, are respectively

$$m\ddot{x} \quad \text{and} \quad m\ddot{y}.$$

Therefore, if the sums of the components of all the applied forces are P and Q, acting in directions parallel to OX and OY respectively,

$$P = \Sigma m\ddot{x} \quad \text{and} \quad Q = \Sigma m\ddot{y},$$

where $\Sigma m\ddot{x}$ denotes the sum of all the effective forces acting parallel to OX, and $\Sigma m\ddot{y}$ denotes the sum of all the effective forces acting parallel to OY.

Since G is the centre of gravity of the body,

$$\Sigma mx = \mathrm{M}\bar{x} \quad \text{and} \quad \Sigma my = \mathrm{M}\bar{y}.$$

Differentiating twice with respect to time,

$$\Sigma m\ddot{x} = \mathrm{M}\ddot{\bar{x}} \quad \text{and} \quad \Sigma m\ddot{y} = \mathrm{M}\ddot{\bar{y}}.$$

Therefore $$\mathrm{P} = \mathrm{M}\ddot{\bar{x}} \quad \text{and} \quad \mathrm{Q} = \mathrm{M}\ddot{\bar{y}} \quad . \quad . \quad (1),$$

and the centre of gravity G moves as though the whole mass were collected at G and the components P and Q of the applied forces were transferred to G.

Now let axes GX′ and GY′ be drawn parallel to OX and OY, respectively (Fig. 117), and let GX′ and GY′ move with the body but always remain parallel to the fixed axes.

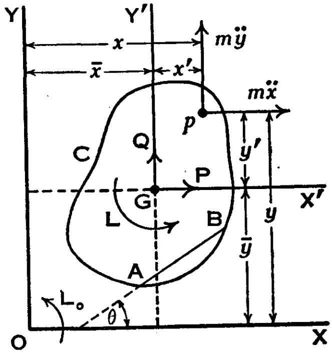


FIG. 117.

Let the co-ordinates of the particle p be x' and y' with respect to the axes GX′ and GY′, then

$$x = \bar{x} + x' \quad \text{and} \quad y = \bar{y} + y',$$

and differentiating twice with respect to time,

$$\ddot{x} = \ddot{\bar{x}} + \ddot{x}' \quad \text{and} \quad \ddot{y} = \ddot{\bar{y}} + \ddot{y}'.$$

Let the sum of the moments about G of the applied forces be L, then these applied forces may be replaced by the forces P and Q acting at G together with a couple L; also let the sum of the moments about O of the applied forces be $\mathrm{L_O}$, then, taking moments about O,

$$\mathrm{L_O} = \mathrm{L} + \mathrm{Q}\bar{x} - \mathrm{P}\bar{y} \quad . \quad . \quad . \quad (2),$$

assuming the directions of the couples L and $\mathrm{L_O}$ to be anticlockwise.

Equating the moments about O of the applied and effective forces,

$$\begin{aligned}\mathrm{L_O} &= \Sigma m\ddot{y}x - \Sigma m\ddot{x}y \\ &= \Sigma m\{(\ddot{\bar{y}} + \ddot{y}')(\bar{x} + x') - (\ddot{\bar{x}} + \ddot{x}')(\bar{y} + y')\} \\ &= \Sigma m\{(\ddot{\bar{y}}\bar{x} + \ddot{y}'\bar{x} + \ddot{\bar{y}}x' + \ddot{y}'x') - (\ddot{\bar{x}}\bar{y} + \ddot{x}'\bar{y} + \ddot{\bar{x}}y' + \ddot{x}'y')\} \qquad (3).\end{aligned}$$

Now $\Sigma m = M$ and since G is the centre of gravity, $\Sigma m x' = 0$ and $\Sigma m y' = 0$; also, differentiating twice, $\Sigma m \ddot{x}' = 0$ and $\Sigma m \ddot{y}' = 0$. Therefore equation (3) becomes

$$L_O = M(\ddot{\bar{y}}\bar{x} - \ddot{\bar{x}}\bar{y}) + \Sigma m(\ddot{y}'x' - \ddot{x}'y') \quad . \quad . \quad (4).$$

But from (2) and (1)

$$L_O = L + Q\bar{x} - P\bar{y} = L + M(\ddot{\bar{y}}\bar{x} - \ddot{\bar{x}}\bar{y}) \quad . \quad . \quad (5).$$

Therefore from (4) and (5), eliminating L_O,

$$L = \Sigma m(\ddot{y}'x' - \ddot{x}'y') \quad . \quad . \quad . \quad (6).$$

Therefore the sum of the moments about G of the applied forces is equal to the sum of the moments about G of the effective forces, and the body turns as though G were a fixed point, for the variables in (6) are measured with respect to the moving axes GX′ and GY′.

Equation (6) may now be put into a more convenient form by employing polar co-ordinates.

In rectangular co-ordinates the accelerations of the particle p, relative to G, are $\ddot{x}'$ and $\ddot{y}'$ (Fig. 118), and in polar co-ordinates the accelerations are $r\ddot{\phi}$ perpendicular to Gp and $r\dot{\phi}^2$ along pG, where $r = \mathrm{G}p$ and ϕ is the angle pGX′.

Therefore, taking moments about G of the effective forces,

$$m(\ddot{y}'x' - \ddot{x}'y') = mr^2\ddot{\phi} \quad . \quad (7).$$

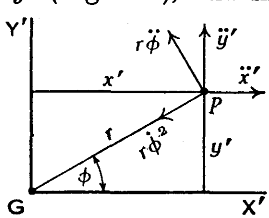


FIG. 118.

This relation could also be obtained by writing $x' = r \cos \phi$ and $y' = r \sin \phi$, differentiating twice with respect to time, and substituting in the left-hand side of (7).

At any instant the angle ϕ will not be the same for all particles in the plane X′GY′, but they will all have the same angular acceleration $\ddot{\phi}$, for if

$$\phi = \theta + \text{a constant},$$

where θ is the angle which any line AB (Fig. 117), fixed on the body, makes with the fixed axis OX at time t, then differentiating twice with respect to t,

$$\ddot{\phi}=\ddot{\theta} \quad . \quad . \quad . \quad . \quad (8),$$

and this is the angular acceleration of the body.

Therefore, from (6), (7), and (8),

$$L=\Sigma mr^2\ddot{\theta},$$

but $\Sigma mr^2=Mk^2=I$, where k is the radius of gyration and I is the moment of inertia of the body about an axis through G perpendicular to the plane X′GY′, therefore

$$L=Mk^2\ddot{\theta}=I\ddot{\theta} \quad . \quad (9).$$

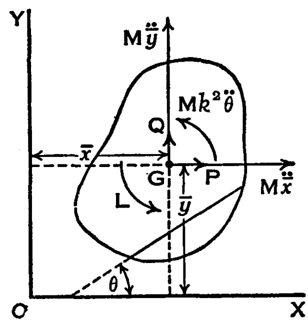


FIG. 119.

The results are summarized in Fig. 119, where P and Q are respectively the sums of the x and y components of the applied forces, L is the sum of the moments about G of the applied forces, $M\ddot{x}$ and $M\ddot{y}$ are the effective forces, and $Mk^2\ddot{\theta}$ is the effective couple.

The equations are

$$P=M\ddot{\bar{x}}, \quad Q=M\ddot{\bar{y}}, \quad \text{and} \quad L=Mk^2\ddot{\theta} \quad . \quad (10).$$

If it is required to take moments about any point other than G, say O, then if L_O is the resultant anticlockwise moment about O of the applied forces,

$$L_O=M\ddot{\bar{y}}\bar{x}-M\ddot{\bar{x}}\bar{y}+Mk^2\ddot{\theta} \quad . \quad . \quad (11).$$

In the Figure the arrow-heads on the effective forces, and on the effective couple, point in the directions in which the variables increase.

Since one of the two fixed axes OX and OY may be drawn in any direction, it follows that the applied forces may be resolved parallel to any direction and the sum of their components may be equated to the sum of the components of the effective forces in the same direction. The examples which follow will make the method clear.

Example 1.—A lorry weighing 2·5 tons when loaded has its centre of gravity 36 inches in front of the back axle centre line, 78 inches behind the front axle centre line, and 40 inches above the road. Given that its speed is 25 miles per hour on a horizontal road and that the coefficient of adhesion is 0·5, it is required to find the minimum distance in which the lorry can be stopped: (*a*) using front brakes; (*b*) using rear brakes; (*c*) using front and rear brakes, assuming that both sets of brakes are applied simultaneously.

(*a*) *Front Brakes.*—Denoting the retardation by f, the effective force is $Mf=\frac{W}{g}f$ acting through G, the centre of gravity, and parallel to the road. The applied forces are the weight W, the reactions R_1 and R_2 at the wheels (Fig. 120), and the horizontal force $0·5R_1$ on the front wheels at the road level. (Of course R_1, R_2, and $0·5R_1$ are each shared by two wheels.) The wheelbase $AB=36+78=114$ inches.

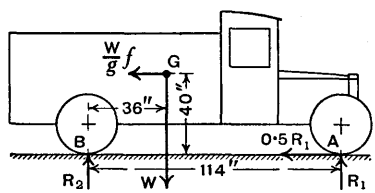


FIG. 120.

Equating applied and effective forces acting horizontally,

$$0·5R_1=\frac{W}{g}f \quad . \quad . \quad . \quad (1).$$

Equating the moments about B of the applied and effective forces,

$$114R_1-36W=40\frac{W}{g}f \quad . \quad . \quad (2).$$

From (1) and (2), eliminating R_1,

$$114\times\frac{2W}{g}f-36W=40\frac{W}{g}f,$$

then $$f=\frac{36\times 32·2}{188}=6·17 \text{ ft./sec.}^2.$$

Care should be taken with the units. In (2) the moments are inches times force, but the result would be the same if each side of the equation were divided by 12. Since g is taken as 32·2, the units of f must be feet and seconds. Note that the value of W does not affect the value of f, but it would be required if R_1 and R_2 had to be determined

For uniform retardation, $v^2=2fs$ or $s=v^2/2f$, where v is the initial velocity and s is the distance travelled.

Now $v=25$ miles per hour $=25\times\frac{88}{60}=\frac{110}{3}$ ft./sec.,

therefore $$s=\left(\frac{110}{3}\right)^2\times\frac{1}{2\times 6\cdot 17}=109 \text{ feet.}$$

(*b*) *Rear Brakes.*—The horizontal applied force is now $0{\cdot}5R_2$ *acting at* B (Fig. 121).

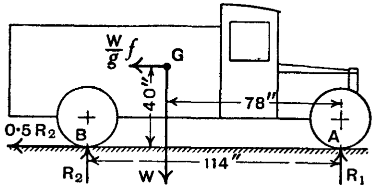


FIG. 121.

Equating applied and effective forces acting horizontally,

$$0{\cdot}5R_2=\frac{W}{g}f \quad . \quad . \quad . \quad (1).$$

Equating moments about A of the applied and effective forces,

$$78W-114R_2=40\frac{W}{g}f \quad . \quad . \quad (2).$$

From (1) and (2)

$$f=\frac{78\times 32\cdot 2}{268}=9\cdot 37 \text{ ft./sec.}^2.$$

Then $$s=\frac{v^2}{2f}=\left(\frac{110}{3}\right)^2\times\frac{1}{2\times 9\cdot 37}=71\cdot 7 \text{ feet.}$$

(c) *Front and Rear Brakes.*—The forces are as shown in Fig. 122.

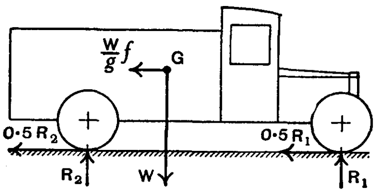


FIG. 122.

Equating applied and effective forces acting horizontally,

$$0{\cdot}5R_1 + 0{\cdot}5R_2 = \frac{W}{g}f \quad . \quad . \quad . \quad (1).$$

Equating applied and effective forces acting vertically, the latter forces being zero,

$$R_1 + R_2 - W = 0 \quad . \quad . \quad . \quad (2).$$

From (1) and (2) $\qquad 0{\cdot}5W = \frac{W}{g}f,$

therefore $\qquad f = 0{\cdot}5 \times 32{\cdot}2 = 16{\cdot}1$ ft./sec.^{2}.

Then $\qquad s = \frac{v^2}{2f} = \left(\frac{110}{3}\right)^2 \times \frac{1}{2 \times 16{\cdot}1} = 41{\cdot}8$ feet.

Example 2.—A pair of wheels and an axle, weighing 1500 lb. and having a radius of gyration $k = 1{\cdot}3$ feet, start from rest and travel on greasy rails down a uniform incline which makes an angle $\beta = 10°$ with the horizontal (Fig. 123). Each wheel has a radius $r = 1{\cdot}5$ feet, and the coefficient of friction between the wheels and the rails is $\mu = 0{\cdot}07$.

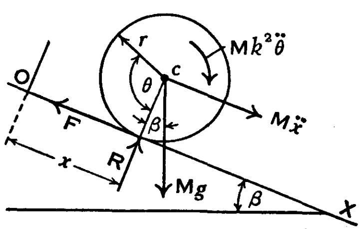


FIG. 123.

It is required to find if slip will occur and to calculate the time taken to travel 300 feet down the incline.

Let the contact be at O when the motion begins, and assume that at time t the centre c has travelled a distance x and the wheels have turned through an angle θ.

Let M be the total mass and let R and F be the normal reaction and the frictional force, respectively, between the wheels and the rails. (Actually $\frac{1}{2}$R and $\frac{1}{2}$F will act on each wheel.) At time t the effective force is $M\ddot{x}$, the effective couple is $Mk^2\ddot{\theta}$, and the applied forces are R, F, and Mg, all acting as shown. Taking $M=W/g$, the value of Mg is $W=1500$ lb.

If there is no slip,

$$x=r\theta \quad \text{and therefore} \quad \ddot{x}=r\ddot{\theta} \qquad . \quad (1).$$

If slip occurs, or is about to occur,

$$F=\mu R \quad . \quad . \quad . \quad . \quad (2).$$

Equating applied and effective forces by resolving perpendicular to the track,

$$R-Mg\cos\beta=0 \quad . \quad . \quad . \quad (3),$$

since there is no effective force in this direction.

Equating applied and effective forces by resolving parallel to the track,

$$Mg\sin\beta-F=M\ddot{x} \quad . \quad . \quad . \quad (4).$$

Equating moments about c of applied and effective forces,

$$Fr=Mk^2\ddot{\theta} \quad . \quad . \quad . \quad (5).$$

It should be noted that equations (3) to (5) are true whether there is slip or pure rolling.

From (4) and (5),

$$\ddot{x}=g\sin\beta-\frac{k^2}{r}\ddot{\theta} \quad . \quad . \quad . \quad (6).$$

Assuming no slip.

Since $\ddot{x}=r\ddot{\theta}$, substituting in (6) and solving for $\ddot{\theta}$,

$$\ddot{\theta}=\frac{gr\sin\beta}{k^2+r^2}.$$

Then from (5), $F = \frac{Mk^2}{r}\ddot{\theta} = \frac{Mgk^2 \sin \beta}{k^2 + r^2}$.

Now $Mg = 1500$ lb., $k = 1{\cdot}3$ ft., $\sin \beta = \sin 10° = 0{\cdot}1736$, and $r = 1{\cdot}5$ ft., therefore

$$F = \frac{1500 \times 1{\cdot}3^2 \times 0{\cdot}1736}{1{\cdot}3^2 + 1{\cdot}5^2} = 112 \text{ lb.}$$

Assuming slip.
From (2) and (3),

$$F = \mu R = \mu Mg \cos \beta \quad . \quad . \quad . \quad (7).$$

Since $\mu = 0{\cdot}07$ and $\cos \beta = \cos 10° = 0{\cdot}9848$, therefore

$$F = 0{\cdot}07 \times 1500 \times 0{\cdot}9848 = 103 \text{ lb.}$$

This value is less than 112 lb., therefore slip occurs. Owing to the low coefficient of friction, the force F does not reach the value which is large enough to cause pure rolling.

From (4) $\qquad \ddot{x} = g \sin \beta - \frac{F}{M}$,

and from (7) $\qquad \frac{F}{M} = \mu g \cos \beta$,

therefore $\qquad \ddot{x} = g(\sin \beta - \mu \cos \beta)$.

Substituting numerical values,

$$\ddot{x} = 32{\cdot}2(0{\cdot}1736 - 0{\cdot}07 \times 0{\cdot}9848) = 3{\cdot}37 \text{ ft./sec.}^2.$$

If s is the distance travelled in time t with constant acceleration $\ddot{x}$, then $s = \frac{1}{2}\ddot{x}t^2$, or $t = \sqrt{(2s/\ddot{x})}$.

Putting $s = 300$ feet, $t = \sqrt{\frac{2 \times 300}{3{\cdot}37}} = 13{\cdot}3$ sec.

76. Compound Pendulum.—A body of mass M swings about an axis O which is horizontal and perpendicular to the plane of the paper (Fig. 124). The centre of gravity is at G, the length $OG = a$, and the radius of gyration about G is k. Assuming the swings to be small, it is required to find the periodic time and the length of the equivalent simple pendulum.

Let OG make an angle θ with the vertical OY at time t, then if G has travelled a distance s from its mean position, $s = a\theta$ and, differentiating twice with respect to time, the acceleration $\ddot{s} = a\ddot{\theta}$.

The accelerations of G are $a\dot{\theta}^2$ towards O and $a\ddot{\theta}$ perpendicular to OG, and the corresponding effective forces are $\mathrm{M}a\dot{\theta}^2$ and $\mathrm{M}a\ddot{\theta}$. The effective couple is $\mathrm{M}k^2\ddot{\theta}$.

The applied forces are Mg acting vertically downwards at G and the reaction R at O.

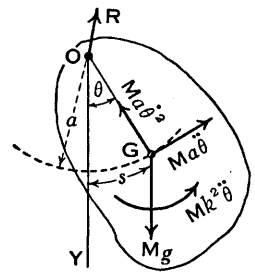


FIG. 124.

Equating the moments about O of the effective and applied forces and so avoiding the unknown reaction at O,

$$\mathrm{M}k^2\ddot{\theta} + \mathrm{M}a^2\ddot{\theta} = -\mathrm{M}ga\sin\theta,$$

therefore $$\ddot{\theta}(k^2 + a^2) + ga\sin\theta = 0,$$

or $$\ddot{\theta} + \frac{ga}{k^2 + a^2}\sin\theta = 0 \quad . \quad . \quad . \quad (1).$$

For small values of θ, $\sin\theta = \theta$ approximately, then

$$\ddot{\theta} + \frac{ga}{k^2 + a^2}\theta = 0 \quad . \quad . \quad . \quad (2).$$

This equation represents simple harmonic motion (Art. 60, p. 104) and may be written as

$$\ddot{\theta} + \omega^2\theta = 0, \quad \text{where} \quad \omega = \sqrt{\frac{ga}{k^2 + a^2}}.$$

The periodic time is

$$\mathrm{T} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{k^2 + a^2}{ga}} \quad . \quad . \quad (3).$$

Now the periodic time of a simple pendulum (Art. 61) is $2\pi\sqrt{(l/g)}$ where l is its length. Therefore a simple pendulum of length $l = (k^2 + a^2)/a$ will have the same periodic time as the compound pendulum and it is called the *equivalent simple pendulum.*

A closer approximation to the value of the periodic time may be obtained in the manner shown in Art. 63, p. 112. For the compound pendulum

$$T=2\pi\sqrt{\frac{k^2+a^2}{ga}}\times F,$$

where F is the amplitude factor.

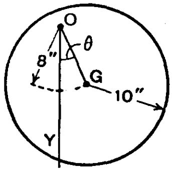


FIG. 125.

Example.—The body is a disc (Fig. 125) of radius $r=10$ inches and $OG=a=8$ inches. The horizontal axis through O is perpendicular to the face of the disc and the swings are small. To find l and T.

$$k^2=\frac{r^2}{2}=\frac{10^2}{2}=50 \text{ in.}^2, \qquad l=\frac{k^2+a^2}{a}=\frac{50+64}{8}=14{\cdot}25 \text{ in.}$$

$$T=2\pi\sqrt{\frac{k^2+a^2}{ga}}=2\pi\sqrt{\frac{l}{g}}=2\pi\sqrt{\frac{14{\cdot}25}{32{\cdot}2\times 12}}=1{\cdot}21 \text{ sec.}$$

77. Centre of Oscillation.—As shown in the preceding Art., the periodic time of a compound pendulum is $T=2\pi\sqrt{\frac{k^2+a^2}{ga}}$ and the length of the equivalent simple pendulum is $l=(k^2+a^2)/a$.

Referring to Fig. 126, O is the centre of suspension and G is the centre of gravity. Join OG and produce to O_1, making $OO_1=l$, then the point O_1 is called the *centre of oscillation.*

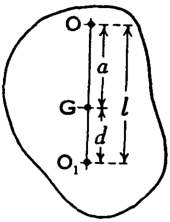


FIG. 126.

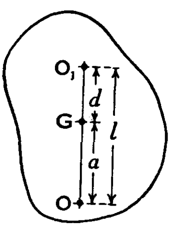


FIG. 127.

Let $O_1G=d$, then $OG+O_1G=a+d=l=(k^2+a^2)/a$, therefore

$$a(a+d)=k^2+a^2, \quad \text{or} \quad ad=k^2, \quad \text{or} \quad d=k^2/a.$$

It will now be shown that if the pendulum is suspended at O_1 (Fig. 127), the point O becomes the centre of oscillation.

When O_1 is the centre of suspension, $T = 2\pi\sqrt{\dfrac{k^2+d^2}{gd}}$,

but $$\frac{k^2+d^2}{d} = \frac{k^2+(k^2/a)^2}{k^2/a} = \frac{a^2+k^2}{a} = l = OO_1,$$

therefore the point O is the centre of oscillation and the periodic time has the same value as before.

Kater's Pendulum.—Kater used a pendulum with two adjustable knife-edges and an adjustable mass for determining the value of g. The positions of the mass and the knife-edges are adjusted until the time of oscillation is the same about each knife-edge, then if l is the distance between the knife-edges, $T = 2\pi\sqrt{(l/g)}$ and the value of g can be calculated.

78. Compound Pendulum Reactions.—Given that the compound pendulum (Fig. 128) has a maximum angular displacement β from its mean position, it is required to find the components R_1 and R_2 of the reaction at the support O when the displacement is θ. The directions of the components R_1 and R_2 are respectively perpendicular to and along GO.

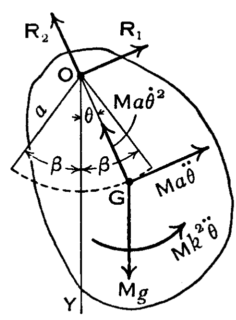


FIG. 128.

The angular acceleration and angular velocity of the pendulum will be wanted in terms of the angular displacement θ.

From Art. 76, equation (1),

$$\ddot{\theta} = -\frac{ga}{k^2+a^2}\sin\theta \quad . \quad . \quad . \quad (1).$$

Multiplying each side by $2\dot{\theta}$,

$$2\dot{\theta}\ddot{\theta} = -\frac{2ga}{k^2+a^2}\sin\theta\,\dot{\theta}.$$

Integrating, $$\dot{\theta}^2 = \frac{2ga}{k^2+a^2}\cos\theta + C,$$

where C is a constant of integration.

When $\theta=\beta$, $\dot{\theta}=0$, therefore the constant

$$C=-\frac{2ga}{k^2+a^2}\cos\beta,$$

and
$$\dot{\theta}^2=\frac{2ga}{k^2+a^2}(\cos\theta-\cos\beta) \quad . \quad . \quad (2).$$

To find R_1, equate the moments about G of the applied and effective forces, then

$$R_1 a=-Mk^2\ddot{\theta},$$

and substituting the value of $\ddot{\theta}$ from (1) and simplifying,

$$R_1=\frac{Mgk^2}{k^2+a^2}\sin\theta \quad . \quad . \quad . \quad (3).$$

To find R_2, equate the applied and effective forces by resolving along GO, then

$$R_2-Mg\cos\theta=Ma\dot{\theta}^2,$$

and substituting the value of $\dot{\theta}^2$ from (2) and simplifying,

$$R_2=Mg\{\cos\theta+\frac{2a^2}{k^2+a^2}(\cos\theta-\cos\beta)\} \quad . \quad (4).$$

When the swings are small, $\sin\theta=\theta$ approximately and $\cos\theta=\cos\beta=1$ approximately, then

$$R_1=\frac{Mgk^2}{k^2+a^2}\theta \quad \text{and} \quad R_2=Mg$$

approximately.

It should be noted that for small swings the maximum value of the angular velocity $\dot{\theta}$ is small. This is evident from equation (2), for $\dot{\theta}^2$ is a maximum when $\theta=0$, and when β is small, $\cos 0°-\cos\beta$ is small and therefore $\dot{\theta}^2$ and $\dot{\theta}$ are small.

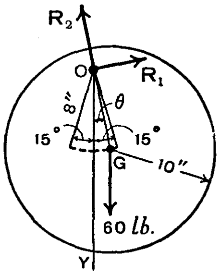


FIG. 129.

Example.—The body is a circular disc (Fig. 129) weighing 60 lb., of radius $r=10$ inches, and $OG=a=8$ inches. The horizontal axis through O is perpendicular to the face of the disc. The maximum angular displacement from the mean position is $\beta = 15°$. It is required to find the components R_1 and R_2 of the reaction at O when θ has each of the values 0°, 5°, 10°, and 15°.

Now $$R_1 = \frac{Mgk^2}{k^2+a^2}\sin\theta,$$

and $$R_2 = Mg\{\cos\theta + \frac{2a^2}{k^2+a^2}(\cos\theta - \cos\beta)\}.$$

Mass $M = W/g$, therefore $Mg = W = 60$ lb.

$$\cos\beta = \cos 15° = 0{\cdot}9659.$$

$$k^2 = \tfrac{1}{2}r^2 = \tfrac{1}{2}\times 10^2 = 50 \text{ in.}^2. \qquad a^2 = 8^2 = 64 \text{ in.}^2.$$

$$\frac{Mgk^2}{k^2+a^2} = \frac{60\times 50}{50+64} = \frac{3000}{114} = \frac{1000}{38} \text{ lb.}$$

$$\frac{2a^2}{k^2+a^2} = \frac{2\times 64}{50+64} = \frac{64}{57}.$$

Therefore $$R_1 = \frac{1000}{38}\sin\theta \text{ lb.}$$

and $$R_2 = 60\left\{\cos\theta + \frac{64}{57}(\cos\theta - 0{\cdot}9659)\right\} \text{ lb.}$$

Substituting the given values of θ, then it is found that R_1 and R_2 have the values shown in the table.

θ	0°	5°	10°	15°
R_1 lb.	0	2·29	4·57	6·81
R_2 lb.	62·3	61·8	60·4	58·0

79. Work and Energy.—From Art. 75, equations (10),

$$P = M\ddot{x}, \quad Q = M\ddot{y}, \quad \text{and} \quad L = Mk^2\ddot{\theta}.$$

Denoting the velocities $\dot{x}$, $\dot{y}$, and $\dot{\theta}$ by u, v, and ω respectively, then

$$P = M\dot{u}, \quad Q = M\dot{v}, \quad \text{and} \quad L = Mk^2\dot{\omega} \quad . \quad (1),$$

as indicated in Fig. 130.

Suppose that in time δt the centre of gravity of the body has small displacements $\delta\bar{x}$ and $\delta\bar{y}$, and the body has a small angular displacement $\delta\theta$, then

Work done by applied forces = Work done by effective forces.

$$P\delta\bar{x} + Q\delta\bar{y} + L\delta\theta = M\dot{u}\delta\bar{x} + M\dot{v}\delta\bar{y} + Mk^2\dot{\omega}\delta\theta.$$

Now $\qquad \delta\bar{x} = u\delta t \quad \text{and} \quad \dot{u}\delta t = \delta u,$

therefore $\qquad M\dot{u}\delta\bar{x} = M\dot{u}u\delta t = Mu\delta u.$

Similarly,

$$M\dot{v}\delta\bar{y} = Mv\delta v \quad \text{and} \quad Mk^2\dot{\omega}\delta\theta = Mk^2\omega\delta\omega.$$

Therefore

$$P\delta\bar{x} + Q\delta\bar{y} + L\delta\theta = Mu\delta u + Mv\delta v + Mk^2\omega\delta\omega.$$

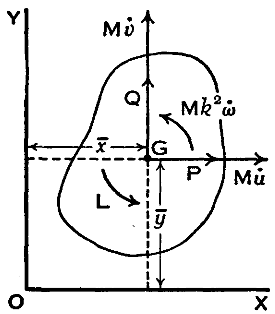


FIG. 130.

If during time t the displacements are $\bar{x}_1 - \bar{x}_0$, $\bar{y}_1 - \bar{y}_0$, and $\theta_1 - \theta_0$, and the corresponding velocity changes are $u_1 - u_0$, $v_1 - v_0$, and $\omega_1 - \omega_0$, then integrating between the appropriate limits,

$$\int_{\bar{x}_0}^{\bar{x}_1} Pd\bar{x} + \int_{\bar{y}_0}^{\bar{y}_1} Qd\bar{y} + \int_{\theta_0}^{\theta_1} Ld\theta = \int_{u_0}^{u_1} Mudu + \int_{v_0}^{v_1} Mvdv + \int_{\omega_0}^{\omega_1} Mk^2\omega d\omega$$

$$= \tfrac{1}{2}M(u_1^2 - u_0^2) + \tfrac{1}{2}M(v_1^2 - v_0^2) + \tfrac{1}{2}Mk^2(\omega_1^2 - \omega_0^2)$$

$$= \text{change in kinetic energy} \quad . \quad . \quad . \quad (2).$$

Therefore during a given time the work done by the applied forces is equal to the change in the kinetic energy.

It must be understood here that the work done by the applied forces is to be interpreted as the work done by the forces which produce acceleration or retardation. For example, if a force P pushes a mass along a straight path against a frictional resistance F, then it is the work done by the accelerating force $P-F$ which is equated to the change of kinetic energy of the mass.

Example 1.—A solid circular cylinder begins to roll up a slope of 1 in 5 (measured as a sine) with a linear velocity of 6 feet per second. It is required to find the distance travelled by the cylinder before coming to rest.

Let W be the weight and r the radius of the cylinder. Let s be the distance travelled and h the corresponding rise (Fig. 131). If k is the radius of gyration about the axis of the cylinder, then $k^2=\frac{1}{2}r^2$.

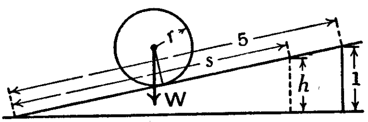


FIG. 131.

Denoting the initial linear velocity by v and the initial angular velocity by ω, then $\omega=v/r$.

The work done in lifting the cylinder a distance h is Wh.

The initial kinetic energy is made up of two parts—

(1) Kinetic Energy of Translation $=\dfrac{W}{2g}v^2$.

(2) Kinetic Energy of Rotation $=\dfrac{W}{2g}k^2\omega^2$.

Therefore the total Kinetic Energy $=\dfrac{W}{2g}(v^2+k^2\omega^2)$.

Work done = Loss of Kinetic Energy,

therefore $$Wh=\frac{W}{2g}(v^2+k^2\omega^2)$$

or $$h=\frac{1}{2g}\left(v^2+\frac{r^2}{2}\frac{v^2}{r^2}\right)=\frac{3v^2}{4g}.$$

Substituting numerical values,

$$h=\frac{3\times 6^2}{4\times 32\cdot 2}=0\cdot 839 \text{ foot}$$

and $$s=5h=5\times 0\cdot 839=4\cdot 20 \text{ feet}.$$

Example 2.—A railway truck is being lowered down a gradient by means of a wire rope coiled on a winding drum at the top of the incline, the rope being supported parallel to the incline on frictionless rollers.

The total mass of the truck is M, the moment of inertia of each of its two pairs of wheels and axles is mk^2, and the radius of the wheels is r. The moment of inertia of the drum excluding the wire rope is I, and its radius is R. The total length of the wire is l, and its mass is ρ per unit length. The angle of the incline is α.

Consider the case when the truck is descending the incline, and winding wire off the drum which is perfectly free to rotate. Show that, when the length of wire paid off from the drum is s, the acceleration of the truck is

$$\frac{(\mathrm{M}+\rho s)g\sin\alpha}{\mathrm{M}+\rho l+\dfrac{2mk^2}{r^2}+\dfrac{\mathrm{I}}{\mathrm{R}^2}}. \qquad \text{[C.U.]}$$

Let v be the velocity of the truck when the length of wire paid off is s (Fig. 132). Then the angular velocity of the wheels is v/r and the angular velocity of the drum is v/R. The mass of rope paid off is ρs and the mass of the whole rope is ρl. Also the magnitude of the linear velocity of the whole rope is v, the same as that of the truck.

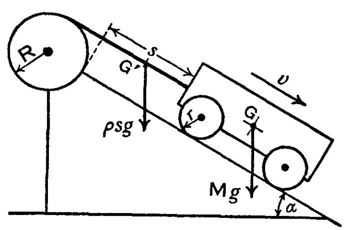


FIG. 132.

Kinetic Energy of Translation (including rope) $=\frac{1}{2}(\mathrm{M}+\rho l)v^2$.

Kinetic Energy of Rotation $=\frac{1}{2}\left(2mk^2\frac{v^2}{r^2}\right)+\frac{1}{2}\mathrm{I}\frac{v^2}{\mathrm{R}^2}$.

Total Kinetic Energy $=\frac{v^2}{2}\left\{\mathrm{M}+\rho l+\frac{2mk^2}{r^2}+\frac{\mathrm{I}}{\mathrm{R}^2}\right\}$ (1).

When the truck has travelled a distance s, its centre of gravity G has been lowered a height $s\sin\alpha$, and the centre of gravity G′ of the length s of the rope has been lowered a height $\frac{1}{2}s \sin \alpha$, therefore

$$\text{Work done by gravity} = \mathrm{M}gs \sin \alpha + \rho sg(\tfrac{1}{2}s \sin \alpha) \qquad (2).$$

Equating (1) and (2) and solving for v^2,

$$v^2 = \frac{(2\mathrm{M}s + \rho s^2)g \sin \alpha}{\mathrm{M} + \rho l + \dfrac{2mk^2}{r^2} + \dfrac{\mathrm{I}}{\mathrm{R}^2}} \qquad . \quad . \quad (3).$$

Since v^2 depends on s^2 as well as on s, it is evident that the formula $v^2 = 2fs$ cannot be used to obtain the acceleration f. The acceleration is obtained by differentiating, with respect to t, each side of (3), then

$$2v\frac{dv}{dt} = \frac{(2\mathrm{M} + 2\rho s)\dfrac{ds}{dt}g \sin \alpha}{\mathrm{M} + \rho l + \dfrac{2mk^2}{r^2} + \dfrac{\mathrm{I}}{\mathrm{R}^2}} \qquad . \quad . \quad (4).$$

Now $\dfrac{ds}{dt} = v$ and dividing (4) by $2v$ gives

$$\text{Acceleration } \frac{dv}{dt} = \frac{(\mathrm{M} + \rho s)g \sin \alpha}{\mathrm{M} + \rho l + \dfrac{2mk^2}{r^2} + \dfrac{\mathrm{I}}{\mathrm{R}^2}} \qquad . \quad . \quad (5).$$

80. Impulse, Momentum, and Moment of Momentum.—From Art. 75, equations (10),

$$\mathrm{P} = \mathrm{M}\ddot{x}, \qquad \mathrm{Q} = \mathrm{M}\ddot{y}, \quad \text{and} \quad \mathrm{L} = \mathrm{M}k^2\ddot{\theta},$$

or using the notation of Art. 79, equations (1), and referring to Fig. 133,

$$\mathrm{P} = \mathrm{M}\dot{u}, \qquad \mathrm{Q} = \mathrm{M}\dot{v}, \quad \text{and} \quad \mathrm{L} = \mathrm{M}k^2\dot{\omega}$$

or

$$\mathrm{P} = \mathrm{M}\frac{du}{dt}, \quad \mathrm{Q} = \mathrm{M}\frac{dv}{dt}, \quad \text{and} \quad \mathrm{L} = \mathrm{M}k^2\frac{d\omega}{dt} \qquad . \quad (1).$$

Let the velocities be u_0, v_0, and ω_0 at time t_0, and u_1, v_1, and ω_1 at time t_1, then integrating (1),

$$\int_{t_0}^{t_1} \mathrm{P}dt = \mathrm{M}\int_{u_0}^{u_1} du = \mathrm{M}(u_1 - u_0), \qquad \int_{t_0}^{t_1} \mathrm{Q}dt = \mathrm{M}(v_1 - v_0),$$

and

$$\int_{t_0}^{t_1} \mathrm{L}dt = \mathrm{M}k^2(\omega_1 - \omega_0) \qquad . \quad . \quad (2).$$

$\int_{t_0}^{t_1} \mathrm{P}dt$ is the *impulse* of the force P during the time $t_1 - t_0$ and is equal to the *change of momentum* $\mathrm{M}(u_1 - u_0)$. Also $\int_{t_0}^{t_1} \mathrm{Q}dt$ is the impulse of the force Q and is equal to the change of momentum $\mathrm{M}(v_1 - v_0)$.

$\int_{t_0}^{t_1} \mathrm{L}dt$ is the impulse of the couple or torque L and is equal to the *change of angular momentum*, or *change of the moment of momentum*, $\mathrm{M}k^2(\omega_1 - \omega_0)$.

The expression $\mathrm{M}k^2(\omega_1 - \omega_0)$ is the change of the moment of momentum about the centre of gravity of the body, and in many cases the change of the moment of momentum about some other point is required, say about O (Fig. 133).

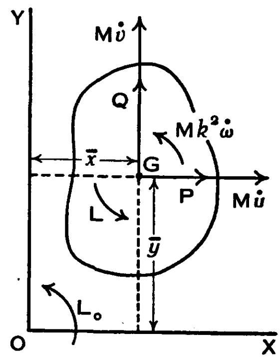


FIG. 133.

Let $\mathrm{L_O}$ be the moment about O of the applied forces, then

$$\begin{aligned}\mathrm{L_O} &= \mathrm{Q}\bar{x} - \mathrm{P}\bar{y} + \mathrm{L} \\ &= \mathrm{M}\dot{v}\bar{x} - \mathrm{M}\dot{u}\bar{y} + \mathrm{M}k^2\dot{\omega} \\ &= \frac{d}{dt}\{\mathrm{M}v\bar{x} - \mathrm{M}u\bar{y} + \mathrm{M}k^2\omega\} \qquad (3).\end{aligned}$$

(Checking by differentiation,

$$\frac{d}{dt}\{\mathrm{M}v\bar{x}\} = \mathrm{M}v\dot{\bar{x}} + \mathrm{M}\dot{v}\bar{x} = \mathrm{M}vu + \mathrm{M}\dot{v}\bar{x},$$

$$-\frac{d}{dt}\{\mathrm{M}u\bar{y}\} = -\mathrm{M}u\dot{\bar{y}} - \mathrm{M}\dot{u}\bar{y} = -\mathrm{M}uv - \mathrm{M}\dot{u}\bar{y},$$

and the terms $\mathrm{M}vu - \mathrm{M}uv$ cancel.)

Equation (3) shows that the sum of the moments of the applied forces about any fixed point is equal to the rate of change of the moment of momentum about the same point.

Also, integrating over the time interval $t_1 - t_0$,

$$\int_{t_0}^{t_1} \mathrm{L_O}dt = \text{Change in the moment of momentum during the time } t_1 - t_0 \quad . \qquad (4).$$

When two bodies collide, or when a point on a moving body becomes fixed suddenly, there is an impulsive force (see also Art. 41, p. 72) which acts for a very short time and usually its mean value is large compared with the value of any ordinary force which may be acting at the same time. In most cases the mean value of an impulsive force cannot be determined, but its impulse is equal to the change of momentum produced and this can be measured. When considering the effect of an impulsive force, any ordinary force acting at the same time may generally be disregarded.

The theorems given below follow from equations (2) and (4); they have already been given in Art. 42, p. 73, but are of sufficient importance to be restated here.

Conservation of Linear Momentum.—If, in any direction, the sum of the components of the applied forces acting on a system of bodies is zero, the total momentum of the system is constant in that direction.

Conservation of Moment of Momentum or Angular Momentum.—If, in a system of bodies, the sum of the moments of the applied forces about any fixed axis is zero, the moment of momentum of the system about that axis is constant.

Example.—In Fig. 134, AB and CB are inclined planes. A solid cylinder is placed on AB and released, its axis being at right angles to the line of maximum slope. Show that it will finally come to rest after a time $30\sqrt{\frac{h}{3g}}$ where h is the initial height of the centre above its position when the cylinder is at rest touching both planes. [C.U.]

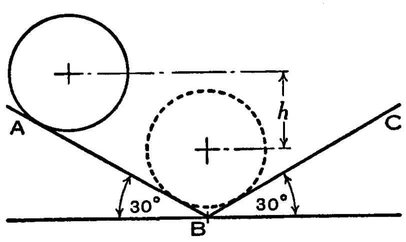


FIG. 134.

Let M be the mass of the cylinder, then Mg is its weight. Let k be the radius of gyration of the cylinder about its axis and let r be its radius, then $k^2=\frac{1}{2}r^2$.

Descending the Left-hand Slope.—Let v be the velocity of the centre O just before impact occurs at c (Fig. 135) and let ω be the corresponding angular velocity, then $\omega = v/r$.

Work done in descending height h = Gain in kinetic energy.

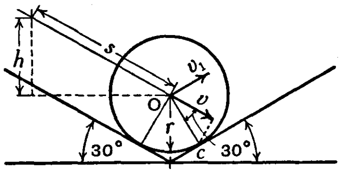


FIG. 135.

$$Mgh = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}Mk^2\omega^2$$

$$= \tfrac{1}{2}Mv^2 + \tfrac{1}{2}M\frac{r^2}{2}\frac{v^2}{r^2}.$$

Therefore $$h = \frac{3v^2}{4g} \quad \text{or} \quad v = 2\sqrt{\frac{gh}{3}}.$$

Let s be the distance travelled by the centre O and let t be the time taken, then

$$s = \frac{h}{\sin 30°} = \frac{h}{0{\cdot}5} = 2h; \quad \text{also} \quad s = \frac{v}{2}t; \quad \text{therefore} \quad \frac{vt}{2} = 2h.$$

Therefore $$t = \frac{4h}{v} = \frac{4h}{2}\sqrt{\frac{3}{gh}} = 2\sqrt{\frac{3h}{g}} \quad . \quad . \quad (1),$$

or $$t = \frac{4h}{v} = \frac{4}{v} \times \frac{3v^2}{4g} = \frac{3v}{g} \quad . \quad . \quad . \quad (2).$$

Impact.—Let v_1 and ω_1 be the initial linear velocity and angular velocity respectively up the right-hand slope, then $\omega_1 = v_1/r$.

The moment of momentum about c, the point of impact (actually a line of impact), is unchanged, because the impulsive force acting at c cannot have any moment about c; therefore

$$Mv_1r + Mk^2\omega_1 = Mvr \sin 30° + Mk^2\omega,$$

$$v_1r + \frac{r^2}{2}\frac{v_1}{r} = \frac{vr}{2} + \frac{r^2}{2}\frac{v}{r},$$

from which $$v_1 = \frac{2}{3}v \quad . \quad . \quad . \quad . \quad (3).$$

Ascending the Right-hand Slope.—Let t_1 be the time taken to travel up the right-hand slope, then it follows from (2) that

$$t_1 = \frac{3v_1}{g}$$

or

$$\frac{t_1}{t} = \frac{v_1}{v}.$$

But from (3), $v_1 = \frac{2}{3}v$, therefore $\frac{t_1}{t} = \frac{2}{3}$ or $t_1 = \frac{2}{3}t$.

The cylinder will take an equal time t_1 to descend the right-hand slope, therefore the time for the return journey on this slope is $2t_1$.

To obtain the Total Time.—If $2t_2$, $2t_3$, $2t_4$, and so on, are the times of subsequent return journeys, then it follows that $t_2 = \frac{2}{3}t_1 = \left(\frac{2}{3}\right)^2 t$, $t_3 = \frac{2}{3}t_2 = \left(\frac{2}{3}\right)^3 t$, and so on.

If T is the total time before the cylinder comes to rest, then

$$\begin{aligned} T &= t + 2t_1 + 2t_2 + 2t_3 + \ . \ . \ . \\ &= (2t + 2t_1 + 2t_2 + 2t_3 + \ . \ . \ .) - t \\ &= 2t\left\{1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \ . \ . \ . \right\} - t. \end{aligned}$$

The series in the brackets is a geometrical progression and its sum is $\frac{1}{1 - \frac{2}{3}} = 3.$

Therefore $$T = 2t \times 3 - t = 5t,$$

but from (1), $$t = 2\sqrt{\frac{3h}{g}},$$

therefore $$T = 10\sqrt{\frac{3h}{g}} = 30\sqrt{\frac{h}{3g}}$$

as was to be proved.

81. Centre of Percussion.—Let G be the centre of gravity of a rigid body which is free to turn about a fixed axis passing through O perpendicular to the plane of the paper (Fig. 136). Suppose an impulsive force P is applied to the body, its line of action being in the plane of the paper and meeting OG produced at O_1, then if there is no impulsive reaction at O, the point O_1 is called the *centre of percussion*. It should be noted, however, that the subsequent motion of the body will produce a reaction at O.

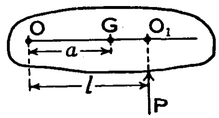


FIG. 136.

It is evident that the line of action of the impulsive force P must be perpendicular to OO_1, otherwise there would be a component parallel to OO_1 and consequently a reaction at O along OO_1.

The position of the point O_1 will now be determined. Let $OG = a$ and $OO_1 = l$; let M be the mass of the body and let k be its radius of gyration about an axis through G parallel to the axis at O. Let ω be the angular velocity of the body produced by the impulsive force P, then the velocity of G is $a\omega$ perpendicular to OG.

Suppose the force P acts for a very short time t, then equating the impulse to the change of momentum perpendicular to OO_1, remembering that there is to be no impulsive force at O,

$$Pt = Ma\omega \quad . \quad . \quad . \quad (1).$$

Taking moments about O, the impulse of the moment of the applied force is equal to the change of the moment of momentum, therefore

$$Plt = Mk^2\omega + Ma^2\omega \quad . \quad . \quad (2).$$

From (1) and (2) $\quad l = (k^2 + a^2)/a,$

and this fixes the position of O_1, the centre of percussion.

If the body were swung as a pendulum about the axis O, then the point O_1 would be the centre of oscillation, as shown in Art. 77—that is, the centre of oscillation of a pendulum is also the centre of percussion.

Exercises IX

1. A cage weighing 1·5 tons is raised by means of a rope coiled round a drum of 5 feet diameter mounted on a horizontal shaft. The drum and shaft weigh 2000 pounds and their radius of gyration is 28 inches. A motor supplies a constant torque of 9000 pound-feet to the shaft. Assuming that the rope is tight when the motor begins to revolve, find

(*a*) The acceleration of the cage;
(*b*) The time required to raise it 40 feet from rest;
(*c*) The tension of the rope.

What torque must be applied to the shaft in order that the cage may descend at a uniform speed of 2 feet per second? Neglect friction. [B.E.]

2. The loaded cage of a goods hoist is raised by a rope which passes round a drum and is connected to a balance weight at the other end. The loaded cage weighs 1·5 tons and the balance weight is 1·1 tons (Fig. 137); the drum weighs 1000 lb., its diameter is 3 feet 6 inches, and its radius of gyration is 16 inches.

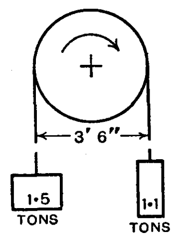


FIG. 137.

Calculate the torque which must be applied to the drum to raise the cage with an acceleration of 4 feet per second per second. What horse-power is required to give this acceleration at the instant the speed is 8 feet per second? Friction of bearings is to be neglected.

3. The centre of gravity of a motor car is at a height h above the road level, at a distance a behind the front axle and at a distance b in front of the back axle. The coefficient of friction between the tyres and road is μ. Neglecting rotational inertia, calculate the greatest retardation which can be produced by the brakes without skidding the wheels on a level road

(1) If the front wheels only are braked;
(2) If the rear wheels only are braked.

Prove that front-wheel braking will be the more efficient provided $a < b + \mu h$. If the car is descending an incline making an angle θ with the horizontal, show that this same condition still holds true. [C.U.]

4. A circular cylinder rolls without slipping down a plane inclined to the horizontal at an angle of 30°. (*a*) Find the linear acceleration of the cylinder; (*b*) What minimum coefficient of friction is necessary to prevent slipping? (*Note.*—$k^2 = \frac{1}{2}r^2$.)

5. A pair of wheels and an axle, weighing 1500 lb. and having a radius of gyration of 1·3 feet, start from rest and travel on rails down a uniform incline which makes an angle of 10° with

the horizontal. The radius of each wheel is 1·5 feet and the coefficient of friction between the wheels and the rails is 0·25. Prove that slip will not occur, and find the time taken to travel 300 feet.

6. A circular face-plate is oscillated about a horizontal axis through O (Fig. 138) perpendicular to its face, O being $8\frac{1}{4}$ inches from the centre of gravity G. The plate makes 100 complete oscillations or cycles in 117 seconds. Find the radius of gyration about an axis through G perpendicular to the face of the plate.

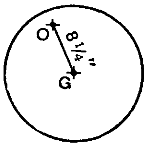


FIG. 138.

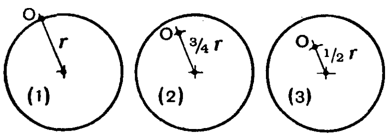


FIG. 139.

7. A flat circular plate of radius r swings in its own plane about a horizontal axis through the point O (Fig. 139). Find the length of the equivalent simple pendulum when the point O is (1) at the circumference, (2) a distance $\frac{3}{4}r$ from the centre, (3) mid-way between the centre and the circumference.

8. A flat square plate, having sides of length $2a$, swings in its own plane about a horizontal axis through a point O (Fig. 140). Find the length of the equivalent simple pendulum for each of the positions of O shown at (1), (2), and (3).

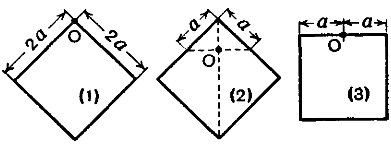


FIG. 140.

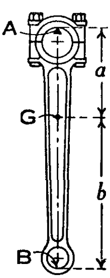


FIG. 141.

9. A connecting-rod AB (Fig. 141) was suspended on a horizontal knife-edge at A and allowed to oscillate. It was found that the rod made 60 complete oscillations or cycles per minute. Next the rod was suspended at B and made 54 cycles per minute. If the centre of gravity G of the rod is a distance a from A and a distance b from B and if $a+b=13{\cdot}75$ inches, find a and b and the radius of gyration k about G in the plane of oscillation.

10. Referring to the preceding exercise, if the rod makes 515 cycles in 10 minutes when suspended at A and 476 cycles in 10 minutes when suspended at B, and if $a+b=18$ inches, find a, b, and k.

11. A rectangular plate, 20 inches by 30 inches, weighing 100 lb., swings about a horizontal axis through O (Fig. 142) perpendicular to the face of the plate. If the swing is 12° on each side of the vertical OY, calculate the values of the reactions R_1 and R_2 when the angular displacement from the vertical is $\theta = 6°$. Also find the maximum value of the angular velocity.

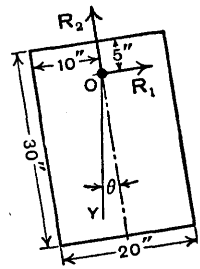


FIG. 142.

12. One end of a uniform heavy spar of length l is attached to the ground by a hinge, and the other end is raised in a vertical plane until the angle made by the spar with the horizontal is θ. It is then allowed to fall under the action of gravitational force, turning about the lower end. Show that if the diameter of the spar is small compared with its length, the direction of the initial reaction at the hinge makes an angle ϕ with the spar given by $\cot \phi = 4 \tan \theta$. [C.U.]

13. Two equal homogeneous solid spheres each of radius r are fixed together by a light rigid bar whose direction passes through the centres which are a distance $2l$ apart. The system is caused to oscillate as a compound pendulum about a point P in the bar distant x from the centre of the bar. Find the period of a small oscillation. Show that the period would be least if it were possible for the distance of P from the centre of the bar to be $\sqrt{\dfrac{5l^2 + 2r^2}{5}}$. Show that this point is inside one of the spheres. [U.L.]

14. A uniform disc can rotate in its own plane, which is vertical, about a smooth hinge at one end of a diameter. It is allowed to fall from the position in which this diameter is horizontal. Prove that, when the horizontal component of the reaction at the hinge is a maximum, the vertical component is $\frac{4}{3}$ of the weight of the disc. [C.U.]

15. A rigid body makes small oscillations about a horizontal axis. Prove that the periodic time is $2\pi\sqrt{k^2/hg}$, where k is the radius of gyration of the body about the axis and h the distance of the axis from the centre of gravity of the body.

A flywheel weighing 3 tons is suspended so as to oscillate about an axis perpendicular to its plane and 3 feet distant from the centre of the wheel. Find the radius of gyration of the wheel about its axis if the time of a small oscillation is 2·5 seconds, and calculate the work required when the wheel is spinning about its axis to increase its velocity from 50 to 100 revolutions per minute. [U.L.]

16. A solid cylinder rolls down a plane inclined at a slope of 1 vertical in 4 horizontal. Use the energy method to find the

velocity of the cylinder after it has rolled down a distance of 20 feet.

17. A four-wheeled truck is running down a slope which makes an angle β with the horizontal (Fig. 143). The total mass of the truck is M, the mass of each pair of wheels and axle is m and their radius of gyration, about their axis, is k. The radius of each wheel is r. Show that the acceleration of the truck is

$$\frac{Mg \sin \beta}{M + 2mk^2/r^2}.$$

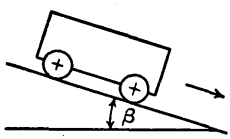


FIG. 143.

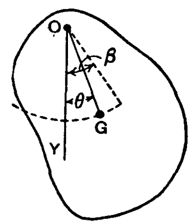


FIG. 144.

18. A compound pendulum has a maximum angular displacement β from its mean position. Show by the principle of the conservation of energy that

$$\dot{\theta}^2 = \frac{2ga}{k^2 + a^2}(\cos \theta - \cos \beta),$$

then by differentiation obtain the equation

$$\ddot{\theta} + \frac{ga}{k^2 + a^2} \sin \theta = 0,$$

where $a = \text{OG}$ (Fig. 144), the distance of the centre of gravity G from the axis of suspension O, k is the radius of gyration about an axis through G parallel to the axis of suspension, and θ is the angular displacement at time t from the vertical OY. Compare this method with that used in Art. 76, p. 141.

19. A pair of wheels and axle of mass M and moment of inertia Mk^2 stand on a horizontal surface. The radius of the wheels is a, and concentrated masses m are attached to parallel spokes at a distance b from the centre.

The wheels are held so that these spokes are turned through an angle α from their lowest position and the system is then released.

Calculate the angular velocity when these spokes make an angle θ with the vertical.

Thence prove that the period of a small oscillation is

$$2\pi \sqrt{\frac{M(a^2 + k^2) + 2m(a - b)^2}{2mgb}}. \qquad \text{[C.U.]}$$

20. Two toothed wheels (1) and (2) (Fig. 145), running freely on parallel shafts, are suddenly put into mesh. For wheel (1), radius $r_1=6$ inches, radius of gyration $k_1=4{\cdot}8$ inches, and weight $W_1=15$ lb. For wheel (2), $r_2=8$ inches, $k_2=6{\cdot}7$ inches, and $W_2=25$ lb. The speeds before meshing are $N_1=50$ r.p.m. and $N_2=30$ r.p.m. Find N_1' and N_2', the speeds after meshing, and also find the loss of energy in inch-pounds, taking N_1' and N_2' to three significant figures.

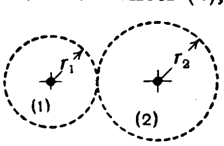


FIG. 145.

21. Using the data from the preceding exercise, but taking $N_1=100$ r.p.m. and $N_2=0$, find N_1' and N_2' and the loss of energy in inch-pounds, taking N_1' and N_2' to three significant figures.

22. Two wheels, A and B, rotate about parallel axes. Initially the wheels are not in contact; A rotates at 300 r.p.m. and B is at rest. By a suitable lever arrangement the wheel B is caused to press with its rim against that of A, the normal pressure between the wheels being 400 pounds; coefficient of friction is 0·06; the moments of inertia of A and B are 3600 lb. inch² and 2000 lb. inch² respectively; diameter of each wheel is 18 inches.

Neglecting the friction of the bearings, determine (i) the angular acceleration of B and the angular retardation of A; (ii) the common angular velocity of the wheels when slipping has ceased; and the time taken to attain this common velocity. [B.E.]

23. A flywheel (A) having a moment of inertia I_A and a radius r_A is coupled by means of a belt drive to another flywheel (B) having a moment of inertia I_B and a radius r_B.

When the belt is thrown on, B is at rest and A has an angular velocity ω.

Show that when slipping of the belt has ceased, the angular velocity acquired by B is

$$\frac{r_A r_B I_A \omega}{r_B^2 I_A + r_A^2 I_B},$$

and that the time during which slip of the belt occurs is

$$\frac{I_A I_B r_A \omega}{(T_1 - T_2)(r_B^2 I_A + r_A^2 I_B)},$$

where T_1 and T_2 are the tensions of the tight and slack sides of the belt during the period of slipping. [C.U.]

24. A sphere, of radius a, rolls, with constant angular velocity ω, on a horizontal plane, directly towards a step of height h. Supposing $a > h$, find the least value of ω that will enable the sphere to mount the step; and, supposing ω greater than this,

find the angular velocity with which it will roll on the upper plane. Assume no slip or rebound when the sphere strikes the step. [B.E.]

25. A concentrated mass M is rigidly attached by a light rod to a horizontal shaft which can rotate in frictionless bearings. The mass is hanging in the equilibrium position, when there is applied to the shaft a constant axial torque D. Show that the shaft and mass will continue to rotate always in the same direction only if D be greater than $0{\cdot}725Mgl$, where l is the distance from M to the axis of the shaft.

[The equation, $\theta \sin \theta = 1 - \cos \theta$, is satisfied by $\theta = 2{\cdot}33$ radians.] [C.U.]

26. A plumb-line consisting of a small mass suspended from a string of length l is attached to the roof of a railway carriage. With the carriage travelling along a straight track at a uniform speed v, the plumb-line is vertical and stationary relative to the carriage. If the carriage suddenly enters a curve of radius r, show that the plumb-line will start to oscillate, the deflection from the vertical being given by

$$\theta = \frac{v^2}{rg}\left[1 - \cos\sqrt{\frac{g}{l}}t\right]. \qquad \text{[C.U.]}$$

27. Two rigidly connected rods make a right angle at their point of junction, and a third rod moves in contact with them, one end sliding on each. If the whole be revolving about one of the first two rods, with angular velocity ω, prove that the kinetic energy of the third rod will be

$$\tfrac{1}{6}ma^2\left\{\left(\frac{d\theta}{dt}\right)^2 + \omega^2 \sin^2 \theta\right\},$$

where m is the mass of the rod, a its length, and θ the angle which it makes with the rod about which the system revolves.

Find also an expression for the angular momentum of the third rod about the axis of revolution of the system. [B.E.]

CHAPTER X

FRICTION

82. Coefficient of Friction.—When one body slides over another the motion is resisted by friction, and the force, acting parallel to the surfaces in contact, which will cause sliding is known as the *force of limiting friction.* If R_N is the mutual normal force between the surfaces and F is the force of limiting friction,

$$\frac{F}{R_N}=\mu,$$

and μ is called the *coefficient of friction.*

If the force F is built up gradually from zero, its value at any instant is equal to the force of friction preventing motion, but the limiting value when motion begins is often briefly described as the *force of friction* or the *friction force.*

The value of the coefficient of friction depends on the materials of the bodies in contact, the condition of the surfaces, the relative sliding velocity, the mutual normal force, and whether the surfaces are dry or lubricated. *Statical friction*, the friction when a body is about to slide, is greater than friction during sliding.

Published values of μ vary considerably, and only a general idea of the magnitudes in a few cases can be given here, assuming moderate normal forces and low speeds. For wood on wood, μ varies from 0·25 to 0·5 when the wood is dry, and from 0·02 to 0·1 when it is greased. For metal on metal, μ varies from 0·15 to 0·3 when there is no lubrication, and from 0·04 to 0·06 when there is lubrication.

83. Laws of Friction.—The following so-called laws of friction are approximately true when the mutual normal force between the surfaces in contact is moderate and the speed of sliding is low.

The force of friction is: (1) Directly proportional to the mutual normal force between the surfaces in contact. (2) Independent of the areas of the surfaces in contact. (3) Independent of the velocity of sliding.

84. A Body on a Horizontal Plane.—Consider a body of weight W supported on a fixed horizontal plane (Figs. 146 to 148), and suppose F is the force, acting parallel to the plane, which will cause sliding and that it is applied gradually. When the force is zero (Fig. 146) the reaction on the body is normal to the plane and it is $R_N = W$. When the force has any intermediate value F′ (Fig. 147) the

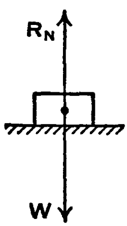


FIG. 146.

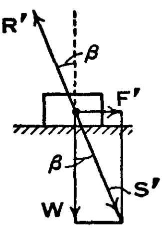


FIG. 147.

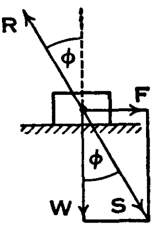


FIG. 148.

resultant of W and F′ is S′ inclined at an angle β to the vertical and the reaction on the body is R′, which is equal and opposite to S′. When F′ is increased to F, so that the body begins to slide (Fig. 148), the resultant of the forces W and F is S inclined at an angle ϕ to the vertical and the reaction on the body is R, which is equal and opposite to S.

The angle ϕ which the reaction R makes with the normal to the plane when sliding begins is known as the *limiting angle of friction*, or briefly as the *angle of friction*. The direction of the angle ϕ from the normal is always such that the component of the reaction R along the plane opposes motion.

From Fig. 148,

$$\frac{F}{W} = \tan \phi,$$

and since W is the normal force on the plane,

$$\frac{F}{W}=\mu,$$

therefore $$\tan\phi=\mu.$$

Also $$R=\sqrt{W^2+F^2}=W\sec\phi.$$

85. Angle of Repose.—Suppose a body of weight W is resting on a plane inclined at an angle β to the horizontal (Fig. 149), and that β is increased until the body begins to slide down the plane due to the force of gravity. The forces acting on the body are its weight W and the reaction R, which is the resultant of the force normal to the plane and the frictional resistance of the plane. It follows that R and W are equal and opposite and inclined at angles β to the normal.

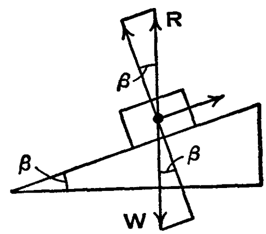


FIG. 149.

The components of W parallel and perpendicular to the plane are W sin β and W cos β respectively, and since there is sliding the frictional force is

$$W\sin\beta=\mu W\cos\beta$$

or $$\tan\beta=\mu.$$

But $\mu=\tan\phi$, therefore sliding begins when $\beta=\phi$. This value of β is called the *angle of repose* and, as shown, it is equal to the angle of friction.

86. A Body Sliding Up an Inclined Plane.—Suppose a body of weight W is being pushed up a plane, inclined at an angle a to the horizontal (Fig. 150), by a force P inclined at an angle θ to the plane as shown.

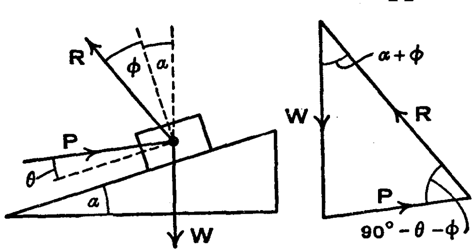


FIG. 150.

The ratio of P to W will be obtained on the assumption that the speed of the body is uniform.

Friction acts down the plane on the body, therefore the reaction R is inclined to the normal to the plane at an angle ϕ measured anticlockwise. The forces P, R and W are in equilibrium, and it follows from the triangle of forces in Fig. 150 that

$$\frac{P}{W}=\frac{\sin(\alpha+\phi)}{\sin(90^\circ-\theta-\phi)}=\frac{\sin(\alpha+\phi)}{\cos(\theta+\phi)}.$$

The value of this ratio will be examined for the cases when $\theta=0$ and $\theta=\alpha$.

When $\theta=0$, the force P is parallel to the plane, then

$$\frac{P}{W}=\frac{\sin(\alpha+\phi)}{\cos\phi}=\frac{\sin\alpha\cos\phi+\cos\alpha\sin\phi}{\cos\phi}=\sin\alpha+\mu\cos\alpha.$$

From the right-angled triangle ABC (Fig. 151), where l is the length of the plane AC, h is the height BC and b is the length of the base AB, $\sin\alpha=h/l$ and $\cos\alpha=b/l$, therefore

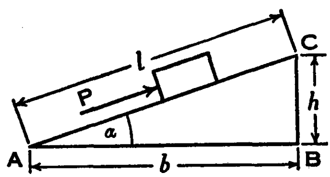


FIG. 151.

$$\frac{P}{W}=\frac{h}{l}+\mu\frac{b}{l}$$

or $$Pl=Wh+\mu Wb.$$

Therefore the work done in sliding the body up the plane AC is equal to the work done in lifting it through the height BC plus the work done in sliding it against friction along the horizontal AB.

When $\theta=\alpha$, the force P in Fig. 150 becomes horizontal, and the equation

$$\frac{P}{W}=\frac{\sin(\alpha+\phi)}{\cos(\theta+\phi)}$$

becomes $$\frac{P}{W}=\tan(\alpha+\phi),$$

which will be required when screw threads are considered (Art. 88).

87. A Body Sliding Down an Inclined Plane.—Suppose the body of weight W is being pulled down the plane by a force P inclined at an angle θ to the plane as shown in Fig. 152. As before, the ratio of P to W will be determined on the assumption that the speed of the body is uniform.

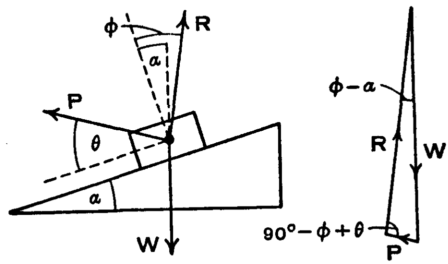


FIG. 152.

Friction acts up the plane on the body, therefore the reaction R is inclined to the normal to the plane at an angle ϕ measured clockwise. From the triangle of forces

$$\frac{P}{W}=\frac{\sin(\phi-\alpha)}{\sin(90^\circ-\phi+\theta)}=\frac{\sin(\phi-\alpha)}{\cos(\phi-\theta)}.$$

When $\theta=0$, the force P is parallel to the plane and

$$\frac{P}{W}=\frac{\sin(\phi-\alpha)}{\cos\phi}.$$

When $\theta=\alpha$, the force P is horizontal and

$$\frac{P}{W}=\frac{\sin(\phi-\alpha)}{\cos(\phi-\alpha)}=\tan(\phi-\alpha).$$

From these equations it can be seen that the force P becomes negative when α is greater than ϕ, that is P must act as a resistance to prevent the body accelerating down the incline.

88. Friction of Square-Threaded Screws.—Consider a helix of lead or pitch p traced on a vertical cylinder of diameter d and height p (Fig. 153), then the development of the cylindrical surface is the rectangle ABCD (Fig. 154) and the helix becomes the diagonal AC. If the *helix angle* CAB is denoted by α, then $\tan\alpha=p/\pi d$. The action of a nut on a square-threaded screw may be represented by one wedge sliding on another when loaded as shown in

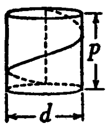


FIG. 153.

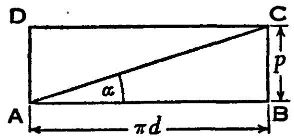


FIG. 154.

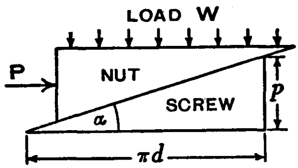


FIG. 155.

Fig. 155, where W is the axial load on the nut and P is the horizontal tangential force acting at the mean radius of the thread. If d_0 is the outside diameter of a square-threaded screw and d is the mean diameter of the thread,

$$d = d_0 - \tfrac{1}{2}\,(\textit{pitch of screw}).$$

In a single-start screw the terms pitch and lead have the same meaning, but in a double-start screw the pitch of the screw is half the lead or pitch p of the helix.

If the nut is being raised by a tangential force Q applied at the circumference of a wheel of diameter D or at the end of a lever or spanner of effective length $\frac{1}{2}$D, the torque on the nut is $\frac{1}{2}QD = \frac{1}{2}Pd$; also from Art. 86

$$\frac{P}{W} = \tan(\alpha + \phi) = \frac{\tan\alpha + \tan\phi}{1 - \tan\alpha\tan\phi}$$

$$= \frac{\dfrac{p}{\pi d} + \mu}{1 - \dfrac{p}{\pi d}\mu} = \frac{p + \mu\pi d}{\pi d - \mu p}.$$

When the nut is given one turn the work done by the effort P is $P\pi d$ and the work done against the resistance W is Wp, therefore the efficiency is

$$\eta = \frac{Wp}{P\pi d} = \frac{\tan\alpha}{\tan(\alpha + \phi)},$$

which has its maximum value when $d\eta/d\alpha = 0$, therefore

$$\frac{\tan(\alpha+\phi)\sec^2\alpha - \tan\alpha\sec^2(\alpha+\phi)}{\tan^2(\alpha+\phi)}=0,$$

$$\tan(\alpha+\phi)\sec^2\alpha = \tan\alpha\sec^2(\alpha+\phi),$$

$$\sin(\alpha+\phi)\cos(\alpha+\phi)=\sin\alpha\cos\alpha$$

$$\sin 2(\alpha+\phi)=\sin 2\alpha,$$

therefore $2\alpha+\phi=90°$ (see Fig. 156)

or $\alpha=45°-\frac{1}{2}\phi.$

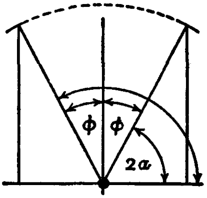


FIG. 156.

$$\eta_{\max}=\frac{\tan(45°-\frac{1}{2}\phi)}{\tan(45°+\frac{1}{2}\phi)}=\frac{(1-\tan\frac{1}{2}\phi)^2}{(1+\tan\frac{1}{2}\phi)^2},$$

which may also be written as

$$\frac{(\cos\frac{1}{2}\phi-\sin\frac{1}{2}\phi)^2}{(\cos\frac{1}{2}\phi+\sin\frac{1}{2}\phi)^2}=\frac{1-2\sin\frac{1}{2}\phi\cos\frac{1}{2}\phi}{1+2\sin\frac{1}{2}\phi\cos\frac{1}{2}\phi}=\frac{1-\sin\phi}{1+\sin\phi}.$$

The maximum value of the efficiency may be obtained by a much shorter but, to some students, less obvious trigonometrical method (see Ex. 5, p. 197).

Suppose the helix angle α (Fig. 155) is greater than the angle of friction ϕ and the loaded nut is turning back against the force P. The load W becomes the effort and the force P (acting in the same direction as when it was the effort) becomes the resistance. The ratio of the work $P\pi d$ to the work Wp is called the *reversed efficiency*, and this will be denoted by η_R.

From Art. 87 $$\frac{P}{W}=\tan(\alpha-\phi),$$

writing $\alpha-\phi$ instead of $\phi-\alpha$, since α is greater than ϕ, and regarding P as positive.

$$\eta_R=\frac{P\pi d}{Wp}=\frac{\tan(\alpha-\phi)}{\tan\alpha},$$

which is a maximum when $2\alpha-\phi=90°$ and the maximum value is the same as that of η. The proof is similar to the one already given.

The nut cannot turn back on the screw under the action of the load W unless α is greater than ϕ. The limit occurs when $\alpha = \phi$, then the forward efficiency is $\eta = \tan \phi / \tan 2\phi = \frac{1}{2}$ approximately. Therefore reversal cannot occur unless the forward efficiency is greater than about 50 per cent.

89. Friction of Vee-Threaded Screws.—When a screw thread is of triangular section the normal reaction at any point is inclined to the screw axis in two mutually perpendicular planes, due to the thread sloping in two directions.

Fig. 157 shows part of the section of the thread by a plane containing the screw axis and β is half the angle of the Vee. It will be assumed that the screw axis is vertical and that the reaction is concentrated at the point O on the thread. The axes OX and OY are mutually perpendicular, with OY parallel to the screw axis, and the angle XOK is equal to β. The normal to OK at O is OU inclined at the angle β to OY; OU is made any convenient length and its projection on OY is OV.

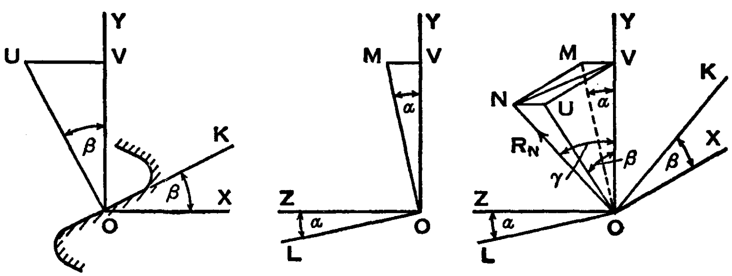


Fig. 157. Fig. 158. Fig. 159.

In Fig. 158 OZ is perpendicular to OY and the plane YOZ is perpendicular to the plane XOY in Fig. 157. The line OL represents the helix inclined at an angle α to OZ and the normal to OL at O is OM inclined at the angle α to OY. The length OM is such that its projection on OY is OV.

Figs. 157 and 158 are combined in Fig. 159, where the axes OX, OY, OZ are mutually perpendicular. Complete the rectangle UVMN and join ON and NV. Since UV and MV are perpendicular to OY, NV must also be perpendicular to OY. Let the angle NOY be γ.

The normal at O to the plane KOL is ON, inclined at the angle γ to the vertical axis OY. The lengths OU and OM are the projections of ON on the planes XOY and YOZ respectively, and OV is the projection of ON, OU, and OM on OY.

Let R_N be the normal reaction on the thread at the point O and let it be represented by ON, then if R_M denotes the component of R_N in the vertical plane YOZ, R_M is represented by OM and

$$R_M = \frac{R_N \cos \gamma}{\cos \alpha}.$$

The relation between the angles γ, α, and β is obtained as follows:—

$NV = OV \tan \gamma$, $\quad NU = MV = OV \tan \alpha$, $\quad UV = OV \tan \beta$,

also $\qquad NV^2 = NU^2 + UV^2$,

therefore $\qquad \tan^2 \gamma = \tan^2 \alpha + \tan^2 \beta$.

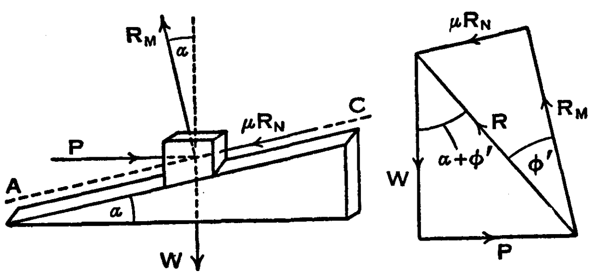


FIG. 160.

Consider the forces acting in the vertical plane on the nut when the load is being raised (Fig. 160). These are the effort P, the load W, the component R_M of the normal reaction R_N, and the friction force μR_N. From the force diagram, if R is the resultant of R_M and μR_N and is inclined at an angle ϕ' to R_M,

$$\tan \phi' = \frac{\mu R_N}{R_M} = \frac{\mu \cos \alpha}{\cos \gamma} = n\mu = n \tan \phi,$$

where $n = \cos \alpha / \cos \gamma$. Also

$$\frac{P}{W} = \tan (\alpha + \phi') = \frac{\tan \alpha + \tan \phi'}{1 - \tan \alpha \tan \phi'} = \frac{\tan \alpha + n \tan \phi}{1 - n \tan \alpha \tan \phi}.$$

Alternatively this result follows from the equation for P/W for a square-threaded screw by writing $n \tan \phi$ instead of $\tan \phi$; or it may be obtained by resolving forces in Fig. 160 parallel and perpendicular to the incline AC.

Most writers simplify the problem by assuming that the normal reaction to the thread is in the vertical plane XOY (Fig. 159). This is equivalent to assuming $\alpha = 0$ and $\gamma = \beta$, in the equation for n, then

$$n = \frac{\cos \alpha}{\cos \gamma} = \frac{1}{\cos \beta},$$

and in general, since α is usually small, this value of n will be very close to its true value. For example, with a two-inch Whitworth screw the two values of n are the same to four significant figures. In spite of this, the present writer regards the above analysis as preferable, particularly as some students fail to understand the approximate method.

90. Friction of Pivots and Collars.—Examples of pivot and collar bearings to take axial thrust are shown in Fig. 161.

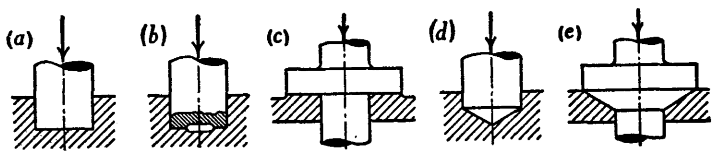


FIG. 161.

A pivot bearing is at the end of a shaft and is also known as a footstep bearing; at (*a*) the pivot is flat, at (*b*) it is recessed, and at (*d*) it is conical. The collar at (*c*) has a flat bearing surface and at (*e*) the bearing surface is conical, but the flat surface is more usual. There may be several collars on a shaft to reduce the intensity of the pressure.

In modern practice ball and roller thrust bearings are generally used where power is being transmitted, and when thrusts are large, as for example on the propeller shafts of ships, they are taken by Michell bearings (Arts. 101 and 102).

An exact theory for a plain thrust bearing cannot be given, as it is uncertain how the pressure is distributed over the rubbing surfaces. In a new bearing the contact may be good over the whole surface, and it is reasonable to assume the pressure is uniformly distributed. Since all parts of the surface are not moving with the same velocity, the wear may be different at different radii and this would cause the pressure distribution to alter; there may also be variations in the coefficient of friction.

Two assumptions will be made, first that the pressure is uniformly distributed, and then that the wear is uniform or proportional to pressure and to velocity. Since velocity is proportional to radius, uniform wear occurs when the product of pressure and radius is constant. In each case it will also be assumed that the coefficient of friction is constant. It will be sufficient to consider a collar bearing with a flat seat and one conical bearing; actually all the cases could be derived from the latter.

91. Collar Bearing — *Uniform Pressure.*—Let P be the axial load, p the uniform pressure per unit area of the rubbing surfaces, and r_1 and r_2 the external and internal radii respectively (Fig. 162), then

$$p=\frac{\mathrm{P}}{\pi(r_1^2-r_2^2)}.$$

The area of a ring of radius r and width δr is $2\pi r\delta r$.

Load on ring is $2p\pi r\delta r$.

Friction torque on ring is $2\mu p\pi r^2\delta r$.

Total friction torque is

$$\mathrm{T}=\frac{2\mu\mathrm{P}}{r_1^2-r_2^2}\int_{r_2}^{r_1}r^2dr=\frac{2\mu\mathrm{P}(r_1^3-r_2^3)}{3(r_1^2-r_2^2)}.$$

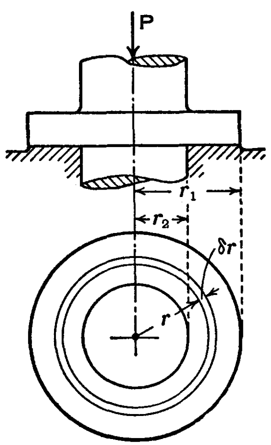


FIG. 162.

For a flat pivot (Fig. 161 (a)) $r_2=0$, then $\mathrm{T}=\frac{2}{3}\mu\mathrm{P}r_1$.

Uniform Wear.—In this case let p be the variable pressure and let $pr=c$.

Load on ring is $2p\pi r\delta r=2\pi c\delta r$.
Total load is

$$P=2\pi c\int_{r_2}^{r_1} dr=2\pi c(r_1-r_2) \quad \text{from which} \quad c=\frac{P}{2\pi(r_1-r_2)}.$$

Friction torque on ring is $2\pi\mu c r\delta r=\dfrac{\mu P r\delta r}{r_1-r_2}$.

Total friction torque is

$$T=\frac{\mu P}{r_1-r_2}\int_{r_2}^{r_1} r\,dr=\frac{\mu P}{r_1-r_2}\cdot\frac{r_1^2-r_2^2}{2}=\tfrac{1}{2}\mu P(r_1+r_2).$$

For a flat pivot (Fig. 161 (a)) $r_2=0$, then $T=\tfrac{1}{2}\mu P r_1$.

It should be noted that the equation $pr=c$ cannot be true when $r=0$, for then p would be infinite. Therefore in this case the value of T is probably too low.

92. Conical Bearing.—Let α be the semi-vertex angle of the cone (Fig. 163), p' be the uniform pressure per unit area of the rubbing surface or, in the case of uniform wear, the variable pressure, and let p be the corresponding uniform or variable pressure on planes perpendicular to the axis.

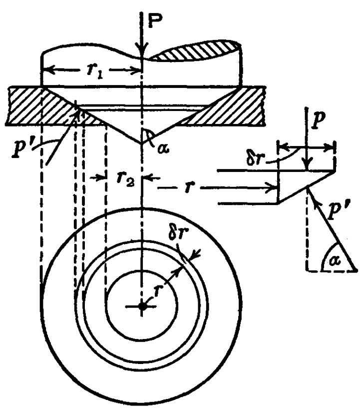


FIG. 163.

Consider a ring of radius r and width δr (see also the enlarged view in Fig. 163). The area of the sloping surface is $2\pi r\delta r/\sin\alpha$, and the force on it is $2p'\pi r\delta r/\sin\alpha$. Resolving forces vertically,

$$2p'\pi r\frac{\delta r}{\sin\alpha}\cdot\sin\alpha=2p\pi r\delta r,$$

therefore $p' = p$ and the pressure is independent of the angle α. Also the area of the rubbing surface of the cone is $1/\sin\alpha$ times that of a flat bearing having the same radial dimensions.

Therefore the formulæ for the conical bearing may be obtained from the collar bearing formulæ by dividing by $\sin\alpha$ in each case.

Uniform Pressure.

$$T = \frac{2\mu P(r_1^3 - r_2^3)}{3(r_1^2 - r_2^2)\sin\alpha}.$$

When $r_2 = 0$, $$T = \frac{2\mu P r_1}{3\sin\alpha}.$$

Uniform Wear.

$$T = \frac{\mu P(r_1 + r_2)}{2\sin\alpha}.$$

When $r_2 = 0$, $$T = \frac{\mu P r_1}{2\sin\alpha}.$$

93. Comparison of the Theories.—For each bearing the ratio of the torque with uniform pressure to the torque with uniform wear is

$$\frac{4(r_1^3 - r_2^3)}{3(r_1^2 - r_2^2)(r_1 + r_2)} = \frac{4}{3}\left\{1 - \frac{r_1 r_2}{(r_1 + r_2)^2}\right\},$$

which has its maximum value when $r_2 = 0$ and is then equal to $\frac{4}{3}$.

If $r_2 = \frac{1}{2}r_1$ the ratio is $\frac{28}{27} = \frac{103\cdot7}{100}$.

If $r_2 = \frac{3}{4}r_1$ the ratio is $\frac{148}{147} = \frac{100\cdot7}{100}$.

As r_2 approaches r_1 the ratio approaches unity.

The above results indicate that it will make little difference which theory is used in practice when r_2 is equal to or greater than $\frac{1}{2}r_1$. In other cases the uniform pressure theory (the larger torque) might be used, for it is probably better to overestimate than underestimate a frictional torque which involves a loss of power.

94. Friction Clutches.—Plate, cone, and centrifugal clutches are discussed in the following Arts., and the theories of the first two types are based on those of pivot and collar bearings. Slip during the engagement of a clutch enables the power transmitted from one shaft to another to be increased gradually until both are revolving at the same speed. In designing a clutch, the maximum torque which can be transmitted without slip has to be determined and, for a plate or a cone clutch, since it is better to underestimate rather than overestimate this torque, it is reasonable to work on the uniform wear theory. It may be arranged in some cases that the torque to produce slip has a value which is up to about 50 per cent. in excess of the maximum working torque, but this figure depends on the type of machine being driven and can be settled only by experience. Some clutches are designed to slip at a predetermined overload in order to protect gears and machines from damage.

95. Plate Clutches.—The plate clutch illustrated in Fig. 164 is one manufactured by Messrs Cooper Roller Bearings Co. Ltd., who kindly supplied the writer with a blueprint from which the drawing was made. It is a multiplate clutch with eight pairs of friction surfaces. Power is transmitted from the driving shaft S_1 to the driven shaft S_2 through two sets of friction plates pressed together by springs. The upper part of the illustration shows the clutch engaged and the lower part shows it disengaged.

The plates B are driven by feathers on the driving boss A which is keyed to the shaft S_1. The driven part of the clutch consists of the boss M keyed to the shaft S_2, the shell L, the cover K, and the plates C and D connected by feathers to the shell L. The plates B and C are arranged alternately, and when the clutch is disengaged they and the plate D are free to slide on the feathers.

The clutch is engaged by pressing the plates together, the pressure being provided by a number of springs P, initially compressed in spring boxes N which are screwed into the cover K. The pressure is applied by sliding the sleeve H to the left along the shaft S_1 (by operating gear not shown) so that the links J push the rollers F radially outwards, between the plate D and the rams G, and slightly increase the compression of the springs. The initial force in a spring depends on the position of the retaining nut R and this is settled by the makers; adjustment is effected

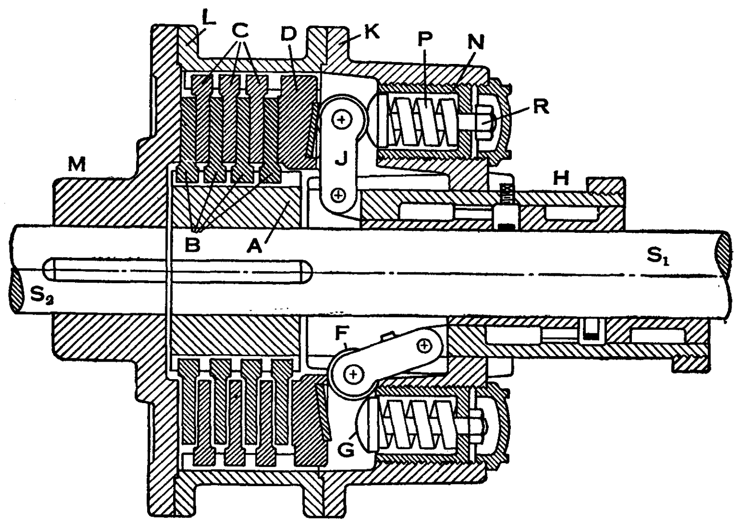


FIG. 164.

by screwing each spring box into the clutch cover K to the required extent. It will be seen on examining the upper spring shown in the Figure that there is no load on the nut R when the clutch is engaged. The clutch is disengaged by moving the sleeve H to the right and removing the pressure from the plates.

When the sleeve H is in the fully engaged position, the centres of the hinged joints between the links J and the sleeve are beyond the centres of the rollers F, and the clutch is held engaged without the application of external force.

The surfaces of the friction plates are kept thoroughly oily by splash lubrication from oil inside the shell L. Also lubrication is provided for the bearings in the sleeve H, which come into action when the clutch is disengaged and the shaft S_1 revolves in the sleeve.

The face of each ram G is spherical and the middle portion of each roller F is made concave to fit the ram; the outer portions are cylindrical and roll on the flat surface of a plate attached to the end plate D. The rollers and the surfaces in contact with them are made of hardened steel. The number of spring boxes varies from two in the smallest clutch up to ten in a clutch having friction plates of 30 inches diameter. The smallest standard clutch is designed to transmit 7·2 horse-power at 100 r.p.m. and the largest 770 horse-power at 100 r.p.m.

The theory of a plate clutch is similar to that given for a collar bearing in Art. 91 where two friction surfaces are in contact. If it is agreed to accept the uniform wear theory (see Arts. 90, 91, and 93), and if there are *n pairs of friction surfaces* in the clutch, the maximum torque T which can be transmitted is

$$T = \tfrac{1}{2} n \mu P (r_1 + r_2),$$

where μ is the coefficient of friction, P is the *total* force provided by all the springs, and r_1 and r_2 are the outside and inside radii respectively of the plates.

In a multiplate clutch the pressure intensity is approximately the same for each pair of surfaces in contact, and its advantage lies in the increased torque which is n times that for a single pair of friction surfaces.

Plate clutches in which the surfaces in contact are Ferodo and metal are shown in Figs. 165 and 166, and the writer is indebted to Messrs Ferodo Ltd. for the pictorial drawings from which these illustrations were prepared. The single plate clutch (Fig. 165) has a casing keyed to the driving shaft, a pressure plate which slides on splines on the inside of the casing, and a plate with a Ferodo facing riveted on each side of it which slides on splines on the driven shaft and is between the casing and the pressure plate. When the clutch is engaged a central spring acts on three levers and causes the Ferodo-faced plate to be held between the pressure plate and the casing. Alternatively there may be a number of springs, equally spaced circumferentially, acting directly on the pressure plate. In this clutch there are two pairs of mating friction surfaces.

Rigid Ferodo clutch facings having considerable mechanical strength and free machining properties are manufactured

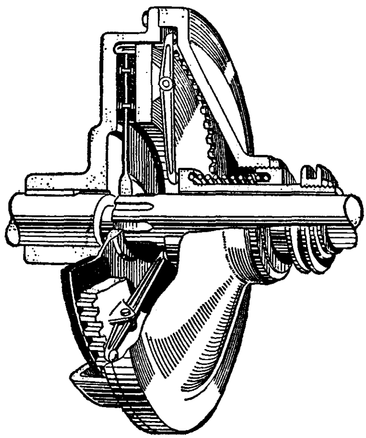


Fig. 165.

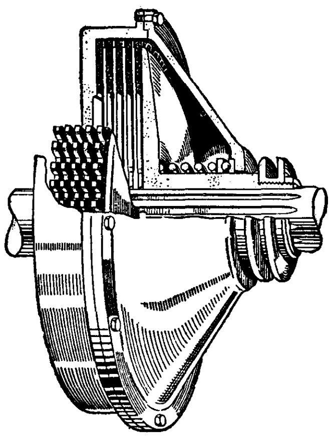


Fig. 166.

for the combined purposes of friction material and clutch plates. The multiplate clutch shown in Fig. 166 has its shell internally splined, preferably with stub gear teeth, and the Ferodo plates are gear cut to correspond. The metal plates have gear cut bores and are mounted on the driven shaft. The number of pairs of mating surfaces in the clutch illustrated is nine; it will be noted that only one side of the Ferodo plate on the left is used.

It is claimed that this design reduces weight and axial dimensions, also that wear of the teeth in the clutch shell is prevented, and of course the riveting of the facings is eliminated. In many cases the Ferodo plates are in sections, so that renewal may be effected without completely dismantling the clutch.

Standard clutch facings are generally unsuitable for running in oil on account of the solvent action of most lubricating oils, but special facings are obtainable which are resistant to this action. The coefficient of friction is considerably less under oil-immersed conditions than with dry plates, and allowance must be made for this in the clutch design.

96. **Cone Clutches.**—Although cone clutches are no longer used in cars and lorries they have many other applications. The sectional plan in Fig 167 illustrates a cone clutch

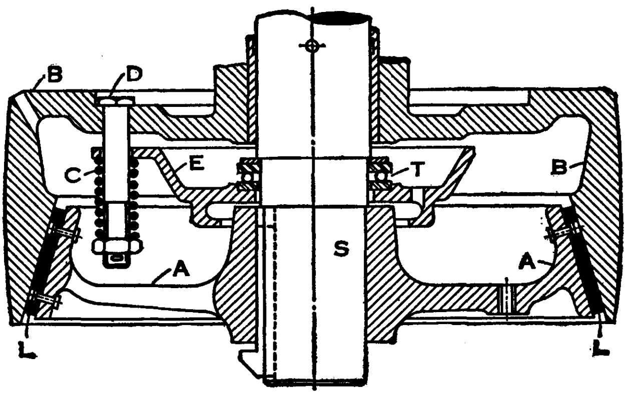


FIG. 167.

manufactured by Messrs Marshall, Sons & Co. Ltd., to whom the writer is indebted for considerable information. It is interesting to note that over 20,000 clutches of this type are in service, some in diesel tractors and others in diesel road rollers.

When a road roller is consolidating a surface there should be no dwell, as this would cause a depression in the material, therefore two clutches are fitted to provide for ahead and reverse movement without gear manipulation. Also it is essential that when a clutch is disengaged it should run free without rubbing and generating heat. This is accomplished more easily with a cone clutch than with one of the plate type.

The clutch for the tractor consists of an inner cone A driven by a shaft S to which it is keyed, and a pulley B which is free to turn on a bush on the shaft. When the clutch is engaged, the interior conical surface of the pulley makes contact with Ferodo lining L riveted to the inner cone A, and pressure between the surfaces is maintained by four helical springs C, one of which is shown in the drawing. Each spring C is compressed between a pressure plate E and a nut on a bolt D, the bolt being fixed in the pulley and passing through a hole in the pressure plate. The springs force the pulley against the Ferodo lining L and tend to push the pressure plate towards the pulley. This thrust is taken by a ball thrust bearing T on the shaft.

When the clutch is disengaged, the pulley is moved axially until there is no contact at the conical surfaces, by gear not shown, and the inner cone continues to rotate with the shaft, but the pulley and the pressure plate come to rest. This retardation is assisted by a clutch brake which has been omitted to simplify the illustration.

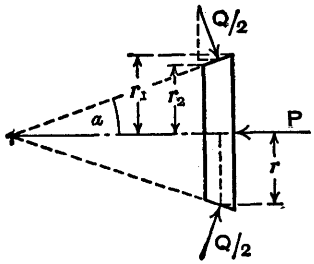


Fig. 168.

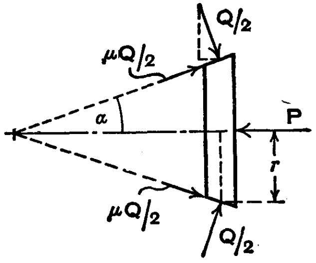


Fig. 169.

The inner member of a cone clutch is shown diagrammatically in Fig. 168, where α is the semi-vertex angle, r_1 and r_2 are the outer and inner radii respectively, r is the mean radius and P is the total axial force. If Q is the normal force distributed over the whole friction surface of the cone, then for equilibrium in Fig. 168 this force may be replaced by two forces $\frac{1}{2}$Q as shown.

Applying the equation for the torque on a conical bearing when the wear is uniform (Art. 92), the maximum torque the clutch will transmit is

$$T=\frac{\mu P(r_1+r_2)}{2\sin\alpha}=\frac{\mu Pr}{\sin\alpha} \quad . \quad . \quad (1).$$

The same result may be obtained immediately from Fig. 168 if the forces $\frac{1}{2}Q$ are assumed to act at the mean radius r. Resolving forces in the direction parallel to the axis,

$$2\times\tfrac{1}{2}Q\sin\alpha=P \quad \text{or} \quad Q=\frac{P}{\sin\alpha}$$

and
$$T=\mu Qr=\frac{\mu Pr}{\sin\alpha}.$$

Another theory sometimes advanced is as follows. In addition to the forces already mentioned, assume that two friction forces $\frac{1}{2}\mu Q$ act along the sloping surface of the cone as indicated in Fig. 169, then resolving forces in the direction parallel to the axis,

$$2(\tfrac{1}{2}Q\sin\alpha+\tfrac{1}{2}\mu Q\cos\alpha)=P$$

from which
$$Q=\frac{P}{\sin\alpha+\mu\cos\alpha}$$

and
$$T=\mu Qr=\frac{\mu Pr}{\sin\alpha+\mu\cos\alpha} \quad . \quad . \quad (2).$$

This theory assumes that limiting friction forces act in two mutually perpendicular directions simultaneously, whereas limiting friction can act only in the direction of any resultant motion. Equation (2) could be modified to allow for this, but although slip may occur in the two directions, the component towards the vertex of the cone would be extremely small and the slip would quickly become entirely circumferential. Therefore it seems probable that equation (1) gives a reasonable value for the maximum torque which can be transmitted.

It is stated in Kent's *Mechanical Engineers' Handbook* * that Prof. H. Bonte of Karlsruhe conducted experiments in 1915 to determine which formula should be used in the design of cone clutches. He found that when the semi-

* 12th Edition, Design and Production Vol., Sec. 17, p. 41.

vertex angle $\alpha = 15°$, the error introduced by using equation (2) was large, but equation (1) gave results which agreed very closely with those obtained by experiment. He concluded that equation (2) was apparently incorrect and should no longer be used in the design of cone clutches.

97. Centrifugal Clutches.—The clutches shown in Figs. 170 and 171 are of the centrifugal type manufactured by Messrs Thomas Broadbent & Sons, Ltd., to whom the writer

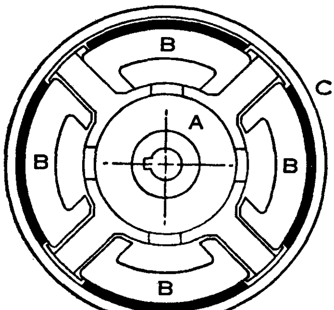


FIG. 170.

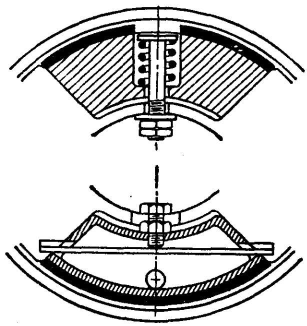


FIG. 171.

is indebted for information. Fig. 170 illustrates a free shoe type clutch. Power is transmitted from a spider A to a pulley C through four free shoes B. Each shoe has a fabric lining on its outer face (represented by a thick black line in the illustration) and is carried in a pocket in the spider. As the spider speed increases, the shoes engage the interior of the rim of the pulley C, which is gradually accelerated until its speed is equal to that of the spider. It will be noted that a gap is shown between the top shoe and the pulley but that the other shoes are touching the pulley; this is because the shoes are free and each is pulled to its lowest position by gravity when the clutch is stationary.

Spring-controlled shoes are also used and two examples of these are shown in Fig. 171. For drives by A.C. single-phase motors and D.C. shunt-wound motors, the shoes are controlled by flat springs which can be adjusted to prevent engagement until the motor has attained any predetermined speed up to 75 per cent. of the maximum, and in this way the starting difficulties are overcome. Shoes controlled by helical springs are used in a narrow-faced clutch designed for operation with motors of small power.

If F lb. is the centrifugal force when a shoe makes contact with the pulley, P lb. is the corresponding spring force, w lb. is the weight of a shoe, r feet is the radius to its centre of gravity, N r.p.m. is the angular speed, and g ft./sec.^{2} is the accelerating effect of gravity, then the force in pounds between a shoe and the pulley is

$$\mathrm{F}-\mathrm{P}=\frac{w}{g}\left(\frac{\mathrm{N}\pi}{30}\right)^2 r-\mathrm{P}.$$

If T lb. ft. is the clutch torque, n is the number of shoes, R feet is the internal radius of the pulley, and μ is the coefficient of friction,

$$\mathrm{T}=n\mu(\mathrm{F}-\mathrm{P})\mathrm{R}=n\mu\left\{\frac{w}{g}\left(\frac{\mathrm{N}\pi}{30}\right)^2 r-\mathrm{P}\right\}\mathrm{R}.$$

When there are no springs, $\mathrm{P}=0$.

98. **Friction Gearing.**—An example of a friction drive between parallel shafts is shown diagrammatically in Fig. 172, where the cylindrical surfaces of two wheels A and B, having diameters d and D respectively, are pressed together by forces P. Theoretically the contact between the wheels is a straight line, but in practice the surfaces are deformed and the contact is over a small area. The driving wheel (say A) has its working surface covered by a continuous fabric facing or lining and the other wheel has a metal surface.

Messrs Ferodo state that fine-grained high tensile cast or alloy iron is the ideal metal surface, and they recommend several linings for driving wheels, depending on the type of duty to be performed by the gearing. It will be realised that when slip occurs, less damage is done by a rotating fabric-lined wheel than by a metal one, and the fabric-lined wheel should be the driver if the risk of forming flats is to be avoided.

The maximum tangential force between the wheels is μP, the corresponding torque required on the driving wheel A, neglecting bearing friction, is

$$T_A = \mu P \cdot \tfrac{1}{2}d.$$

If T_A is in lb. ft. and N r.p.m. is the speed of A, the horse-power transmitted is

$$\frac{2\pi N T_A}{33{,}000} = \frac{\pi N \mu P d}{33{,}000}.$$

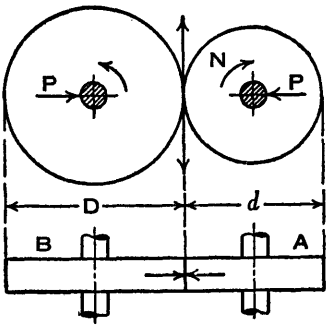


FIG. 172.

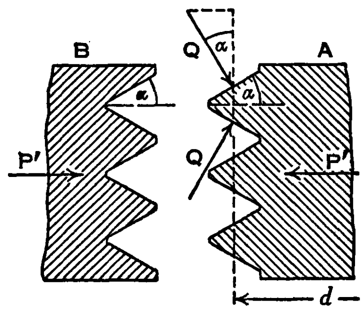


FIG. 173.

Suppose that Vee-shaped grooves are cut circumferentially in the surfaces of the wheels as shown in the enlarged sections in Fig. 173, and clearance is allowed at the bottoms of the grooves so that all the contact is on the sides. It will be shown that this enables the pressure between the wheels to be reduced while the value of T_A remains unchanged. Let P′ be the force perpendicular to the axis of each wheel and let Q be the normal force on each face of each groove. If there are n of these faces on a wheel and 2α is the included angle of each Vee,

$$nQ \sin \alpha = P' \quad \text{or} \quad nQ = \frac{P'}{\sin \alpha}.$$

The maximum torque on wheel A is

$$T_A = \mu n Q \frac{d}{2} = \frac{\mu P' d}{2 \sin \alpha},$$

where d is the mean diameter. But when there are no grooves $T_A = \frac{1}{2}\mu P d$, therefore if the torque is the same in each case and the coefficient of friction is unchanged,

$$\frac{P'}{\sin \alpha} = P \quad \text{or} \quad P' = P \sin \alpha.$$

For example if $\alpha = 15°$, $P' = 0{\cdot}2588P$, or the force between the wheels is reduced to nearly one-quarter of its former value, and this results in less wear on the shaft bearings. Generally the grooved wheels are both of metal, and in this case a smaller coefficient of friction would have to be taken into account. The obvious drawback to the grooved wheels is that there is sliding at all points of contact except at the mean radius, and this causes wear; also the cost would be greater.

A bevel wheel drive is shown in Fig. 174, where the wheels A and B have mean diameters d and D respectively. The axes of the shafts are mutually perpendicular, but they may be inclined to one another at any angle provided the vertices of the cones are at the point of intersection of the axes.

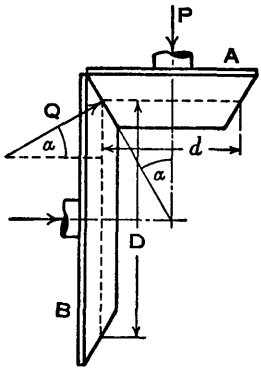


FIG. 174.

It will be assumed that the pressure between the wheels is provided by axial forces and that A is the driving wheel. Let P be the axial force on A and α be the semi-vertex angle of the cone of A, then if Q is the total normal force between the wheels,

$$Q \sin \alpha = P.$$

The maximum torque required on the wheel A is

$$T_A = \mu Q \frac{d}{2} = \frac{\mu P d}{2 \sin \alpha}.$$

The maximum pressure in pounds per inch width recommended by Messrs Ferodo for cylindrical wheels on parallel shafts, or for bevel wheels, is

$$\frac{p_B d}{2\left(1+\frac{d}{D}\right)},$$

where p_B is the basic pressure which varies from 75 to 30 pounds per inch width, depending on the type of fabric, and the diameters d and D are in inches and refer to the friction wheel and the metal wheel respectively, as already defined. The coefficient of friction $\mu = 0{\cdot}25$. In the case of an internal cylindrical wheel drive, the positive sign in the denominator of the above empirical expression is changed to minus.

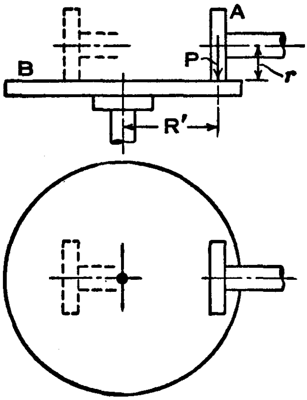


FIG. 175.

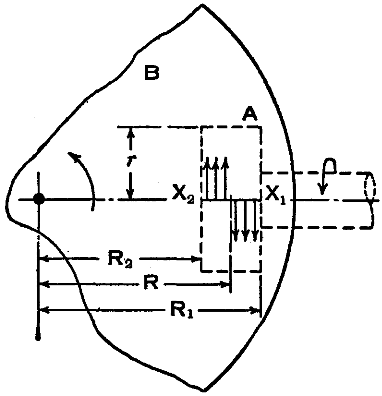


FIG. 176.

In the wheel and disc friction drive shown in Fig. 175 the shafts are mutually perpendicular and the edge of the wheel A is pressed against the face of the wheel or disc B. With this arrangement either shaft may be the driver and the speed of the driven shaft may be altered gradually by moving A axially, towards or away from the centre of B. It will be assumed in the remarks which follow that A is the driver.

If P is the force between the wheels, it will be proved that the maximum torque required on the driving wheel A is $T_A = \mu Pr$ and the corresponding torque given to the disc B is $T_B = \mu PR'$, where r and R' are the radii indicated in Fig. 175, R' being measured to the central plane of A.

In this type of drive, slip occurs the whole time. Consider the enlarged view (Fig. 176) where the wheel A, shown thicker and dotted, makes contact with the disc B along the line X_1X_2 and the points X_1 and X_2 are at radii R_1 and R_2 respectively on B. Assume there is no slip at a point in X_1X_2 at radius R, then the point X_2 on the driving wheel A is moving faster than the corresponding point X_2 on B, and the point X_1 on A is moving more slowly than the point X_1 on B. The arrows perpendicular to X_1X_2 indicate the friction forces acting on the wheel B.

The normal pressure per unit length on X_1X_2 is $\dfrac{P}{R_1 - R_2}$ and the torques are

$$T_A = \frac{\mu P}{R_1 - R_2}\{(R - R_2) - (R_1 - R)\}r$$
$$= \frac{\mu P}{R_1 - R_2}\{2R - (R_1 + R_2)\}r.$$

$$T_B = \frac{\mu P}{R_1 - R_2}\left\{(R - R_2)\frac{(R + R_2)}{2} - (R_1 - R)\frac{(R_1 + R)}{2}\right\}$$
$$= \frac{\mu P}{R_1 - R_2}\left\{R^2 - \frac{R_1^2 + R_2^2}{2}\right\}.$$

If T_A is known, the value of R may be calculated. When T_A is a maximum, $R = R_1$ and X_1 is the point of no slip. Substituting $R = R_1$ in each of the equations and simplifying, the maximum values are

$$T_A = \mu Pr \quad \text{and} \quad T_B = \tfrac{1}{2}\mu P(R_1 + R_2) = \mu PR',$$

where $R' = \frac{1}{2}(R_1 + R_2)$.

In practice the continuous sliding causes wear, resulting in variations in speed and torque, therefore suitable widths for wheel treads can be settled only by experience. Recommended values for widths of Ferodo wheel treads vary from $\frac{3}{4}$ inch for wheel diameters of 4 to 8 inches, to $1\frac{1}{2}$ inches for diameters of 30 inches and over. The maximum permissible pressure in pounds per inch width varies from $\frac{75}{4 \cdot 12}d$ to $\frac{30}{4 \cdot 12}d$, depending on the type of friction material used, where d is the diameter of the friction wheel in inches. The coefficient of friction is $\mu = 0{\cdot}25$.

99. Friction Circle.—A journal or pin A fixed to a wheel B carries a load W and turns clockwise in a bearing C, as shown in Fig. 177 where the clearance is greatly exaggerated.

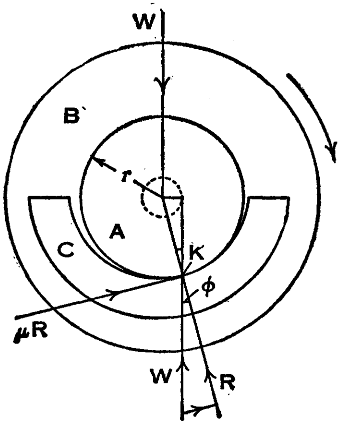


Fig. 177.

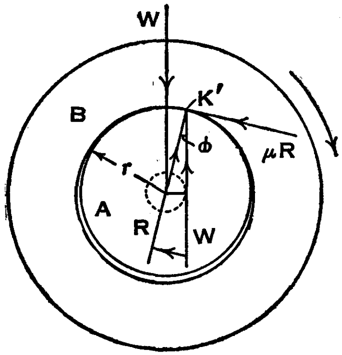


Fig. 178.

It will be assumed that there is line contact and ordinary dry friction between the pin and the bearing. When the pin is at rest the reaction to the load W is vertically below the centre, but during motion it is at K and has components consisting of a normal force R and a tangential force μR opposing motion. Since there can be no resultant force on the pin, the resultant of the forces R and μR, which is inclined at an angle ϕ to the normal, is an upward force W acting parallel to the given load W. The two forces W form an anticlockwise friction couple which opposes motion and is equal to $Wr \sin \phi$, where r is the radius of the pin.

A clockwise couple of the same magnitude would rotate the pin in its bearings at a steady speed.

The circle shown dotted has a radius equal to $r \sin \phi$ and is known as a *friction circle.* The reaction to the load on the pin is tangential to this circle, therefore its line of action may be determined.

If the wheel B turns on the pin A (Fig. 178), contact between B and A is at K′, but the reaction on the moving body is in the same line and has the same magnitude as before. Therefore the friction couple is the same in magnitude and in direction, whether the pin rotates or is fixed.

Since ϕ is small, the values of $\sin \phi$ and $\tan \phi$ are not very different, therefore the friction couple $Wr \sin \phi$ is approximately equal to $Wr \tan \phi$, that is to $Wr\mu$.

Therefore if a shaft of radius r feet is rotating at N r.p.m. and the total load on its bearings is W lb., the horse-power lost in friction is approximately

$$\frac{2\pi NWr\mu}{33{,}000}.$$

100. Friction Axis.—The line of thrust or pull in a moving link which has pin joints at its ends does not coincide with the geometrical axis of the link, the line passing through the centres of the pins; it is along a line called the *friction axis* which is tangential to the two friction circles. Four tangents may be drawn to these two circles, and the position of the one which is the friction axis in a given case depends on the directions of the external forces on the link and on the directions of the motion of this link relative to the links to which it is connected.

Examples are illustrated in Figs. 179 and 180, where the crank and connecting-rod mechanism is shown diagrammatically. In Fig. 179 a thrust P applied to the piston pivoted to the end A of the connecting-rod AB causes the crank CB to rotate clockwise. At A the rod is moving anti-clockwise relatively to the piston against a clockwise friction moment, and this is overcome by the line of thrust towards A in the rod being tangential to the friction circle at a point above its centre. At B the rod is moving anticlockwise relatively to the crank, and the friction moment is overcome by the line of thrust towards B in the rod being tangential to the friction circle at a point below its centre. The friction axis *ab* is obtained by drawing the common tangent to the two friction circles in the position shown.

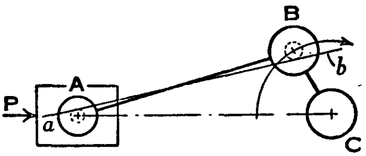


FIG. 179.

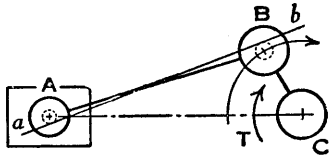


FIG. 180.

In Fig. 180 a torque T turns the crank clockwise and the piston is pulled towards C by the connecting-rod. The relative motions are exactly as before, but the rod is now in tension instead of compression. The pull in the rod at A has to provide an anticlockwise moment and its line of action must be tangential to the friction circle at a point below the centre. The pull in the rod at B provides an anticlockwise moment and its line of action is tangential to the friction circle at a point above its centre. The friction axis *ab* is drawn as shown.

101. Ball and Roller Bearings.—When a loaded steel ball rolls on a smooth steel surface, both are compressed and there are hysteresis losses; also during the rolling a certain amount of slip takes place due to the differences in the changes in the dimensions of the surfaces in contact. The consequent resistance to motion is known as *rolling friction.* Osborne Reynolds developed a theory on this subject and published his conclusions in a paper in 1875, which was reprinted in his *Scientific Papers,* Vol. I. It is interesting to read of the invariable failure of numerous attempts to use india-rubber tyres for wheels, said to be due to the excessive resistance to rolling.

The subject of ball and roller bearings has been widely studied and written about by scientists since those early days, and in particular an extensive work by Dr Arvid Palmgren, based on years of research, contains a great deal of theoretical and practical information.* A few rolling friction coefficients given by Dr Palmgren for rough calculations, which can be applied to normal conditions without serious error, are quoted below.

Values of Rolling Friction Coefficients μ_B

Self-aligning ball bearings	0·0010
Cylindrical roller bearings with flange-guided short rollers	0·0011
Thrust ball bearings	0·0013
Single row deep groove ball bearings . . .	0·0015
Tapered and spherical roller bearings with flange-guided rollers	0·0018
Needle bearings	0·0045

If F is the frictional tangential force, assumed to act at the surface of the bore, and W is the load on the bearing,

$$F = \mu_B W.$$

A ball or roller bearing consists, in general, of two races separated by balls or rollers in a cage. One race is fixed in a housing, the other is secured to the moving part, and the object of the cage is to prevent the balls or rollers rubbing each other. The two main types are journal and thrust bearings to carry radial and axial loads respectively, although some bearings are designed to take both radial and axial loads.

The writer is indebted to the firms mentioned below for their kind co-operation.

The drawings for Figs. 181 to 186 were specially prepared by The Skefko Ball Bearing Co. Ltd. A deep groove ball bearing for radial loading, which will carry some thrust, is shown in Fig. 181, and a double-row self-aligning ball bearing in which the outer track is spherical is represented in Fig. 182, the self-aligning feature being essential when any misalignment is expected. Fig. 183 illustrates a double-row spherical roller bearing intended for heavy

* *Ball and Roller Bearing Engineering,* by Arvid Palmgren, Dr Eng., published by SKF Industries, Inc., Philadelphia. 2nd Edition 1946.

loads, as in rolling mills, railway axle-boxes, etc. The rollers are barrel-shaped and their axes converge to two

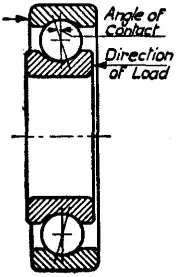


FIG. 181.

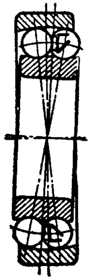


FIG. 182.

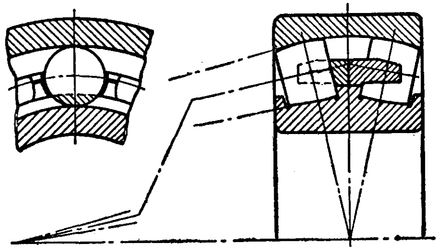


FIG. 183.

points on the main axis; in addition to radial loads this bearing can take thrust loads in either direction.

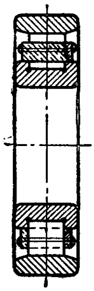


FIG. 184.

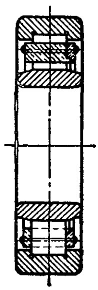


FIG. 185.

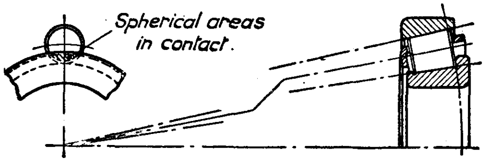


FIG. 186.

Cylindrical roller bearings are made with either a flanged inner race (Fig. 184) or a flanged outer race (Fig. 185), but the former is more common. The tapered roller bearing (Fig. 186) takes thrust in one direction in addition to radial loads.

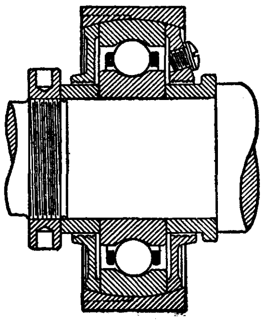


FIG. 187.

Two of the bearings made by The Hoffman Manufacturing Co. Ltd. are illustrated in Figs. 187 and 188. The single-row self-aligning ball journal, wide type (Fig. 187), has the periphery of the outer race accurately ground spherical to fit into a seating of corresponding shape in an outer shell or housing. As this bearing is of the same internal construction as the single-row rigid bearing and the self-aligning feature does not affect the standard curvature of the ball tracks, it has the same load-carrying capacity as the corresponding standard rigid ball journal.

The single-thrust bearing with spherical seating ring, complete with housing (Fig. 188), has the races placed side by side and the balls, separated from one another by means of a cage, are sandwiched between them. One race is provided with a spherical seating face which rests on a correspondingly shaped seating ring to compensate for out of squareness of the abutment of this race. These bearings are also supplied with both races having flat seatings; those for very light duties having flat tracks.

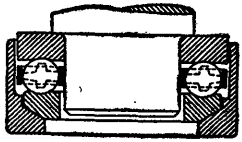


FIG. 188.

A split roller bearing complete with a self-aligning housing and pedestal, manufactured by Messrs Cooper Roller Bearings Co. Ltd., is shown in Fig. 189. The inner race is clamped to the shaft and the outer race is held in a housing which is spherically seated in the pedestal base and cap. The split cage containing the rollers can also be seen. It will be noticed that the joints of the two halves of the races are inclined to the axis of the shaft so the rollers will run over the junctions as though the races were not split. The obvious advantage with this type of bearing is that it does not have to be passed over a long length of shaft, on which there may be obstacles, to reach its working position.

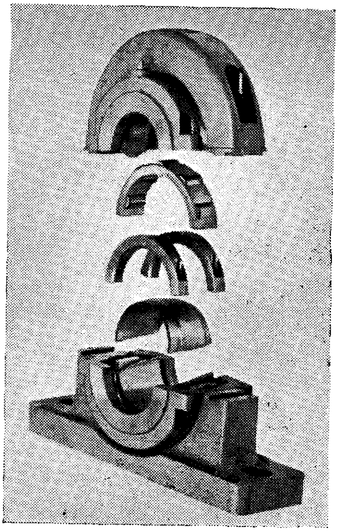
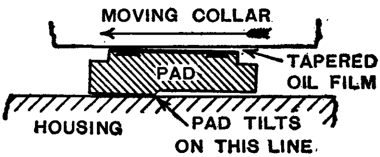


FIG. 189.

102. Michell Bearings.*—The Michell bearing makes use of the principle that bearing loads should be carried completely by the lubricating oil, instead of the oil acting merely to reduce friction between metallic surfaces. The bearing takes its name from the inventor, Mr A. G. M. Michell.

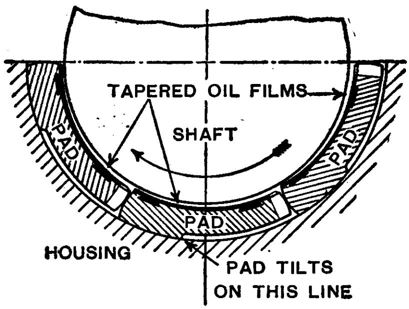
In all Michell bearings there are two elements: the shaft (either journal or thrust collar), and the stationary parts which are pivoted and known as *pads*. These two elements do not come into contact when running, being forcibly separated and kept apart by automatically generated tapered oil films drawn from the normal oil supply. There is no metallic friction, no wear, and no renewal of parts so long as good clean oil is present in sufficient volume to carry the load.

Figs. 190 and 191 illustrate the action in a Michell thrust block and a Michell journal bearing respectively. The tapered pressure oil film, or wedge of lubricant, is self-generated by the motion of the shaft or collar and is not dependent on any extraneous pressure from an oil pump. The pads are so designed that they tilt and float the shaft on the oil film when a suitable relative speed has been attained. They have white metal faces, as white metal is less liable to damage from foreign matter which may be present in the oil.



THRUST PAD

FIG. 190.



JOURNAL PAD

FIG. 191.

In the thrust bearing, as the thrust collar revolves in its oil bath, the oil adhering to its surface is carried

* Based on information obtained from Messrs Michell Bearings Ltd. For further information on these and other bearings, see *Lubrication*, by A. G. M. Michell, F.R.S. Blackie & Son, Ltd.

round and lifts every pad at its leading edge to admit the tapered oil film. Thus every pad generates a tapered pressure oil film of a thickness appropriate to the load, the speed, and the viscosity of the lubricating medium. For maximum efficiency, the pivot is offset from the centre of the circumferential width of the pad, and this offset is right-handed or left-handed to suit the direction of rotation. When necessary, the pads can be pivoted centrally to suit both directions of rotation, with a slight reduction in efficiency.

The Michell thrust bearing is a simple single-collar unit capable of carrying at least twenty times the load per square inch of a flat multi-collar thrust bearing. Thrust shafts may be disposed at any angle and standardised designs are available.

The Michell journal bearing usually has six pads surrounding the shaft journal. Each pad is free to tilt, and is prevented from cross-winding by suitable flanges. Oil is automatically introduced between each pair of pads from an annulus in the housing.

Load-carrying Capacity.—The working load depends on several factors, mainly diameter, length, peripheral speed, and viscosity of the oil. The capacity increases with the speed, and loads of several thousand pounds per square inch have been sustained in prolonged tests. For bearings which do not start under full load, up to 500 lb. per sq. in. may be used, but 350 lb. per sq. in. is taken as the limit when starting under full load. However, it is to be noted that bearings will work satisfactorily with considerable overloads.

Friction.—Experiments with a Michell bearing loaded to 560 lb. per sq. in. gave a coefficient of friction $\mu = 0{\cdot}0020$, and the calculated figure was 0·0022. The coefficient of friction of a good ordinary bearing is $\mu = 0{\cdot}036$, *about eighteen times as much.*

Exercises X

1. A block B weighing 100 lb. is pulled along a horizontal plane by a force P inclined at an angle θ to the horizontal (Fig. 192). Show that

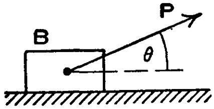


Fig. 192.

$$\frac{P}{100}=\frac{\sin\phi}{\cos(\theta-\phi)},$$

where ϕ is the angle of friction.

Given that the coefficient of friction $\mu=0{\cdot}3$, find the minimum value of P and the corresponding value of θ.

2. A block weighing 500 lb. rests on a plane which is inclined at 10° to the horizontal and the coefficient of friction between the surfaces in contact is 0·25. Find the force required to push the block up the plane if its line of action is (*a*) parallel to the plane and (*b*) horizontal.

3. In Figs. 193 and 194 the body is moving down the plane at a uniform rate. If ϕ is greater than α (Fig. 193), show that

$$\frac{P}{W}=\frac{\sin(\phi-\alpha)}{\cos\phi}$$

and

$$Pl+Wh=\mu Wb.$$

If ϕ is less than α (Fig. 194), show that

$$Wh=\mu Wb+Pl.$$

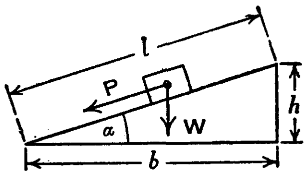


Fig. 193.

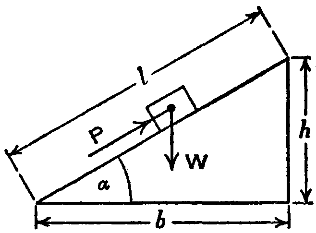


Fig. 194.

4. A square-threaded screw, $1\frac{1}{2}$ inches external diameter, $\frac{1}{4}$ inch pitch, is used to operate a valve $4\frac{1}{2}$ inches diameter against a pressure of 200 lb. per sq. in. Find the torque required to close the valve if the coefficient of friction is 0·15. [Inst. C.E.]

5. Show that the efficiency of a square-threaded screw is

$$\frac{\tan \alpha}{\tan (\alpha+\phi)}=\frac{\sin \alpha \cos (\alpha+\phi)}{\cos \alpha \sin (\alpha+\phi)}=\frac{1-\dfrac{\sin \phi}{\sin (2\alpha+\phi)}}{1+\dfrac{\sin \phi}{\sin (2\alpha+\phi)}}$$

and that its maximum value occurs when $\alpha=45^\circ-\frac{1}{2}\phi$ and is

$$\frac{1-\sin \phi}{1+\sin \phi} \quad \text{or} \quad \frac{\tan (45^\circ-\frac{1}{2}\phi)}{\tan (45^\circ+\frac{1}{2}\phi)}.$$

6. Express the efficiency of a square-threaded screw in the form

$$\frac{p(\pi d-\mu p)}{\pi d(p+\mu\pi d)},$$

and show that it is a maximum when

$$\frac{p}{\pi d}=\sqrt{\mu^2+1}-\mu=\frac{1-\sin \phi}{\cos \phi}.$$

Use this trigonometrical value of $p/\pi d$ to obtain the maximum efficiency in the forms quoted in Ex. 5.

7. Derive an expression for the efficiency of a square thread of mean diameter d and pitch p, the coefficient of friction being μ.

Compare the magnitudes of the torques required to raise and to lower a screw jack when carrying a given load, if the screw has a single-start square thread of $\frac{1}{2}$ inch pitch with an outside diameter of $2\frac{1}{4}$ inches. The coefficient of friction is 0·02. [U.L.]

8. An axial load of 4 tons is carried by a 3-inch B.S. Whitworth screw which is vertical and rotated in a fixed nut by a tangential force Q applied to a lever at a point 30 inches from the screw axis. The minor or inside diameter of the screw is 2·634 inches, there are 3·5 threads per inch, and the angle of the Vee is 55° measured in an axial plane. Assuming the load rotates with the screw and is being raised, find the value of the force Q in pounds. Take $\mu=0{\cdot}15$ and $n=1/\cos \beta$. (See Art. 89.)

9. If a 3-inch square-threaded screw having 3·5 threads per inch is used in the preceding exercise, what would be the value of Q? Take the thread depth to be equal to half the pitch.

10. A turnbuckle with right- and left-hand threads is used to couple two railway coaches. The threads, which are square, have a pitch of 0·5 inch on a mean diameter of 1·5 inches and are of the single-start type. Taking the coefficient of friction as 0·1, find the work to be done in drawing the coaches together a

distance of 6 inches, (*a*) against a steady load of 0·25 ton; (*b*) if the load increases uniformly over the 6 inches from 0·25 to 0·75 ton. [U.L.]

11. A flat collar bearing having an external diameter of 6 inches and an internal diameter of 3 inches supports an axial load of 1000 lb. If the coefficient of friction is 0·05, find the friction torque, assuming (*a*) uniform pressure, (*b*) uniform wear.

12. The loading device of a small testing machine consists of a spur wheel, driven through suitable gearing, which acts as a nut on the vertical square-threaded screw which applies the load directly to the test-piece. The screw has an outside diameter of 3 inches and a single thread of $\frac{3}{4}$ inch pitch; and the thrust is taken by a collar bearing 8 inches outside and 4 inches inside diameter. The coefficient of friction for both screw and thrust bearing is 0·15. Find the torque on the pinion required to apply a load of 2 tons to the test-piece, assuming that the pressure is uniformly distributed on the thrust bearing. [Inst. C.E.]

13. In the calculation of the power lost in a footstep bearing it is sometimes assumed that the wear at a point on the bearing surface is proportional to the product of the pressure at the point and the rubbing velocity there, and it is also assumed that the wear is constant all over the surface. On this basis, obtain the power lost in friction in the footstep bearing of a vertical shaft, the bearing consisting of a flat pad pressing against the end of the shaft.

The shaft runs at 120 r.p.m., has a weight of 30 cwt., a diameter of 5 inches, and the coefficient of friction between the surfaces is 0·1.

Do you consider that the assumption of "constant wear" is reasonable in practice? [Inst. C.E.]

14. A multiplate clutch is used to connect two shafts in line. In this, one set of plates can slide axially in a shell attached to one shaft, while the other set of plates can slide along the second shaft. Sketch the arrangement showing an operating mechanism for pressing the plates together.

If the inner and outer diameters of the contact surfaces are 3·5 inches and 5·5 inches respectively and there are six contacts, find the axial thrust required to transmit 10 H.P. at 750 r.p.m. The coefficient of friction is 0·3, and it is assumed that the contact pressure times radius is a constant over each surface. Find also the contact pressures at the inner and outer radii. [I.Mech.E.]

15. The movable jaw of a bench vice is at the upper end of a hinged arm 18 inches long, the centre line of the screw being 14 inches above the hinge. The screw is 1 inch outside diameter, 4 threads per inch, square thread. The mean radius of the thrust collar is $1\frac{1}{4}$ inches. Find the tangential force to be applied to the screw at a radius of 11 inches, to produce a force of 1200 lb. at the jaw. Coefficient of friction on screw and thrust collar is 0·08.

What is the mechanical efficiency of the vice? [U.L.]

16. A friction clutch is required to transmit 45 H.P. at 2000 r.p.m. It is to be of single-plate disc type with both sides of the plate effective, the pressure being applied axially by means of springs and limited to 10 lb. per sq. in.

If the outer diameter of the plate is to be 12 inches, find the required inner diameter of the clutch ring and the total force exerted by the springs. Assume the wear to be uniform and a coefficient of friction of 0·3. [U.L.]

17. A cone clutch in which the semi-vertex angle is 15° is to transmit 10 H.P. at 450 r.p.m. when on the point of slipping.

(*a*) If the mean radius of the inner member is 5·5 inches and the coefficient of friction is 0·3, find the total spring force required.

(*b*) Calculate the width of the friction surface if the normal pressure is limited to 10 lb. per sq. in.

18. Assume that in the cone clutch shown in Fig. 167, p. 179, the mean diameter of the internal member is 15·25 inches, the semi-vertex angle is 15° and the total spring load is 600 lb.

(*a*) Find the horse-power which can be transmitted at 750 r.p.m. if the torque is to be two-thirds of the torque which would produce slip and $\mu = 0{\cdot}3$.

(*b*) If the width of the lining is equal to one-fifth of the mean diameter of its friction surface, find the normal pressure in lb. per sq. in.

19. (*a*) Find what the width of the lining should be in the cone clutch shown in Fig. 167, p. 179, if the mean diameter of the friction surface is 17·5 inches, the semi-vertex angle is 15°, the total spring force is 500 lb., and the normal pressure between the surfaces is 20 lb. per sq. in.

(*b*) Calculate the speed at which 35 H.P. can be transmitted when the torque is equal to two-thirds of the torque which would produce slip. Take $\mu = 0{\cdot}3$.

20. A centrifugal friction clutch (Fig. 195) has a driving member consisting of a spider carrying four shoes which are kept from contact with the clutch case by means of the flat springs until increase of centrifugal force overcomes the resistance of the springs and power is transmitted by friction between shoes and case.

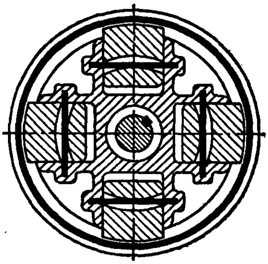


FIG. 195.

Determine the necessary weight of each shoe if 30 H.P. is to be transmitted at 750 r.p.m., with engagement beginning at 75 per cent. of the running speed. The inside diameter of the drum is 12 inches and the radial distance of the centre of gravity of each shoe from the shaft axis is 5 inches. Assume a coefficient of friction of 0·25. [U.L.]

21. A centrifugal clutch has four blocks which slide radially in a spider keyed to the driving shaft and make contact with the internal cylindrical surface of a drum keyed to the driven shaft. When the clutch is at rest each block is pulled against a stop by a spring so as to leave a radial clearance of $\frac{1}{4}$ inch between the block and the drum. The pull exerted by the spring is then 100 lb. and the mass-centre of the block is 8 inches from the axis of the clutch.

If the internal diameter of the drum is 20 inches, the weight of each block is 15 lb., the stiffness of each spring is 200 lb. per in. and the coefficient of friction between block and drum is 0·3, find the maximum horse-power the clutch can transmit at 500 r.p.m. [I.Mech.E.]

22. If in the wheel and disc friction drive (Figs. 175 and 176), the disc B drives the wheel A, prove that the maximum power is transmitted when the point of no slip is at X_2 and that the values of the torques are then

$$T_B = \tfrac{1}{2}\mu P(R_1 + R_2) \quad \text{and} \quad T_A = \mu P r.$$

Show also that when A drives B the maximum efficiency is $\dfrac{R_1 + R_2}{2R_1}$ and when B drives A it is $\dfrac{2R_2}{R_1 + R_2}$.

23. A lifting jack with differential screw threads is shown diagrammatically in Fig. 196. The portion B screws into the fixed base C and carries a right-handed square thread of pitch 0·375 inch, the mean diameter of the thread being 2·25 inches. The part A is prevented from rotating and carries a right-handed thread of 0·25 inch pitch on a mean diameter of 1·25 inches, screwing into the part B. If the coefficient of friction for each thread is 0·15, find the torque necessary to be applied to the part B to raise a load W of 1000 lb. [U.L.]

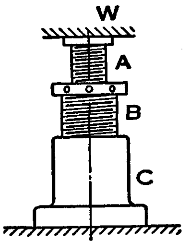


FIG. 196.

24. A sluice gate weighing 6 tons is subjected to a normal pressure of 250 tons. It is raised by means of a vertical screw which engages with a screwed bush fixed to the top of the gate. The screw is rotated by a 50 b.h.p. motor running at a maximum speed of 600 r.p.m., a bevel pinion on the motor shaft gearing with a bevel wheel of 80 teeth keyed to the vertical screw. The screw is 5 inches mean diameter and 1 inch pitch. The coefficient of friction for the screw in the nut is 0·08 and between the gate and its guides is 0·10.

If friction losses, additional to those mentioned above, amount to 15 per cent. of the total power available, determine the maximum number of teeth for the bevel pinion. [U.L.]

CHAPTER XI

TURNING MOMENTS AND FLYWHEELS

103. Piston Effort.—In a crank and connecting-rod mechanism operated by a piston, the axial force on the piston at any instant is known as the *piston effort,* and this may be regarded as positive when it drives the crank and negative when it opposes motion.

In a double-acting heat engine, piston effort is the difference between the forces on the two sides of a piston. These varying forces are calculated from indicator diagrams in which the pressures, usually in lb. per sq. inch, are automatically plotted vertically on a stroke base. If at any instant the pressures on the opposite sides of the piston are p_1 and p_2 respectively and the corresponding areas are a_1 and a_2, then the opposing forces are p_1a_1 and p_2a_2.

When a cylinder is vertical, the piston effort is increased on the down stroke by a force equal to the weight of the reciprocating parts, and decreased by an equal amount on the up stroke.

104. Inertia Forces of Reciprocating Parts.—From Art. 27, p. 42, the acceleration of a piston is given approximately by

$$f=\omega^2 r\left\{\cos\theta+\frac{r}{l}\cos 2\theta\right\},$$

where r is the length of the crank, l is the length of the connecting-rod, θ is the angle turned through by the crank from its position at the beginning of the stroke (crank and connecting-rod in line, but not overlapping), and ω is the angular velocity of the crank. Graphical methods for obtaining the piston acceleration are given in Arts. 32 to 34.

If M is the mass and W is the weight of the reciprocating parts, the inertia force is approximately

$$F = Mf = \frac{W}{g}\omega^2 r\left\{\cos\theta + \frac{r}{l}\cos 2\theta\right\}.$$

The inertia effect of the connecting-rod is considered in detail in Arts. 110 to 113. A proportion of the mass m of the connecting-rod, equal to $m\frac{GB}{AB}$, may be assumed to reciprocate with the piston, where AB is the length of the rod between centres, G is its centre of gravity and B is at the crank pin (Fig. 197). In some cases this mass is roughly one-third of the mass m. When desired, the angular acceleration of the rod may also be taken into account, but it is neglected in this chapter until Art. 110 is reached.

The reciprocating parts are accelerated from rest at the beginning of each stroke until the piston attains its maximum velocity and then retarded gradually until the end of the stroke is reached. Work is done to give kinetic energy to the reciprocating parts as the velocity increases, and this energy is given out as the velocity decreases. Therefore the inertia force F has to be subtracted from the piston effort; the alteration of sign during the stroke is taken care of by the equation for F. The total work done and the mean effort during a stroke are not affected by the inertia forces of the reciprocating parts.

The algebraic difference between the piston effort and the inertia force will be called the *net piston effort*. In solving any problem it should be stated whether or not the reciprocating part of the connecting-rod has been allowed for in the inertia force; this affects the value of the net piston effort as defined here.

105. Crank Effort.—If P is the net piston effort and Q is the force in the connecting-rod, Q may be resolved in two directions at the crank pin B, along the crank BC and perpendicular to it (Fig. 197). The component P_C perpendicular to BC is the *crank-pin effort*, and the other component produces a thrust or pull in the crank. The product $P_C r$, where $r = BC$, is the *crank effort*, or the *torque* or *turning moment* on the crankshaft. From Fig. 197,

$$P = Q \cos \phi \quad \text{and} \quad P_C = Q \sin (\theta + \phi),$$

therefore $$\frac{P_C}{P} = \frac{\sin (\theta + \phi)}{\cos \phi} = \frac{\sin (\theta + \phi)}{\sin (90^\circ - \phi)}.$$

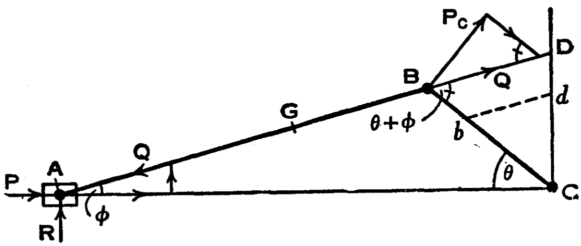


Fig. 197.

Draw CD perpendicular to AC to intersect AB produced at D, then from the triangle BCD,

$$\frac{\sin (\theta + \phi)}{\sin (90^\circ - \phi)} = \frac{CD}{CB},$$

therefore $$\frac{P_C}{P} = \frac{CD}{CB}$$

or $$P_C . CB = P . CD.$$

Therefore, for any value of θ, the crank effort is the product of the net piston effort P and the length CD.

If the net piston effort is represented by Cb marked off along the crank and bd is drawn parallel to AB to cut CD at d, then, since the triangles BCD and bCd are similar, the length Cd represents the crank pin effort.

The relation between the net piston effort and the crank pin effort may also be obtained very simply by equating the rates of work at A and B, since these are equal when friction is neglected. Consider Fig. 198

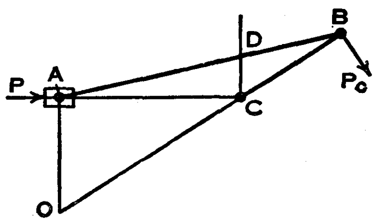


Fig. 198.

where O is the instantaneous centre for the connecting-rod when it is in the position shown (Art. 17, p. 19). Let v_a and v_b be the velocities of the points A and B respectively, then

$$Pv_a = P_C v_b$$

or

$$\frac{P_C}{P} = \frac{v_a}{v_b} = \frac{OA}{OB} = \frac{CD}{CB}$$

and

$$P_C . CB = P . CD.$$

106. Turning Moment Diagram.—An indicator diagram from a single-acting four-stroke gas engine is given at the top of Fig. 199, and it is required to draw the turning moment, crank effort, or torque diagram on an angle base, as shown at the bottom. The original drawings were made as accurately as possible to larger scales, but the procedure will be explained without reference to numerical values. The data are referred to in Ex. 5, p. 219.

Vertical lines are drawn on the indicator diagram corresponding to 15° angular intervals of the crank and numbered from 0 to 12 on the out stroke; on the return stroke the lines are numbered from 12 to 23 and the final line is at the starting point 0.

The piston effort diagram, due to the varying gas pressure, is obtained by multiplying each ordinate on the indicator diagram by the piston area and plotting the results on an angle base. Since there are two revolutions in one cycle, the base is divided into 48 equal parts and each stroke corresponds to 12 of these parts. During the suction, compression, and exhaust strokes, the motion of the piston is opposed by the gas forces, therefore the piston effort due to these forces is plotted as negative.

The inertia force is calculated at 15° intervals from 0° to 180° from the equation in Art. 104 (the weight of the reciprocating part of the connecting-rod is added to the main reciprocating weight) and the curves are plotted as shown. The curve from 180° to 360° is the same as the curve from 180° to 0° inverted, and of course the second revolution is identical with the first. At the beginning of each stroke the inertia force hinders motion, or is negative, and towards the end of the stroke it assists motion, or is positive. It will be noted that these curves are all inverted to enable the net piston effort to be measured direct from the diagram.

The crank effort or torque diagram is obtained by

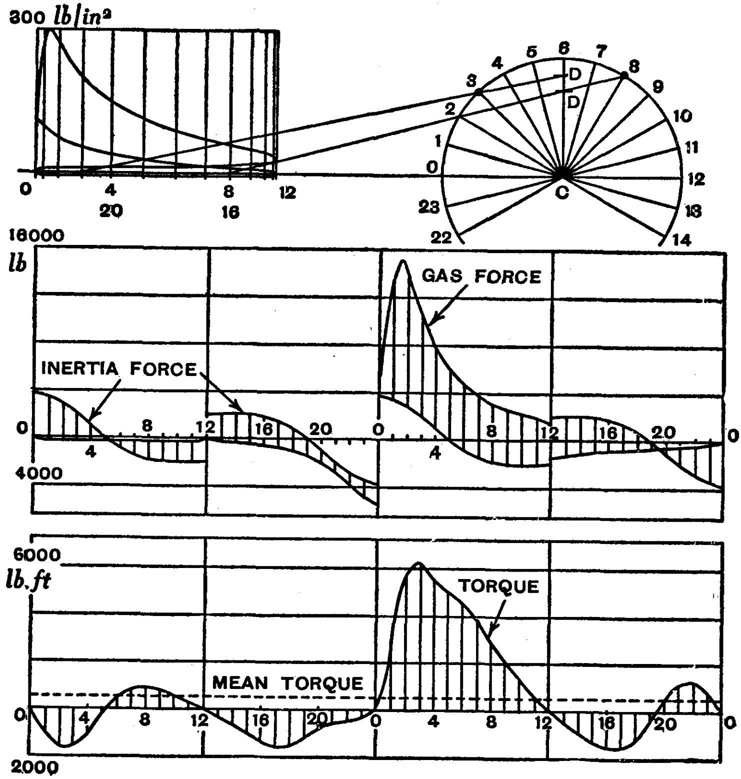


Fig. 199.

multiplying each net piston effort ordinate by the corresponding value of CD. To avoid confusion only two positions of the point D are shown—when the crank pin is at the points 3 and 8. The length CD corresponding to point 3 is used with ordinate 3 on the suction stroke and ordinate 3 on the firing stroke. The same length CD is used with ordinate 21 on the compression stroke and on the exhaust stroke, since ordinates 3 and 21 on the indicator diagram are one and the same ordinate. Similarly the length CD corresponding to point 8 is used with ordinates 8 and 16. At the ends of each stroke, CD is of course zero.

The mean torque is found by dividing the net area of the torque diagram, which is positive above the base-line and negative below it, by the length of the base-line. One method is to measure the positive and negative areas with a planimeter and divide the difference by the length of the base-line; another method is to divide by 48 the algebraic sum of the 44 ordinates. Actually there are 48 intervals and 49 ordinates, but five of the ordinates are zero.

107. Fluctuation of Energy.—In the example illustrated in the preceding Art. the turning moment or torque varies considerably during the cycle, and this variation may be reduced by having an engine with two or more cylinders. In practice the resisting torque to be overcome is generally nearly uniform and may be taken as equal to the mean crank effort, represented by the horizontal dotted line in Fig. 199.

Suppose the ordinates of the curve KL (Fig. 200) represent the total crank effort of an engine during one cycle, plotted on an angle base OH, and that the ordinates of the horizontal line MN represent the resisting torque. The work done during the cycle is represented by the area bounded by the crank effort curve, the base-line and the extreme ordinates at O and H, and also by the area of the rectangle OMNH. Where the crank effort curve is above the line MN the supply of energy exceeds the demand, and where it is below the line MN the demand exceeds the supply. This lack of equality between supply and demand is taken care of by corresponding changes in the kinetic energy of the moving parts.

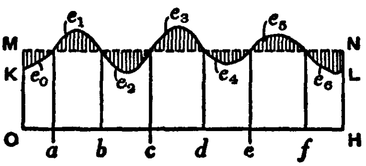


FIG. 200.

Let E be the kinetic energy of the moving parts when the crank effort and the resisting torque are equal and represented by the ordinate at a; let the areas labelled e_0, e_1, e_2, etc. denote amounts of work which are alternately negative and positive.

At ordinate b the energy in the moving parts is $E+e_1$, at ordinate c it is $E+e_1-e_2$, at ordinate d it is $E+e_1-e_2+e_3$, and so on.

The angular speed of the cranks is a maximum when the kinetic energy is greatest and a minimum when the kinetic energy is least.

The energy of the moving parts is fluctuating continuously, and the *maximum fluctuation of energy* is the difference between the values of the kinetic energy at the maximum and minimum speeds.

The *coefficient of fluctuation of energy* is the ratio of the maximum fluctuation of energy to the work done per cycle. Numerical values of this coefficient should be treated with caution, because the definition is sometimes based on the work "per revolution" or on the work "per stroke."

Example.—Given that the fluctuations of energy in ft. lb. in a turning moment diagram for one cycle are as indicated in Fig. 201 and that the work done during the cycle is 1400 ft. lb., to find the maximum fluctuation of energy and the coefficient of fluctuation of energy.

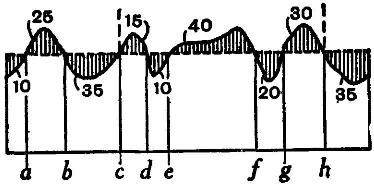


Fig. 201.

This example has been designed to show that the maximum fluctuation of energy is not necessarily proportional to the largest single area above the mean torque line.

If E is the kinetic energy of the moving parts at the ordinate at a, then their kinetic energy at each of the other ordinates is as follows:—

At b,	$E+25$.	At f,	$E-5+40=E+35$.
c,	$E+25-35=E-10$.	g,	$E+35-20=E+15$.
d,	$E-10+15=E+5$.	h,	$E+15+30=E+45$.
e,	$E+5-10=E-5$.	a,	$E+45-35-10=E$.

The greatest kinetic energy is $E+45$ and occurs at ordinate h, the least is $E-10$ at ordinate c. The maximum fluctuation of energy is

$$E+45-(E-10)=55 \text{ ft. lb.},$$

or, considering the separate values between the ordinates c and h,

$$15-10+40-20+30=55 \text{ ft. lb.}$$

The angular speed is a maximum at ordinate h and a minimum at ordinate c.

The coefficient of fluctuation of energy is

$$\frac{55}{1400}=0{\cdot}039.$$

108. Internal Combustion Engine.—In a single-cylinder single-acting four-stroke engine, the work done by the gases during the firing stroke overcomes the external resistance and the friction of the engine, and also increases the kinetic energy of the flywheel on the crankshaft. During the other three strokes the work is done by the flywheel. The maximum fluctuation of energy is obtained from a crank effort diagram drawn on an angle base, as in Fig. 199.

Since the maximum and minimum speeds of the crankshaft occur when the piston is near the end and the beginning respectively of the working stroke, it may be assumed, as an approximation, that the maximum fluctuation of energy is equal to the work done during the working stroke minus one-quarter of the net work done during the cycle.

The net work per cycle is the work done by the gases during the working stroke minus the work done on the gases in the other three strokes; it is also equal, in foot-pounds, to the indicated horse-power multiplied by 33,000 and divided by the number of cycles per minute.

109. Flywheels.—The object of using a flywheel is to reduce the fluctuation of speed during each cycle of an engine or machine. Since the turning moment on an engine crankshaft fluctuates whilst the external resisting moment remains nearly uniform, the speed of the flywheel and the kinetic energy stored in it increase when the supply of energy exceeds the demand and decrease when the demand exceeds the supply.

By providing a suitable flywheel, the variation in the speed can be kept within desired limits. Generally the kinetic energy of the other moving parts is neglected, and the procedure is to equate the maximum fluctuation of energy to the change of kinetic energy of the flywheel.

The difference between the maximum and minimum speeds during a cycle is called the *fluctuation of speed*. The ratio of the fluctuation of speed to the mean speed is called the *coefficient of fluctuation of speed*.

Let W lb. be the weight of the flywheel, k ft. the radius of gyration, and I the moment of inertia about the axis of rotation. Let ω_1 and ω_2 be the maximum and minimum speeds respectively, and ω the mean speed, in radians per second. Let N_1, N_2, and N be the corresponding values in revolutions per minute.

It is usually assumed that

$$\omega = \tfrac{1}{2}(\omega_1 + \omega_2) \quad \text{or} \quad N = \tfrac{1}{2}(N_1 + N_2).$$

The coefficient of fluctuation of speed is

$$c = \frac{\omega_1 - \omega_2}{\omega} = \frac{N_1 - N_2}{N}.$$

$$\begin{aligned}
\text{Max. fluctuation of energy} &= \text{Change of kinetic energy} \\
&= \tfrac{1}{2}I(\omega_1^2 - \omega_2^2) \\
&= \tfrac{1}{2}I\left(\frac{2\pi}{60}\right)^2(N_1^2 - N_2^2) \\
&= I\frac{\pi^2}{900}\,\frac{(N_1 - N_2)(N_1 + N_2)}{2} \\
&= I\frac{\pi^2}{900}\,\frac{(N_1 - N_2)}{N}N^2 \\
&= \frac{Wk^2}{g}\,\frac{\pi^2}{900}cN^2.
\end{aligned}$$

Example.—Given that the maximum fluctuation of energy is 85 ft. lb. and the mean speed is 1400 r.p.m., to find the weight of the flywheel so that the coefficient of fluctuation of speed is 1 per cent. The radius of gyration is to be 6 inches.

$$\frac{Wk^2}{g}\frac{\pi^2}{900}cN^2=85.$$

$$W=\frac{85g\times 900}{k^2\pi^2cN^2}$$

$$=\frac{85\times 32{\cdot}2\times 900}{0{\cdot}5^2\pi^2\times 0{\cdot}01\times 1400^2}$$

$$=50{\cdot}9 \text{ lb.}$$

110. Dynamically Equivalent Connecting-Rod.—In considering the inertia forces of reciprocating parts (Art. 104), a proportion of the mass of the connecting-rod was included. The inertia effect of the connecting-rod will now be examined in detail, and in doing this it is convenient to assume the mass of the rod is replaced by a dynamically equivalent system consisting of two concentrated masses. Two such arrangements will be considered.

Let a connecting-rod AB, of length l between centres and moment of inertia I and radius of gyration k about an axis through its centre of gravity G, perpendicular to the plane of motion, have its mass m replaced by a mass m_A at A and a mass m_X at X (Fig. 202). Let AG $=a$, GB $=b$, and GX $=x$.

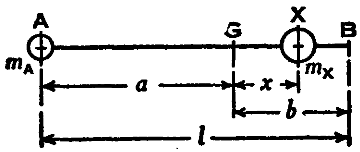


FIG. 202.

For the systems to be dynamically equivalent to one another,

$$m_A+m_X=m,$$

$$m_A a=m_X x,$$

$$I=m_A a^2+m_X x^2=mk^2,$$

that is, the total mass, the centre of gravity, and the moment of inertia about the axis through G must be unchanged.

From these equations,

$$m_A = \frac{mx}{a+x}, \qquad m_X = \frac{ma}{a+x},$$

$$I = m_A a^2 + m_X x^2 = \frac{m}{a+x}(xa^2 + ax^2) = max.$$

But $I = mk^2$, therefore $ax = k^2$ or $x = \frac{k^2}{a}$.

The length x may be calculated or obtained graphically as shown in Fig. 203. Draw GH perpendicular to AB and make $GH = k$. Join AH and draw HX at right angles to AH to intersect AB at X, then X is the required point.

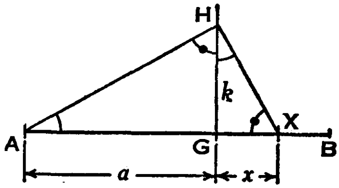


FIG. 203.

From similar triangles,

$$\frac{a}{k} = \frac{k}{x}, \quad \text{therefore} \quad ax = k^2.$$

It will be seen that the position of one of the two concentrated masses could be selected anywhere in the line AB, and the corresponding position of the other mass and the value of each mass could be calculated.

Suppose now that the mass m of the connecting-rod is replaced by concentrated masses at A and B. Let these masses be m_1 at A and m_2 at B (Fig. 204).

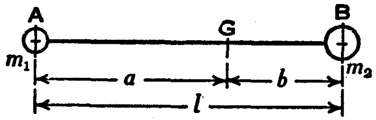


FIG. 204.

If the total mass and the centre of gravity are unchanged,

$$m_1 + m_2 = m \quad \text{and} \quad m_1 a = m_2 b,$$

therefore $$m_1 = m\frac{b}{l}, \qquad m_2 = m\frac{a}{l}$$

and the moment of inertia about the axis through G becomes

$$I' = m_1 a^2 + m_2 b^2 = \frac{m}{l}(ba^2 + ab^2) = mab,$$

which differs from $I = mk^2$, since $k^2 = ax$ and x is not, in general, equal to b.

The moment of inertia I' is too large by an amount

$$I' - I = m(ab - k^2).$$

Suppose the rod with masses concentrated at A and B is given a clockwise angular acceleration a, by an applied torque T_1, then

$$T_1 = I'a = maba,$$

and this is greater than the torque $T = mk^2a$ which would be required to give the original rod the same angular acceleration.

By assuming that an anticlockwise effective couple $m(ab - k^2)a$ acts on the two-mass rod (Fig. 205), it becomes dynamically equivalent to the original rod, for, if T is the clockwise torque which will produce the acceleration a,

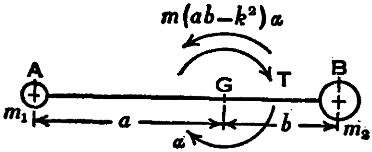


FIG. 205.

$$\begin{aligned} T &= I'a - m(ab - k^2)a \\ &= maba - m(ab - k^2)a \\ &= mk^2a. \end{aligned}$$

111. Inertia Effect of a Connecting-Rod.—In the slider-crank mechanism (Fig. 206) ω and Ω are the angular

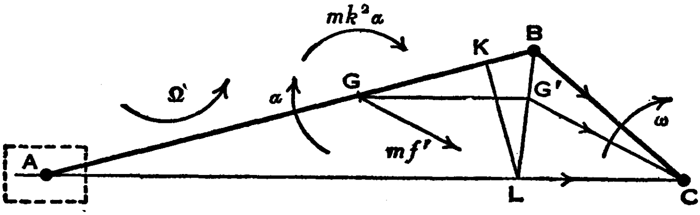


FIG. 206.

velocities of the crank BC and connecting-rod AB respectively, ω being uniform and Ω variable, a is the angular acceleration of AB, f is the acceleration of A, and

f' is the acceleration of the centre of gravity G of AB. There is an effective couple $I\alpha = mk^2\alpha$ on the rod AB and an effective force mf' at G.

To obtain the acceleration f' of the point G, draw the line KL by Klein's construction (Arts. 32 and 33, pp. 50–52) and join LB, then $f = \omega^2$LC is the acceleration of A, ω^2BC is the acceleration of B, and LB is the acceleration image of the rod AB. Draw GG′ parallel to AC to intersect LB at G′ and join G′C, then the acceleration of G is $f' = \omega^2$G′C, along a line drawn parallel to G′C through G.

The angular acceleration of AB is $\alpha = \omega^2 \frac{LK}{AB}$ (Art. 32).

There are various ways of determining the turning moment required at the crank to overcome the inertia effects of the connecting-rod, and an analytical method is considered first. Assume that the mass m of the rod is replaced by masses m_1 and m_2 at A and B respectively (Fig. 207), and that an effective couple $m(ab - k^2)\alpha$ acts on the rod in the opposite direction to that of the acceleration α, as explained in the preceding Art. The two-mass rod is then dynamically equivalent to the original rod.

The mass m_1 has an acceleration f along AC with the reciprocating parts; the mass m_2 has an acceleration along BC and therefore has no effect on the turning moment.

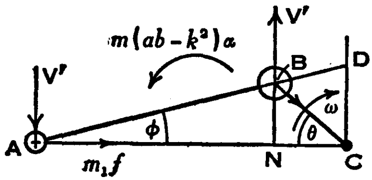


FIG. 207.

To produce the effective couple $m(ab - k^2)\alpha$, assume that parallel and opposite forces V′ act *on the rod* at A and B, as shown, perpendicular to AC. Drop a perpendicular BN on to AC, then AN $= l \cos\phi$ and NC $= r\cos\theta$, where $l =$ AB and $r =$ BC.

Equating the applied and effective couples acting on the rod,

$$V'l\cos\phi = m(ab - k^2)\alpha$$

and

$$V' = \frac{m(ab - k^2)\alpha}{l\cos\phi}.$$

The turning moment required at the crank to overcome the total inertia effect of the connecting-rod is

$$m_1 f.\mathrm{CD} + \mathrm{V}'.\mathrm{NC}$$

or $$m_1 f.\mathrm{CD} + \frac{m(ab - k^2)\alpha}{l \cos \phi} r \cos \theta \quad . \quad . \quad (1).$$

As an approximation the second term is often neglected, and in this case the first term may be dealt with by adding m_1 to the mass of the reciprocating parts. This procedure was followed in obtaining the turning moment diagram in Fig. 199, Art. 106.

If the value of the second term is calculated, an expression for $\dot{\Omega}$ or $\frac{d^2\phi}{dt^2}$ is quoted in Ex. 21, p. 45, and $\alpha = -\dot{\Omega}$.

The turning moment to overcome the total inertia effect of the rod may be put into another form, as follows.

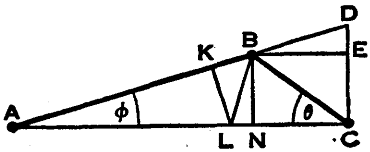


FIG. 208.

In Fig. 208 the lines KL, BL, BN, and CD are drawn as already explained, then BE is drawn parallel to AC to cut CD at E. AB $=l$ and BC $=r$.

Since $\alpha = \omega^2 \frac{\mathrm{LK}}{\mathrm{AB}} = \omega^2 \frac{\mathrm{AL} \sin \phi}{l}$ and $r \cos \theta = \mathrm{NC} = \mathrm{BE}$,

$$\frac{\alpha r \cos \theta}{l \cos \phi} = \omega^2 \frac{\mathrm{AL} \sin \phi}{l} \cdot \frac{\mathrm{BE}}{l \cos \phi} = \frac{\omega^2 \mathrm{AL}}{l^2} \mathrm{BE} \tan \phi = \frac{\omega^2 \mathrm{AL}}{l^2} \mathrm{DE}.$$

Also $$f = \omega^2 \mathrm{LC}.$$

Therefore the turning moment required at the crank to overcome the total inertia effect of the rod is

$$m_1 \omega^2 \mathrm{LC}.\mathrm{CD} + \frac{m(ab - k^2)}{l^2} \omega^2 \mathrm{AL}.\mathrm{DE} \quad . \quad (2).$$

If the gravity effect on the rod is to be taken into account, let w_2 be the weight corresponding to the mass m_2 at the crank pin B. The additional turning moment required, assuming the line of stroke to be horizontal, is $w_2 r \cos \theta$.

112. Inertia Effect of Connecting-Rod — Graphical Solution.—A graphical method for determining the turning moment required at the crank to overcome the total inertia effect of the connecting-rod is shown in Fig. 209.

Assume the mass m of the rod is replaced by a mass m_A at A and a mass m_X at X, as explained in Art. 110, the point X being found as shown in Fig. 203, then the two-mass rod is dynamically equivalent to the original rod. Find

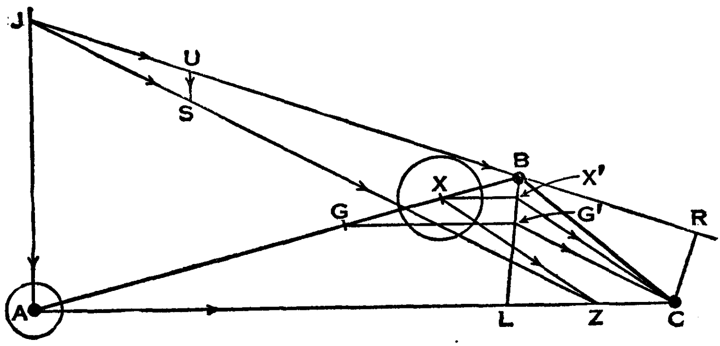


Fig. 209.

the point L by Klein's construction and join BL. Draw GG′ and XX′ parallel to AC, to cut BL at G′ and X′ respectively. Join G′C and X′C, then $m\omega^2$G′C is the total effective force on the rod and $m_X\omega^2$X′C is the effective force at X.

Draw XZ parallel to X′C to cut AC at Z, then the effective force at X acts along XZ. Also the effective force at A is $m_A\omega^2$LC and it acts along AZ. It follows that $m\omega^2$G′C, the resultant of the effective forces at A and X, must act through Z.

Therefore draw ZJ parallel to CG′ to cut at J a line AJ drawn perpendicular to AC and join JB. Then an applied force along JZ, equal to $m\omega^2$G′C, would give the rod the necessary effective force at G and the effective couple mk^2a. As a matter of interest it may be pointed out that the perpendicular distance from G to the line JZ is $\dfrac{k^2a}{\omega^2 G'C}$.

The applied force on the rod along JZ is produced by a force at A along JA and a force at the crank pin B along JB. From J mark off JS = G′C along JZ to represent $m\omega^2$G′C and draw SU parallel to AJ to cut JB at U, then JU and US are the components of JS. Finally draw CR at right angles to JB, intersecting JB produced at R.

The force to be applied along JB at the crank pin B is equal to $m\omega^2$JU, and the turning moment required at the crank to overcome the total inertia effect of the rod is

$$m\omega^2\text{JU}\,.\,\text{CR},$$

the lengths JU and CR being measured with the linear scale.

Suppose, for example, m is in pounds, the drawing is half full size, the actual lengths of JU and CR are measured in inch units, and the turning moment is to be in pound-feet. Divide m by 32·2; multiply each length by 2 and divide each by 12.

113. Forces on Engine Frame due to Connecting-Rod Inertia.—To find the forces acting on the engine frame, due to the inertia effect of the connecting-rod, consider again the effective couple and the effective forces acting on the rod AB (Fig. 210) when its mass m is assumed to be replaced by masses m_1 and m_2 concentrated at A and B respectively. The corresponding applied forces acting on the rod may be indicated as shown in Fig. 211.

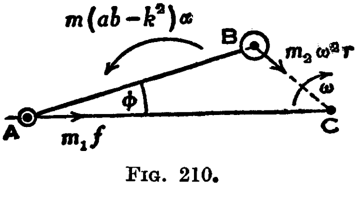


FIG. 210.

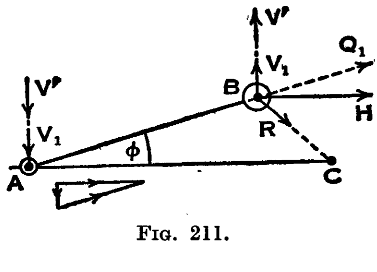


FIG. 211.

The parallel and opposite forces V′ produce the effective couple (Art. 111), and

$$\text{V}' = \frac{m(ab - k^2)\alpha}{l \cos \phi}.$$

The radial force along BC on the mass m_2 is

$$R = m_2\omega^2 r.$$

A force Q_1, provided by the crank, along the rod AB, and a downward force V_1 at A, perpendicular to AC, give the required force along AC on m_1. At B the force Q_1 is shown resolved into the components V_1 and H which are respectively perpendicular and parallel to AC.

$$V_1 = m_1 f \tan\phi \quad \text{and} \quad H = m_1 f.$$

Let $V = V_1 + V'$, then

$$V = m_1 f \tan\phi + \frac{m(ab - k^2)a}{l\cos\phi}.$$

It follows that the forces on the engine frame, due to the inertia effect of the connecting-rod, are as shown in Fig. 212. The force R may be balanced by a revolving mass fixed to an extension of the crank BC, then there remains an unbalanced force H and a couple V.AC.

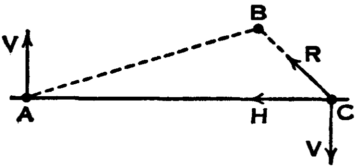


FIG. 212.

The effect of the inertia of the reciprocating parts may be taken into account by adding their mass to the mass m_1 at A when obtaining the values of V and H from the equations given above. (See also Arts. 194 and 195, on primary and secondary balance.)

Exercises XI

1. In a direct-acting horizontal steam engine the cylinder diameter is 18 inches and the stroke 24 inches, the difference in pressure on the two sides of the piston is 120 lb. per sq. in., the weight of the reciprocating parts 350 lb., and the speed 250 r.p.m. The connecting-rod is 5 feet long. Find the crank effort when the crank is 45° after inner dead centre. [U.L.]

2. A vertical internal combustion engine has a cylinder bore of 7 inches and a stroke of 8 inches. The speed is 500 r.p.m. The connecting-rod is 16 inches long and the weight of the parts moving with the piston is 45 lb. On the working stroke the gas pressure is 176 lb. per sq. in. when the piston has moved downwards a distance corresponding to a rotation of 30° of the crank. Determine graphically the velocity and acceleration of the piston for this position. Also find the turning moment exerted on the crankshaft, taking into account the weight and inertia of the piston. [I.Mech.E.]

3. Prove that the relation between the crank pin effort P_C and the net piston effort P, $P_C . CB = P . CD$, obtained in Art. 105, is also true if the line of stroke is offset from the crankshaft axis.

Find the value of the crank effort in pound-feet given that the offset is $\frac{1}{2}$ inch, the lengths of the connecting-rod and crank are respectively $AB = 10{\cdot}5$ inches and $CB = 3$ inches, and $P = 1000$ lb. when the crank angle $\theta = 30°$ measured from a datum line parallel to the line of stroke (as in Fig. 61, p. 61).

4. The axis of the crankshaft of a vertical, single cylinder, four-stroke petrol engine is offset $\frac{3}{4}$ inch from the line of stroke so as to reduce the side thrust of the piston on the firing stroke. The crank is $2\frac{1}{2}$ inches and the connecting-rod is $9\frac{1}{2}$ inches long, the cylinder is $3\frac{1}{2}$ inches diameter, the weight of the reciprocating parts is 3 lb. and the speed is 2400 r.p.m.

If, when the crank has turned through 60° from the t.d.c., the pressure of the gases in the cylinder is 160 lb. per sq. in. gauge, find the effective turning moment on the crankshaft.

[I.Mech.E.]

5. The indicator diagram in Fig. 199, p. 206, is from a single-acting four-stroke gas engine running at 220 r.p.m. The bore is 8 inches, the stroke is 20 inches, the connecting-rod is 34 inches long between centres, and its centre of gravity is 10·5 inches from the big-end. The piston weighs 180 lb. and the connecting-rod weighs 90 lb. Assume that the part of the mass of the rod which reciprocates with the piston may be calculated by taking moments about the big-end (Art. 104, p. 202).

Replot the indicator diagram on a larger scale as accurately as possible (the suction pressure is -5 lb. per sq. in.) on a base 10 inches long. Draw curves of piston effort and inertia force on a scale of 1 inch to 4000 lb. and a torque curve on a scale of 1 inch to 2000 lb. ft. Compare the curves with those in Fig. 199.

6. A shaft is driven by a motor which exerts a constant torque while the load taken off the shaft fluctuates from zero to a maximum during each revolution. The load cycle can be defined by a graph of load-torque plotted against the angle turned through by the shaft, reckoned from the instant at which the torque is zero. If the scales chosen for such a diagram are 1 inch = 10 lb. ft. of torque and 1 inch = 60° of shaft angle, the graph is an equilateral triangle, the torque being zero at 0° and 360° of shaft angle.

Find the moment of inertia of the flywheel to be mounted on the shaft so that the speed fluctuation shall not be more than 3 r.p.m. above or below a mean speed of 200 r.p.m.

[Inst. C.E.]

7. A turning moment diagram for an engine is drawn to the following scales:—

Turning moment	1 in. = 1000 lb. in.
Crank angle	1 in. = 30°.

The turning moment diagram repeats itself every $\frac{1}{2}$ revolution of the engine, and the areas above and below the mean torque line, taken in order, are 0·39, 0·82, 0·08, 0·44, 1·15, 0·36 sq. in.

The rotating parts are equal to a mass of 75 lb. at a radius of 6 inches. Determine the coefficient of fluctuation of speed if the engine runs at 1200 r.p.m. [Inst. C.E.]

8. A 20 H.P. gas engine working on the four-stroke cycle runs at 200 r.p.m., firing every cycle. The speed is not to vary more than $\pm 2\frac{1}{2}$ per cent.

Assuming that the maximum fluctuation of energy is three-quarters of the useful work from one explosion, find the moment of inertia of the flywheel required. Suggest a suitable cross-section for the rim, assuming it to be 5 feet mean diameter and made of cast iron weighing 0·26 lb. per cu. in. [U.L.]

9. An electric motor capable of 360 H.P. drives the flywheel of a rolling mill at 90 r.p.m. The mill takes a peak load of 450 H.P. for 10 seconds only. If the radius of gyration of the wheel is 7 feet, what should be its weight if the drop in speed on overload is not to exceed 10 per cent. of its normal speed?

[Inst. C.E.]

10. A shaft fitted with a flywheel rotates at 250 r.p.m. and drives a machine, the resisting torque of which varies in a cyclic manner over a period of three revolutions. The torque rises from 500 lb. ft. to 2000 lb. ft. in a uniform manner during $\frac{1}{2}$ revolution and remains constant for the following 1 revolution. It then falls uniformly to 500 lb. ft. during the next $\frac{1}{2}$ revolution and remains constant for 1 revolution, the cycle being then repeated.

If the driving torque applied to the shaft is constant and the flywheel has a weight of 1000 lb. and a radius of gyration 2 feet, find the horse-power necessary to drive the machine and the percentage fluctuation of speed. [U.L.]

11. A flywheel used in a rolling-mill drive weighs 6 tons and its radius of gyration is 4 feet. It was observed that during the passage of a billet through the mill, the speed of the flywheel decreased from 120 r.p.m. to 90 r.p.m. in 15 seconds. Find the loss of kinetic energy in this time and the average torque due to the flywheel. [Inst. C.E.]

12. A machine press is worked by an electric motor, delivering 3 H.P. continuously. At the commencement of an operation, a flywheel of moment of inertia 1200 lb. ft.^{2} on the machine is rotating at 250 r.p.m. The pressing operation requires 3500 ft. lb. of energy and occupies 0·75 sec. Find the maximum number of pressings that can be made in 1 hour and the reduction in speed of the flywheel after each pressing. Neglect friction losses. [U.L.]

13. A connecting-rod AB, of length 12·5 inches between centres, has a mass of 6½ lb. The centre of gravity G is 9·1 inches from A, and the radius of gyration about an axis through G, perpendicular to the plane of motion, is $k=4{\cdot}82$ inches.

(a) A dynamically equivalent two-mass rod is to have one mass m_A at A and the other mass m_X at a point X between G and B. Find the length GX in inches and the values of m_A and m_X in pounds.

(b) If the masses of the dynamically equivalent rod are to be m_1 at A and m_2 at B, and the rod is to be given a clockwise angular acceleration $\alpha=2600$ rad./sec.^{2}, find the values of m_1, m_2, and the necessary anticlockwise effective couple $m(ab-k^2)\alpha$.

14. A connecting-rod AB is connected to a piston at A and driven by a crank BC. Find the torque required at the crank-shaft to overcome the inertia of the connecting-rod when the crank angle ACB is 60°, given the following particulars.

AB $=12{\cdot}5$ in., BC $=3{\cdot}5$ in.; total mass of rod is $m=6{\cdot}5$ lb., the reciprocating part of this mass is $m_1=1{\cdot}77$ lb. at A, and the revolving part is m_2 at B; the radius of gyration about an axis through the centre of gravity G, perpendicular to the plane of motion, is $k=4{\cdot}82$ in.; AG $=9{\cdot}1$ in. and N $=1500$ r.p.m.

It is recommended that the torque should be obtained in three ways: from each of expressions (1) and (2), Art. 111, and by the graphical method of Art. 112.

15. Prove that the area intercepted between one positive portion of the curve $y=h\sin\theta$, and the line $y=\lambda h$, is

$$2h\left[\sqrt{1-\lambda^2}-\frac{\lambda}{2}(\pi-2\sin^{-1}\lambda)\right].$$

Hence, assuming the crank effort diagram of a single cylinder steam engine to consist of the positive portions of two equal sine curves, prove that the flywheel must be designed to store and return 20 per cent. of the work done in one stroke.

Prove also that for a two-cylinder engine with cranks at right angles, the flywheel must store and return 4 per cent. of the work done in one stroke of one cylinder. [C.U.]

CHAPTER XII

BELT AND ROPE DRIVES

114. Length of Belt Connecting Two Pulleys on Parallel Shafts.—Let the symbols be as shown in Figs. 213 and 214, and let the angles α and β be in radians.

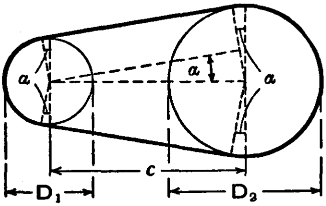


FIG. 213.

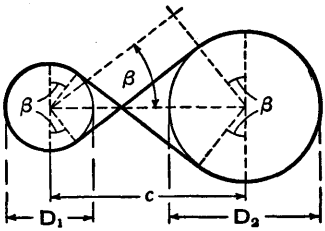


FIG. 214.

Open Belt (Fig. 213).—Let l_o be the total length of the open belt, then

$$l_o = \tfrac{1}{2}D_1(\pi - 2\alpha) + \tfrac{1}{2}D_2(\pi + 2\alpha) + 2\sqrt{c^2 - (\tfrac{1}{2}D_2 - \tfrac{1}{2}D_1)^2}$$
$$= \tfrac{1}{2}\pi(D_1 + D_2) + \alpha(D_2 - D_1) + \sqrt{4c^2 - (D_2 - D_1)^2},$$

and $\quad \sin\alpha = \dfrac{\frac{1}{2}(D_2 - D_1)}{c} \quad$ or $\quad \alpha = \sin^{-1}\left(\dfrac{D_2 - D_1}{2c}\right)$ radian.

Crossed Belt (Fig. 214).—Let l_c be the total length of the crossed belt, then

$$l_c = \tfrac{1}{2}D_1(\pi + 2\beta) + \tfrac{1}{2}D_2(\pi + 2\beta) + 2\sqrt{c^2 - (\tfrac{1}{2}D_1 + \tfrac{1}{2}D_2)^2}$$
$$= \tfrac{1}{2}(\pi + 2\beta)(D_1 + D_2) + \sqrt{4c^2 - (D_1 + D_2)^2},$$

and $\quad \sin\beta = \dfrac{\frac{1}{2}(D_1 + D_2)}{c} \quad$ or $\quad \beta = \sin^{-1}\left(\dfrac{D_1 + D_2}{2c}\right)$ radian.

If $D_1 + D_2$ and c are constants, the angle β is constant, therefore the length l_c of the crossed belt is constant, that is to say, the crossed belt will fit any number of pairs of pulleys on two parallel shafts provided the sum of the diameters of each pair has the same value.

115. Velocity Ratio of Pulleys.—Suppose that power is transmitted by an open belt from one pulley to another as shown in Fig. 215, and that the diameters of the driver and follower are D_1 and D_2 respectively. Let the corresponding speeds be N_1 and N_2 in revolutions per unit time, and let the belt speed be v. In the Figure the *tight side* of the belt, the length in which the tension is greatest, is below the pulleys, and the *slack side*, the length in which the tension is least, is above the pulleys.

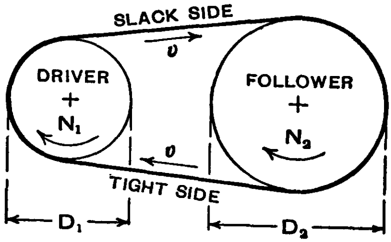


FIG. 215.

Assuming there is no slip, and that the belt is very thin and inextensible,

$$v = \pi D_1 N_1 = \pi D_2 N_2,$$

and the velocity ratio of the pulleys is

$$\frac{N_1}{N_2} = \frac{D_2}{D_1}.$$

If the belt is crossed, the direction of rotation of the follower is of course reversed.

When a belt is bent over a pulley the outer surface is stretched and the inner surface is compressed, but the neutral or middle layer does not alter in length. During motion, the outer and inner surfaces of the belt travel at different speeds when on a pulley, since they are at different radii, but they travel at one speed when between the pulleys; the middle layer travels at a uniform speed throughout its journey. (This statement is not strictly true on account of creep, which is discussed in Arts. 120 and 121.) It follows that the effective radius of each pulley is the radius to the middle layer of the belt.

Let t be the belt thickness, then the effective diameters of the pulleys are D_1+t and D_2+t. Therefore, if the belt thickness is taken into account,

$$\frac{N_1}{N_2}=\frac{D_2+t}{D_1+t}.$$

For example, suppose $D_1=12$ inches, $D_2=24$ inches, and $t=\frac{1}{4}$ inch, then $N_1/N_2=2$ or $1{\cdot}98$ according as t is neglected or included. In practice t is generally neglected.

116. Ratio of Belt Tensions.—Consider a belt *which is about to slip* round a fixed pulley in the anticlockwise direction (Fig. 216), over an arc of contact AB which subtends an angle θ radians at the centre of the pulley; this angle is called the *angle of embrace* or *angle of lap*. Let the tensions be T_1 at B and T_2 at A, T_1 being greater than T_2. Let mn be a very short length of the belt subtending an angle $\delta\theta$ at the centre of the pulley, and let the tensions be T at m and $T+\delta T$ at n. Let the resultant of the radial force distributed on the length mn of the belt be P, and let the coefficient of friction between the belt and the pulley be μ.

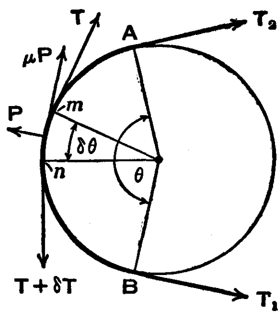


FIG. 216.

Since the belt is about to slip, resolving along the tangent at the middle point of mn,

$$(T+\delta T)\cos\frac{\delta\theta}{2}-T\cos\frac{\delta\theta}{2}=\mu P.$$

When $\delta\theta$ is very small, $\cos\frac{\delta\theta}{2}$ is approximately equal to 1, therefore

$$(T+\delta T)-T=\mu P$$

or

$$\delta T=\mu P \quad . \quad . \quad . \quad (1).$$

Resolving along the radius at the middle point of mn,

$$P=T\sin\frac{\delta\theta}{2}+(T+\delta T)\sin\frac{\delta\theta}{2}.$$

When $\delta\theta$ is very small, $\sin\frac{\delta\theta}{2}$ is approximately equal to $\frac{\delta\theta}{2}$ and $\delta T\frac{\delta\theta}{2}$ is negligible, therefore

$$P = T\delta\theta \quad . \quad . \quad . \quad . \quad (2).$$

From equations (1) and (2),

$$\delta T = \mu T\delta\theta.$$

Therefore
$$\int_{T_2}^{T_1}\frac{dT}{T} = \int_0^{\theta}\mu d\theta.$$

Integrating,
$$\log_e\frac{T_1}{T_2} = \mu\theta$$

or
$$\frac{T_1}{T_2} = e^{\mu\theta},$$

where $e = 2{\cdot}71828\ldots$ is the base of Napierian logarithms and, as already stated, θ is in radians.

If the belt is not about to slip round the pulley, the tension ratio may have any value between 1 and $e^{\mu\theta}$. This ratio is discussed further in Art. 120, where creep is considered.

Example.—Given that the maximum tension is 350 lb., the angle of embrace is 170°, and the coefficient of friction is 0·27, it is required to find the value of the minimum tension, assuming the belt is about to slip.

From the formula the minimum tension is

$$T_2 = \frac{T_1}{e^{\mu\theta}}.$$

Substituting numerical values,

$$\theta = \tfrac{170}{180}\pi \text{ radians}, \qquad \mu\theta = 0{\cdot}27 \times \tfrac{17}{18}\pi = 0{\cdot}255\pi,$$

$$e^{\mu\theta} = 2{\cdot}7183^{0\cdot255\pi} = 2{\cdot}228,$$

and
$$T_2 = \frac{350}{2{\cdot}228} = 157 \text{ lb.}$$

117. Ratio of Rope or V Belt Tensions.—The section of a rope in a V groove on a pulley is shown in Fig. 217.

Let R be the resultant normal reaction on each of the surfaces of contact over a short length of the groove, corresponding to the arc *mn* in Fig. 216 in the preceding Art. It follows that the equation $\delta T = \mu P$ which applies to a flat belt becomes for a rope in a V groove

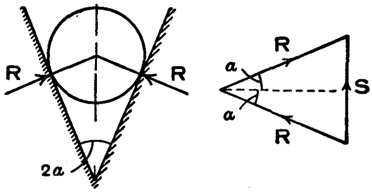


Fig. 217.

$$\delta T = 2\mu R \quad . \quad . \quad . \quad (1).$$

From the triangle of forces shown in Fig. 217 the resultant force due to the two forces R is $S = 2R \sin \alpha$, where α is equal to half the angle between the sides of the groove. Therefore the equation $P = T\delta\theta$ for the flat belt becomes for the rope in the V groove $S = T\delta\theta$, or

$$2R \sin \alpha = T\delta\theta \quad . \quad . \quad . \quad (2).$$

From equations (1) and (2), eliminating R,

$$\delta T = \frac{\mu T \delta\theta}{\sin \alpha}.$$

Therefore
$$\int_{T_2}^{T_1} \frac{dT}{T} = \frac{\mu}{\sin \alpha} \int_0^{\theta} d\theta$$

and
$$\frac{T_1}{T_2} = e^{\frac{\mu\theta}{\sin \alpha}}.$$

It will be noted that the difference between this equation and the one obtained in the preceding Art. for a flat belt is that $\mu\theta$ is now divided by $\sin \alpha$. The equation applies also to a V belt.

Example.—To find the value of $e^{\frac{\mu\theta}{\sin \alpha}}$, given that $\mu = 0{\cdot}27$, $\theta = 170°$, and the groove angle is 45°, that is $\alpha = 22{\cdot}5°$.

In the example in the preceding Art. with the same values of μ and θ, it was found that $\mu\theta = 0{\cdot}255\pi$ and $e^{\mu\theta} = 2{\cdot}228$.

Therefore
$$\frac{\mu\theta}{\sin \alpha} = \frac{0{\cdot}255\pi}{\sin 22{\cdot}5°} = \frac{0{\cdot}255\pi}{0{\cdot}3827} = 2{\cdot}093$$

and
$$e^{\frac{\mu\theta}{\sin \alpha}} = 2{\cdot}7183^{2{\cdot}093} = 8{\cdot}109.$$

Comparing this result with the value 2·228 obtained for a flat belt, it is evident that a rope on a pulley with a V groove has a greatly increased limiting ratio for its tensions. For a V-belt drive the groove angle in the pulley is usually less than it is for a rope drive, and consequently the ratio of the tensions is greater for the V belt than for the rope.

In the Arts. which follow, it may be assumed in general that the term "belt" includes the ordinary flat belt, the V belt, and the rope drive.

118. Centrifugal Tension in a Belt.—The formula for the relation between the maximum and minimum tensions in a belt wrapped round a pulley was obtained on the assumption that the pulley was fixed and slip was about to occur. Suppose now that power is transmitted from one pulley to another by a belt, then centrifugal force causes an additional tension which is proportional to the square of the speed and has to be taken into account in cases where the speed and mass of the belt are not negligible.

Consider a very short length of belt mn on a pulley of radius r (Fig. 218), and let $\delta\theta$ be the angle subtended by mn at the centre of the pulley. Let the weight of the belt be w per unit length and let its speed be v. The weight of the length mn is $wr\delta\theta$, and the centrifugal force on mn is

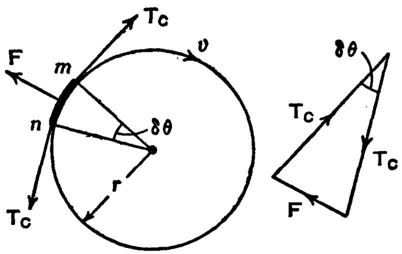


FIG. 218.

$$F = \frac{wr\delta\theta}{g}\frac{v^2}{r} = \frac{w\delta\theta}{g}v^2.$$

For equilibrium, assume that equal tangential forces T_C act at m and n, then from the triangle of forces in Fig. 218

$$T_C\delta\theta = F = \frac{w\delta\theta}{g}v^2 \quad \text{or} \quad T_C = \frac{wv^2}{g},$$

and this force is called the *centrifugal tension.* (See Art. 43, p. 73, for remarks on centrifugal force.)

Let T_1 and T_2 be the tensions in the tight and slack sides respectively when the belt is transmitting power and is on the point of slipping, then the effective tensions are $T_1 - T_C$ and $T_2 - T_C$, and the ratio of tensions obtained in Art. 116 becomes

$$\frac{T_1 - T_C}{T_2 - T_C} = e^{\mu\theta}.$$

In the case of a rope drive on pulleys with **V** grooves, or a **V**-belt drive, the value of this ratio is $e^{\frac{\mu\theta}{\sin\alpha}}$ (Art. 117).

119. Power Transmitted by a Belt.—Let H be the horse-power transmitted by a belt, v be the belt speed in feet per second, w be the weight of the belt in pounds per foot length, and let the forces T_1, T_2, and T_C be in pounds.

The effective tensions in the belt are $T_1 - T_C$ and $T_2 - T_C$, the driving force is

$$(T_1 - T_C) - (T_2 - T_C) = T_1 - T_2$$

and
$$H = \frac{(T_1 - T_2)v}{550}.$$

It will be shown that the driving force $T_1 - T_2$ is proportional to $T_1 - T_C$, and since T_C increases as the speed increases it follows, assuming T_1 to be constant, that the driving force decreases as the speed increases. The product $(T_1 - T_2)v$ is zero when $v = 0$ and again when $T_1 - T_2 = 0$, and at an intermediate stage the horse-power has its maximum value. This maximum value will now be determined.

Assume that the belt is always on the point of slipping and let $e^{\mu\theta} = k$, then the ratio of the effective tensions is

$$\frac{T_1 - T_C}{T_2 - T_C} = k,$$

from which
$$T_2 = \frac{T_1 - T_C}{k} + T_C$$

and $$T_1 - T_2 = T_1 - \frac{T_1 - T_C}{k} - T_C$$

$$= \left(1 - \frac{1}{k}\right)(T_1 - T_C).$$

Substitution in the horse-power equation gives

$$H = \frac{k-1}{550k}(T_1 - T_C)v$$

$$= \frac{k-1}{550k}\left(T_1 v - \frac{wv^3}{g}\right), \quad \text{since } T_C = \frac{wv^2}{g}.$$

H is a maximum when $\frac{dH}{dv} = 0$, therefore

$$\frac{dH}{dv} = \frac{k-1}{550k}\left(T_1 - \frac{3wv^2}{g}\right) = 0,$$

from which $$T_1 - \frac{3wv^2}{g} = 0 \quad \text{or} \quad T_1 - 3T_C = 0.$$

Therefore, for maximum power,

$$v = \sqrt{\frac{T_1 g}{3w}} \quad \text{and} \quad T_C = \tfrac{1}{3}T_1.$$

$$H_{max} = \frac{k-1}{550k}(T_1 - \tfrac{1}{3}T_1)\sqrt{\frac{T_1 g}{3w}}$$

$$= \frac{k-1}{550k} \cdot \frac{2T_1}{3}\sqrt{\frac{T_1 g}{3w}}.$$

If in the above analysis k denotes $e^{\frac{\mu\theta}{\sin\alpha}}$, the equations may be applied to a drive on pulleys with V grooves.

Example.—Given $T_1 = 350$ lb., $w = 0{\cdot}4$ lb./ft., and $e^{\mu\theta} = 2$, it is required to plot horse-power against speed and to find the maximum horse-power and the speed at which this occurs.

This type of example should be worked from first principles, but to save space the results already obtained will be used here.

$$T_C = \frac{wv^2}{g} = \frac{0 \cdot 4v^2}{32 \cdot 2} = \frac{2v^2}{161}. \qquad k = e^{\mu\theta} = 2.$$

$$H = \frac{k-1}{550k}(T_1 - T_C)v = \frac{1}{550 \times 2}\left(350 - \frac{2v^2}{161}\right)v = \frac{1}{550}\left(175 - \frac{v^2}{161}\right)v.$$

$H = 0$ when $\left(175 - \frac{v^2}{161}\right)v = 0$, from which $v = 0$ or

$$v = \sqrt{175 \times 161} = 167 \cdot 9 \text{ ft./sec.}$$

H is a maximum when $v = \sqrt{\frac{T_1 g}{3w}} = \sqrt{\frac{350 \times 32 \cdot 2}{3 \times 0 \cdot 4}}$

$$= 96 \cdot 91 \text{ ft./sec.}$$

$$H_{max} = \frac{k-1}{550k} \cdot \frac{2T_1}{3}\sqrt{\frac{T_1 g}{3w}} = \frac{1}{550 \times 2} \times \frac{2 \times 350}{3} \times 96 \cdot 91 = 20 \cdot 56.$$

Values of H have been calculated at intervals of 10 ft./sec. and the graph is shown in Fig. 219.

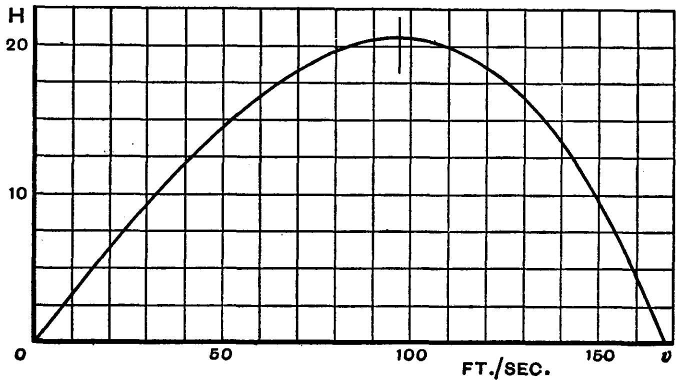


FIG. 219.

120. Creep.—Suppose power is transmitted from a driving pulley AB to a follower CD, and in the first place let the pulleys be of equal diameter (Fig. 220). To simplify the following explanation, the centrifugal tension in the belt will be regarded as negligible.

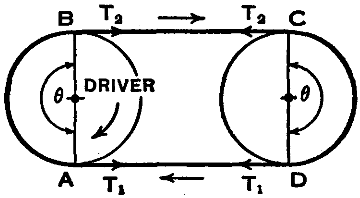


FIG. 220.

Let T_1 and T_2 be the tensions in the tight and slack sides respectively, then with the direction of motion as indicated the tight side is below the pulleys. If θ is the angle of embrace and the belt is on the point of slipping bodily, the relation between the tensions is

$$\frac{T_1}{T_2}=e^{\mu\theta}$$

and the driving force T_1-T_2 has its maximum value.

On the driver the tension decreases gradually from T_1 at A where the belt first meets the pulley, to T_2 at B where it leaves the pulley, and the strain must vary in the same way; therefore every little bit of the belt decreases in length as it moves round in the clockwise direction with the pulley from A to B. It follows that the belt is slipping back slightly in the anticlockwise direction from B to A relative to the pulley. This small amount of slip is known as *creep*, and its direction is towards the tight side of the belt.

Similarly, on the follower the tension increases gradually from T_2 at C to T_1 at D, and the strain increases in the same way. Therefore the belt is slipping forward slightly in the clockwise direction from C to D relative to the pulley, and creep is again towards the tight side of the belt.

The question which now arises is, what happens when the driving force T_1-T_2 has less than its maximum value? The belt cannot be on the point of slipping bodily round either pulley, and the ratio of the tensions must be less than $e^{\mu\theta}$, therefore let

$$\frac{T_1}{T_2}=e^{\mu\beta},$$

where β is less than θ.

When the power transmission is about to begin, the tensions T_1 and T_2 are equal, that is $e^{\mu\beta}=1$ and $\beta=0$. As the tension ratio increases, β increases until ultimately $\beta=\theta$, the angle of embrace. When β is less than θ, the belt creeps over an arc subtending an angle β at the centre of either pulley, and over the remaining arc subtending an angle $\theta-\beta$ there is no relative motion between the belt and the pulley. This arc over which there is no creep is called an *idle arc*.

For the value of β shown in Fig. 221, creep occurs on the driver over the arc BE from B to E (towards the tight side) and on the follower over the arc FD from F to D (towards the tight side). The idle arcs are AE and CF, and are shown shaded as though the belt were fixed to the pulleys.

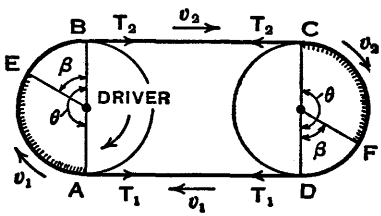


FIG. 221.

It remains to explain why one end of the idle arc on each pulley is assumed to be at the point where the belt goes on to the pulley. Assuming that the rate at which the mass of the belt passes every point is constant and remembering that the stretch is greater on the tight side than on the slack side, it follows that the belt speed is greater on the tight side than on the slack side. Let these speeds be v_1 and v_2 respectively. Now the peripheral speed of a pulley is equal to the speed of the idle arc on it and for the driver this speed is either v_1 or v_2 according as the idle arc begins at A or ends at B; similarly for the follower the peripheral speed is either v_2 or v_1 according as the idle arc begins at C or ends at D. But on account of creep the peripheral speed of the follower is less than that of the driver, since the torque is the same for each pulley and a loss of power necessitates a drop in speed. Therefore the peripheral speeds of the driver and follower are v_1 and v_2 respectively, and in each case the idle arc begins at the point where the belt goes on to the pulley.

The peripheral speeds of the driver and follower are equal to the speeds of the tight and slack sides of the belt respectively, even when there are no idle arcs. Let the angles β become greater and greater and the idle arcs become smaller and smaller. Right up to the moment when the idle arcs vanish the peripheral speeds of the driver and follower are v_1 and v_2 respectively, therefore these are the speeds when there are no idle arcs.

When the pulleys differ in size as in Fig. 222 the angle β has the same value on each, but the idle arc CF on the larger pulley subtends a larger angle at the centre of the pulley than does the idle arc AE on the smaller pulley.

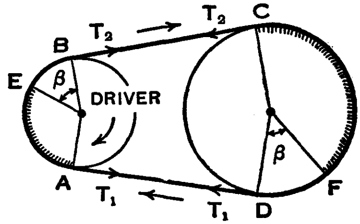


FIG. 222.

121. Effect of Creep on the Velocity Ratio.—Let the diameters of the driver and follower be D_1 and D_2 respectively, and let the corresponding speeds be N_1 and N_2 in revolutions per unit time, as indicated in Fig. 223. Let l be the span of each unsupported length of belt between the pulleys, v_1 be the speed on the tight side and v_2 be the speed on the slack side, then, as explained in the preceding Art., v_1 and v_2 are the peripheral speeds of the driver and follower respectively. Let e_1 and e_2 be the strains in the tight and slack sides respectively, then the corresponding lengths become $l(1+e_1)$ and $l(1+e_2)$.

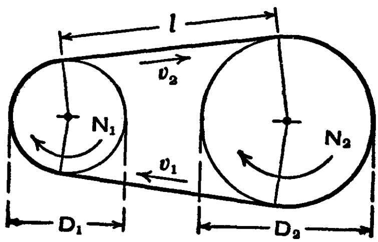


FIG. 223.

Therefore, assuming that the rate at which the mass of the belt passes every point is constant,

$$\frac{v_1}{v_2}=\frac{l(1+e_1)}{l(1+e_2)}=\frac{1+e_1}{1+e_2}.$$

But $v_1 = \pi D_1 N_1$ and $v_2 = \pi D_2 N_2$, therefore

$$\frac{N_1}{N_2} = \frac{D_2(1+e_1)}{D_1(1+e_2)}.$$

The fractional loss of belt speed is a measure of the creep and is equal to

$$\frac{v_1 - v_2}{v_1} = 1 - \frac{v_2}{v_1} = 1 - \frac{1+e_2}{1+e_1} = \frac{e_1 - e_2}{1+e_1} = e_1 - e_2$$

to the first order of small quantities.

The law connecting load with extension or stress with strain is not a linear one, although it is sometimes assumed to be linear for the purposes of calculation; the modulus of elasticity increases as the tension increases and varies with the belt material. It has been shown by J. G. Jagger and F. Sykes that in some cases the creep is greater than would be expected from the above simple theory, and the reader should refer to the paper mentioned below for further information and theory.* Numerous curves are given in the paper, and it is shown that stress-strain graphs form hysteresis loops which are one of the causes of loss of power.

122. Mean Tension in a Belt.—Let T_0 be the initial tension in a belt when it is stationary and T_1 and T_2 be the tensions when power is transmitted, then if it is assumed that the total length of the belt does not alter and stress is proportional to strain, it follows that the mean tension is constant, therefore

$$T_0 = \tfrac{1}{2}(T_1 + T_2),$$

an equation sometimes used in calculations. As stated in the preceding Art., stress is not proportional to strain and the modulus of elasticity for a belt increases as the tension increases. Therefore the mean tension increases as T_1 increases.

* "Power Transmission by Belting," by J. G. Jagger and F. Sykes, *Proc. I. Mech. E.*, 1940, vol. 143, p. 318; 1941, vol. 145, p. 231.

The curves in Fig. 224 are from an illustration in a paper by Prof. H. W. Swift,* and show the relations between the mean tension $\frac{1}{2}(T_1+T_2)$ and the tension difference or driving force T_1-T_2, for a horizontal drive with a compound belt on pulleys arranged at 15-foot centres. The dotted curves allow for the varying modulus of elasticity, and show that the mean tension is not constant but increases as the tension difference increases. The straight line represents the limiting condition when $T_2=0$ and the mean tension is equal to $\frac{1}{2}T_1$; as the mean tension cannot be less than $\frac{1}{2}T_1$ the curves cannot cross this line.

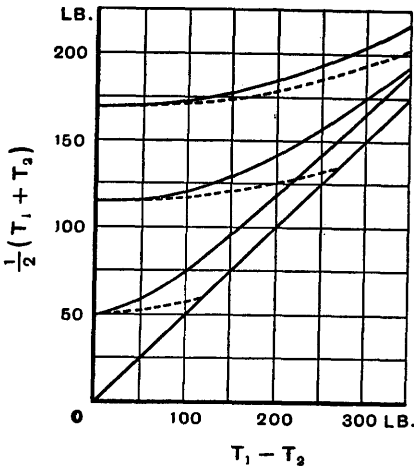


FIG. 224.

The full line curves allow for the catenary effect or the sagging of the belt between the pulleys, and this is touched on in the following Art. There is a great deal of information in Prof. Swift's paper which interested readers would do well to study.

123. Long Horizontal Drives.—Prof. Swift showed that with a horizontal drive the power transmitted can be increased if the centre distance between the pulleys is increased. This is due to the greater belt sag between the pulleys increasing the tensions and also affecting the angle of embrace.

It can be proved that the difference between the length and the span of each unsupported portion of the belt, assuming the curve to be a parabola, is approximately

* "Power Transmission by Belts: An Investigation of Fundamentals," by H. W. Swift, *Proc. I. Mech. E.*, 1928, vol. II, p. 659.

equal to $\dfrac{w^2l^3}{24T^2}$, where T is the tension at each end, w is the weight per unit length (of span), and l is the span.

When the belt is stationary and the tension is T_0, the sum of the differences between the belt length and the span on the two sides is

$$\frac{w^2l^3}{24}\cdot\frac{2}{T_0^2}.$$

When the tension is T_1 on the tight side and T_2 on the slack side, the sum of the differences is

$$\frac{w^2l^3}{24}\left\{\frac{1}{T_1^2}+\frac{1}{T_2^2}\right\}.$$

Therefore the increase in the length of the belt due to its sag is

$$\frac{w^2l^3}{24}\left\{\frac{1}{T_1^2}+\frac{1}{T_2^2}-\frac{2}{T_0^2}\right\}.$$

The stretch in the whole belt is approximately

$$(l+\tfrac{1}{2}\pi d)(e_1+e_2-2e_0),$$

where d is the mean diameter of the two pulleys, and e_1, e_2, and e_0 are the strains produced by the tensions T_1, T_2, and T_0 respectively, therefore

$$\frac{w^2l^3}{24}\left\{\frac{1}{T_1^2}+\frac{1}{T_2^2}-\frac{2}{T_0^2}\right\}=(l+\tfrac{1}{2}\pi d)(e_1+e_2-2e_0).$$

If the relation between tension and strain is known for a particular belt, then, for any given value of the initial tension T_0, values of T_1 can be calculated by substituting various values of T_2 in the above equation. Prof. Swift shows in his paper how this work may be done graphically. The full line curves in Fig. 224 show, for a compound belt on pulleys arranged at 15-foot centres, that the mean tension $\frac{1}{2}(T_1+T_2)$ increases considerably as the tension difference T_1-T_2 increases.

In a horizontal drive the mean tension increases for a given value of $T_1 - T_2$, as the centre distance increases, therefore T_1 and T_2 both increase and the tension ratio T_1/T_2 decreases. Therefore the value of $T_1 - T_2$ and the power transmitted can be increased without exceeding the allowable tension ratio. The student should now work out Ex. 25 and Ex. 26.

Exercises XII

1. Two pulleys with diameters of 40 inches and 24 inches have their centres 10 feet apart. Find the length of the belt required to connect the pulleys if it is (*a*) open, (*b*) crossed.

2. A hemp rope which is wound twice round a post has a force of 5 tons applied to it at one end. If the coefficient of friction is 0·4, find what force must be applied to the other end of the rope to prevent slip.

3. A steel wire cable passes round a fixed post and the end is made fast to the standing part of the cable by two clamps in the manner shown in Fig. 225. When the pull in the cable is 4 tons the clamps begin to slip. Calculate what this tendency to slip amounts to, measured in tons. The coefficient of friction between the post and cable is 0·25. The angle of contact may be taken as 180° and the obliquity of the two portions of the loop may be neglected. [C.U.]

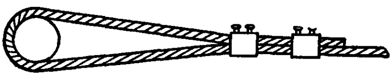


FIG. 225.

4. A body is pulled by a rope which passes 3 times round a power-driven capstan, of 2 feet diameter, developing 10 H.P. at 20 r.p.m. Find the value of the pull on the body and the force required at the free end of the rope to prevent slip, given that the coefficient of friction is 0·3.

5. Ten H.P. is to be transmitted by a flat belt from a shaft running at 200 r.p.m. to a parallel shaft running at the same speed. The allowable tension in the belt is 600 lb. and the coefficient of friction between belt and pulley is 0·2. Find what diameter of pulleys will be required. [Inst. C.E.]

6. A belt connects two pulleys, each 3 feet diameter, on parallel shafts. It is found that the belt slips when the moment of resistance is 450 pound-feet. Find the tension in the tight side of the belt, assuming that the coefficient of friction is 0·25.

7. The flywheel of an engine is 4 feet external diameter and carries a rope brake consisting of 2 ropes, each 1 inch diameter, lying side by side. During a test the horse-power absorbed by the brake is 12·6 at 210 r.p.m. and the tension on the slack end of the brake is 14·3 lb. The angle of lap of the ropes on the wheel is 360°. What is the average coefficient of friction between the ropes and the wheel?

Establish the formula used for determining the ratio of the tensions at the ends of the ropes. [U.L.]

8. An open belt transmits power from a pulley 1 foot in diameter rotating at 200 r.p.m. to a pulley 4 feet in diameter on a parallel shaft. The distance between the two shafts is 12 feet. If the coefficient of friction between the belt and the pulleys is 0·5 and the maximum tension in the belt is 200 lb., what is the greatest horse-power which can be transmitted? Neglect effects of centrifugal force. Why in practice would the speed of the driven pulley be slightly less than 50 r.p.m.? [C.U.]

9. Derive an expression for the centrifugal tension in a belt used for the transmission of power. Show also that when the belt is transmitting its maximum power, the centrifugal tension is one-third of the maximum allowable tension. [Inst. C.E.]

10. A belt weighs $\frac{1}{2}$ lb. per foot run and passes over a pulley 3 feet in diameter. If the greatest permissible tension is 160 lb., find the maximum horse-power which can be transmitted and the corresponding r.p.m. of the pulley, assuming that the tension on the tight side of the belt is twice that on the slack side. [Inst. C.E.]

11. Two parallel shafts are to be connected by an open belt $\frac{1}{4}$ inch thick. The speed of the driving shaft is 400 r.p.m. and the diameter of the driving pulley is 2 feet. Assuming an angle of lap of 180° and a coefficient of friction 0·3, find the width of belt necessary to transmit 10 H.P., if the maximum allowable stress is 250 lb. per sq. in. The belt material weighs 0·05 lb. per cu. in. [Inst. C.E.]

12. A rope drive is required to transmit 50 H.P. at 150 r.p.m. from a grooved pulley whose mean diameter to the rope centres is 4 ft. 6 in. The included groove angle is 45°, the angle of embrace of the ropes 180° and the coefficient of friction 0·25.

Allowing a maximum tension of 160 lb. for each rope, determine the number of ropes required. [U.L.]

13. Repeat the preceding exercise, taking centrifugal tension into account, given that the rope weighs 0·35 lb. per foot.

14. A leather belt transmits 40 H.P. from a pulley 30 inches diameter which runs at 500 r.p.m. The angle of embrace is 160° and the coefficient of friction is 0·3. If the weight of 1 cubic inch of leather is 0·035 lb. and the stress in the belt is not to exceed 350 lb. per sq. in., find the minimum cross-sectional area of the belt. [I.Mech.E.]

15. Find an expression for the ratio of the effective tensions on the tight and slack sides of a belt, allowing for centrifugal force, when slipping between the belt and pulley is about to occur.

Twelve H.P. is to be transmitted by a belt on a pulley, 3 feet in diameter, running at 220 r.p.m., the angle of lapping being 170°. The belting used weighs 1·9 lb. per sq. ft., the coefficient of friction between belt and pulley is 0·27, and the maximum permissible tension in the belt is 70 lb. per inch of width. Find the smallest permissible width of the belt and the approximate tension when the pulley is stationary. [U.L.]

16. A line shaft is driven by belt from a motor of 70 H.P. output. The belt weighs 2·75 lb. per foot length and has an angle of contact of 160° on the motor pulley. The linear speed of the belt is 4500 ft./min. It is proposed to change the pulleys so that the belt speed is reduced to 3000 ft./min., the same belt being used with the same angle of contact. If the coefficient of friction is 0·3, compare the maximum tensions in the belt, and the loads on the pulley bearings in the two cases. Any belt formula used should be established. [U.L.]

17. A flat belt, $\frac{1}{4}$ inch thick, $4\frac{1}{2}$ inches wide, and weighing 0·68 lb. per foot, is used to transmit power from a shaft running at 600 r.p.m. to a parallel shaft which is to run at 300 r.p.m.

Find the horse-power which can be transmitted, given that the centrifugal tension is equal to one-third of the greatest tension in the belt, and that the latter is not to exceed 250 lb. per sq. in. The smaller angle of lap may be assumed to be 170° and the coefficient of friction $\mu = 0{\cdot}3$.

Find also the necessary initial tension in the belt and the approximate pulley diameters. [U.L.]

18. A machine shaft is to be driven from an engine by means of a rope drive. The engine develops 100 H.P. and runs at 150 r.p.m. The driving pulley on the engine shaft is 6 feet diameter. The ropes to be used are 1 inch diameter and weigh 0·35 lb. per foot. The angle of the grooves for the ropes is to be 45°. If the maximum tension in each rope is not to exceed 160 lb., determine the number of ropes required, assuming that the angle of lap of the ropes on the driven pulley is to be 150° and the coefficient of friction between rope and pulley surface is to be 0·3. Allowance must be made for centrifugal stress. [U.L.]

19. Derive the expression

$$\frac{T_1 - \dfrac{wv^2}{g}}{T_2 - \dfrac{wv^2}{g}} = e^{\mu\theta},$$

connecting the pulls on the two sides of a belt subjected to centripetal action, T_1 and T_2 being the total pulls.

A V-belt having a lap of 180° has a cross-sectional area of

1 sq. in. and runs in a groove of included angle $2\alpha = 45°$. The density of the belt is 0·05 lb. per cu. in., and the maximum stress is limited to 600 lb. per sq. in., the coefficient of friction being 0·15.

Find the maximum horse-power that can be transmitted if the wheel has a mean diameter of 12 inches and runs at 1000 r.p.m. [U.L.]

20. Two pulleys, with diameters of 20 inches and 50 inches and centres 100 inches apart, are connected by an open belt.

(*a*) Find the angle (in degrees and minutes) subtended at the centre of the larger pulley by the idle arc when the belt is on the point of slipping on the smaller pulley.

(*b*) Find the angles (in degrees and minutes) subtended at the centres of the pulleys by the idle arcs when $\frac{T_1}{T_2} = 1{\cdot}5$ and $\mu = 0{\cdot}25$.

21. A rope drive is to transmit 80 H.P. from a driving pulley rotating at 160 r.p.m. and the rope speed is to be such that this power is a maximum. Find the effective diameter of the driving pulley and the number of $1\frac{1}{2}$ inch diameter ropes required. Assume that the angle of embrace is 180°, each groove angle is 45°, the coefficient of friction is 0·2, the maximum stress in the rope is 200 lb. per sq. in., and the weight is $0{\cdot}3d^2$ lb. per foot, where d inch is the diameter.

22. Use the data in Ex. 21 and find the number of ropes required to transmit 80 H.P. if the driving pulley rotating at 160 r.p.m. has its effective diameter reduced to 5 feet. The rope speed will of course be reduced and will no longer be such that the horse-power is a maximum.

23. Determine the width of an open belt for the following drive. The pulleys are 1 foot and 4 feet diameter, at a centre distance of 10 ft. 9 in.; horse-power transmitted, 12; speed of larger pulley, 300 r.p.m.; safe tension in belt, 100 lb. per inch width; coefficient of friction, 0·2; weight of 1 foot length of belt 1 inch wide, 0·12 lb. [U.L.]

24. Prove that the velocity of the rim of a belt pulley is equal to the velocity of the belt at the point where it runs on to the pulley unless there is slipping over the whole arc of embrace.

An open belt connecting two equal pulleys is elastic and is such that the extension per unit length is proportional to the tension. A pull of 700 lb. in the belt produces an extension of 2 inches on a 15-foot length. If the difference of tensions on the two sides of the belt is 600 lb., neglecting inertia, find approximately the percentage loss of power due to creep. [C.U.]

25. An open belt, connecting two equal pulleys of 2 feet diameter on parallel shafts, is running at 825 ft. per min. and

transmitting 2 H.P. The coefficient of friction is 0·3 and the centre distance between the pulleys is short. Neglecting centrifugal tension, show that the driving force $T_1 - T_2 = 80$ lb., $T_1 = 131$ lb., $T_2 = 51$ lb., and the mean tension is 91 lb., to the nearest pound in each case.

Find the mean tension and the percentage increase in the horse-power if a long horizontal drive is used with the pulley centre distance equal to 15 feet and it is assumed that the tension ratio is $e^{0\cdot 3\pi}$ as before. The relation between the tensions and strains (see Art. 123) is

$$\frac{w^2 l^3}{24}\left\{\frac{1}{T_1^2}+\frac{1}{T_2^2}-\frac{2}{T_0^2}\right\}=(l+\tfrac{1}{2}\pi d)(e_1+e_2-2e_0).$$

To eliminate the strains e_1, e_2, and e_0, assume that $\dfrac{T}{ae}=63{,}000$ lb. per sq. in. and that a, the cross-sectional area of the belt, is 1 sq. in. Take $w=0\cdot 6$ lb. per ft.; substitute $l=15$ ft. and $d=2$ ft.; assume $T_0=91$ lb., the mean tension in the short drive. Put $T_1=e^{0\cdot 3\pi}T_2$, then the equation contains one unknown quantity T_2 and may be solved by trial. It only remains to find T_1 and $T_1 - T_2$, then the mean tension and the percentage increase in the horse-power can be obtained.

Note.—It is unfortunate that the symbol e has two meanings, the base of Napierian logarithms and strain, but this is unlikely to mislead the student.

26. In the long horizontal drive in Ex. 25, find T_2 and the mean tension if $T_1=350$ lb. and it is assumed there is no restriction on the tension ratio.

27. A small pulley of radius r_1 on a lineshaft drives a large pulley of radius r_2 on a machine vertically below it, the centre distance being d. Show that slipping is equally likely to occur at either pulley if the tension in the belt where it runs on to the large pulley is given by

$$T=\frac{e^{\mu(\pi-2\theta)}-1}{e^{\mu(\pi+2\theta)}-e^{\mu(\pi-2\theta)}}wd,$$

where w = weight of belt per unit length and $\sin\theta=(r_2-r_1)/d$. [U.L.]

28. A heavy belt hangs over a wheel of radius a, and the ends, of lengths l_1 and l_2, hang vertically downwards. Prove that when slipping is about to occur,

$$l_2=l_1e^{\mu\pi}+\frac{2a\mu}{1+\mu^2}(1+e^{\mu\pi}). \qquad \text{[U.L.]}$$

CHAPTER XIII

TOOTHED GEARING

124. Types of Gears.—Toothed wheels or gears are used to provide a positive drive between shafts which are, in general, near to one another. Spur, helical, spiral, bevel and worm gears are discussed in this chapter.

Parallel shafts may be connected by spur gears or helical gears, the former having teeth which are parallel to the axes of the shafts and the latter having each tooth cut on a helix. Helical gears on parallel shafts are sometimes known as helical spur gears. Bevel wheels connect shafts whose axes intersect. Spiral gears and worm gears are used to connect shafts whose axes are not parallel and do not intersect.

125. Definitions.—Before discussing toothed wheels and some of the essential theory, it will be convenient to begin with a few definitions. Other terms will be defined as required.

Gears, *gear wheels*, *wheels*, and *pinions* all describe toothed wheels, but when two toothed wheels of different diameters are in mesh, the larger may be referred to as the wheel and the smaller as the pinion. A *rack* is part of a toothed wheel of infinite diameter.

The *pitch cylinders* of a pair of gears in mesh are the imaginary friction cylinders which will roll together without slip while one gear drives the other.

A *pitch circle* (Fig. 226) is a section of a pitch cylinder by a plane at right angles to the axis. The diameter of this circle is the *pitch diameter*.

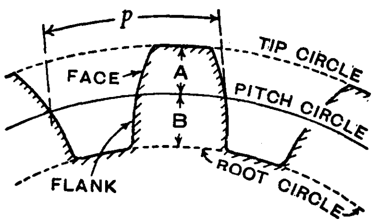


Fig. 226.

The *circular pitch* p (Fig. 226) is the length of the arc of the pitch circle between corresponding points on adjacent teeth.

The *diametral pitch* P is the number of teeth divided by the pitch diameter, or the number of teeth per inch of pitch diameter.

If T is the number of teeth and D is the pitch diameter,

$$p=\frac{\pi D}{T} \quad \text{and} \quad P=\frac{T}{D}$$

from which

$$pP=\pi,$$

or the product of the circular and diametral pitches is equal to π.

The *module* m is the pitch diameter divided by the number of teeth, or the reciprocal of the diametral pitch P.

$$m=\frac{D}{T}=\frac{1}{P}=\frac{p}{\pi}.$$

Although it is understood that D is in inches, m may be in any unit of length, generally inches or millimetres, and the unit must be stated.

The *addendum* A (Fig. 226) is the radial height of a tooth from the pitch circle to the tip.

The *dedendum* B (Fig. 226) is the radial depth of a tooth from the pitch circle to the root.

The *working depth* is the sum of the addenda of the two gears in mesh.

The *addendum circle* or *tip circle* (Fig. 226) is the circle drawn through the tips of the teeth.

The *dedendum circle* or *root circle* (Fig. 226) is the circle drawn through the bottoms of the tooth spaces.

The *bottom clearance* is the least distance between the tip of a tooth and the bottom of its mating space.

The *face* of a tooth (Fig. 226) is the acting surface of the addendum.

The *flank* of a tooth (Fig. 226) is the acting surface of the dedendum plus the surface of the fillet at the root.

126. Condition for Constant Velocity Ratio of Toothed Wheels.—Let the thick line curves (Fig. 227) be portions of two teeth, one on wheel 1 and the other on wheel 2, in contact at the point Q, and suppose the wheels are turning in the directions indicated. Let TT be the common tangent and MN be the common normal to the curves at the point Q. From the centres O_1 and O_2 of the wheels draw O_1M and O_2N perpendicular to MN.

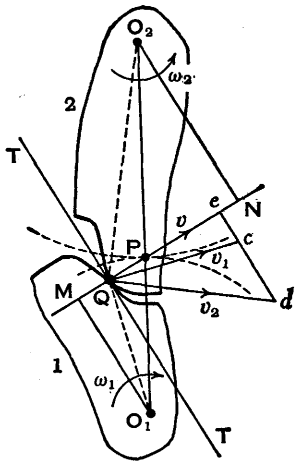


FIG. 227.

The point Q is moving in the direction Qc, perpendicular to O_1Q, when considered as a point on wheel 1, and in the direction Qd, perpendicular to O_2Q, when considered as a point on wheel 2.

Let the lengths Qc and Qd represent the velocities of Q, which will be denoted by v_1 and v_2 respectively. The components of v_1 and v_2 in the direction of the common normal MN must be equal, for there can be no relative motion between the tooth curves along the normal if contact is maintained. Relative motion can occur only along the common tangent TT. Therefore, if de is drawn perpendicular to MN, it will pass through c, and Qe will represent the component velocity v along MN of v_1 and v_2.

Join O_1O_2, intersecting MN at P.

Let ω_1 and ω_2 be the angular velocities of the wheels 1 and 2 respectively, then

$$\omega_1 = \frac{v}{O_1M}, \qquad \omega_2 = \frac{v}{O_2N}$$

and

$$\frac{\omega_1}{\omega_2} = \frac{O_2N}{O_1M} = \frac{O_2P}{O_1P}.$$

Therefore in order to have a constant velocity ratio for all positions of the wheels, P must be a fixed point.

Now pitch circles with centres at O_1 and O_2 and rolling contact at P have a constant angular velocity ratio

O_2P/O_1P. Therefore, provided O_1O_2 and MN always intersect at P, called the *pitch point*, the velocity ratio of the toothed wheels is constant.

The fundamental condition to be observed when designing wheel teeth is as follows. *The common normal at the point of contact between a pair of teeth must always pass through the pitch point.*

Suppose a pinion having a pitch circle diameter $d=2O_1P$ and t teeth drives a wheel having a pitch circle diameter $D=2O_2P$ and T teeth. The pinion and the wheel must have the same circular pitch p, therefore

$$p=\frac{\pi d}{t}=\frac{\pi D}{T}, \quad \text{from which} \quad \frac{d}{D}=\frac{t}{T},$$

and the velocity ratio may be expressed as

$$\frac{\omega_1}{\omega_2}=\frac{O_2P}{O_1P}=\frac{D}{d}=\frac{T}{t}.$$

If one tooth shape is decided upon and another tooth is designed to mesh correctly with it, the second tooth is said to be *conjugate* to the first.

127. Velocity of Sliding of Teeth.—Sliding between a pair of teeth in contact at Q occurs along the common tangent TT to the tooth curves. (See Fig. 228, where the essential lines in Fig. 227 are redrawn.)

The velocity of Q, considered as a point on wheel 1, along TT, is represented by ec. From the similar triangles Qec and O_1MQ,

$$\frac{ec}{MQ}=\frac{v_1}{O_1Q}=\omega_1,$$

from which $ec=\omega_1 . MQ$.

Similarly the velocity of Q, considered as a point on wheel 2, along TT, is represented by ed, and $ed=\omega_2 . QN$.

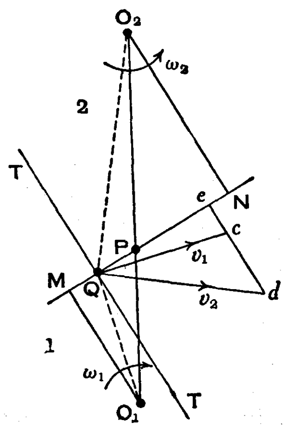


FIG. 228.

The velocity of sliding at Q is

$$v_S = ed - ec = \omega_2 . QN - \omega_1 . MQ$$
$$= \omega_2(QP + PN) - \omega_1(MP - QP)$$
$$= (\omega_1 + \omega_2)QP + \omega_2 . PN - \omega_1 . MP.$$

Since $\qquad \dfrac{\omega_1}{\omega_2} = \dfrac{O_2P}{O_1P} = \dfrac{PN}{MP}, \qquad \omega_1 . MP = \omega_2 . PN,$

therefore $\qquad v_S = (\omega_1 + \omega_2)QP.$

Therefore the velocity of sliding is proportional to the distance of the point of contact from the pitch point. Since the angular velocity of wheel 2, relative to wheel 1, is $\omega_1 + \omega_2$, and P is the instantaneous centre for this relative motion, the value of v_S could have been written down at once without the above analysis. It should be noted, however, as already stated, that the separate velocities along the tangent TT of Q, considered first as a point on wheel 1 and then on wheel 2, are $\omega_1 . MQ$ and $\omega_2 . QN$ respectively.

128. Cycloidal Teeth.—The profiles of teeth on a wheel can be settled arbitrarily, and then the teeth on a second wheel can be designed to mesh correctly with those on the first wheel. In practice, the profiles are either of cycloidal or involute form, but nearly always the latter. Before discussing cycloidal teeth three curves will be defined.

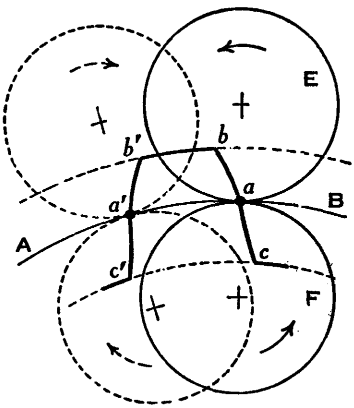


FIG. 229.

A *cycloid* is a curve traced by a point on a circle which rolls without slipping on a straight line. An *epicycloid* is a curve traced by a point on a circle which rolls without slipping on the *outside* of a fixed circle. A *hypocycloid* is a curve traced by a point on a circle which rolls without slipping on the *inside* of a fixed circle.

In Fig. 229 AB is the pitch circle of a wheel. The circle E is rolled on the outside of AB, as indicated, and the point *a* on E traces the epicycloid *ab*. The circle F is rolled on the inside of AB and the point *a* on F traces the hypocycloid *ac*. The two curves, extending from the addendum circle to the root circle, form the profile *bac* of one side of a *cycloidal tooth*. Similarly the two curves *a′b′* and *a′c′* forming the profile of the opposite side of the tooth are traced by the point *a′*. In practice, the junctions with the root circle at *c* and *c′* are made by smooth curves known as *fillets*.

In Fig. 230 AB and A′B′ are the pitch circles of wheels 1 and 2 respectively, and P is the pitch point. Consider the profiles of teeth U and V which make contact with one

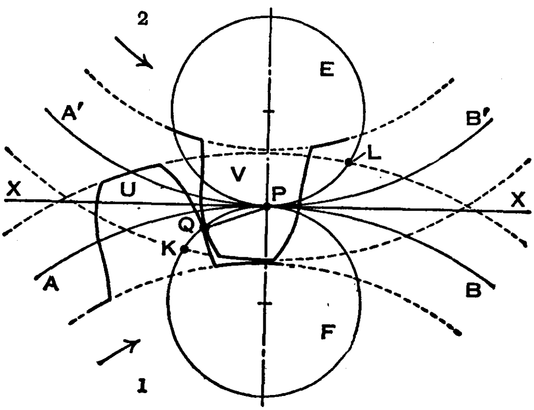


FIG. 230.

another. A point on the circle F would trace the flank of tooth U when F rolls on the inside of AB and the face of tooth V when F rolls on the outside of A′B′. Similarly, a point on the circle E would trace the face of tooth U and the flank of tooth V. The rolling circles E and F may have unequal diameters, but if several wheels are to be interchangeable they must have rolling circles of equal diameters.

It will now be shown that the common normal at the point of contact between two cycloidal teeth always passes through the pitch point, which is the fundamental condition for a constant velocity ratio.

Let the teeth U and V be in contact at the point Q, then the tooth curves have a common normal at Q. From the method of tracing an epicycloid and a hypocycloid, it is evident that a normal at the tracing point, to either of these curves, passes through the point of contact of the rolling circle and the pitch circle, which is an instantaneous centre. Since the normal at the point Q is common to both tooth profiles, the rolling circle F, which passes through Q, must make contact with the two pitch circles at a common point, that is at the pitch point P. This is true for all positions of the point of contact Q, therefore the common normal at the point of contact Q is QP, that is, it always passes through the pitch point P.

It will be assumed that wheel 1 is turning clockwise and driving wheel 2. Contact between the flank of tooth U and the face of tooth V begins at K, one of the points of intersection of the addendum circle of wheel 2 with circle F, and the point of contact moves round the circle F to the pitch point P; then the face of tooth U meets the flank of tooth V and the point of contact moves round the circle E from P to L, one of the points of intersection of the addendum circle of wheel 1 with circle E.

The *path of contact* is the path travelled by the point of contact, and it is the arc KP plus the arc PL.

The inclination of QP to XX, the common tangent at the pitch point P, is called the *pressure angle* or *angle of obliquity*. It can be seen that this angle is continually varying, being a maximum when Q is at K, zero when Q is at P, and another maximum when Q is at L.

129. Internal Cycloidal Teeth.—A wheel with teeth on the inside of its rim is an *internal wheel, internal gear*, or *annular gear*, and it can mesh only with an external gear. Fig. 229, p. 246, illustrates an internal gear if *cabb'a'c'* is regarded as a tooth space. The root circle is *bb'*, the addendum circle is *cc'*. The faces are *ac* and *a'c'*, and the flanks are *ab* and *a'b'*.

130. Involute Teeth.—An involute of a circle is a plane curve generated by a point on a tangent which rolls on the circle without slipping, or by a point on a taut string which is unwrapped from a reel (Fig. 231). In connection with toothed wheels, the circle is called a *base circle*. One position of the rolling tangent, or unwrapped taut string, is PQ, which is equal in length to the arc PA, and the curve AQR is the involute.

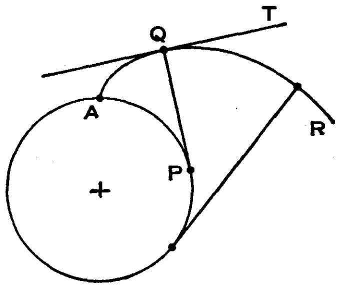


FIG. 231.

Since, for an instant, PQ is turning about P, the point Q is moving in the direction perpendicular to PQ; therefore the tangent QT to the involute at Q is perpendicular to PQ and PQ is the normal to the involute.

In Fig. 232 S_1 and S_2 are base circles with centres at O_1 and O_2 respectively. Involutes *ab* and *a′b′* of the circles S_1 and S_2 respectively are in contact at Q. Tangents MQ and NQ to the circles are the normals to the involutes at Q, therefore MN is the common normal at Q and the common tangent to the base circles. If the circles are turned about their centres so that the involutes remain in contact, the point of contact Q will remain in the common tangent MN.

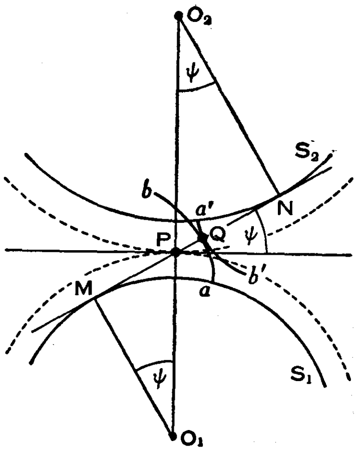


FIG. 232.

Draw the line of centres O_1O_2 intersecting MN at P, then P is a fixed point, and if pitch circles are drawn through P with centres O_1 and O_2, P is the pitch point. If the involutes in contact at Q are profiles of teeth, the common normal at Q passes through the pitch point and the velocity ratio of the wheels is O_1P/O_2P; therefore involute teeth satisfy the fundamental condition.

From the similar triangles O_1MP and O_2NP,

$$\frac{O_1M}{O_2N}=\frac{O_1P}{O_2P},$$

which determines the ratio of the radii of the base circles S_1 and S_2.

The angle ψ between the common tangent to the base circles and the common tangent to the pitch circles is the *pressure angle* or *angle of obliquity*. The angles PO_1M and PO_2N are equal to ψ, and the radii of the base circles are given by

$$O_1M = O_1P \cos \psi \quad \text{and} \quad O_2N = O_2P \cos \psi.$$

If the distance between the centres O_1 and O_2 is altered (by design or otherwise), the radii of the pitch circles alter, but their ratio is unchanged because it is equal to the ratio of the radii of the base circles. Also the common normal at the point of contact still passes through the pitch point, therefore the wheels continue to work correctly; this is not the case with cycloidal teeth. It should be noted, however, that the pressure angle increases when the centre distance O_1O_2 is increased.

131. Internal Involute Teeth.—Internal gears have the addendum circle and the tooth faces inside the pitch circle and the dedendum circle and the tooth flanks outside it. The teeth of an internal gear correspond in shape to the spaces of an external gear and the involute surfaces are concave. The centre distance between an internal gear and the external gear with which it meshes is equal to half the difference between the diameters of the pitch circles. Therefore this type of drive is used when compactness is essential.

132. Path of Contact.—Suppose that the pinion 1 is driving the wheel 2 (Fig. 233), then the point of contact between a pair of teeth is on the common tangent MN to the base circles, and MN is called the *line of action*. Draw arcs of the addendum circles to intersect MN at K and L, then the part KL of the line of action is the *path of contact*.

If the driving pinion 1 is turning clockwise, contact between a pair of teeth begins at K and ends at L. If the wheel 2 is made the driver and the directions of motion are reversed, contact between a pair of teeth begins at L and ends at K.

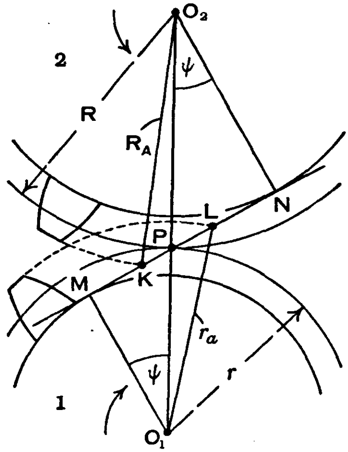


FIG. 233.

On a driving tooth the contact begins on the flank near the root circle and finishes at the outer end of the tooth face; on the corresponding driven tooth the contact begins at the outer end of the tooth face and finishes on the flank near the root circle.

The length of the path of contact depends on the radii O_1L and O_2K of the addendum circles. Let these radii be denoted by r_a and R_A respectively (Fig. 233), and let the corresponding pitch circle radii be r and R. The length of the part KP of the path of contact is obtained as follows:—

$$\begin{aligned} KP &= KN - PN \\ &= \sqrt{O_2K^2 - O_2N^2} - O_2P \sin \psi \\ &= \sqrt{R_A^2 - R^2 \cos^2 \psi} - R \sin \psi. \end{aligned}$$

Similarly

$$\begin{aligned} PL &= ML - MP \\ &= \sqrt{O_1L^2 - O_1M^2} - O_1P \sin \psi \\ &= \sqrt{r_a^2 - r^2 \cos^2 \psi} - r \sin \psi. \end{aligned}$$

Therefore the length of the path of contact is

$$\begin{aligned} KL &= KP + PL \\ &= \sqrt{R_A^2 - R^2 \cos^2 \psi} + \sqrt{r_a^2 - r^2 \cos^2 \psi} - (R + r) \sin \psi. \end{aligned}$$

133. Arc of Contact.—One involute profile of a tooth on a pinion 1 is shown in three positions in Fig. 234, when contact begins at K (Art. 132), when it is at the pitch point P, and when it ends at L. The pitch circle intersects the involute, in its three positions, at k', P, and l', and the corresponding points on the base circle are k, p, and l respectively. The *arc of contact* is $k'Pl'$, the path of the point of intersection of the involute and the pitch circle during contact between two teeth. When one gear drives another, the two pitch circles roll together, therefore a pinion and a wheel in mesh have arcs of contact of equal length.

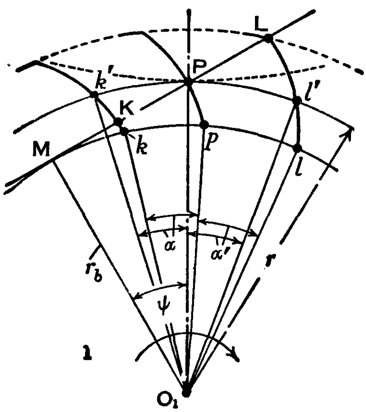


FIG. 234.

The arc of contact is divided into two parts by the pitch point P; the first part $k'P$ is the *arc of approach* and the second part Pl' is the *arc of recess*, and the angles they subtend at O_1 are called the *angle of approach* and the *angle of recess* respectively.

If the tooth is turned about O_1 from one position to another, all radial lines turning with it move through equal angles. Therefore the arcs $k'P$ and kp subtend equal angles at O_1, denoted by α in Fig. 234. Similarly the arcs Pl' and pl subtend angles α' at O_1.

Since the involutes kK, pP, and lL may be generated by rolling the tangent ML on the base circle Mkl, arc $kp = KP$ and arc $pl = PL$. Therefore, denoting the radius O_1M of the base circle by r_b,

$$\textit{Angle of approach } \alpha = \frac{kp}{r_b} = \frac{KP}{r_b} \text{ radian;}$$

$$\textit{Angle of recess } \alpha' = \frac{pl}{r_b} = \frac{PL}{r_b} \text{ radian.}$$

If the pitch circle radius is r, $\cos\psi = r_b/r$, therefore

$$\textit{Length of arc of contact} = \left\{\frac{\mathrm{KP}+\mathrm{PL}}{r_b}\right\}r = \frac{\mathrm{KL}}{\cos\psi}$$

$$= \frac{\textit{Length of path of contact}}{\cos\psi}.$$

In order that the driven wheel may turn continuously, at least one pair of teeth must always be in contact, therefore the length of the arc of contact must not be less than the circular pitch, and the number of pairs of teeth in contact is equal to the arc divided by the circular pitch.

The *base circle pitch* is the pitch measured along the base circle and will be denoted by p_b. The ratio p_b/p is equal to the ratio of the radii of the base and pitch circles, that is to $\cos\psi$, or

$$p_b = p\cos\psi.$$

$$\textit{Number of pairs of teeth in contact} = \frac{\textit{Length of arc of contact}}{\textit{Circular pitch}}$$

$$= \frac{\mathrm{KL}}{\cos\psi}\cdot\frac{1}{p} = \frac{\mathrm{KL}}{p_b}$$

$$= \frac{\textit{Length of path of contact}}{\textit{Base circle pitch}}.$$

Usually this is not a whole number. For example, suppose it is between 1 and 2, then the interpretation is that the number of pairs of teeth in contact at one time varies from 1 to 2.

134. Interference—Wheel and Pinion.—A pinion 1 turns clockwise and drives a wheel 2 (Fig. 235). The common tangent to the base circles is MN and the part KL is the path of contact between two teeth *a* and *b*, starting at K and ending at L.

If the radius of the addendum circle of pinion 1 is increased, the point L moves along PN towards N and coincides with N when the radius is equal to O_1N. If the radius exceeds this value, the point of contact L will be inside the base circle of wheel 2 and not on the involute profile of tooth *b*. The tip of tooth *a* will then undercut tooth *b* at the root and remove part of the involute profile of *b*. This effect is known as *interference*, and it occurs when the teeth are being cut. Similarly, increasing the radius of the addendum circle of wheel 2 beyond the value O_2M causes the tip of tooth *b* to interfere with tooth *a*. The points M and N are called *interference points*, and if the path of contact does not extend beyond either of these points, interference is avoided.

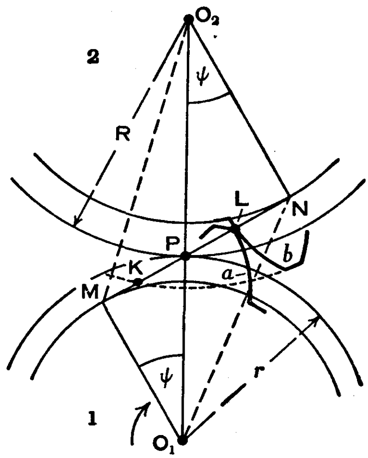


FIG. 235.

The limiting value of the radius of the addendum circle of the pinion is O_1N and of the wheel is O_2M; but interference is more likely to occur on the pinion than on the wheel, and in this case the critical radius is O_2M. To determine whether there will be interference when two gears are in mesh, each addendum circle radius is compared with its limiting value and the blank diameter of either gear can be reduced if necessary.

The lengths O_2M and O_1N may be obtained in terms of the pitch circle radii *r* and R and the pressure angle ψ as follows. From the right-angled triangle O_2NM,

$$
\begin{aligned}
O_2M^2 &= O_2N^2 + MN^2 \\
&= O_2N^2 + (MP + PN)^2 \\
&= (O_2P \cos \psi)^2 + (O_1P \sin \psi + O_2P \sin \psi)^2 \\
&= R^2 \cos^2 \psi + (r + R)^2 \sin^2 \psi,
\end{aligned}
$$

$$O_2M = \sqrt{R^2 + (r^2 + 2rR) \sin^2 \psi} \quad . \quad . \quad . \quad (1).$$

From the right-angled triangle O_1MN,

$$\begin{aligned} O_1N^2 &= O_1M^2 + MN^2 \\ &= O_1M^2 + (MP + PN)^2 \\ &= (O_1P \cos \psi)^2 + (O_1P \sin \psi + O_2P \sin \psi)^2 \\ &= r^2 \cos^2 \psi + (r + R)^2 \sin^2 \psi, \end{aligned}$$

$$O_1N = \sqrt{r^2 + (2rR + R^2) \sin^2 \psi} \quad . \quad . \quad . \quad (2).$$

Theoretically the minimum number of teeth possible on the pinion can be calculated in the following way.

Suppose the addendum A of the wheel is equal to $1/P$ where P is the diametral pitch, then, since A must not exceed $O_2M - R$ (see Fig. 235),

$$\frac{1}{P} < O_2M - R.$$

Let t be the number of teeth on the pinion, then $t = 2rP$; therefore

$$\frac{2r}{t} < O_2M - R,$$

and the minimum value of t is given by

$$t = \frac{2r}{O_2M - R} = \frac{2}{(O_2M/r) - (R/r)}.$$

The value of O_2M is substituted from (1) and then t can be calculated for any value of R/r.

In practice this method is not used. The British Standards Institution give empirical rules in B.S. 436 for the proportions of teeth in order that interference may be avoided. The addendum varies not only with the pitch but also with the numbers of teeth in the mating gears. It is less on the wheel than on the pinion (see Art. 136).

135. Interference—Rack and Pinion.—A rack and a pinion in mesh are shown in Fig. 236, where $OP = r$ is the radius of the pitch circle, OM is the radius of the base circle, and ψ is the pressure angle.

As already defined, a rack is part of a toothed wheel of infinite diameter, therefore its base circle diameter is infinite and the profiles of the involute teeth are straight lines. Since these straight profiles are tangential to the pinion profiles at points of contact, they are perpendicular to the line of action.

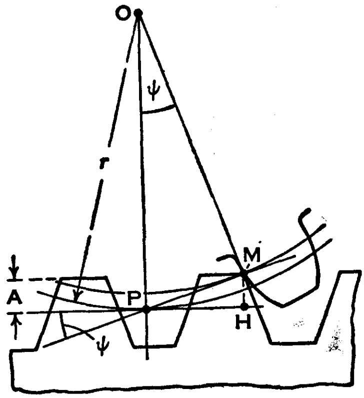


FIG. 236.

In Fig. 236, PH is the pitch line of the rack and PM is the line of action inclined at the angle ψ to PH. The rack addendum as drawn is $A = HM$, and M is the interference point, the point at which the line of action is tangential to the base circle.

The limiting value of A occurs when

$$A = HM = PM \sin \psi.$$

But $PM = OP \sin \psi = r \sin \psi$, therefore

$$A = r \sin^2 \psi$$

is the maximum value of A if interference is to be avoided.

136. Proportions of Teeth.—The nominal tooth thickness is a length of arc on the pitch circle equal to half the circular pitch.

In the *British Standard system* the pressure angle is 20°. Complete rules are given in B.S. 436 (Machine Cut Gears—Helical and Straight Spur), and the following are intended as an indication only of the proportions used. Other factors involved, which are not considered here, are the total number of teeth, the centre distance, easing of tooth face, fillet radii, and the class of gear required.

The wheel addendum and dedendum are denoted by A and B respectively, and the corresponding symbols for the pinion are a and b. The numbers of teeth on the wheel and the pinion are T and t respectively.

$$A=\frac{p}{\pi}\left\{1-0\cdot4\left(1-\frac{t}{T}\right)\right\}. \qquad B=0\cdot7162p-A.$$

$$a=\frac{p}{\pi}\left\{1+0\cdot4\left(1-\frac{t}{T}\right)\right\}. \qquad b=0\cdot7162p-a.$$

When the wheel and pinion are the same size, $t=T$, then

$$A=\frac{p}{\pi}=0\cdot3183p=\frac{1}{P},$$

$$B=0\cdot7162p-\frac{p}{\pi}=0\cdot3979p=\frac{1\cdot25}{P},$$

and these are also the values used for a rack. For precision ground gears, the value $B=0\cdot3979p$ is increased to $0\cdot4583p$.

In the original *Brown and Sharpe system* the pressure angle is $14\frac{1}{2}°$.

$$\text{Addendum}=\frac{1}{P}=0\cdot3183p.$$

$$\text{Dedendum}=\frac{1\cdot1571}{P}=0\cdot3683p.$$

$$\text{Total height}=\frac{2\cdot1571}{P}=0\cdot6866p.$$

$$\text{Bottom clearance}=\frac{0\cdot1571}{P}=0\cdot05p.$$

$$\text{Blank diameter}=D+\frac{2}{P}=\frac{T+2}{P},$$

where D is the pitch circle diameter and T is the number of teeth.

The reader is referred to *Involute Gears* by W. Steeds (Longmans, Green & Co.) for further information regarding the above and other systems.

137. Helical Gears.—A helical toothed wheel has each tooth cut on a helix, and two such wheels may be used in place of ordinary straight-toothed spur wheels to connect parallel shafts, the helices being right-handed on one wheel and left-handed on the other. The teeth engage gradually and give a smooth drive of high efficiency.

Imagine ABCD to be part of a strip of paper wrapped round a base cylinder of radius R_b (Fig. 237), then if the strip is held taut and unwrapped from the cylinder, points

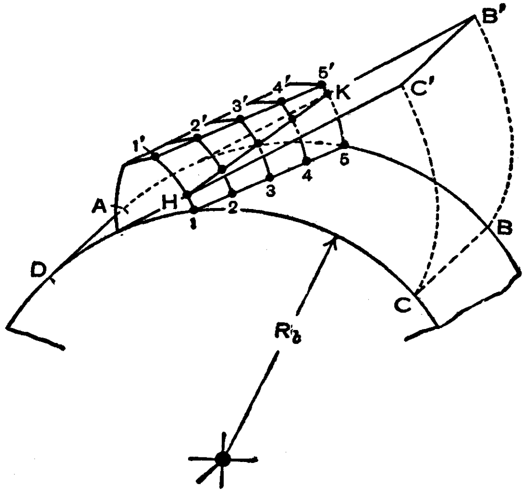


FIG. 237.

on it will describe involutes. For example, when the strip arrives at the position AB′C′D, the points B and C have traced out the involutes BB′ and CC′ respectively.

A straight line HK on the strip, inclined to the edges AB′ and DC′, becomes the helix 12345 (actually drawn as a straight line) if the strip is wrapped on the cylinder and, starting from this position, HK generates one side of a helical tooth when the strip is unwrapped from the cylinder. The curves 11′, 22′, 33′, 44′, and 55′ are all similar involutes described by points on the straight line HK. The opposite side of the tooth is generated by another strip of paper, not shown.

If the strip AB′C′D is run off one base cylinder on to another, the straight line HK describes simultaneously the mating surfaces of two teeth in contact. Alternatively, the mating surfaces may be generated separately by cutting the paper along the generating line HK and then wrapping and unwrapping each part on its own cylinder.

The contact between the two teeth is along a straight line, beginning as a point at 5 (if the mating addendum extends as far as that and the tooth profile illustrated is the driver), getting longer and longer until it extends across the teeth, then becoming shorter and shorter until finally contact ceases at the point 1′.

The gears discussed in this Art. are the *single helical* type, in which there is some axial thrust, due to the teeth not being parallel to the axes of rotation. The forces on the teeth are discussed in Art. 139. In *double helical* gears each wheel is equivalent to two single helical gears of equal lead and opposite hand, and the resulting axial thrust is zero. In both types, if a pinion having t teeth is in mesh with a wheel having T teeth and the speeds are respectively n and N r.p.m., then

$$\frac{n}{N}=\frac{T}{t}.$$

A helical wheel and its pitch cylinder are shown diagrammatically in the upper part of Fig. 238, where each sloping line represents part of a helix which is the intersection of one side of a tooth with the pitch cylinder. One turn *su* of one helix has been drawn on the cylinder. Let D be the diameter of the pitch cylinder and L the lead of each helix.

In the lower part of the Fig. the rectangle STUV, in which ST = L and TU = πD, is the development of the pitch cylinder. The diagonal SU becomes the helix *su* if the rectangle is wrapped round the cylinder.

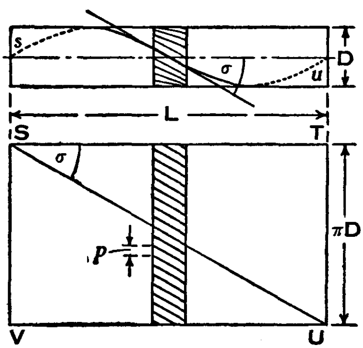


Fig. 238.

The angle TSU is the *spiral angle* and is denoted by σ (sigma). It is the complement of VSU, the *lead angle*, the angle used in connection with the friction of screw threads and then often called the helix angle (Art. 88). The relation between σ, D, and L is

$$\tan \sigma = \frac{\pi D}{L}.$$

The *circular pitch* p is measured on an arc of the pitch circle, as in the case of a spur wheel (see Fig. 238 and enlarged view in Fig. 239). If the pitch circle diameter is D and there are T teeth,

$$p = \frac{\pi D}{T}.$$

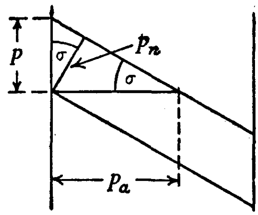


FIG. 239.

The *normal pitch* p_n is the distance between similar faces of adjacent teeth, along a helix on the pitch cylinder normal to the teeth.

$$p_n = p \cos \sigma.$$

The *axial pitch* p_a is the distance, parallel to the axis, between similar faces of adjacent teeth.

$$p_a = p \cot \sigma.$$

There is also the *diametral pitch* P already defined (p. 243).

$$P = \frac{T}{D} \quad \text{and} \quad pP = \pi.$$

The *normal diametral pitch* P_n is the diametral pitch corresponding to p_n and $p_n P_n = \pi$, therefore

$$P_n = \frac{\pi}{p_n} = \frac{\pi}{p \cos \sigma} = \frac{P}{\cos \sigma}.$$

The significance of the pitch P_n becomes obvious when it is understood that the normal pitch p_n is also the circular pitch of the approximately equivalent straight-toothed spur wheel, the radius of which is obtained as follows.

The plane EF (Fig. 240) through the centre of the wheel and normal to a tooth helix, intersects the pitch cylinder in an ellipse, as shown in the sectional plan where E′F′ and GH are the major and minor axes respectively.

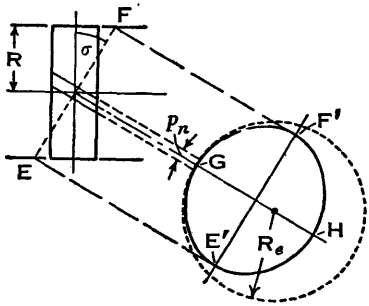


FIG. 240.

The radius of curvature of the ellipse at the point G can be shown to be *

$$R_e = \frac{R}{\cos^2 \sigma},$$

where R is the radius of the pitch cylinder. The radius R_e is approximately that of the equivalent spur wheel, the circular pitch of which is p_n.

Suppose there are T teeth in the helical wheel and T_e teeth in the equivalent spur wheel, then

$$T = \frac{2\pi R}{p} = 2PR \quad \text{and} \quad T_e = \frac{2\pi R_e}{p_n} = 2P_n R_e.$$

Since $P_n = P/\cos \sigma$ and $R_e = R/\cos^2 \sigma$,

$$T_e = \frac{2PR}{\cos^3 \sigma} = T \sec^3 \sigma.$$

138. Pressure Angles in Helical Gears.—Let ABCD (Fig. 241) be a rectangle on the plane tangential to the pitch cylinders at their line of contact CD. Let P be the pitch point at the centre of CD and NPQ be the tangent to a tooth helix passing through P.

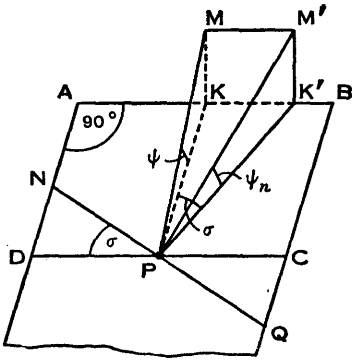


FIG. 241.

Let the angle KPM $= \psi$ be the pressure angle in the plane through P perpendicular to CD, and let the angle K′PM′ $= \psi_n$ be the *normal*

* The method is given in Art. 200 in the author's *Mathematics*.

pressure angle, that is the pressure angle in the plane through P normal to the side of the tooth or perpendicular to NQ.

To obtain the relation between ψ_n and ψ, Fig. 241 has been drawn with K and K′ in the line AB and KM and K′M′ perpendicular to the plane ABCD. Also MM′ is parallel to KK′, therefore KK′M′M is a rectangle.

Since PK is perpendicular to DC and PK′ is perpendicular to NQ, the angle KPK′ is equal to the spiral angle σ.

It follows from the Fig. that

$$\tan \psi_n = \frac{M'K'}{PK'} = \frac{MK}{PK'} = \frac{PK \tan \psi}{PK'} = \cos \sigma \tan \psi.$$

139. Tooth Forces in Helical Gears.—To estimate the forces on a tooth, friction between teeth is neglected and the resultant of the forces on the teeth in contact is assumed to act in the direction normal to the tooth surface at the pitch point at the centre of the face width.

In Fig. 242, which is redrawn from Fig. 241, suppose the tooth represented by the tangent NPQ is being driven in the direction of the arrow X, in the plane tangential to the pitch cylinders, and that the total force at P is F represented by M′P. Since M′K′P and K′KP are right-angled triangles, M′K′ and K′P are components of M′P, and K′K and KP are components of K′P.

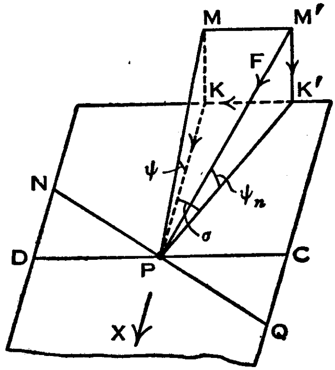


FIG. 242.

Let the component forces at P be F_r (radial), F_a (axial) and F_t (tangential or driving), represented by M′K′, K′K, and KP respectively.

Let T_q be the torque on the wheel and R be the pitch cylinder radius, then, using = for "is represented by" where necessary,

$$F_t = \frac{T_q}{R} = KP.$$

$$F_a = K'K = KP \tan \sigma = F_t \tan \sigma.$$

$$F_r = M'K' = K'P \tan \psi_n = F_t \sec \sigma \tan \psi_n.$$

$$F = M'P = K'P \sec \psi_n = F_t \sec \sigma \sec \psi_n.$$

140. Spiral Gears.—Two shafts which are not parallel and do not intersect may be connected by two helical wheels, which are then called *spiral*, *skew*, or *screw gears*, the first of these terms being the most popular in this country. Strictly, "spiral" is incorrect as applied to these gears, since a spiral is a curve in a plane, but it is advisable to distinguish between helical gears on parallel shafts and helical gears on mutually inclined shafts which do not intersect. (American writers warn their readers against using the word "spiral" in this way. They talk of helical gears on shafts which are neither parallel nor intersecting, and of helical spur gears on parallel shafts.)

Helical gears, being on parallel shafts, are of opposite hand; spiral gears may be of the same hand or opposite hand. Helical gears have line contact between the teeth, but spiral gears have point contact (theoretically). Consequently spiral gears cannot be used to transmit more than light loads.

Two shafts, AB and DE, which are not parallel and do not intersect, are shown in elevation and plan in Fig. 243. The *shaft angle*, or angle between the shafts, may be defined as the angle between lengths of the shafts which are rotating in opposite directions when viewed from the point of intersection of their axes in the plan. The lengths OB and OE are rotating in opposite directions when viewed from O, therefore the shaft angle θ is the angle BOE. If the axis of DE could be turned through the

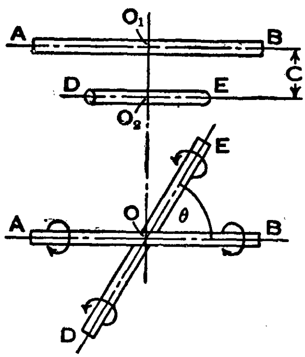


FIG. 243.

angle θ so as to coincide, in the plan, with the axis of AB, the drive from one shaft to the other could be done by helical gears.

The centre distance of spiral gears required to connect the shafts is equal to the shortest distance between the axes, and this is equal to the length of the common perpendicular O_1O_2 (Fig. 243) denoted by C.

When the gears are of the same hand, as for example in Fig. 244 where both are right-handed, the shaft angle is the

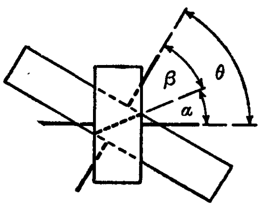


FIG. 244.

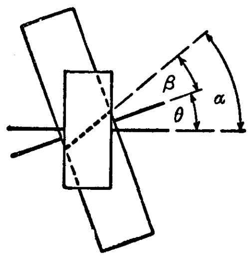


FIG. 245.

sum of the spiral angles, in this case $\theta = \alpha + \beta$. When the gears are of opposite hand, as in Fig. 245 where the pinion is right-handed and the wheel is left-handed, the shaft angle is the difference between the spiral angles, or $\theta = \alpha - \beta$, assuming each spiral angle is regarded as positive. If right-hand and left-hand spiral angles are given opposite signs, the shaft angle is always the algebraic sum of these angles.

If there are t teeth on the pinion and T teeth on the wheel, and the speeds are n and N r.p.m. respectively,

$$\frac{n}{N} = \frac{T}{t}.$$

It has been shown for helical gears that

$$p_n = p \cos \sigma,$$

where p_n is the normal pitch, p is the circular pitch, and σ is the spiral angle.

For spiral gears, the pinion and wheel have the same normal pitch, but the circular pitches depend on the spiral angles.

Pinion with pitch cylinder diameter d and spiral angle α—

$$\text{Circular pitch} = \frac{p_n}{\cos\alpha} = \frac{\pi d}{t},$$

therefore
$$d = \frac{p_n t}{\pi \cos\alpha}.$$

Wheel with pitch cylinder diameter D and spiral angle β—

$$\text{Circular pitch} = \frac{p_n}{\cos\beta} = \frac{\pi \mathrm{D}}{\mathrm{T}},$$

therefore
$$\mathrm{D} = \frac{p_n \mathrm{T}}{\pi \cos\beta}.$$

From the equations for d and D,

$$\frac{d\cos\alpha}{t} = \frac{\mathrm{D}\cos\beta}{\mathrm{T}}.$$

The centre distance is

$$\mathrm{C} = \tfrac{1}{2}(d + \mathrm{D}) = \frac{p_n}{2\pi}\left\{\frac{t}{\cos\alpha} + \frac{\mathrm{T}}{\cos\beta}\right\}.$$

From Art. 137 it follows that the numbers of teeth in the equivalent spur wheels are

$$t_e = t\sec^3\alpha \quad \text{and} \quad \mathrm{T}_e = \mathrm{T}\sec^3\beta$$

for the pinion and wheel respectively.

141. Efficiency of Spiral Gears.—An approximation to the efficiency of a pair of spiral gears will now be obtained. The gears in Fig. 246 are shown separately to a larger scale in Figs. 247 and 248, with the forces acting on each of a pair of teeth in contact. The tooth profile on the driver is underneath and is represented by a dotted line, whereas the mating profile, being above on the follower, is shown as a full line. The forces are assumed to act at the centre of the width of each tooth and to be in the plane which is tangential to the pitch cylinders; actually the normal force on each tooth is inclined at the normal pressure angle to this plane. The teeth slide lengthways on each other as the gears turn, and there is a corresponding frictional force on each tooth. Losses due to friction in the bearings are neglected.

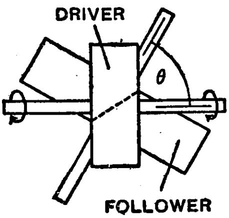


Fig. 246.

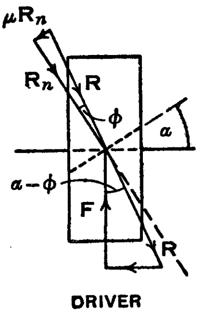


Fig. 247.

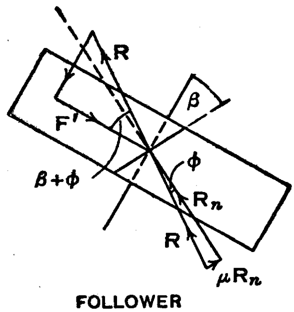


Fig. 248.

On the driver the applied force is F and the spiral angle is α; on the follower the useful resisting force is F′ and the spiral angle is β. On each tooth the normal reaction in the tangential plane is R_n, the frictional resistance is μR_n, ϕ is the angle of friction, and the resultant reaction is R. The shaft angle is $\theta = \alpha + \beta$, both gears being of the same hand.

Since the actual normal reaction on the tooth is inclined at the pressure angle ψ_n to the tangential plane, the coefficient of friction μ may be taken as equal to $\mu_n \sec \psi_n$, where μ_n is the ordinary coefficient of friction. In general this refinement is not of importance, since the expression for the efficiency is only an approximation; also the coefficient of friction is not usually known accurately.

From Fig. 247,

$$F = R \cos (\alpha - \phi),$$

and from Fig. 248,

$$F' = R \cos (\beta + \phi).$$

If the angular speed of the driver is n and the pitch cylinder diameter is d, the work done per unit time is $F.\pi dn$. If the angular speed of the follower is N and the pitch cylinder diameter is D, the work given out per unit time is $F' . \pi DN$. Therefore the efficiency is

$$\eta = \frac{F'DN}{Fdn}.$$

From the preceding Art.,

$$\frac{D}{d} = \frac{T \cos \alpha}{t \cos \beta} \quad \text{and} \quad \frac{N}{n} = \frac{t}{T}, \quad \text{or} \quad \frac{DN}{dn} = \frac{\cos \alpha}{\cos \beta}.$$

Therefore $$\eta = \frac{\cos (\beta + \phi) \cos \alpha}{\cos (\alpha - \phi) \cos \beta} \quad . \quad . \quad (1),$$

where, as already stated, α and β are the spiral angles of the driver and follower respectively, and the shaft angle is $\theta = \alpha + \beta$. (See Ex. 13, p. 274, for the efficiency when $\theta = \alpha - \beta$ and when $\theta = \beta - \alpha$.)

If $\cos (\beta + \phi)$ and $\cos (\alpha - \phi)$ are expanded, the above expression for the efficiency may then be simplified and it becomes

$$\eta = \frac{1 - \mu \tan \beta}{1 + \mu \tan \alpha} \quad . \quad . \quad . \quad (2),$$

where $\mu = \tan \phi$.

For a given shaft angle θ, the values of α and β when the efficiency is a maximum are most easily obtained by using the identity

$$\cos A \cos B = \tfrac{1}{2}\{\cos (A + B) + \cos (A - B)\},$$

then from (1),

$$\eta = \frac{\cos (\alpha + \beta + \phi) + \cos (\alpha - \beta - \phi)}{\cos (\alpha + \beta - \phi) + \cos (\alpha - \beta - \phi)}.$$

This is a maximum when

$$\cos (\alpha - \beta - \phi) = 1 \quad \text{or} \quad \alpha - \beta - \phi = 0.$$

Using this equation and the relation $\alpha + \beta = \theta$ gives

$$\alpha = \tfrac{1}{2}(\theta + \phi) \quad \text{and} \quad \beta = \tfrac{1}{2}(\theta - \phi)$$

for maximum efficiency.

142. Bevel Gears.—Two shafts with axes OM and ON intersecting at O (Fig. 249) are connected by friction wheels 1 and 2, which are frusta of right cones OAB and OBC. Assuming there is no slip, the shafts will have a constant velocity ratio.

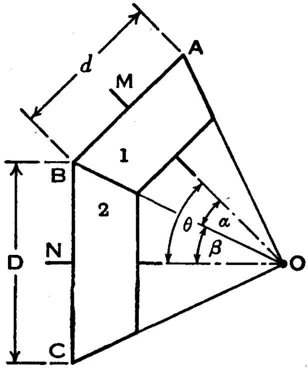


Fig. 249.

When the drive is transmitted from one wheel to the other by teeth, the wheels are *bevel gears* and the cones OAB and OBC are the *pitch cones.* The outer end faces of the teeth intersect the pitch cones in circles called the *pitch circles*; in Fig. 249 these are AB and BC. The *pitch point* is the point of contact of the pitch circles, in this case the point B.

The semi-vertex angles of the pitch cones are the *pitch angles*; these are denoted by α and β in Fig. 249. The shaft angle is the sum of the pitch angles, or $\theta=\alpha+\beta$.

Let d and D be the pitch circle diameters of the pitch cones OAB and OBC respectively, then

$$\tfrac{1}{2}d=\text{OB}\sin\alpha \quad \text{and} \quad \tfrac{1}{2}\text{D}=\text{OB}\sin\beta$$

or

$$\frac{d}{\text{D}}=\frac{\sin\alpha}{\sin\beta}.$$

The linear speeds of points on the pitch circles are equal, therefore, denoting the angular velocities of the wheels by ω_1 and ω_2, $\tfrac{1}{2}d\omega_1=\tfrac{1}{2}\text{D}\omega_2$ and

$$\frac{\omega_1}{\omega_2}=\frac{\text{D}}{d}=\frac{\sin\beta}{\sin\alpha}.$$

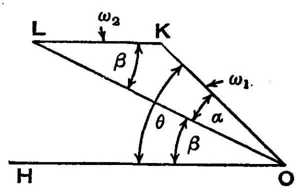


Fig. 250.

Since $\theta=\alpha+\beta$, the pitch angles α and β can be determined when ω_1, ω_2, and θ are known.

A graphical solution is shown in Fig. 250, where HOK is the given angle θ. OK is made equal to ω_1 on a suitable scale, KL is drawn parallel to OH and equal to ω_2, and OL is joined. In the triangle OKL, since the ratio of sin β and sin α is equal to the ratio of ω_1 and ω_2, α and β are the required angles.

Cycloidal teeth may be generated on a cylindrical wheel by rolling two smaller cylinders in contact with the pitch cylinder, one inside it and the other outside it. A straight line is drawn on each rolling cylinder, parallel to its axis, then the face of a tooth is generated by the line on the outer cylinder and the flank is generated by the line on the inner cylinder.

The teeth on a bevel wheel may be generated in the same way by using cones instead of cylinders. The vertices of the cones coincide, therefore any point on the surface of either rolling cone moves in a path which is on the surface of a sphere, and the teeth are known as *spherical cycloidal teeth.*

Similarly, by rolling a plane ORS (Fig. 251) on a cone OAB, a *spherical involute tooth* profile $cee'c'$ is generated by a line $c'e'$ drawn on the plane through the vertex O of the cone. The axis of the cone is OM. Since any point fixed on the plane is at a constant distance from O, its path lies on the surface of a sphere. For example, the curve cc' is on a sphere of radius Oc and the curve ee' is on another sphere of radius Oe. The cone on which the plane rolls is a *base cone*, and it corresponds to a base cylinder in a spur wheel.

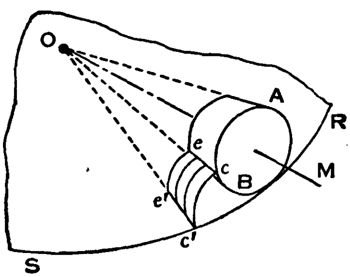


FIG. 251.

A *crown wheel* is a bevel gear in which the pitch angle is 90°. The pitch cone in this case becomes a flat disc XOX as shown in Fig. 252, where two teeth only are drawn. The addendum is the height A above the pitch cone and the dedendum is the depth B below it.

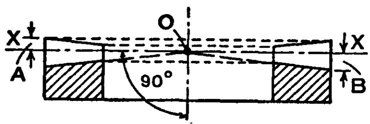


FIG. 252.

It is evident that with spherical involute teeth the outer ends of crown wheel teeth lie on the surface of a sphere and their profiles are curved surfaces; the semi-vertex angle of the base cone is of course less than 90°. In general the outer ends are machined to a cylindrical surface and the profiles are made flat surfaces. The teeth are then called *octoid teeth*, the name being derived from the shape of the path of contact which, although practically straight over the actual path, is a figure 8 over the complete theoretical path. It is beyond the scope of this book to discuss this matter further.

The teeth of bevel wheels are cut so that they mesh with a crown wheel, and in general they are octoid teeth. It should be noted, however, that the difference between spherical involute teeth and octoid teeth is only slight; also, when the tooth faces are modified at the tips by a few thousandths of an inch, the profiles are approximations to octoid teeth.

Mitre wheels are equal bevel wheels which connect shafts whose axes are mutually perpendicular.

143. Tredgold's Approximation—Equivalent Spur Gears. —Theoretically, the outer ends of the teeth of a bevel wheel ABEF (Fig. 253) lie on the surface of a sphere AHB. Tredgold's approximation consists of drawing the tooth

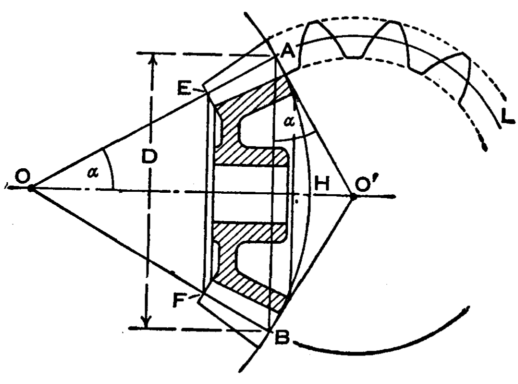


FIG. 253.

outlines on the *back cone* O′AB which envelops the sphere, intersecting the pitch cone OAB in the pitch circle AB. The common axis of the two cones is OO′ and the angles OAO′ and OBO′ are right angles.

An arc of a circle AL is drawn with its centre at O′, and O′AL is the development of part of the back cone. The arc AL is part of the pitch circle of a spur wheel of radius O′A called the *equivalent spur wheel*. The teeth on this spur wheel are then designed in the usual way, and their profiles are very approximately the same as those of the outer ends of the bevel wheel teeth.

As already explained, the bevel wheel teeth are generated so that they are conjugate to crown wheel teeth, but the point is that factors such as the number of teeth in contact, interference, strength of teeth, etc. are determined by considering the teeth on the equivalent spur wheel.

If D and D_e are the pitch circle diameters of the bevel wheel and the equivalent spur wheel respectively, and a is the pitch angle, then the angle O′AB is equal to a and

$$D_e = \frac{D}{\cos a}.$$

144. Worm Gears.—Shafts which are neither parallel nor intersecting may be connected by a worm A and worm wheel B (Fig. 254). Generally the axes are mutually perpendicular. The worm has helical teeth or threads which mesh with curved teeth on the worm wheel, and there is line contact, consequently greater loads can be carried by worm gears than by spiral gears. When the worm has more than one thread it is called a *multi-thread* or *multi-start worm*. If n is the angular speed of the worm and N is that of the worm wheel,

$$\frac{N}{n} = \frac{\text{Number of threads on worm}}{\text{Number of teeth on wheel}}.$$

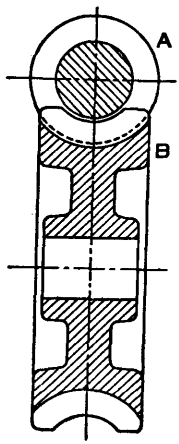


FIG. 254.

If λ is the lead angle of the worm and ϕ is the friction angle, the efficiency, when the worm is the driver, is

$$\eta = \frac{\tan \lambda}{\tan (\lambda + \phi)},$$

which is the value obtained for a screw and nut in Art. 88, p. 166, where the angle λ is denoted by α.

There is considerable sliding between the worm and the worm wheel teeth and, to make the friction losses as low as possible, the worm may be of case-hardened steel and the worm wheel of phosphor-bronze. In large wheels the teeth are cut on a phosphor-bronze rim which is secured to a steel wheel.

145. References.—In this chapter it has been possible to give only a brief outline of the theory of toothed gears, and the reader is advised to refer to the following books for further information.

Involute Gears. By W. Steeds. Longmans, Green.
Gears. By H. E. Merritt. Pitman.
Gears: Spur, Helical, Bevel and Worm. By P. S. Houghton. The Technical Press.

The following British Standards, published by the British Standards Institution, should also be consulted.

B.S. 436. *Machine-Cut Gears.* A. *Helical and Straight Spur.*
B.S. 545. *Bevel Gears (Machine-Cut).*
B.S. 721. *Machine-Cut Gears. Worm Gearing.*

Exercises XIII

1. A gear wheel has 30 teeth and the diametral pitch is 5. Calculate the values of the pitch diameter, the circular pitch, and the module.

2. (*a*) What is the condition that must be satisfied if the velocity ratio between a pair of spur gears is to remain constant during the period of contact between a pair of teeth?

(*b*) A pair of gears, having 40 and 20 teeth respectively, are rotating in mesh, the speed of the smaller being 2000 r.p.m. Determine the speed of sliding between the gear teeth faces at the point of engagement, at the pitch point, and at the point of disengagement, if the smaller gear is the driver. Assume that the gear teeth are of 20° involute form, that the addendum length is 0·200 inch, and that the diametral pitch is 5. [U.L.]

3. Two spur wheels having 20 and 40 teeth of involute form mesh externally. The diametral pitch is 4 and the addendum on both wheels is 0·25 inch, the pressure angle being 20°.

Determine the maximum velocity of sliding of one tooth over another if the larger wheel runs at 200 r.p.m. [U.L.]

4. Two gear wheels of 6 diametral pitch have 24 and 33 teeth respectively. The pressure angle is 20° and each wheel has a standard addendum of 1 module. Find the length of the arc of contact and the maximum sliding velocity if the speed of the smaller wheel is 120 r.p.m. [U.L.]

5. Two mating gear wheels have 20 and 40 involute teeth of 2 diametral pitch and 20° obliquity, and the addendum on each wheel is to be of such a length that the line of contact on each side of the pitch point has half the maximum possible length.

Determine the addendum height for each gear wheel and the length of the line of contact.

If the smaller wheel rotates at 250 r.p.m., find (*a*) the velocity of the point of contact along the surface of each tooth at the instant when the tip of a tooth on the smaller wheel is in contact, and (*b*) the velocity of sliding at this instant. [U.L.]

6. Two gears in mesh have a centre distance of 12 inches and a gear ratio of 2·2 to 1. The teeth are of Brown and Sharpe proportions with a diametral pitch of 4. Find the pitch diameters, the blank diameters, and the number of teeth on each gear and their full depth.

7. A gear wheel having 20 involute teeth of $\frac{1}{2}$ inch circular pitch is to be generated by means of a straight-sided rack cutter. The addendum of the cutter and of the wheel is 0·159 inch. What is the smallest pressure angle which may be employed if undercutting is to be avoided?

Calculate from first principles the length of the arc of contact when two such wheels, each of 20 teeth, mesh together correctly. [U.L.]

8. A wheel and pinion in mesh have pitch circle radii of R and r respectively. Suppose the wheel, whose teeth have addenda equal to A, causes interference on the pinion. Reference to Fig. 235 will show that this interference can be avoided by increasing the pressure angle ψ. Prove that the minimum value of ψ which will prevent interference is given by

$$\sin \psi = \sqrt{\frac{A^2 + 2AR}{r^2 + 2rR}}.$$

9. One method of avoiding interference is to extend the distance between the gear centres. This increases the pitch

diameters and the pressure angle ψ, but the gears are cut with standard cutters and the base circles are not altered. Since the pitch diameters are increased and the numbers of teeth are unchanged, the circular pitch p and the diametral pitch P are altered.

Let e be the extension of the centre distance C, ψ_e be the increased pressure angle, and p_e and P_e be the new circular and diametral pitches respectively.

Show that

$$\cos \psi_e = \frac{C \cos \psi}{C+e},$$

$$p_e = \left\{\frac{C+e}{C}\right\} p,$$

and

$$P_e = \frac{\pi C}{(C+e)p} = \frac{CP}{C+e}.$$

10. Find the least number of teeth a pinion may have if interference by the wheel with which it meshes is to be avoided, given that the pressure angle is 20°, the addendum of a wheel tooth is 0·72/P, and the ratio of the pitch diameters of the wheel and the pinion is 4.

11. The torque on the pinion of a helical gear drive is 500 lb. ft. and the pitch diameter of the pinion is 2 feet. The spiral angle is 30° and the normal pressure angle is 20°. Find the component forces on a tooth.

12. A spiral wheel reduction gear, of ratio 3 to 2, is to be used on a machine, with the angle between the shafts 80°. The approximate centre distance between the shafts is 5 inches, the normal pitch of the teeth is 0·4 inch, and the wheel diameters are equal. Find the number of teeth on each wheel, the pitch circle diameter, and the spiral angles. Find the efficiency of the drive if the friction angle is 5°. [U.L.]

13. Two spiral gears A and B of opposite hand, with spiral angles α and β respectively, are in mesh, and A is the driver. Show that the efficiency, when α is greater than β, is

$$\eta = \frac{\cos(\beta - \phi)\cos\alpha}{\cos(\alpha - \phi)\cos\beta} = \frac{1 + \mu \tan\beta}{1 + \mu\tan\alpha},$$

and when α is less than β,

$$\eta = \frac{\cos(\beta + \phi)\cos\alpha}{\cos(\alpha + \phi)\cos\beta} = \frac{1 - \mu \tan\beta}{1 - \mu\tan\alpha}.$$

14. Two spiral gear wheels A and B have 45 and 15 teeth at spiral angles 20° and 50° respectively. Both wheels are of the same hand. A is 6 inches diameter.

Find the distance and the angle, between the shafts. If the teeth are of 20° involute form and the coefficient of friction is 0·08, find the efficiency of the gears (*a*) if A is the driver, (*b*) if B is the driver. Derive any formulæ you use. [U.L.]

15. A screw gear has teeth with a helix angle α, and the number of teeth on the gear is n. Show that the number of teeth n_1 for which the gear cutter should be chosen is given by $n_1 = n \sec^3 \alpha$.

[*Note.*—Take the helix angle to be the spiral angle defined in the text.]

Two horizontal shafts are connected by a pair of screw gears having a normal tooth section of 4 D.P. The driving gear has 16 teeth and a helix angle of 45°, right-handed. The driven wheel has 12 teeth and a helix angle of 15°, and also is right-handed.

Make a sketch of the arrangement, stating the angle between the shafts and the normal distance between them. [U.L.]

16. Two bevel wheels, 1 and 2, are to connect two shafts whose axes are mutually inclined at 75°. Wheel 1 is to have a pitch diameter of 8 inches, run at 400 r.p.m., and drive wheel 2 at 250 r.p.m.

Find, for each wheel, the pitch angle and the pitch diameter of the equivalent spur wheel.

17. (*a*) If a pair of spiral gears, A and B, rotate at rates N_A and N_B respectively, have pitch diameters D_A and D_B respectively, and spiral angles α and β respectively, show that the sliding velocity along tooth helices at the pitch point is equal to

$$\pi(N_A D_A \sin \alpha + N_B D_B \sin \beta).$$

(*b*) Hence show that, if the gear ratio r, the centre distance C, the normal diametral pitch P, and the shaft angle θ are fixed, this sliding velocity will be a minimum when

$$\cot \alpha = \frac{r + \cos \theta}{\sin \theta}.$$

[U.L.]

18. Two shafts are to be connected by spiral gears with a velocity ratio of 3 to 1. The angle between the shafts is 45° and the least distance between the shaft axes is to be 9 inches. The normal diametral pitch is to be 5 and the pinion is to have 20 teeth. Determine the pitch circle diameters and the spiral angles, which are to have the same hand.

If the pinion rotates at 240 r.p.m., what will be the speed of rubbing between the teeth? [U.L.]

CHAPTER XIV

GEAR TRAINS

146. Gear Trains.—When motion is transmitted from one shaft to another by toothed wheels, these wheels form a *gear train, wheel train,* or *train of wheels.* In an *ordinary* type of gear train the wheels turn about fixed axes, but in an epicyclic train, to be dealt with later, at least one axis moves. In a *simple train* (Figs. 255 and 256)

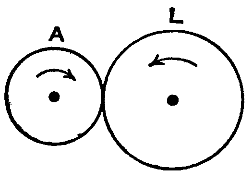


FIG. 255.

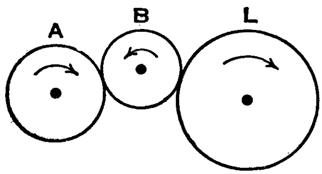


FIG. 256.

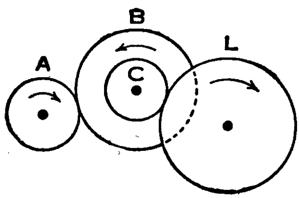


FIG. 257.

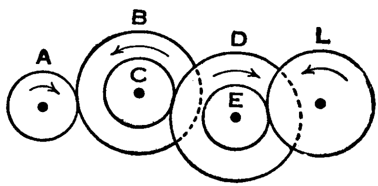


FIG. 258.

there is one wheel on each shaft, and in a *compound train* at least one shaft carries a *compound wheel,* that is two wheels which are fixed to one another (Figs. 257 and 258).

The *velocity ratio* or the *value of a train of wheels* is defined or used by various authorities in two different ways: it is (1) the ratio of the speeds of the first and last wheels or shafts, or (2) the ratio of the speeds of the last and first wheels or shafts. To avoid confusion the ratio should be plainly stated, for example N_1/N_2 or N_2/N_1. The sign of the ratio is positive when both wheels rotate in the same direction, and negative when they rotate in opposite directions; it may be put in by inspection when required.

Let the numbers of teeth in the wheels A, B, C, etc. (Figs. 255 to 258) be described by these same letters, and let the corresponding wheel speeds be N_A, N_B, N_C, etc. Let A be the first wheel or driver and L be the last wheel or final driven wheel.

Let the pitch circle diameters of wheels A and L (Fig. 255) be d_A and d_L respectively, then if v is the linear speed of a point on either of these circles,

$$v = N_A \pi d_A = N_L \pi d_L \quad \text{or} \quad \frac{N_A}{N_L} = \frac{d_L}{d_A}.$$

But the numbers of teeth in the wheels are proportional to their pitch circle diameters, therefore

$$\frac{N_A}{N_L} = \frac{L}{A},$$

and since there are only two wheels they turn in opposite directions.

The intermediate wheel B in Fig. 256 is known as an *idle wheel* or *idler*, and it enables the wheel L to turn in the same direction as the wheel A. An idler is driven and it drives the following wheel; it reverses the direction of rotation but does not affect the numerical value of the velocity ratio. There may be several idlers in a gear train, particularly where the input and output shafts are some distance apart; one alternative would be to use fewer and larger wheels, but these would probably occupy more space.

In Fig. 256, $$\frac{N_A}{N_L} = \frac{N_A}{N_B} \cdot \frac{N_B}{N_L},$$

but $$\frac{N_A}{N_B} = \frac{B}{A} \quad \text{and} \quad \frac{N_B}{N_L} = \frac{L}{B},$$

therefore $$\frac{N_A}{N_L} = \frac{B}{A} \cdot \frac{L}{B} = \frac{L}{A},$$

which is independent of the size of the idler.

In the compound train in Fig. 257, B and C form the compound wheel, therefore $N_B = N_C$ and

$$\frac{N_A}{N_L} = \frac{N_A}{N_B} \cdot \frac{N_C}{N_L} = \frac{B}{A} \cdot \frac{L}{C},$$

and it can be seen from the Fig. that the first and last wheels turn in the same direction.

In the compound train in Fig. 258 there are two compound wheels, B with C and D with E; the first and last wheels turn in opposite directions.

$$N_B = N_C \quad \text{and} \quad N_D = N_E.$$

$$\frac{N_A}{N_L} = \frac{N_A}{N_B} \cdot \frac{N_C}{N_D} \cdot \frac{N_E}{N_L} = \frac{B}{A} \cdot \frac{D}{C} \cdot \frac{L}{E}.$$

It follows from these examples that the velocity ratio

$$\frac{N_A}{N_L} = \frac{\text{Product of teeth in driven wheels}}{\text{Product of teeth in driving wheels}}.$$

Also in a train in which the wheel axes are parallel and in which there are idlers, the first and last wheels turn in the same direction when the number of idlers is odd, and in opposite directions when the number of idlers is even. In general it is advisable to decide the directions of rotation of the various wheels by inspection rather than by a rule. This is particularly true if the wheel axes are not all parallel, for instance when a train includes bevel wheels or a worm and worm wheel.

When the axes of the first and last wheels of a train coincide, it is called a *reverted train.* An example is shown in Fig. 259, where A is the driving wheel, L is the final driven wheel, and BC is a compound wheel.

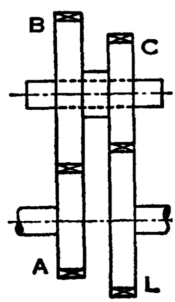


FIG. 259.

147. Centre Distance between Two Gear Wheels on Parallel Shafts.—If A and B are the numbers of teeth on two wheels in mesh on parallel shafts and the diametral pitch is denoted by P, then the pitch circle diameters are A/P and B/P, therefore the centre distance is

$$\frac{A+B}{2P}.$$

When the centre distance is specified and a standard diametral pitch is used, it may not be possible to produce a given velocity ratio. Either the centre distance or the velocity ratio may have to be modified slightly. See exercises 4 to 7 at the end of this chapter.

148. Epicyclic Trains.—When one or more of the axes in a gear train move about another axis, it is called an *epicyclic train*. A simple example is shown in Fig. 260, where a wheel S and an arm R have a common axis at O about which they can turn; a second wheel P meshes

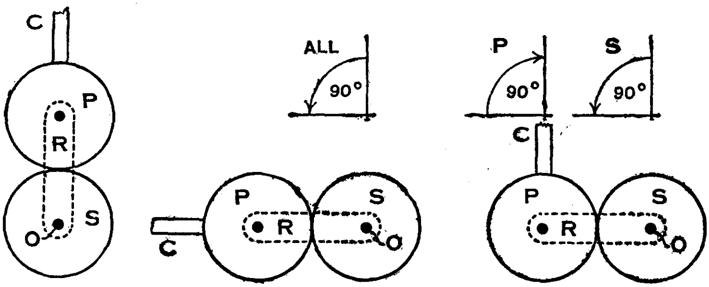


FIG. 260. FIG. 261. FIG. 262.

with S and has its axis in the arm R. If the arm R is fixed, the train is an ordinary one and S could drive P or vice versa, but if S is fixed and the arm R is rotated, the gear becomes an epicyclic train, for the axis of the wheel P moves and the wheel itself turns round on S.

If P is prevented from turning and is made to move round S, both R and S will rotate and the epicyclic train is then the *sun and planet motion* devised by Watt as a substitute for the ordinary crank in his steam engine. The *planet* wheel P was bolted to the connecting-rod C and the *sun* wheel S was fixed to the flywheel shaft. If P and S are the same size, then when the arm R does one revolution the sun S does two revolutions in the same direction, as will now be demonstrated.

Starting with the arm R and the connecting-rod C in the vertical positions, lock the wheels and turn all through 90° (Fig. 261), then fix R and turn P back through 90° (Fig. 262). The result is that P and C are in their original angular positions, the arm R has turned through 90° and the sun S has turned through $90° + 90°$ or 180°, therefore S rotates twice as fast as the arm R.

It is interesting to note that Watt made a model of a crank mechanism to convert reciprocating motion to rotary motion, but neglected to take out a patent and a workman passed on the idea to a rival, who patented it in 1780.* Watt decided to use another method and patented his sun and planet motion in 1781. Instead of using the arm R to keep P and S in gear, as shown in Fig. 260, he arranged the pin at the centre of P to move in a stationary groove, but both methods produce the same motion of S.

Actually the connecting-rod C does not remain vertical during one cycle of the engine, and the planet P oscillates whilst moving round the sun, but after one revolution of the arm R the sun has turned through two revolutions.

Another arrangement of an epicyclic train is shown in Fig. 264, where the planet P meshes with an internally toothed or annular wheel A in addition to meshing with the sun S. In practice there may be two, three, or more planets arranged round the sun at equiangular intervals to reduce the loads on the teeth and to balance the radial forces. Before dealing with the general methods of finding velocity ratios in epicyclic trains, it is important to understand that if a sun, a planet, and an arm are locked together and turned one revolution about the axis of the sun, then the planet turns one revolution about its own axis, although it does not move relatively to the arm. This fact is illustrated in Fig. 263, where the wheels and arm are shown in positions at intervals of 90°, and an arrow has been

* See *Text-Book on the Steam Engine*, by T. M. Goodeve (Crosby Lockwood & Co., 1879), pp. 104–107.

drawn on the planet to emphasize that it has turned one revolution about its own axis when it gets back to its starting position.

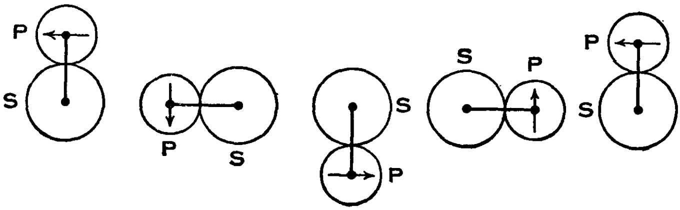


FIG. 263.

Three ways of finding velocity ratios in epicyclic gears are described below. These are tabular, semi-graphical, and algebraic methods, but it will be seen that all make use of the velocity ratio obtained when the arm is fixed. Referring to Fig. 264, two cases will be considered: first when the sun (or the annulus) is fixed and the arm is turning, and second when the sun, annulus, and arm are all turning. The letters S, P, and A describing the sun, planet, and annulus respectively will also be used to denote the numbers of teeth on them. The speeds of S, P, A, and the arm R will be denoted by N_S, N_P, N_A, and N_R respectively.

149. Epicyclic Trains—Tabular Method.

Case i. S fixed, R and A turning (Fig. 264). To find the relation between N_A and N_R.

	R	S	P	A
(1) Fix R. Turn S, +1 rev. . .	0	+1	$-\frac{S}{P}$	$-\frac{S}{A}$
(2) Lock wheels. Turn all, −1 rev. .	−1	−1	−1	−1
(3) Add (1) and (2)	−1	0	$-\left(\frac{S}{P}+1\right)$	$-\left(\frac{S}{A}+1\right)$

Line (1) shows that when R is fixed and S does one revolution, P does $-\frac{S}{P}$ revolution and A does $-\frac{S}{A}$ revolution.

When R is fixed,

$$\frac{N_A}{N_S} = -\frac{S}{A}.$$

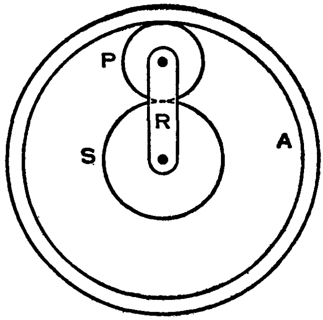


FIG. 264.

Line (2) brings S back to its starting position, so that when in line (3) the first two lines are added, the relation between the speeds of A and R can be found when S is fixed.

When S is fixed,

$$\frac{N_A}{N_R} = \frac{-\left(\frac{S}{A}+1\right)}{-1} = \frac{S}{A}+1.$$

If A is to be fixed, it would be given $+1$ revolution in line (1) of the table, then P would do $+\frac{A}{P}$ and S would do $-\frac{A}{S}$. Add (1) and (2), then the results would be R, -1; S, $-\frac{A}{S}-1$; P, $\frac{A}{P}-1$; and A, 0.

Case ii. R, S, and A turning. To find N_R given N_S and N_A. The lines in the table are completed in the order indicated.

	R	S	A
(1) Fix R. Turn S, +1 rev. . . .	0	$+1$	$-\frac{S}{A}$
(4)	0	$N_S - N_R$	$-\frac{S}{A}(N_S - N_R)$
(3) Lock wheels. Turn all, N_R revs. . .	N_R	N_R	N_R
(2) Write down the results . . .	N_R	N_S	N_A

Complete line (4) so that lines (4)+(3) give line (2) as follows: Give R, 0. Give S, $N_S - N_R$, then it follows from line (1) that A must have $-\frac{S}{A}(N_S - N_R)$.

From the column headed A, by addition

$$N_R - \frac{S}{A}(N_S - N_R) = N_A,$$

from which
$$N_R = \frac{N_A + \frac{S}{A}N_S}{1 + \frac{S}{A}}.$$

Example 1.—Given $S=45$ and $A=115$, it is required to find N_R when A is fixed and $N_S=200$ r.p.m. (Fig. 264).

	R	S	A
(1) Fix R. Turn A, -1 rev.	0	$+\frac{115}{45}=\frac{23}{9}$	-1
(2) Lock wheels. Turn all, $+1$ rev. . . .	$+1$	$+1$	$+1$
(3) Add (1) and (2)	$+1$	$+\frac{32}{9}$	0

It is immaterial whether A is given $+1$ or -1 revolution, but it was noticed that by using the negative sign the figures in line (3) would be positive.

From line (3)
$$\frac{N_S}{N_R} = \frac{32}{9},$$

therefore $\quad N_R = \frac{9}{32}N_S = \frac{9}{32} \times 200 = 56{\cdot}25$ r.p.m.

Example 2.—Using data from Ex. 1, it is required to find N_R when $N_S=200$ r.p.m. and $N_A=-60$ r.p.m.

	R	S	A
(1) Fix R. Turn A, -1 rev. . . .	0	$+\frac{23}{9}$	-1
(4)	0	$\frac{23}{9}(60+N_R)$	$-60-N_R$
(3) Lock wheels. Turn all, N_R revs. . .	N_R	N_R	N_R
(2) Write down the results . . .	N_R	200	-60

Complete lines (1), (2), and (3) as shown, then fill in line (4) so that lines (4)+(3) give line (2) as follows: Give R, 0. Give A, $-60-N_R$, then from line (1) it follows that S must have $-\frac{23}{9}(-60-N_R)$ or $\frac{23}{9}(60+N_R)$.

From column headed S, by addition

$$\tfrac{23}{9}(60 + N_R) + N_R = 200,$$

from which $$N_R = 13{\cdot}125 \text{ r.p.m.}$$

The student should now work out this example again by giving +1 revolution to S in line (1) and obtaining the final equation from the column headed A.

150. Epicyclic Trains—Semi-Graphical Method.*—In Fig. 265 there are two planets P, but this does not affect the speed relations between the members of the train. When R is fixed and S does +1 revolution, P does $-\frac{S}{P}$ and A does $-\frac{S}{A}$ revolution.

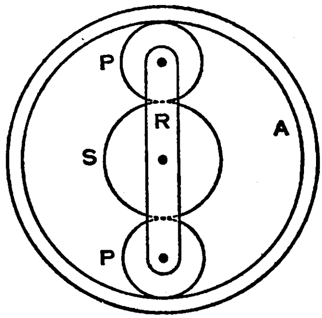


FIG. 265.

From a point r mark off a vector rs in the positive direction, equal to unity on a convenient scale (Fig. 266 (1)), to represent one positive revolution of S relative to R. Also mark off a vector ra in the negative direction, equal to $\frac{S}{A}$ on the same scale, to represent $-\frac{S}{A}$ of a revolution of A relative to R. The point r has also been labelled O to

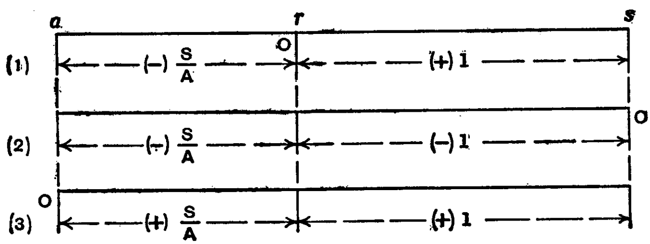


FIG. 266.

* See letter in *Engineering*, April 6, 1951, p. 408, from Mr W. McHutchison, who states that this method was introduced by Prof. William Kerr at the Royal Technical College, Glasgow.

emphasize that it is the origin or reference point. A vector rp could be marked off in the negative direction to represent the motion of each planet P relative to R, but this has been omitted to simplify the illustration.

Let O be shifted to s as shown in Fig. 266 at (2), then this is equivalent to adding one negative revolution to each of the motions of S, R, and A. The result is that S is at rest, R does -1 revolution and A does $-1-\frac{S}{A}$ revolutions.

Next let O be shifted to a as shown at (3), then A is at rest, R does $+\frac{S}{A}$ and S does $+\frac{S}{A}+1$ revolutions.

In solving a problem, this vector diagram is drawn once only and the new reference point is settled as required. Lengths are regarded as positive or negative according as they are measured to the right or left of the new reference point, which may be anywhere on the vector diagram and this may be produced in either direction. This type of vector diagram is made use of again in Art. 153, where torques in epicyclic trains are considered.

The examples already worked out in Art. 149 will now be repeated.

Example 1.—Given $S=45$ and $A=115$, it is required to find N_R when A is fixed and $N_S=200$ r.p.m. (Fig. 265).

Fix R. Let S turn $+1$ revolution, then A does $-\frac{45}{115}=-\frac{9}{23}$ revolution.

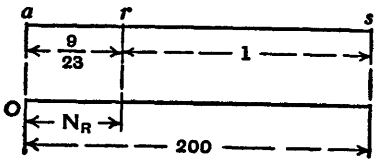


FIG. 267.

Mark off $rs=+1$ and $ra=-\frac{9}{23}$ as indicated in Fig. 267; the negative sign is not required on the diagram. Since A is fixed, the new reference point O is at a and the known and unknown speeds, 200 r.p.m. and N_R r.p.m., are proportional to the lengths as and ar respectively.

It follows that

$$\frac{N_R}{200}=\frac{\frac{9}{23}}{\frac{9}{23}+1}=\frac{9}{32}$$

and $$N_R=\frac{9}{32}\times 200=56{\cdot}25 \text{ r.p.m.}$$

It will be seen that the ratio of the vectors *ar* and *as* would remain unaltered whether A or S were given ± 1 turn in the first place.

Example 2.—Using data from Ex. 1 it is required to find N_R when $N_S = 200$ and $N_A = -60$ r.p.m.

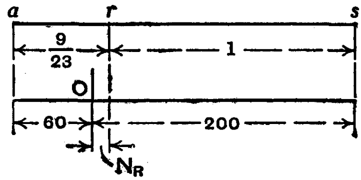


FIG. 268.

Since N_A is negative and N_S is positive, the reference point O is now between *a* and *s* (Fig. 268) at the position shown, and this is $\frac{60}{260}$ of the length *as* from *a*. The speed N_R is proportional to the length indicated.

Equating ratios

$$\frac{60 + N_R}{\frac{9}{23}} = \frac{200 - N_R}{1},$$

from which $$23(60 + N_R) = 9(200 - N_R)$$

and $$N_R = 13{\cdot}125 \text{ r.p.m.}$$

151. Epicyclic Trains—Algebraic Method.—As in the other methods, the first step is to consider the motion of each member of the train relative to the arm. When the arm R is fixed (Fig. 265) and the sun S does $+1$ revolution, P does $-\frac{S}{P}$ and A does $-\frac{S}{A}$ revolution.

These relative motions may be expressed in another way, for the speeds of S, P, and A relative to R are $N_S - N_R$, $N_P - N_R$, and $N_A - N_R$ respectively.

Therefore $$\frac{N_S - N_R}{N_P - N_R} = \frac{1}{-S/P} = -\frac{P}{S}$$

and $$\frac{N_S - N_R}{N_A - N_R} = \frac{1}{-S/A} = -\frac{A}{S}.$$

The examples previously worked out are repeated below.

Example 1.—Given S = 45 and A = 115, it is required to find N_R when A is fixed and $N_S = 200$ r.p.m.

Since
$$\frac{N_S - N_R}{N_A - N_R} = -\frac{A}{S},$$

substituting $N_S = 200$, $N_A = 0$, $A = 115$, and $S = 45$,

$$\frac{200 - N_R}{0 - N_R} = -\frac{115}{45} = -\frac{23}{9},$$

from which $$N_R = 56{\cdot}25 \text{ r.p.m.}$$

Example 2.—Using data from Ex. 1, it is required to find N_R when $N_S = 200$ and $N_A = -60$ r.p.m.

Substituting the known values in the equation used in the preceding example,

$$\frac{200 - N_R}{-60 - N_R} = -\frac{115}{45} = -\frac{23}{9},$$

from which $$N_R = 13{\cdot}125 \text{ r.p.m.}$$

152. Compound Epicyclic Gear—Example.—Fig. 269 shows a compound epicyclic gear. If the shaft P is driven at 500 r.p.m. while the annulus A_2 rotates at 500 r.p.m.

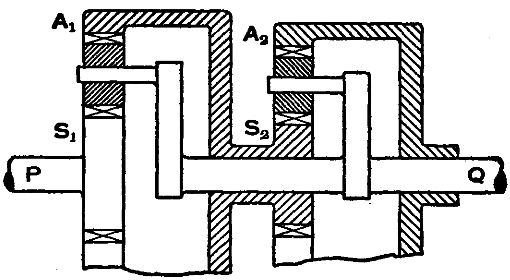


FIG. 269.

in the opposite direction, determine the speed and direction of rotation of the shaft Q. The numbers of teeth in the wheels are: $S_1 = 30$, $S_2 = 25$, $A_1 = 80$, $A_2 = 70$. [Inst. C.E.]

The general procedure in a compound epicyclic gear problem is to consider separately the simple epicyclic trains of which the gear is composed and to note the motion one train derives from another. The example given above is an exception in which the whole gear should be considered at one time.

The problem will be solved by the semi-graphical method, and it will be left to the reader to check the answer by the other methods. Let the shaft Q be fixed and the annulus A_2 be given $+1$ revolution, then the shaft P will turn $-\frac{A_2}{S_2} \times -\frac{A_1}{S_1} = \frac{70}{25} \times \frac{80}{30} = \frac{112}{15}$ revolutions.

Draw the vector diagram qa_2p (Fig. 270), making $qa_2 = 1$ and $qp = \frac{112}{15}$ to a convenient scale. If P is driven at $+500$ r.p.m. while A_2 rotates at -500 r.p.m., the reference point O must be midway between a_2 and p, and the speed of the shaft Q is represented by the length marked N_Q.

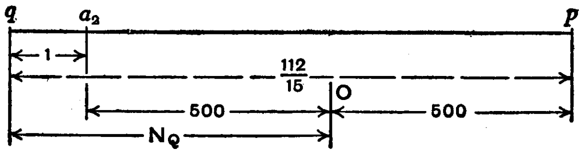


FIG. 270.

From the diagram, considering all lengths as positive,

$$\frac{N_Q - 500}{1} = \frac{1000}{\frac{112}{15} - 1} = \frac{15{,}000}{97},$$

$$N_Q = 500 + 154{\cdot}6 = 654{\cdot}6 \text{ r.p.m.},$$

but since the direction from O is to the left, the sign is negative, that is $N_Q = -654{\cdot}6$ r.p.m. The shaft Q and the annulus A_2 rotate in the same direction and the shaft P rotates in the opposite direction.

When an epicyclic gear forms part of a train, the problem is generally simplified if the epicyclic part is considered separately. For instance, in the example solved above, if the shaft P is driven by two or more wheels, its speed should be calculated before dealing with the epicyclic train.

153. Torques in an Epicyclic Train.—The velocity diagram for an epicyclic train may be used when finding the torques acting on the train and its casing. (See the reference on p. 284 to a letter in *Engineering* where the ideas are explained by Mr W. McHutchison and credited to Prof. William Kerr.)

Referring to Fig. 271, if the arm R is fixed and the sun S turns +1 revolution, the planet P turns $-\frac{S}{P}$ and the annulus A turns $-\frac{S}{A}$ revolutions. The velocity diagram is shown in Fig. 272.

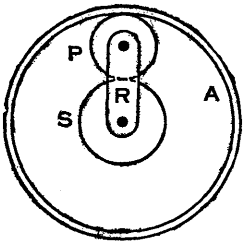


FIG. 271.

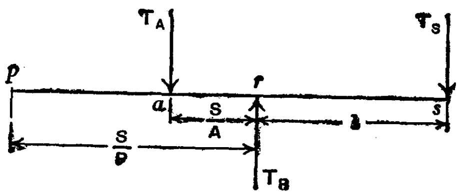


FIG. 272.

Assuming the epicyclic train runs at constant speed, then, neglecting losses, there is no external work done on it and there is no resultant torque, therefore

$$\Sigma T\omega = 0 \quad \text{and} \quad \Sigma T = 0,$$

where T is the torque on any part of the train and ω is the angular velocity of that part, and each symbol is given the correct sign.

The conditions are similar to those which apply when a lever is in equilibrium under a system of loads P at distances x from a reference point. There is no resultant moment and no resultant force, therefore

$$\Sigma Px = 0 \quad \text{and} \quad \Sigma P = 0.$$

It follows that the torques may be applied to the vector diagram as though they were loads acting on a lever. In

Fig. 272, T_A, T_R, and T_S are the torques applied to the annulus, the arm, and the sun respectively, and from the lever analogy the algebraic sum of the moments about any point is zero. For instance, taking moments about *a*,

$$T_S\left(\frac{S}{A}+1\right)-T_R\frac{S}{A}=0,$$

or taking moments about *r*,

$$T_S.1+T_A\left(-\frac{S}{A}\right)=0.$$

Also the algebraic sum of the "loads" is zero, therefore

$$T_A+T_S-T_R=0.$$

When drawing the diagram, the directions of the torques T_A, T_S, and T_R are determined by inspection.

Example.—In the epicyclic train shown in Fig. 273, the wheel A is fixed and the input at the arm R is 4 horse-power at 600 r.p.m. It is required to find the speed of the wheel D, the torque on it, and the torque to hold the wheel A. Frictional losses are to be neglected. As indicated, the numbers of teeth are A = 47, B = 35, C = 42, D = 40.

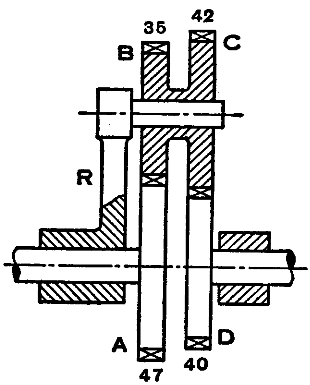


FIG. 273.

If R is fixed and A turns +1 revolution, then D turns $+\frac{47}{35}\times\frac{42}{40}=+1{\cdot}41$ revolutions. The velocity diagram *rad* is shown in Fig. 274. Since A is to be fixed, the new reference point O is at *a*, the speed of the arm R is proportional to *ar* and it is -600 r.p.m. The speed of the wheel D is proportional to *ad* and it is denoted by N_D,

$$N_D=600(1{\cdot}41-1)=246 \text{ r.p.m.},$$

and the direction of motion of D is opposite to that of R.

Since $\quad \text{H.P.}=\dfrac{2\pi NT}{33{,}000}, \quad T_R=\dfrac{4\times 33{,}000}{2\pi\times 600}=35{\cdot}01 \text{ lb. ft.}$

If the torque T_R is drawn on the diagram as a force acting downwards at the point r, then it is obvious from the lever analogy that for equilibrium T_D is downwards at d and T_A is upwards at a.

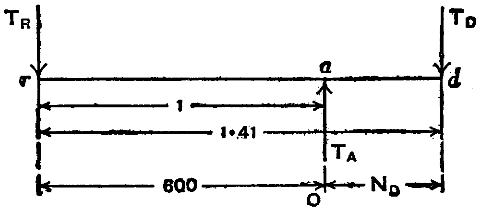


FIG. 274.

Taking moments about d,

$$0{\cdot}41T_A = 1{\cdot}41T_R = 1{\cdot}41 \times 35{\cdot}01,$$

$$T_A = \frac{1{\cdot}41 \times 35{\cdot}01}{0{\cdot}41} = 120{\cdot}4 \text{ lb. ft.},$$

and if T_R is clockwise, T_A is anticlockwise.

From the diagram, $T_A = T_R + T_D$,

therefore $T_D = 120{\cdot}4 - 35{\cdot}01 = 85{\cdot}4$ lb. ft.

As a check, taking moments about a,

$$0{\cdot}41T_D = 1 \times T_R,$$

therefore $$T_D = \frac{35{\cdot}01}{0{\cdot}41} = 85{\cdot}4 \text{ lb. ft.}$$

Alternative Methods.—Referring to the above example and Fig. 273, find the velocity ratio ω_D/ω_R of D to R with A fixed, by either the tabular or the algebraic method, then

$$T_D\omega_D = T_R\omega_R.$$

Similarly, find the ratio ω_A/ω_R of A to R with D fixed, then

$$T_A\omega_A = T_R\omega_R.$$

An examination of these equations shows that they can be written down immediately from the diagram in Fig. 274, so the chief difference in the two methods is in the calculation of the velocity ratios.

Torques may also be calculated by considering the forces, on pairs of teeth, acting tangentially to the pitch circles of the wheels in mesh. This is a fundamental method, but perhaps the longest and most difficult one.

154. Torque to Accelerate a Gear Train.—Consider a train of wheels A, B, C, D, etc. driven by A, and suppose it is required to find the torque T which would give A a definite uniform acceleration and accelerate the whole train. The special case of a planet wheel is discussed in Art. 155.

Let ω_A, ω_B, etc. be the wheel velocities at any instant, α_A, α_B, etc. be the accelerations, and I_A, I_B, etc. be the moments of inertia of the wheels. Let T_A, T_B, T_C, etc. be the torques required to accelerate the individual wheels, when applied about their own axes, and t_B, t_C, etc. be the torques on the wheel A corresponding to T_B, T_C, etc., then the total torque on A is

$$T = T_A + t_B + t_C + \ . \ . \ .$$

Consider any wheel, say C, then

$$T_C = I_C \alpha_C.$$

Neglecting losses, $$t_C \omega_A = T_C \omega_C,$$

therefore $$t_C = I_C \alpha_C \frac{\omega_C}{\omega_A}.$$

Since the wheels are in a train, the accelerations are proportional to the velocities, so

$$\frac{\alpha_C}{\alpha_A} = \frac{\omega_C}{\omega_A},$$

therefore $$t_C = I_C \left(\frac{\omega_C}{\omega_A} \right)^2 \alpha_A.$$

The total torque required on wheel A is

$$\begin{aligned} T &= T_A + t_B + t_C + \ . \ . \ . \\ &= I_A \alpha_A + I_B \left(\frac{\omega_B}{\omega_A} \right)^2 \alpha_A + I_C \left(\frac{\omega_C}{\omega_A} \right)^2 \alpha_A + \ . \ . \ . \\ &= \left\{ I_A + I_B \left(\frac{\omega_B}{\omega_A} \right)^2 + I_C \left(\frac{\omega_C}{\omega_A} \right)^2 + \ . \ . \ . \right\} \alpha_A. \end{aligned}$$

The expression in brackets is the *equivalent moment of inertia* of the system referred to the wheel A.

155. Torque to Accelerate a Planet Wheel.*—A planet wheel P, of mass m and moment of inertia I_P about its own axis (Fig. 275), turns on a pin C at the outer end of an arm R of length $OC = r$. The arm R turns about a fixed pin at O with varying angular velocity ω_R and uniform angular acceleration α_R. The corresponding angular velocity of the planet P is ω_P and its acceleration is α_P; the ratio of the velocities may be either positive or negative. Note that ω_P and α_P are actual values and are *not* relative to the arm.

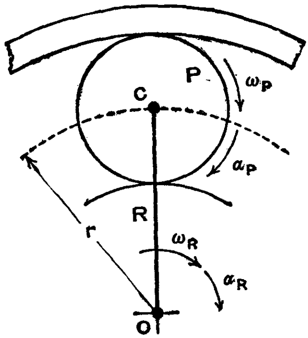


FIG. 275.

It is required to find the torque on the arm which would accelerate the planet.

Since P and R are parts of an epicyclic train, the relation between ω_P and ω_R is known and the accelerations are proportional to the velocities.

The torque to accelerate P about its own centre is

$$T_P = I_P \alpha_P.$$

The torque to accelerate the centre of gravity of P about O is

$$T_O = m r^2 \alpha_R.$$

Let t be the torque on the arm which would produce the torque T_P, then neglecting losses,

$$t\omega_R = T_P \omega_P,$$

therefore the total torque on the arm to accelerate the planet is

$$\begin{aligned} T &= t + T_O \\ &= T_P \frac{\omega_P}{\omega_R} + T_O \\ &= I_P \alpha_P \frac{\omega_P}{\omega_R} + m r^2 \alpha_R. \end{aligned}$$

* It is assumed that either the sun or the annulus is fixed.

Since $$\frac{a_P}{a_R} = \frac{\omega_P}{\omega_R},$$

$$T = \left\{ I_P \left(\frac{\omega_P}{\omega_R} \right)^2 + mr^2 \right\} a_R.$$

The expression in brackets is the *equivalent moment of inertia* of the planet referred to the arm R.

Example.—In the epicyclic train shown in Fig. 276, all the teeth are of 4 d.p. Wheel F (40T), mounted on shaft A, is stationary. D (50T) and E (30T) are made of one piece of metal, and can rotate freely on a pin C attached to plate B. A similar pair of pinions D′ and E′ are mounted on a pin C′. B can rotate freely on shaft A. Wheel G is mounted on the driving-shaft H. The polar moment of inertia of the disc B, including the two pins C and C′, is 1200 lb. in.^{2}. D and E together weigh 7 lb., with a polar radius of gyration about their own centre line of 3 inches. G has a polar moment of inertia of 200 lb. in.^{2}.

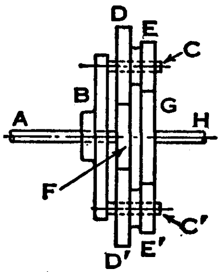


FIG. 276.

Find the torque required on shaft H to accelerate the system so that B has an angular acceleration of 5 rad. per sec.^{2}. [U.L.]

Since all the teeth are of 4 diametral pitch,

$$G + E = F + D \quad \text{or} \quad G = 40 + 50 - 30 = 60 \text{ teeth.}$$

Also, if r is the perpendicular distance between the axes of C and H, from Art. 147,

$$r = \frac{G + E}{2 \times 4} = \frac{60 + 30}{8} = 11{\cdot}25 \text{ in.}$$

The mass of D + E will be denoted by $m_{DE} = 7$ lb.

The moments of inertia are

$$I_B = 1200 \text{ lb. in.}^2, \qquad I_{DE} = 7 \times 3^2 = 63 \text{ lb. in.}^2,$$

and $$I_G = 200 \text{ lb. in.}^2.$$

The relations between the velocities of B, DE, and G are obtained by one of the usual methods. The tabular method is given here.

	B	G	DE	F
(1) Fix B. Turn G, -1 rev.	0	-1	$+2$	$-2\frac{1}{2}$
(2) Lock wheels. Turn all, $+2\frac{1}{2}$ rev.	$+2\frac{1}{2}$	$+2\frac{1}{2}$	$+2\frac{1}{2}$	$+2\frac{1}{2}$
(3) Add (1) and (2)	$+2\frac{1}{2}$	$+1\frac{1}{2}$	$+4\frac{1}{2}$	0
Multiply by 2, for convenience	$+5$	$+3$	$+9$	0

The acceleration of G is

$$a_G = a_B\frac{\omega_G}{\omega_B} = 5 \times \tfrac{3}{5} = 3 \text{ rad./sec.}^2.$$

If T_G is the torque on G and H, then

$$T_G = \left\{ I_G + I_B\left(\frac{\omega_B}{\omega_G}\right)^2 + 2\left[I_{DE}\left(\frac{\omega_{DE}}{\omega_B}\right)^2 + m_{DE}r^2 \right]\left(\frac{\omega_B}{\omega_G}\right)^2 \right\} a_G$$

$$= \left\{ I_G + (I_B + 2m_{DE}r^2)\left(\frac{\omega_B}{\omega_G}\right)^2 + 2I_{DE}\left(\frac{\omega_{DE}}{\omega_G}\right)^2 \right\} a_G$$

$$= \left\{ 200 + (1200 + 2 \times 7 \times 11{\cdot}25^2)\left(\frac{5}{3}\right)^2 + 2 \times 63\left(\frac{9}{3}\right)^2 \right\} \frac{3}{32{\cdot}2 \times 12} \text{ lb. in.}$$

$$= \{200 + 8255{\cdot}2 + 1134\} \frac{1}{32{\cdot}2 \times 4} = 74{\cdot}4 \text{ lb. in.}$$

Exercises XIV

1. Assume that in the compound train shown in Fig. 258 the driver A runs at 500 r.p.m. Find the speed of the wheel L, given that the numbers of teeth in the wheels are A = 30, B = 70, C = 35, D = 84, E = 38, L = 72.

2. Find suitable wheels for the compound train in the preceding exercise, if the speed of A is 500 r.p.m. and the speed of L is to be 25 r.p.m. The wheels available have from 20 to 100 teeth in steps of 5 and there is one of each size. The wheels in the train are to be the smallest possible.

3. A wheel of 40 teeth on a lathe spindle gears with a wheel of 50 teeth to which is fixed one of 25 teeth, and this gears with a wheel of 95 teeth on the lead screw which is right-handed and has four threads per inch.

Find the number of threads per inch there would be in a screw cut in the lathe, and say what alteration would be necessary in the train if the screw is to be left-handed.

4. The distance between the centre lines of two parallel shafts is 8·5 inches, and a wheel A keyed to one shaft drives a wheel B keyed to the other. The driver rotates at 400 r.p.m. and the driven shaft is to rotate at 296 r.p.m. approximately. If the diametral pitch of each wheel is to be 5, find the number of teeth on each and the actual speed of the driven shaft.

5. A shaft running at 1000 r.p.m. carries two gear wheels A and B. A second shaft, parallel to the first, carries two sliding gears C and D which can mesh with A and B respectively, thus giving two speeds to the second shaft.

Taking the diametral pitch of all gears to be 4, find the number of teeth on each wheel, if the speeds of the second shaft are to be 750 r.p.m. exactly and 450 r.p.m. approximately, and the distance between the shaft axes is approximately 10 inches. What is then the lower speed of the second shaft?

[Inst. C.E.]

6. A driving shaft running at 120 r.p.m. drives a parallel shaft at 480 r.p.m. through a pair of gear wheels. The smaller pinion has 14 teeth and the diametral pitch is 3. It is proposed to replace the gear wheels with another pair having the same diametral pitch so that the speed of the driven shaft shall be as nearly as possible 520 r.p.m. Find suitable numbers of teeth for the wheels and the actual speed of the driven shaft. Find also what alteration would have to be made in the distance between the centre lines of the shafts if the new speed were strictly adhered to. [Inst. C.E.]

7. The distance between the centre lines of two parallel shafts is to be 21 inches $\pm 0{\cdot}2$ inch. They are to be connected by a pair of toothed wheels so that the speed ratio is 4 : 1. The wheels are to have a standard diametral pitch of a value between 1 and 3 varying by $\frac{1}{4}$ and are to have as few teeth as possible. Find the actual distance between the shaft centre lines.

[Inst. C.E.]

8. An epicyclic gear consists of a sun S, a planet P, an annulus A, and an arm R. The numbers of teeth are $S=35$, $P=25$, and $A=85$. If the sun is fixed and the arm turns clockwise at 255 r.p.m., find the speed and direction of motion of the annulus. Obtain the answer by each of the three methods explained in the text.

9. Using data from Ex. 8, find the speed and direction of motion of the arm if the annulus is fixed and the sun turns clockwise at 255 r.p.m.

10. Using data from Ex. 8, find the speed and direction of motion of the arm if the sun turns clockwise at 255 r.p.m. and the annulus turns anticlockwise at 110 r.p.m.

11. In the epicyclic gear shown in Fig. 277 (modified from the original illustration) the driving shaft A and the driven shaft B are in line. The wheel C is keyed to the driving shaft and gears

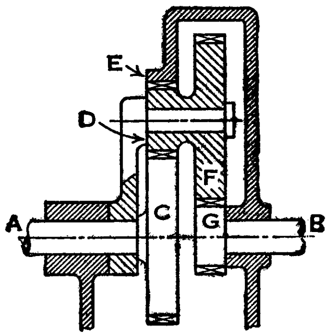


FIG. 277.

with the pinion D, which in turn gears with the fixed internal wheel E. Wheel F is compound with D and gears with the pinion G, which is keyed to the driven shaft B. If the numbers of teeth on the wheels are C 86, D 28, F 90, and G 24, find the speed of the shaft B in terms of that of the shaft A. [I.Mech.E.]

12. A fixed shaft carries (i) a gear wheel which has 25 teeth and is free to rotate on the shaft, (ii) a wheel keyed to the shaft and having 35 teeth, the pitch of which is half that on the loose wheel, and (iii) an arm, free to rotate on the shaft and carrying a compound pinion one half of which has 45 teeth in mesh with the fixed wheel and the other half is in mesh with the loose wheel. Find the number of turns the loose wheel must be driven to cause the arm to make 10 complete revolutions. [Inst. C.E.]

13. In a geared pulley block the hauling chain passes round a sprocket wheel of 24 inches diameter keyed to the spindle A. A sleeve B, free to rotate on the spindle, has an arm fitted with a stud on which a compound pinion CD—C, 40 teeth, D, 16 teeth—turns. Pinion C gears with E, 16 teeth, keyed to the spindle, and D gears internally with F, 64 teeth, which is fixed to the frame of the block and is concentric with the

spindle. The lifting chain passes round a sprocket wheel of 8 inches diameter keyed to the sleeve. Present an outline sketch showing the arrangement, and determine the velocity ratio of the block. [U.L.]

14. A compound epicyclic gear box is arranged as shown in Fig. 278. Both sun wheels are keyed to the driven shaft B.

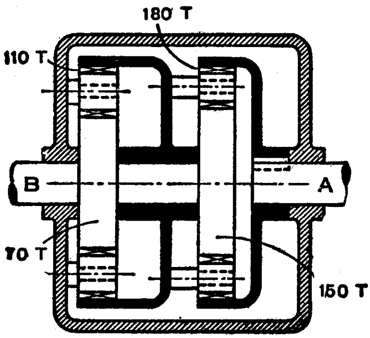


FIG. 278.

Prove that when the outside casing is fixed the driven shaft B rotates at half the speed of the driving shaft A in the opposite direction. [C.U.]

15. The indicating mechanism for showing the difference between the speeds of two turbines has the following construction: keyed to a shaft A is a pulley B at one end and a gear wheel C at the other end; a hollow shaft D concentric with A has a pulley E keyed to one end and a disc F to the other end; F carries two pins on which equal pinions G can rotate freely and gear with C, and with an annular wheel H keyed to a third shaft J in line with A and D; the rotation of a pointer keyed to J indicates the difference of speed between the two turbines. The pulleys B and E, 6 inches and 21 inches diameter respectively, are driven in the same direction by belts connecting equal pulleys on the turbine shafts, and when the speeds of the turbines are equal the pointer remains stationary.

Give a sketch of the arrangement and determine suitable tooth numbers for all gear wheels if the number of teeth on the smallest wheel must not be less than 20, and keeping the total number of teeth as small as possible.

If the turbine pulleys are each of 12 inches diameter and the turbine speeds are 750 and 760 r.p.m., find the speed of the pointer. [U.L.]

16. An epicyclic gear (Fig. 279) consists of a sun wheel S attached to the driving shaft, a planet wheel P of the same size, a fixed annulus A and an arm R attached to the driven shaft.

If the driving shaft transmits 2 H.P. at 240 r.p.m., find the speed of the driven shaft, the torque transmitted by it, and the torque required to hold the annulus.

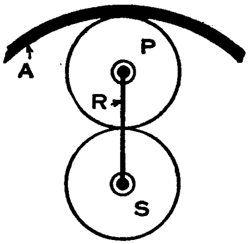


FIG. 279.

17. In the epicyclic gear (Humpage's) shown in Fig. 280 the bevel wheel A is keyed to the driving shaft F and the wheel B is keyed to the driven shaft G. D and E are compound, and revolve on pins carried by a sleeve which is free to revolve about the common axis of the shafts. C is a fixed wheel.

The numbers of teeth on the wheels are A, 15; B, 21; C, 32; D, 37; E, 16.

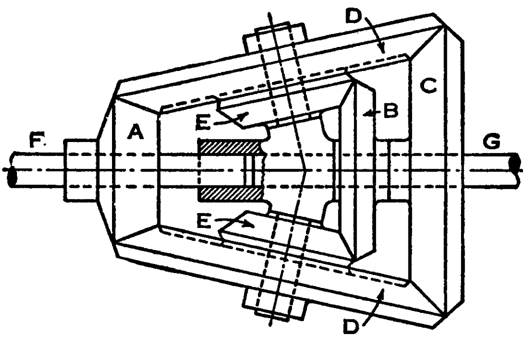


FIG. 280.

Find the speed of G when F makes 150 r.p.m., and the torque necessary to hold C when the gear transmits 5 H.P. at the given speed of the driving shaft.

If B is fixed and C free to rotate, find the speed of C when F makes 150 r.p.m. [C.U.]

18. In the gear system shown diagrammatically in Fig. 281 the sun wheel A (40 teeth) is fixed to the driving shaft P. The planets are carried on a bracket B fixed to the driven shaft Q and mesh with the annulus D (100 teeth), which is supported from a sleeve free to rotate on P. D is geared externally (120 teeth) and engages with the pinion C (30 teeth) on the shaft R. The speed of R is controlled to be twice that of Q and both rotate in the same direction.

Assuming no loss of power, prove that, if the input at P is 7·6 H.P., the output is 5·6 H.P. at Q and 2 H.P. at R. [C.U.]

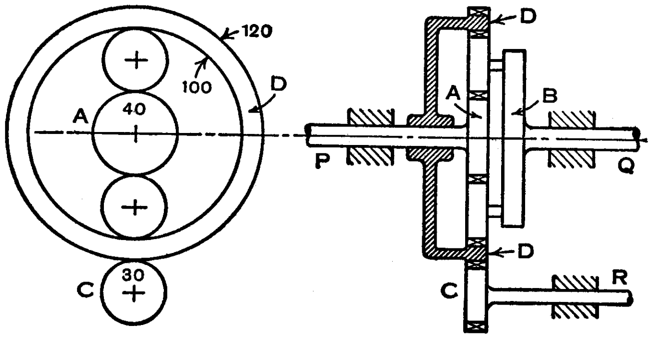


FIG. 281.

19. A gear box A (Fig. 282), with coaxial input and output shafts, B and C, is mounted in bearings in a fixed frame so as to be capable of rotating about their common axis, and is geared to a third parallel shaft D by gears E. When the shaft D is

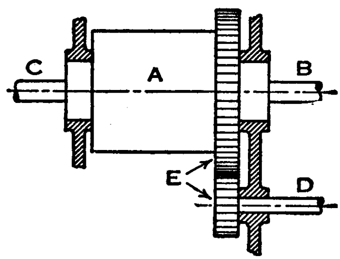


FIG. 282.

held fixed, the angular velocity of C is n times that of B, in the same direction. If the gear box is rotated with angular velocity ω_A, prove that the angular velocities of B and C are connected by the relation

$$\omega_C = n\omega_B + (1 - n)\omega_A.$$

In a particular case $n = 0{\cdot}4$, 3 H.P. is supplied at B at 500 r.p.m., the shaft D is rotated at 200 r.p.m. in the opposite direction to the direction of rotation of B, and the gears E have 20 and 60 teeth respectively. Find the rotational speed of C, the torque on D, and the output at C. Neglect all losses. [C.U.]

20. The diagram (Fig. 283) shows an epicyclic gear in which the sun wheel A and the two planet wheels B are all of 6 inches pitch line diameter and weigh 2 lb. each. The planet wheels are carried on pins in the disc D, which is 18 inches diameter and weighs 18 lb. The annular gear C is fixed. Treating the wheels and disc as solid discs of 6 inches and 18 inches diameter respectively and neglecting the weight of the pins, calculate the torque which must be applied to the shaft of wheel A in order that this wheel shall have an acceleration of 100 rad./sec.^{2} $\left(k^2 \text{ for disc} = \frac{d^2}{8}\right)$. [Inst. C.E.]

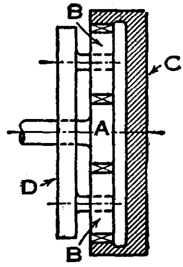


FIG. 283.

21. In the epicyclic gear shown in Fig. 284, a disc P attached to shaft X carries three pins on which three sets of compound wheels, A and B, revolve freely. Wheels A mesh with an

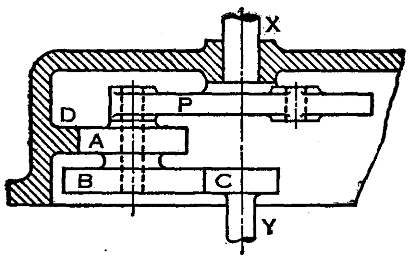


FIG. 284.

annular wheel D fixed to the casing, and wheels B mesh with wheel C attached to shaft Y.

Find (*a*) the number of teeth on D, (*b*) the velocity ratio between X and Y, (*c*) the constant torque applied to X to raise the speed of Y to 2500 r.p.m. in 5 seconds.

Wheel.	Number of Teeth.	Diametral Pitch.	Weight.	Radius of Gyration.
Disc P with shaft X	—	—	5·4 lb.	2·80 in.
A	16	5	} 1·2 lb. total	1·45 in.
B	32	8		
C with shaft Y	16	8	2·8 lb.	1·60 in.

[U.L.]

22. Fig. 285 shows a valve gear drive for a radial cylinder engine. The eccentric D is keyed to the rotating crankshaft and rotates freely in A, the cam disc C rides freely on the shaft and the internally toothed gear B is fixed to the engine frame.

Find the relation between the rotations of the shaft and

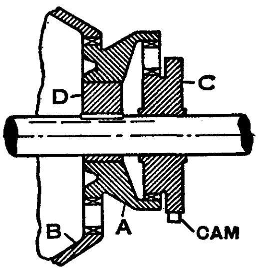


FIG. 285.

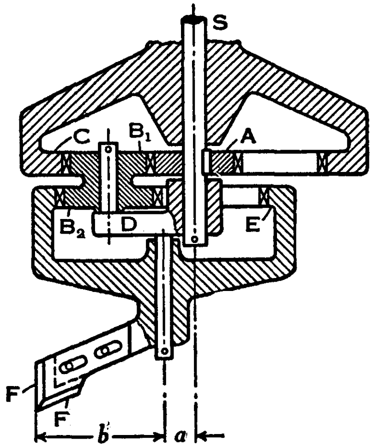


FIG. 286.

cam disc for the following gears: A external teeth, 68; A internal teeth, 68; B internal teeth, 72; C external teeth, 64.

Show that with this set of gears four inlet cams equally spaced on C will operate correctly the inlet valves for a nine-cylinder engine working on the four-stroke cycle, the firing order being 1, 3, 5, etc. [C.U.]

23. Fig. 286 shows the drilling head of a machine for cutting holes of various profiles. S is the driving spindle to which the sun wheel A is keyed and on which the arm D rotates freely: the internally toothed wheel C is fixed to the frame: the tool with cutting edges at F, F rotates on a spindle attached to the arm D and is driven by the double wheel B_1, B_2, and the internally toothed wheel E.

Show, graphically or otherwise, that with the gears specified and distances a and b 0·41 inch and 2·41 inches respectively, the tool will cut a hole approximately square and of side about 4 inches.

Show further that for $b > a$ and within the limits $3a < b < 9a$, a four-sided figure will be cut of which the circumscribing square has a side

$$2\sqrt{\frac{(b+3a)^3}{27a}}.$$

The numbers of teeth in the wheels are A, 60; B_1, 80; B_2, 96; C, 220; E, 198. [C.U.]

CHAPTER XV

BRAKES AND DYNAMOMETERS

156. Band Brakes.—The band brake shown diagrammatically in Fig. 287 consists of a flexible band of leather, rope, or steel, or steel lined with friction fabric, which embraces part of the circumference of a wheel or drum. The ends of the band are jointed at A and B to a lever AOC pivoted on a fixed pin at O. When a force P is applied to the lever at C, the lever turns about the pin O and tightens the band on the drum; consequently, if the drum is rotating its speed is reduced by friction. In Fig. 287 the moment about O of the force P is clockwise, and for the band to tighten it is essential for the length OA to be greater than OB.

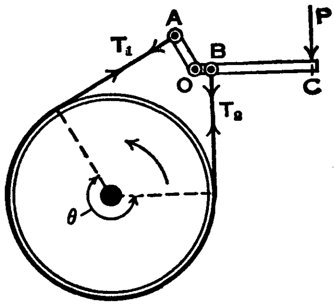


FIG. 287.

Let T_1 and T_2 be the forces in the band at A and B respectively, then, assuming the drum is rotating anticlockwise, the relation between T_1 and T_2 (Art. 116) is

$$\frac{T_1}{T_2}=e^{\mu\theta} \quad . \quad . \quad . \quad . \quad (1),$$

where e is the base of Napierian logarithms, μ is the coefficient of friction, and θ is the angle of embrace.

Considering the equilibrium of the lever AOC and taking moments about O, assuming the lines of action of the forces P, T_1, and T_2 are perpendicular to OC, OA, and OB respectively,

$$P.OC=T_1.OA-T_2.OB \quad . \quad . \quad (2).$$

Let r be the effective radius of the drum, that is the radius measured to the middle of the thickness of the band, and T_q be the braking torque on the drum, then

$$T_q = (T_1 - T_2)r \quad . \quad . \quad . \quad (3).$$

If T_q is known, the values of T_1, T_2, and P may be determined from the above three equations.

Either end of the band could be anchored at the point O, or to a fixed point on the machine which is driving the drum. If the greater tension T_1 is held in this way, the direction of the force P would have to be reversed in order to tighten the band on the drum, then

$$P.OC = T_2.OB.$$

If, with the arrangement shown in Fig. 287, the direction of rotation of the drum is reversed, then the greater tension T_1 acts at B and equation (2) becomes

$$P.OC = T_2.OA - T_1.OB.$$

In this case $T_2.OA$ would have to be greater than $T_1.OB$, otherwise the force T_1 would tighten the band on the drum without the application of the force P.

157. Band Brakes lined with Blocks.—The band brake discussed in the preceding Art. may be lined with blocks of wood or other material, as in Fig. 288, where two of the blocks are shown. Suppose there are n blocks, each subtending an angle 2θ at the centre of the brake drum. Let the greatest and least tensions in the band be T_0 and T_n respectively, and assume the direction of rotation of the drum is anticlockwise. The ratio of the tensions T_0 and T_n may be obtained as follows.

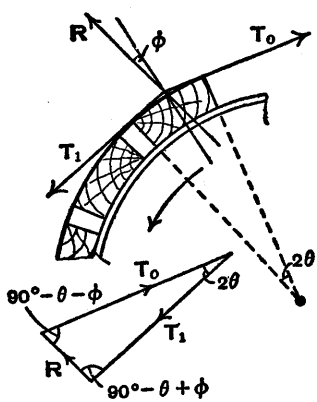


Fig. 288.

Numbering the blocks in the anticlockwise direction, let the tensions in the band be T_1 between the first and second blocks, T_2 between the second and third blocks, and so on. Assume that the resultant reaction of the drum on the first block is a force R, inclined at an angle ϕ to the bisector of the angle 2θ, where ϕ is the angle of friction and $\tan\phi=\mu$. For equilibrium of this block, the lines of action of the forces T_0, T_1, and R intersect at a point, and from the triangle of forces

$$\frac{T_0}{T_1}=\frac{\sin\{(90^\circ-\theta)+\phi\}}{\sin\{(90^\circ-\theta)-\phi\}}=\frac{\cos\theta\cos\phi+\sin\theta\sin\phi}{\cos\theta\cos\phi-\sin\theta\sin\phi}.$$

Dividing numerator and denominator by $\cos\theta\cos\phi$,

$$\frac{T_0}{T_1}=\frac{1+\mu\tan\theta}{1-\mu\tan\theta}.$$

Similarly it follows that this is also the value of each of the ratios T_1/T_2, T_2/T_3, . . ., T_{n-1}/T_n, therefore

$$\frac{T_0}{T_n}=\frac{T_0}{T_1}\times\frac{T_1}{T_2}\times\frac{T_2}{T_3}\times\ .\ .\ .\ \times\frac{T_{n-1}}{T_n}=\left\{\frac{1+\mu\tan\theta}{1-\mu\tan\theta}\right\}^n.$$

If T_q is the braking torque on the drum and r is the radius measured to the middle of the thickness of the band,

$$T_q=(T_0-T_n)r.$$

158. Standard Integrals.—To simplify the Arts. which follow, the values of several integrals are quoted here for reference.

$$\int_{-a}^{a}\sin\theta\,d\theta=0. \qquad \int_{-a}^{a}\cos\theta\,d\theta=2\sin a.$$

$$\int_{-a}^{a}\sin 2\theta\,d\theta=0. \qquad \int_{-a}^{a}\cos 2\theta\,d\theta=\sin 2a.$$

$$\int_{-a}^{a}\sin^2\theta\,d\theta=a-\tfrac{1}{2}\sin 2a. \qquad \int_{-a}^{a}\cos^2\theta\,d\theta=a+\tfrac{1}{2}\sin 2a.$$

159. Brakes with Internal-Expanding Shoes.—A well-known type of brake consists of two shoes, with friction fabric riveted to their outer circumferential faces, pivoted on one or two stationary pins inside a rotating drum. The shoes are arranged so that they can be pushed outwards against the rim of the drum and then friction reduces the speed of the latter. The shoe BC (Fig. 289) is pivoted at C and a force Q applied at B pushes it against the inside of the rim of the drum D. When the force is removed, the shoe is pulled away from the rim by a spring.

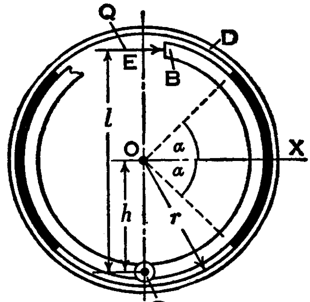


FIG. 289.

It will be assumed that the shoe does not bend and that the radial pressure at any point of contact is proportional to the radial component of the virtual displacement at that point. These assumptions are reasonable if the shoe is fairly substantial and the compression of the friction fabric follows Hooke's law.

Let the working face of the shoe subtend an angle 2α at the centre O of the drum, and let r and b be its radius and breadth respectively. Let the axis OX, drawn through the centre of the shoe face, be perpendicular to CO and let $CO = h$. Let $CE = l$ be the perpendicular distance from C of the line of action of the force Q.

Since the shoe turns about C (Fig. 290), the virtual displacement of any point A on its working face is proportional to CA and is represented by AA′ drawn perpendicular to CA. Several positions of the point A are shown, and each length AA′ is proportional to the corresponding length CA. The radial component of each displacement is represented by Aa, the projection of AA′ on the radius OA produced.

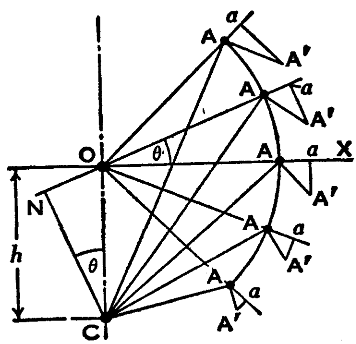


FIG. 290.

Consider any radius OA inclined at an angle θ to OX.

Draw CN perpendicular to OA, then the right-angled triangles CNA and AaA' are similar. Also the angle OCN is equal to θ. Therefore

$$\frac{Aa}{AA'}=\frac{CN}{CA} \quad \text{and} \quad CN=h\cos\theta.$$

The radial component of the virtual displacement of A is represented by

$$Aa=AA'\frac{CN}{CA}=k'.CA\frac{CN}{CA}=k'h\cos\theta,$$

where k' is a constant.

Since the radial pressure is proportional to the radial displacement, it is equal to

$$kh\cos\theta \quad . \quad . \quad . \quad . \quad (1),$$

where k is another constant.

The pressure is a maximum when $\cos\theta$ is a maximum, that is, when $\theta=0$, or the radius OA is perpendicular to CO. The pressure varies as indicated in Fig. 291, where the

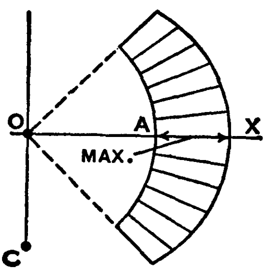


FIG. 291.

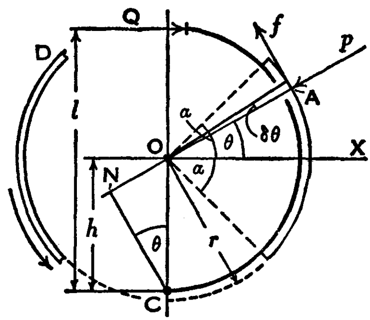


FIG. 292.

lengths of the radiating lines, measured from the circular arc, are proportional to $\cos\theta$. The general shape of the pressure curve is the same for each shoe but, as will be seen below, the pressures are greater on the left-hand shoe pivoted at C (Fig. 289), assuming the drum is rotating anticlockwise and the applied moment Ql is the same for each shoe.

The radial force p at A (Fig. 292) on a short length $r\delta\theta$ of shoe face of breadth b is

$$p=khbr\cos\theta\delta\theta \quad . \quad . \quad . \quad (2).$$

The moment of the force p about the pin C is

$$ph \cos \theta,$$

and the total moment about C due to radial forces is

$$M_p = kh^2br\int_{-\alpha}^{\alpha} \cos^2 \theta d\theta$$

$$= kh^2br(\alpha + \tfrac{1}{2} \sin 2\alpha) \quad . \quad . \quad (3).$$

The tangential force f at A on the shoe is

$$f = \mu p = \mu khbr \cos \theta \delta\theta \quad . \quad . \quad (4).$$

The moment of f about C is

$$f . \mathrm{NA} = \mu khbr(r + h \sin \theta) \cos \theta \delta\theta,$$

and the total frictional moment about C is

$$M_f = \mu khbr\int_{-\alpha}^{\alpha} (r \cos \theta + \tfrac{1}{2}h \sin 2\theta)d\theta$$

$$= 2\mu khbr^2 \sin \alpha \quad . \quad . \quad . \quad . \quad (5).$$

If the drum D is rotating anticlockwise, M_f is anticlockwise. The moment of the force Q about C is Ql clockwise and M_p is anticlockwise. Therefore, for the shoe in Fig. 292,

$$M_p + M_f = Ql \quad . \quad . \quad . \quad (6).$$

Substituting from (3) and (5) and solving for k, denoting its value by k_t,

$$k = k_t = \frac{Ql}{hbr\{h(\alpha + \frac{1}{2} \sin 2\alpha) + 2\mu r \sin \alpha\}} \quad . \quad (7).$$

If the drum is rotating clockwise, M_f becomes negative in (6), and the value of k, which will be denoted by k_l, is then

$$k = k_l = \frac{Ql}{hbr\{h(\alpha + \frac{1}{2} \sin 2\alpha) - 2\mu r \sin \alpha\}} \quad . \quad (7a),$$

and this is also the value of k for the left-hand shoe pivoted at C (Fig. 289) when the drum is rotating anticlockwise.

The constant k is positive, and comparing (7) and (7a)

it is seen that (7a) gives the larger value, assuming Ql has the same value in each case.

The radial pressure at any point on the face of a shoe is found by substituting the appropriate values of k and θ in (1), and since k is larger for the left-hand shoe, the pressure at any point on the face of this shoe is greater than the pressure at the corresponding point on the right-hand shoe.

The braking torque supplied by a short length $r\delta\theta$ of shoe face is fr. Substituting for f from (4) and integrating, the torque T due to one shoe is

$$\mathrm{T}=\mu khbr^2\int_{-a}^{a}\cos\theta\, d\theta=2\mu khbr^2\sin a \qquad . \qquad (8).$$

For two shoes the separate braking torques are added together. With the arrangement shown in Fig. 293,

$$\mathrm{T}=2\mu(k_t+k_l)hbr^2\sin a \qquad . \qquad . \qquad (9),$$

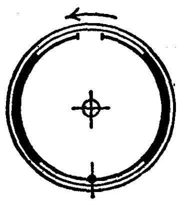


FIG. 293.

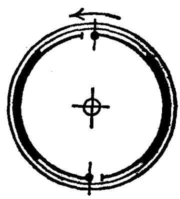


FIG. 294.

where k_t and k_l are the values of k obtained from (7) and (7a) respectively. With the arrangement shown in Fig. 294, the shoes are pivoted at the opposite ends of a diameter, therefore

$$\mathrm{T}=4\mu khbr^2\sin a \qquad . \qquad . \qquad . \qquad (10),$$

where k is obtained from (7a), or from (7) if the direction of rotation of the drum is reversed.

For the resultant radial force on a shoe see Ex. 5, p. 325; for the unbalanced force on the drum see Ex. 6, p. 325.

160. Leading and Trailing Shoes.—Consideration of Fig. 295 shows that when the drum is rotating anticlockwise the moment about the pin C of the friction force f_l, acting at any point on the face of the shoe l, pushes the shoe against the drum. The corresponding moment of the friction force f_t on the shoe t tends to pull this shoe away from the drum. The shoe l is called a *leading* or *primary* shoe and the shoe t is called a *trailing* or *secondary* shoe. If the direction of rotation of the drum is reversed, the right-hand shoe becomes the leading shoe. In Fig. 294 both shoes are leading shoes.

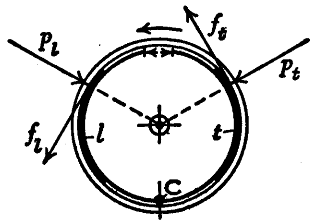


FIG. 295.

The resultant radial force on a leading shoe is greater than on a trailing shoe. In general, a shoe may be defined as leading or trailing according as it provides the larger or the smaller braking torque.

161. Brakes with External-Contracting Fixed Shoes.*— When brake shoes are applied to the outside of a drum, they may be either fixed or pivoted to the posts or arms which operate them. External-contracting shoes fixed to the brake posts are shown diagrammatically in Fig. 296. Pivoted shoes are dealt with in Art. 162.

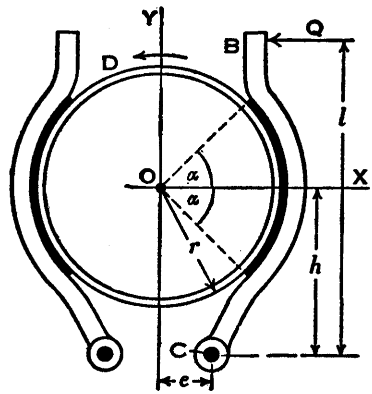


FIG. 296.

The axis OX passes through the centre of the right-hand shoe, and the position of the pin C at the bottom of the brake post BC, relative to the axes OX and OY, is as indicated. The shoe lining, of radius r and breadth b, subtends an angle 2α at the centre O of the drum D. The moment about C of the applied force Q is Ql.

* For a different treatment of brakes with external-contracting shoes, see "An Analysis of the Forces and Pressure Distribution in Brake Shoes," by J. Hirschhorn, *Engineering*, April 4, 1952.

Consider any radius OA inclined at an angle θ to OX (Fig. 297). The virtual displacement of the point A on the shoe face is represented by AA′ drawn perpendicular to CA and the radial component is represented by the projection Aa on the radius OA. The lines CH and CN are drawn perpendicular to OX and OA respectively, then the angle HCN is equal to θ.

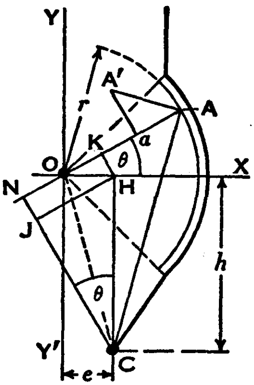


Fig. 297.

The right-angled triangles CNA and AaA′ are similar, therefore the radial component of the virtual displacement of A is represented by

$$Aa = AA'\frac{CN}{CA} = k' . CA\frac{CN}{CA} = k' . CN,$$

where k' is a constant.

Draw HJ and HK perpendicular to CN and OA respectively, then

$$CN = CJ + JN = CJ + HK = h \cos \theta + e \sin \theta.$$

Since the outward radial pressure at A on the shoe is proportional to the radial displacement, it is equal to

$$k(h \cos \theta + e \sin \theta) \quad . \quad . \quad . \quad (1),$$

where k is another constant.

The pressure is a maximum when

$$\frac{d}{d\theta}(h \cos \theta + e \sin \theta) = 0$$

$$-h \sin \theta + e \cos \theta = 0$$

$$\tan \theta = \frac{e}{h},$$

and this occurs when OA is perpendicular to CO. Therefore the smaller the angle COY′, the nearer the point of maximum pressure is to the centre of the shoe face.

The general shape of the pressure curve, plotted radially outwards from the drum circle, is shown in Fig. 298, but the actual values of the pressures depend, of course, on

k, h, and e. The curve for the left-hand shoe is similar, but it will be seen later that k has a smaller value, therefore the pressures are less, assuming the drum is rotating anti-clockwise and that Ql has the same value for each shoe.

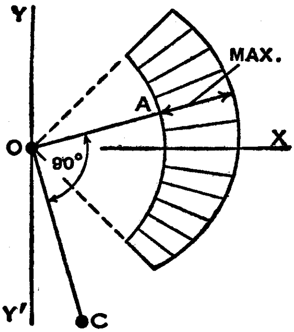


FIG. 298.

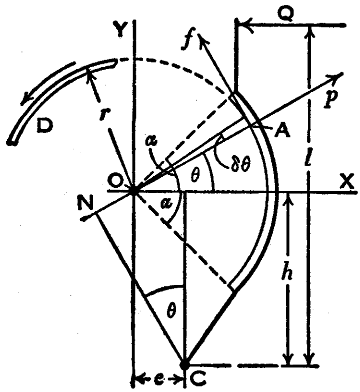


FIG. 299.

The outward radial force p at A on a short length $r\delta\theta$ of shoe face of breadth b (Fig. 299) is

$$p = kbr(h \cos \theta + e \sin \theta)\delta\theta \quad . \quad . \quad (2).$$

The moment of this force about C is p.CN, and the total moment of the radial forces about C is

$$M_p = kbr\int_{-\alpha}^{\alpha} (h \cos \theta + e \sin \theta)^2 d\theta.$$

Squaring and integrating (see Art. 158),

$$M_p = kbr\{h^2(\alpha + \tfrac{1}{2} \sin 2\alpha) + e^2(\alpha - \tfrac{1}{2} \sin 2\alpha)\} \quad . \quad (3).$$

The tangential force f at A on the shoe is

$$f = \mu p = \mu kbr(h \cos \theta + e \sin \theta)\delta\theta \quad . \quad . \quad (4),$$

and the moment of f about C is f.NA.

Referring back to Fig. 297,

$$\begin{aligned} NA &= OA + NO = OA + NK - OK \\ &= OA + JH - OK \\ &= r + h \sin \theta - e \cos \theta, \end{aligned}$$

therefore

$$f.\mathrm{NA} = \mu kbr(h \cos\theta + e \sin\theta)(r + h \sin\theta - e \cos\theta)\delta\theta.$$

The total frictional moment about C (Fig. 299) is

$$\mathrm{M}_f = \mu kbr\int_{-a}^{a}(h \cos\theta + e \sin\theta)(r + h \sin\theta - e \cos\theta)d\theta.$$

Multiplying and integrating (see Art. 158),

$$\mathrm{M}_f = \mu kbrh(2r \sin a - e \sin 2a) \quad . \quad . \quad (5).$$

If the drum is rotating anticlockwise,

$$\mathrm{M}_p - \mathrm{M}_f = \mathrm{Q}l \quad . \quad . \quad . \quad (6).$$

Substituting from (3) and (5) and solving for k, denoting its value by k_l,

$$k_l = \frac{\mathrm{Q}l}{br\{h^2(a + \frac{1}{2}\sin 2a) + e^2(a - \frac{1}{2}\sin 2a) - \mu h(2r \sin a - e \sin 2a)\}} \quad (7).$$

For the left-hand shoe, both the moments M_p and M_f act in the anticlockwise direction about the brake-post pin, therefore

$$\mathrm{M}_p + \mathrm{M}_f = \mathrm{Q}l \quad . \quad . \quad . \quad (6a),$$

and the value of k, which will be denoted by k_t, is

$$k_t = \frac{\mathrm{Q}l}{br\{h^2(a + \frac{1}{2}\sin 2a) + e^2(a - \frac{1}{2}\sin 2a) + \mu h(2r \sin a - e \sin 2a)\}} \quad (7a).$$

The braking torque supplied by a short length $r\delta\theta$ of a shoe face is fr, and for a whole shoe, substituting for f from (4), it is

$$\mathrm{T} = \mu kbr^2\int_{-a}^{a}(h \cos\theta + e \sin\theta)d\theta$$

$$= 2\mu kbr^2h \sin a \quad . \quad . \quad . \quad . \quad (8).$$

The braking torque due to two shoes is

$$\mathrm{T} = 2\mu(k_l + k_t)br^2h \sin a \quad . \quad . \quad (9),$$

where k_l and k_t are the values of k obtained from (7) and (7a) respectively.

When the drum is rotating anticlockwise, the leading shoe is on the right and the trailing shoe is on the left. It was assumed above in (6) that M_f acts in the anticlockwise direction about the pin C, and since the value of M_f in (5) is positive, with reasonable values of r and e, the assumption is correct. Therefore friction pulls the right-hand shoe against the drum; similarly it tends to pull the other shoe away from the drum.

A larger value of k is obtained from (7) than from (7a), therefore the right-hand shoe provides the greater braking torque.

For the resultant radial force on a shoe see Ex. 8, p. 325, and for the unbalanced force on the drum see Ex. 9, p. 326.

162. Brakes with External-Contracting Pivoted Shoes.—External-contracting shoes pivoted to the brake posts are shown diagrammatically in Fig. 300. The axis OX passes through the centre of the right-hand shoe and through the pin S, and the position of the pin C relative to the axes OX and OY is as indicated. The shoe lining, of radius r and breadth b, subtends an angle 2α at the centre O of the drum D. The moment about C of the applied force Q is Ql.

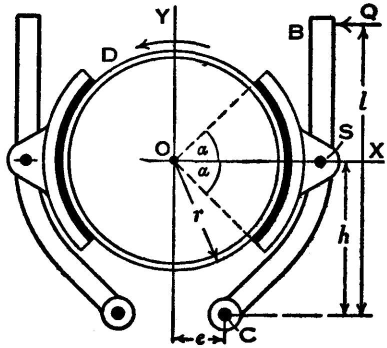


FIG. 300.

Consider any radius OA inclined at an angle θ to OX (Fig. 301). The virtual displacement of the point A is equal to the displacement of S plus the displacement of A relative to S. Let $OS = d$, $CS = u$, and the varying length $SA = v$. Draw SS′ perpendicular to CS and AA′ perpendicular to SA to represent the displacement of S and the displacement of A relative to S respectively.

Draw SO′ parallel to AO, then the projection Ss of SS′ on SO′, plus the projection Aa of AA′ on AO, would represent the radial component of the virtual displacement of A if Ss and Aa could be correctly drawn to the same scale. It will be found later that for positive values of θ, the displacement Aa is outwards on one shoe and inwards on the other, but this does not affect the argument.

Draw CH perpendicular to OX and SL perpendicular to OA. Let the angle HCS $=\epsilon$ and the angle SAL $=\phi$. It can be seen that the angle HSO$'=\theta$, the angle OSS$'=\epsilon$, and the angle AA$'a=\phi$.

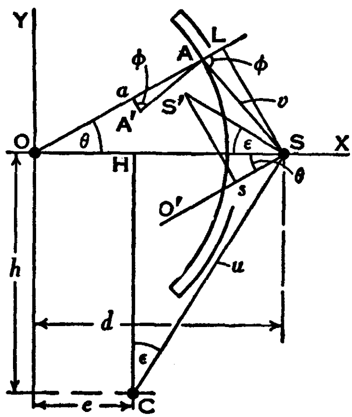


FIG. 301.

The radial displacement of A is equal to

$$c_1 . u \frac{\mathrm{S}s}{\mathrm{SS}'} + c_2 . v \frac{\mathrm{A}a}{\mathrm{AA}'} \quad \text{or} \quad c_1 . u \cos(\theta+\epsilon) + c_2 . v \sin\phi,$$

where c_1 and c_2 are constants, and, since

$$\cos(\theta+\epsilon) = \cos\theta\cos\epsilon - \sin\theta\sin\epsilon \quad \text{and} \quad v\sin\phi = d\sin\theta,$$

the displacement is equal to

$$k'_1 \cos\theta + k'_2 \sin\theta,$$

where k'_1 and k'_2 are constants.

The radial pressure at A is proportional to the radial displacement, and is equal to

$$k_1 \cos\theta + k_2 \sin\theta \quad . \quad . \quad . \quad (1),$$

where k_1 and k_2 are constants replacing k'_1 and k'_2.

The pressure is a maximum when

$$\frac{d}{d\theta}(k_1\cos\theta + k_2\sin\theta) = 0$$

$$-k_1\sin\theta + k_2\cos\theta = 0$$

$$\tan\theta = \frac{k_2}{k_1} \quad . \quad . \quad (2).$$

The outward radial force p at A on a short length $r\delta\theta$ of shoe face of breadth b (Fig. 302) is

$$p = br(k_1 \cos\theta + k_2 \sin\theta)\delta\theta \quad . \quad . \quad (3),$$

and the corresponding friction force f on the shoe is

$$f = \mu p \quad . \quad . \quad . \quad . \quad (4).$$

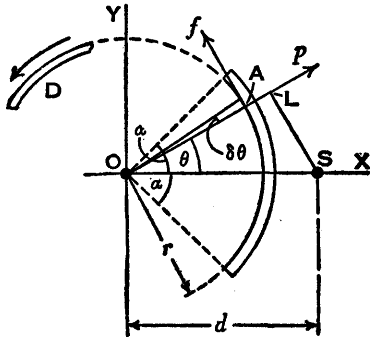


FIG. 302.

The algebraic sum of the moments of p and f about the pin S, assuming the drum D is rotating anticlockwise, is

$$p.\mathrm{SL} + f.\mathrm{AL},$$

or

$$pd \sin\theta + \mu p(d \cos\theta - r) \quad . \quad . \quad (5).$$

The resultant moment about the pin S must be zero, therefore

$$\int_{-a}^{a} br(k_1 \cos\theta + k_2 \sin\theta)(d \sin\theta + \mu d \cos\theta - \mu r)d\theta = 0.$$

Multiplying and integrating (see Art. 158),

$$\mu k_1\{d(a + \tfrac{1}{2} \sin 2a) - 2r \sin a\} + k_2 d(a - \tfrac{1}{2} \sin 2a) = 0 \quad (6),$$

from which

$$\frac{k_2}{k_1} = -\frac{\mu\{d(a + \frac{1}{2} \sin 2a) - 2r \sin a\}}{d(a - \frac{1}{2} \sin 2a)} \quad . \quad (7).$$

The denominator on the right in (7) is positive, since a is greater than $\frac{1}{2}\sin 2\alpha$ or $\sin\alpha\cos\alpha$, and the numerator is positive for values of d/r which usually occur in practice. The value of k_1 is positive, as the pressure given by (1) must be positive when $\theta=0$, therefore it follows from (7) that k_2 is negative.

For the left-hand shoe, changing the sign preceding μ in (5) and therefore in (7),

$$\frac{k_2}{k_1}=\frac{\mu\{d(\alpha+\frac{1}{2}\sin 2\alpha)-2r\sin\alpha\}}{d(\alpha-\frac{1}{2}\sin 2\alpha)} \quad . \quad (7a),$$

and in this case both k_1 and k_2 are positive. The numerical value of the ratio is the same for each shoe and it increases if μ or d increases.

At this stage it is convenient to examine the pressure distribution on the shoes. Suppose $k_2/k_1=-\frac{1}{3}$ in (7) or $\frac{1}{3}$ in (7a), and $\alpha=\pi/4$; then if $\mu=\frac{1}{2}$, $d=1{\cdot}29r$, and if $\mu=\frac{1}{3}$, $d=1{\cdot}41r$. These values of d give possible positions for the shoe pivots and show that the assumed numerical value of the ratio k_2/k_1 is reasonable. In practice, μ, d, r, and α would be given definite values, and k_1 and k_2 obtained by calculation.

Although k_1 has not been determined numerically, by putting $k_1=1$ and $k_2=-\frac{1}{3}$ in (1) the shape of the pressure curve for the right-hand shoe can be plotted as shown by the polar curve on the right in Fig. 303. For the left-hand shoe, putting $k_1=1$ and $k_2=\frac{1}{3}$ in (1), the dotted curve is obtained. Actually, with anticlockwise rotation of the drum and equal moments

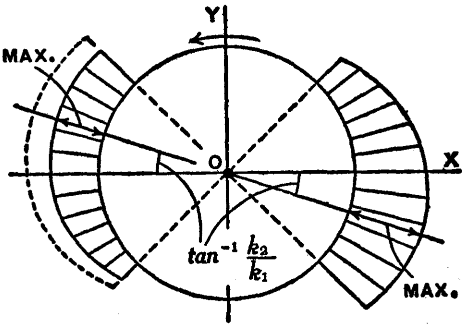


Fig. 303.

Ql on the brake posts (Fig. 300), k_1 is less for the left-hand shoe than for the right-hand one, and the pressure distribution is as shown by the full-line curve, but of course the reduction of the pressures would have to be calculated in any particular case.

The ratio of the pressures at the opposite ends of a diameter is equal to a constant (see Ex. 13, p. 327), and it follows from (2) that the positions of the points of maximum pressure are obtained by drawing the diameter inclined to OX at an angle $\tan^{-1}(-\frac{1}{3})$, since it has been assumed that $k_2/k_1 = -\frac{1}{3}$.

To evaluate k_1 and k_2 a second equation is required for each shoe. Since there is no resultant couple on a shoe, shift the forces p and f (Fig. 302) to S, omitting the moments which this transference necessitates, where S (Fig. 304) is the pivot on the right-hand shoe, and take moments about the brake-post pin C. This can be done conveniently by first resolving along and perpendicular to OX.

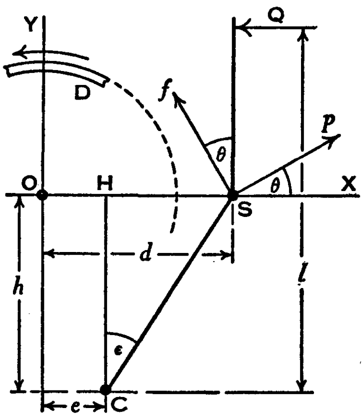


FIG. 304.

The length HS $= d - e$ will be written as $h \tan \epsilon$.

The clockwise moment about C due to p and f, with anticlockwise rotation of the drum, is

$$(p\cos\theta)h - (f\cos\theta)h\tan\epsilon - (p\sin\theta)h\tan\epsilon - (f\sin\theta)h \quad (8),$$

or $$ph\{(1-\mu\tan\epsilon)\cos\theta - (\tan\epsilon+\mu)\sin\theta\}.$$

For the whole shoe, substituting for p from (3) and equating to Ql,

$$Ql = hbr\int_{-\alpha}^{\alpha}(k_1\cos\theta + k_2\sin\theta)\{(1-\mu\tan\epsilon)\cos\theta - (\tan\epsilon+\mu)\sin\theta\}d\theta$$

$$= hbr\{k_1(\alpha + \tfrac{1}{2}\sin 2\alpha)(1-\mu\tan\epsilon) - k_2(\alpha - \tfrac{1}{2}\sin 2\alpha)(\tan\epsilon+\mu)\} \quad (9).$$

Substituting for k_2 in terms of k_1 for the right-hand shoe from (7), simplifying and solving for k_1,

$$k_1 = \frac{Ql}{hbr\left\{(a + \frac{1}{2}\sin 2a)(1+\mu^2) - \frac{2r\mu}{d}(\sin a)(\tan\epsilon + \mu)\right\}} \quad (10),$$

where $\tan\epsilon = (d-e)/h$.

For the left-hand shoe, with anticlockwise rotation of the drum, $-f$ becomes $+f$ in (8) and this changes the signs preceding μ in (9); also the negative ratio in (7) becomes positive in (7*a*). These alterations are equivalent to changing the signs preceding μ in (10). Therefore, for the left-hand shoe,

$$k_1 = \frac{Ql}{hbr\left\{(a + \frac{1}{2}\sin 2a)(1+\mu^2) + \frac{2r\mu}{d}(\sin a)(\tan\epsilon - \mu)\right\}} \quad (10a).$$

After obtaining k_1 for each shoe, the corresponding values of k_2 are found from (7) and (7*a*) where, as already mentioned, the numerical value of the ratio is the same for each shoe.

Pressures are calculated from (1) after substituting corresponding values of k_1 and k_2.

The braking torque due to a short length $r\delta\theta$ of shoe face is fr or μpr. Substituting for p from (3) and integrating, the torque due to a whole shoe is

$$\begin{aligned} T &= \mu br^2 \int_{-a}^{a} (k_1 \cos\theta + k_2 \sin\theta)\,d\theta \\ &= 2\mu br^2 k_1 \sin a \quad . \quad . \quad . \quad . \quad (11), \end{aligned}$$

and the torque due to both shoes is obtained by substituting the sum of the two values of k_1 in (11).

For a given value of $\tan\epsilon$, assuming Ql has the same value for each shoe, a larger value of k_1 is obtained from (10) than from (10*a*). This may be more evident if the brackets are removed from the expressions $\tan\epsilon \pm \mu$. Therefore, with anticlockwise rotation of the drum, the right-hand shoe provides the greater braking torque and is the leading shoe.

If $\tan \epsilon = 0$, that is if CS is perpendicular to OX (Fig. 300 or Fig. 304), the two values of k_1 are equal and the shoes provide equal braking torques. Increasing $\tan \epsilon$, that is putting the brake-post pivots nearer either of the axes OX and OY, increases the torque due to the leading shoe and reduces the torque due to the trailing shoe.

For the resultant radial force on a shoe see Ex. 11, p. 326, and for the unbalanced force on the drum see Ex. 12, p. 327.

163. Dynamometers.—A dynamometer is an apparatus used for measuring forces or torques, in order that the work done by them in a given time may be calculated. The two main types are *absorption dynamometers* and *transmission dynamometers*; in the first type the work is wasted, usually through being converted into heat by friction, and in the second type the work is transmitted through the dynamometer with only a small loss.

There are many varieties of dynamometers, and a few of these are mentioned below. Further information is given in D. A. Low's *Applied Mechanics*.

164. Rope Brake Dynamometer.—One of the simplest absorption dynamometers consists of a rope brake as shown in Fig. 305. Two or three lengths of rope are wrapped once round the rim of a driven pulley or drum fixed on a horizontal shaft and the ropes are held in position by several wooden blocks. (For smaller powers the ropes may be arranged to embrace half the circumference of the pulley.)

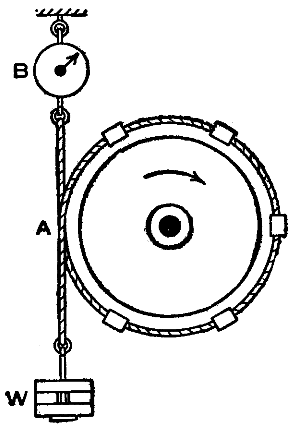


Fig. 305.

The upper ends of the ropes are connected to a spring balance B, and the lower ends support a load consisting of slotted weights, the number of which may be varied as required. Let W lb. be the hanging load, including the weight of the rope, etc. below the point A; S lb. be the spring-balance reading less the weight of the rope, etc. above the point A; r inches be the effective radius of the pulley, measured to the centre of the rope, and N r.p.m. be the speed of the pulley.

The torque is $(W-S)r$ lb. in. and the brake horse-power is

$$\frac{2\pi N(W-S)r}{33{,}000 \times 12}.$$

The friction between the ropes and the rim of the pulley converts the work done into heat, and this may be carried away by water circulating in a channel inside the rim. The water is supplied to the channel by one pipe and removed by another pipe with a flattened end to form a scoop. Due to centrifugal action the water is held in the rim provided the pulley speed does not fall below a certain value.

165. Froude Hydraulic Dynamometer.—Messrs Heenan & Froude manufacture widely used hydraulic dynamometers in which resistance is provided by water circulating between vanes or pockets in a rotor and in an outer casing or stator. The rotor is driven by the engine to be tested, and the stator is prevented from revolving by a load carried on an arm. The horse-power is calculated from the speed of the rotor and the torque on the stator.

166. Belt Dynamometer.—The Tatham dynamometer shown in Fig. 306 is an example of the transmission type, and it is used to obtain the value of $T_1 - T_2$, the difference between the tensions on the tight and slack sides of the belt which is transmitting power from the pulley A to the pulley B.

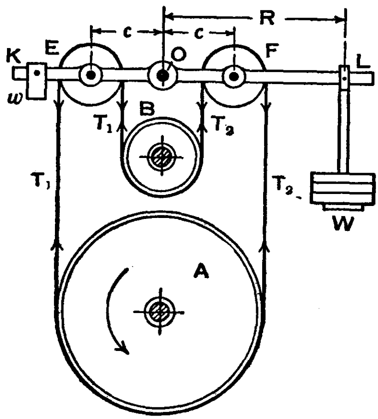


FIG. 306.

The belt passes over loose pulleys E and F, rotating on pins attached to the lever KL which can turn about a fixed pin O; the axes of the pulleys E and F are on opposite sides of and equidistant from the axis of the pin O. The lever KL carries a small adjustable weight w at K and a load W at L, the object of the weight w being to balance the lever before the load W is applied. In designing the dynamometer it is arranged that the straight portions of the belt are vertical.

The downward forces on the lever at the axes of the pulleys E and F are $2T_1$ and $2T_2$ respectively. From the Fig., taking moments about O, using lb. and inch units,

$$WR = 2T_1c - 2T_2c = 2c(T_1 - T_2),$$

therefore
$$T_1 - T_2 = \frac{WR}{2c} \text{ lb.}$$

Let r inches be the effective radius of the pulley A, that is the radius measured to the centre of the belt thickness, and let N r.p.m. be its speed. Neglecting the work lost in friction in the dynamometer, the horse-power transmitted is

$$\text{H.P.} = \frac{2\pi N(T_1 - T_2)r}{33{,}000 \times 12} = \frac{\pi Nr}{33{,}000 \times 12} \cdot \frac{WR}{c}.$$

167. Torsion Meters.—When the power to be measured is very large, as in the case of a steam turbine, the ordinary types of dynamometers are unsuitable, and an instrument known as a *torsion meter* may be used to measure the angle of twist in a known length of the shaft. The horse-power is calculated from the angle of twist, the speed, and the dimensions of the shaft. There are various types of torsion meters and one is described in Art. 168.

It is shown in books on Strength of Materials * that

$$\frac{T}{J} = \frac{C\theta}{l},$$

where T is the torque which twists a length l of a shaft through an angle θ radian, C is the modulus of rigidity,

$$J = \frac{\pi d^4}{32}$$

* See *Strength of Materials,* by B. B. Low, Longmans, Green & Co. Ltd., pp. 21–23.

for a solid shaft of diameter d, and

$$J=\frac{\pi}{32}(D^4-d^4)$$

for a hollow shaft of external and internal diameters D and d respectively.

The horse-power is $\text{H.P.}=\dfrac{2\pi NT}{33{,}000}$,

where N r.p.m. is the speed and T is in lb. ft.

Therefore for a solid shaft

$$\text{H.P.}=\frac{2\pi^2 NCd^4}{33{,}000\times 12\times 32l}\theta,$$

and for a hollow shaft

$$\text{H.P.}=\frac{2\pi^2 NC(D^4-d^4)}{33{,}000\times 12\times 32l}\theta,$$

where $C=12\times 10^6$ lb./in.^{2} for steel; D, d, and l are in inches; and the angle of twist θ radian is obtained from the scale reading on the torsion meter.

168. The Thring Torsion Meter.—An instrument for measuring the angle of twist in a length of shaft which is transmitting power is shown diagrammatically in Fig. 307. In its original form it was known as the Hopkinson-Thring torsion meter, and at a later date it was modified by Mr Thring. The following description does not include all the improvements.

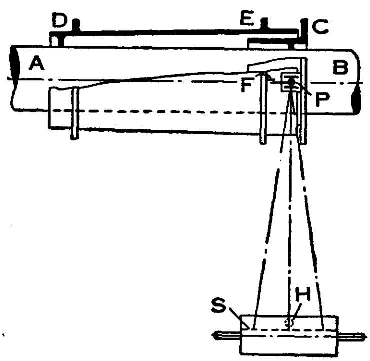


FIG. 307.

A collar C and a sleeve DE, both in two parts with longitudinal joints, are clamped to the shaft AB, the contact in each case being made by one internal gripping flange. The external lugs and bolts have been omitted to simplify the illustration. The length of shaft in which the angle of twist is measured is equal to the distance between the gripping flanges.

The external flange on the collar C carries a bracket on which two mirrors, back to back, are supported on a radial pivot P. A horizontal arm, between the mirrors, turns with them about the radial pivot when the shaft twists. This arm is operated by a leaf spring F fixed to the external flange E on the sleeve (details of the arm and spring are omitted from the drawing). When the end A of the shaft twists clockwise relatively to the end B (viewed from A), the mirrors turn, and the reflection from the front mirror, of the metallic filament of a lamp H in a scale box, is moved to the right on a scale S marked on a ground-glass screen. The scale box is supported on trunnions and its position can be adjusted.

When the shaft has turned half a revolution, the rear or second mirror is facing the lamp H and the reflection of the filament appears on the left half of the scale. Since the shaft is turning quickly, the two lines are seen on the scale simultaneously and the distance between them is a measure of the angle of twist in the shaft.

Two zero or fixed mirrors are also provided, mounted back to back on the bracket which carries the working mirrors and immediately above them. When power is not being transmitted, the four mirrors are parallel to the shaft axis and the four reflections of the lamp filament all give the same reading on the scale if the shaft is rotated.

Exercises XV

1. Assume that the effective diameter of the drum of the band brake shown in Fig. 287, p. 303, is 20 inches, the angle of embrace is 240°, $\mu = 0{\cdot}2$, OA $= 6{\cdot}5$ inches, OB $= 1{\cdot}5$ inches, and OC $= 25$ inches. Find the value of the applied force P, if the braking torque is to be 3000 lb. in. when the drum is rotating (*a*) anticlockwise, (*b*) clockwise.

2. Referring to the data in Ex. 1, suppose one end of the band is connected to the lever at A and the other end is pivoted to the frame of the machine. Find the value of the force P which will produce the braking torque when the drum is rotating (*a*) anticlockwise, (*b*) clockwise.

3. A band brake is lined with 10 blocks each of which subtends an angle of 18° at the centre of the brake drum (Fig. 288, p. 304). The drum diameter is 20 inches, the radial thickness of a block plus half the thickness of the band is $1\frac{1}{2}$ inches, and $\mu = 0{\cdot}35$. The lever is arranged as in Fig. 287; its dimensions are OA = 6·5 inches, OB = 1·5 inches, and OC = 25 inches. Find the value of the force P if the braking torque is to be 4000 lb. in. and the drum is rotating clockwise.

4. The hand-brake shown in Fig. 308 is used on a wall crane. The brake drum is 24 inches diameter, the lifting drum is 16 inches diameter and carries a load of 900 lb. If the coefficient of friction of the brake band is 0·3, find the *least* force at the end of the 2-foot lever to support the load. [U.L.]

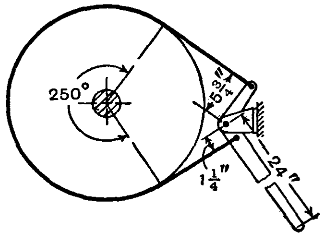


Fig. 308.

5. Referring to Art. 159 on brakes with internal-expanding shoes, prove that the resultant radial force on a shoe is

$$P = khbr(\alpha + \tfrac{1}{2}\sin 2\alpha)$$

and the resultant frictional force is

$$F = \mu khbr(\alpha + \tfrac{1}{2}\sin 2\alpha).$$

Denoting P by P_l and P_t for the leading shoe and trailing shoe respectively, find the value of the ratio P_l/P_t, given that $h = 0{\cdot}8r$, $\alpha = \pi/4$ rad., and $\mu = 0{\cdot}4$. Assume that Ql has the same value for each shoe.

6. Show that the unbalanced force *on the drum*, when the internal expanding shoes turn on a common pivot C, as in Fig. 289, is

$$R = (k_l - k_t)hbr(\alpha + \tfrac{1}{2}\sin 2\alpha)\sqrt{1+\mu^2},$$

and that if the drum is rotating anticlockwise the direction of this force is as indicated in Fig. 309, where $\gamma = \tan^{-1}\mu$. The symbols k_l and k_t are the values of k for the leading shoe and trailing shoe respectively.

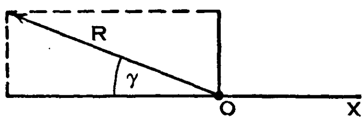


Fig. 309.

7. Find the maximum pressure in lb. per sq. in. on each shoe (Fig. 289), given Q = 50 lb., $l = 8$ inches, $h = 4$ inches, $r = 5$ inches, $b = 1\frac{1}{2}$ inches, $\alpha = \pi/4$ rad., and $\mu = 0{\cdot}4$.

8. Show that, for the brake with external-contracting fixed shoes, considered in Art. 161 and illustrated in Fig. 296 where the drum is rotating anticlockwise, the resultant radial force on a shoe is

$$P = kbr\sqrt{h^2(\alpha + \tfrac{1}{2}\sin 2\alpha)^2 + e^2(\alpha - \tfrac{1}{2}\sin 2\alpha)^2}.$$

The two values of P are denoted by P_l and P_t in Fig. 310; show that the line of action of each resultant is inclined at an angle β to the axis through the centre of the shoe, where

$$\tan\beta = \frac{e(\alpha - \frac{1}{2}\sin 2\alpha)}{h(\alpha + \frac{1}{2}\sin 2\alpha)}.$$

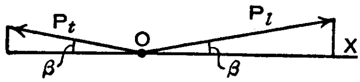


FIG. 310.

9. Prove that, for the brake with external-contracting fixed shoes (Fig. 296), the unbalanced force R *on the drum*, which is rotating anticlockwise, has horizontal and vertical components H and V respectively, as indicated in Fig. 311, where

$$H = br\{(k_l - k_t)h(\alpha + \tfrac{1}{2}\sin 2\alpha) - \mu(k_l + k_t)e(\alpha - \tfrac{1}{2}\sin 2\alpha)\},$$

$$V = br\{\mu(k_l - k_t)h(\alpha + \tfrac{1}{2}\sin 2\alpha) + (k_l + k_t)e(\alpha - \tfrac{1}{2}\sin 2\alpha)\},$$

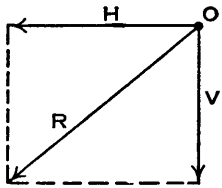


FIG. 311.

and k_l and k_t are the values of k obtained in Art. 161 for the leading shoe and trailing shoe respectively.

10. Find the value of θ when the pressure is a maximum on each shoe of the brake with external-contracting fixed shoes (Fig. 296), then calculate the maximum pressure on the leading shoe and the pressures when θ is 45° and −45°, in lb. per sq. in. Assume that $Q = 100$ lb. and $l = 15$ inches for each brake post, and that $h = 8$ inches, $e = 2$ inches, $r = 5$ inches, $b = 2$ inches, $\alpha = \pi/4$ rad., and $\mu = 0{\cdot}4$.

What is the fraction by which the leading shoe pressures must be multiplied to give the corresponding pressures on the trailing shoe?

11. Referring to the brake with external-contracting pivoted shoes, Art. 162 and Fig. 300, where the drum is rotating anticlockwise, show that the resultant radial force on a shoe is

$$P = br\sqrt{k_1^2(\alpha + \tfrac{1}{2}\sin 2\alpha)^2 + k_2^2(\alpha - \tfrac{1}{2}\sin 2\alpha)^2},$$

and its line of action makes an angle β with the axis through the centre of the shoe, where

$$\tan\beta = \frac{k_2(\alpha - \frac{1}{2}\sin 2\alpha)}{k_1(\alpha + \frac{1}{2}\sin 2\alpha)}.$$

In Fig. 312 the force P is denoted by P_l and P_t for the leading shoe and trailing shoe respectively, and it will be seen that β is negative for the former and positive for the latter. This follows from the signs of k_1 and k_2 which are discussed in the text.

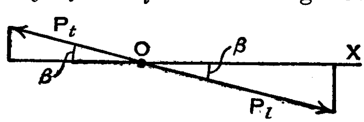


FIG. 312.

12. In the brake with external-contracting pivoted shoes, Art. 162 and Fig. 300, the drum is rotating anticlockwise. Show that the horizontal component H and the vertical component V of the unbalanced force R *on the drum* (Fig. 313) are given by

$$H = br\{(k_{l_1} - k_{t_1})(\alpha + \tfrac{1}{2}\sin 2\alpha) + \mu(k_{l_2} - k_{t_2})(\alpha - \tfrac{1}{2}\sin 2\alpha)\},$$

$$V = br\{\mu(k_{l_1} - k_{t_1})(\alpha + \tfrac{1}{2}\sin 2\alpha) - (k_{l_2} - k_{t_2})(\alpha - \tfrac{1}{2}\sin 2\alpha)\},$$

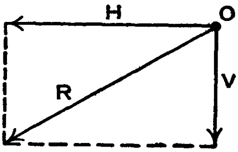


Fig. 313.

where the suffixes $_l$ and $_t$ denote the leading shoe and trailing shoe respectively.

13. Prove that for the brake with external-contracting pivoted shoes (Fig. 300) the ratio of the pressures on the leading and trailing shoes, at points which are diametrically opposite one another, is equal to k_{l_1}/k_{t_1}, where the suffixes $_l$ and $_t$ denote the leading shoe and trailing shoe respectively. Then assuming Ql has the same value for each shoe, show that

$$\frac{k_{l_1}}{k_{t_1}} = \frac{d(\alpha + \frac{1}{2}\sin 2\alpha)(1+\mu^2) + 2r\mu \sin\alpha(\tan\epsilon - \mu)}{d(\alpha + \frac{1}{2}\sin 2\alpha)(1+\mu^2) - 2r\mu \sin\alpha(\tan\epsilon + \mu)}.$$

14. Show that in a brake with internal-expanding shoes (Art. 159) the leading shoe would tend to become self-locking if M_f exceeded M_p or if

$$\frac{\mu r}{h} > \frac{\alpha + \frac{1}{2}\sin 2\alpha}{2\sin\alpha}.$$

Then show that if $\mu = 0{\cdot}5$ and $\alpha = 90°$ (these values are larger than is usual in practice), r/h would have to exceed $1{\cdot}57$ approximately before self-locking occurred, or about $1{\cdot}82$ if α is reduced to $45°$.

15. The diagram (Fig. 314) shows a friction brake in which the curved lever BC is pivoted at the fixed point C and carries a friction lining which presses on the rotating drum D over an arc of 90°. The diameter of the drum is 10 inches. The distances OB and OC are each 6 inches. The pressure exerted by the friction lining on the drum at any point P is $50 \sin\theta$ lb. per sq. in. The lining is 2 inches wide and the coefficient of friction is 0·3.

Calculate the braking torque exerted on the drum, also the amount of the force F which is required at B to apply the brake. The drum rotates in the clockwise direction. [U.L.]

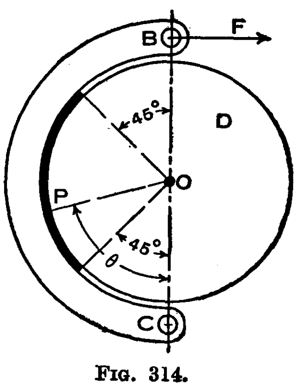


Fig. 314.

CHAPTER XVI

GOVERNORS

169. Function of a Governor.—A governor is used to regulate the mean speed of a machine or a prime mover over a long period when there are variations in the load. For instance, if the load on an engine increases it becomes necessary to increase the supply of working fluid, and if the load decreases less working fluid is required. The governor automatically controls the supply to the engine when the load varies, and keeps the mean speed within certain limits. When the load increases the engine speed decreases, the configuration of the governor changes and a valve is moved to increase the supply of working fluid; conversely, when the load decreases the engine speed increases, and the governor decreases the supply of working fluid.

The function of a flywheel in an engine is entirely different, as it is employed to control the speed variations caused by the fluctuations of the engine turning moment during each cycle of operations, and it does not control speed variations caused by a varying load. The varying demand for power is met by the governor regulating the supply of working fluid.

170. Centrifugal Governors.—The *centrifugal governor* is the type most commonly used, and the analyses of several forms of it are given in the Arts. which follow. In every case there are radial inertia forces depending on the square of the angular velocity, and the reader might like to refresh his memory at this stage by referring back to Art. 43, p. 73, where centripetal and centrifugal forces are discussed and illustrated by an example on a conical pendulum which is closely related to a centrifugal governor.

Before analysing any particular form of centrifugal governor the general idea of its operation in an engine will be given. In the form shown diagrammatically in Fig. 315 equal masses A and A′, known as the *governor balls*, attached to arms BC and BC′ respectively, revolve with a vertical spindle which is driven by the engine. The upper ends of the arms are pivoted at B to the spindle, so that the balls may rise or fall as they revolve about the vertical axis. Links CD and C′D′ connect the arms to a sleeve S which is keyed to the spindle; this sleeve revolves with the spindle but can slide up and down it. The balls and the sleeve rise or fall according as the spindle speed increases or decreases. The sleeve is connected by links to the mechanism operating the supply of working fluid, so that this is decreased when the sleeve rises and increased when it falls.

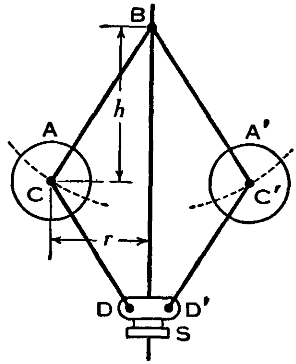


FIG. 315.

If the load on the engine increases, the engine and governor speeds decrease, the balls and sleeve descend, the supply of working fluid is increased and the engine speed is increased; thus the extra power output is provided to balance the increased load. If the load decreases, the engine and governor speeds increase, the balls and sleeve rise, the supply of working fluid is decreased, the engine speed is decreased and the power output is reduced.

The governor runs steadily when the forces acting on a ball are in equilibrium; one of these forces is the centrifugal force, which is really a reversed effective force, as explained in Art. 43.

The *height* of the governor is defined as the vertical distance of the centre of a ball below the point where the axes of the arms (or arms produced) intersect on the spindle axis, and it will be denoted by h (Fig. 315). Often the arms are pivoted at points offset from the spindle axis and sometimes they cross one another.

The radius of the path of the centre of a ball will be denoted by r.

171. Simple Centrifugal Governor.—One ball of a simple centrifugal governor with its arm and link is shown in Fig. 316. At (a) the pivot B is on the spindle axis at O, and at (b) and (c) it is offset from the axis. At (b) the arm produced intersects the axis at O and at (c) the arm crosses the axis at O.

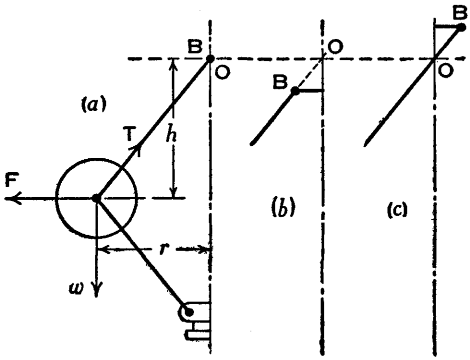


FIG. 316.

Let w be the weight of the ball, T the tension in the arm and F the centrifugal force, when the radius to the centre of the ball is r and the angular velocity of the arm and ball about the spindle axis is ω. It will be assumed that the weights of the arm, the link and the sleeve are negligible, then, provided the height h is not varying, the forces w, T, and F are in equilibrium.

$$F = \frac{w}{g}\omega^2 r,$$

and taking moments about O,

$$Fh = wr$$

or

$$\frac{w}{g}\omega^2 rh = wr,$$

from which

$$h = \frac{g}{\omega^2}.$$

If g is in feet per second per second and ω is in radians per second, then h is in feet.

Let N be the speed in revolutions per minute, then

$$\omega=\frac{2\pi N}{60}=\frac{\pi N}{30}$$

and
$$h=\frac{900g}{\pi^2 N^2}\text{ feet.}$$

If the numerical value of g is taken as 32·2,

$$h=\frac{2936}{N^2}\text{ feet}=\frac{35{,}230}{N^2}\text{ inches.}$$

Since h is inversely proportional to N^2, it is small at high speeds, and at such speeds the change in h corresponding to a small change in speed is insufficient to enable a governor of the simple type shown in Fig. 316 to operate the mechanism to give the necessary change in the fuel supply. For example, the values of h corresponding to speeds of 100, 200, 300, and 400 r.p.m. are 3·52, 0·88, 0·39, and 0·22 inches respectively. Consequently, this governor could only work satisfactorily at relatively low speeds.

172. Porter Governor.—The Porter governor shown diagrammatically in Fig. 317 (*a*) is an example of a *loaded governor*. A load attached to the sleeve slides up and down the central spindle, and the additional downward force increases the speed of revolution required to enable the balls to rise to any predetermined level.

Let w be the weight of each ball and W be the weight of the central load. Consider the forces acting on one-half of the governor, and for the configuration shown let T_1 be the force in the arm and T_2 be the force in the suspension link; let θ and ϕ be the inclinations to the vertical of the arm and link respectively. The effect of friction is taken into account in Art. 174.

There are several ways of determining the relation between the height h and the speed ω, the simplest of which probably consists of imagining the link to be removed and considering the equilibrium of the arm. Let Q be the horizontal component of the force T_2 in the link, then, since the vertical component is $\frac{1}{2}$W, it can be seen from the triangle of forces for the lower joint of the link (Fig. 317 (*b*)) that

$$Q = \tfrac{1}{2}W \tan \phi.$$

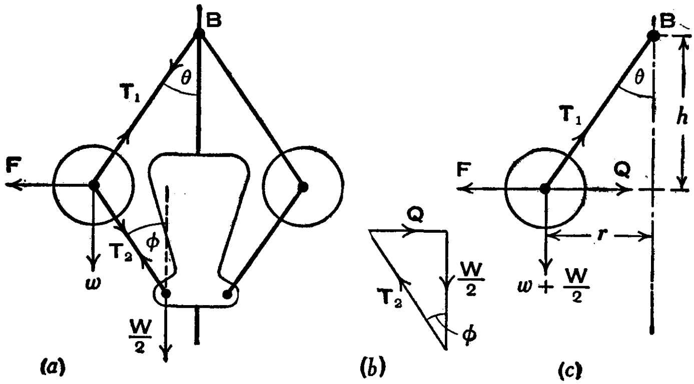


FIG. 317.

Therefore the action of the arm will be unaltered if the link is removed and the forces Q and $\frac{1}{2}$W are applied to the ball as shown at (*c*).

Taking moments about B,

$$Fh = (w + \tfrac{1}{2}W)r + Qh$$

or
$$\frac{w}{g}\omega^2 rh = (w + \tfrac{1}{2}W)r + \tfrac{1}{2}Wh \tan \phi.$$

Now $\tan \theta = \dfrac{r}{h}$. Let $\tan \phi = q \tan \theta = \dfrac{qr}{h}$, then

$$\frac{w}{g}\omega^2 rh = (w + \tfrac{1}{2}W)r + \tfrac{1}{2}Wqr,$$

from which
$$h = \frac{w + \frac{1}{2}W(1 + q)}{w} \cdot \frac{g}{\omega^2}.$$

If $q = 1$, as is often the case,

$$h = \frac{w + W}{w} \cdot \frac{g}{\omega^2}.$$

Alternatively, *h* may be determined by resolving vertically and horizontally the forces acting at the centre of the ball. From Fig. 317 (c),

$$T_1 \cos \theta = w + \tfrac{1}{2}W$$

and

$$F = T_1 \sin \theta + Q.$$

Eliminating T_1 and substituting $\frac{1}{2}Wq \tan \theta$ for Q,

$$F = (w + \tfrac{1}{2}W) \tan \theta + \tfrac{1}{2}Wq \tan \theta,$$

from which the equation for h is easily obtained.

Another method is to consider the equilibrium of the forces acting on the link CD (Fig. 318); these forces are F, w, and T_1 at C, and $\frac{1}{2}W$ and Q at D. Produce BC to intersect at I a line through D perpendicular to the spindle axis, and draw CM perpendicular to ID, then by taking moments about I the forces T_1 and Q are avoided. It will be left to the reader to complete the solution by this method. The point I is, of course, the instantaneous centre of the link CD.

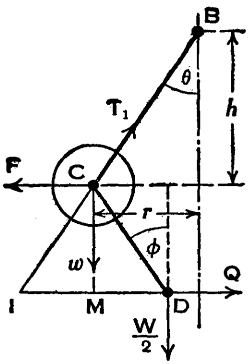


FIG. 318.

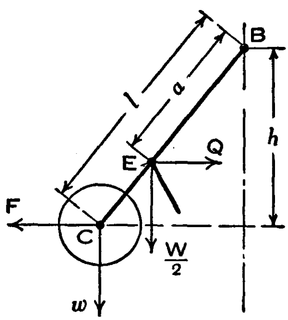


FIG. 319.

Consider now the effect of connecting the suspension link to an intermediate point E on the arm (Fig. 319) instead of to the end C. This reduces the moment about B of the force in the suspension link to a/l of its former value, the lengths a and l being as indicated. The effect is equivalent to reducing the load W to Wa/l, and the value of the height h becomes

$$h = \frac{w + \frac{Wa}{2l}(1+q)}{w} \cdot \frac{g}{\omega^2},$$

and when $q=1$,

$$h = \frac{w + \frac{Wa}{l}}{w} \cdot \frac{g}{\omega^2}.$$

If this is not obvious, the reader should produce the equation by taking moments about B and simplifying. Ex. 3, p. 364, where the point B is not on the spindle axis, is worth consideration.

Example.—The dimensions of a Porter governor are shown in Fig. 320 (a); each ball weighs 4 lb. and the central load is 56 lb. When the sleeve is in its lowest position the arms are inclined at 30° to the vertical. Given that the sleeve travel is 2 inches, it is required to find the maximum and minimum speeds, in revolutions per minute, at which the governor operates.

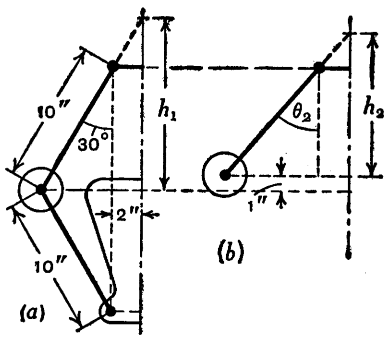


FIG. 320.

It will be seen that the arms and suspension links are of equal length, and the axes of the upper joints of the arms and the lower joints of the suspension links are offset equal distances from the spindle axis. Therefore $q=1$, and

$$h = \frac{w+W}{w} \cdot \frac{g}{\omega^2} = \frac{4+56}{4} \cdot \frac{g}{\omega^2} = \frac{15g}{\omega^2}.$$

The speed in revolutions per minute is

$$N=\frac{60\omega}{2\pi}=\frac{30}{\pi}\sqrt{\frac{15g}{h}}=\frac{30}{\pi}\sqrt{\frac{15\times 12\times 32{\cdot}2}{h}},$$

where h is in inches.

Let h_1 and h_2 denote the governor heights when the sleeve is in its lowest and highest positions respectively, and let N_1 and N_2 be the corresponding speeds. Let θ_2 be the inclination of each arm when the height is h_2.

From Fig. 320 (*a*),

$$\begin{aligned} h_1 &= 10\cos 30^\circ + 2\cot 30^\circ \\ &= 10\times 0{\cdot}8660 + 2\times 1{\cdot}7321 \\ &= 8{\cdot}660 + 3{\cdot}464 \\ &= 12{\cdot}12 \text{ inches.} \end{aligned}$$

When the sleeve rises 2 inches the balls rise 1 inch.

From Fig. 320 (*b*),

$$\cos\theta_2=\frac{10\cos 30^\circ - 1}{10}=0{\cdot}7660, \quad \text{from which } \theta_2=40^\circ.$$

$$\begin{aligned} h_2 &= 10\cos\theta_2 + 2\cot\theta_2 \\ &= 7{\cdot}660 + 2\times 1{\cdot}1918 \\ &= 10{\cdot}04 \text{ inches.} \end{aligned}$$

$$N_1=\frac{30}{\pi}\sqrt{\frac{180\times 32{\cdot}2}{12{\cdot}12}}=209 \text{ r.p.m.}$$

$$N_2=\frac{30}{\pi}\sqrt{\frac{180\times 32{\cdot}2}{10{\cdot}04}}=229 \text{ r.p.m.}$$

173. Proell Governor.—In the Proell governor, one half of which is shown diagrammatically in Fig. 321, the ball arm CD carries the ball A at its upper end and one half the central load at its lower end. The link BE is pivoted to the arm at an intermediate point E. The difference between this governor and the Porter governor is that the ball is fixed to an extension of DE instead of being at the point E.

Consider the forces acting on the arm CD and indicated in Fig. 321; these are F and w at C, $\frac{1}{2}$W and H at D, and the force at E acting in the direction EB. The arm CD tends to bend, consequently the resultant forces at the ends C and D do not act along it. For equilibrium the lines of action of these resultant forces and the force at E must intersect at a point in EB or EB produced. Therefore it is an easy matter to find the resultant force at D and the horizontal component H, when the forces F and w and the configuration of the governor are known. The point is that the value of H is unknown in the first place; it might even be negative, that is it could act from right to left, or it might be zero. In the analysis which follows, the force H and the force in the link EB are avoided. Friction is considered in Art. 174.

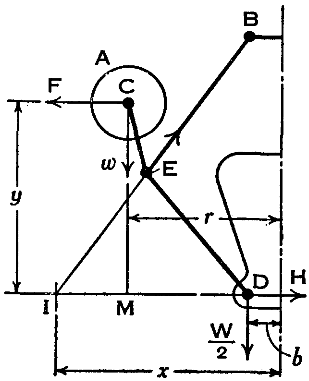


FIG. 321.

Produce BE to intersect at I a line through D at right angles to the spindle axis, then I is the instantaneous centre of the arm CD. Drop a perpendicular CM on to ID.

Taking moments about I, using the symbols shown,

$$\mathrm{F.CM} = w.\mathrm{IM} + \tfrac{1}{2}\mathrm{W.ID},$$

$$\frac{w}{g}\omega^2 ry = w(x-r) + \tfrac{1}{2}\mathrm{W}(x-b),$$

$$\omega^2 = \frac{\{w(x-r) + \frac{1}{2}\mathrm{W}(x-b)\}g}{wry}.$$

From this equation the speed ω may be obtained for any given configuration of the governor.

174. Governor Friction.—As a governor sleeve rises or falls, the motion is opposed by friction in the governor and in the links the sleeve operates. These frictional resistances may be reduced to a single force R at the sleeve, acting downwards when the sleeve is ascending and upwards when it is descending. In a loaded governor this is equivalent to altering the central load from W to $W \pm R$, the positive or negative sign being used according as the governor speed is increasing or decreasing.

For the Porter governor when friction is neglected (Art. 172),

$$\omega^2 = \frac{w + \frac{1}{2}W(1+q)}{w} \cdot \frac{g}{h},$$

therefore, with friction,

$$\omega^2 = \frac{w + \frac{1}{2}(W \pm R)(1+q)}{w} \cdot \frac{g}{h}.$$

For the Proell governor with friction (see equation at the end of the preceding Art. where friction is neglected),

$$\omega^2 = \frac{\{w(x-r) + \frac{1}{2}(W \pm R)(x-b)\}g}{wry}.$$

Example.—In the Porter governor example on p. 334, $w = 4$ lb., $W = 56$ lb., and $q = 1$; the sleeve travel is 2 inches and the ball lift is 1 inch. It was shown that the lowest and highest speeds were $N_1 = 209$ r.p.m. and $N_2 = 229$ r.p.m., corresponding to heights $h_1 = 12{\cdot}12$ inches and $h_2 = 10{\cdot}04$ inches respectively.

Suppose now that the governor friction is equivalent to a force $R = 3$ lb. at the sleeve. It is required to find the speeds corresponding to the above values of h_1 and h_2.

Since $q = 1$,

$$\omega = \sqrt{\frac{w + W \pm R}{w} \cdot \frac{g}{h}} = \sqrt{\frac{60 \pm 3}{4} \times \frac{32{\cdot}2 \times 12}{h}} \text{ rad./sec.},$$

where h is in inches, and

$$N = \frac{30\omega}{\pi} = \frac{30}{\pi}\sqrt{(15 \pm 0{\cdot}75) \times \frac{386{\cdot}4}{h}} \text{ r.p.m.}$$

Consider *sleeve ascending.*

When $h = 12{\cdot}12$ inches, $N = \dfrac{30}{\pi}\sqrt{\dfrac{15{\cdot}75 \times 386{\cdot}4}{12{\cdot}12}} = 214$ r.p.m.

When $h = 10 \cdot 04$ inches, $N = \frac{30}{\pi}\sqrt{\frac{15 \cdot 75 \times 386 \cdot 4}{10 \cdot 04}} = 235$ r.p.m.

Consider *sleeve descending.*

When $h = 10 \cdot 04$ inches, $N = \frac{30}{\pi}\sqrt{\frac{14 \cdot 25 \times 386 \cdot 4}{10 \cdot 04}} = 224$ r.p.m.

When $h = 12 \cdot 12$ inches, $N = \frac{30}{\pi}\sqrt{\frac{14 \cdot 25 \times 386 \cdot 4}{12 \cdot 12}} = 204$ r.p.m.

The relations between speed and ball lift are shown in Fig. 322 with the calculated speeds plotted horizontally, and an examination of the curves will explain the effect of governor friction. One ball in its extreme positions is included in the diagram.

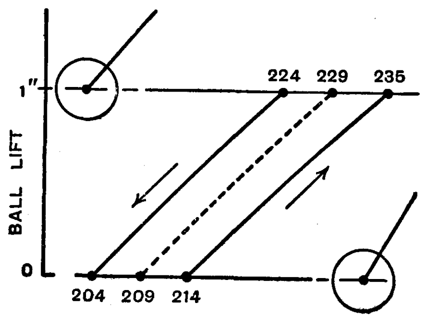


FIG. 322.

Assume the ball is in its lowest position and that the governor is revolving at 214 r.p.m., then if the speed increases to 235 r.p.m. the ball ascends to its highest position. If the speed now decreases to 204 r.p.m. the ball remains at the upper level until 224 r.p.m. is reached, then it descends until the speed is 204 r.p.m. An increase of speed to 214 r.p.m. is required before the ball begins to rise again. The ball does not necessarily move from one extreme position to another, since at any moment the speed may become steady or it may increase or decrease. The dotted curve between the two full line curves shows the variation of lift with speed when friction is assumed to be negligible, the two speeds plotted being those obtained in the example on p. 334.

175. Effect of Masses of Arms and Links.—In this Art. an arm is considered, but similar reasoning may be applied to a link. It will be assumed that the arm is of uniform cross-section; let w' be its weight and l be its length, measured from the axis of the pivot B (Fig. 323) to the centre C of the ball. As before, let h be the height and let r be the radius to the centre of the ball.

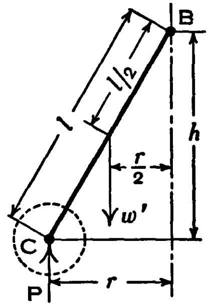


FIG. 323.

Gravity Effect.—Suppose an upward vertical force P is applied at C so that the vertical forces on the arm are in equilibrium. Since the arm is of uniform cross-section its centre of gravity is at a distance $\frac{1}{2}l$ from B and at a radius $\frac{1}{2}r$ from the spindle axis. Taking moments about B,

$$Pr = w' \times \tfrac{1}{2}r$$

and $$P = \tfrac{1}{2}w'.$$

Therefore, to allow for the gravity effect of the arm, a mass weighing $\frac{1}{2}w'$ must be assumed to be added to the mass of the ball. The gravity effect is the same whether the pivot B is on the spindle axis or offset from it.

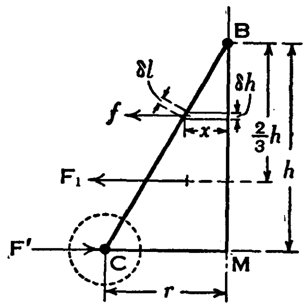


FIG. 324.

Centrifugal Effect.—The position of the pivot B has to be taken into account when allowing for the centrifugal effect of the arm. First, it will be assumed that B is on the spindle axis. Let f be the centrifugal force acting on any short length δl of the arm, and let this short length be at a radius x from the spindle axis (Fig. 324). Let δh be the projection on the spindle axis of the length δl.

The mass of the length δl is $\dfrac{w'}{g}\dfrac{\delta l}{l}$ or $\dfrac{w'}{g}\dfrac{\delta h}{h}$, and the centrifugal force on it is

$$f = \frac{w'}{g}\frac{\delta h}{h}\omega^2 x.$$

The resultant centrifugal force on the arm is

$$F_1 = \frac{w'}{g}\frac{\omega^2}{h}\sum_0^h x\delta h = \frac{w'}{g}\frac{\omega^2}{h} \times (\text{Area of triangle BCM}),$$

where BM $= h$ and CM $= r$. The force F_1 acts through the centre of area of the triangle, and this is a vertical distance $\frac{2}{3}h$ below B.

Let a horizontal force F′ be applied at C so that the horizontal forces are in equilibrium, then taking moments about B,

$$F'h = F_1 \times \tfrac{2}{3}h = \frac{w'\omega^2}{gh}\frac{hr}{2}\frac{2h}{3},$$

from which
$$F' = \frac{w'}{3g}\omega^2 r.$$

Therefore, to allow for the centrifugal effect of the arm, a mass weighing $\frac{1}{3}w'$ must be assumed to be added to the mass of the ball.

Consider next the case where the pivot B is offset a distance AB $= c$ from the spindle axis as shown in Fig. 325. Let the vertical length BM $=$ AN be denoted by h' since the height h is now equal to ON; then the projection of δl on AN becomes $\delta h'$.

The centrifugal force on the short length δl is

$$f = \frac{w'}{g}\frac{\delta h'}{h'}\omega^2(x+c),$$

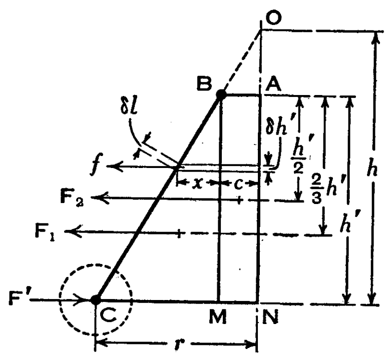


FIG. 325.

and the resultant centrifugal force on the whole arm is proportional to the area of the trapezium ABCN, which is the sum of the areas of the triangle BCM and the rectangle ABMN.

Let F_1 and F_2 be the resultant forces acting through the centres of area of the triangle BCM and the rectangle ABMN respectively. As before, let F′ be the horizontal force acting at C for the purpose of equilibrium, then taking moments about B,

$$F'h' = F_1 \times \tfrac{2}{3}h' + F_2 \times \tfrac{1}{2}h'$$
$$= \frac{w'\omega^2}{gh'} \frac{h'(r-c)}{2} \frac{2h'}{3} + \frac{w'\omega^2}{gh'} h'c\frac{h'}{2},$$

from which
$$F' = \frac{w'}{g}\omega^2\left(\frac{r-c}{3} + \frac{c}{2}\right)$$
$$= \frac{w'}{3g}\omega^2\left(r + \frac{c}{2}\right)$$
$$= \frac{w'}{3g}\left(1 + \frac{c}{2r}\right)\omega^2 r.$$

Therefore, to allow for the centrifugal effect of the arm, a mass weighing $\frac{w'}{3}\left(1 + \frac{c}{2r}\right)$ must be assumed to be added to the mass of the ball. When necessary the additional mass can be calculated for particular values of r, but in some cases it is sufficiently accurate to use the mean value of r.

If the arm crosses the spindle axis, the constant c is negative, and the weight of the mass assumed to be added to that of the ball becomes $\frac{w'}{3}\left(1 - \frac{c}{2r}\right)$.

176. Spring-Loaded Governors.—In a spring-loaded governor one or more springs provide the necessary forces to control the balls at the various speeds. The Hartnell governor shown in Fig. 326 is an example of the spring-loaded type. There are two bell-crank levers L pivoted at points O to a frame B which is attached to the governor spindle and therefore rotates with it. Each lever carries a ball A at the end of the outer arm and a roller R at the end of the other arm. A helical spring in compression

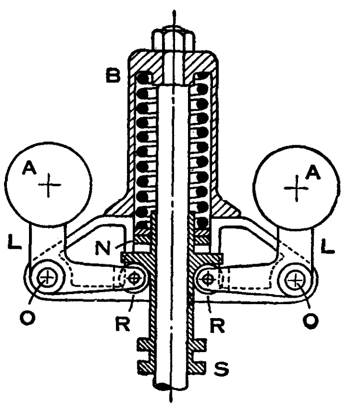


FIG. 326.

provides equal downward forces on the two rollers through a collar on the sleeve S and, since the bell-crank levers are pivoted at the points O, inward radial forces act on the balls at the ends of the outer arms. The spring force may be adjusted by screwing a nut N up or down on the sleeve, but the details are not shown in the Figure.

In analysing this and other spring-loaded governors, the inward radial forces or centripetal forces acting on the balls will be reversed; then the centrifugal and other forces acting on a bell-crank lever may be regarded as being in equilibrium when the governor is running steadily.

The forces acting at the extremities of one bell-crank lever are shown in Fig. 327, where w is the weight of the ball and r is the distance of its centre from the spindle axis, P is the spring force, F is the centrifugal force, and a and b are the lengths of the arms. It will be assumed that the arms are mutually perpendicular and that when the sleeve is at its mid-point of travel the ball arm is vertical. When these conditions do not hold, it is a simple matter to modify the equations.

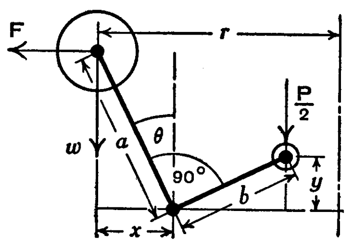


FIG. 327.

Let x be the outward travel of the ball and y the upward travel of the roller when the bell-crank lever turns through an angle θ from its mean position, as indicated.

From similar triangles,

$$\frac{x}{y}=\frac{\sqrt{a^2-x^2}}{\sqrt{b^2-y^2}}=\frac{a}{b}.$$

Taking moments about the axis of the pin round which the bell-crank lever turns,

$$\tfrac{1}{2}\mathrm{P}\sqrt{b^2-y^2}=\mathrm{F}\sqrt{a^2-x^2}+wx,$$

from which $$\mathrm{P}=\frac{2\mathrm{F}a}{b}+\frac{2wx}{\sqrt{b^2-y^2}}.$$

Also $$F = \frac{w}{g}\omega^2 r.$$

When the ball is to the right of its mean position, the gravity moment wx becomes negative. If the weight of the sleeve is W and is taken into account, it must be added to the spring force P when obtaining the relation between P and F. If the friction force at the sleeve is considered, it is added to or subtracted from the spring force according as the speed is increasing or decreasing.

Let S be the spring rate or stiffness, that is the force per unit compression, then if the spring force increases from P_1 to P_2 when the sleeve moves up a distance y',

$$S = \frac{P_2 - P_1}{y'}.$$

If force is measured in pounds and y' in inches, then S is in pounds per inch.

If P_1 is the value of P when the sleeve is in its lowest position, then

$$\text{Initial spring compression} = \frac{P_1}{S}.$$

Example.—One of the bell-crank levers of a Hartnell governor is shown in Fig. 328 and its arms are mutually perpendicular. The spindle axis is vertical, and when the ball arm is parallel to it, the ball rotates at a radius of 5·5 inches and the sleeve is in its mean position. The lengths of the ball arm and the roller arm are 5 inches and 4 inches respectively, and the ball weighs 6 lb. The total lift of the sleeve is 2 inches, and the lowest and highest speeds between which the governor operates are 350 r.p.m. and 370 r.p.m. respectively. Neglecting friction and the weight of the sleeve, it is required to find the value of the spring force at each of these speeds, the stiffness of the spring and the amount it is compressed when the sleeve is in its lowest position.

Let F_1 be the centrifugal force, r_1 the radius to the centre of the ball from the spindle axis, and P_1 the spring force (Fig. 329) when the speed is 350 r.p.m., that is, when the

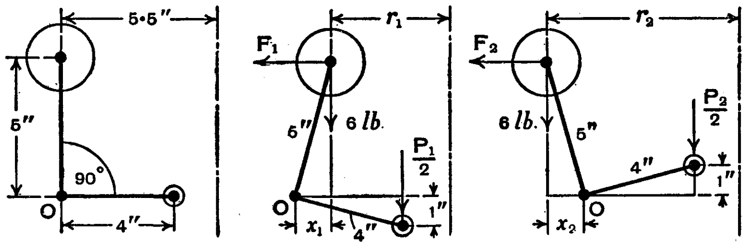


FIG. 328. FIG. 329. FIG. 330.

sleeve is in its lowest position. The vertical displacement of the roller below its mean position is 1 inch, and the force on it is $\frac{1}{2}P_1$. Let x_1 be the corresponding horizontal displacement of the ball.

From similar triangles,

$$\frac{x_1}{5}=\frac{1}{4} \quad \text{and} \quad x_1=1{\cdot}25 \text{ inches,}$$

therefore $$r_1=5{\cdot}5-1{\cdot}25=4{\cdot}25 \text{ inches}$$

and $$F_1=\frac{6}{32{\cdot}2}\times\left(\frac{350\pi}{30}\right)^2\times\frac{4{\cdot}25}{12}=88{\cdot}7 \text{ lb.}$$

Taking moments about O,

$$\tfrac{1}{2}P_1\sqrt{4^2-1}=F_1\sqrt{5^2-1{\cdot}25^2}-6\times1{\cdot}25$$

and $$P_1=2F_1\times\frac{5}{4}-\frac{2\times6\times1{\cdot}25}{\sqrt{15}}.$$

Substituting the value of F_1, then $P_1=218$ lb.

Referring to Fig. 330 and using the suffix 2 with the symbols to denote their values when the speed is 370 r.p.m., then since the roller is now 1 inch above its mean position,

$$x_2=x_1=1{\cdot}25 \text{ inches, therefore } r_2=5{\cdot}5+1{\cdot}25=6{\cdot}75 \text{ inches.}$$

$$F_2=\frac{6}{32{\cdot}2}\times\left(\frac{370\pi}{30}\right)^2\times\frac{6{\cdot}75}{12}=157{\cdot}4 \text{ lb.}$$

Taking moments about O,

$$\tfrac{1}{2}P_2\sqrt{4^2-1}=F_2\sqrt{5^2-1{\cdot}25^2}+6\times 1{\cdot}25.$$

Substituting the value of F_2 and solving for P_2 gives $P_2=397$ lb.

The spring force increases from $P_1=218$ lb. to $P_2=397$ lb. when the sleeve rises 2 inches, therefore the spring stiffness is

$$S=\frac{397-218}{2}=89{\cdot}5 \text{ lb. per inch.}$$

When the sleeve is in its lowest position,

$$\text{Spring compression}=\frac{218}{89{\cdot}5}=2{\cdot}44 \text{ inches.}$$

177. Spring-Loaded Governors (*continued*).—A governor in which the balls are connected by two springs in tension, one behind the other, is shown in Fig. 331. As drawn the spindle is horizontal and the path of the centres of the balls is in a vertical plane.

Let P be the sum of the tensions in the two springs and F be the centrifugal force when the radius to the centre of each ball is r, then

$$F=P.$$

The effect of gravity on the balls is to produce varying equal and opposite forces on the sleeve as the spindle rotates.

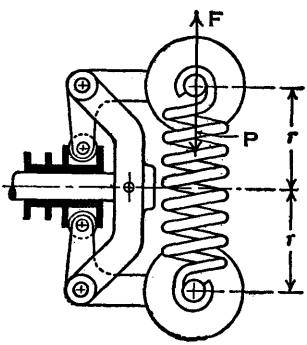


Fig. 331.

If S is the rate or stiffness of each spring, and if the tension $\tfrac{1}{2}P$ increases from $\tfrac{1}{2}P_1$ to $\tfrac{1}{2}P_2$ as the spring length increases from $2r_1$ to $2r_2$, then

$$S=\frac{\tfrac{1}{2}P_2-\tfrac{1}{2}P_1}{2r_2-2r_1}=\frac{P_2-P_1}{4(r_2-r_1)}.$$

When $P=P_1$ the extension is

$$\frac{\tfrac{1}{2}P_1}{S}=\frac{2(r_2-r_1)P_1}{P_2-P_1}.$$

Since the spring force cannot be adjusted in this governor, it is usual to have a supplementary adjustable spring which acts on a lever and provides a force at the sleeve. In this case the force P is less than F, and the relation between the forces may be obtained by taking moments about the axis of the pivot of the bell-crank lever. (See Fig. 348 and Ex. 16 on p. 367.)

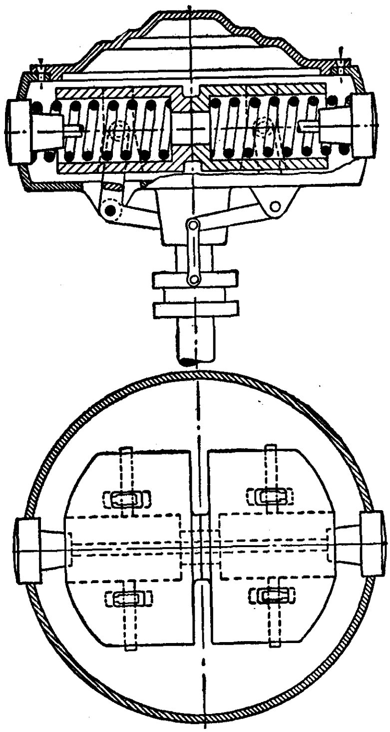


FIG. 332.

An example of a type of governor in which there are two helical springs in compression, one acting on each mass, is illustrated in Fig. 332; this is the Hartung governor. For simplicity the springs have been omitted from the plan view. The rod which passes through the masses and the springs is screwed at each end so that the springs may be adjusted to provide the required initial forces on the masses. When the governor is operating and running at a steady speed, the centrifugal force and the spring force acting on a mass are equal to one another.

Another type of spring-loaded governor is illustrated in Fig. 333. The bell-crank levers L are pivoted at points O to a head H which is attached to the sleeve S and rises or falls with it, whilst rotating with the spindle T which turns in bearings in the fixed tube shown in black. The rollers R bear on the enlarged top of the spindle, therefore when the balls A move outwards, the head H and the sleeve S rise and the helical spring inside the sleeve is compressed. The initial force in the spring may be adjusted by turning the plug D which is screwed into the lower end of the sleeve.

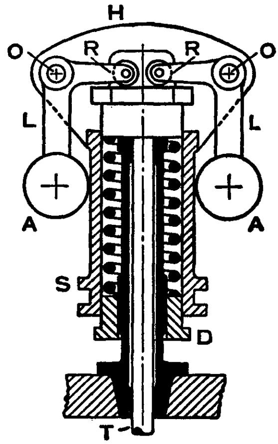


Fig. 333.

The forces acting on one bell-crank lever when the governor is running at a steady speed are shown in Fig. 334; W is the weight of the sleeve and head, P is the total spring force, w is the weight of the ball, and F is the centrifugal force.

At C, the centre of the ball, there is the horizontal force F and the vertical force w. At the pivot O there is the vertical force $\frac{1}{2}(W+P)$. There is also a horizontal reaction at O and a vertical reaction at the roller R.

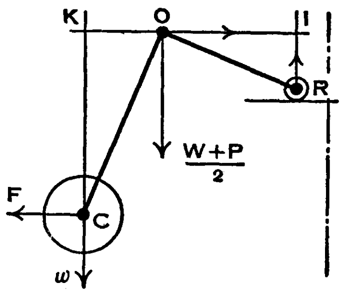


Fig. 334.

Since the roller moves horizontally and the pivot moves vertically when the speed changes, the instantaneous centre I of the bell-crank lever is at the intersection of a vertical line through the centre of the roller R and a horizontal line through O. Draw a vertical line through C to intersect IO produced at K.

Taking moments about I,

$$F.CK = w.KI + \tfrac{1}{2}(W + P).OI.$$

When the various dimensions are known it is a simple matter to solve a particular problem.

A variation of the type known as the Pickering governor is shown diagrammatically in Fig. 335. This illustration is from a clockwork motor for driving a gramophone, but

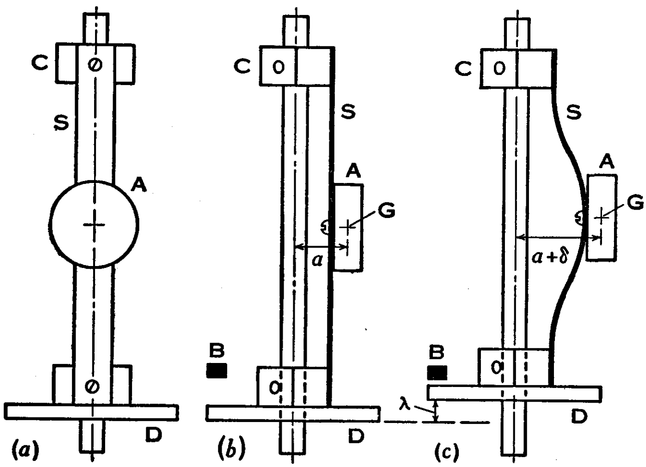


Fig. 335.

the governor has other uses. There are three straight leaf springs (only one is shown) arranged at equal angular intervals round the spindle, with a mass attached to the centre of each; the masses move outwards and the springs bend as they rotate about the spindle axis with increasing speed.

Views of the governor at rest with two springs removed are shown at (*a*) and (*b*). At (*c*) the governor is assumed to be rotating and the spring S is deflected together with the mass A. The upper end of the spring is attached by a screw to a hexagonal nut C fixed to the governor spindle, and the lower end of the spring is attached to a sleeve D which is free to slide on the spindle and consists of a hexagonal nut secured to a disc. The spindle runs in a bearing at each end and is driven through gearing by the motor. The sleeve D can rise until it reaches a stop or brake B, the position of which is adjustable.

The problem is to determine the speed at which the governor will run at the moment when the sleeve D reaches the brake B, and the solution depends partly on the relation between the deflection of the centre of the spring and the lift of the sleeve.

Let the distance from the spindle axis to the centre of gravity G of the mass A be a when the governor is at rest and $a+\delta$ when it is rotating at any speed ω, the increase δ being the deflection of the centre of the spring. Let λ be the lift of the sleeve corresponding to the deflection δ.

Each leaf spring is equivalent to a uniform beam with direction-fixed ends, and the maximum deflection of such a beam when it carries a load at the centre is *

$$\delta=\frac{\mathrm{W}l^3}{192\mathrm{EI}} \quad . \quad . \quad . \quad (1),$$

where W is the load, l is the distance between the ends of the beam, I is the moment of inertia or second moment of area of its cross-section, about the neutral axis, and E is the modulus of elasticity. It is assumed in this equation that δ is small compared with l.

In the case of the leaf spring the central load is the centrifugal force or reversed effective force

$$\mathrm{F}=\frac{w}{g}\omega^2(a+\delta) \quad . \quad . \quad . \quad (2),$$

where w is the weight of the mass A and, as already stated, ω is the angular speed and $a+\delta$ is the radius to the centre of the mass.

Therefore it follows from (1) and (2) that the central deflection of the leaf spring is given by

$$\delta=\frac{w}{g}\omega^2(a+\delta)\frac{l^3}{192\mathrm{EI}} \quad . \quad . \quad . \quad (3).$$

* See the author's *Strength of Materials*, Longmans, Green & Co. Ltd.

It can be shown that the relation between the lift λ and the deflection δ is

$$\lambda=\frac{2 \cdot 4\delta^2}{l} \text{ approximately} \quad . \quad (4).$$

This result is obtained by considering the equations of the deflected spring and its curved length. These equations are

$$\frac{d^2y}{dx^2}=-\frac{W}{2EI}\left(x-\frac{l}{4}\right) \quad . \quad . \quad . \quad (5),$$

and

$$s=4\int_0^{l/4}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}dx \quad . \quad . \quad (6),$$

where distances x are measured along the undeflected centre line of the spring from the fixed end, deflections y are in the perpendicular direction, s is measured along the curved centre line, l is the distance between the ends, and the other symbols are as already defined. The procedure is to integrate equation (5) and substitute the value of $\frac{dy}{dx}$ in (6); then expand the expression in brackets by the binomial theorem, neglecting all terms after the second, and integrate. The result is an approximate expression for the curved length s.

Finally,

$$\lambda=s-l \quad . \quad . \quad . \quad . \quad (7),$$

and the value of λ may then be expressed in the form given in equation (4).

It should perhaps be emphasized that, for a given spring, s is a fixed length and l is a variable length, therefore care must be exercised when substituting numerical values of l in equations (3) and (4).

Example.—A gramophone is driven by a clockwork motor which has a governor of the type shown in Fig. 335. It is required to find the speed reached by the turntable when the sleeve on the governor has risen $\frac{1}{32}$ inch. The data are as follows: Weight of one mass is 0·65 oz.; each spring is 0·2 inch wide and 0·005 inch thick, and its effective length is 1·6 inches; a, the radius to the centre of gravity G when the governor is at rest, is 0·42 inch; the ratio of the governor speed to the turntable speed is 10·5; $E = 30 \times 10^6$ lb. per sq. in.

The distance between the ends of a spring is

$$l = s - \lambda = 1{\cdot}6 - \tfrac{1}{32} = 1{\cdot}57 \text{ inches.}$$

From equation (4),

$$\delta = \sqrt{\frac{\lambda l}{2{\cdot}4}} = \sqrt{\frac{1{\cdot}57}{32 \times 2{\cdot}4}} = 0{\cdot}143 \text{ inch.}$$

$$a + \delta = 0{\cdot}42 + 0{\cdot}143 = 0{\cdot}563 \text{ inch.}$$

$$I = \tfrac{1}{12}bh^3 = \tfrac{1}{12} \times 0{\cdot}2 \times 0{\cdot}005^3 = 20{\cdot}8 \times 10^{-10} \text{ inch}^4$$

(p. 88, Ex. 29).

From equation (3),

$$\omega = \sqrt{\frac{192 EI\delta g}{w(a+\delta)l^3}}$$

$$= \sqrt{\frac{192 \times 30 \times 10^6 \times 20{\cdot}8 \times 10^{-10} \times 0{\cdot}143 \times 32{\cdot}2 \times 12 \times 16}{0{\cdot}65 \times 0{\cdot}563 \times 1{\cdot}57^3}}$$

$$= 86{\cdot}5 \text{ rad./sec.}$$

$$\text{Turntable speed} = \frac{86{\cdot}5 \times 60}{10{\cdot}5 \times 2\pi} = 78{\cdot}7 \text{ r.p.m.}$$

178. Sensitiveness of Governors.—Consider two governors A and B running at the same speed, and suppose that when this speed increases or decreases by a certain amount the movement of the sleeve of A is greater than the movement of the sleeve of B, then governor A is more sensitive than governor B. In general, the greater the sleeve movement corresponding to a given change of speed, measured as a fraction, the greater is the sensitiveness of the governor. Put in another way, it may be said that for a given sleeve movement the sensitiveness of the governor increases as the speed range decreases.

There is no universally accepted definition of sensitiveness, but it is often defined as $(N_2 - N_1)/N$, or, alternatively and more logically, as $N/(N_2 - N_1)$, where N_1 and N_2 are the lowest and highest speeds respectively of the governor, corresponding to the extreme positions of the sleeve, and N is the mean speed.

The most sensitive governor would be one in which for a particular speed the sleeve would remain in any position within its range; the slightest change of speed would cause the sleeve to move to an extreme position. This governor is said to be *isochronous*; it would be over-sensitive and of no practical use because it would oscillate between its extreme positions. When a governor oscillates because it is over-sensitive, it is said to *hunt*.

179. Stability of Governors.—A governor is said to be *stable* when for every speed in its range there is a definite configuration, friction being neglected, and a change of speed causes the radius of the path of the balls to alter. When friction is taken into account the configuration at a particular speed depends on whether it has been arrived at by a rising or falling speed.

Since
$$\mathrm{F}=\frac{w}{g}\omega^2 r,$$
$$\frac{\mathrm{F}}{r}=\frac{w}{g}\omega^2,$$
and when ω increases the ratio F/r must also increase, therefore F must increase more rapidly than r. Expressed in symbols,
$$\frac{d\mathrm{F}}{dr}>\frac{\mathrm{F}}{r},$$
or, when F is plotted against r, the slope of the curve at any point is greater than F/r. Stability is considered further with the aid of curves in the Art. which follows.

180. Controlling Force.—When a body travels in a circular path there is, of course, an inward radial force or centripetal force acting on it. In the case of a governor running at a steady speed, the inward force on a ball is known as the *controlling force*, and it is equal and opposite to the centrifugal reaction. The controlling force is provided indirectly by gravity (as in the Porter governor), or by one or two springs acting directly on the ball, or by a combination of spring and gravity forces (as in the Hartnell governor). The moment produced by the weight of a ball in the Hartnell governor usually has only a small effect on the controlling force, and it may be positive or negative, according to the position of the bell-crank lever.

The controlling force curve AB (Fig. 336) shows the relation between the force and the radius for a ball in a Porter governor when friction is neglected. The values used in constructing this curve are from the example on p. 334, where $w=4$ lb., $W=56$ lb., and

$$h=\frac{w+W}{w}\cdot\frac{g}{\omega^2}.$$

From Fig. 320 (a), if r is the radius and h is the height, for any configuration, it follows from similar triangles that

$$\frac{r}{h}=\frac{r-2}{\sqrt{100-(r-2)^2}}.$$

Also the centrifugal force is

$$F=\frac{w}{g}\omega^2 r.$$

From these equations the controlling force, which is equal and opposite to the centrifugal force, is

$$F=\frac{(w+W)r}{h}=\frac{60(r-2)}{\sqrt{100-(r-2)^2}},$$

and the curve AB in Fig. 336 is the graph of this equation for values of r from 2 inches to 10 inches. These values extend, of course, beyond the limits possible in one governor.

The curves CD and HE have been drawn on the assumption that there is a frictional force $R=5$ lb. at the sleeve and that the limits of the radius r are 7 inches and 8·5 inches. The equation for the controlling force becomes

$$F=\frac{(w+W\pm R)r}{h}=\frac{(60\pm 5)(r-2)}{\sqrt{100-(r-2)^2}}.$$

The curve CD applies when the speed is increasing and the sleeve is rising (R is positive); the curve HE applies when the speed is decreasing and the sleeve is falling (R is negative). As indicated by the arrows, the path of the controlling force variation is from C to D and from E to H.

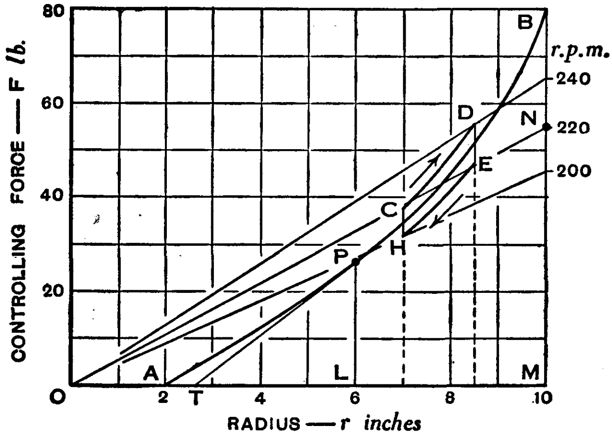


FIG. 336.

Consider any point P on the curve AB, say where $r=6$ inches. Draw the tangent PT intersecting the axis of r at T and draw the ordinate PL; then for the point P, $r=\mathrm{OL}$ and $\mathrm{F}=\mathrm{PL}$.

The slope of the curve at P is $\frac{\mathrm{PL}}{\mathrm{TL}}$, which is greater than $\frac{\mathrm{PL}}{\mathrm{OL}}$, that is,

$$\frac{d\mathrm{F}}{dr} > \frac{\mathrm{F}}{r},$$

the condition for stability (Art. 179), and this condition holds for all points on the curve AB and also on the curves CD and HE. The proof of the stability of the governor may also be obtained by differentiating with respect to r the equations for F.

The speed of the governor corresponding to any point on the curves or, alternatively, the points corresponding to any speed, may be found by calculation. Another way to determine such points is to draw a speed line as follows.

Calculate the controlling force F for a speed N (r.p.m.) from the equation

$$F = \frac{wr}{g}\left(\frac{\pi}{30}\right)^2 N^2,$$

using any convenient radius. In the present case take $r = 10$ inches and suppose $N = 220$ r.p.m., then

$$F = \frac{4 \times 10}{32{\cdot}2 \times 12}\left(\frac{\pi}{30}\right)^2 \times 220^2 = 54{\cdot}9 \text{ lb.}$$

Mark off the ordinate MN, as shown, equal to 54·9 on the force scale, but label the point N as 220 r.p.m. Join N to the origin O, then ON is a speed line and the points where it intersects the curves give the values of the radius r and the controlling force F when the governor speed is 220 r.p.m. That this is true is evident from the equation, since F is proportional to r when N is given a particular value. Speed lines for 200 r.p.m. and 240 r.p.m. are also shown in the diagram, and any number of such lines may be drawn.

Consider next the controlling force curves for a Hartnell governor. The tabulated values of the radius r and the controlling force F are from the example on p. 343. The extreme values of F are worked out in that example, and it is a simple matter to calculate the intermediate value when $r = 5{\cdot}5$ inches by taking moments about the pivot of the bell-crank lever.

r inches	4·25	5·5	6·75
F lb.	88·7	123	157·4

For practical purposes the relation between r and F is a linear one represented by the straight line AB (Fig. 337), the equation of which is of the form

$$F = cr - d,$$

where c and d are constants.

The straight lines CD and HE have been drawn on the assumption that there is a frictional force of 15 lb. at the sleeve, this force being added to or subtracted from the spring force according to whether the sleeve is rising or falling, but the calculations need not be given here. The controlling force increases from C to D and decreases from E to H.

A speed line ON has been drawn through the point H. The slope of this line is the value of F/r at H and it is less than the slope of HE. If another speed line were drawn through the point D its slope would be less than that of CD, but to avoid confusion this speed line has been omitted. Therefore the governor having CD and HE for its controlling force "curves" is stable.

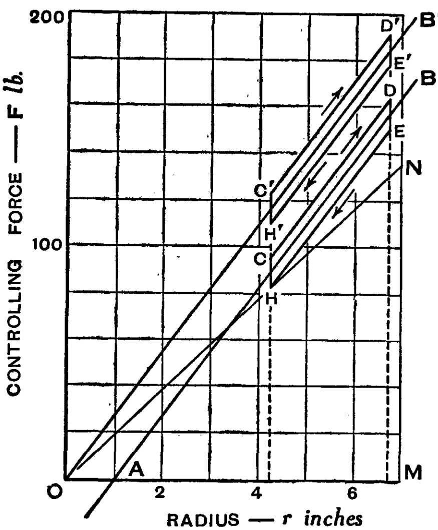


FIG. 337.

Alternatively the proof of stability may be obtained by differentiation. The equations of the straight lines CD and HE are of the form

$$F = cr - d',$$

from which

$$\frac{dF}{dr} = c \quad \text{and} \quad \frac{F}{r} = c - \frac{d'}{r}.$$

The slope $\frac{dF}{dr}$ is greater than $\frac{F}{r}$, therefore the governor is stable.

Suppose now that the initial spring force, the force when the sleeve is in its lowest position, is increased so that the straight line AB is moved up parallel to itself until it passes through the origin O and becomes the straight line OB′, the equation of which is of the form

$$F = cr.$$

This line OB′ is a controlling force line when friction is neglected, and since it passes through the origin it is also a speed line, therefore the governor is isochronous. This fact is also obvious since

$$\frac{dF}{dr} = c \quad \text{and} \quad \frac{F}{r} = c.$$

When friction is taken into account the controlling force lines are C′D′ and H′E′ and the governor is not isochronous. It is unstable as the sleeve rises and stable as the sleeve falls, for the slope of C′D′ is less than the slope of a speed line through C′ or D′, and the slope of H′E′ is greater than the slope of a speed line through H′ or E′.

For stability the slope of a controlling force line must be greater than the slope of any speed line which intersects it, but the less the difference between these slopes the greater is the sensitiveness of the governor. Therefore, by increasing the initial spring force and raising the controlling force lines CD and HE, the sensitiveness of the governor is increased, but of course care must be taken to see that the condition for stability is maintained.

181. Effort and Power of a Governor.—The *effort* of a governor is the force it exerts at the sleeve when there is a given percentage change of speed. When the governor is running steadily there is no force at the sleeve, but when the speed changes there is a resistance at the sleeve which opposes its motion. It is usual to assume that this resistance, which is equal to the effort, varies uniformly from a maximum value to zero whilst the governor moves into its new position of equilibrium. The work done at the sleeve is known as the *power* of the governor, and it is equal to the product of the mean value of the effort and the distance through which the sleeve moves during the given percentage change of speed. For the purpose of comparison it is convenient to take the change of speed as 1 per cent.

Consider the effort and power of a Hartnell governor. Assume that the arms of each bell-crank lever are mutually perpendicular and the moment of the weight w of each ball is negligible, so that the relations between the centrifugal force F, the spring force P, and the radius r of the ball path are $\mathrm{F}a=\frac{1}{2}\mathrm{P}b$ and $\mathrm{F}=cr-d$, where a and b are the lengths of the arms and c and d are constants (Fig. 338).

Let the speed be ω, when the radius is r, the centrifugal force is F and the spring force is P.

Let the speed increase to $(1+k)\omega$, where k is small, let F increase to F′, and let Q be the initial downward force at the sleeve opposing motion (Fig. 339). The radius r is unchanged, and the two values of the centrifugal force are

$$\mathrm{F}=\frac{w}{g}\omega^2 r \quad \text{and} \quad \mathrm{F}'=\frac{w}{g}(1+k)^2\omega^2 r.$$

By division $\dfrac{\mathrm{F}'}{\mathrm{F}}=(1+k)^2=1+2k$ approximately, since k is small, therefore

$$\mathrm{F}'-\mathrm{F}=(1+2k)\mathrm{F}-\mathrm{F}=2k\mathrm{F}.$$

Taking moments about the pivot O,

$$\mathrm{F}a=\tfrac{1}{2}\mathrm{P}b \quad \text{and} \quad \mathrm{F}'a=\tfrac{1}{2}(\mathrm{P}+\mathrm{Q})b.$$

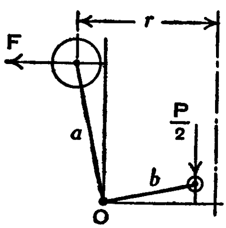


Fig. 338.

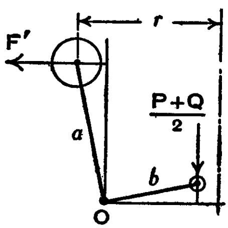


Fig. 339.

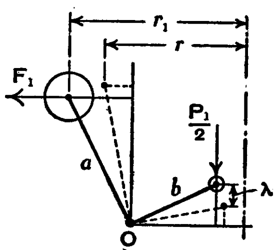


Fig. 340.

By subtraction $\qquad (\mathrm{F}'-\mathrm{F})a=\frac{1}{2}\mathrm{Q}b$

and
$$\mathrm{Q}=\frac{2a}{b}(\mathrm{F}'-\mathrm{F})=\frac{4ak}{b}\mathrm{F}.$$

Assume that the governor now moves into its new position of equilibrium, and let r, F′, and P increase to r_1,

F_1, and P_1 respectively (Fig. 340). The effort Q decreases uniformly to zero, and the mean effort is

$$Q_m = \tfrac{1}{2}Q = \frac{2ak}{b}F.$$

The value of F_1 is given by

$$F_1 = cr_1 - d$$

and by
$$F_1 = \frac{w}{g}(1+k)^2\omega^2 r_1.$$

Eliminating F_1,

$$cr_1 - d = \frac{w}{g}(1+k)^2\omega^2 r_1,$$

from which r_1 may be determined.

Denoting the sleeve lift by λ, then from the two pairs of similar triangles in Fig. 340 it follows that

$$\lambda = \frac{b}{a}(r_1 - r).$$

Let U be the power, then

$$U = \lambda Q_m = \frac{b}{a}(r_1 - r) \times \frac{2ak}{b}F = 2(r_1 - r)kF.$$

Example.—A Hartnell governor is running at 350 r.p.m. and the radius of the ball path is 4·25 inches. It is required to find the mean effort and the power when the speed increases by 1 per cent. Assume $F = cr - d$; also when $r = 4{\cdot}25$ inches, $F = 88{\cdot}7$ lb., and when $r = 6{\cdot}75$ inches, $F = 157{\cdot}4$ lb.; take $w = 6$ lb., $a = 5$ inches and $b = 4$ inches.

Substituting the pairs of values of r and F in the equation $F = cr - d$ and solving the two simultaneous equations for c and d, then

$$F = 27{\cdot}5r - 28{\cdot}2.$$

When $r = 4{\cdot}25$ inches and $F = 88{\cdot}7$ lb., the speed increases by 1 per cent. Since the radius is at first unchanged, F increases to $F' = 88{\cdot}7 \times 1{\cdot}01^2$ lb.

$$F' - F = 88{\cdot}7 \times 2 \times 0{\cdot}01 = 1{\cdot}774 \text{ lb.}$$

The effort at the sleeve is

$$Q = 1{\cdot}774 \times \frac{2a}{b} = 1{\cdot}774 \times \frac{10}{4} = 4{\cdot}435 \text{ lb.}$$

Mean effort is $Q_m = \frac{1}{2}Q = 2{\cdot}22$ lb.

When the radius increases to r_1,

$$F_1 = 27{\cdot}5r_1 - 28{\cdot}2.$$

Also $F_1 = \dfrac{6}{32{\cdot}2 \times 12} \times 1{\cdot}01^2 \times \left(\dfrac{350 \times \pi}{30}\right)^2 r_1 = 21{\cdot}28r_1.$

Therefore $27{\cdot}5r_1 - 28{\cdot}2 = 21{\cdot}28r_1,$

from which $r_1 = 4{\cdot}53$ inches.

Increase of radius is

$$r_1 - r = 4{\cdot}53 - 4{\cdot}25 = 0{\cdot}28 \text{ inch.}$$

Sleeve lift is

$$\lambda = 0{\cdot}28\frac{b}{a} = 0{\cdot}28 \times \frac{4}{5} = 0{\cdot}224 \text{ inch.}$$

Power is

$$U = \lambda Q_m = 0{\cdot}224 \times 2{\cdot}22 = 0{\cdot}497, \quad \text{say } 0{\cdot}50 \text{ in. lb.}$$

182. **Inertia Governors.**—In a centrifugal governor the configuration is altered by radial forces after the speed has changed. In an inertia governor, in addition to radial forces, there are tangential forces which affect the configuration as soon as acceleration or retardation begins and enable the governor to operate more rapidly.

There have been numerous designs of inertia governors, and one is shown diagrammatically in Figs. 341 and 342. A disc D is fixed to a horizontal driving shaft O, the plane of the disc being perpendicular to the axis of the shaft. An arm AB, pivoted to the disc at A and carrying a mass at the end B, is connected to an eccentric (not shown) which operates the valve controlling the engine. A tension spring S is pivoted at one end to the disc at C and at the other end to the arm at E. When the arm AB moves relatively to the disc it shifts the position of the eccentric. In practice there are often two such arms arranged symmetrically with respect to the axis of the shaft, and the gravity effects on the masses neutralize one another.

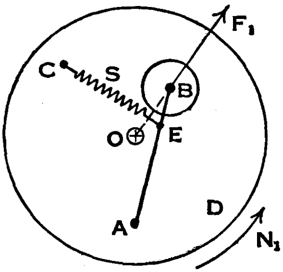


FIG. 341.

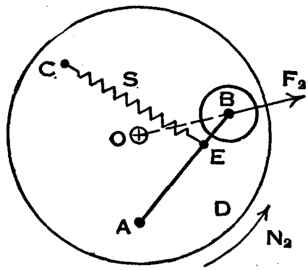


FIG. 342.

It will be assumed that the mass of the arm AB is negligible, that the mass at B is concentrated at that point and that the gravity effect on it is balanced. When the governor is running at a steady speed N_1 (Fig. 341) there is a centrifugal force F_1 acting radially at B and a force at E due to the spring tension. The moments about A of these forces are in equilibrium. When the speed has increased and reached a steady value N_2 (Fig. 342), the arm AB has turned about A relatively to the disc D so that the mass at B is further from the shaft O, the centrifugal force becomes F_2 and its moment about A is equal to that of the increased spring tension.

When the governor speed is increasing, let ω be the velocity of the disc D at any instant, in the anticlockwise direction, and $\dot{\omega}$ be its acceleration (Fig. 343); let Ω and $\dot{\Omega}$ be the clockwise velocity and acceleration respectively of the arm AB *relative* to the disc D; let AB $=a$, OB $=r$, and let m be the mass concentrated at B.

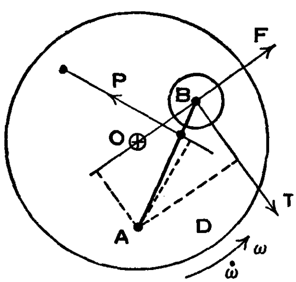


FIG. 343.

Due to the motion of the disc, the mass at B has an acceleration

$\omega^2 r$ along BO and an acceleration $\dot{\omega} r$ perpendicular to BO. As indicated, the reversed effective forces are

$$F = m\omega^2 r \quad \text{and} \quad T = m\dot{\omega} r.$$

Due to the motion of the arm relative to the disc, the mass at B has accelerations $\Omega^2 a$ and $2u\omega = 2\Omega a\omega$ (the Coriolis acceleration), along BA and AB respectively, and an acceleration $\dot{\Omega} a$ perpendicular to BA. The corresponding reversed effective forces are $m\Omega^2 a$, $2m\Omega a\omega$, and $m\dot{\Omega} a$. The first two of these forces act through A and therefore have no moment about this point. Initially the arm is accelerated clockwise relative to the disc, and the moment about A of the reversed effective force $m\dot{\Omega} a$ opposes the moments of F and T, when the arm AB is arranged as shown. Actually the acceleration $\dot{\Omega}$ is small and the force $m\dot{\Omega} a$ is usually neglected.

It will be realized from Fig. 343 that with the arm AB in the position shown, it is essential for the direction of rotation of the disc to be anticlockwise, so that the force T assists the arm to move outwards as the speed increases.

The values of Ω and $\dot{\Omega}$ are in general unknown, and a complete solution of the problem cannot be obtained. When the governor speed alters, the arm oscillates about its final position of equilibrium, and a vibration damper may be fitted to make the oscillations die out quickly.

An equation giving the approximate relation between the spring force P and the forces F and T is obtained by taking moments about the point A. The reader should now refer to Ex. 21, p. 368, where $\dot{\omega}$ is given and ω is to be calculated.

In some forms of inertia governors the arm is heavy and forms the mass which moves relatively to the disc. The pivot is near the centre of gravity of the arm, and the moment of the reversed effective forces about the pivot is increased by an amount $I\dot{\omega}$, where I is the moment of inertia of the mass about its centre of gravity. There is also a couple $-I\dot{\Omega}$, but this is usually neglected since, as already stated, $\dot{\Omega}$ is small. The radius r is measured from the centre of the shaft to the centre of gravity of the mass, and the forces F and T act at the centre of gravity of the mass.

183. Variable-Speed Hydraulic Governor.—The following description of the basic idea of a direct-acting hydraulic governor is abstracted from a paper * by Dr J. Z. Bujak, and Fig. 344, which has been redrawn from the same source, shows the governor diagrammatically in its simplest form. Readers who are interested should refer to the original paper, where the development of a production type hydraulic governor meant primarily for road transport compression-ignition engines is fully reported. This governor underwent many thousands of miles of road testing. An abbreviation of the mathematical theory is printed in the paper, and the complete theory is in an unpublished appendix in the library of the Institution of Mechanical Engineers.

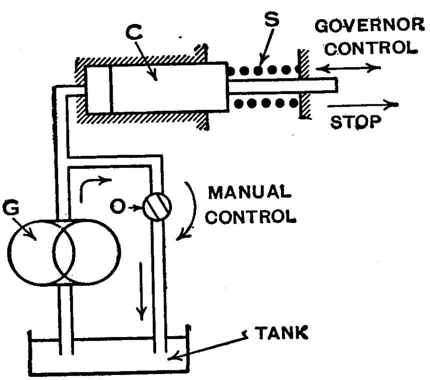


FIG. 344.

In the simplest form of the governor (Fig. 344) a positive displacement pump G, driven at a speed proportional to the engine speed, pumps the fluid in a closed circuit in which there is a restricting orifice O. Due to the flow through the orifice a pressure P is built up, acting on the control plunger C which is loaded by a spring S. This

* "The Variable-Speed Hydraulic Governor," by J. Z. Bujak, Dr. Inż., *Proc. I. Mech. E.* (1945), vol. 153, p. 193.

plunger is linked to the control rod of the fuel injection pump. For a fixed size of the orifice O, the pressure P varies as the square of the engine speed. Increase in speed causes an increase in the pressure and a displacement of the plunger which decreases the supply of fuel to the engine. The speed setting of the governor can be altered by changing the size of the orifice O which, as shown in the diagram, is hand controlled.

Exercises XVI

1. (*a*) Show that in a conical pendulum (Fig. 345) the height h in feet is equal to g/ω^2, where ω is the angular velocity of the mass about the axis in radians per second and g is in foot and second units.

(*b*) Find the value of h in inches when the angular velocity is 80 r.p.m. and $g=32{\cdot}2$ ft./sec.^{2}, and then calculate the value of the radius r if $l=6{\cdot}5$ inches.

(*c*) If l is doubled and the speed is still 80 r.p.m., find the new value of r.

(*d*) If r and l in (*b*) are doubled, find the speed at which the pendulum would have to revolve.

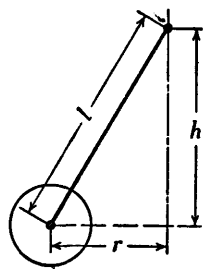


FIG. 345.

2. A simple governor has two balls, each weighing 5 lb. carried by four equal arms each of length 10 inches from the pivots on the vertical centre line to the centres of the balls. The rise of the sleeve is limited to $\frac{3}{4}$ inch from the lowest position in which each of the arms makes an angle of 35° with the vertical spindle. Calculate the limits of speed for this governor.

What central load applied at the sleeve would increase the lower limit to 178 r.p.m.?

Friction and the mass of the arms are to be neglected. [C.U.]

3. The suspension link ED of the loaded governor shown in Fig. 346 is connected to the arm BC at the intermediate point E. Consider the equilibrium of the arm and show (*a*) by taking moments

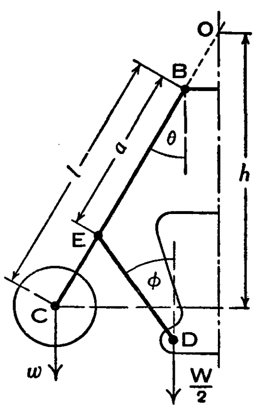


FIG. 346.

about B, and (*b*) by taking moments about O, the point where the arm axis intersects the spindle axis, that

$$h = \frac{w + \frac{Wa}{2l}(1+q)}{w} \cdot \frac{g}{\omega^2},$$

where q is determined by the equation $\tan \phi = q \tan \theta$.

Note.—Since the suspension link is connected to the arm at the intermediate point E, the arm is equivalent to a loaded beam and the reaction at B is not along its axis. This reaction is avoided when moments are taken about B, but it enters into the equation when moments are taken about O.

4. The arms and suspension links of a Porter governor are of equal length and the axes of the upper and lower joints intersect the spindle axis. Each ball weighs 4 lb. and the central load is 45 lb. (*a*) Find the height in inches at which this governor will run when the speed is 250 r.p.m. (*b*) What increase in speed would enable the sleeve to rise 1 inch?

5. Using the data in Ex. 4 and allowing for a frictional force of 5 lb. at the sleeve, find the height, assuming that (*a*) the speed rises to 250 r.p.m., and (*b*) the speed falls to 250 r.p.m.

6. In a governor of the Porter type the weight of the central load is 40 lb. and that of each of the revolving masses 5 lb. The upper and lower arms are each 8 inches in length, and the points of suspension are $1\frac{1}{2}$ inches from the axis of the spindle. When the sleeve is in mid-position, the arms of the governor are at right angles.

Determine for this position the speed of the governor.

[Inst. C.E.]

7. Deduce an expression for the height of a Porter governor which has equal arms intersecting on the axis, showing that the height is not affected by the length of the arms.

In such a governor having a central weight of 45 lb. and balls of 5 lb. each, the range of speed for a given configuration is from 245 to 255 r.p.m. Find the effective friction force acting at the sleeve.

If all the arms are 9 inches long, what angle do they make with the vertical axis? [U.L.]

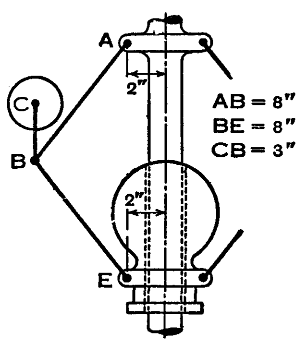


FIG. 347.

8. A governor of the Proell type is shown in Fig. 347. When in mid-position the portion BC of the arm EBC is vertical and the radius of the ball path is 7 inches. The weight of each ball is 7 lb.

Find the magnitude of the central load if the speed of the governor is to be 120 r.p.m. when the sleeve is in mid-position. [Inst. C.E.]

9. A governor of the Hartnell type runs at a mean speed of 360 r.p.m. and a 3 per cent. reduction of speed causes a sleeve movement of 0·5 inch. Each ball weighs 5 lb. and is carried on a bell-crank lever with arms at right angles, the length of the ball arm is 6 inches, the length of the sleeve arm is 3 inches, the lever pivot is $4\frac{1}{2}$ inches from the axis of rotation. The ball arm is vertical in the mean position. Neglecting the effects of gravity, determine the necessary stiffness of the spring, in lb. per inch, acting on the sleeve. Derive any formula used. [Inst. C.E.]

10. The following particulars refer to a governor of the Hartnell type. Each ball of mass 8 lb. is attached to the vertical arm of a bell-crank lever which pivots about a fixed fulcrum. The horizontal arm of each lever presses against the sleeve, movement of which is controlled by a spring surrounding the spindle. The vertical arms are 5 inches and the horizontal arms 3 inches long and, when the sleeve is in its lowest position, the ball centres are vertically above the pivots of the bell-cranks and 4 inches from the axis of the governor spindle.

If the sleeve is to begin to lift at 270 r.p.m. and is to lift $\frac{1}{2}$ inch for an increase of speed of 6 per cent., find the initial force exerted by the spring and the stiffness of the spring. [I.Mech.E.]

11. In a Hartnell governor the arms are at right angles, the vertical arm is 6 inches long and the horizontal arm 5 inches long. When the sleeve is in mid-position, the arms are vertical and horizontal, and the radius of the ball path is 7 inches. Each of the two balls weighs 12 lb., and the lift of the sleeve is $2\frac{1}{4}$ inches.

If the range of the governor is to be from 290 r.p.m. to 310 r.p.m., calculate, neglecting friction, the stiffness and initial compression of the spring. [Inst. C.E.]

12. A Hartnell type spring-loaded governor rotates about a vertical axis. The two rotating masses weigh $2\frac{1}{4}$ lb. each, and move at a radius of $4\frac{3}{4}$ inches when the speed is 550 r.p.m.; at this speed the arms, 4 inches and 3 inches effective length, are respectively vertical and horizontal. The equilibrium speed is 575 r.p.m. when the rotating masses are at their maximum radius of $5\frac{3}{4}$ inches.

Determine the stiffness, or rate, of the spring, the compression of the spring at 550 r.p.m., and the radius at which the weights rotate when the equilibrium speed is 525 r.p.m. [U.L.]

13. Discuss the effects on the sensitiveness and stability of a spring-loaded governor of—

(*a*) Friction.

(*b*) Alteration of initial spring compression. [Inst. C.E.]

14. In a governor of the Hartnell type each bell-crank lever has mutually perpendicular arms and the two masses are supplied entirely by the "ball-arms," which are 3 inches long, with rectangular cross-sections 1 inch by $\frac{1}{2}$ inch; the other arms are $1\frac{1}{2}$ inches long. The lengths are measured to the axes of the bell-crank lever pivots, which are $2\frac{1}{2}$ inches from the centre line of the governor spindle. The movement of the sleeve is $\frac{3}{4}$ inch, and when it is in its mid-position the "ball-arms" are vertical.

Given that the weight of 1 cubic inch of the metal is 0·284 lb., and considering the centrifugal effect of the mass of a "ball-arm," find the mass in pounds of the equivalent ball when the sleeve is in (*a*) its lowest position, (*b*) its mid-position and (*c*) its highest position.

15. If the sleeve of the governor described in the preceding exercise begins to lift at 800 r.p.m. and reaches its upper limit when the speed increases by 5 per cent., find the initial force exerted by the spring and the stiffness of the spring, neglecting the gravity effect on the arms. Also calculate the speed at which the governor runs when the sleeve is in its mid-position.

16. A spring-loaded governor is shown in the diagram (Fig. 348). The two balls, each of weight 12 lb., are connected across by two springs A. A supplementary spring B provides an additional force at the sleeve through the medium of a lever which pivots about a fixed centre at its left-hand end. In the mean position, the radius of the governor balls is 6 inches and the speed is 600 r.p.m. The tension in each spring A is then 250 lb. Find the tension in the spring B for this position.

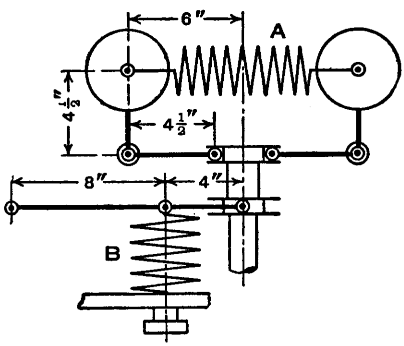


FIG. 348.

If when the sleeve moves up $\frac{3}{4}$ inch the speed is to be 630 r.p.m., find the necessary stiffness of the spring B in lb. per inch if the stiffness of each spring A is 45 lb. per inch.

Neglect the moments produced by the weights of the balls.

[U.L.]

17. In the governor described in the preceding exercise, let the weight of each ball be reduced from 12 lb. to 11 lb. Show that for equilibrium the speed is 627 r.p.m. when the sleeve is in its mean position (Fig. 348) and 658 r.p.m. when it moves up $\frac{3}{4}$ inch from this position, neglecting the moments produced by the weights of the balls.

Instead of reducing the weight of each ball, the speeds can be raised by increasing the tension in the spring B which is adjustable. Find what this tension should be when the sleeve is in its mean position if the governor is then to run at 627 r.p.m., and calculate the speed when the sleeve moves up $\frac{3}{4}$ inch.

Show that the governor is more sensitive when the spring tension is increased than when the ball weights are reduced.

18. In the governor shown in Fig. 349 the pivots for the bell-crank levers are carried by the sleeve, which is capable of an axial movement relatively to the governor spindle A, which has a cap fixed to its upper end. The spring is compressed between the sleeve and the cap, and the outward movement of the balls with increase of speed causes the sleeve to be raised against the compression of the spring due to pressure of the rollers on the ends of the short arms of the bell-crank levers.

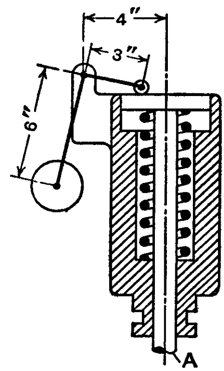


FIG. 349.

If the mass of the sleeve is 30 lb. and of each ball is 6 lb., and the minimum radius of rotation of the balls is 4 inches, find the initial compression in the spring and also the stiffness of the spring in order that the sleeve shall begin to rise at 240 r.p.m. and rise 0·25 inch when the speed increases to 264 r.p.m. [U.L.]

19. Find the mean effort, sleeve lift, and power of the governor described in the preceding exercise when the speed increases by 1 per cent. from 240 r.p.m. Assume that the relation between the spring force P (lb.) and the sleeve lift λ (inch) is given by $P = 115 + 219\lambda$.

20. (*a*) Find what the thickness of the leaf springs should be in the governor illustrated in Fig. 335, given that the breadth is 1 inch and the effective length is 12 inches, the sleeve lift is $\frac{1}{4}$ inch when the spindle speed reaches its maximum value of 300 r.p.m., the weight of each mass is 1 lb., $a = 1$ inch and $E = 30 \times 10^6$ lb. per sq. in.

(*b*) Find the governor speed in revolutions per minute when the sleeve lift is 0·1 inch.

21. The diagram, Fig. 350, shows the arrangement of a governor. Two weighted arms, A, A′, are pivoted on pins B, B′, $2\frac{1}{2}$ inches apart, attached to a plate C which rotates about its centre. Each weighted arm weighs $\frac{1}{2}$ lb. and the centres of gravity G, G′ are 2 inches from the centres of the pivots. Points S, S′ in the arms, at 1 inch distance from the pivots, are connected

by a spring. A linkage (not shown) ensures that the two angles θ and θ' are equal. The stiffness of the spring is 4 lb. per inch.

(*a*) Find the tension required in the spring so that the angles θ, θ' shall be 30° when the governor speed is 300 r.p.m.

(*b*) If the governor, rotating in the anticlockwise direction, accelerates at the rate of 50 rad. per sec. per sec., at what speed of rotation will the angles be 45°? [U.L.]

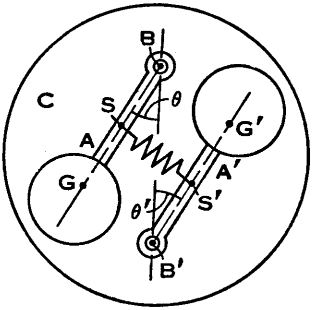


FIG. 350.

22. (*a*) Shaft governors may be of the centrifugal or of the inertia type. Distinguish between the actions of these two types in controlling the speed of an engine.

(*b*) A shaft inertia governor consists of an arm AB pivoted at C, C being a fixed point on a disc concentric with and rigidly attached to the engine shaft. C is offset from the shaft axis O by 3 inches. The arm AB is 14 inches long and is symmetrical about C, that is AC = CB = 7 inches. A weight of 15 lb. is attached to each end of the arm at A and B, these weights being in the form of circular discs, each 6 inches in diameter, with their axes parallel to that of the shaft. In the normal position the arm ACB is at right angles to the radius OC.

If the speed of the engine increases by 15 r.p.m. in 2 seconds, this increase being at a uniform rate, determine the torque about C needed to hold the arm stationary relative to the concentric disc. Neglect the weight of the arm, but explain carefully the reasoning behind any equations you may employ. [U.L.]

CHAPTER XVII

BALANCING

184. Centrifugal Force.—A mass of weight W is attached to an arm OA at A and moves in a circular path of radius r about an axis O with an angular velocity ω (Fig. 351). The inward radial acceleration of the mass is $\omega^2 r$ (Art. 28, p. 47), and the arm OA supplies the centripetal force $(W/g)\omega^2 r$ which acts on the mass (Art. 43, p. 73). The reaction is the centrifugal force and this acts outwards on the axis O.

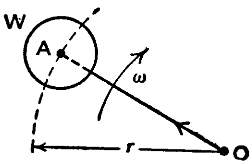


FIG. 351.

For a given angular velocity the force at O is proportional to the product Wr. If, for example, the weight of the mass at A is doubled and the radius OA is halved, the force is unaltered since $2W.\frac{1}{2}r = Wr$.

To balance the force on the axis O, let the arm OA be produced through O to B, let $OB = r_1$ and suppose a mass of weight W_1 is attached to the arm at B (Fig. 352). There are now two opposing forces acting at O, one proportional to Wr in the direction OA and the other proportional to $W_1 r_1$ in the direction OB. If it is arranged that $W_1 r_1 = Wr$, the resultant force at O is zero and the two centrifugal forces balance one another.

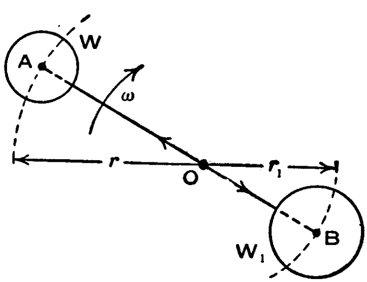


FIG. 352.

185. Static and Dynamic Forces.—It should be noted that the directions of static and dynamic forces are different. The static forces on the arms OA and OB (Fig. 353, top), due to the masses at A and B, are W at A, W_1 at B and the reaction at O, all of which act vertically. For equilibrium $W_1r_1 = Wr$, and the reaction at O is $W + W_1$.

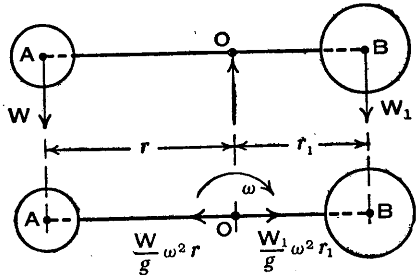


FIG. 353.

The dynamic forces at O, due to the masses at A and B, are $(W/g)\omega^2 r$ and $(W_1/g)\omega^2 r_1$, acting along the arms as shown (Fig. 353, bottom). For equilibrium $W_1r_1 = Wr$.

The static forces are constant in magnitude and direction, whereas the dynamic forces are proportional to the square of the speed of rotation ω and act along the moving arms.

186. Static and Dynamic Balance.—A body which can be turned about an axis is in *static balance* when its centre of gravity is in the axis of rotation. Suppose the body shown diagrammatically in Fig. 354 has an out of balance mass 1, of weight w_1, at a radius r_1 from the axis of rotation OX, then after the body has been rotated slowly it will

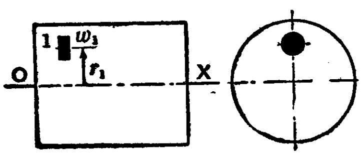


FIG. 354.

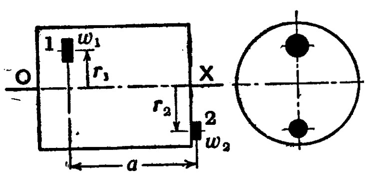


FIG. 355.

always come to rest with the mass 1 vertically below the axis, assuming friction to be negligible. If a mass 2, of weight w_2, is fixed to the body in the axial plane passing through the centre of gravity of mass 1, but on the opposite side of and at a radius r_2 from the axis OX (Fig. 355), and the values of w_2 and r_2 are arranged so that $w_2r_2 = w_1r_1$, the centre of gravity of the system will be in the axis OX and the body will be in static balance. If the body is now rotated slowly about OX, it may be brought to rest in equilibrium with the axial plane containing the masses 1 and 2 in any angular position. The mass 2 is shown at one end of the body, as this is a convenient position in which to attach it.

A rotating body is in *dynamic balance* when the resultant of the centrifugal forces is zero and there is no resultant couple. Suppose the body shown in Figs. 354 to 356 is rotating with an angular velocity ω. In Fig. 354 there is an unbalanced centrifugal force $(w_1/g)\omega^2 r_1$, that is a force proportional to $w_1 r_1$. In Fig. 355 the resultant centrifugal force is zero since $w_2 r_2 = w_1 r_1$, but there is an unbalanced couple (clockwise in the position shown) of moment proportional to $w_1 r_1 a$ or $w_2 r_2 a$, where a is the perpendicular distance between the planes of revolution of the masses 1 and 2. Therefore although the body in Fig. 355 is in static balance, it is not in dynamic balance.

To produce dynamic balance let masses 3 and 4 of weights w_3 and w_4 be fixed at radii r_3 and r_4 respectively, in the axial plane passing through the masses 1 and 2 and on opposite sides of OX as shown in Fig. 356. Let $w_3 r_3 = w_4 r_4$, and let b be the perpendicular distance between the planes of revolution of the masses 3 and 4, then the centrifugal forces due to these masses produce a couple (anticlockwise in the position shown) of moment proportional to $w_3 r_3 b$ or $w_4 r_4 b$. If it is arranged that $w_3 r_3 b = w_1 r_1 a$, the two opposing couples will be equal for all angular positions of the rotating system and the body will be in dynamic balance.

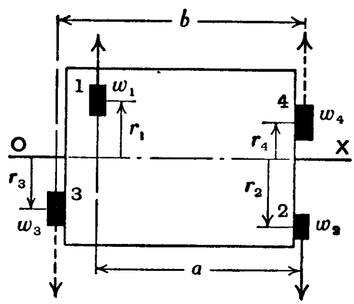


FIG. 356.

187. Balancing Several Masses Revolving in One Plane.—In Fig. 357 W_1, W_2, and W_3 are the weights of masses attached to the outer ends of rigidly connected arms of lengths r_1, r_2, and r_3 respectively, which are revolving in a plane with angular velocity ω about an axis O. The forces at O act outwards and are proportional to the products W_1r_1, W_2r_2, and W_3r_3. Let W be the weight of a balancing mass in the same plane at the end of an arm of length r, then this mass provides an outward force at O proportional to Wr. It is required to find the magnitude of Wr and the relative angular position of the arm.

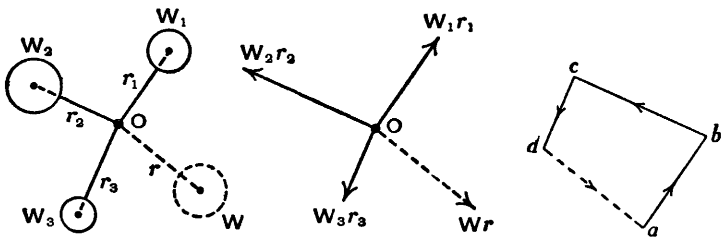


FIG. 357.

Draw the force polygon *abcd* as shown, with sides *ab*, *bc*, and *cd* representing in magnitude and direction the products W_1r_1, W_2r_2, and W_3r_3 respectively, then the closing side *da* represents the equilibrant Wr.

A convenient value is selected for, say, the radius, and the value of the weight is calculated, then a mass of weight W attached to the end of an arm, which is parallel to *da* and of length r, will provide a force at O to balance the given forces, and there will be no resultant dynamic force on the axis.

188. Forces in Parallel Planes in Equilibrium.—Since many dynamic balancing problems involve arranging that forces in parallel planes are in equilibrium, two such examples in statics will first be considered. In Professor Dalby's method * the forces are transferred to one of the parallel planes, then a force polygon and a couple polygon are drawn. In an alternative method the solution is obtained from two force polygons. The first example concerns the simple case in which the forces are parallel,

* *The Balancing of Engines*, by W. E. Dalby, Edward Arnold & Co.

consequently the required vector sums are obtained without drawing the polygons, each of which consists of two lines.

(1) A force P acts in a direction perpendicular to a shaft AB at a point C (Fig. 358), and it is required to find the values of the parallel forces Q and R at A and B respectively which would produce equilibrium. The lengths are AB $=l$, AC $=a$, and CB $=b$. In this example the forces are all in the plane of the paper, and the parallel planes are those perpendicular to the paper which contain the lines of action of the forces.

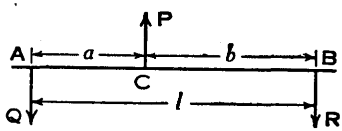


FIG. 358.

Consider the effect in the plane at A, called a *reference plane*, perpendicular to AB, of the forces P and R (Figs. 359 and 360). The effect of the force P on the point A is the

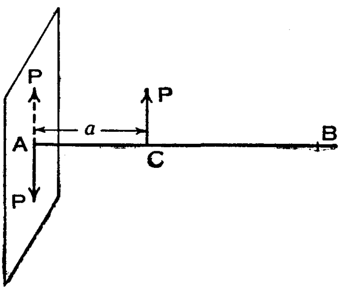


FIG. 359.

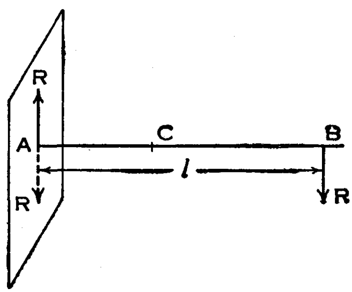


FIG. 360.

parallel upward force P, shown dotted in Fig. 359, and an anticlockwise couple Pa. Similarly, the effect of the force R on the point A is the parallel downward force R, shown dotted in Fig. 360, and a clockwise couple Rl.

The couples are in equilibrium if their vector sum is zero, that is if

$$Rl - Pa = 0, \quad Rl = Pa, \quad \text{or} \quad R = \frac{Pa}{l}.$$

In the reference plane at A there are unequal forces P and R, and for equilibrium a downward force Q is required

(Fig. 361). The vector sum of the three forces must be zero, therefore

$$Q + R - P = 0 \quad \text{or} \quad Q = P - R.$$

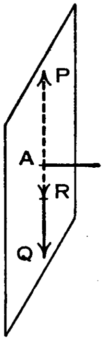


FIG. 361.

It is evident that when the forces on the shaft are in equilibrium, the *resultant couple is zero* and the *resultant force is zero.*

The procedure for obtaining the values of Q and R (Fig. 358) may be put concisely as follows.

Let the reference plane be at A and take moments about A, then

$$Rl - Pa = 0, \quad \text{from which} \quad R = \frac{Pa}{l}.$$

Since the resultant force in the reference plane is zero,

$$Q + R - P = 0, \quad \text{from which } Q = P - R.$$

Alternatively, to find the value of Q let the reference plane be at B and take moments about B, then

$$Ql - Pb = 0, \quad \text{from which} \quad Q = \frac{Pb}{l}.$$

(2) Forces P_1, P_2, and P_3 act in directions perpendicular to a shaft AB of length l, as shown in Fig. 362, at distances a, b, and c respectively from A, and it is required to find the

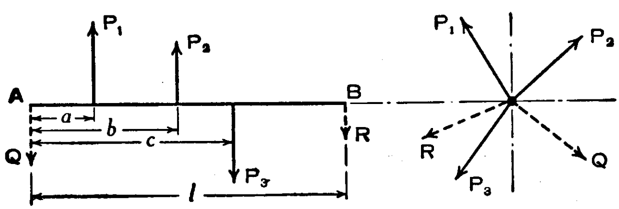


FIG. 362.

forces Q and R which would produce equilibrium if placed in planes perpendicular to the shaft at A and B respectively. In this case the forces are not in the plane of the paper but are inclined to it as shown in the end view (looking from A towards B).

Let the reference plane be at A and take moments about A. As before, the resultant couple must be zero for equilibrium, therefore

$$\text{Vector sum of } Rl + P_1a + P_2b + P_3c = 0$$

or $$R = -\frac{1}{l}(\text{Vector sum of } P_1a + P_2b + P_3c) \quad . \quad (1).$$

Also the resultant force in the reference plane must be zero, therefore

$$\text{Vector sum of } Q + R + P_1 + P_2 + P_3 = 0$$

or $$Q = -(\text{Vector sum of } R + P_1 + P_2 + P_3) \quad . \quad (2).$$

Vector sums have to be found in equations (1) and (2) because the couples are in several planes and the lines of action of the forces in the reference plane are inclined to one another. The solutions are easily obtained by drawing a couple polygon and a force polygon. A numerical example is worked out in Art. 190.

Alternatively, let the reference plane be at B and take moments about B, then

$$Q = -\frac{1}{l}\{\text{Vector sum of } P_1(l-a) + P_2(l-b) + P_3(l-c)\} \quad (3),$$

and the value of Q may be found from a couple polygon.

The values of R and Q may also be obtained without using a couple polygon. Rewriting equations (1) and (3) as follows, dividing the terms on the right by l,

$$R = -\left\{\text{Vector sum of } \frac{P_1a}{l} + \frac{P_2b}{l} + \frac{P_3c}{l}\right\} \quad . \quad . \quad (4),$$

and

$$Q = -\left\{\text{Vector sum of } \frac{P_1(l-a)}{l} + \frac{P_2(l-b)}{l} + \frac{P_3(l-c)}{l}\right\} \quad (5).$$

If the equations are used in this form, the values of R and Q are found by drawing two force polygons; this is equivalent to replacing each of the forces P_1, P_2, and P_3 by forces at A and B and finding the equilibrant in each of these planes. It is obvious that these force polygons are similar figures to the couple polygons which may be drawn to solve equations (1) and (3), but the numerical example given in Art. 190 will assist in emphasizing this fact.

189. Balancing One Revolving Mass by Two Masses Revolving in Parallel Planes.—Let W be the weight of a given revolving mass and W_C and W_D be the weights of balancing masses which are to be determined (Figs. 363 and 364). Let the distances of the centres of gravity of the masses from the axis of revolution be r, r_C, and r_D as shown, and let the planes of revolution of the unknown

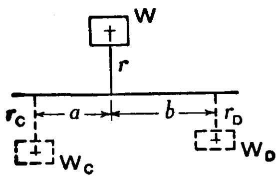


FIG. 363.

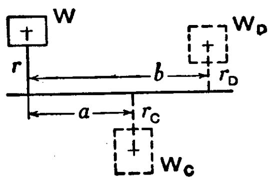


FIG. 364.

masses be at distances a and b from that of the given mass. In Fig. 363 the unknown masses are on opposite sides of the plane of revolution of the given mass, but in Fig. 364 they are on the same side.

The forces produced by the revolving masses are proportional to Wr, W_Cr_C, and W_Dr_D, as indicated in Figs. 365 and 366, and for equilibrium these forces must act in a

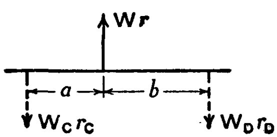


FIG. 365.

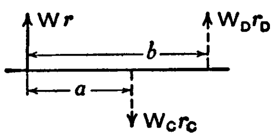


FIG. 366.

revolving plane containing the axis of revolution. This plane has been taken as the plane of the paper in the Figs. The problem has now been reduced to one to be solved as explained in the first part of the preceding Art.

For the case shown in Fig. 365,

$$W_C r_C(a+b) - Wrb = 0, \qquad W_D r_D(a+b) - Wra = 0$$

or

$$W_C r_C = \frac{Wrb}{a+b} \quad \text{and} \quad W_D r_D = \frac{Wra}{a+b}.$$

For the case shown in Fig. 366,

$$W_C r_C(b-a) - Wrb = 0, \qquad W_D r_D(b-a) - Wra = 0$$

or

$$W_C r_C = \frac{Wrb}{b-a} \quad \text{and} \quad W_D r_D = \frac{Wra}{b-a}.$$

If the radii r_C and r_D are decided on, the values of W_C and W_D may be calculated.

190. Balancing Several Masses Revolving in Parallel Planes.—In Fig. 367 W_1 and W_2 are the weights of unbalanced masses at the ends of arms of lengths r_1 and r_2 respectively, attached to a shaft revolving about its axis

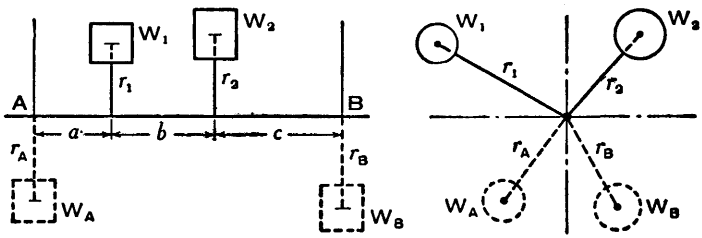


FIG. 367.

AB, and the relative angular positions of the arms are as shown in the end view. Let the masses be balanced by a mass of weight W_A at the end of an arm of length r_A in the plane at A and a mass of weight W_B at the end of an arm of length r_B in the plane at B. It is required to determine the weights and relative positions of the balancing masses.

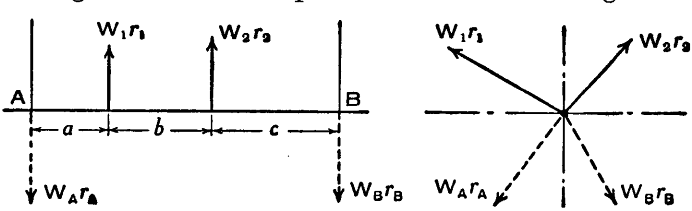


FIG. 368.

The forces produced by the revolving masses are proportional to W_1r_1, W_2r_2, W_Ar_A, and W_Br_B (Fig. 368). The problem involves finding the magnitudes and directions of the products W_Ar_A and W_Br_B in order that the forces in the parallel planes may be in equilibrium. A numerical example follows.

Example.—In Fig. 369 there are three unbalanced revolving masses in parallel planes labelled 1, 2, and 3, and

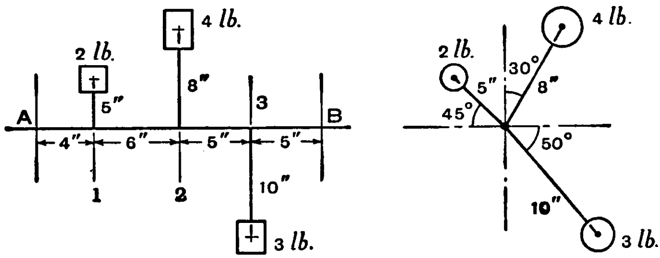


FIG. 369.

it is required to arrange balancing masses in the planes A and B. The weights of the given masses and the linear and angular dimensions are as shown; the end view is as seen when looking from A towards B.

The known forces are proportional to the values given in Fig. 370; the units are lb.-in. The actual forces in pounds are equal to these values multiplied by $\omega^2/12g$, where ω is the angular speed of the shaft in radians per second and g is the accelerating effect of gravity in feet per second per second.

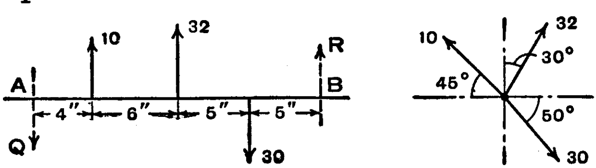


FIG. 370.

First Solution.—Let Q and R be proportional to the required forces in the planes A and B respectively, acting in unknown directions.

Let the reference plane be at A and take moments about A, then in pound and inch units

$$R \times 20 = -(\text{Vector sum of } 10 \times 4 + 32 \times 10 + 30 \times 15) \quad (1)$$
$$= -(\text{Vector sum of } 40 + 320 + 450).$$

The terms in this equation may also be obtained by arranging the data from Fig. 369 and the required products as shown in the table. The symbols w, r, and l denote weight, radius or length of arm, and distance from the reference plane at A, respectively.

Plane	w lb.	r in.	wr lb.-in.	l in.	wrl lb.-in.^{2}
A	—	—	Q	0	0
1	2	5	10	4	40
2	4	8	32	10	320
3	3	10	30	15	450
B	—	—	R	20	20R

The couple polygon is shown in Fig. 371. For convenience the direction of each couple is drawn parallel

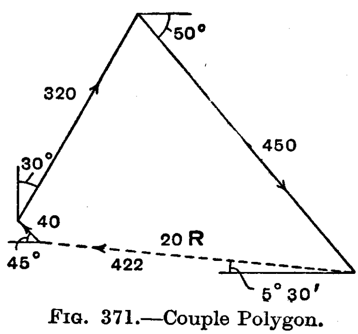


FIG. 371.—Couple Polygon.

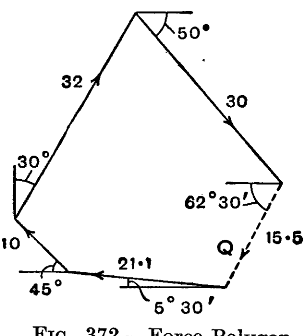


FIG. 372.—Force Polygon.

to the direction of the corresponding force on the shaft, although the axes of these couples are respectively perpendicular to these directions. The closing side represents 20R and measures to scale 422 units, therefore

$$R = \frac{422}{20} = 21 \cdot 1 \text{ lb.-in.},$$

and the direction is 5° 30′ above the horizontal (to the nearest half degree).

Since the sum of the forces in the reference plane A must be zero, it follows from Fig. 370 or from the fourth column of the table on p. 380 that

$$Q = -(\text{Vector sum of } R + 10 + 32 + 30) \quad . \qquad (2).$$

The force polygon is shown in Fig. 372. Each side is parallel to the direction of the corresponding force on the shaft and the closing side measures to scale 15·5 units, therefore

$$Q = 15{\cdot}5 \text{ lb.-in},$$

and the direction is 62° 30′ below the horizontal (to the nearest half degree).

Alternatively, Q may be obtained by drawing a couple polygon. Let the reference plane be at B and take moments about B (Fig. 370), then in pound and inch units,

$$Q \times 20 = -(\text{Vector sum of } 30 \times 5 + 32 \times 10 + 10 \times 16) \quad (3),$$
$$= -(\text{Vector sum of } 150 + 320 + 160).$$

The data from Fig. 369 and the required products are also given in the table. The symbol l' denotes distance from the reference plane at B.

Plane	w lb.	r in.	wr lb.-in.	l' in.	wrl' lb.-in.^{2}
B	—	—	R	0	0
3	3	10	30	5	150
2	4	8	32	10	320
1	2	5	10	16	160
A	—	—	Q	20	20Q

The couple polygon is shown in Fig. 373. The closing side represents 20Q and measures to scale 310 units, therefore

$$Q = \tfrac{310}{20} = 15{\cdot}5 \text{ lb.-in.},$$

and the direction is 62° 30′ below the horizontal, as before.

Suppose the balancing masses are at the ends of arms 10 inches long, then the values of their weights are

$$\frac{R}{10}=\frac{21\cdot 1}{10}=2\cdot 11 \text{ lb. in plane B}$$

and
$$\frac{Q}{10}=\frac{15\cdot 5}{10}=1\cdot 55 \text{ lb. in plane A.}$$

The directions of the arms are as shown in Fig. 374.

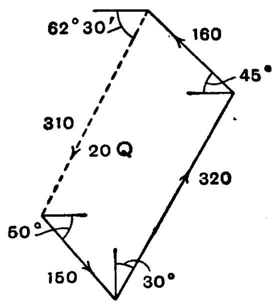


FIG. 373.—Couple Polygon.

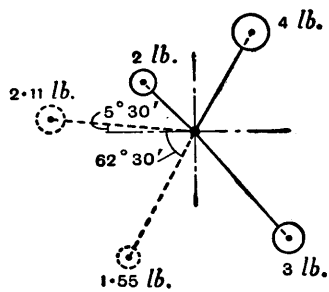


FIG. 374.—End View of Masses.

Second Solution.—Take moments about A (Fig. 370) as in equation (1), then in pound and inch units

$$R \times 20 = -(\text{Vector sum of } 10 \times 4 + 32 \times 10 + 30 \times 15).$$

Divide through by 20, then

$$R = -(\text{Vector sum of } 2 + 16 + 22\cdot 5),$$

and the terms represent forces in the plane B.

The data from Fig. 369 and the required products are also shown in the first six columns of the table. The symbol l denotes distance from the reference plane at A.

Plane	w lb.	r in.	wr lb.-in.	l in.	$wrl/20$ lb.-in.	l' in.	$wrl'/20$ lb.-in.
A	—	—	—	0	—	20	Q
1	2	5	10	4	2	16	8
2	4	8	32	10	16	10	16
3	3	10	30	15	22·5	5	7·5
B	—	—	—	20	R	0	—

Draw the force polygon (Fig. 375), making the direction of each side parallel to the direction of the corresponding force on the shaft (Fig. 370). The closing side measured to scale gives

$$R = 21 \cdot 1 \text{ lb.-in.},$$

and its direction is as indicated.

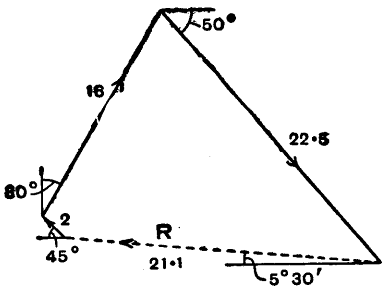


FIG. 375.—Force Polygon in Plane B.

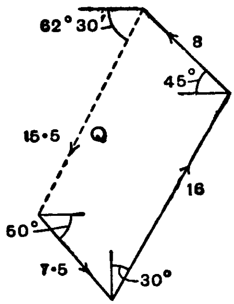


FIG. 376.—Force Polygon in Plane A.

Take moments about B (Fig. 370) as in equation (3), then in pound and inch units

$$Q \times 20 = -(\text{Vector sum of } 30 \times 5 + 32 \times 10 + 10 \times 16).$$

Divide through by 20, then

$$Q = -(\text{Vector sum of } 7 \cdot 5 + 16 + 8),$$

and the terms represent forces in the plane A. These values are also arrived at in the final column of the table on p. 382, where the symbol l' denotes distance from the reference plane at B.

Draw the force polygon (Fig. 376), making the direction of each side parallel to the direction of the corresponding force on the shaft (Fig. 370). The closing side measured to scale gives

$$Q = 15 \cdot 5 \text{ lb.-in.}$$

and its direction is as indicated.

The weights of the balancing masses are obtained, as already explained, when the lengths of the arms have been settled.

The force polygons in Figs. 375 and 376 are similar figures to the couple polygons in Figs. 371 and 373 respectively, for the numerical value of each force is $\frac{1}{20}$th that of the corresponding couple. It is immaterial which type of polygon is used; the problem has been solved in both ways because some students fail to realize there is no fundamental difference between the solutions.

In solving an example of this type, the diagrams should be drawn to large scales on one sheet of paper on a drawing-board, then it is possible to draw lines parallel to one another and to measure lengths with a fair degree of accuracy. In a book it is inevitable that related diagrams should appear on different pages, but an attempt has been made to overcome this drawback by indicating the various angles in each diagram.

Where greater accuracy is required than can be obtained by graphical methods, the length of the closing side of a polygon may be determined by calculation.

191. Direction of a Couple Vector.—Where all the forces are on one side of a reference plane, each couple vector is drawn parallel to the direction of the corresponding force, that is radially outwards, as in the example in the preceding Art. Where there are forces on each side of a reference plane, the couple vectors are drawn radially outwards for the moments on one side and radially inwards for those on the other side.

192. Balancing a Reciprocating Mass.—The acceleration of the piston A (Fig. 377) is approximately

$$\omega^2 r\left\{\cos\theta + \frac{r}{l}\cos 2\theta\right\},$$

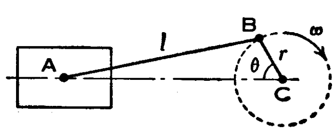


FIG. 377.

where ω is the angular velocity of the crank, θ is the crank angle measured from the line of stroke AC, and r and l are the lengths of the crank and connecting-rod respectively (Art. 26, p. 39).

If W is the weight of the reciprocating mass, which includes part of the connecting-rod in addition to the piston, the inertia force is approximately

$$F = \frac{W}{g}\omega^2 r\left\{\cos\theta + \frac{r}{l}\cos 2\theta\right\},$$

and this is also the force at the point C, in the direction CA for positive values of F.

First suppose the connecting-rod to be infinitely long, then r/l is zero. This is equivalent to assuming that the piston has simple harmonic motion and the equation for the force reduces to

$$F = \frac{W}{g}\omega^2 r\cos\theta.$$

Imagine the piston and connecting-rod to be removed and replaced by a mass of weight W attached to the crank pin at B (Fig. 378). The centrifugal force at C is in the direction CB and equal to $\frac{W}{g}\omega^2 r$ and may be represented by the radius CB$=r$; then the components of this force,

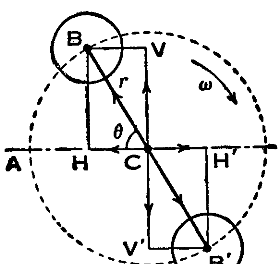


FIG. 378.

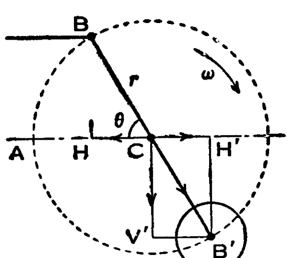


FIG. 379.

$\frac{W}{g}\omega^2 r\cos\theta$ along CA and $\frac{W}{g}\omega^2 r\sin\theta$ perpendicular to CA, are represented by CH and CV respectively, as shown. The first of these components (CH) is equal to the inertia force of the reciprocating parts, but the second (CV) is an additional force. Suppose now that the mass at B is

N

balanced by an equal mass at B′ diametrically opposite to B and at the same radius, then the balancing force along CB′ has components, represented by CH′ and CV′, which are respectively equal and opposite to those of the force along CB.

Next let the mass at B be removed and the piston and connecting-rod replaced, then the forces are as shown in Fig. 379. Since CH′ = CH the force at C is zero along the line of stroke AC, but the downward force $\frac{W}{g}\omega^2 r \sin(\theta + 180°)$ represented by CV′ is unbalanced. Therefore although the reciprocating mass moving with simple harmonic motion may be balanced by the revolving mass at B′, this is only achieved by introducing an out of balance force at C in the direction perpendicular to the line of stroke. Consequently, the effect is merely to turn the direction of the simple harmonic disturbing force through a right angle and its maximum value $\frac{W}{g}\omega^2 r$ is unchanged.

Reciprocating masses having simple harmonic motion can, in general, only be completely balanced by other reciprocating masses. In practice, as for example in the case of a locomotive, reciprocating masses may be partially balanced by revolving masses, with the result that there is a lack of balance both in the lines of stroke and in the perpendicular directions in the planes in which the balancing masses revolve. In this case the problem is one of compromise.

Before examining further the subject of reciprocating masses, the distribution of the mass of a connecting-rod will be considered.

193. Distribution of the Mass of a Connecting-Rod.—The small end of a connecting-rod reciprocates with the piston and the big end revolves with the crank pin. It is sometimes assumed that one-third of the total mass of the rod reciprocates and the remaining two-thirds revolves. A closer approximation may be obtained by calculation if the position of the centre of gravity of the rod is ascertained by balancing it on a knife-edge. Suppose AB is the rod, shown diagrammatically in Fig. 380, G is its centre of gravity and w is its total weight; then if AB denotes the length measured between the centres of the ends, the weight of the reciprocating mass is $\frac{\text{GB}}{\text{AB}}w$ and the weight of the revolving mass is $\frac{\text{AG}}{\text{AB}}w$.

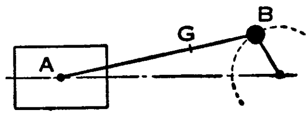


FIG. 380.

The two weights may also be found by weighing each end of the rod in turn. This is done by supporting the rod horizontally on a knife-edge at each end, one on a weighing machine and the other on a table, and then reversing the rod end for end. As a check the sum of the two separate weights should of course be equal to the total weight. It should be noted that this mass distribution does not give a system dynamically equivalent to that of the actual rod. (See also Arts. 110 to 113.)

194. Primary Balance.—For each cylinder of an engine the equation for the approximate value of the inertia force of the reciprocating parts is

$$F=\frac{W}{g}\omega^2 r\left\{\cos\theta+\frac{r}{l}\cos 2\theta\right\},$$

which may be written

$$F=\frac{W}{g}\omega^2 r\cos\omega t+\frac{W}{g}\omega^2\frac{r^2}{l}\cos 2\omega t,$$

where t denotes time and $\omega t=\theta$.

The first term on the right-hand side,

$$\frac{W}{g}\omega^2 r\cos\omega t,$$

is called the *primary force*, and it is the inertia force when the piston is assumed to move with simple harmonic motion.

When examining the primary balance of an engine with two or more cylinders in line, it is convenient to imagine the reciprocating masses to be transferred to their respective crank pins and to treat the problem as one of revolving masses.

For instance, considering the primary forces in a four-cylinder in-line engine, let *abcd* (Fig. 381) be the force polygon representing the centrifugal forces produced by the four imaginary masses shown fixed to the cranks in the end view and suppose the polygon closes, thus indicating that the centrifugal forces are in equilibrium. Assume the cylinders are horizontal (this in no way affects the argument), and let AC represent the line of stroke in the reference plane, then the projections of *bc* and *cd* on AC neutralize those of *da* and *ab*, that is

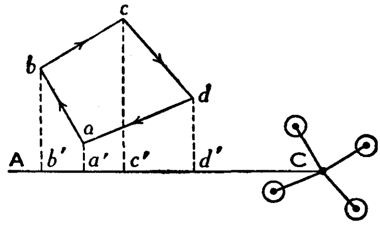


FIG. 381.

$$b'c' + c'd' = -(d'a' + a'b').$$

The projections on AC represent the primary forces for the given positions of the cranks, therefore these primary forces balance one another. The force polygon turns as the cranks turn, and the primary forces are in balance for all positions of the cranks.

If the imaginary revolving masses are not in equilibrium, the closing side of the force polygon is drawn and its projection on AC gives the out of balance primary force for the given positions of the cranks. Since the polygon turns with the cranks, the length of its closing side represents the maximum value of the out of balance primary force.

Similarly, the corresponding couple polygon turns with the cranks, and it follows that if it closes, the primary couples are in equilibrium. If the couples are not in equilibrium, the projection on AC of the closing side of the polygon represents the out of balance primary couple for the given positions of the cranks. Also the length of the closing side represents the maximum value of the out of balance primary couple.

The two conditions for primary balance are—

(1) The primary force polygon must close.

(2) The primary couple polygon must close.

These polygons are so named merely for convenience; it is the projections of the sides on the line of stroke in the reference plane which represent the primary forces and couples.

It should be noted that when the resultant primary force is not zero, the value of the resultant primary couple depends on the position of the reference plane. This, of course, is because when the force is moved, parallel to itself, from one reference plane to another, a couple is introduced; consequently the resultant couple depends on the position of the force, but the effect on the engine frame of the combination of the force and the resultant couple is exactly the same in each case.

195. Secondary Balance.—From the preceding Art. the approximate value of the inertia force of the reciprocating parts for each cylinder is

$$F=\frac{W}{g}\omega^2 r\cos\omega t+\frac{W}{g}\omega^2\frac{r^2}{l}\cos 2\omega t.$$

The second term on the right-hand side is called the *secondary force*; it is approximate only, but sufficiently accurate for many purposes. (To obtain the value of this force to any required degree of accuracy see Art. 200, p. 405.) In dealing with the primary force, the vector representing the centrifugal force $\frac{W}{g}\omega^2 r$ created by an imaginary mass of weight W attached to the crank pin was projected on to the line of stroke, and the secondary force may be dealt with in a similar way. Since the angle $2\omega t$ is double the crank angle ωt, it is convenient to write the secondary force in the form

$$\frac{W}{g}(2\omega)^2\frac{r^2}{4l}\cos 2\omega t$$

by multiplying and dividing by 4.

Suppose a mass of weight W is fixed to the outer end of an imaginary crank, of length $r^2/4l$, which has an angular velocity 2ω or twice the velocity of the actual crank. Then the centrifugal force is

$$\frac{W}{g}(2\omega)^2\frac{r^2}{4l}$$

and its component along the line of stroke is the secondary force.

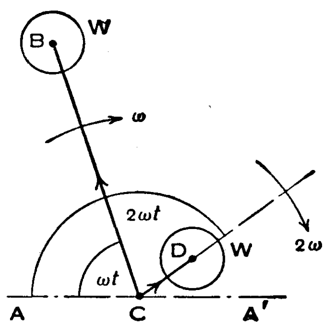


FIG. 382.

Let CB be the actual crank of length r (Fig. 382) and CD be the imaginary crank of length $r^2/4l$. Let the imaginary masses, each of weight W, be fixed to these cranks at B and D as shown. The angular velocity of CB is ω and of CD is 2ω. If the line of stroke is AC, the angle ACB is ωt and the angle ACD is $2\omega t$.

The imaginary centrifugal forces acting at C are

$$\frac{W}{g}\omega^2 r \text{ along CB} \quad \text{and} \quad \frac{W}{g}(2\omega)^2\frac{r^2}{4l} \text{ along CD,}$$

and the components of these forces along the line of stroke give the primary and secondary forces respectively. For the configuration shown, the primary force acts along CA and the secondary force acts in the opposite direction along CA′.

When an engine has two or more cylinders in line, each imaginary secondary crank with a mass attached to its outer end is inclined to the line of stroke at twice the corresponding actual crank angle at every instant, and the values of the resultant secondary forces and couples may be obtained by considering the revolving masses. The problem is similar to that of primary balance discussed in the preceding Art.

The conditions for secondary balance are—

(1) The secondary force polygon must close.
(2) The secondary couple polygon must close.

The polygons represent centrifugal forces and centrifugal couples respectively, and it is the projections of their sides on the line of stroke in the reference plane which represent the secondary forces and couples.

The statement in the preceding Art. about the position of the reference plane applies also here if "secondary" is substituted for "primary"—When the resultant secondary force is not zero the value of the resultant secondary couple depends on the position of the reference plane.

Several simple examples are considered in the following Art.

196. Examples of Primary and Secondary Forces and Couples.—These examples concern reciprocating parts only, the object being to find the primary and secondary forces and couples they produce in different designs of engine. For the purposes of comparison it is assumed in each case that the cylinders are vertical and the reciprocating masses are equal.

If the actual revolving masses are not balanced, they may be combined with the imaginary masses at the crank pins when primary balance is examined. This takes into account the effect produced by the actual revolving masses in the plane of reciprocation, but not in the perpendicular plane containing the crankshaft axis. The effect in this transverse plane is obtained by projecting on to it the force and couple polygons due to the actual revolving masses.

(1) *Two-Cylinder In-Line Engine.*—The crankshaft has two cranks 180° apart as shown diagrammatically in Fig. 383, where the crank angles are θ and $\theta+180°$. The dotted lines indicate the connecting-rods which are assumed to be removed, and the black dots on the crank pins are imaginary masses, each of the same weight W as the corresponding reciprocating parts. The length of each crank is r, and the distances between the centre lines shown are c and a as indicated.

Let the reference plane be at the centre of the left-hand main bearing. The primary force polygon closes and consists of equal and opposite centrifugal forces proportional to Wr, represented in Fig. 383 by separate lines for clearness. The resultant centrifugal force is zero, therefore there is no vertical component in the reference plane and the primary forces are balanced.

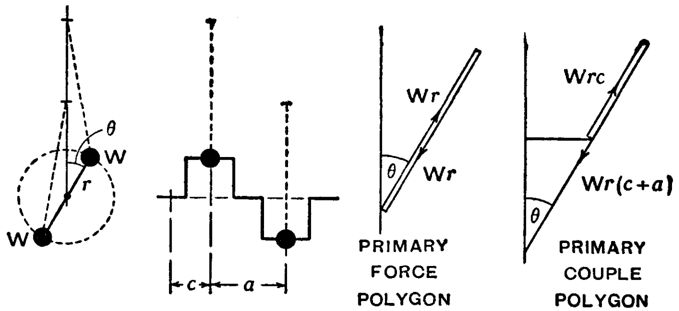


FIG. 383.

Take moments about the reference plane, then the centrifugal couples are proportional to Wrc and W$r(c+a)$ and the primary couple polygon is drawn as shown. It does not close and the resultant centrifugal couple is proportional to W$r(c+a)$ − Wrc, that is to Wra. The resultant primary couple is proportional to the vertical component Wra cos θ.

The resultant primary couple for any crank angle θ is equal to

$$\frac{\mathrm{W}}{g}\omega^2 ra \cos\theta,$$

and its maximum values are

$$\pm\frac{\mathrm{W}}{g}\omega^2 ra,$$

which occur when $\theta = 0°$ and $180°$ respectively. (Mathematically, of course, the second value is a minimum.)

Consider the secondary forces and couples. The crankshaft with its imaginary cranks of length $r^2/4l$, denoted by r', is shown in Fig. 384 with a mass of weight W attached to each crank pin. Since these cranks revolve twice as fast as the actual cranks, the crank pins must be in line;

for when the actual crank angles of the left-hand and right-hand cranks are θ and $\theta+180°$ respectively, the secondary crank angles are twice these values, that is 2θ and $2\theta+360°$.

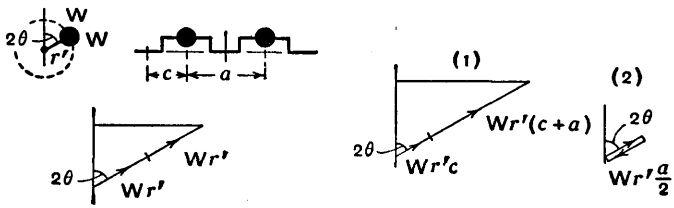


FIG. 384.

Let the reference plane be at the centre of the left-hand main bearing as before. The secondary force polygon does not close and the resultant centrifugal force is proportional to $2Wr'$. The resultant secondary force is proportional to the vertical component $2Wr' \cos 2\theta$ and is equal to

$$2\frac{W}{g}(2\omega)^2\frac{r^2}{4l}\cos 2\theta.$$

The maximum values are

$$\pm 2\frac{W}{g}(2\omega)^2\frac{r^2}{4l} \quad \text{or} \quad \pm 2\frac{W}{g}\omega^2\frac{r^2}{l},$$

and they occur when $\cos 2\theta = \pm 1$, that is when $\theta = 0$, $90°$, $180°$, and $270°$, or four times per revolution of the actual cranks. The sign is alternately positive and negative.

Take moments about the reference plane, then the secondary couple polygon shown in Fig. 384 at (1) does not close. The resultant centrifugal couple is proportional to

$$Wr'c + Wr'(c+a) \quad \text{or} \quad Wr'(2c+a)$$

and the resultant secondary couple is proportional to the vertical component

$$Wr'(2c+a)\cos 2\theta.$$

This value depends on the position of the reference plane, since the secondary forces are not balanced, and if $2c = -a$ or $c = -\frac{1}{2}a$, that is if the reference plane is at the centre of the engine, the resultant secondary couple is zero. With the reference plane in this position the secondary couple polygon closes as shown at (2), for the two couples are of opposite sign, one being clockwise and the other anti-clockwise. As previously stated (Art. 191, p. 384), the rule for constructing a couple polygon, where moments are taken on opposite sides of a reference plane, is that the vectors are drawn radially outwards for the moments on one side and radially inwards for those on the other side.

It should be noted that since r' is small compared with r, the corresponding diagrams in Figs. 383 and 384 are not drawn to the same scales.

(2) *Three-Cylinder In-Line Engine.*—The crankshaft has three cranks 120° apart and is shown in Fig. 385 with No. 1 crank turned through an angle θ from the vertical. The connecting-rods and reciprocating parts are removed

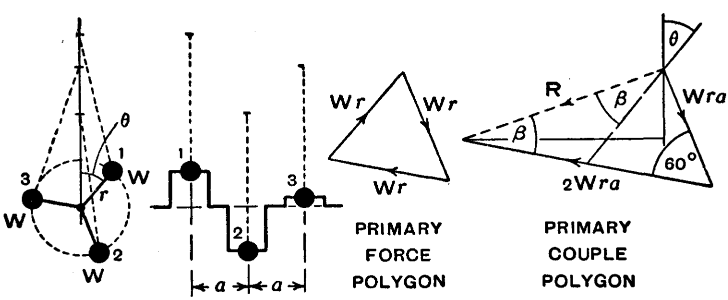


Fig. 385.

and an imaginary mass of weight W is attached to each crank pin. Each crank length is r, and the distance between adjacent parallel cylinder centre lines is a.

Let the reference plane be at the centre of No. 1 crank pin. The primary force polygon closes, so there is no resultant centrifugal force and consequently no vertical component. Therefore the primary forces are balanced.

Take moments about the reference plane, then the centrifugal couples are proportional to Wra and 2Wra and the primary couple polygon is drawn as shown in Fig. 385. It is easy to see that each angle marked β is equal to 30° and the resultant centrifugal couple is proportional to

$$R = 2Wra \cos 30° = \sqrt{3}Wra.$$

The resultant primary couple is proportional to the vertical component of R, that is to

$$\sqrt{3}Wra \cos(\theta + 30°),$$

and is equal to

$$\sqrt{3}\frac{W}{g}\omega^2 ra \cos(\theta + 30°).$$

The maximum values are

$$\pm\sqrt{3}\frac{W}{g}\omega^2 ra,$$

and occur when $\theta + 30° = 0°$ and 180°, that is when $\theta = -30°$ and 150° respectively. It will be seen that when No. 1 crank is in each of these positions, No. 2 crank is horizontal.

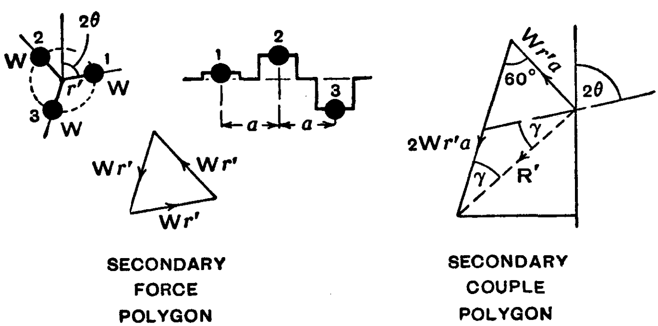


Fig. 386.

The secondary crankshaft with its imaginary cranks of length $r' = r^2/4l$ is shown in Fig. 386, with No. 1 crank inclined at an angle 2θ to the vertical and a mass of weight W is attached to each crank pin. The positions of the secondary cranks Nos. 2 and 3 should be noted. When the crank angles of the actual cranks Nos. 1, 2, and 3 are θ, $\theta + 120°$ and $\theta + 240°$ respectively, the corresponding crank angles of the secondary cranks Nos. 1, 2, and 3 must be twice these values, that is 2θ, $2\theta + 240°$ and $2\theta + 480°$ respectively.

Let the reference plane be at the centre of No. 1 crank pin as before. The secondary force polygon closes, therefore the secondary forces are balanced (Fig. 386).

Take moments about the reference plane, then the centrifugal couples are proportional to $Wr'a$ and $2Wr'a$ and the secondary couple polygon is drawn as shown in Fig. 386. It can be seen that each of the angles marked γ is equal to 30°. The resultant centrifugal couple is proportional to

$$R' = 2Wr'a \cos 30° = \sqrt{3}Wr'a.$$

The resultant secondary couple is proportional to the vertical component of R′, that is to

$$\sqrt{3}Wr'a \cos (2\theta - 30°),$$

and is equal to

$$\sqrt{3}\frac{W}{g}(2\omega)^2\frac{r^2}{4l}a \cos (2\theta - 30°).$$

The maximum values are

$$\pm\sqrt{3}\frac{W}{g}(2\omega)^2\frac{r^2}{4l}a \quad \text{or} \quad \pm\sqrt{3}\frac{W}{g}\omega^2\frac{r^2}{l}a,$$

and occur when

$$2\theta - 30° = 0, 180°, 360°, \text{ and } 540°$$

or

$$\theta = 15°, 105°, 195°, \text{ and } 285°,$$

which is four times per revolution of the actual crankshaft. The sign is alternately positive and negative.

(3) *Four-Cylinder In-Line Engine.*—The crankshaft arrangement is shown in the upper part of Fig. 387, where the crank angle is θ for the end cranks and $\theta + 180°$ for the other two cranks. The connecting-rods are removed and an imaginary mass of weight W, the weight of the corresponding reciprocating parts, is attached to each

crank pin. The length of each crank is r and the distance between adjacent cylinder centre lines is a.

The secondary crankshaft with its imaginary cranks of length $r' = r^2/4l$ is shown in the lower part of Fig. 387 with a mass of weight W attached to each crank pin. As in the case of the two-cylinder engine in example (1), the crank pins are in line and the common crank angle is 2θ when the actual crank angles are θ and $\theta + 180°$.

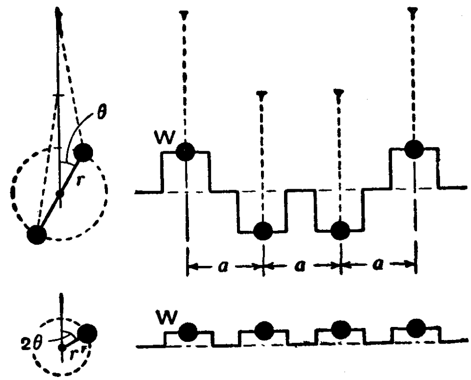


FIG. 387.

Since the four-cylinder engine is equivalent to two two-cylinder engines, the discussion which follows should be evident without the aid of further diagrams. First consider the primary forces and couples. The centrifugal forces, each proportional to Wr, due to the imaginary masses on the actual crankshaft, are in equilibrium, therefore there is no resultant vertical component and the primary forces are balanced. There are two equal and opposite centrifugal couples proportional to Wra in the plane of the actual cranks, therefore there is no resultant couple in the vertical plane and the primary couples are balanced.

Next consider the secondary effects. The resultant centrifugal force (Fig. 387, bottom) is proportional to 4Wr' and equal to

$$4\frac{W}{g}(2\omega)^2\frac{r^2}{4l},$$

and so there is a resultant secondary force equal to

$$4\frac{W}{g}(2\omega)^2\frac{r^2}{4l}\cos 2\theta$$

acting in a reference plane at the centre of the crankshaft. The maximum values occur when $\cos 2\theta = \pm 1$, that is when $\theta = 0$, $90°$, $180°$, and $270°$, or four times per revolution of the actual crankshaft, and they are alternately positive and negative.

There is no resultant moment, in the longitudinal vertical plane (Fig. 387, bottom), about the centre of the crankshaft, and therefore the resultant secondary couple is zero when the reference plane is taken in this position. As already explained, shifting the force parallel to itself alters the couple, but the combination of the two has the same effect on the engine frame for all positions of the reference plane.

197. Balancing of Locomotives.—In a locomotive the connecting-rods are long compared with the cranks, and the pistons are regarded as having simple harmonic motion, that is only primary balance is considered. All the revolving parts and up to two-thirds of the reciprocating parts are balanced by revolving masses which are usually placed in the wheels near the rims.

In general there are two, three, or four cylinders, and each one is an inside or outside cylinder according as it is between or outside the frames. A two-cylinder locomotive has its two cranks at right angles to one another. In a coupled locomotive two or more pairs of wheels are coupled together to increase adhesion and enable a greater tractive force to be used. The coupling rods are connected to parallel cranks and may be balanced by revolving masses in the wheels.

Revolving masses have been dealt with in Art. 190 and will not be considered further here. It has been shown in Art. 192 that when a reciprocating mass is balanced by a revolving mass a disturbing force is introduced, in the direction at right angles to the line of stroke, in the plane of revolution. Since in a locomotive only part of each reciprocating mass is balanced and the balancing masses are in the wheels, it is evident that there is a lack of balance in planes parallel and perpendicular to the track. The forces and couples involved will be investigated in a simple example.

The two cranks of an inside cylinder locomotive are shown diagrammatically in Fig. 388, and as the cranks are mutually perpendicular the crank angles * may be denoted by θ and $\theta+90°$. The centre lines of the cylinders are at distances a on either side of the centre line XX, the length of each crank is r and the angular velocity is ω.

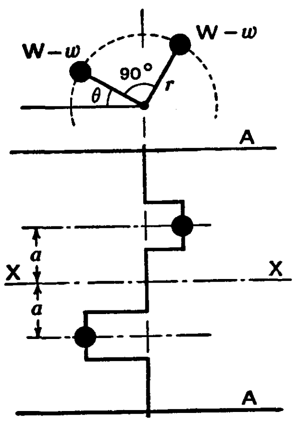


FIG. 388.

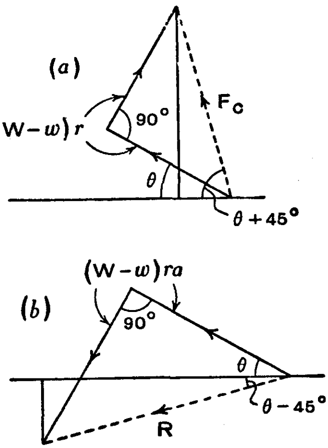


FIG. 389.

Let W be the weight of each reciprocating mass and let w be the weight of the part of each of these masses which is balanced by revolving masses in the wheels A, A, then $W-w$ is the weight of the unbalanced part.

Assume that an imaginary mass of weight $W-w$ is attached to each crank pin and take the reference plane at the centre line XX. The force polygon is shown in Fig. 389 (a). The resultant centrifugal force is proportional to

$$F_C=\sqrt{2}(W-w)r;$$

* Assuming the cranks are rotating clockwise and the cylinders are in front, that is to the right of the cranks in Fig. 388, then θ is the supplement of the usual crank angle.

the resultant unbalanced primary force is equal to

$$\frac{\sqrt{2}(W-w)}{g}\omega^2 r \cos(\theta+45^\circ),$$

and since it acts along XX it produces a variation of equal value in the tractive effort.

The maximum values of this variation are

$$\pm\frac{\sqrt{2}(W-w)}{g}\omega^2 r,$$

and occur when $\theta = -45^\circ$ and 135° respectively, that is when the vector F_C in Fig. 389 (*a*) is in each of its horizontal positions.

The couple polygon is shown in Fig. 389 (*b*). The resultant centrifugal couple is proportional to

$$R = \sqrt{2}(W-w)ra;$$

the resultant unbalanced primary couple is equal to

$$\frac{\sqrt{2}(W-w)}{g}\omega^2 ra \cos(\theta-45^\circ),$$

and produces a swaying effect on the locomotive in a horizontal plane.

The maximum values of this couple are

$$\pm\frac{\sqrt{2}(W-w)}{g}\omega^2 ra,$$

and occur when $\theta = 45^\circ$ and 225° respectively, that is when the vector R in Fig. 389 (*b*) is in each of its horizontal positions.

If the line of action of the drawbar pull is above or below that of the resultant unbalanced primary force, there is also a couple, usually small, in the central longitudinal vertical plane. This couple is equal to the product of the unbalanced force and the vertical distance between the lines of action of the two forces.

Suppose now that the partial balancing of the reciprocating masses has been obtained by a mass of weight w_1

fixed at a radius r_1 in each driving wheel, as shown diagrammatically in Fig. 390. The mass in the driving wheel in the plane of the paper is blacked in and its angular position is assumed to be $180° + \theta'$ from the line of stroke, measured in the direction of rotation, when the crank angles are θ and $\theta + 90°$. The corresponding mass in the other driving wheel and the two cranks with an imaginary mass of weight w attached to each crank pin are shown dotted. As previously stated, w is the weight of the part of each reciprocating mass which is balanced by the masses in the driving wheels. Since the actual revolving masses also have to be balanced by masses in the driving wheels, in practice the two sets of balancing masses are combined to produce the required forces, but this does not affect the present discussion.

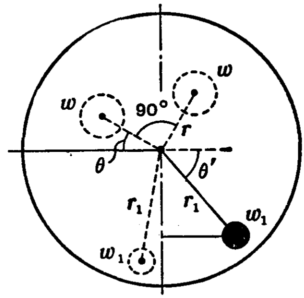


FIG. 390.

The vertical component of the centrifugal force due to the mass of weight w_1 in the driving wheel in the plane of the paper is

$$\frac{w_1}{g}\omega^2 r_1 \sin \theta',$$

and its maximum value,

$$\frac{w_1}{g}\omega^2 r_1,$$

is known as the *hammer blow*, but since the change in the vertical component of the force is continuous there is no actual blow. The same load variation takes place in the other driving wheel, but of course the maximum values in the two wheels do not occur simultaneously. These varying vertical components of the forces in the wheels produce oscillation of the locomotive about the longitudinal horizontal axis.

Let W_S be the static load on a driving wheel, then the maximum and minimum values of the load on the rail are

$$W_S \pm \frac{w_1}{g}\omega^2 r_1$$

and occur alternately when the centre of gravity of the balancing mass is respectively below or above the centre of the wheel.

If the centrifugal force should exceed the static load the wheel would lift and impact would occur when it returned to the rail.

In a coupled locomotive the masses which partially balance the reciprocating masses may be distributed between the coupled wheels. This does not affect the horizontal forces but it reduces the hammer blow per wheel.

198. Direct and Reverse Cranks.—For the single cylinder engine shown diagrammatically in Fig. 391 at (*a*) the primary force is equal to $\frac{W}{g}\omega^2 r \cos\theta$ and proportional to $Wr\cos\theta$. As already explained this force may be regarded as the component in the line of stroke of the centrifugal force produced by an imaginary mass of weight W fixed to the crank pin, as indicated at (*b*).

Suppose that two cranks of length r are revolving with equal angular speeds ω in opposite directions as at (*c*), so that when one crank angle is θ the other is $-\theta$, and let each crank carry a mass of weight $\frac{1}{2}W$ at its outer end. Each centrifugal force is proportional to $\frac{1}{2}Wr$ and the sum of the components in the line of stroke is proportional to $2 \times \frac{1}{2}Wr\cos\theta$ or $Wr\cos\theta$, which is proportional to the primary force. The components in

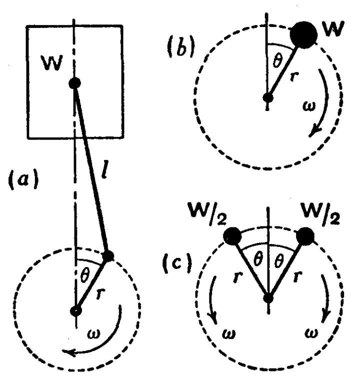


Fig. 391.

the direction perpendicular to the line of stroke balance one another.

Therefore the two masses at the ends of the cranks revolving in opposite directions produce a resultant force equal to the primary force. These two cranks are called the *direct* and *reverse cranks*, the former being the actual crank of the engine.

In a similar way it follows that the secondary force

$$\frac{W}{g}(2\omega)^2\frac{r^2}{4l}\cos 2\theta$$

may be produced by direct and reverse cranks of length $r' = r^2/4l$, carrying masses each of weight $\frac{1}{2}$W and revolving in opposite directions with equal angular speeds 2ω, as shown in Fig. 392.

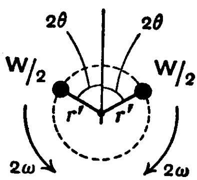


FIG. 392.

The application of direct and reverse cranks is illustrated in the following Art., where the primary and secondary forces in a radial engine are considered.

199. Radial Engine.—A five-cylinder radial engine with cylinders spaced at regular angular intervals of 72° is shown diagrammatically in Fig. 393. The cylinder centre lines are in one plane and it is assumed that the crank is joined to the five connecting-rods by a common crank pin. The primary and secondary forces will be examined with the aid of direct and reverse cranks (Fig. 394).

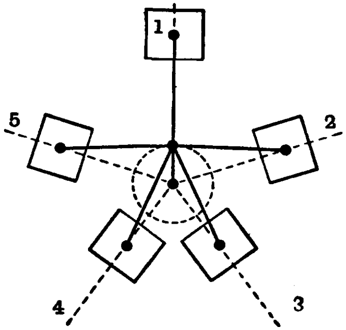


FIG. 393.

First consider the primary direct and reverse cranks, revolving in opposite directions with angular speeds ω. There is one direct crank (or five direct cranks which coincide) for the five cylinders, therefore it is labelled 1 2 3 4 5; it is shown in Fig. 394 in line with the centre line of No. 1 cylinder. Denoting the angular intervals between the centre lines by a, it follows that since the direct crank makes anticlockwise angles of 0, a, $2a$, $3a$, and $4a$ with the centre lines Nos. 1, 2, 3, 4, and 5 respectively, the five reverse cranks Nos. 1, 2, 3, 4, and 5 make clockwise angles of 0, a, $2a$, $3a$, and $4a$ with these same centre lines respectively.

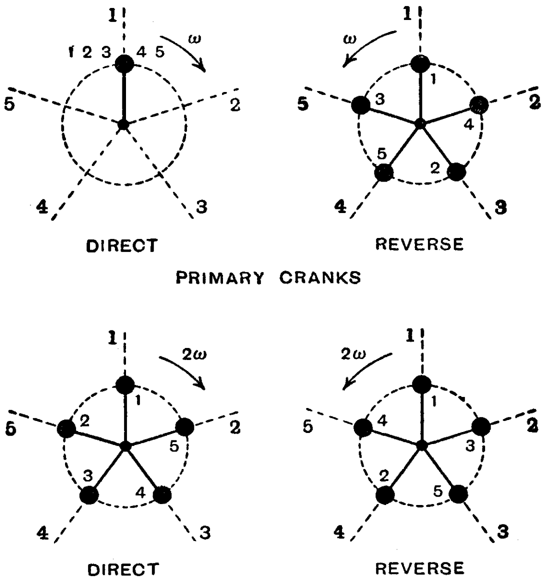


FIG. 394.

Let W be the weight of the reciprocating mass in each of the five cylinders, then the one direct crank carries a mass of weight $\frac{5}{2}$W at the crank pin and each reverse crank carries a mass of weight $\frac{1}{2}$W. Since the reverse cranks are spaced at regular intervals, the five masses attached to them are balanced. Therefore the primary lack of balance is proportional to $\frac{5}{2}Wr$, where r is the crank length, and this force may be balanced by a mass arranged diametrically opposite the crank pin.

Next consider the secondary direct and reverse cranks, revolving in opposite directions with angular speeds 2ω.

Each of these cranks must be inclined to its own cylinder centre line at twice the inclination of the corresponding primary crank. Therefore when the primary direct crank is inclined at anticlockwise angles of 0, α, 2α, 3α, and 4α to centre lines Nos. 1, 2, 3, 4, and 5 respectively, the secondary direct cranks Nos. 1, 2, 3, 4, and 5 are inclined at anticlockwise angles of 0, 2α, 4α, 6α, and 8α to these same centre lines respectively. Similarly the secondary reverse cranks Nos. 1, 2, 3, 4, and 5 are inclined at clockwise angles of 0, 2α, 4α, 6α, and 8α to centre lines Nos. 1, 2, 3, 4, and 5 respectively.

It will be seen from Fig. 394 that the secondary direct and reverse cranks are spaced at regular angular intervals, therefore the five forces, each equal to $\frac{W}{2g}(2\omega)^2\frac{r^2}{4l}$, are balanced in each case and there is no resultant secondary force.

200. Fourier Series for the Acceleration of a Piston.—In dealing with primary and secondary balance (Arts. 194 and 195) the approximate acceleration of the reciprocating parts was used, that is

$$\omega^2 r\left(\cos\theta+\frac{r}{l}\cos 2\theta\right).$$

This approximation was obtained (Art. 26, p. 39) by including only two terms in the expansion of $(1-\sin^2\phi)^{\frac{1}{2}}$. By continuing the expansion and expressing the displacement of the piston as a Fourier series, that is in terms of

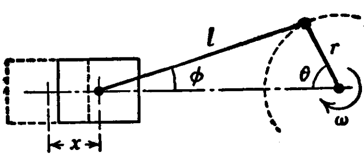


FIG. 395.

$\cos\theta$, $\cos 2\theta$, $\cos 4\theta$, $\cos 6\theta$, and so on, and differentiating twice, the values of the higher harmonics may be found to any required degree of accuracy. A few terms in the series are obtained in the analysis which follows.

The displacement of the piston from the beginning of the stroke (Fig. 395) is

$$x = l + r - l\cos\phi - r\cos\theta$$
$$= r(1-\cos\theta) + l(1-\cos\phi) \quad . \quad . \quad (1).$$

Let $\frac{r}{l} = m$, then $lm = r$.

Now $\quad l\sin\phi = r\sin\theta \quad$ or $\quad \sin\phi = m\sin\theta$,

therefore $\quad \cos\phi = (1-\sin^2\phi)^{\frac{1}{2}} = (1-m^2\sin^2\theta)^{\frac{1}{2}}$.

Expanding by the binomial theorem

$$\cos\phi = 1 - \tfrac{1}{2}m^2\sin^2\theta - \tfrac{1}{8}m^4\sin^4\theta - \tfrac{1}{16}m^6\sin^6\theta$$
$$- \tfrac{5}{128}m^8\sin^8\theta - \tfrac{7}{256}m^{10}\sin^{10}\theta - \ldots$$

Substituting in (1) and simplifying,

$$x = r(1 - \cos\theta + \tfrac{1}{2}m\sin^2\theta + \tfrac{1}{8}m^3\sin^4\theta + \tfrac{1}{16}m^5\sin^6\theta$$
$$+ \tfrac{5}{128}m^7\sin^8\theta + \tfrac{7}{256}m^9\sin^{10}\theta + \ldots) \qquad (2).$$

It can be shown that

$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$.

$\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$.

$\sin^6\theta = \frac{5}{16} - \frac{15}{32}\cos 2\theta + \frac{3}{16}\cos 4\theta - \frac{1}{32}\cos 6\theta$.

$\sin^8\theta = \frac{35}{128} - \frac{7}{16}\cos 2\theta + \frac{7}{32}\cos 4\theta - \frac{1}{16}\cos 6\theta + \frac{1}{128}\cos 8\theta$.

$\sin^{10}\theta = \frac{63}{256} - \frac{105}{256}\cos 2\theta + \frac{15}{64}\cos 4\theta - \frac{45}{512}\cos 6\theta$
$\qquad + \frac{5}{256}\cos 8\theta - \frac{1}{512}\cos 10\theta$.

Substituting these values in (2),

$$x = r(A_0 - \cos\theta + A_2\cos 2\theta + A_4\cos 4\theta$$
$$+ A_6\cos 6\theta + \ldots) \qquad (3),$$

where

$$A_0 = 1 + \tfrac{1}{4}m + \tfrac{3}{64}m^3 + \tfrac{5}{256}m^5 + \tfrac{5\times35}{128\times128}m^7 + \tfrac{7\times63}{256\times256}m^9 + \ldots$$
$$A_2 = -\tfrac{1}{4}m - \tfrac{1}{16}m^3 - \tfrac{15}{512}m^5 - \tfrac{5\times7}{128\times16}m^7 - \tfrac{7\times105}{256\times256}m^9 - \ldots$$
$$A_4 = \tfrac{1}{64}m^3 + \tfrac{3}{256}m^5 + \tfrac{5\times7}{128\times32}m^7 + \tfrac{7\times15}{256\times64}m^9 + \ldots$$
$$A_6 = -\tfrac{1}{512}m^5 - \tfrac{5}{128\times16}m^7 - \tfrac{7\times45}{256\times512}m^9 - \ldots$$

Differentiating (3) twice with respect to time,

$$\frac{d^2x}{dt^2}=\omega^2 r\,(\cos\,\theta+\mathrm{B}_2\cos\,2\theta+\mathrm{B}_4\cos\,4\theta$$
$$+\mathrm{B}_6\cos\,6\theta+\,.\,.\,.) \qquad (4),$$

where $\mathrm{B}_2=-4\mathrm{A}_2,\quad \mathrm{B}_4=-16\mathrm{A}_4,\quad \mathrm{B}_6=-36\mathrm{A}_6,\ .\ .\ .$

The numerical values of the constants B_2, B_4, and B_6 have been calculated for three values of m, using the expressions for A_2, A_4, and A_6 with the terms as given above and the results are tabulated.

m	B_2	B_4	B_6
$\frac{1}{3}$	0·343108	−0·010099	0·000334
$\frac{1}{4}$	0·254025	−0·004098	0·000074
$\frac{1}{5}$	0·202038	−0·002062	0·000024

Substituting one set of constants in equation (4), say when $m=\frac{1}{5}$, the acceleration is

$$\frac{d^2x}{dt^2}=\omega^2 r\,(\cos\,\theta+0{\cdot}202038\cos\,2\theta-0{\cdot}002062\cos\,4\theta$$
$$+0{\cdot}000024\cos\,6\theta-\,.\,.\,.).$$

It will be seen that succeeding harmonics become smaller rapidly; also each one is reduced when m is decreased, that is when the connecting-rod/crank ratio is increased.

The inertia force due to the reciprocating parts is

$$\mathrm{F}=\frac{\mathrm{W}}{g}\omega^2 r\,(\cos\,\theta+\mathrm{B}_2\cos\,2\theta+\mathrm{B}_4\cos\,4\theta$$
$$+\mathrm{B}_6\cos\,6\theta+\,.\,.\,.) \qquad (5),$$

and this equation may be applied to each cylinder of an engine to obtain the force acting along each line of stroke.

Exercises XVII

1. A uniform circular disc is free to rotate without friction about a horizontal axis through its centre perpendicular to its plane. Two holes, $\frac{1}{2}$ inch and 1 inch diameter, are drilled through the disc 4 inches and 3 inches respectively from the axis of rotation, the centres of the holes subtending an angle of 90° at the axis. Find the angular position of the disc when it has rotated into its position of stable equilibrium, and determine the diameter of the hole to be drilled 4 inches from the axis of rotation to allow the disc to rest in any angular position. [C.U.]

2. The spindle AC shown in Fig. 396 has vertical support only at A and is supported horizontally by collars at B and C. DE is a stiff arm rigidly attached at D to the spindle and carrying a weight of 20 lb. at E. If AB = BD = DC = 4 inches and DE = 12 inches, find the horizontal forces at B and C with the spindle at rest.

Also find these forces if the spindle is rotating freely at 120 r.p.m. and determine the speed of rotation for which the reaction at B is zero. [Inst. C.E.]

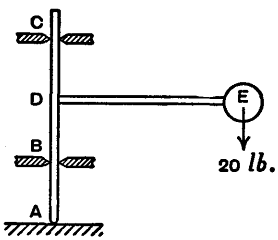


FIG. 396.

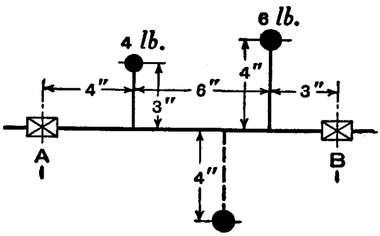


FIG. 397.

3. A single cylinder vertical engine which runs at 150 r.p.m. has a stroke of 1 foot 6 inches and a connecting-rod length of 4 feet. The weight of the reciprocating parts is 350 lb., and that of the rotating parts 200 lb., the latter assumed to be acting at the crank radius. It is desired to balance the whole of the rotating and two-thirds of the reciprocating masses by two balance weights in the planes of the crank webs, assuming their centres of gravity to be 8 inches from the crankshaft axis.

Find the magnitude of the balance weights and the maximum horizontal unbalanced force due to their introduction. [Inst. C.E.]

4. Define *centrifugal force*. A shaft, running in bearings A and B, has two concentrated masses of 4 and 6 lb. rigidly fixed to it as shown in Fig. 397, the masses and the axis of the shaft all being in the same plane. Find the reactions at the bearings A and B when the shaft is rotating at 240 r.p.m.

In order to reduce the reactions at A and B to zero, a mass (m) is fixed to the shaft as shown dotted in the figure. Find the magnitude of this mass and its distance from the bearing A.

[Inst. C.E.]

5. A three-cylinder engine has the cranks arranged at angular intervals of 120°. The cylinder pitch is 18 inches and the equivalent revolving mass per cylinder at crank radius is 150 lb. In order to give dynamic balance, revolving balance weights are to be placed at twice the crank radius in planes which lie 15 inches from, and on opposite sides of, the centre crank. Find the magnitudes and positions of these balance weights. [I.Mech.E.]

6. A revolving shaft carries four eccentric masses A, B, C, and D in this order along its axis. The magnitudes, radii of mass centres from the shaft axis, and the distances of the planes of revolution from A are given below.

Find the weight of B and the relative angular positions of the masses in order to obtain complete balance.

Mass.	Weight (lb.).	Radius (inches).	Distance from A (feet).
A	6	2·0	0
B	W	2·5	1
C	16	1·0	2
D	7	1·5	3

[U.L.]

7. A horizontal shaft AE, 7 feet long, is supported in bearings at the end A and at a point B, 1 foot from the end E. The shaft carries a pulley C of weight 40 lb. and a pulley D of weight 20 lb., placed at distances 2 feet and 5 feet respectively from A. In addition a pulley of weight 20 lb. is placed at the end E.

The centres of gravity of the pulleys C, D, and E are $\frac{1}{8}$ inch, $\frac{1}{8}$ inch, and $\frac{1}{4}$ inch respectively from the axis of the shaft.

If the pulleys are mounted so as to give static balance, find the forces on the bearings when the shaft rotates at 240 r.p.m.

[Inst. C.E.]

8. Explain carefully the meaning of the terms *static balance* and *dynamic balance* when applied to a rotating shaft carrying several eccentric masses.

Attached to a uniformly rotating shaft are four discs A, B, C, and D, spaced at equal intervals along the shaft, of weight 15 lb., 25 lb., 14 lb., and 12 lb. respectively; the mass centres of the discs are at 0·20 inch, 0·15 inch, 0·25 inch, and 0·40 inch respectively from the axis of rotation. An additional mass M may be attached to D at an effective radius of 3 inches from the axis of rotation. Find the minimum value of the weight of M, and the relative angular positions of the mass centres of all the weights to ensure complete dynamic balance for the rotating shaft.

[U.L.]

9. A shaft carries four wheels, A, B, C, D, equally pitched 10 inches apart. The unbalance (Wr) values for A and C are respectively 5 and 6 lb.-in., and the line of unbalance in C is at 90° to that of A, which may be taken as the reference direction. The out-of-balance amounts for B and D are initially unknown, but the complete rotor is dynamically balanced by adding a weight of 0·81 lb. to wheel B at a radius of 24 inches and at an angle of 215° to the reference direction, and by removing material of weight 0·20 lb. from D at a radius of 18 inches and an angle of 120°. Determine the initial and final unbalance values for B and D.

[U.L.]

10. An inside cylinder locomotive is to be balanced by balance weights placed in the driving wheels. The engine has two cylinders, the cranks being at right angles. The stroke is 20 inches, the distance between the centre lines of the cylinders 24 inches, and the distance between the planes containing the balance weights is 60 inches. The weight of the reciprocating parts per cylinder is 500 lb., and that of the revolving parts 600 lb. per cylinder. The radius of the mass centre of each balance weight is 30 inches.

Find the magnitude of the balance weights and their angular positions, in order that the whole of the revolving and two-thirds of the reciprocating masses shall be balanced.

[Inst. C.E.]

11. The crankshaft of a locomotive has two cranks at right angles, and they are symmetrically placed at a centre distance of $\frac{1}{2}c$ on each side of the engine centre line. The equivalent weight at each crank pin is w lb., and balance is achieved by two balance weights each of W lb. placed one in each driving wheel at a radius of R inches. The distance between the wheels is l inches and the radius of each crank r inches.

Show that for perfect balance

$$W = w\frac{r}{R}\frac{\sqrt{l^2+c^2}}{l\sqrt{2}},$$

and also show that the angular position of W with respect to the adjacent crank is given by $\tan^{-1}\dfrac{c-l}{c+l}$. [U.L.]

12. The three cranks of a three-cylinder locomotive are all on the same axle and are set at 120°. The pitch of the cylinders is 3·5 feet and the stroke of each piston is 26 inches. The reciprocating masses are 600 lb. for the inside cylinder and 520 lb. for each outside cylinder, and the planes of rotation of the balance weights are 2·75 feet from the inside crank.

If 40 per cent. of the reciprocating parts are to be balanced, find (*a*) the magnitude and position of the balance weights required at a radius of 24 inches, (*b*) the hammer blow per wheel when the axle makes six revolutions per second. [I.Mech.E.]

13. A single-cylinder horizontal gas engine has a stroke of 17 inches and the crank makes 270 r.p.m. The revolving parts are equivalent to 150 lb. at crank radius, the reciprocating parts weigh 120 lb., and the connecting-rod weighs 160 lb.; the connecting-rod is 40 inches long between centres, and its centre of gravity is 13 inches from the big end centre. Revolving balance weights are to be introduced on extensions of the crank webs in order to balance all the revolving parts and one-half of the reciprocating parts. Find the magnitude and position of the balance weights required at a radius of 9 inches.

If the effect of the obliquity of the connecting-rod is ignored, what is the nature and magnitude of the residual unbalanced force on the engine frame? [I.Mech.E.]

14. A locomotive has two inside cylinders, 2 feet apart, centre to centre, and the stroke of each is 24 inches. The cranks are at right angles, and the reciprocating parts for each cylinder weigh 500 lb. Primary horizontal balance of 70 per cent. of the reciprocating masses is secured by providing balance weights on the driving wheels at a radius of 3 feet, the distance between the planes of the balance weights being 5 feet. Find the hammer blow on the rails when the locomotive travels at 60 m.p.h., the driving wheels being 6 feet 9 inches in diameter. [Inst. C.E.]

15. A crankshaft has three cranks of 5 inches radius set at 120° to each other and equally spaced with a pitch of 12 inches. The revolving mass for each crank is equivalent to 35 lb. at crank

radius. The shaft is supported in two bearings symmetrically placed relatively to the cranks and 38 inches apart. If the shaft rotates at 400 r.p.m., determine the dynamical load at each bearing.

Determine the magnitudes of two weights required to balance the shaft, one at a radius of 7 inches in the plane of the left-hand crank and the other at a radius of 9 inches rotating in a plane 8 inches beyond the right-hand bearing. State the angular positions relative to the left-hand crank. [U.L.]

16. A compound steam engine has cylinder centre lines A, B (Fig. 398), 60 inches apart. The cranks, A, B, of 8 inches radius, overhang the bearings and are set at 90°. The rotating masses at

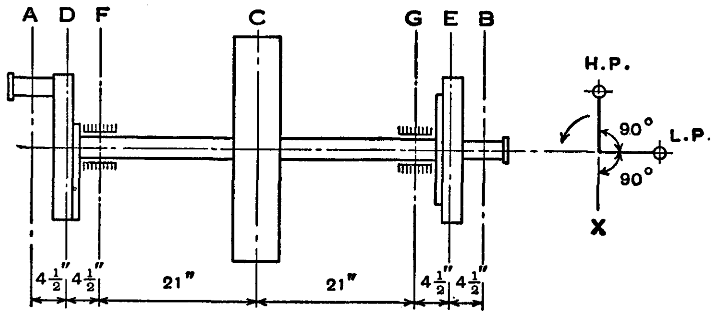


FIG. 398.

crank radius are: H.P., 100 lb., L.P., 150 lb, Disc type crank webs D, E are used, and are filled in to provide balance weights equivalent to 80 lb. at crank radius, opposite to their respective cranks. The discs are the same for both sides, and the central planes of the balancing masses are $4\frac{1}{2}$ inches inside the engine centre lines. There is a rope flywheel, C, midway between the bearings, F, G, which is out of balance, on a radial line (marked X in Fig.) opposite the H.P. crank, to the extent of 50 lb. at 20 inches radius. Determine the dynamical reactions on the bearings at a speed of 180 r.p.m. and the change that would be produced in these by balancing the flywheel. [U.L.]

17. The centre lines of the two cylinders of a V twin engine lie in one plane and the included angle is 60°. The pistons each weigh $1\frac{1}{2}$ lb., the stroke is 4 inches, and the connecting-rods are attached to a common crank pin. A rotating balance weight is provided, the centre of gravity of which has a radius of $2\frac{1}{2}$ inches. Find the least possible out-of-balance primary force when the engine runs at 1600 r.p.m. [Inst. C.E.]

18. An engine having five cylinders in line has successive cranks 144° apart, the distance between cylinder centre lines being 15 inches. The reciprocating mass for each cylinder is 35 lb., the crank radius is $4\frac{1}{2}$ inches and the connecting-rod length is 18 inches. The engine runs at 600 r.p.m.

Examine the engine for balance of primary and secondary forces and couples. Determine the maximum values of these and the position of the central crank at which these maximum values occur. [U.L.]

19. In a six-cylinder in-line engine used in automobile practice, the normal design of the crankshaft is such that when cranks Nos. 1 and 6 are at top dead centre, cranks Nos. 2 and 5 are at 120° and cranks Nos. 3 and 4 are at 240°.

Make a sketch of the crankshaft and show that the engine has primary and secondary balance, by drawing the primary and secondary force and couple polygons.

CHAPTER XVIII

VIBRATIONS I

201. Simple Harmonic Motion.—Simple harmonic motion has been discussed in Chap. VII, pp. 104–116, where it was shown that, denoting the displacement of a particle from its mid-point of travel by x at time t, the acceleration, being proportional to the displacement and directed towards the mid-point of travel, may be written as

$$\frac{d^2x}{dt^2} = -\omega^2 x,$$

or

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad . \quad . \quad . \quad (1).$$

The solution of this equation is

$$x = \text{A} \cos \omega t + \text{B} \sin \omega t \quad . \quad . \quad (2),$$

where the values of the arbitrary constants A and B depend on the conditions in the problem. The solution may also be expressed in either of the forms

$$x = \text{C} \cos (\omega t + \phi) \quad . \quad . \quad . \quad (3),$$

or

$$x = \text{C} \sin (\omega t + \phi') \quad . \quad . \quad . \quad (4).$$

The *amplitude* is C, the *periodic time* is $\text{T} = 2\pi/\omega$, and the *frequency* is $f = 1/\text{T} = \omega/2\pi$. Frequency is measured in oscillations, vibrations, or cycles per unit time. A body has made one *oscillation*, *vibration*, or *cycle* when it has passed once through its whole path and is then about to make another journey, moving in the same direction as at first. For example, if a pendulum swings from rest at A to rest at B and then swings back to rest at A, it has made one oscillation and its path is from A to B and back to A.

It must be understood, however, that the length of the path will gradually decrease when the motion is damped, but then the motion will not be simple harmonic motion.

202. Motion of a Mass Suspended by a Helical Spring.— The analysis which follows may be applied to any system in which the restoring force is proportional to the displacement, the helical spring being merely a particular example.

Consider a mass of weight W suspended by a helical spring, as shown in Fig. 399, where OO is the level of statical equilibrium. Let k be the stiffness or the force which produces unit extension in the spring, then since the extension is proportional to the load, it follows that the statical extension due to the load W is W/k. The mass of the spring will be neglected.

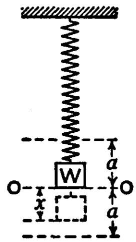


FIG. 399.

Let the mass be pulled down a distance a from the statical level OO and released, then, disregarding resistance due to air, etc., the mass will vibrate with simple harmonic motion and the extremities of its travel will be a distance a below and above the level OO, a being the amplitude.

The case in which the vibrations gradually die away on account of resistance will be considered in the next Art.

Let the displacement of the mass from OO be x at a time t, regarding x as positive when measured below OO, then the additional force extending or compressing the spring is kx, the sign of this force being positive or negative according as x is positive or negative. The corresponding force acting on the mass is $-kx$, and this force always accelerates the mass towards the level OO.

Since mass × acceleration = accelerating force,

therefore
$$\frac{W}{g}\frac{d^2x}{dt^2} = -kx,$$

which may be written as

$$\frac{d^2x}{dt^2} + \frac{kg}{W}x = 0 \quad . \quad . \quad . \quad (1).$$

Comparing this equation with the general form

$$\frac{d^2x}{dt^2}+\omega^2x=0$$

and its solution $x=\mathrm{A}\cos\omega t+\mathrm{B}\sin\omega t,$

it can be seen that $\omega=\sqrt{\frac{kg}{\mathrm{W}}}$

and that the solution of (1) is

$$x=\mathrm{A}\cos\sqrt{\frac{kg}{\mathrm{W}}}t+\mathrm{B}\sin\sqrt{\frac{kg}{\mathrm{W}}}t \quad . \quad . \quad (2).$$

The arbitrary constants A and B are found by using two conditions. The mass is released and begins to move with zero velocity when $x=a$, therefore, measuring time from this instant, the conditions are $x=a$ and $dx/dt=0$ when $t=0$.

Putting $t=0$ and $x=a$ in (2), it follows that $\mathrm{A}=a$.

Differentiating (2),

$$\frac{dx}{dt}=\sqrt{\frac{kg}{\mathrm{W}}}\left\{-\mathrm{A}\sin\sqrt{\frac{kg}{\mathrm{W}}}t+\mathrm{B}\cos\sqrt{\frac{kg}{\mathrm{W}}}t\right\}.$$

Putting $t=0$ and $dx/dt=0$, it follows that $\mathrm{B}=0$.

Substituting $\mathrm{A}=a$ and $\mathrm{B}=0$ in (2),

then $$x=a\cos\sqrt{\frac{kg}{\mathrm{W}}}t \quad . \quad . \quad . \quad (3).$$

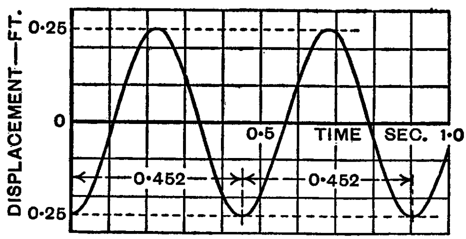


Fig. 400.

As a numerical example, taking $a=0{\cdot}25$ ft., $k=120$ lb. per ft., $\mathrm{W}=20$ lb., and $g=32{\cdot}2$ ft./sec.^{2}, then

$$\omega=\sqrt{\frac{kg}{W}}=\sqrt{\frac{120\times 32{\cdot}2}{20}}=13{\cdot}90 \text{ rad./sec.},$$

and the displacement at time t is

$$x=0{\cdot}25 \cos 13{\cdot}90t,$$

the graph of which is shown in Fig. 400.

The periodic time $T=\frac{2\pi}{\omega}=\frac{2\pi}{13{\cdot}90}=0{\cdot}452$ sec.

The frequency $f=\frac{1}{T}=\frac{13{\cdot}90}{2\pi}=2{\cdot}21$ cycles/sec.

Since $\frac{1}{\omega}=\sqrt{\frac{W}{kg}}$ and $\frac{W}{k}$ is the statical deflection of the spring due to the load W, the periodic time may also be expressed as

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{\text{statical deflection}}{g}} \qquad . \qquad (4).$$

Energy Method of Obtaining the Equation of Motion.—The sum of the kinetic energy and the potential energy of the mass plus the potential energy or strain energy of the spring must be constant. The equation of motion may be obtained by writing down this equality and differentiating with respect to time.

Mass—Kinetic Energy (K.E.) is $\frac{W}{2g}\left(\frac{dx}{dt}\right)^2$.

Potential Energy (P.E.) is $W(a-x)$, taking the lowest position as the datum level.

Spring—Strain Energy (S.E.) is $\frac{1}{2}$(force × extension), or

$$\tfrac{1}{2}(W+kx)\left(\frac{W}{k}+x\right)=\frac{1}{2k}(W+kx)^2.$$

Mass and Spring—Total Energy, K.E. + P.E. + S.E., is

$$\frac{W}{2g}\left(\frac{dx}{dt}\right)^2+W(a-x)+\frac{1}{2k}(W+kx)^2=\text{const.} \qquad . \qquad (1).$$

P.E. + S.E. becomes $Wa+W^2/2k+\frac{1}{2}kx^2$, but Wa is the P.E. and $W^2/2k$ is the S.E. when $x=0$, therefore

$$\frac{W}{2g}\left(\frac{dx}{dt}\right)^2 + (\text{P.E.} + \text{S.E.})_{x=0} + \tfrac{1}{2}kx^2 = \text{const.} \quad . \qquad (2),$$

or
$$\frac{W}{2g}\left(\frac{dx}{dt}\right)^2 + \tfrac{1}{2}kx^2 = \text{const.} \quad . \qquad (3).$$

Differentiating, $$\frac{W}{g}\frac{dx}{dt}\frac{d^2x}{dt^2} + kx\frac{dx}{dt} = 0$$

from which $$\frac{d^2x}{dt^2} + \frac{kg}{W}x = 0. \quad . \quad . \quad . \qquad (4).$$

The strain energy is $\frac{1}{2}kx^2$ if $W = 0$ and extension $= x$; actually $\frac{1}{2}kx^2$ is the change in P.E. + S.E. The combined graphs of the three energies (Fig. 400a) were plotted by substituting the data from the Ex. on p. 416, using inch and lb. units, in the P.E., S.E., and K.E. expressions. Horizontal lengths measured from the dotted line are, to the left $(\text{P.E.} + \text{S.E.})_{x=0}$, and to the right $\frac{1}{2}kx^2 + \text{K.E.}$ *Note:* K.E. reduces to $(W/2g)\omega^2(a^2 - x^2) = \frac{1}{2}k(a^2 - x^2)$.

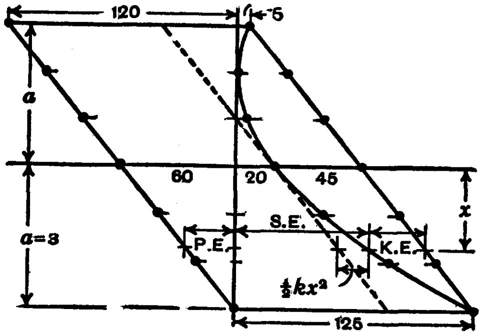


FIG. 400a.

203. Damped Vibrations.—Suppose the motion of a vibrating mass is opposed by a force which is proportional to the velocity, and let this force be denoted by $-b\frac{dx}{dt}$, where b is a constant and the minus sign indicates that the force opposes the motion.

Using the notation of the preceding Art., the equation

$$\text{mass} \times \text{acceleration} = \text{accelerating force}$$

becomes $$\frac{W}{g}\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx,$$

from which $$\frac{d^2x}{dt^2}+\frac{bg}{W}\frac{dx}{dt}+\frac{kg}{W}x=0,$$

or, for convenience, putting $\frac{bg}{W}=2c$ and $\frac{kg}{W}=\omega^2$,

$$\frac{d^2x}{dt^2}+2c\frac{dx}{dt}+\omega^2x=0 \quad . \quad . \quad . \quad (1),$$

which represents the damped vibrations such as would occur when a mass oscillates at the end of a spring, and the amplitude of the motion gradually decreases on account of air resistance, etc., until finally the mass comes to rest. The time which elapses before the motion ceases depends on the value of the damping constant c, and if c is of sufficient magnitude the mass will not oscillate, but will return towards the level of statical equilibrium without quite reaching it.

The mass will oscillate provided c is less than ω. In this case the solution of (1) is *

$$x=e^{-ct}(\mathrm{A}\cos\sqrt{\omega^2-c^2}\,t+\mathrm{B}\sin\sqrt{\omega^2-c^2}\,t) \quad . \quad (2),$$

which can also be written in the form

$$x=\mathrm{C}e^{-ct}\cos(\sqrt{\omega^2-c^2}\,t-\phi) \quad . \quad . \quad (3),$$

where C and ϕ are arbitrary constants.

To determine C and ϕ, suppose $t=0$ and $\frac{dx}{dt}=0$ when $x=a$, where a is the maximum displacement (Fig. 399).

Putting $t=0$ and $x=a$ in (3),

$$a=\mathrm{C}\cos(-\phi)=\mathrm{C}\cos\phi,$$

from which $$\mathrm{C}=\frac{a}{\cos\phi}.$$

Differentiating (3),

$$\frac{dx}{dt}=\mathrm{C}e^{-ct}\{-\sqrt{\omega^2-c^2}\sin(\sqrt{\omega^2-c^2}\,t-\phi)\}$$
$$-\mathrm{C}ce^{-ct}\cos(\sqrt{\omega^2-c^2}\,t-\phi).$$

* The method of solving equations of this type is explained in the author's *Mathematics*, Longmans, Green & Co., or in any book on Differential Equations.

Putting $t=0$ and $\dfrac{dx}{dt}=0$, and remembering that

$$\sin(-\phi)=-\sin\phi \quad\text{and}\quad \cos(-\phi)=\cos\phi,$$

$$0=C\sqrt{\omega^2-c^2}\sin\phi-Cc\cos\phi,$$

from which

$$\tan\phi=\frac{c}{\sqrt{\omega^2-c^2}} \quad\text{or}\quad \phi=\tan^{-1}\frac{c}{\sqrt{\omega^2-c^2}}.$$

The angle ϕ can be represented as shown in Fig. 401. Since $C=a/\cos\phi$, taking the value of $\cos\phi$ from the triangle it can be seen that $C=\omega a/\sqrt{\omega^2-c^2}$.

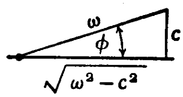


FIG. 401.

Equation (3) may now be written as

$$x=\frac{\omega a}{\sqrt{\omega^2-c^2}}e^{-ct}\cos(\sqrt{\omega^2-c^2}\,t-\phi) \quad . \quad (4),$$

where

$$\phi=\tan^{-1}\frac{c}{\sqrt{\omega^2-c^2}}.$$

As a numerical example, taking $a=0{\cdot}25$ ft., $k=120$ lb. per ft., $W=20$ lb., and $g=32{\cdot}2$ ft./sec.^{2}, then

$$\omega=\sqrt{\frac{kg}{W}}=\sqrt{\frac{120\times32{\cdot}2}{20}}=13{\cdot}90 \text{ rad./sec.},$$

as in the preceding Art.

Taking the damping force $b\dfrac{dx}{dt}=2\dfrac{dx}{dt}$ lb., the velocity being in ft./sec., then $2c=\dfrac{bg}{W}=\dfrac{2\times32{\cdot}2}{20}$ and $c=1{\cdot}61$ rad./sec.

Therefore

$$\sqrt{\omega^2-c^2}=\sqrt{13{\cdot}90^2-1{\cdot}61^2}=13{\cdot}81 \text{ rad./sec.}$$

and
$$\frac{\omega a}{\sqrt{\omega^2-c^2}}=\frac{13{\cdot}90\times0{\cdot}25}{13{\cdot}81}=0{\cdot}252 \text{ ft.};$$

also $\tan\phi=\dfrac{c}{\sqrt{\omega^2-c^2}}=\dfrac{1{\cdot}61}{13{\cdot}81}=0{\cdot}117$, from which $\phi=6°\,40'$.

The value of $\sqrt{\omega^2 - c^2}$ becomes $13{\cdot}81 \times \dfrac{180}{\pi} = 791$ when expressed in degrees/sec.

Substituting numerical values in (4),

$$x = 0{\cdot}252 e^{-1{\cdot}61t} \cos (791t - 6^\circ\ 40') \qquad . \qquad (5),$$

the graph of which is shown by the thick line curve in Fig. 402.

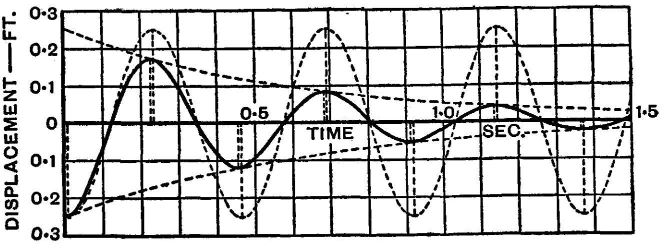


FIG. 402.

The graphs of $0{\cdot}252 \cos (791t - 6^\circ\ 40')$ and $\pm\ 0{\cdot}252e^{-1{\cdot}61t}$ are also given, and it can be seen how the damping factor $e^{-1{\cdot}61t}$ quickly damps out the motion. It is of interest to notice that the highest and lowest points (*turning-points*) on the damped curve occur slightly before the highest and lowest points on the cosine curve. This shows that the mass takes less time to move from the level of statical equilibrium to an extreme position than it takes to make the return journey.

The periodic time is $\mathrm{T} = \dfrac{2\pi}{\sqrt{\omega^2 - c^2}} = \dfrac{2\pi}{13{\cdot}81} = 0{\cdot}455$ sec.

When there is no damping, $c = 0$, and the ratio

$$\frac{\text{Periodic time with damping}}{\text{Periodic time without damping}} = \frac{\omega}{\sqrt{\omega^2 - c^2}} = \frac{13{\cdot}90}{13{\cdot}81} = 1{\cdot}007,$$

which is greater than unity, showing that damping increases the periodic time.

The displacements at successive turning-points will now be examined. It has been stated that the displacement x

at any time t is obtained from equation (3),

$$x = Ce^{-ct} \cos\left(\sqrt{\omega^2 - c^2}\, t - \phi\right).$$

Differentiating,

$$\frac{dx}{dt} = Ce^{-ct}\{-\sqrt{\omega^2 - c^2} \sin\left(\sqrt{\omega^2 - c^2}\, t - \phi\right)\} - Cce^{-ct} \cos\left(\sqrt{\omega^2 - c^2}\, t - \phi\right).$$

Now $\frac{dx}{dt} = 0$ at turning-points, therefore by equating to zero it follows that

$$\tan\left(\sqrt{\omega^2 - c^2}\, t - \phi\right) = -\frac{c}{\sqrt{\omega^2 - c^2}},$$

or $$\sqrt{\omega^2 - c^2}\, t - \phi = \tan^{-1}\left\{-\frac{c}{\sqrt{\omega^2 - c^2}}\right\}.$$

There are many angles θ_1, θ_2, etc., at intervals of π radians whose tangent is $-c/\sqrt{\omega^2 - c^2}$, therefore, denoting the successive turning-points by the suffixes, 1, 2, etc.,

$$\sqrt{\omega^2 - c^2}\, t_1 - \phi = \theta_1$$

and $$\sqrt{\omega^2 - c^2}\, t_2 - \phi = \theta_2 = \theta_1 + \pi.$$

From these equations,

$$t_2 - t_1 = \frac{\pi}{\sqrt{\omega^2 - c^2}}.$$

Therefore $$\frac{x_2}{x_1} = \frac{e^{-ct_2} \cos\left(\sqrt{\omega^2 - c^2}\, t_2 - \phi\right)}{e^{-ct_1} \cos\left(\sqrt{\omega^2 - c^2}\, t_1 - \phi\right)}$$

$$= \frac{e^{-ct_2} \cos(\theta_1 + \pi)}{e^{-ct_1} \cos \theta_1}$$

$$= -e^{-c(t_2 - t_1)} = -e^{-c\pi/\sqrt{\omega^2 - c^2}}$$

or, in general terms,

$$\frac{x_n}{x_{n-1}} = -e^{-c\pi/\sqrt{\omega^2 - c^2}}$$

the ratio being negative because the cosines have the same numerical value but differ in sign. The values x_1, x_2, etc., form a series in geometrical progression and the common ratio is $-e^{-c\pi/\sqrt{\omega^2-c^2}}$. If x_1 is positive, then x_2 is negative and x_3 is positive, and so on.

Changing the sign of the common ratio,

$$-\frac{x_n}{x_{n-1}}=e^{-c\pi/\sqrt{\omega^2-c^2}}$$

then it is evident that $-c\pi/\sqrt{\omega^2-c^2}$ is the logarithm to the base e of the positive ratio $-x_n/x_{n-1}$. The quantity $c\pi/\sqrt{\omega^2-c^2}$ is known as the *logarithmic decrement.*

204. Forced and Damped Vibrations.—A vibrating mass whose motion is forced as well as damped will now be examined. Consider the helical spring carrying a mass of weight W as in Art. 202, but let the top of the spring be given simple harmonic motion with a vertical amplitude a_1 as indicated in Fig. 403, where O_1O_1 is the mean position of the top of the spring and OO is the level of the mass when in statical equilibrium. The amplitude of the motion of the mass is labelled a, but this value is as yet unknown.

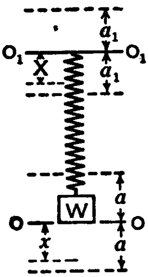


FIG. 403.

Suppose the mass descends a distance x when the top of the spring descends a distance X. If x is greater than X, then the extension of the spring is $x-X$, and if k is the force per unit extension, the corresponding force acting upwards on the mass is $k(x-X)$. Since the positive direction is downwards, this force may be expressed as $-k(x-X)$.

Taking $-b\dfrac{dx}{dt}$ as the damping force, as in the preceding Art., the equation of motion becomes

$$\frac{W}{g}\frac{d^2x}{dt^2}=-b\frac{dx}{dt}-k(x-X) \quad . \quad . \quad (1),$$

or, dividing by $\dfrac{W}{g}$ and for convenience putting $\dfrac{bg}{W}=2c$ and

$\frac{kg}{W} = \omega^2$, the equation may be written in the form

$$\frac{d^2x}{dt^2} + 2c\frac{dx}{dt} + \omega^2 x = \omega^2 X.$$

Since the top of the spring moves with simple harmonic motion with an amplitude a_1, let $X = a_1 \sin pt$ as indicated in Fig. 404, then the equation of motion becomes

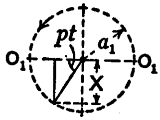


FIG. 404.

$$\frac{d^2x}{dt^2} + 2c\frac{dx}{dt} + \omega^2 x = \omega^2 a_1 \sin pt \quad . \quad (2).$$

The complete solution of this equation * is the sum of two solutions called the *complementary function* and the *particular integral.* The complementary function is obtained by equating the left-hand side of the equation to zero and it is the solution given in equation (3), p. 419, that is

$$x = Ce^{-ct} \cos\left(\sqrt{\omega^2 - c^2}\, t - \phi\right) \quad . \quad . \quad (3).$$

The particular integral is a particular solution which satisfies the whole of equation (2) and it can be shown to be

$$x = \frac{\omega^2 a_1}{\sqrt{(\omega^2 - p^2)^2 + 4c^2p^2}} \sin(pt - \epsilon) \quad . \quad (4),$$

where $\epsilon = \tan^{-1}\frac{2cp}{\omega^2 - p^2}$. The angle ϵ is the phase difference † between the motion of the mass and the motion of the top of the spring, the mass lagging by this angle.

The complete solution of (2) is the sum of (3) and (4), but the damped natural vibrations represented by (3) eventually die away and after a time the motion of the mass is represented by (4).

The amplitude of the motion of the mass is therefore

$$\frac{\omega^2 a_1}{\sqrt{(\omega^2 - p^2)^2 + 4c^2p^2}},$$

* This type of equation is solved in the author's *Mathematics*, Longmans, Green & Co., or in any book on Differential Equations.

† For definition see p. 107.

which was denoted by a in Fig. 403, and the forced amplitude of the top of the spring is a_1.

Therefore the ratio

$$\frac{\text{Amplitude of mass}}{\text{Amplitude of top of spring}}=\frac{\omega^2 a_1}{\sqrt{(\omega^2-p^2)^2+4c^2p^2}}\cdot\frac{1}{a_1},$$

which may be called the *magnification of amplitude* or *magnification factor* and written as

$$\text{Magnification of amplitude}=\frac{1}{\sqrt{\left(1-\frac{p^2}{\omega^2}\right)^2+\left(\frac{2c}{\omega}\right)^2\frac{p^2}{\omega^2}}}.$$

This ratio is plotted (Fig. 405) against the ratio of the applied frequency $p/2\pi$ to the natural frequency $\omega/2\pi$,

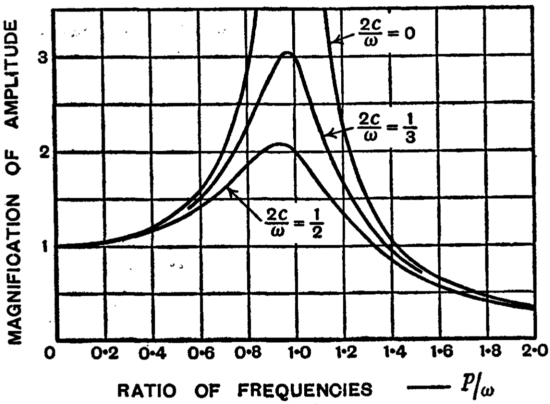


FIG. 405.

—that is, the ratio p/ω—for three values of $2c/\omega$. Since c is dependent on the damping force, a comparison of the curves shows how the ratio of the amplitudes is affected by the damping; the greater the damping, the less the magnification of the amplitude. The curves also show that when the ratio p/ω reaches a certain value, which is different for each curve, the magnification of the amplitude becomes less than unity—that is to say, the amplitude is actually reduced.

If there is no damping, $c=0$, then the magnification becomes infinite when $p=\omega$—that is, when the applied and natural frequencies are equal. This equality of the frequencies is called *resonance*. In practice there is always some damping, so that c is not zero although it may be very small. It can be seen from the two lower curves that when the applied and natural frequencies are equal, or nearly equal, the amplitude of the motion of the mass may become large, but it will not be infinite. It should be noticed that when there is damping, then the maximum amplitude occurs when p is less than ω.

To understand how the relation between p and ω affects the phase difference $\epsilon=\tan^{-1}\dfrac{2cp}{\omega^2-p^2}$ and also to understand resonance, it is worth making a simple experiment. Secure a mass to one end of a piece of elastic or a light helical spring and hold the other end in the hand.

(*a*) Move the hand up and down very slowly and the mass will follow the motion of the hand.

(*b*) Move the hand up and down rapidly and it will be found that the mass goes down when the hand goes up and *vice versa*. If the hand is moved fast enough the mass will be almost motionless.

(*c*) Move the hand at some intermediate speed, found by trial, and the amplitude of the motion of the mass will become large. This is what happens at resonance.

Now consider these effects from a mathematical standpoint. The phase difference is $\epsilon=\tan^{-1}\dfrac{2cp}{\omega^2-p^2}$. When $p=\omega$, $\epsilon=\tan^{-1}\infty=\pi/2$, and this is the value of ϵ at resonance. When p is less than ω, ϵ is less than $\pi/2$ (Fig. 406, top). If p is nearly zero, then ϵ is nearly zero, and this is what happened in experiment (*a*) when the hand and the mass moved together. When p is greater than ω, ϵ is greater than $\pi/2$ (Fig. 406, bottom) because ω^2-p^2 is negative, and the greater p is compared with ω the closer ϵ approaches to the value π. This is what happened

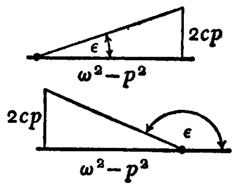


FIG. 406.

when the hand and the mass moved in opposite directions in experiment (*b*).

In the preceding analysis a spring loaded at one end was given a known *periodic displacement* at the other end. The equations also apply when a system is disturbed by a known *periodic force*. For example, consider an engine, of weight W, with an out-of-balance flywheel and supported on a spring as shown diagrammatically in Fig. 407.

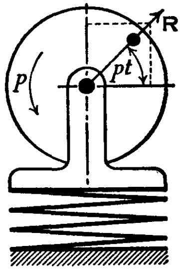


FIG. 407.

Let the vertical periodic disturbing force due to the out-of-balance flywheel be R sin pt, where R is a variable force depending on p^2, p being the speed of rotation in radians per second, then equation (1) becomes

$$\frac{W}{g}\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + R \sin pt,$$

the upward direction being taken as positive, and this equation reduces to

$$\frac{d^2x}{dt^2} + 2c\frac{dx}{dt} + \omega^2 x = \frac{Rg}{W} \sin pt,$$

using the same notation as before.

Now suppose that R, the maximum value of the disturbing force at a particular speed p, would give the spring a deflection a_1 if applied statically, then

$$R = ka_1 \quad \text{and} \quad \frac{Rg}{W} = \frac{kg}{W}a_1 = \omega^2 a_1,$$

therefore
$$\frac{d^2x}{dt^2} + 2c\frac{dx}{dt} + \omega^2 x = \omega^2 a_1 \sin pt,$$

which is the same as equation (2), p. 424.

Since R is a variable depending on p^2, it must be remembered that the statical deflection a_1 is also a variable.

The magnification factor can now be interpreted as the ratio

$$\frac{\text{Amplitude of the mass}}{\text{Deflection due to a statical force R}}.$$

205. Torsional Vibrations—Shaft Carrying One Flywheel.—A shaft held at one end and carrying a flywheel at the other end is shown diagrammatically in Fig. 408. If the flywheel is turned through a small angle θ about OX, the axis of the shaft, and then released, it will oscillate with simple harmonic motion, for the torque in the shaft is proportional to the angle of twist. It will be assumed that the mass of the shaft is negligible compared with the mass of the flywheel, and damping will be neglected. Periodic motion may be superimposed on the system when it is driven at the end O instead of being fixed. In practice, the object is to reduce the oscillations as much as possible.

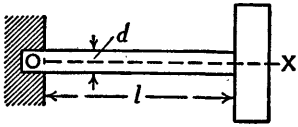


FIG. 408.

Considering the shaft, let l be the length, d the diameter, θ the angle of twist caused by a torque T_q at any time t, then if C is the modulus of rigidity it is known that $T_q = \frac{C\theta}{l}\frac{\pi d^4}{32}$. This torque opposes the motion of the flywheel, and therefore the torque on the flywheel is

$$T_q = -\frac{C\theta}{l}\frac{\pi d^4}{32} \quad . \quad . \quad . \quad (1),$$

the sign being negative because the torque acts towards the unstrained position—that is, when θ is positive the torque is negative and *vice versa*.

Considering the motion of the flywheel,

$$T_q = I\frac{d^2\theta}{dt^2} \quad . \quad . \quad . \quad (2),$$

where I is the moment of inertia of the flywheel about the axis of rotation.

From (1) and (2), $$I\frac{d^2\theta}{dt^2} = -\frac{C\theta}{l}\frac{\pi d^4}{32},$$

or $$\frac{d^2\theta}{dt^2} + \frac{C\pi d^4}{32Il}\theta = 0 \quad . \quad . \quad . \quad (3).$$

Writing ω^2 for $C\pi d^4/32Il$, then

$$\frac{d^2\theta}{dt^2}+\omega^2\theta=0 \quad . \quad . \quad . \quad (4),$$

which is the equation of simple harmonic motion, and the general solution is

$$\theta=A\cos\omega t+B\sin\omega t \quad . \quad . \quad (5).$$

The periodic time is

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{32Il}{C\pi d^4}} \quad . \quad . \quad . \quad (6).$$

This formula may be expressed in another way. Since the torque in the shaft is $C\theta\pi d^4/32l$, therefore, putting $\theta=1$,

$$\text{Torque per unit angle of twist}=C\pi d^4/32l.$$

Therefore

$$T=2\pi\sqrt{\frac{\text{Moment of inertia of flywheel about axis}}{\text{Torque per unit angle of twist}}} \quad . \quad (7).$$

Example.—To find the periodic time from the following data—Weight of flywheel, $W=100$ lb.; radius of gyration of flywheel, $k=8$ in.; $g=32{\cdot}2$ ft./sec.^{2}; $C=11{\cdot}7\times10^6$ lb./in.^{2}; $l=10$ in.; and $d=1{\cdot}5$ in.

The calculations will be made in pound, inch, and second units.

$$I=\frac{W}{g}k^2=\frac{100}{32{\cdot}2\times12}\times8^2=16{\cdot}56\ \frac{\text{lb. in.}^2}{\text{in./sec.}^2}.$$

$$T=2\pi\sqrt{\frac{32Il}{C\pi d^4}}=2\pi\sqrt{\frac{32\times16{\cdot}56\times10}{11{\cdot}7\times10^6\times\pi\times1{\cdot}5^4}}=0{\cdot}0335\text{ sec.}$$

Checking the units:

$$\sqrt{\frac{\text{lb. in.}^2}{\text{in./sec.}^2}\times\frac{1}{\text{lb./in.}^2}\times\frac{\text{in.}}{\text{in.}^4}}=\text{sec.}$$

206. Torsional Vibrations—Shaft Diameter Not Uniform. —The results obtained in the preceding Art. require modification if the diameter of the shaft is not uniform.

As before, the shaft is held at one end and carries a flywheel at the other end, but it will be assumed now that the total length is $l_1 + l_2$, the length l_1 having a diameter d_1 and the length l_2 having a diameter d_2, as shown in Fig. 409.

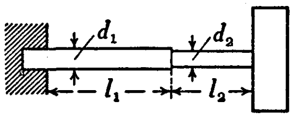


FIG. 409.

Let θ_1 and θ_2 be the angles of twist in the lengths l_1 and l_2 respectively, then the total twist is $\theta = \theta_1 + \theta_2$.

The torque is the same at every section of the shaft and denoting the torque on the flywheel by T_q,

$$T_q = -\frac{C\theta_1}{l_1}\frac{\pi d_1^4}{32} = -\frac{C\theta_2}{l_2}\frac{\pi d_2^4}{32},$$

from which

$$\theta_1 = -T_q\frac{32}{C\pi}\frac{l_1}{d_1^4} \quad \text{and} \quad \theta_2 = -T_q\frac{32}{C\pi}\frac{l_2}{d_2^4},$$

then
$$\theta = \theta_1 + \theta_2 = -T_q\frac{32}{C\pi}\left(\frac{l_1}{d_1^4} + \frac{l_2}{d_2^4}\right),$$

or
$$T_q = -\frac{C\pi\theta}{32\left(\frac{l_1}{d_1^4} + \frac{l_2}{d_2^4}\right)} \quad . \quad . \quad (1).$$

Also
$$T_q = I\frac{d^2\theta}{dt^2} \quad . \quad . \quad . \quad (2).$$

From (1) and (2)

$$\frac{d^2\theta}{dt^2} + \frac{C\pi\theta}{32I\left(\frac{l_1}{d_1^4} + \frac{l_2}{d_2^4}\right)} = 0 \quad . \quad . \quad . \quad (3),$$

and the periodic time is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{32I}{C\pi}\left(\frac{l_1}{d_1^4} + \frac{l_2}{d_2^4}\right)} \quad . \quad . \quad (4).$$

Suppose the shaft is made up of lengths l_1, l_2, l_3, etc., having diameters d_1, d_2, d_3, etc., then, denoting

$$\left(\frac{l_1}{d_1^4} + \frac{l_2}{d_2^4} + \frac{l_3}{d_3^4} + \ . \ . \ .\right) \text{ by } \sum\frac{l}{d^4},$$

the periodic time becomes

$$T=2\pi\sqrt{\frac{32I}{C\pi}\sum\frac{l}{d^4}} \quad . \quad . \quad . \quad (5).$$

207. Torsional Vibrations—Shaft Carrying Two Flywheels. —A shaft of length l and diameter d carries flywheels 1 and 2, one at each end as shown in Fig. 410. It is required to find the periodic time of the torsional vibrations when the flywheels are oscillating in opposite directions.

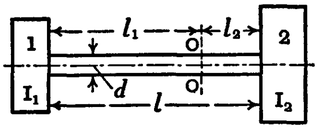


FIG. 410.

Since the flywheels are oscillating in opposite directions, there will be no movement at some section OO and the periodic times of the flywheels will be equal. The centre of the section OO is called a *node*, or the section may be called a *nodal section.* Once the position of OO is known, then either flywheel with its part of the shaft up to OO may be treated as shown in Art. 205.

Let the section OO be l_1 from flywheel 1 and l_2 from flywheel 2, then $l_1+l_2=l$. Let I_1 and I_2 be the moments of inertia of the flywheels about the axis of rotation and let T_1 and T_2 be their periodic times, which, as already stated, are equal.

It was shown in Art. 205 that the periodic time $T=2\pi\sqrt{32Il/C\pi d^4}$, and this formula may be applied to each part of the system.

Therefore $$T_1=2\pi\sqrt{\frac{32I_1l_1}{C\pi d^4}} \quad . \quad . \quad . \quad (1),$$

and $$T_2=2\pi\sqrt{\frac{32I_2l_2}{C\pi d^4}} \quad . \quad . \quad . \quad (2).$$

But $T_1=T_2$, therefore

$$I_1l_1=I_2l_2 \quad . \quad . \quad . \quad (3).$$

Also $$l_1+l_2=l \quad . \quad . \quad . \quad (4).$$

Solving the simultaneous equations (3) and (4) gives

$$l_1=\frac{I_2l}{I_1+I_2} \quad \text{and} \quad l_2=\frac{I_1l}{I_1+I_2}.$$

Substituting the value of l_1 in (1) or the value of l_2 in (2),

$$\text{Periodic time } T_1 = T_2 = 2\pi\sqrt{\frac{32 I_1 I_2 l}{C\pi d^4 (I_1 + I_2)}} \quad . \quad (5).$$

Example.—The shaft shown in Fig. 411 carries two heavy masses at A and B and is driven by a light gear situated at CC. The weight of A is 800 lb. and its radius of gyration is 27 in.; the corresponding values for B are 1200 lb. and 33 in. The shaft diameter between CC and B, marked X in., is undecided. Assuming it to be $3\frac{1}{2}$ in., determine the frequency of free torsional oscillations of the system. Thereafter, determine what X should be if the node of the vibration is to be in the plane, CC, of the drive. Deduce any formula used.

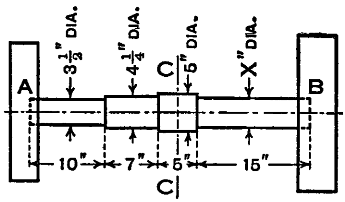


FIG. 411.

Rigidity Modulus $= 12 \times 10^6$ lb. per sq. in. [U.L.]

Taking $X = 3\frac{1}{2}$ in., the first step is to find the position of the node of the vibration. Assuming that the node is in the 15-inch length on the right, let it be at a distance x from the end B of the shaft. Let I_A and I_B be the moments of inertia of the masses about the axis of rotation, and let the mass of the shaft be neglected.

If I_A were equal to I_B and if the shaft were of uniform diameter, the node would be midway between the masses, or 18·5 inches from the end B. But $I_B = \frac{1200}{g} \times 33^2$ being considerably greater than $I_A = \frac{800}{g} \times 27^2$ causes the node to be nearer B than A, and the greater diameters on the 7-inch and 5-inch lengths have a similar effect.

Therefore it seems highly probable that x is less than 15 inches, as already assumed.

$$\text{Periodic time of mass B is } T_B = 2\pi\sqrt{\frac{32 I_B x}{C\pi X^4}} \quad (1).$$

From equation (5) of the preceding Art., it follows that

$$\text{Periodic time of mass A is } T_A = 2\pi\sqrt{\frac{32I_A}{C\pi}\sum\frac{l}{d^4}} \qquad (2).$$

But $\quad T_B = T_A, \quad$ therefore $\quad I_B\dfrac{x}{X^4} = I_A\sum\dfrac{l}{d^4},$

or, substituting numerical values, cancelling g,

$$1200 \times 33^2\frac{x}{3{\cdot}5^4} = 800 \times 27^2\left(\frac{10}{3{\cdot}5^4} + \frac{7}{4{\cdot}25^4} + \frac{5}{5^4} + \frac{15 - x}{3{\cdot}5^4}\right).$$

Multiplying each side by $\dfrac{3{\cdot}5^4}{1200 \times 33^2}$ and simplifying,

$$x = \tfrac{54}{121}(10 + 3{\cdot}22 + 1{\cdot}20 + 15 - x)$$

from which $x = 9{\cdot}08$ inches, and the assumption that this value would be less than 15 inches is correct.

$$\text{Frequency} \qquad f = \frac{1}{T_B} = \frac{1}{2\pi}\sqrt{\frac{C\pi X^4}{32I_B x}}.$$

Substituting numerical values and taking $g = 32{\cdot}2 \times 12$ in./sec.^{2},

$$f = \frac{1}{2\pi}\sqrt{\frac{12 \times 10^6 \times \pi \times 3{\cdot}5^4 \times 32{\cdot}2 \times 12}{32 \times 1200 \times 33^2 \times 9{\cdot}08}} = 12{\cdot}1 \text{ oscillations/sec.}$$

The value of X will now be found so that the node of the vibration may be in the plane CC.

The periodic times of the masses A and B are equal, therefore

$$I_A\sum\frac{l}{d^4}\text{ (from A to CC)} = I_B\sum\frac{l}{d^4}\text{ (from CC to B)}.$$

Substituting numerical values, cancelling g,

$$800 \times 27^2\left(\frac{10}{3{\cdot}5^4} + \frac{7}{4{\cdot}25^4} + \frac{2{\cdot}5}{5^4}\right) = 1200 \times 33^2\left(\frac{2{\cdot}5}{5^4} + \frac{15}{X^4}\right).$$

Simplifying and solving for X, gives $X = 4{\cdot}48$ inches.

208. Torsional Vibrations—Unloaded Shaft Fixed at One End.—Consider a shaft OX (Fig. 412), of diameter d and length l, fixed at O and making torsional vibrations about its longitudinal axis. Let P be a section at a distance x from O, and let Q be a section near to P and at a distance $x+\delta x$ from O.

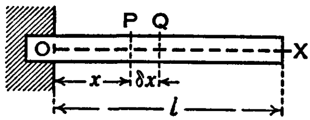


FIG. 412.

When a uniform shaft of length l is twisted through an angle θ by a torque T_q, then $T_q=I_pC\theta/l$, where $I_p=\pi d^4/32$ is the polar moment of inertia, or second moment of area about a perpendicular axis through the centre, of a cross-section and C is the modulus of rigidity, and the torque has the same value at every section.

In the case to be considered the torque varies from section to section on account of the inertia of the vibrating shaft. Let $\delta\theta$ be the angle of twist in the length δx, then $T_q=I_pC\delta\theta/\delta x$, and if δx is made indefinitely small the torque at the section P is ultimately obtained.

Torque at P is $$T_q=I_pC\frac{\partial\theta}{\partial x} \quad . \quad . \quad . \quad (1),$$

the symbols of partial differentiation being used because time t is also a variable.

Torque at Q is

$$T_q+\frac{\partial T_q}{\partial x}\delta x=I_pC\frac{\partial\theta}{\partial x}+I_pC\frac{\partial^2\theta}{\partial x^2}\delta x \quad . \quad . \quad (2),$$

the terms on the right of the equality sign being obtained from (1).

From (1) and (2), by subtraction, the change of torque in the length δx is

$$I_pC\frac{\partial^2\theta}{\partial x^2}\delta x \quad . \quad . \quad . \quad (3).$$

The change of torque given by (3) produces angular acceleration in the mass of length δx and is equal to $I\partial^2\theta/\partial t^2$ or

$$\frac{wI_p\delta x}{g}\frac{\partial^2\theta}{\partial t^2} \quad . \quad . \quad . \quad (4),$$

where w is the weight per unit volume and $I = wI_p\delta x/g$ is the moment of inertia of the mass about the axis of rotation.

Equating (4) to (3) and simplifying, then

$$\frac{\partial^2\theta}{\partial t^2} = a^2\frac{\partial^2\theta}{\partial x^2} \quad . \quad . \quad . \quad (5),$$

where the substitution $a^2 = Cg/w$ is made for convenience.

Suppose that the free end of the shaft is twisted through an angle θ_0 and released, then, measuring time from the instant the shaft is released and assuming that the vibrations are harmonic at every section,

let $$\theta = X \cos \omega t \quad . \quad . \quad . \quad (6),$$

where X is a function of x only.

From (6), by partial differentiation,

$$\frac{\partial^2\theta}{\partial t^2} = -\omega^2 X \cos \omega t = -\omega^2\theta,$$

and $$\frac{\partial^2\theta}{\partial x^2} = \frac{d^2X}{dx^2}\cos \omega t = \frac{d^2X}{dx^2}\,\frac{\theta}{X}.$$

Substituting these values in (5),

$$-\omega^2\theta = a^2\frac{d^2X}{dx^2}\,\frac{\theta}{X},$$

or $$\frac{d^2X}{dx^2} + \frac{\omega^2}{a^2}X = 0 \quad . \quad . \quad . \quad (7).$$

The solution of this equation is

$$X = A \cos \frac{\omega}{a}x + B \sin \frac{\omega}{a}x,$$

and substituting this value of X in (6), then

$$\theta = \left(A \cos \frac{\omega}{a}x + B \sin \frac{\omega}{a}x\right) \cos \omega t \quad . \quad (8)$$

is the solution of (5).

The values of the arbitrary constants A and B will now be determined.

At the fixed end, $x = 0$ and $\theta = 0$ for all values of t,

therefore $0 = (A + 0) \cos \omega t$, from which $A = 0$.

Putting $A=0$ in (8),

$$\theta = B \sin \frac{\omega}{a}x \cos \omega t \quad . \quad . \quad . \quad (9).$$

At the free end,

$$x=l \quad \text{and} \quad \theta=\theta_0 \quad \text{when} \quad t=0,$$

therefore

$$\theta_0 = B \sin \frac{\omega}{a}l, \quad \text{from which} \quad B=\theta_0/\sin \frac{\omega}{a}l.$$

Substituting this value of B in (9), then

$$\theta=\theta_0 \frac{\sin \frac{\omega}{a}x}{\sin \frac{\omega}{a}l} \cos \omega t \quad . \quad . \quad . \quad (10).$$

Differentiating, $\dfrac{\partial\theta}{\partial x} = \dfrac{\omega}{a}\theta_0 \dfrac{\cos \frac{\omega}{a}x}{\sin \frac{\omega}{a}l} \cos \omega t.$

Now at the free end of the shaft the torque is always zero, so that when $x=l$, $\partial\theta/\partial x=0$ for all values of t, therefore

$$0=\frac{\omega}{a}\theta_0 \cot \frac{\omega}{a}l \cos \omega t, \quad \text{from which} \quad \cot \frac{\omega}{a}l=0,$$

and so $\qquad \dfrac{\omega}{a}l=\dfrac{\pi}{2}, \quad 3\dfrac{\pi}{2}, \quad 5\dfrac{\pi}{2}, \quad$ etc.,

and the fundamental vibration is given by putting $\dfrac{\omega}{a}l=\dfrac{\pi}{2}$, or $\sin \dfrac{\omega}{a}l=1$, in (10).

Therefore $\qquad \theta=\theta_0 \sin \dfrac{\omega}{a}x \cos \omega t,$

or

$$\theta=\theta_0 \sin \frac{\pi x}{2l} \cos \frac{\pi a}{2l}t \quad . \quad . \quad . \quad (11).$$

The periodic time of this vibration is

$$T=2\pi \Big/ \frac{\pi a}{2l} = \frac{4l}{a} = 4l\sqrt{\frac{w}{Cg}}.$$

The frequency is $f=\frac{1}{T}=\frac{1}{4l}\sqrt{\frac{Cg}{w}}.$

As a numerical example suppose the shaft is 5 ft. long, and take $C=11{\cdot}7\times10^6$ lb./in.^{2}, $w=490$ lb./ft.^{3} (for steel), and $g=32{\cdot}2$ ft./sec.^{2}.

Working in lb., ft., and sec. units, then

$$f=\frac{1}{4\times5}\sqrt{\frac{11{\cdot}7\times10^6\times144\times32{\cdot}2}{490}}=526 \text{ oscillations/sec.,}$$

which is seen to be very high. It will be noticed that the diameter of the shaft does not enter into the calculations, and provided the cross-section is circular, the shaft may be solid or hollow.

209. Transverse Vibrations of a Beam Carrying One Concentrated Load.—If a beam is deflected and suddenly released it will make transverse vibrations. The simplest case occurs when the beam carries a concentrated load of such magnitude that the mass of the beam may be neglected. A particular example is illustrated in Fig. 413. The deflection of the beam is proportional to the load W, and if the beam is deflected beyond the position of statical equilibrium and released, then, disregarding damping, the load will vibrate with simple harmonic motion, as did the load supported by a helical spring (Art. 202, p. 415). Since the beam acts as a spring the mathematics is the same in each case.

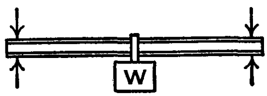


FIG. 413.

$$\text{Periodic time } T=2\pi\sqrt{\frac{\delta}{g}},$$

where δ is the statical deflection due to the load W.

$$\text{Frequency } f=\frac{1}{T}=\frac{1}{2\pi}\sqrt{\frac{g}{\delta}}.$$

The formulæ are not strictly correct if, as the beam vibrates, the load turns about an axis perpendicular to the plane of vibration, as shown diagrammatically in

Fig. 414, where the two short lines, one continuous and one dotted, indicate two positions of the load. However, the angular motion and the rotational energy involved are usually small and may be neglected.

FIG. 414.

Example. — To find the natural frequency of vibration when a load of 1500 lb. is carried at the centre of a beam, 20 ft. long, simply supported at each end. The statical deflection for a central load W is $Wl^3/48EI$ and it is given that $I = 122{\cdot}3$ in.^{4} and $E = 30 \times 10^6$ lb./in.^{2}.

$$\text{Frequency } f = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{1}{2\pi}\sqrt{\frac{48EIg}{Wl^3}}.$$

Using lb., inch, and sec. units,

$$f = \frac{1}{2\pi}\sqrt{\frac{48 \times 30 \times 10^6 \times 122{\cdot}3 \times 32{\cdot}2 \times 12}{1500 \times 20^3 \times 12^3}}$$

$$= 9{\cdot}12 \text{ oscillations/sec.}$$

210. Transverse Vibrations of a Beam Carrying More than One Concentrated Load.—An approximate method which may be applied to a beam carrying any number of concentrated loads will now be considered. A particular example is illustrated in Fig. 415, where a beam carries loads W_1 and W_2 and the statical deflections at these loads are δ_1 and δ_2 respectively.

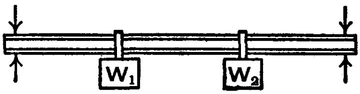


FIG. 415.

It will be assumed that the weight of the beam may be neglected, that the loads vibrate with simple harmonic motion of the same frequency, that the amplitude of the motion of each load is r times its statical deflection, and that each passes through its position of statical equilibrium at the same time.

The method consists of applying the energy equation (equation (3), p. 418) and equating the sum of the kinetic energies of the moving loads at their mean positions to the sum of the energy terms $\frac{1}{2}kx^2$ when each load is at its maximum deflected position. The rotational movement of each load about an axis perpendicular to the plane of vibration will be neglected.

First consider one load W having a statical deflection δ and therefore, by assumption, an amplitude $r\delta$. Since deflection is proportional to load, it follows that a deflection $r\delta$ could be produced by a force rW. In Fig. 416 the load W is represented diagrammatically in its mean position by a black dot and the amplitude $r\delta$ is shown greatly magnified. As indicated in the Fig., if a radius OA $=r\delta$ rotates with an angular velocity ω rad./sec., then the projection P of A on the diameter YOY′ will have a maximum velocity $v=\omega r\delta$ when it is at the midpoint O, and this is the maximum velocity of the load W.

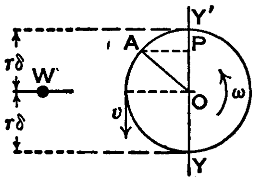


FIG. 416.

The kinetic energy of the load W when in its mid-position is

$$\frac{W}{2g}v^2=\frac{W}{2g}\omega^2r^2\delta^2.$$

When the load is in an extreme position the energy term $\frac{1}{2}kx^2$ in equation (3), p. 418, becomes $\frac{1}{2}(rW/r\delta)(r\delta)^2$ or $\frac{1}{2}Wr^2\delta$.

Equating these energies,

$$\frac{W}{2g}\omega^2r^2\delta^2=\tfrac{1}{2}Wr^2\delta,$$

from which

$$\omega=\sqrt{\frac{gW\delta}{W\delta^2}}\text{ rad./sec.,}$$

and for one load this leads to the results obtained in the preceding Art.

For a number of loads the frequency becomes

$$f=\frac{\omega}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{g\Sigma W\delta}{\Sigma W\delta^2}}\text{ oscillations/sec.}$$

For the particular example shown in Fig. 415,

$$f=\frac{1}{2\pi}\sqrt{\frac{g(W_1\delta_1+W_2\delta_2)}{W_1\delta_1^2+W_2\delta_2^2}}\text{ oscillations/sec.}$$

In applying this method to a numerical example, the major part of the work often depends on finding the values of the deflections under the loads. Another way of solving this problem is by Dunkerley's formula given in Art. 213 on the Whirling of Shafts.

211. Transverse Vibrations of an Unloaded Beam Simply Supported at Each End.—Consider a uniform beam of length l, weighing w lb. per unit length, simply supported at each end (Fig. 417 (a)). When the beam is vibrating let y be the deflection from the mean position, at time t, of any point on the centre line of the beam at a distance x from one end, as shown at (b) where the deflection is exaggerated. The deflection of the beam due to its weight need not be considered, and it is to be understood that the deflection y is entirely due to vibration. It will be supposed that the vibration is caused by a force, or series of forces, deflecting the beam and then being removed suddenly. Time t will be measured from the instant when the applied forces are removed.

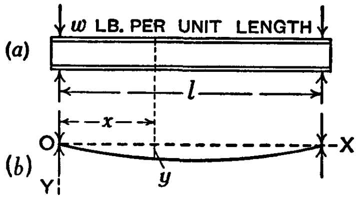


FIG. 417.

The inertia force on a length δx at time t is $\dfrac{w\delta x}{g}\dfrac{\partial^2 y}{\partial t^2}$.

The elastic force on the same length depends on the deflection and can be shown to be $-\mathrm{EI}\dfrac{\partial^4 y}{\partial x^4}\delta x$, the sign being negative because the elastic force is always directed towards the mean position. The constant E is the modulus of elasticity of the material, and I is the moment of inertia or second moment of area, about the neutral axis, of a cross-section of the beam.

Equating the elastic and inertia forces and dividing by δx,

$$-\mathrm{EI}\frac{\partial^4 y}{\partial x^4}=\frac{w}{g}\frac{\partial^2 y}{\partial t^2},$$

or
$$\frac{\partial^4 y}{\partial x^4}+\frac{w}{g\mathrm{EI}}\frac{\partial^2 y}{\partial t^2}=0 \quad . \quad . \quad . \quad (1).$$

Assume that every point on the beam moves with harmonic motion and let

$$y=\mathrm{X}\cos \omega t \quad . \quad . \quad . \quad (2),$$

where X is a function of x only, ω is in rad./sec., and t is time in sec., then from (2), by partial differentiation,

$$\frac{\partial^4 y}{\partial x^4}=\frac{d^4\mathrm{X}}{dx^4}\cos \omega t \quad \text{and} \quad \frac{\partial^2 y}{\partial t^2}=-\omega^2\mathrm{X}\cos \omega t.$$

Substituting these values in (1),

$$\frac{d^4\mathrm{X}}{dx^4}\cos \omega t-\frac{w}{g\mathrm{EI}}\omega^2\mathrm{X}\cos \omega t=0,$$

then, dividing by $\cos \omega t$ and putting $(w/g\mathrm{EI})\omega^2=a^4$ for convenience in the work which follows,

$$\frac{d^4\mathrm{X}}{dx^4}-a^4\mathrm{X}=0 \quad . \quad . \quad . \quad (3).$$

The general solution of this equation * is

$$\mathrm{X}=\mathrm{A}\cosh ax+\mathrm{B}\sinh ax+\mathrm{C}\cos ax+\mathrm{D}\sin ax \quad (4).$$

Substituting this value of X in (2),

$$y=(\mathrm{A}\cosh ax+\mathrm{B}\sinh ax+\mathrm{C}\cos ax+\mathrm{D}\sin ax)\cos \omega t \quad (5),$$

and the arbitrary constants A, B, C, and D may be found by considering the following conditions, which are true for all values of t: y and $\frac{\partial^2 y}{\partial x^2}$ are zero at each end of the beam, $\frac{\partial^2 y}{\partial x^2}$ being proportional to the bending moment.

Differentiating (5) twice with respect to x,

$$\frac{\partial^2 y}{\partial x^2}=a^2(\mathrm{A}\cosh ax+\mathrm{B}\sinh ax-\mathrm{C}\cos ax-\mathrm{D}\sin ax)\cos \omega t \quad (6).$$

* See the author's *Mathematics* or any book on Differential Equations.

Putting $x=0$ and $y=0$ in (5),

$$0=A+C \quad . \quad . \quad . \quad (7).$$

Putting $x=0$ and $\frac{\partial^2 y}{\partial x^2}=0$ in (6),

$$0=A-C \quad . \quad . \quad . \quad (8).$$

From (7) and (8), by addition and subtraction, it follows that $A=0$ and $C=0$.

Putting $x=l$, $y=0$, and $A=C=0$ in (5),

$$0=B \sinh al + D \sin al \quad . \quad . \quad (9).$$

Putting $x=l$, $\frac{\partial^2 y}{\partial x^2}=0$, and $A=C=0$ in (6),

$$0=B \sinh al - D \sin al \quad . \quad . \quad (10).$$

From (9) and (10), by addition,

$$2B \sinh al=0.$$

Either $B=0$ or $\sinh al=0$. If $\sinh al=0$ then $al=0$. But al cannot be zero when the beam is vibrating because $(w/gEI)\omega^2=a^4$ and ω is not zero; therefore $B=0$.

Equation (5) now reduces to

$$y=D \sin ax \cos \omega t \quad . \quad . \quad (11).$$

From (9) and (10), by subtraction,

$$2D \sin al=0.$$

Either $D=0$ or $\sin al=0$. If $D=0$, then y is always zero in (11) and this means that the beam is not vibrating. Therefore $\sin al=0$.

Therefore $\qquad al=\pi,\ 2\pi,\ 3\pi$, etc.,

or $\qquad a=\pi/l,\ 2\pi/l,\ 3\pi/l$, etc.

But $\qquad a^4=\frac{w}{gEI}\omega^2$ from which $\omega=a^2\sqrt{\frac{gEI}{w}}$.

Therefore ω may have the following series of values:

$$\omega_1=\frac{\pi^2}{l^2}\sqrt{\frac{gEI}{w}}, \qquad \omega_2=4\frac{\pi^2}{l^2}\sqrt{\frac{gEI}{w}}=4\omega_1,$$

$$\omega_3=9\frac{\pi^2}{l^2}\sqrt{\frac{gEI}{w}}=9\omega_1, \quad \text{and so on.}$$

Substituting the lowest values of a and ω in (11),

$$y = \mathrm{D} \sin \frac{\pi x}{l} \cos \frac{\pi^2}{l^2}\sqrt{\frac{g\mathrm{EI}}{w}}t \quad . \quad . \quad (12),$$

and this gives the fundamental or first mode of vibration. The first three modes of vibration are shown in Fig. **418.**

The fundamental periodic time is

$$\mathrm{T} = \frac{2\pi}{\omega_1} = \frac{2l^2}{\pi}\sqrt{\frac{w}{g\mathrm{EI}}} \text{ sec.}$$

The fundamental frequency is

$$f = \frac{1}{\mathrm{T}} = \frac{\pi}{2l^2}\sqrt{\frac{g\mathrm{EI}}{w}} \text{ oscillations/sec.}$$

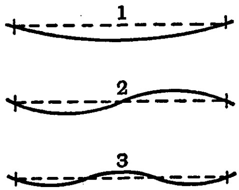


FIG. 418.

Example.—Given $l = 10$ ft., $w = 9$ lb. per ft., $\mathrm{I} = 10{\cdot}91$ in.^{4}, and $\mathrm{E} = 30 \times 10^6$ lb./in.^{2}, to find the fundamental frequency.

Working in lb., ft., and sec. units,

$$f = \frac{\pi}{2 \times 10^2}\sqrt{\frac{32{\cdot}2 \times 30 \times 10^6 \times 12^2 \times 10{\cdot}91}{9 \times 12^4}}$$

$$= 44{\cdot}8 \text{ oscillations/sec.}$$

212. Whirling of Shafts—Single Loads.—When a shaft rotates there are various speeds at which it deflects, and these deflections become dangerous unless the speed of rotation is quickly altered. These particular speeds are called *critical speeds* or *whirling speeds.* When a shaft *whirls,* the centrifugal forces (see Art. 43, p. 73) just exceed the elastic righting forces in the shaft itself, and the whirling speeds may be found by equating these forces. The simplest case occurs when a shaft carries a flywheel which is heavy compared with the weight of the shaft, then, neglecting the latter weight, the shaft will have one whirling speed.

A particular case is shown in Fig. 419, where a flywheel of weight W is secured to the centre of a shaft, of length l, which is supported at each end in a short bearing. A short bearing

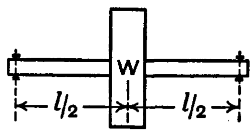


FIG. 419.

will be interpreted as one which allows the shaft to deflect as if freely supported.

Let ω be the whirling speed, let y be the deflection of the flywheel from the position of statical equilibrium, and let k be the force which would cause unit deflection.

Now deflection is proportional to load, and a deflection y would be caused by a force ky and this is the elastic force in the shaft. The centrifugal force is $\frac{W}{g}\omega^2 y$.

Equating these forces,

$$\frac{W}{g}\omega^2 y = ky,$$

from which

$$\omega = \sqrt{\frac{kg}{W}}.$$

This result may be written as

$$\omega = \sqrt{\frac{g}{\delta}},$$

where $\delta = W/k$ is the static deflection due to the load W. If g is in in./sec.^{2} and δ is in inches, then ω will be in rad./sec.

The formula may also be expressed as

$$\text{Whirling speed} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} \text{ rev./sec.},$$

and it will be noticed that this is the same form as that obtained in Art. 209 for the transverse frequency of a beam carrying one load.

Since shaft speeds are generally reckoned in revolutions per minute, it is more convenient to give the whirling speed as

$$N = \frac{30}{\pi}\sqrt{\frac{g}{\delta}} \text{ rev./min.}$$

The formula may be applied to any shaft carrying one flywheel which is heavy compared with the weight of the shaft.

For the case shown in Fig. 419, the static deflection of the shaft at the central load W is $\delta = Wl^3/48EI$, where E is the modulus of elasticity and I is the moment of inertia, about a diameter, of a cross-section of the shaft.

Example.—Suppose $W = 500$ lb., $l = 5$ ft., $E = 30 \times 10^6$ lb./in.², and the diameter of the shaft is $d = 2$ in., then $I = \pi d^4/64 = \pi/4$ in.⁴,

$$\delta = \frac{Wl^3}{48EI} = \frac{500 \times 5^3 \times 12^3 \times 4}{48 \times 30 \times 10^6 \times \pi} = 0{\cdot}0955 \text{ in.},$$

and $$N = \frac{30}{\pi}\sqrt{\frac{g}{\delta}} = \frac{30}{\pi}\sqrt{\frac{32{\cdot}2 \times 12}{0{\cdot}0955}} = 607 \text{ rev./min.}$$

213. Whirling of Shafts—More than One Load.—It has been shown in the preceding Art. that the whirling speed of a shaft carrying one load has the same numerical value as the natural frequency of transverse vibrations. This also applies to shafts carrying more than one load and to unloaded shafts. Therefore, referring to Art. 210, the first whirling speed of a shaft carrying several loads may be given, approximately, as

$$\text{Whirling speed} = \frac{1}{2\pi}\sqrt{\frac{g\Sigma W\delta}{\Sigma W\delta^2}} \text{ rev./sec.},$$

or $$N = \frac{30}{\pi}\sqrt{\frac{g\Sigma W\delta}{\Sigma W\delta^2}} \text{ rev./min.} \quad . \quad . \quad (1).$$

There is also an empirical formula due to Dunkerley which may be used. Suppose a shaft carries loads W_1, W_2, W_3, etc., and let N_1, N_2, N_3, etc., be the corresponding whirling speeds when the loads are considered one at a time. Also let N be the whirling speed when the shaft carries all the loads, then, according to Dunkerley's formula,

$$\frac{1}{N^2} = \frac{1}{N_1^2} + \frac{1}{N_2^2} + \frac{1}{N_3^2} + \ldots \quad . \quad . \quad (2).$$

The two methods are illustrated by the example which follows.

Example.—To find the whirling speed in revolutions per minute of a steel shaft which is loaded and supported in short bearings as shown in Fig. 420, where the linear dimensions are in inches. The weight of the shaft is to be neglected.

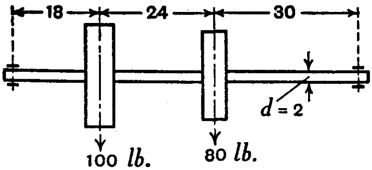


FIG. 420.

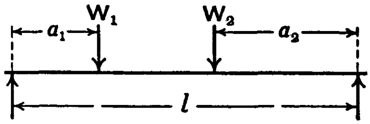


FIG. 421.

First Method.—The static deflection of the shaft is required at each load when both loads are acting. In many complicated examples these deflections have to be found graphically, but in the present case formulæ will be used.

Referring to Fig. 421 and denoting the deflections under the loads W_1 and W_2 by δ_1 and δ_2 respectively, it can be shown for a shaft with freely supported ends that

$$\delta_1 = \frac{a_1}{6EIl}\{2a_1W_1(l-a_1)^2 + a_2W_2(l^2 - a_1^2 - a_2^2)\},$$

and then, from symmetry, interchanging the suffixes 1 and 2, it follows that

$$\delta_2 = \frac{a_2}{6EIl}\{2a_2W_2(l-a_2)^2 + a_1W_1(l^2 - a_1^2 - a_2^2)\}.$$

The value of

$$I = \pi d^4/64 = \pi \times 2^4/64 = \pi/4 \text{ in.}^4 \quad \text{and} \quad E = 30 \times 10^6 \text{ lb./in.}^2.$$

Substituting numerical values, working in lb. and inch units,

$$l^2 - a_1^2 - a_2^2 = 72^2 - 18^2 - 30^2 = 3960,$$

$$\delta_1 = \frac{18 \times 4}{6 \times 30 \times 10^6 \times \pi \times 72}\{2 \times 18 \times 100 \times 54^2 + 30 \times 80 \times 3960\}$$

$$= 0{\cdot}0354 \text{ in.}$$

and

$$\delta_2=\frac{30\times 4}{6\times 30\times 10^6\times\pi\times 72}\{2\times 30\times 80\times 42^2+18\times 100\times 3960\}$$

$$=0{\cdot}0460 \text{ in.}$$

Whirling speed

$$N=\frac{30}{\pi}\sqrt{\frac{g\Sigma W\delta}{\Sigma W\delta^2}}=\frac{30}{\pi}\sqrt{\frac{g(W_1\delta_1+W_2\delta_2)}{W_1\delta_1^2+W_2\delta_2^2}} \text{ rev./min.}$$

$$=\frac{30}{\pi}\sqrt{\frac{32{\cdot}2\times 12(100\times 0{\cdot}0354+80\times 0{\cdot}0460)}{100\times 0{\cdot}0354^2+80\times 0{\cdot}0460^2}}$$

$$=929 \text{ rev./min.}$$

Second Method.—When using this method the loads are considered one at a time and a formula is required for the deflection, under the load, of a shaft carrying one load.

In Fig. 422 (A) a load W is carried on a shaft of length $a+b$ at a distance a from one end and the shaft is freely supported at each end. Denoting the deflection under the load by δ, it can be shown that

$$\delta=\frac{Wa^2b^2}{3(a+b)EI},$$

and, as before, $E=30\times 10^6$ lb./in.^{2} and $I=\pi/4$ in.^{4}.

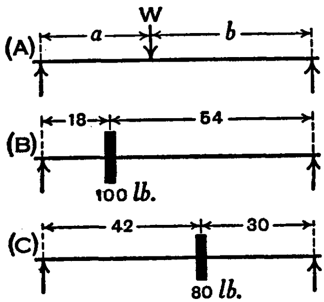


FIG. 422.

Considering the 100-lb. load (Fig. 422 (B)) and denoting the deflection under it by δ_1, then substituting the numerical values in the formula,

$$\delta_1=\frac{Wa^2b^2}{3(a+b)EI}=\frac{100\times 18^2\times 54^2\times 4}{3\times 72\times 30\times 10^6\times\pi}=0{\cdot}0186 \text{ in.}$$

If N_1 is the whirling speed when the shaft carries the 100-lb. load, then

$$N_1=\frac{30}{\pi}\sqrt{\frac{g}{\delta_1}}=\frac{30}{\pi}\sqrt{\frac{32{\cdot}2\times 12}{0{\cdot}0186}}=1380 \text{ rev./min.}$$

Considering the 80-lb. load (Fig. 422 (C)) and denoting the deflection under it by δ_2, then substituting numerical values in the formula,

$$\delta_2 = \frac{Wa^2b^2}{3(a+b)EI} = \frac{80 \times 42^2 \times 30^2 \times 4}{3 \times 72 \times 30 \times 10^6 \times \pi} = 0{\cdot}0250 \text{ in.}$$

If N_2 is the whirling speed when the shaft carries the 80-lb. load, then

$$N_2 = \frac{30}{\pi}\sqrt{\frac{g}{\delta_2}} = \frac{30}{\pi}\sqrt{\frac{32{\cdot}2 \times 12}{0{\cdot}0250}} = 1190 \text{ rev./min.}$$

The whirling speed N when the shaft carries both loads is given by

$$\frac{1}{N^2} = \frac{1}{N_1^2} + \frac{1}{N_2^2} = \frac{1}{1380^2} + \frac{1}{1190^2},$$

from which

$$N = \sqrt{\frac{1380^2 \times 1190^2}{1380^2 + 1190^2}} = 901 \text{ rev./min.}$$

and this result is about 3 per cent. lower than that obtained by the first method.

214. Whirling Speeds of an Unloaded Shaft with a Short Bearing at Each End.—Theoretically, an unloaded shaft will whirl at an infinite number of speeds, but in practice it is not possible to run the shaft fast enough to reach more than a few of these critical speeds, and in many cases it is only the first whirling speed which need be considered.

In Fig. 423 an unloaded shaft, of length l and of uniform diameter d, is shown supported at each end in a short bearing. As previously explained, it will be assumed that a short bearing allows a shaft to deflect as if it were simply supported.

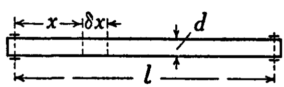


Fig. 423.

Let y be the deflection at a distance x from one end,
E the modulus of elasticity of the material of the shaft,
I the moment of inertia of a cross-section about its neutral axis,
w the weight of the shaft per unit length,
ω the whirling speed,
and g the acceleration due to gravity.

The whirling speed may be found by equating the elastic force and the centrifugal force acting on a length δx. The elastic force is $\mathrm{EI}\frac{d^4y}{dx^4}\delta x$ and the centrifugal force is $\frac{w\delta x}{g}\omega^2 y$, therefore

$$\mathrm{EI}\frac{d^4y}{dx^4}=\frac{w}{g}\omega^2 y,$$

or

$$\frac{d^4y}{dx^4}-\frac{w\omega^2}{g\mathrm{EI}}y=0 \quad . \quad . \quad . \quad (1).$$

Denoting $\frac{w\omega^2}{g\mathrm{EI}}$ by a^4, for convenience in the work which follows, then

$$\frac{d^4y}{dx^4}-a^4y=0 \quad . \quad . \quad . \quad (2),$$

which is the same form as equation (3), Art. 211, obtained when considering the transverse vibrations of a beam, and the solution is

$$y=\mathrm{A}\cosh ax+\mathrm{B}\sinh ax+\mathrm{C}\cos ax+\mathrm{D}\sin ax \qquad (3).$$

The arbitrary constants A, B, C, and D are found as in Art. 211; y and $\frac{d^2y}{dx^2}$ are zero at each end of the shaft, $\frac{d^2y}{dx^2}$ being proportional to the bending moment.

Differentiating (3) twice,

$$\frac{d^2y}{dx^2}=a^2(\mathrm{A}\cosh ax+\mathrm{B}\sinh ax-\mathrm{C}\cos ax-\mathrm{D}\sin ax) \quad (4).$$

Putting $x=0$ and $y=0$ in (3),

$$0=A+C \qquad . \quad . \quad . \quad (5).$$

Putting $x=0$ and $\dfrac{d^2y}{dx^2}=0$ in (4),

$$0=A-C \qquad . \quad . \quad . \quad (6).$$

From (5) and (6), by addition and subtraction, it follows that $A=0$ and $C=0$.

Putting $x=l$, $y=0$, and $A=C=0$ in (3),

$$0=B\sinh al+D\sin al \qquad . \quad . \quad (7).$$

Putting $x=l$, $\dfrac{d^2y}{dx^2}=0$, and $A=C=0$ in (4),

$$0=B\sinh al-D\sin al \qquad . \quad . \quad (8).$$

From (7) and (8), by addition,

$$2B\sinh al=0.$$

Either $B=0$ or $\sinh al=0$. If $\sinh al=0$, then $al=0$, but l cannot be zero and a cannot be zero when the shaft is whirling, therefore $B=0$.

Equation (3) now reduces to

$$y=D\sin ax \qquad . \quad . \quad . \quad (9).$$

From (7) and (8), by subtraction,

$$2D\sin al=0.$$

Either $D=0$ or $\sin al=0$. If $D=0$, then y is zero in (9) and this is impossible for the shaft is whirling, therefore $\sin al=0$.

Therefore $$al=\pi,\ 2\pi,\ 3\pi,\ \text{etc.},$$

and $$a=\pi/l,\ 2\pi/l,\ 3\pi/l,\ \text{etc.}$$

But $$a^4=\frac{w\omega^2}{gEI}, \quad \text{from which} \quad \omega=a^2\sqrt{\frac{gEI}{w}}.$$

Therefore ω may have the following values:—

$$\omega_1=\frac{\pi^2}{l^2}\sqrt{\frac{gEI}{w}}, \qquad \omega_2=4\frac{\pi^2}{l^2}\sqrt{\frac{gEI}{w}}=4\omega_1,$$

$$\omega_3=9\frac{\pi^2}{l^2}\sqrt{\frac{gEI}{w}}=9\omega_1,$$

and so on, the units being rad./sec.

Expressing these whirling speeds in rev./min., then

$$N_1 = \frac{60}{2\pi}\omega_1 = 30\frac{\pi}{l^2}\sqrt{\frac{gEI}{w}}, \qquad N_2 = 4N_1, \qquad N_3 = 9N_1,$$

and so on.

Substituting values of a in (9), then

$$y = D \sin\frac{\pi x}{l}, \qquad y = D \sin\frac{2\pi x}{l}, \qquad y = D \sin\frac{3\pi x}{l},$$

and so on. From these equations it is evident that when the shaft whirls it will bend as shown in Fig. 418, p. 443, where 1, 2, and 3 now indicate the first, second, and third whirling speeds respectively.

Example 1.—To find the first and second whirling speeds of a steel shaft 6 feet long, $\frac{3}{4}$ inch diameter, supported in a short bearing at each end. Assume that a cubic inch of steel weighs 0·28 lb. and take $E = 30 \times 10^6$ lb./in.^{2}.

The weight of an inch length of shaft of diameter d inches is

$$w = 0{\cdot}28\frac{\pi}{4}d^2 = 0{\cdot}07\pi d^2 \text{ lb.}$$

Also $g = 32{\cdot}2 \times 12$ in./sec.^{2} and $I = \pi d^4/64$.

Substituting numerical values in the formula

$$N_1 = 30\frac{\pi}{l^2}\sqrt{\frac{gEI}{w}},$$

then

$$N_1 = 30\frac{\pi}{l^2}\sqrt{\frac{32{\cdot}2 \times 12 \times 30 \times 10^6}{0{\cdot}07\pi d^2} \times \frac{\pi d^4}{64}} = 4{\cdot}8 \times 10^6\frac{d}{l^2} \text{ rev./min.}$$

where d and l are in inches.

Putting $d = \frac{3}{4}$ and $l = 72$, then

$$N_1 = 4{\cdot}8 \times 10^6 \times \tfrac{3}{4} \times \frac{1}{72^2} = 694 \text{ rev./min.}$$

The second whirling speed is

$$N_2 = 4N_1 = 4 \times 694 = 2776 \text{ rev./min.}$$

Since the value of E was given to two figures only, these results are probably correct to two figures only.

Example 2.—To find the first whirling speed of the shaft described in Ex. 1 if a mass weighing 5 lb. is attached to it midway between the bearings.

First suppose the shaft to be weightless, then the whirling speed due to the 5-lb. load is given by the formula

$$N = \frac{30}{\pi}\sqrt{\frac{g}{\delta}} \text{ rev./min.} \quad \text{(Art. 212)},$$

where, in this case,

$$\delta = \frac{Wl^3}{48EI} = \frac{Wl^3}{48E} \times \frac{64}{\pi d^4}.$$

Substituting numerical values, using lb. and inch units,

$$\delta = \frac{5 \times 72^3}{48 \times 30 \times 10^6} \times \frac{64 \times 4^4}{\pi \times 3^4} = 0{\cdot}0834 \text{ in.},$$

and $$N = \frac{30}{\pi}\sqrt{\frac{32{\cdot}2 \times 12}{0{\cdot}0834}} = 650 \text{ rev./min.}$$

The first whirling speed of the shaft due to its own weight when there is no other load was found in Ex. 1 to be $N_1 = 694$ rev./min.

The two results may be combined now by Dunkerley's empirical formula given in Art. 213. Denoting the whirling speed by N_c when the 5-lb. load and the weight of the shaft are both taken into account, then

$$\frac{1}{N_c^2} = \frac{1}{N^2} + \frac{1}{N_1^2} = \frac{1}{650^2} + \frac{1}{694^2},$$

from which $$N_c = 474 \text{ rev./min.}$$

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Mechanics Applied to Vibrations and Balancing. By D. Laugharne Thornton. Chapman & Hall.
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Exercises XVIII

When required, assume, unless otherwise stated, that for steel $E = 30 \times 10^6$ *lb./in.*^{2} *and* $C = 11{\cdot}7 \times 10^6$ *lb./in.*^{2}, *and take* $g = 32{\cdot}2$ *ft./sec.*^{2}.

1. A light helical spring, which stretches 1 inch with a load of 2 pounds, is fixed at its upper end. From the lower end hangs another light helical spring which stretches 1 inch with a load of $1\frac{1}{4}$ pounds. To the lower end of the latter spring is attached a body weighing 5 pounds.

Find the period of an oscillation of the body along the axis of the springs. Prove the formula you use. [B.E.]

2. It is found that a given mass, suspended from a fixed point by a light helical spring, falls a distance of 2 inches before rising if it is carefully attached, without stretching the spring, and is then allowed to fall under gravity. Form the equation of motion and deduce the number of vibrations per minute.

The mass when suspended from a second light spring makes 300 vibrations per minute. If the two springs are now joined end on and the mass again suspended, how many vibrations per minute will be made?

If, when the mass is at rest, an equal mass be dropped on to it from a height of one foot, without rebound, find how far the two masses will descend before coming momentarily to rest.

[B.E.]

3. A body weighing 5 lb. vibrates with a frequency of 4·9 oscillations per second in a vacuum and a frequency of 4·8 oscillations per second when the motion is damped by a resistance which is proportional to the velocity. Find the resistance per unit velocity.

4. A weight of 20 lb. is attached to a spring requiring 10 lb. to extend it by 1 in. The weight is deflected 6 in. from its equilibrium position and is then allowed to vibrate under the action of a periodic force 60 cos $\frac{\pi}{5}t$ lb., where t is measured from the instant of release.

Draw a curve showing its position at each instant during the first 3 seconds, (*a*) if vibrating in air, (*b*) if vibrating in a very viscous fluid which offers a resistance to motion equal to 10 lb. when the velocity is 1 ft. per second. [U.L.]

5. A body of mass 80·5 lb. has a simple harmonic motion under a force of 31·25 lb. per foot displacement. Find the period.

If the body be retarded by a frictional force proportional to the speed, and equal to 3·75 lb. per ft./sec., find the new period and the log. dec. of the amplitude.

If, further, there is in action a force R sin pt, of period 2 sec., find R in order that the amplitude of the forced oscillation may be 0·5 ft. [B.E.]

6. Show that the inclination θ to the vertical, at any time t, of a simple pendulum of length l making small oscillations in a medium in which the resistance per unit mass is k times the linear velocity is given by

$$l\frac{d^2\theta}{dt^2}+kl\frac{d\theta}{dt}+g\theta=0.$$

The time of a complete oscillation of a pendulum making small oscillations *in vacuo* is 2 seconds; if the angular retardation due to the air is 0·04 times the angular velocity of the pendulum, and the initial amplitude is 1°, show that in ten complete oscillations the amplitude will be reduced to approximately 40′. [U.L.]

7. A flywheel weighing 55 lb. is suspended by a steel wire 10 feet long and of 0·144 inch diameter (Fig. 424). It is found by experiment that the flywheel makes 20 complete torsional oscillations in 147 seconds. Find the radius of gyration of the flywheel.

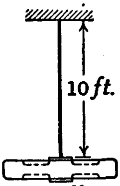

FIG. 424.

8. The crankshaft of an engine carries two disc flywheels 4 feet diameter and each weighing 1 ton. These are mounted 3 feet 6 inches apart on the shaft, which is 4 inches diameter. Determine the time of natural torsional oscillation of the system. Take the modulus of rigidity of the material as $12{\cdot}5\times10^6$ lb. per sq. inch. [U.L.]

9. The shaft shown in Fig. 425 is fixed at one end and carries at the other end a flywheel which weighs 50 lb. and has a radius of gyration of $7\frac{1}{2}$ inches. The diameters and lengths indicated are in inches. Find the natural frequency of torsional oscillation of the system.

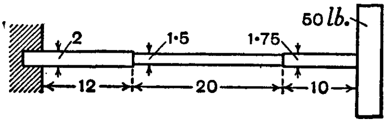

FIG. 425.

10. A shaft, of length l and circular cross-section, tapers uniformly from a diameter d_1 to a diameter d_2. If it is fixed at one end and has a heavy flywheel of moment of inertia I attached to it at the other end, find the value of the periodic time of torsional oscillation.

11. Calculate the lowest frequency of torsional vibration of a steel shaft 20 feet long which is held at one end. Assume the weight of a cubic foot of steel to be 490 lb.

12. A uniform circular plate of 18 inches diameter, mass 12 lb., is suspended horizontally by a vertical wire attached to its centre and fixed at the upper end. The period of a complete torsional oscillation is 2·2 seconds.

Find the couple required to turn the plate through one revolution.

A small body of mass *m* lb. rests in a groove on the plate. Assuming that the effect of the groove and the added mass on the inertia of the plate is negligible, find the greatest frictional force brought into play to prevent the body sliding when the amplitude of the oscillation is 90°, (i) when the groove is radial and *m* is 6 inches from the centre, (ii) when the groove is circular, concentric with the plate, and of 6 inches radius. [B.E.]

13. A simply supported girder has a span of 10 feet and its dimensions are such that a single load of 1 ton at mid-span produces a deflection of 1/10th inch. A single-cylinder engine, weighing 300 lb., is mounted on the girder at the centre of the span and is out of balance by an amount equivalent to a weight of 1 lb. at a radius of 4 inches. The frictional and other resistances to vertical movement of the system are proportional to velocity and have a value of 10 lb. for a velocity of 1 foot per second. Find (*a*) the speed of the engine at which maximum deflection occurs in the girder, and (*b*) the maximum mid-span deflection when the engine runs at 200 r.p.m. [U.L.]

14. A disc weighing 9 lb. is mounted with its axis horizontal at the middle of a light beam which deflects 3·85 inches under the weight. Calculate the natural period of vibration of the system.

If the disc is lifted, so that the weight is just taken from the beam, and then released, draw a curve showing the position of the weight during the first two seconds of movement (1) when there is no damping, (2) when the motion is damped by a resistance proportional to the velocity, having a value of 4·2 lb. at 1 ft. per sec.

If the C.G. of the disc is $1\frac{1}{3}$ inches out of truth, and the disc is rotated at speeds increasing from zero to 200 r.p.m., the motion being damped, draw a curve showing the maximum amplitude of vibration at each speed. Draw a curve showing how the difference of phase between the out-of-balance force and the displacement of the disc varies with the speed. [U.L.]

15. An unloaded beam is fixed at one end and is quite free at the other end. Using the notation of Art. 211, show that $\cosh al = -\sec al$ when transverse vibrations occur; then, solving this equation graphically, show that the lowest value of al is 1·875. [*Hint.*—When $x=0$, then $y=0$ and $\frac{\partial y}{\partial x}=0$; when $x=l$, then $\frac{\partial^2 y}{\partial x^2}=0$ and $\frac{\partial^3 y}{\partial x^3}=0$.]

Given $l=10$ ft., $w=9$ lb. per ft., $I=10{\cdot}91$ in.^{4}, and $E=30\times10^6$ lb./in.^{2}, find the fundamental frequency of vibration. Compare this frequency with that obtained in the example worked out on p. 443.

16. A steel shaft 8 feet long and 3 inches diameter is supported at each end in a short bearing and carries a disc weighing $\frac{1}{2}$ ton at a distance of 3 feet from one end. Find the whirling speed, neglecting the weight of the shaft. (The deflection formula required is $\delta=\frac{Wa^2b^2}{3(a+b)EI}$.)

17. A steel shaft is loaded and supported in short bearings as shown in Fig. 426, where the linear dimensions are in inches. Neglecting the weight of the shaft, find the lowest whirling speed by each of the methods described in Art. 213.

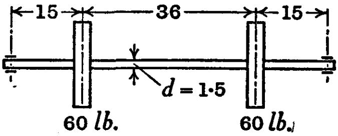

FIG. 426.

18. Obtain an expression for the first whirling speed of a uniform shaft supported freely at its ends and unloaded; also an expression for the whirling speed of a weightless uniform shaft with a central load; then combine the two expressions.

Find the first whirling speed of a shaft, 0·5 inch in diameter and 24 inches in length, carrying a load of 2·25 pounds at its middle point. The metal of the shaft weighs 0·3 lb./in.^{3} and $E=30\times10^6$ lb./in.^{2}. [U.L.]

19. If the ends of a shaft are constrained in long bearings so that the slope at each end is zero, show, using the notation of Art. 214, that the shaft will whirl when $\cosh al=\sec al$. By solving this equation graphically, show that at the lowest whirling speed $al=4{\cdot}73$ approximately.

20. In order to find the moment of inertia of an accurately balanced turbine disc of mass M lb. keyed to a shaft of radius r ft., the disc is placed with the shaft horizontal and resting on level surfaces on either side, and a small mass m lb. is fixed to the wheel at a distance h ft. from the axis of the shaft. If T seconds is the time of a complete small oscillation, prove that the moment of inertia of the wheel is given by

$$I=m\left\{\frac{ghT^2}{4\pi^2}-(h-r)^2\right\}-Mr^2. \qquad \text{[U.L.]}$$

21. Prove that the period of oscillation of the sleeve of a Porter governor in which the speed of revolution is ω rad./sec. is

$$T=\frac{2\pi}{\omega}\left\{\frac{m+2M\sin^2\alpha}{m}\cdot\frac{l\sin\alpha}{l\sin^3\alpha+c}\right\}^{\frac{1}{2}}$$

if the sleeve is slightly disturbed from its position of equilibrium corresponding to this speed. The four links are each equal in length to l, and are pivoted at points distance c from the axis of rotation, and are, for the configuration corresponding to the speed ω, inclined at α to the vertical. The mass of each ball is m and of the central load M. [U.L.]

CHAPTER XIX

VIBRATIONS II

215. Torsional Stiffness and Equivalent Shafts.—The natural vibrations of the shaft and flywheel system shown in Fig. 427 have been discussed in Art. 205, p. 428. The torque in the shaft resisting the motion of the flywheel is

$$T_q = \frac{C\theta J}{l}$$

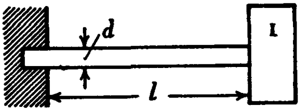

Fig. 427.

where $J = \pi d^4/32$, assuming the shaft to be solid (Art. 167, p. 322).

Let s be the value of the torque when $\theta = 1$, then s is the torsional stiffness or torque per unit angle of twist of the shaft and

$$s = \frac{CJ}{l} \quad . \quad . \quad . \quad . \quad (1).$$

The equation of motion is

$$I\frac{d^2\theta}{dt^2} = -\frac{C\theta J}{l} = -s\theta,$$

or

$$\frac{d^2\theta}{dt^2} + \frac{s}{I}\theta = 0 \quad . \quad . \quad . \quad (2),$$

where I is the moment of inertia of the flywheel about the axis of rotation; $\omega^2 = s/I$, which may be compared with $\omega^2 = k/M = kg/W$ when a mass M is suspended by a spring of stiffness k.

Also $$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{s}} \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{s}{I}} \quad . \quad (3).$$

The value of the periodic time has been given in the same form in equation (7), p. 429.

Consider a shaft made up of several lengths each with its own uniform diameter. In Fig. 428 (a) the shaft has three uniform diameters and it is labelled 1, 2 and 3; corresponding suffixes are used to show to which lengths of the

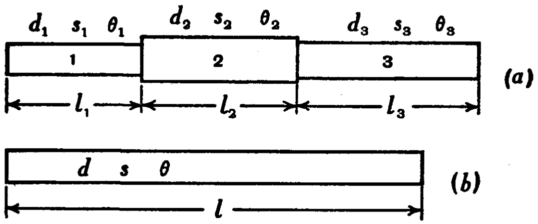

FIG. 428.

shaft the various symbols refer. It will be shown that the shaft may be replaced (theoretically) by one of uniform diameter d and length l and having an equivalent torsional stiffness s (Fig. 428 (b)).

The two shafts must have the same total angle of twist θ when equal opposing torques T_q are applied at their opposite ends, therefore

$$\theta = \theta_1 + \theta_2 + \theta_3 \qquad . \qquad . \qquad . \qquad (4),$$

and
$$T_q = s\theta = s_1\theta_1 = s_2\theta_2 = s_3\theta_3$$

from which
$$\theta_1 = \frac{s}{s_1}\theta, \quad \theta_2 = \frac{s}{s_2}\theta, \quad \theta_3 = \frac{s}{s_3}\theta.$$

Substituting in (4)

$$\theta = \frac{s}{s_1}\theta + \frac{s}{s_2}\theta + \frac{s}{s_3}\theta$$

therefore
$$\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} \qquad . \qquad . \qquad (5),$$

which may be written

$$\frac{1}{s} = \frac{32l}{C\pi d^4} = \frac{32}{C\pi}\left(\frac{l_1}{d_1^4} + \frac{l_2}{d_2^4} + \frac{l_3}{d_3^4}\right) \qquad . \qquad . \qquad (6).$$

A convenient value is assigned to d, say $d = d_1$, then

$$l = l_1 + l_2\left(\frac{d_1}{d_2}\right)^4 + l_3\left(\frac{d_1}{d_3}\right)^4 \qquad . \qquad . \qquad (7)$$

is the equivalent length of shaft of diameter d_1. The equivalent torsional stiffness is

$$s=\frac{C\pi d_1^4}{32l} \quad . \quad . \quad . \quad . \quad . \quad (8).$$

The above procedure is a simple way of obtaining the value of s, but an examination of equations (6) shows that this value can be arrived at without using d and l. In subsequent calculations, l could be given any suitable value, as for example the length of the actual shaft; in this case d would not, of course, be equal to the assumed value d_1.

Example.—The following data for the shaft and flywheels shown in Fig. 429 are from the worked example on p. 432.

Lengths: $l_1=10$ in.; $l_2=7$ in.; $l_3=5$ in.; $l_4=15$ in.
Diameters: $d_1=3{\cdot}5$ in.; $d_2=4{\cdot}25$ in.; $d_3=5$ in.; $d_4=3{\cdot}5$ in.

$I_A=800\times 27^2$ lb. in.^{2}; $I_B=1200\times 33^2$ lb. in.^{2}.
$C=12\times 10^6$ lb./in.^{2}.

It is required to find the frequency of free torsional oscillations of the system.

The solution will be obtained by using an equivalent shaft of uniform diameter.

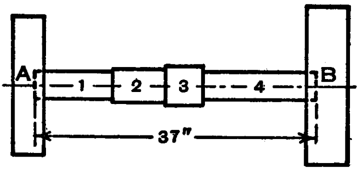

FIG. 429.

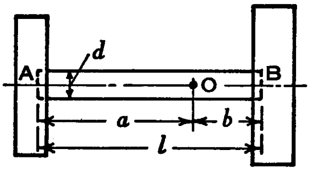

FIG. 430.

Since $d_1=d_4=3{\cdot}5$ in., this is a convenient value to assume for d in equations (6) where a term l_4/d_4^4 must be included inside the brackets. Therefore the equivalent length is

$$l=10+7\left(\frac{3{\cdot}5}{4{\cdot}25}\right)^4+5\left(\frac{3{\cdot}5}{5}\right)^4+15$$

$$=10+3{\cdot}22+1{\cdot}20+15$$

$$=29{\cdot}42 \text{ in.}$$

The equivalent torsional stiffness is

$$s=\frac{C\pi d_1^4}{32l}=\frac{12\times 10^6\pi\times 3{\cdot}5^4}{32\times 29{\cdot}42}=6{\cdot}01\times 10^6 \text{ lb. in./rad.}$$

The shaft with its flywheels may be redrawn as in Fig. 430. Assume the node is at O and let $OA=a$ and $OB=b$. Then $a+b=l$. Let s_a and s_b be the stiffnesses of OA and OB respectively.

Since the periodic times of the flywheels are equal,

$$\frac{I_A}{s_a}=\frac{I_B}{s_b} \quad\text{or}\quad \frac{I_A a}{CJ}=\frac{I_B b}{CJ}.$$

Therefore $\dfrac{a}{b}=\dfrac{I_B}{I_A}=\dfrac{1200\times 33^2}{800\times 27^2}=\dfrac{121}{54}$ and $\dfrac{a+b}{b}=\dfrac{175}{54}.$

Since $l=a+b=29{\cdot}42$, $b=29{\cdot}42\times\dfrac{54}{175}=9{\cdot}08$ in. which gives the position of the node in the equivalent shaft.

Torsional stiffness is inversely proportional to length, therefore

$$s_b=\frac{s(a+b)}{b}=6{\cdot}01\times 10^6\times\frac{175}{54}=19{\cdot}48\times 10^6 \text{ lb. in./rad.}$$

Frequency $f=\dfrac{1}{2\pi}\sqrt{\dfrac{s_b}{I_B}}$

$$=\frac{1}{2\pi}\sqrt{\frac{19{\cdot}48\times 10^6\times 32{\cdot}2\times 12}{1200\times 33^2}}$$

$$=12{\cdot}1 \text{ oscillations/sec.}$$

With regard to the position of the node, the distance $b=9{\cdot}08$ in. applies also to the actual shaft, because d is equal to d_4 and b is less than l_4.

Suppose, with different data, s_b were less than s_4 but greater than the equivalent stiffness of the length l_3+l_4, then the node would be in the length l_3. Let $x+l_4$ be its distance from B, then

$$\frac{1}{s_b}=\frac{x}{s_3 l_3}+\frac{1}{s_4}$$

from which x could be determined.

Using this method, it can be shown that the distance of the node from the end A in Fig. 429 is approximately 27·92 in., but this is a longer process than calculating the distance from B.

216. Elastic Line for a Shaft.—A shaft AB with a rotor at each end (Fig. 431) is assumed to be oscillating and the node is at O. A straight line mOn intersects at m and n perpendiculars to the shaft axis at A and B respectively. If Am represents α radian, the amplitude of the rotor at A, then Bn represents β radian, the amplitude of the rotor at B. If p is any point in mOn, then pP, the perpendicular to the axis, represents the amplitude of the section of the shaft at P. The line mOn is the elastic line for the shaft, assuming the shaft inertia to be negligible.

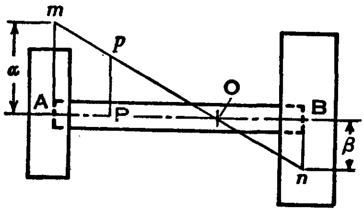

FIG. 431.

217. Torsional Vibrations—Shaft carrying Three Rotors.—When a shaft carries three rotors, as in Fig. 432, natural

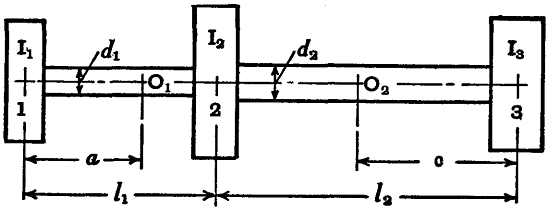

FIG. 432.

torsional oscillations can occur in two ways, that is with either one or two nodes. In each case two rotors turn in one direction whilst the remaining rotor turns in the opposite direction; it will be seen later that in the lower mode of vibration two adjacent rotors turn together whereas in the higher mode the end rotors turn together.

The positions of the nodes depend on the stiffness of each length of shaft and the moments of inertia of the rotors.

These positions and the values of the frequencies may be obtained by considering the higher mode of vibration.

Suppose the nodes are at O_1 and O_2 (Fig. 432) and let the other symbols be as indicated. Also $J_1 = \pi d_1^4/32$ and $J_2 = \pi d_2^4/32$.

From Art. 215 the general equation of motion is

$$\frac{d^2\theta}{dt^2} + \frac{s}{I}\theta = 0, \quad \text{where} \quad s = \frac{CJ}{l}.$$

Rotor 1. Frequency $f = \frac{1}{2\pi}\sqrt{\frac{CJ_1}{I_1 a}}$ . . . (1).

Rotor 2. The resisting torque is supplied by two lengths of shaft, one on each side of the rotor, between the nodes O_1 and O_2. Each length is twisted through the same angle and the combined stiffness is equal to the sum of the separate stiffnesses.

$$f = \frac{1}{2\pi}\sqrt{\frac{C}{I_2}\left(\frac{J_1}{l_1 - a} + \frac{J_2}{l_2 - c}\right)} \quad . \quad . \quad (2).$$

Rotor 3. $f = \frac{1}{2\pi}\sqrt{\frac{CJ_2}{I_3 c}}$ . . . (3).

The frequencies must be equal, therefore, from (1), (2) and (3)

$$\frac{J_1}{I_1 a} = \frac{1}{I_2}\left(\frac{J_1}{l_1 - a} + \frac{J_2}{l_2 - c}\right) = \frac{J_2}{I_3 c} \quad . \quad . \quad (4).$$

From these equations $a/c = J_1 I_3 / J_2 I_1$. Also a quadratic equation in either a or c is obtained, therefore there are two values of a and two of c. One value of a and the corresponding value of c give the positions of the two nodes; but in the other pair of values, one gives the position of the single node and the other is beyond the physical limits of the equation. For instance if a is greater than l_1 it does not give the position of an actual node, although its value can be substituted in a frequency equation to give a correct result and is of use when an elastic line is drawn.

Example.—Three rotors 1, 2 and 3 (Fig. 432), having moments of inertia of 4000, 5000 and 2000 lb. in.^{2} respectively, are carried on a uniform shaft of 3 in. diameter.

The length of shaft between rotors 1 and 2 is 45 in. and between 2 and 3 is 30 in. It is required to find the frequencies of the natural torsional vibrations and also the amplitudes of the rotors 2 and 3, assuming the amplitude of rotor 1 is 0·1° in each case. The modulus of rigidity $C=12\times10^6$ lb./in.².

Since the shaft is uniform, equations (4) may be written

$$\frac{1}{I_1 a}=\frac{1}{I_2}\left(\frac{1}{l_1-a}+\frac{1}{l_2-c}\right)=\frac{1}{I_3 c}.$$

Substituting known values, using lb. and in. units,

$$\frac{1}{4000a}=\frac{1}{5000}\left(\frac{1}{45-a}+\frac{1}{30-c}\right)=\frac{1}{2000c}.$$

Therefore $\quad 4000a=2000c \quad$ and $\quad c=2a.$

Also $$\frac{5}{4a}=\frac{1}{45-a}+\frac{1}{30-2a}$$

which reduces to $\quad 11a^2-450a+3375=0.$

The roots are $\quad a=31{\cdot}02$ in. and $9{\cdot}89$ in.

therefore $\quad c=62{\cdot}04$ in. and $19{\cdot}78$ in.

Of these values, $c=62{\cdot}04$ in. cannot give the position of an actual node, because 62·04 in. is greater than l_2 or 30 in. Therefore $a=31{\cdot}02$ in. gives the position of the single node at the lower frequency; $a=9{\cdot}89$ in. and $c=19{\cdot}78$ in. give the positions of the two nodes at the higher frequency.

The frequencies may be obtained from (1), (2) or (3). From (1), using lb. and in. units,

$$\text{Lower frequency}=\frac{1}{2\pi}\sqrt{\frac{12\times10^6\times32{\cdot}2\times12}{4000}\times\frac{\pi\times3^4}{32\times31{\cdot}02}}$$

$$=86{\cdot}76 \text{ oscillations/sec.}$$

$$\text{Higher frequency}=86{\cdot}76\sqrt{\frac{31{\cdot}02}{9{\cdot}89}}$$

$$=153{\cdot}6 \text{ oscillations/sec.}$$

The nodes and the elastic lines are shown in Fig. 433. For the lower frequency O′ is the actual node; N is the corresponding node given by the value $c=62{\cdot}04$ and it can be explained as being the node, in the shaft between 3 and 2 produced, which would enable rotor 3 to have the same frequency as that of rotor 1, if there were no rotor 2. For the higher frequency the nodes are the points marked O.

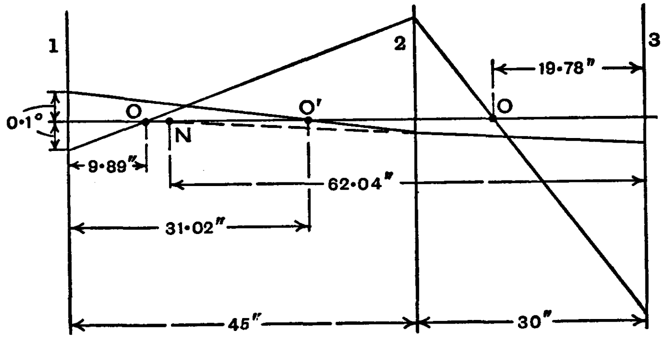

FIG. 433.

The 0·1° amplitudes of rotor 1 are marked off to a suitable scale on the perpendicular to the shaft axis and the elastic lines are drawn through the nodes. The directions of the given amplitudes are unknown, but making one positive and the other negative perhaps aids clarity. The amplitudes of the rotors 2 and 3 are then calculated, using similar triangles. The signs are here disregarded.

Amplitudes.—Lower frequency.

Rotor 2. $$0{\cdot}1 \times \frac{45-31{\cdot}02}{31{\cdot}02} = 0{\cdot}0450^\circ.$$

Rotor 3. $$0{\cdot}0450 \times \frac{62{\cdot}04}{62{\cdot}04-30} = 0{\cdot}0871^\circ.$$

Higher frequency.

Rotor 2. $$0{\cdot}1 \times \frac{45-9{\cdot}89}{9{\cdot}89} = 0{\cdot}3550^\circ.$$

Rotor 3. $$0{\cdot}3550 \times \frac{19{\cdot}78}{30-19{\cdot}78} = 0{\cdot}6871^\circ.$$

218. Torsional Vibrations of a Multi-mass System.—When a shaft carries more than three rotors, the algebraic process of finding the frequencies is involved and it is simpler to use the tabulation method, due to Holzer, which will now be described. There are variations of this method, but they will not be discussed.

Suppose the shaft carries six rotors, labelled 1 to 6 in Fig. 434 and having moments of inertia I_1, I_2, . . . about

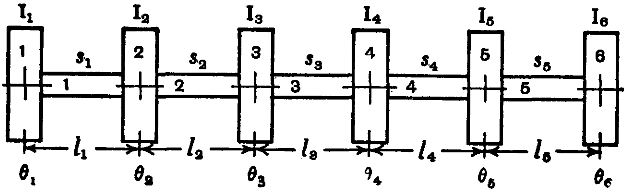

FIG. 434.

the shaft axis. The parts of the shaft between the rotors are denoted by 1, 2, . . . as shown, their lengths are l_1, l_2, . . ., and their torsional stiffnesses are s_1, s_2, . . .

Let $\omega = 2\pi f$ be the common angular frequency of the rotors when natural oscillations are occurring and let θ_1, θ_2, . . . be the amplitudes. The motions are simple harmonic and the rotors arrive at their extreme positions simultaneously; the maximum accelerations are $-\omega^2\theta_1$, $-\omega^2\theta_2$, . . ., and the corresponding inertia torques are $I_1\omega^2\theta_1$, $I_2\omega^2\theta_2$, . . . As in simpler problems, the inertia of the shaft is assumed to be negligible.

Since the oscillations are natural, no external torque is supplied and the sum of the inertia torques must be zero.

Therefore $$\Sigma I\omega^2\theta = 0 \quad \text{or} \quad \Sigma I\theta = 0$$

as all the rotors have the same frequency.

At the lowest frequency there is one node and at the fifth, the highest in this particular example, there are five nodes. In many cases only the lower frequencies are of interest.

Equating the maximum torques in shaft length 1 and rotor 1, due to the inertia of the rotor,

$$s_1(\theta_1 - \theta_2) = I_1\omega^2\theta_1 \quad \text{from which} \quad \theta_2 = \theta_1 - \frac{\omega^2}{s_1}I_1\theta_1.$$

Similarly, considering shaft length 2 and rotors 1 and 2,

$$s_2(\theta_2 - \theta_3) = I_1\omega^2\theta_1 + I_2\omega^2\theta_2$$

from which $$\theta_3 = \theta_2 - \frac{\omega^2}{s_2}(I_1\theta_1 + I_2\theta_2).$$

The other amplitudes are obtained in the same way, the sixth being

$$\theta_6 = \theta_5 - \frac{\omega^2}{s_5}(I_1\theta_1 + I_2\theta_2 + I_3\theta_3 + I_4\theta_4 + I_5\theta_5).$$

A convenient value is assigned to θ_1 and ω is estimated, then $\Sigma I\theta$ for the six rotors can be found and this summation would be zero if the value of ω were correct. The first attempt is made with ω equal to a value arrived at by considering an approximately equivalent two- or three-rotor system, after which several trials may be required before the correct value of ω is obtained. The various results are tabulated and the work is straightforward, although sometimes tedious.

The table may be arranged in various ways, but the following headings are convenient in the worked example on p. 468.

No.	I	θ	$I\theta$	$\Sigma I\theta$	$\frac{\omega^2}{sg}$	$\frac{\omega^2}{sg}\Sigma I\theta$

The units must be considered carefully, particularly those of the moments of inertia I which may or may not include g. If $I = (W/g)k^2$ the final heading should be $(\omega^2/s)\Sigma I\theta$, but if $I = Mk^2$ where M is in say tons or pounds, the final heading should be $(\omega^2/sg)\Sigma I\theta$. If the linear units are inches, $g = 32{\cdot}2 \times 12$ in./sec.^{2} The torsional stiffness $s = CJ/l$.

An important problem which can be solved by this method is that of finding the natural frequencies of torsional vibrations of say a four- or six-cylinder engine driving a generator or a propeller. The crankshaft with its cranks, the connecting-rods, and the pistons have to be converted to the equivalent shaft and rotor system. There are various rules for making the conversion, but it is beyond the scope of this book to discuss these. The reader who requires this information should refer to the work by Dr Ker Wilson.*

Example.—Fig. 435 (i) is assumed to represent a four-cylinder engine with its flywheel and a generator rotor, reduced to the equivalent torsional system. The shaft

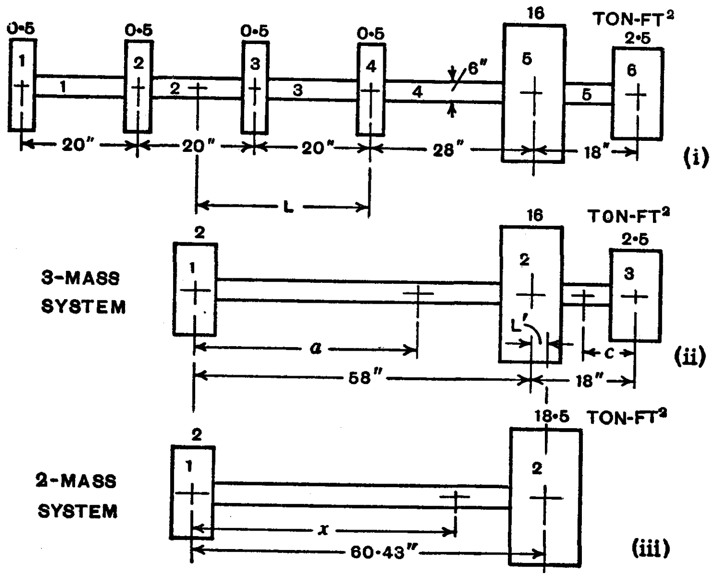

FIG. 435.

diameter is 6 in. and the other linear dimensions and the moments of inertia of the six rotors are as indicated. The modulus of rigidity is 12×10^6 lb./in.². It is required to find the first and second frequencies of the natural torsional oscillations by the tabular method.

The approximately equivalent three-mass system is shown at (ii). The rotors 1 to 4 are replaced by one of moment of inertia $4 \times 0{\cdot}5 = 2$ ton-ft.², at the centre of the engine. Since frequency is a function of moment of inertia times length or Il, it is a reasonable approximation to

* *Practical Solution of Torsional Vibration Problems*, 3rd Ed. 1956, Vol. I. By W. Ker Wilson. Chapman & Hall.

replace two or more rotors by one having a moment of inertia I_e equal to the sum of the separate moments of inertia and positioned so that $I_e L = \Sigma I l$, the lengths being measured from a convenient origin; it should be noted that no allowance is made for the differing amplitudes of the separate rotors. Algebraically the process is similar to that of concentrating several masses at their mass centre.

If lengths are measured from the centre line of rotor No. 4, the equation is

$$(0{\cdot}5 + 0{\cdot}5 + 0{\cdot}5 + 0{\cdot}5)L = 0{\cdot}5 \times 60 + 0{\cdot}5 \times 40 + 0{\cdot}5 \times 20$$

from which $L = 30$ in.

Denoting the rotors in the three-mass system by 1, 2 and 3, the distance between the centre lines of rotors 1 and 2 is $30 + 28 = 58$ in.

To convert the three-mass system to the approximately equivalent two-mass system shown at (iii), a rotor with a moment of inertia $16 + 2{\cdot}5 = 18{\cdot}5$ ton-ft.^{2} is placed at a distance L' from the centre line of the intermediate rotor,

then $$L' = \frac{2{\cdot}5 \times 18}{18{\cdot}5} = 2{\cdot}43 \text{ in.}$$

and the distance between the centre lines of the two rotors (now labelled 1 and 2) is $58 + 2{\cdot}43 = 60{\cdot}43$ in.

With the two-mass system (Fig. 435 (iii)), if x is the distance of the node from the centre line of rotor 1,

$$2x = 18{\cdot}5(60{\cdot}43 - x) \quad \text{from which} \quad x = 54{\cdot}5 \text{ in.,}$$

and the angular frequency is

$$\omega_1 = \sqrt{\frac{CJ}{xI_1}} = \sqrt{\frac{12 \times 10^6 \pi \times 6^4 \times 32{\cdot}2 \times 12}{54{\cdot}5 \times 32 \times 2 \times 2240 \times 12^2}} = 129{\cdot}5 \text{ rad./sec.}$$

With the three-mass system (Fig. 435 (ii)), equation (4), Art. 217, may be written

$$\frac{1}{I_1 a} = \frac{1}{I_2}\left(\frac{1}{l_1 - a} + \frac{1}{l_2 - c}\right) = \frac{1}{I_3 c}$$

since the value of J is the same for all sections. Substituting numerical values (in this case $l_1 = 58$ in. and $l_2 = 18$ in.), using ton and inch units,

$$\frac{1}{2\times 12^2 a}=\frac{1}{16\times 12^2}\left(\frac{1}{58-a}+\frac{1}{18-c}\right)=\frac{1}{2{\cdot}5\times 12^2 c}.$$

From these equations

$$c=0{\cdot}8a \quad \text{and} \quad 1{\cdot}025a^2-73{\cdot}90a+1044=0.$$

The single node is given by $a=52{\cdot}81$ in. and the two nodes by $a=19{\cdot}28$ in. and $c=15{\cdot}42$ in.

The first angular frequency is

$$\omega_1=\sqrt{\frac{\mathrm{CJ}}{a\mathrm{I}_1}}=\sqrt{\frac{12\times 10^6\pi\times 6^4\times 32{\cdot}2\times 12}{52{\cdot}81\times 32\times 2\times 2240\times 12^2}}=131{\cdot}6 \text{ rad./sec.}$$

The second angular frequency is

$$\omega_2=\omega_1\sqrt{\frac{52{\cdot}81}{19{\cdot}28}}=217{\cdot}8 \text{ rad./sec.}$$

From experience it would be expected that the values obtained for ω_1 by the two approximate methods are low. If $\omega=132$ rad./sec. is used in the tabulation method, it is found that $\Sigma\mathrm{I}\theta=219{\cdot}8$ instead of zero. To save space, the first tabulation given here is based on $\omega=140$ rad./sec. which is a guess.

The numbers in the first column of the table (p. 471) refer to the rotors and the intermediate parts of the shaft (Fig. 435 (i)). The units of I have been taken as ton-in.^{2}. In the first line $\theta_1=1$ rad., a convenient angle; $\mathrm{I}_1\theta_1$ and $\Sigma\mathrm{I}\theta=72$; $72\times 0{\cdot}001488$ gives $0{\cdot}1071$ rad. in the final column.

Since $\theta_2=\theta_1-(\omega^2/s_1g)\mathrm{I}_1\theta_1$ (for equations and remarks on units see pp. 466 and 467), $\theta_2=1{\cdot}0000-0{\cdot}1071=0{\cdot}8929$ rad. which is entered in the θ column. Then $\mathrm{I}_2\theta_2=72\times 0{\cdot}8929=64{\cdot}29$ which is added to the 72 in the $\Sigma\mathrm{I}\theta$ column. The work is continued until $\Sigma\mathrm{I}\theta=55{\cdot}48$ is obtained for the six rotors. As this result is not zero, $\omega=140$ rad./sec. is not the correct value of the angular frequency and another guess is necessary.

The second tabulation (p. 471) has been made with $\omega=143$ rad./sec. and, as shown in the table, $\Sigma\mathrm{I}\theta=-3{\cdot}06$

1st Tabulation. $\omega = 140$ rad./sec. $\omega^2 = 19600$.

No.	I ton-in.^{2}	θ rad.	Iθ	ΣIθ	$\dfrac{\omega^2}{sg}$	$\dfrac{\omega^2}{sg}\Sigma$I$\theta$ rad.
1	72	1·0000	72·00	72·00		
					0·001488	0·1071
2	72	0·8929	64·29	136·29		
					0·001488	0·2028
3	72	0·6901	49·69	185·98		
					0·001488	0·2767
4	72	0·4134	29·76	215·74		
					0·002084	0·4496
5	2304	−0·0362	−83·40	132·34		
					0·001340	0·1773
6	360	−0·2135	−76·86	55·48		

Values of ω^2/sg are calculated as follows:

$$\frac{\omega^2}{sg} = \frac{\omega^2 l}{CJg} = \frac{19600 \times 2240 \times 32l}{12 \times 10^6\pi \times 6^4 \times 32{\cdot}2 \times 12} = 0{\cdot}000074418l \text{ ton}^{-1}\text{ in.}^{-2}.$$

When $l = l_1 = l_2 = l_3 = 20$ in., $\omega^2/sg = 0{\cdot}001488$.
$l = l_4 = 28$ in., $\omega^2/sg = 0{\cdot}002084$.
$l = l_5 = 18$ in., $\omega^2/sg = 0{\cdot}001340$.

2nd Tabulation. $\omega = 143$ rad./sec. $\omega^2 = 20449$.

No.	I ton-in.^{2}	θ rad.	Iθ	ΣIθ	$\dfrac{\omega^2}{sg}$	$\dfrac{\omega^2}{sg}\Sigma$I$\theta$ rad.
1	72	1·0000	72·00	72·00		
					0·001553	0·1118
2	72	0·8882	63·95	135·95		
					0·001553	0·2111
3	72	0·6771	48·75	184·70		
					0·001553	0·2868
4	72	0·3903	28·10	212·80		
					0·002174	0·4626
5	2304	−0·0723	−166·58	46·22		
					0·001398	0·0646
6	360	−0·1369	−49·28	−3·06		

$$\frac{\omega^2}{sg} = \frac{\omega^2 l}{CJg} = \frac{20449 \times 2240 \times 32l}{12 \times 10^6\pi \times 6^4 \times 32{\cdot}2 \times 12} = 0{\cdot}000077642l \text{ ton}^{-1}\text{ in.}^{-2}.$$

When $l = l_1 = l_2 = l_3 = 20$ in., $\omega^2/sg = 0{\cdot}001553$.
$l = l_4 = 28$ in., $\omega^2/sg = 0{\cdot}002174$.
$l = l_5 = 18$ in., $\omega^2/sg = 0{\cdot}001398$.

which is not far from zero. Since $\Sigma I\theta$ changes from $+55{\cdot}48$ to $-3{\cdot}06$ as ω increases from 140 to 143 rad./sec., ω is now slightly high in value, but it is correct to three figures, as will be seen later.

The approximate elastic line for the shaft at the first frequency is shown in Fig. 436 where the values of θ are given to two places of decimals and are from the second tabulation.

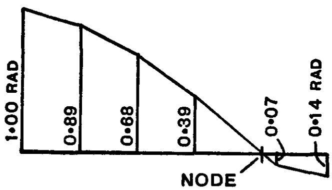

FIG. 436.

By plotting the two values of $\Sigma I\theta$, 55·48 and $-3{\cdot}06$, against ω and assuming a linear law as in Fig. 437, a closer approximation to the value of ω when $\Sigma I\theta = 0$ is obtained. This assumption is permissible when the values of ω are near one another.

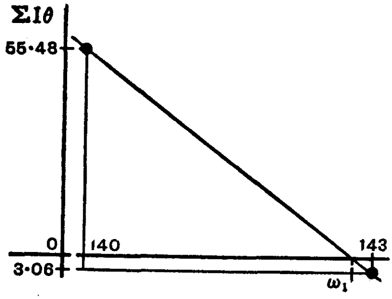

FIG. 437.

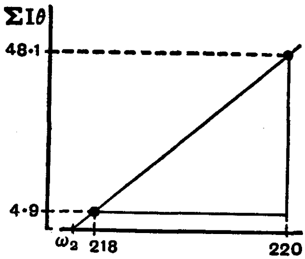

FIG. 438.

By calculation

$$\frac{143-\omega_1}{3{\cdot}06} = \frac{143-140}{55{\cdot}48+3{\cdot}06},$$

from which $\omega_1 = 142{\cdot}8$ rad./sec.

As a check, if the tabulation is made with $\omega = 142{\cdot}8$ rad./sec. it is found that $\Sigma I\theta = 0{\cdot}54$ which can be regarded as very near zero in this instance. It should be noted however that as the entries in the tables are given mainly to four significant figures only, the fourth figure in the value of ω is unreliable.

The frequency $f_1 = \frac{\omega_1}{2\pi} = \frac{142 \cdot 8}{2\pi} = 22 \cdot 73$ vib./sec.

$= 1364$ vib./min.

The second frequency is obtained in a similar way. By the three-rotor method $\omega_2 = 217 \cdot 8$ rad./sec. (p. 470). Therefore a first guess might be 220 rad./sec. With this value the tabulation method gives $\Sigma I\theta = 48 \cdot 1$. If the next guess is 218 rad./sec., $\Sigma I\theta = 4 \cdot 9$.

These values are plotted in Fig. 438. By calculation, assuming a linear law,

$$\frac{218 - \omega_2}{4 \cdot 9} = \frac{220 - 218}{48 \cdot 1 - 4 \cdot 9}$$

from which $\omega_2 = 217 \cdot 8$ rad./sec. The fact that the three-rotor method gives the same result is a coincidence.

The frequency $f_2 = \frac{\omega_2}{2\pi} = \frac{217 \cdot 8}{2\pi} = 34 \cdot 66$ vib./sec.

$= 2080$ vib./min.

Higher frequencies are found by continuing the tabulations and the approximate values of all the angular frequencies are shown in the graph (Fig. 440) in the following Art.

219. Relation between $\Sigma I\theta$ and ω.—When $\Sigma I\theta$ is plotted against ω, a curve of the type drawn in Fig. 439 is obtained, the values of the angular frequencies ω_1, ω_2, etc., being at the points where the curve crosses the ω axis, that is where $\Sigma I\theta = 0$. If $(\omega^2/g)\Sigma I\theta$, the *excess torque*, is plotted against ω, the curve begins at the origin, since every ordinate in Fig. 439 then has to be multiplied by the corresponding value of ω^2 and divided by g.

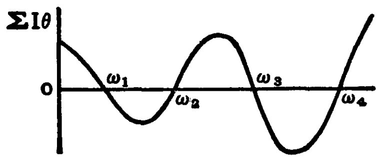

FIG. 439.

The vital parts of the curve, when $\Sigma I\theta$ is plotted against ω for the example given in the preceding Art., are illustrated in Fig. 440. Some of the positive and negative values found by tabulation cannot be plotted because they are too large to fit in with the scale used; the semi-circles and dotted lines are intended to show that the curve is continuous.

The greater the value of ω, the steeper is the curve where it crosses the axis. For instance, near the third angular frequency, as ω increases by 10 from 400 to 410 rad./sec.,

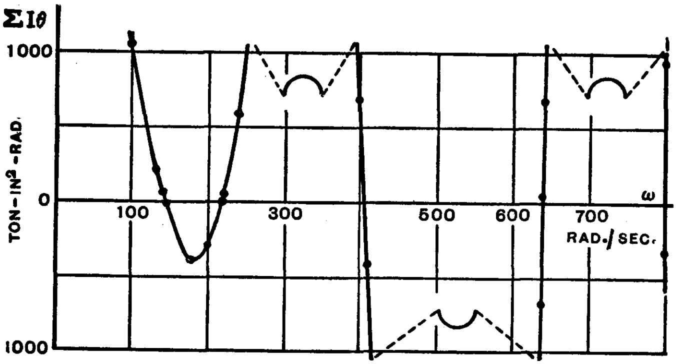

Fig. 440.

$\Sigma I\theta$ changes from $+690$ to -410; near the fifth frequency, as ω increases by only 0·1 from 798·6 to 798·7 rad./sec., a difference too small to show, $\Sigma I\theta$ changes from $+704$ to -331. At $\omega = 800$ rad./sec., $\Sigma I\theta = -6309$.

220. Torsional Vibrations of a Geared System.—A shaft 1 which carries a rotor A (Fig. 441 (*a*)) has an angular velocity N_1 and drives through gear wheels E and F, a shaft 2 and a rotor B at an angular velocity N_2. The diameters and lengths are as indicated. The frequency of the natural vibrations of this geared system may be obtained by considering an equivalent system carrying three rotors 1, 2 and 3 on a shaft of uniform diameter (Fig. 441 (*b*)). This diameter will be taken as d_1.

It will be assumed that the gear teeth are rigid and always in contact and that all the torsional flexibility is in the shafts. In practice, due to backlash the oscillating gears would be noisy and the tooth wear would be rapid.

The values to be determined in the equivalent system are the length l_e of shaft of diameter d_1 to replace the length l_2, and the moments of inertia of the rotors 2 and 3. The rotor 1 is merely rotor A with a new label. The symbols s and θ, with appropriate suffixes, will denote stiffness and angle of twist respectively.

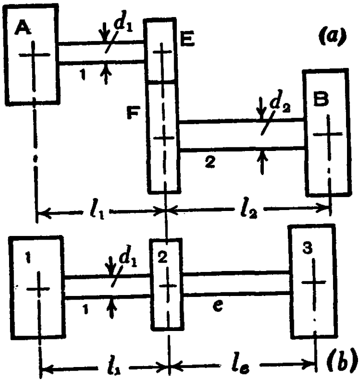

FIG. 441.

Suppose the geared system is at rest and is in equilibrium when applied opposing torques on the rotors A and B have twisted the shafts 1 and 2 through angles θ_1 and θ_2 respectively. The torque $s_2\theta_2$ in shaft 2 is N_1/N_2 times that in shaft 1, and in the equivalent system the torque $s_e\theta_e$ in the length l_e must be equal to the torque in the length l_1,

therefore $$s_2\theta_2 = (N_1/N_2)s_e\theta_e \quad . \quad . \quad . \quad (1).$$

Relative to the rotor B, wheel F turns through an angle θ_2 and wheel E turns through an angle $(N_1/N_2)\theta_2$ and this is the twist required in the equivalent length l_e, that is

$$\theta_e = (N_1/N_2)\theta_2 \quad . \quad . \quad . \quad (2).$$

From (1) and (2) $$s_e = (N_2/N_1)^2 s_2 \quad . \quad . \quad . \quad (3),$$

or $$\frac{CJ_1}{l_e} = \left(\frac{N_2}{N_1}\right)^2 \frac{CJ_2}{l_2}$$

therefore $$l_e = l_2\left(\frac{N_1}{N_2}\right)^2 \frac{J_1}{J_2} = l_2\left(\frac{N_1}{N_2}\right)^2 \left(\frac{d_1}{d_2}\right)^4 \quad . \quad . \quad (4).$$

It follows from Art. 154 that the equivalent moment of inertia of rotor 3 is

$$I_3 = \left(\frac{N_2}{N_1}\right)^2 I_B \quad . \quad . \quad . \quad (5),$$

and for rotor 2

$$I_2 = I_E + \left(\frac{N_2}{N_1}\right)^2 I_F \quad . \quad . \quad . \quad (6).$$

These relations may be arrived at in another way. The kinetic energy of a rotor of moment of inertia I, rotating with angular velocity Ω is $\frac{1}{2}I\Omega^2$ and an equivalent moment of inertia must be such that the kinetic energy is unaltered. Suppose the maximum angular velocities of rotors B and 3, when they are oscillating and the shafts are otherwise at rest, are Ω_B and Ω_3 respectively, then $\Omega_B/\Omega_3 = N_2/N_1$. Therefore the moment of inertia of rotor 3 must satisfy the kinetic energy condition when the velocities are N_2 and N_1, and this also applies to rotor 2.

Therefore $$I_3N_1^2 = I_BN_2^2$$

and $$I_2N_1^2 = I_EN_1^2 + I_FN_2^2$$

which, after dividing through by N_1^2, become the relations given above.

Example.—A motor drives a centrifugal pump through gearing, the pump speed being one-third that of the motor. The shaft from the motor to the pinion is $2\frac{1}{4}$ in. diameter and 10 in. long, the motor having a moment of inertia of 1000 lb. ft.². The impeller shaft is 4 in. diameter and 24 in. long, the moment of inertia of the impeller being 3500 lb. ft.².

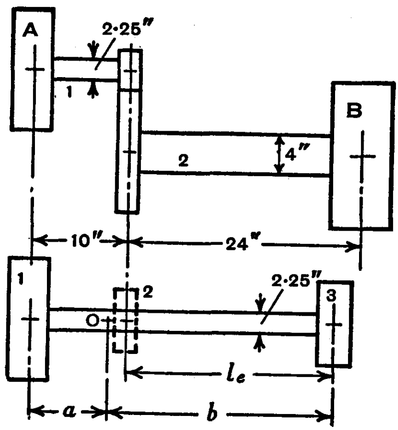

FIG. 442.

Working from first principles, determine the frequency of torsional oscillations of the system. Neglect the inertia and flexibility of the gear and the inertia of the shaft. Modulus of rigidity of shafting $= 12 \times 10^6$ lb. per sq. in.

[U.L.]

In Fig. 442, A represents the motor and B the pump. Since the reduction of a geared system to an equivalent one-shaft system has been explained above, the results will be used here.

$$N_1/N_2 = 3.$$

$$l_e = l_2\left(\frac{N_1}{N_2}\right)^2\left(\frac{d_1}{d_2}\right)^4 = 24 \times 3^2\left(\frac{2{\cdot}25}{4}\right)^4 = 21{\cdot}62 \text{ in.}$$

$$I_3 = \left(\frac{N_2}{N_1}\right)^2 I_B = \frac{3500}{9} = 388{\cdot}9 \text{ lb. ft.}^2$$

$$I_1 = I_A = 1000 \text{ lb. ft.}^2 \quad I_2 \text{ is to be neglected.}$$

Assume the node is at O. $I_3 b = I_1 a$, therefore

$$\frac{b}{a} = \frac{I_1}{I_3} = \frac{1000}{388{\cdot}9} \quad \text{and} \quad \frac{a+b}{a} = \frac{1388{\cdot}9}{388{\cdot}9}.$$

Since

$$a + b = 10 + l_e = 31{\cdot}62 \text{ in.}, \quad a = \frac{388{\cdot}9}{1388{\cdot}9} \times 31{\cdot}62 = 8{\cdot}85 \text{ in.}$$

Frequency
$$f = \frac{1}{2\pi}\sqrt{\frac{CJ_1}{I_1 a}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{12 \times 10^6 \times 32{\cdot}2 \times 12}{1000 \times 12^2} \times \frac{\pi \times 2{\cdot}25^4}{32 \times 8{\cdot}85}} = 15{\cdot}2 \text{ vib./sec.}$$

221. Torsional Vibrations of a Geared System with Three or more Branches.*—A three-branch geared system is shown in Fig. 443, but the analytical method of finding the natural frequencies of torsional vibration which is described is applicable to a system having more than three branches. This method consists of reducing the system to one in which all the shafts run at the same speed and then equating to zero the sum of the torques at the junction of the shafts.

A shaft A carrying a rotor A is connected through gears X, Y, Z to shafts B and C carrying rotors B and C. The moments of inertia of the rotors A, B, and C are I_A, I_B, and

* For a full discussion of this subject, see *Practical Solution of Torsional Vibration Problems*, by W. Ker Wilson. Chapman & Hall.

I_C respectively, the stiffnesses of the corresponding shafts are s_A, s_B, and s_C, their lengths are l_A, l_B, and l_C, and their speeds are N_A, N_B, and N_C respectively. The moments of inertia of the gears are I_X, I_Y, and I_Z.

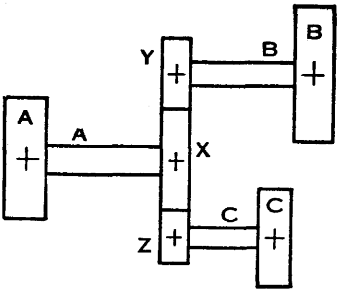

FIG. 443.

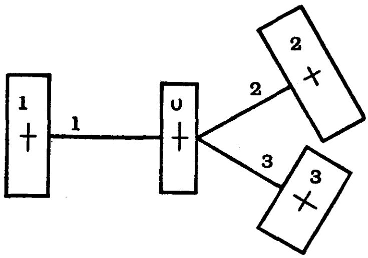

FIG. 444.

In the equivalent system (Fig. 444) the figures 1, 2, 3 correspond to the letters A, B, C in the original system and the rotor 0 replaces the gears X, Y, Z. The amplitudes of the rotors 0, 1, 2, and 3 are denoted by θ_0, θ_1, θ_2, and θ_3 respectively, and Ω is used for angular frequency. The relations between the equivalent and actual values of the moments of inertia and of the stiffnesses are given later.

Equating the maximum torques in shaft 1 and rotor 1, due to the inertia of the rotor,

$$s_1(\theta_1 - \theta_0) = I_1\Omega^2\theta_1.$$

For convenience let $s_1/I_1 = \omega_1^2$, then

$$\omega_1^2(\theta_1 - \theta_0) = \Omega^2\theta_1 \quad . \quad . \quad . \quad (1)$$

and

$$\frac{\theta_1}{\theta_0} = \frac{\omega_1^2}{\omega_1^2 - \Omega^2}.$$

Similarly, putting $s_2/I_2 = \omega_2^2$ and $s_3/I_3 = \omega_3^2$,

$$\frac{\theta_2}{\theta_0} = \frac{\omega_2^2}{\omega_2^2 - \Omega^2} \quad \text{and} \quad \frac{\theta_3}{\theta_0} = \frac{\omega_3^2}{\omega_3^2 - \Omega^2}.$$

When natural oscillations occur, the total torque is zero at the junction of the shafts 1, 2, 3 and the rotor 0, therefore

$$I_0\Omega^2\theta_0 + I_1\Omega^2\theta_1 + I_2\Omega^2\theta_2 + I_3\Omega^2\theta_3 = 0$$

or $$\Omega^2\theta_0\left(I_0+I_1\frac{\theta_1}{\theta_0}+I_2\frac{\theta_2}{\theta_0}+I_3\frac{\theta_3}{\theta_0}\right)=0.$$

But $$I_1\frac{\theta_1}{\theta_0}=\frac{I_1\omega_1^2}{\omega_1^2-\Omega^2}=\frac{s_1}{\omega_1^2-\Omega^2}$$

and similarly for the succeeding terms, therefore

$$\Omega^2\theta_0\left(I_0+\frac{s_1}{\omega_1^2-\Omega^2}+\frac{s_2}{\omega_2^2-\Omega^2}+\frac{s_3}{\omega_3^2-\Omega^2}\right)=0 \quad . \qquad (2).$$

When the system is oscillating, Ω^2 is not zero, therefore, provided θ_0 is not zero,

$$\frac{s_1}{\Omega^2-\omega_1^2}+\frac{s_2}{\Omega^2-\omega_2^2}+\frac{s_3}{\Omega^2-\omega_3^2}=I_0 \quad . \qquad (3),$$

from which the three values of Ω^2 can be found by trial and error and their square roots are the angular frequencies. If I_0 is assumed to be negligible, the equation has two roots only.

In general, if all the values of ω are different, θ_0 is not zero during oscillation. If, however, two or more values of ω are the same, say $\omega_1=\omega_2$, then $\theta_0=0$ is a solution which when substituted in equation (1) gives $\Omega^2=\omega_1^2$ and similarly for other equal values of ω.

Although it may not be immediately obvious that this value of Ω^2 satisfies equation (2), since two of the terms tend to infinity, it does so because these terms cancel. This solution may also be deduced by solving for Ω^2 algebraically, making the values of ω equal after clearing of fractions (see Ex. 12). The values of Ω^2 do not then all satisfy equation (3) from which they have been obtained, but they all satisfy equation (2).

The equivalent values of the stiffnesses and of the moments of inertia are obtained as follows (see Art. 220).

Suppose the common speed selected for the equivalent system is that of shaft A, then

$$s_1=s_A, \quad s_2=\left(\frac{N_B}{N_A}\right)^2 s_B, \quad s_3=\left(\frac{N_C}{N_A}\right)^2 s_C.$$

$$I_1 = I_A,\quad I_2 = \left(\frac{N_B}{N_A}\right)^2 I_B,\quad I_3 = \left(\frac{N_C}{N_A}\right)^2 I_C,$$

$$I_0 = I_X + \left(\frac{N_B}{N_A}\right)^2 I_Y + \left(\frac{N_C}{N_A}\right)^2 I_Z.$$

Example.—It is required to find the three frequencies of torsional vibration of the three-branch system shown in Fig. 445 where the linear dimensions are given. The moments of inertia in lb. in.^{2} are $I_A = 2000$, $I_B = 50$, $I_C = 60$, $I_X = 1200$, $I_Y = 10$, and $I_Z = 80$; the velocity ratios are $N_B/N_A = 4$ and $N_C/N_A = 2$. Modulus of rigidity $= 12 \times 10^6$ lb./in.^{2} and $g = 32{\cdot}2 \times 12$ in./sec.^{2}

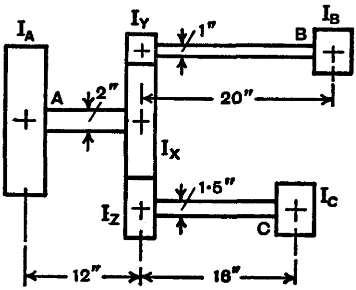

FIG. 445.

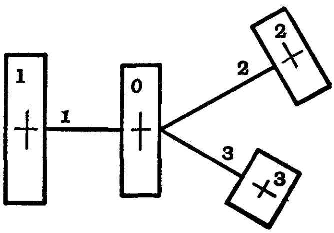

FIG. 446.

In the equivalent system (Fig. 446) the speed of the shafts will be taken as that of shaft A. The various values are calculated as follows.

$$s_1 = s_A = \frac{CJ_A}{l_A} = \frac{12 \times 10^6 \pi \times 2^4}{12 \times 32} = 1{\cdot}5708 \times 10^6 \text{ lb. in./rad.}$$

$$s_2 = 4^2 s_B = \frac{16 \times 12 \times 10^6 \pi \times 1^4}{20 \times 32} = 0{\cdot}9425 \times 10^6 \text{ lb. in./rad.}$$

$$s_3 = 2^2 s_C = \frac{4 \times 12 \times 10^6 \pi \times 1{\cdot}5^4}{16 \times 32} = 1{\cdot}4910 \times 10^6 \text{ lb. in./rad.}$$

$$I_1 = I_A = \frac{2000}{32{\cdot}2 \times 12} = 5{\cdot}1760 \text{ lb. in. sec.}^2.$$

$$I_2 = 4^2 I_B = \frac{16 \times 50}{32{\cdot}2 \times 12} = 2{\cdot}0704 \text{ lb. in. sec.}^2.$$

$$I_3 = 2^2 I_0 = \frac{4 \times 60}{32\cdot2 \times 12} = 0\cdot6211 \text{ lb. in. sec.}^2.$$

$$I_0 = I_X + 4^2 I_Y + 2^2 I_Z$$

$$= \frac{1200 + 16 \times 10 + 4 \times 80}{32\cdot2 \times 12} = \frac{1680}{32\cdot2 \times 12} = 4\cdot3478 \text{ lb. in. sec.}^2.$$

$$\omega_1^2 = \frac{s_1}{I_1} = \frac{1\cdot5708 \times 10^6}{5\cdot1760} = 0\cdot3035 \times 10^6.$$

$$\omega_2^2 = \frac{s_2}{I_2} = \frac{0\cdot9425 \times 10^6}{2\cdot0704} = 0\cdot4552 \times 10^6.$$

$$\omega_3^2 = \frac{s_3}{I_3} = \frac{1\cdot4910 \times 10^6}{0\cdot6211} = 2\cdot4006 \times 10^6.$$

Substituting values in the equation

$$\frac{s_1}{\Omega^2 - \omega_1^2} + \frac{s_2}{\Omega^2 - \omega_2^2} + \frac{s_3}{\Omega^2 - \omega_3^2} = I_0$$

$$\frac{1\cdot5708 \times 10^6}{\Omega^2 - 0\cdot3035 \times 10^6} + \frac{0\cdot9425 \times 10^6}{\Omega^2 - 0\cdot4552 \times 10^6} + \frac{1\cdot4910 \times 10^6}{\Omega^2 - 2\cdot4006 \times 10^6}$$
$$= 4\cdot3478.$$

If a rough graph is plotted for each of the terms on the left-hand side, an idea of the values of Ω^2 where the sum of the ordinates is about 4·3 is soon obtained. By trial and error the roots are

$\Omega^2 = 0\cdot3868 \times 10^6$;	$0\cdot8460 \times 10^6$;	$2\cdot8475 \times 10^6$
$\Omega = 621\cdot9$;	$919\cdot8$;	$1687\cdot5$ rad./sec.
$f = \Omega/2\pi = 99\cdot0$;	146;	269 vib./sec.

Exercises XIX

1. A steel shaft has a diameter of 2 in. for a length of 15 in. and a diameter of 3 in. also for a length of 15 in. One end of the shaft is fixed and the other end carries a heavy disc weighing 2000 lb. and having a radius of gyration of 1·6 ft. Determine the frequency of the free torsional vibrations. Modulus of rigidity $= 12 \times 10^6$ lb. per sq. in. [I.Mech.E.]

2. Two rotors, $W_1 = 50$ lb., $k_1 = 5$ in., and $W_2 = 70$ lb., $k_2 = 6$ in., mounted on shafts 2 in. in diameter are coupled together on the same axis and the frequency of torsional oscillation is 100 cycles per sec. Determine the length of the equivalent shaft 2 in. diameter connecting the rotors. If the actual length is 8 in. and the remaining elasticity is in the coupling, find its value in lb. in. per radian. [I.Mech.E.]

3. For purposes of calculation the engine, flywheel and propeller of a ship's drive may be taken as three rotors, mounted in the above order on a 12 in. diameter solid steel shaft. The equivalent moment of inertia of the engine is estimated to be 6 ton ft.^{2} and the moments of inertia of the flywheel and propeller are 30 and 10 ton ft.^{2} respectively. To allow for the entrained water the inertia of the propeller is to be increased by 20 per cent. The effective distance between the engine and flywheel is 20 ft. and between the flywheel and propeller 100 ft.

Working from first principles find the frequencies of free torsional oscillations and draw the corresponding elastic lines assuming an amplitude of 0·1 degree for the propeller. The modulus of rigidity of steel is 5300 tons per sq. in. [U.L.]

4. A uniform shaft 3·5 in. diameter carries three rotors A, B, and C having moments of inertia of 26, 60, and 36 ton in.^{2} respectively. The distance between A and B is 30 in. and between B and C 55 in. Find the frequencies of the free torsional vibrations. If the rotor A has an amplitude of 1° in each case, find the amplitudes of B and C. The modulus of rigidity of the shaft is 5250 tons per sq. in. [U.L.]

5. A shaft has three gear wheels A, B, and C, keyed rigidly to it, wheels A and C being at the ends of the shaft. The moments of inertia of the gear wheels, in lb. ft.^{2}, are 50 for A, 25 for B, and 15 for C. The torsional stiffness of the shaft between A and B is 1000 lb. ft. per degree, and between B and C is 700 lb. ft. per degree.

Find the two possible frequencies of torsional oscillation of the system if the inertia of the shaft be neglected. [U.L.]

6. In Fig. 447 AB is part of the first-frequency elastic line from the worked example on p. 468, the angular amplitudes in radians being taken from the second tabulation on p. 471. The line AB is produced to intersect the shaft axis at N, which is the node if the shaft carries only the rotor 1 and has the same frequency as that of the complete system. Given that the shaft diameter is 6 in., $I_1 = 72$ ton-in.^{2}, and $C = 12 \times 10^6$ lb./in.^{2}, use the node N to show that the angular frequency is 143 rad./sec. approximately.

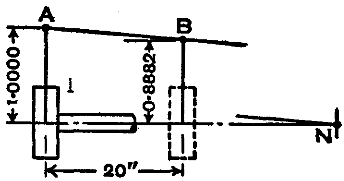

FIG. 447.

7. In the example on the torsional vibrations of a shaft carrying six rotors, p. 468, it was found by assuming a linear relationship between ω and $\Sigma I\theta$ that the second angular frequency is $\omega_2 = 217{\cdot}8$ rad./sec. approximately. Show by the tabulation method that with this value $\Sigma I\theta$ is near zero.

8. Referring again to the example on p. 468, it can be shown that when $\omega = 400$ rad./sec., $\Sigma I\theta = 690$, and when $\omega = 410$ rad./sec., $\Sigma I\theta = -410$. By calculation, assuming a linear law, the curve crosses the axis at the third angular frequency (see graph, Fig. 440) when $\omega = 406{\cdot}3$ rad./sec. approximately. Using this value show by the tabulation method and further interpolation that ω_3 probably lies between 406·4 and 406·5 rad./sec.

9. Fig. 448 shows a system of rotors which are dynamically equivalent, for torsional oscillations, to a four cylinder reciprocating engine and flywheel directly coupled to a rotor. The moment of inertia of the mass at each crank (C_1 to C_4) is 0·6 ton ft.^{2}, that of the flywheel (F) is 20 ton ft.^{2}, and that of the rotor (R) is 8 ton ft.^{2}. The equivalent diameter of the shaft is 6 in.

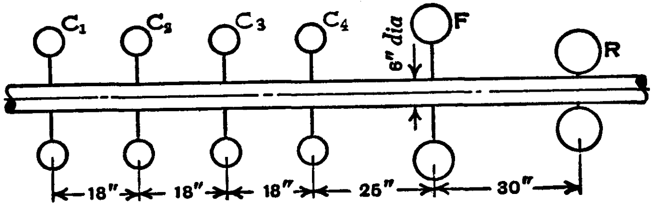

FIG. 448.

Show that the first frequency of natural torsional vibration is approximately 957 vibrations per minute, and draw the elastic line for the shaft, giving the distance of the node from the flywheel. Assume $C = 12 \times 10^6$ lb. per sq. in. [U.L.]

10. Two parallel shafts AB and CD are connected together by toothed gear wheels at B and C, and carry rotors at A and D whose moments of inertia are 1250 lb. in.^{2} and 25,000 lb. in.^{2} respectively; shaft AB is of length 40 in. and diameter $2\frac{1}{2}$ in., and CD is of length 26 in. and diameter $3\frac{1}{4}$ in.; the speed of AB is $2\frac{1}{2}$ times that of CD. The modulus of rigidity of the shaft material is 12×10^6 lb. per sq. in.

Find, from first principles, the natural frequency of vibration of the system, neglecting the masses of the gears at C and B.

Explain concisely how to allow for the moments of inertia of the gear wheels if these were given. [U.L.]

11. An electric motor running at 2250 r.p.m. drives a centrifugal pump running at 650 r.p.m. through a single stage gear reduction.

The motor armature has a moment of inertia of 750 lb. ft.^{2} and the pump impeller one of 2000 lb. ft.^{2}. The shaft from the pump to the gears is $3\frac{1}{2}$ in. diameter and 12 ft. long, and that from the motor to the gears is 2 ft. long.

What should be the diameter of the shaft from the motor to the gears to ensure that the node for natural torsional vibrations is at the gears? Determine the frequency of these vibrations and the amplitude of the impeller vibrations for an amplitude of one degree at the motor.

The inertia of the shafts and the gears may be neglected. The modulus of rigidity for the steel shafts is 12×10^6 lb. per sq. in. [U.L.]

12. The three-branch system in Fig. 449 has two equal branches B and C running at the same speed, and $N_B/N_A = 2$. Assuming that the moments of inertia of the gears can be neglected, equation (3) of the equivalent system (see Art. 221) becomes

$$\frac{s_1}{\Omega^2 - \omega_1^2} + \frac{s_2}{\Omega^2 - \omega_2^2} + \frac{s_3}{\Omega^2 - \omega_3^2} = 0.$$

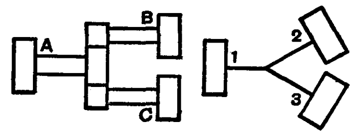

FIG. 449.

Simplifying and then putting $s_3 = s_2$ and $\omega_3 = \omega_2$ (the validity of this step has been shown in Art. 221)

$$(\Omega^2 - \omega_2^2)\{s_1(\Omega^2 - \omega_2^2) + 2s_2(\Omega^2 - \omega_1^2)\} = 0.$$

(a) Show that

$$\Omega^2 = \frac{s_A s_B (I_A + 8I_B)}{I_A I_B (s_A + 8s_B)} \quad \text{and} \quad \frac{s_B}{I_B}.$$

(b) Also obtain the first of these values of Ω^2 by replacing the shafts and rotors B and C, by one shaft of stiffness $2s_B$ carrying a rotor of moment of inertia $2I_B$, and combining it with shaft A to form one equivalent shaft carrying a rotor at each end.

The other value of Ω^2 is found by considering the oscillation of the rotors B and C with the node at the gears, then $\Omega^2 = s_B/I_B$.

CHAPTER XX

VIBRATIONS III

222. Effect of Spring Mass on Vibrations—Rayleigh Method.—In general when considering the vibration of a suspended body, it is sufficiently accurate for practical purposes to neglect the mass of the spring, but its effect will now be considered in several types of suspension, since this refinement is occasionally necessary. It will be assumed that the configuration of the spring is little affected by inertia forces, consequently changes in configuration during vibration may be taken to be practically the same as in the case of a massless spring. Therefore the variations in the sum of the potential energies of the mass and the spring may be regarded as approximately linear (excluding the strain energy which is also called potential energy).

With this assumption the change in the sum of the potential and strain energies is approximately $\frac{1}{2}kx^2$, as in the simpler case in which the suspension is by a massless spring (p. 417), and the energy equation (equation (3), p. 418)

$$\frac{W}{2g}\left(\frac{dx}{dt}\right)^2 + \tfrac{1}{2}kx^2 = \text{const.}$$

is applicable to the system provided the first term is adjusted so that it includes the kinetic energy of the spring. This is done by adding part of the spring mass to the main mass.

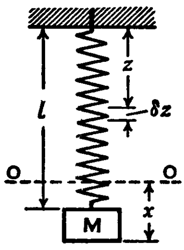

FIG. 450.

1. *Helical Spring.*—In Fig. 450 the spring length is l and the displacement of the mass M of weight W is x at time t, during longitudinal vibration. If w is the weight of the spring per unit length when the length is l, the total weight is wl, a constant.

Suppose v is the velocity of the mass M at time t, then it is assumed that the velocity of a short length δz of the spring, at a distance z from the support, is vz/l. The kinetic energy of the length δz is

$$\frac{w\delta z}{2g}\left(\frac{vz}{l}\right)^2$$

and the kinetic energy of the whole spring is

$$\frac{wv^2}{2gl^2}\int_0^l z^2dz = \frac{wv^2}{2gl^2}\frac{l^3}{3} = \tfrac{1}{2}\left(\frac{wl}{3g}\right)v^2.$$

The kinetic energy of the mass M and the spring is

$$\tfrac{1}{2}\left(\frac{W}{g} + \frac{wl}{3g}\right)v^2$$

therefore the effect of the spring mass during longitudinal vibration may be accounted for by adding one-third of it to the main mass M.

2. *Cantilever Single-Leaf Spring or Beam carrying a Mass* M *at the Free End.*—In this case (Fig. 451) it is assumed that during transverse vibration the deflection curve of the beam is similar to the curve a weightless beam would have if it carried a static load at the free end. The length of the beam is l and its weight per unit length is w.

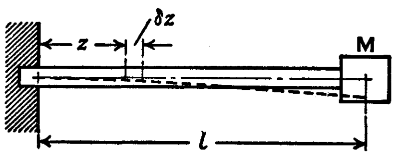

FIG. 451.

It can be shown, for a weightless cantilever beam carrying a static load at the free end, that the ratio of the deflection at a section a distance z from the fixed end, to the deflection at the free end is $(3lz^2 - z^3)/2l^3$.

If v is the transverse velocity at time t of the mass M at the free end, the kinetic energy of the element δz of the spring is

$$\frac{w\delta z}{2g}\left(\frac{3lz^2 - z^3}{2l^3}\right)^2 v^2$$

and the kinetic energy of the whole spring is

$$\frac{wv^2}{8gl^6}\int_0^l(9l^2z^4-6lz^5+z^6)dz=\frac{1}{2}\left(\frac{33wl}{140g}\right)v^2.$$

The kinetic energy of the mass M and the spring is

$$\frac{1}{2}\left(\frac{W}{g}+\frac{33wl}{140g}\right)v^2$$

therefore to allow for the effect of the spring mass during transverse vibration, 33/140 of it is added to the main mass M.

3. *A Beam freely supported at Each End and carrying a Mass* M *at the Centre.*—The beam carrying the mass M at the centre is shown in Fig. 452. The length is l and the weight per unit length is w. During transverse vibration, the deflection curve is assumed to be similar to that of a weightless beam which is freely supported at each end and carries a central static load. For this weightless beam, the ratio of the deflection at a section a distance z from the nearest support, to the deflection at the centre is $(3l^2z-4z^3)/l^3$.

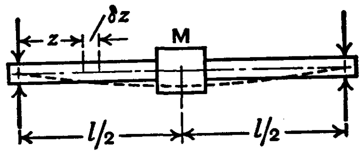

FIG. 452.

The kinetic energy of the whole beam is

$$2\int_0^{l/2}\frac{wv^2}{2gl^6}(3l^2z-4z^3)^2dz=\frac{1}{2}\left(\frac{17}{35}\frac{wl}{g}\right)v^2$$

and in this case 17/35 of the mass of the beam has to be added to the main mass M.

4. *Shaft in Torsion.*—A shaft of length l is fixed at one end and carries a flywheel of moment of inertia I (Fig. 453) at the other end. Torsional vibrations are set up and, neglecting the moment of inertia of the shaft, the energy equation is

$$\frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2+\frac{1}{2}s\theta^2=\text{const.}$$

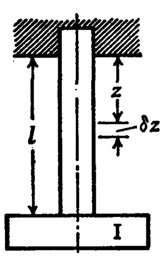

FIG. 453.

where θ is the angle of twist at time t and s is the torsional stiffness. The mass of the shaft may be allowed for as follows.

The angular velocity of a short length δz of the shaft at a distance z from the fixed end is z/l times that of the flywheel. Let i be the moment of inertia of the whole shaft about its axis, then the kinetic energy of the element δz is

$$\frac{i\delta z}{2l}\left(\frac{z}{l}\frac{d\theta}{dt}\right)^2$$

and the kinetic energy of the whole shaft is

$$\frac{i}{2l_3}\left(\frac{d\theta}{dt}\right)^2\int_0^l z^2dz=\tfrac{1}{2}\left(\frac{i}{3}\right)\left(\frac{d\theta}{dt}\right)^2.$$

Therefore one-third of the moment of inertia of the shaft has to be added to the moment of inertia of the flywheel.

In the two-rotor system (Fig. 454), O is the node when the shaft is assumed to be massless and each of the parts into which the system is divided at O is dealt with as already explained. If i is the moment of inertia of the whole shaft, the moments of inertia are adjusted as follows:

Rotor 1. $$\mathrm{I}_1+\frac{ai}{3(a+b)}=\mathrm{I}_1+\frac{\mathrm{I}_2 i}{3(\mathrm{I}_1+\mathrm{I}_2)}.$$

Rotor 2. $$\mathrm{I}_2+\frac{bi}{3(a+b)}=\mathrm{I}_2+\frac{\mathrm{I}_1 i}{3(\mathrm{I}_1+\mathrm{I}_2)}.$$

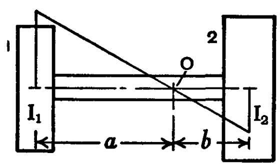

FIG. 454.

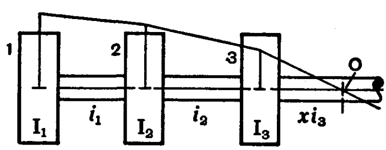

FIG. 455.

In a multi-rotor system the rotors are considered in pairs, 1 and 2, 2 and 3, and so on, then if the two rotors of a pair are in phase and their amplitudes are not very different in value, one half the moment of inertia of the connecting shaft is added to that of each rotor of the pair. For example, using the symbols given in Fig. 455, where part of a system and its elastic line are shown, and assuming xi_3 is the moment of inertia of the shaft length between rotor 3 and the node O, the adjustments to I_1, I_2 and I_3 are made as follows:

Rotor 1.	$I_1 + \frac{1}{2}i_1$.
Rotor 2.	$I_2 + \frac{1}{2}i_1 + \frac{1}{2}i_2$.
Rotor 3.	$I_3 + \frac{1}{2}i_2 + \frac{1}{3}xi_3$.

The reader is here referred to Dr W. Ker Wilson's book *Practical Solution of Torsional Vibration Problems* where the rules for dealing with a multi-rotor system are given. It is also pointed out that "In most practical systems of the type which can be regarded as composed of a series of concentrated masses the influence of shaft inertia is negligible".

223. Forced Vibrations of a Suspended Mass.—Forced and damped vibrations have been dealt with in Art. 204, and by omitting the damping term the effect of forcing without damping is obtained. Since the general equation may not be available when this type of problem has to be solved, the forced vibrations of one suspended mass will now be examined from first principles, on the assumption that damping may be disregarded. As in Art. 204, the forcing action may be applied either at the supported end of the spring or directly on the mass (Figs. 456 and 457).

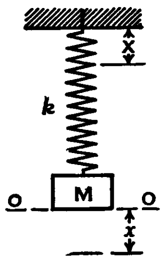

FIG. 456.

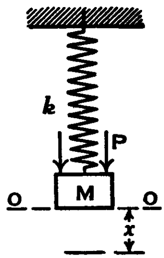

FIG. 457.

In Fig. 456 the mass M (or W/g) is suspended by a spring of stiffness k and the support is given simple harmonic motion, defined by the equation $X = A \sin pt$, in the vertical direction. (It is convenient here to denote the amplitude by A instead of by a_1 as on p. 424.) At time t, X and x are the displacements of the support and mass respectively, measured as positive downwards; then the additional upward force on the mass is $k(x - X)$ and the equation of motion is

$$M\frac{d^2x}{dt^2} = -k(x - X).$$

Dividing by M and denoting k/M by ω^2, the equation may be written

$$\frac{d^2x}{dt^2} + \omega^2 x = \omega^2 A \sin pt \quad . \quad . \quad (1).$$

The natural vibrations are obtained by equating the left-hand side to zero (Art. 204). Assume that the forced vibrations of the mass are given by

$$x = C \sin pt \quad \text{from which} \quad \frac{d^2x}{dt^2} = -p^2 C \sin pt.$$

Substituting in (1) and dividing by $\sin pt$,

$$-p^2 C + \omega^2 C = \omega^2 A$$

from which the amplitude is

$$C = \frac{\omega^2 A}{\omega^2 - p^2}$$

therefore

$$x = \frac{\omega^2 A \sin pt}{\omega^2 - p^2} \quad . \quad . \quad . \quad (2).$$

Resonance occurs when $p = \omega$ and then the amplitude tends to become infinite. If p is less than ω, x is positive and the mass moves in phase with the support. If p is greater than ω, x is negative and the mass is 180° out of phase.

The magnification of amplitude, or ratio of the amplitudes of the motions of the mass and support, is $\omega^2/(\omega^2 - p^2)$ and this value would be obtained by putting $c = 0$ in the expression on p. 425.

Consider now the case where the support is fixed and a periodic force $P = F \sin pt$ is applied to the mass; this force is indicated in Fig. 457 by two arrows.

At time t the additional upward force on the mass is

$$kx - F \sin pt$$

therefore
$$M\frac{d^2x}{dt^2} = -kx + F \sin pt.$$

Suppose the force F would produce a static deflection A in the spring, then $F = kA$ and the equation may be written

$$\frac{d^2x}{dt^2} + \omega^2 x = \omega^2 A \sin pt$$

as in (1). Therefore the motion of the mass is the same in each case provided $F = kA$ and the displacement of the mass when the periodic force $F \sin pt$ is applied may be expressed in the following ways:

$$x = \frac{\omega^2 A \sin pt}{\omega^2 - p^2} = \frac{\omega^2 F \sin pt}{k(\omega^2 - p^2)} = \frac{F \sin pt}{M(\omega^2 - p^2)} \quad . \quad (3).$$

224. Natural Vibrations of Two Suspended Masses.—Two masses M_1 and M_2 (or W_1/g and W_2/g) are shown in Fig. 458 suspended, one under the other, by springs of stiffnesses k_1 and k_2 respectively. Let x_1 and x_2 be the deflections, measured as positive downwards from the positions of rest O_1O_1 and O_2O_2, at time t, when the masses are vibrating freely along the common axis of the springs. The system has two degrees of freedom along the axis, since it can vibrate in two distinct modes.

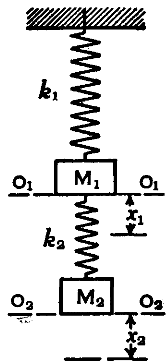

FIG. 458.

The increase in the length of the upper spring is x_1 and that of the lower spring is $x_2 - x_1$; these increases are accompanied by the additional upward forces

$$k_1x_1 - k_2(x_2 - x_1) \quad \text{and} \quad k_2(x_2 - x_1)$$

acting on the upper and lower masses respectively, opposing their motions.

The equations of motion are

$$M_1\frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1)$$

$$M_2\frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1)$$

or

$$M_1\frac{d^2x_1}{dt^2} + (k_1 + k_2)x_1 - k_2x_2 = 0 \quad . \quad . \quad (1),$$

$$M_2\frac{d^2x_2}{dt^2} - k_2x_1 + k_2x_2 = 0 \quad . \quad . \quad (2).$$

Assume $x_1 = a_1 \cos \omega t$ and $x_2 = a_2 \cos \omega t$. (3),

where a_1 and a_2 are the amplitudes.

Differentiating x_1 and x_2 twice,

$$\frac{d^2x_1}{dt^2} = -\omega^2a_1 \cos \omega t, \quad \frac{d^2x_2}{dt^2} = -\omega^2a_2 \cos \omega t \quad . \quad (4).$$

Substituting values from (3) and (4) in (1) and (2), and dividing through by $-\cos \omega t$

$$M_1\omega^2a_1 - (k_1 + k_2)a_1 + k_2a_2 = 0 \quad . \quad . \quad (5),$$

$$M_2\omega^2a_2 + k_2a_1 - k_2a_2 = 0 \quad . \quad . \quad (6).$$

From (5) the ratio of the amplitudes is

$$\frac{a_1}{a_2} = \frac{k_2}{k_1 + k_2 - M_1\omega^2} \quad . \quad . \quad . \quad (7),$$

and from (6)

$$\frac{a_1}{a_2} = \frac{k_2 - M_2\omega^2}{k_2} \quad . \quad . \quad . \quad (8).$$

These values of a_1/a_2 must be equal, therefore

$$\frac{k_2}{k_1 + k_2 - M_1\omega^2} = \frac{k_2 - M_2\omega^2}{k_2}$$

from which

$$\omega^4 - \left(\frac{k_1 + k_2}{M_1} + \frac{k_2}{M_2}\right)\omega^2 + \frac{k_1k_2}{M_1M_2} = 0 \quad . \quad . \quad (9),$$

and the roots of this quadratic equation in ω^2 are

$$\omega^2 = \tfrac{1}{2}\left(\frac{k_1+k_2}{M_1}+\frac{k_2}{M_2}\right) \pm \sqrt{\tfrac{1}{4}\left(\frac{k_1+k_2}{M_1}+\frac{k_2}{M_2}\right)^2 - \frac{k_1k_2}{M_1M_2}} \quad . \quad (10).$$

Let Ω_1 and Ω_2 be the two positive roots, then from (3)

$$\left.\begin{aligned} x_1 &= a_1 \cos \Omega_1 t \\ x_2 &= a_2 \cos \Omega_1 t \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} x_1 &= a_1 \cos \Omega_2 t \\ x_2 &= a_2 \cos \Omega_2 t \end{aligned}\right\} \quad . \quad (11),$$

and the ratio a_1/a_2 has two values which are obtained from either (7) or (8). From (8)

$$\frac{a_1}{a_2} = \frac{k_2 - M_2\Omega_1^2}{k_2} \quad \text{or} \quad \frac{a_1}{a_2} = \frac{k_2 - M_2\Omega_2^2}{k_2} \quad . \quad (12).$$

The two masses vibrate at the same frequency, corresponding to either Ω_1 or Ω_2. At the lower frequency the masses move in the same sense, but at the upper frequency

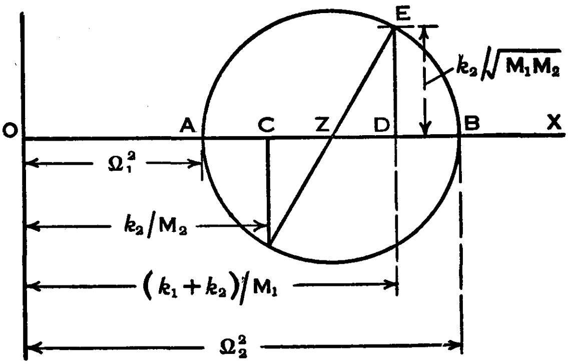

FIG. 459.

they move in opposite senses, the values of a_1/a_2 being positive and negative respectively. This fact is easily proved algebraically, but the application of Mohr's circle given by J. P. Den Hartog in a similar problem * can be used to provide a simple demonstration.

On a line OX mark off $OC = k_2/M_2$ and $OD = (k_1+k_2)/M_1$ to a suitable scale (Fig. 459). Erect a perpendicular at D and

* *Mechanical Vibrations* by J. P. Den Hartog. 2nd Ed. McGraw-Hill

mark off $DE = k_2/\sqrt{M_1 M_2}$. With centre Z, the mid point of CD, and radius ZE, describe the circle AEB intersecting OX at A and B. Then OA and OB represent to scale Ω_1^2 and Ω_2^2 respectively. The proof is briefly as follows:

$$OZ = \tfrac{1}{2}(OD + OC) = \tfrac{1}{2}\left(\frac{k_1 + k_2}{M_1} + \frac{k_2}{M_2}\right).$$

$$AZ^2 = ZB^2 = ZE^2 = ZD^2 + DE^2 = \tfrac{1}{4}\left(\frac{k_1 + k_2}{M_1} - \frac{k_2}{M_2}\right)^2 + \frac{k_2^2}{M_1 M_2}.$$

Squaring the expression in brackets and simplifying, the result can be written as

$$AZ = ZB = ZE = \sqrt{\tfrac{1}{4}\left(\frac{k_1 + k_2}{M_1} + \frac{k_2}{M_2}\right)^2 - \frac{k_1 k_2}{M_1 M_2}}.$$

Therefore the roots given by (10) are represented by

$$\omega^2 = OZ \pm ZE$$

or $$\Omega_1^2 = OZ - AZ = OA$$

and $$\Omega_2^2 = OZ + ZB = OB.$$

From Fig. 459 it can be seen that Ω_1^2 is less than k_2/M_2 and Ω_2^2 is greater than k_2/M_2, or $M_2\Omega_1^2 < k_2$ and $M_2\Omega_2^2 > k_2$. Therefore it follows from (12) that the amplitude ratio a_1/a_2 is positive at the lower frequency and negative at the upper frequency.

The general solution of equation (9) is the sum of the solutions given by (11) and the resulting motion is periodic but not simple harmonic. Further similar examples in which there are two natural modes of vibration are the double pendulum (Art. 227), the coupled pendulums (Art. 228) and a bar supported on two springs (Art. 229).

225. Forced Vibrations of Two Suspended Masses.—The two masses M_1 and M_2 suspended one under the other, as described in the preceding Art., are shown again in Figs. 460 and 461. Suppose that in Fig. 460 the support is given simple harmonic motion defined by the equation $X = A \sin pt$ and in Fig. 461 the support is fixed and a periodic force $P = F \sin pt$ (represented by two arrows) is applied to the mass M_1. The forced vibrations of the masses will be investigated on the assumption that the damping is negligible.

At time t the displacements are X, x_1, and x_2 as shown. For the case where the support is oscillating the increase in the length of the upper spring is $x_1 - X$ and that of the lower spring is $x_2 - x_1$. The additional upward forces are

$$k_1(x_1 - X) - k_2(x_2 - x_1)$$

and $\quad k_2(x_2 - x_1)$

opposing the motions of the upper and lower masses respectively.

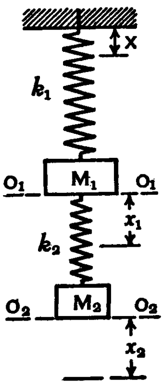

FIG. 460.

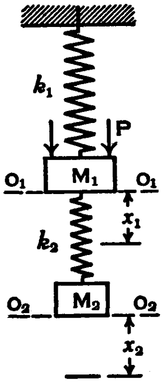

FIG. 461.

These forces are algebraically the same as in the preceding Art. with the addition of the term $-k_1X$ or $-k_1A \sin pt$, and the equations of motion are

$$M_1\frac{d^2x_1}{dt^2} + (k_1 + k_2)x_1 - k_2x_2 = k_1A \sin pt \quad . \quad (1),$$

$$M_2\frac{d^2x_2}{dt^2} - k_2x_1 + k_2x_2 = 0 \quad . \quad . \quad (2).$$

For the case where the periodic force $P = F \sin pt$ is applied to the upper mass, the increase in length of the upper spring is x_1 and that of the lower spring is $x_2 - x_1$. The additional upward forces are

$$k_1x_1 - F \sin pt - k_2(x_2 - x_1) \quad \text{and} \quad k_2(x_2 - x_1)$$

opposing the motions of the upper and lower masses respectively.

If $F \sin pt$ is written as $k_1A \sin pt$, where A is the static deflection the force F would produce in the upper spring, then the equations of motion are as given in (1) and (2).

Assume that the forced vibrations of the masses are given by

$$x_1 = a_1 \sin pt \quad \text{and} \quad x_2 = a_2 \sin pt \qquad . \quad (3).$$

Differentiating twice, substituting values in (1) and (2), and dividing through by $-\sin pt$,

$$M_1 p^2 a_1 - (k_1 + k_2)a_1 + k_2 a_2 = -k_1 A \qquad . \quad (4),$$

$$M_2 p^2 a_2 + k_2 a_1 - k_2 a_2 = 0 \qquad . \quad . \quad (5).$$

From (4)
$$a_1 = \frac{k_1 A + k_2 a_2}{k_1 + k_2 - M_1 p^2} \qquad . \quad . \quad . \quad (6),$$

and from (5)
$$a_1 = \frac{(k_2 - M_2 p^2)a_2}{k_2} \qquad . \quad . \quad . \quad (7).$$

The values of the amplitudes a_1 and a_2 can be obtained from (6) and (7), and substituted in (3) to give x_1 and x_2.

226. Vibration Absorber.—It will now be shown that the mass M_2 can be used to prevent the forced vibrations of the main mass M_1 (Fig. 461).

Dividing (4) and (5) of the preceding Art. by M_1 and M_2 respectively,

$$p^2 a_1 - \left(\frac{k_1}{M_1} + \frac{k_2}{M_1}\right)a_1 + \frac{k_2}{M_1}a_2 = -\frac{k_1}{M_1}A \qquad . \quad (1),$$

$$p^2 a_2 + \frac{k_2}{M_2}(a_1 - a_2) = 0 . \qquad . \quad . \quad (2).$$

Suppose that when the masses M_1 and M_2 are suspended separately, each by its own spring, the natural frequencies of the two systems are $\omega_1/2\pi$ and $\omega_2/2\pi$ respectively, then

$$\omega_1^2 = \frac{k_1}{M_1} \quad \text{and} \quad \omega_2^2 = \frac{k_2}{M_2}.$$

Eliminating M_1 and M_2 in (1) and (2) and solving for a_1 and a_2, it is easy to show that

$$\frac{a_1}{A} = \frac{1 - \dfrac{p^2}{\omega_2^2}}{\left(1 + \dfrac{k_2}{k_1} - \dfrac{p^2}{\omega_1^2}\right)\left(1 - \dfrac{p^2}{\omega_2^2}\right) - \dfrac{k_2}{k_1}} \qquad . \quad . \quad (3),$$

$$\frac{a_2}{A}=\frac{1}{\left(1+\frac{k_2}{k_1}-\frac{p^2}{\omega_1^2}\right)\left(1-\frac{p^2}{\omega_2^2}\right)-\frac{k_2}{k_1}} \quad . \quad . \quad (4).$$

It can be seen from (3) that a_1, the amplitude of the main mass M_1, is zero when

$$1-\frac{p^2}{\omega_2^2}=0, \quad \text{or} \quad \omega_2^2=p^2.$$

Substituting this value in (4), the denominator becomes $-k_2/k_1$ and

$$a_2=-\frac{k_1A}{k_2}=-\frac{F}{k_2},$$

where $F=k_1A$ is the maximum value of the periodic force $F \sin pt$ applied to the mass M_1.

The displacement of the mass M_2 at time t is $x_2=a_2 \sin pt$, as assumed in Art. 225, equation (3), therefore

$$x_2=-\frac{F}{k_2}\sin pt \quad \text{or} \quad x_2k_2=-F \sin pt,$$

that is the force x_2k_2 in the lower spring is equal and opposite to the periodic force $F \sin pt$ applied to the main mass, and the latter remains at rest.

Therefore the vibration of the main mass M_1 can be prevented by arranging the natural angular frequency ω_2 of the absorber system, consisting of the mass M_2 and the spring of stiffness k_2, to be equal to the angular frequency p of the disturbing force $F \sin pt$, then $k_2/M_2=p^2$.

The amplitudes a_1 and a_2 would tend to become infinite and resonance would occur if the denominator in (3) and (4) were zero. See Ex. 10, p. 507.

227. Natural Oscillations of a Double Pendulum.—Two simple pendulums are suspended, as shown in Fig. 462, one below the other. The lengths are l_1 and l_2 and the masses of the bobs are $M_1=W_1/g$ and $M_2=W_2/g$. The natural oscillations of the bobs will be investigated on the assumption that the angles of swing are small; these angles are exaggerated in Fig. 462.

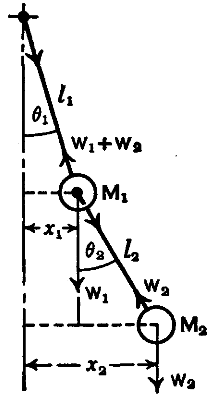

FIG. 462.

Let the horizontal displacements of the masses M_1 and M_2 at time t be x_1 and x_2 respectively, and the corresponding angular displacements of the pendulums be θ_1 and θ_2. Since these angles are small, the tension in the lower link is approximately W_2 and in the upper link is approximately W_1+W_2.

The horizontal component of the tension W_2 is $W_2(x_2-x_1)/l_2$ acting inwards on the mass M_2 and outwards on the mass M_1; also the horizontal component of the tension W_1+W_2 is $(W_1+W_2)x_1/l_1$ acting inwards on the mass M_1.

The equations of motion are as follows:

$$\frac{W_1}{g}\frac{d^2x_1}{dt^2}=-\frac{(W_1+W_2)x_1}{l_1}+\frac{W_2(x_2-x_1)}{l_2} \quad . \quad (1).$$

$$\frac{W_2}{g}\frac{d^2x_2}{dt^2}=-\frac{W_2(x_2-x_1)}{l_2} \quad . \quad . \quad (2).$$

Assume $\quad x_1=a_1\cos\omega t, \quad x_2=a_2\cos\omega t,$

then $\quad \dfrac{d^2x_1}{dt^2}=-\omega^2a_1\cos\omega t, \quad \dfrac{d^2x_2}{dt^2}=-\omega^2a_2\cos\omega t.$

Substituting in (1) and (2) and dividing by $-\cos\omega t$,

$$\frac{W_1}{g}\omega^2a_1=\frac{(W_1+W_2)a_1}{l_1}-\frac{W_2(a_2-a_1)}{l_2} \quad . \quad (3).$$

$$\frac{W_2}{g}\omega^2a_2=\frac{W_2(a_2-a_1)}{l_2}. \quad . \quad . \quad (4).$$

From (4) $$\frac{a_1}{a_2}=\frac{g-\omega^2l_2}{g} \quad . \quad . \quad . \quad (5).$$

The value of a_1/a_2 can also be obtained from (3), but it is simpler to use the sum of (3) and (4), that is

$$\frac{W_1}{g}\omega^2 a_1+\frac{W_2}{g}\omega^2 a_2=\frac{(W_1+W_2)a_1}{l_1}$$

from which
$$\frac{a_1}{a_2}=\frac{W_2\omega^2 l_1}{(W_1+W_2)g-W_1\omega^2 l_1}\quad . \quad . \quad (6).$$

Equating the values of a_1/a_2 in (5) and (6) and simplifying,

$$\omega^4-\frac{(W_1+W_2)(l_1+l_2)g}{W_1l_1l_2}\omega^2+\frac{(W_1+W_2)g^2}{W_1l_1l_2}=0\quad . \quad (7).$$

The roots of this quadratic equation in ω^2 are

$$\omega^2=\frac{(W_1+W_2)(l_1+l_2)g}{2W_1l_1l_2}\pm\sqrt{\frac{(W_1+W_2)^2(l_1+l_2)^2g^2}{4W_1^2l_1^2l_2^2}-\frac{(W_1+W_2)g^2}{W_1l_1l_2}}\quad (8).$$

Alternatively equation (7) could be derived by considering angular displacements and the ratio of the angular amplitudes which, of course, is not equal to a_1/a_2.

Let A and B be the angular amplitudes of the upper and lower pendulums respectively, then the value of A/B may now be obtained as follows. As before the relations are approximately correct when the amplitudes are small.

From Fig. 462, $\theta_1=x_1/l_1$ and $\theta_2=(x_2-x_1)/l_2$, therefore $A=a_1/l_1$ and $B=(a_2-a_1)/l_2$. By division

$$\frac{A}{B}=\frac{a_1l_2}{(a_2-a_1)l_1}=\frac{l_2}{(a_2/a_1-1)l_1}.$$

Substituting for a_2/a_1 from (5) and simplifying

$$\frac{A}{B}=\frac{g-\omega^2l_2}{\omega^2l_1}\quad . \quad . \quad . \quad (9).$$

Example.—Given that the lengths of the pendulums and the masses of the bobs are as shown in Fig. 463 (i), it is required to find the frequencies of small oscillations and the ratios of the amplitudes of the bobs and of the links.

Substituting values in equation (8),

$$\omega^2 = \frac{(5+3)(2+1)g}{2\times 5\times 2\times 1} \pm \sqrt{\frac{8^2\times 3^2 g^2}{4\times 5^2\times 2^2} - \frac{8g^2}{5\times 2}}$$

$$= 1{\cdot}2g \pm \sqrt{1{\cdot}44g^2 - 0{\cdot}8g^2}$$

$$= 0{\cdot}4g \quad \text{or} \quad 2g \text{ rad.}^2\text{/sec.}^2, \text{ if } g \text{ is in ft./sec.}^2$$

$$f = \frac{60\omega}{2\pi} \text{ vib./min.}$$

$$\text{Lower frequency} = \frac{30}{\pi}\sqrt{0{\cdot}4\times 32{\cdot}2} = 34{\cdot}3 \text{ vib./min.}$$

$$\text{Higher frequency} = \frac{30}{\pi}\sqrt{2\times 32{\cdot}2} = 76{\cdot}6 \text{ vib./min.}$$

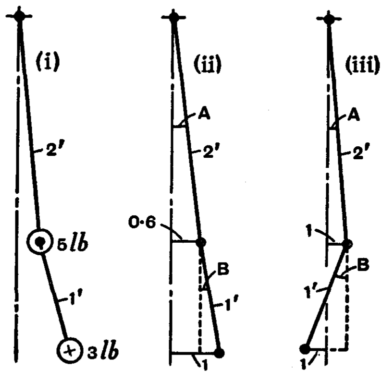

FIG. 463.

From (5) the ratio of the amplitudes of the bobs is

$$\frac{a_1}{a_2} = \frac{g-\omega^2 l_2}{g} = \frac{g-\omega^2}{g}.$$

When $\omega^2 = 0{\cdot}4g$, $a_1/a_2 = 0{\cdot}6$.

When $\omega^2 = 2g$, $a_1/a_2 = -1$.

The two configurations are shown diagrammatically in Figs. 463 (ii) and (iii), where convenient lengths are used for the amplitudes in each case, the actual values being unknown.

The ratios of the angular amplitudes may be obtained from equation (9) or by inspection from Figs. 463 (ii) and (iii).

From (ii) $$\frac{A}{B}=\frac{0\cdot6/2}{(1-0\cdot6)/1}=0\cdot75.$$

From (iii) $$\frac{A}{B}=\frac{1/2}{(-1-1)/1}=-0\cdot25.$$

228. Natural Oscillations of Coupled Pendulums.—Two simple pendulums 1 and 2 (Fig. 464), of length l and mass $M=W/g$, suspended at points O_1 and O_2 on the same level, are connected by a light spring of stiffness k at points a distance a below O_1O_2. It is arranged that there is no force in the spring when the pendulums are at rest. It is required to find the natural frequencies of oscillation when the swings are small, neglecting the mass of the spring.

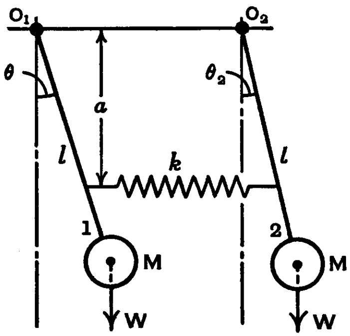

FIG. 464.

If θ_1 and θ_2 are the angular displacements at time t, the change in the length of the spring is approximately $a(\theta_1-\theta_2)$ and the spring force is $ka(\theta_1-\theta_2)$.

For the configuration shown where θ_1 is greater than θ_2, this force is compressive, therefore it opposes the motion of pendulum 1 and assists that of pendulum 2. The equations of motion are

$$\frac{W}{g}l^2\frac{d^2\theta_1}{dt^2}=-Wl\theta_1-ka^2(\theta_1-\theta_2) \quad . \quad . \quad (1),$$

$$\frac{W}{g}l^2\frac{d^2\theta_2}{dt^2}=-Wl\theta_2+ka^2(\theta_1-\theta_2) \quad . \quad . \quad (2).$$

Assume $\theta_1=A\cos\omega t$ and $\theta_2=B\cos\omega t$

then $$\frac{d^2\theta_1}{dt^2}=-\omega^2A\cos\omega t \qquad \frac{d^2\theta_2}{dt^2}=-\omega^2B\cos\omega t.$$

Substituting these values in (1) and (2) and dividing by $-\cos \omega t$,

$$\frac{W}{g}l^2\omega^2 A = WlA + ka^2(A - B) \quad . \quad . \quad (3),$$

$$\frac{W}{g}l^2\omega^2 B = WlB - ka^2(A - B) \quad . \quad . \quad (4).$$

From (3) $$\frac{A}{B} = \frac{-ka^2g}{Wl^2\omega^2 - Wlg - ka^2g} \quad . \quad . \quad (5).$$

From (4) $$\frac{A}{B} = \frac{Wl^2\omega^2 - Wlg - ka^2g}{-ka^2g} \quad . \quad . \quad (6).$$

Therefore $(Wl^2\omega^2 - Wlg - ka^2g)^2 - (ka^2g)^2 = 0.$

Factorizing

$$(Wl^2\omega^2 - Wlg)(Wl^2\omega^2 - Wlg - 2ka^2g) = 0$$

and $$\omega^2 = \frac{g}{l} \quad \text{or} \quad \frac{g}{l} + \frac{2kg}{W}\frac{a^2}{l^2}.$$

Substituting these values of ω^2 in (5) gives $A/B = 1$ or -1.

At the lower angular frequency, $\omega = \sqrt{g/l}$, the two pendulums swing in phase, but at the higher angular frequency they swing alternately away from and towards one another.

229. Natural Vibrations of a Horizontal Bar supported on Two Springs.—A uniform bar of length $2l$ is supported in a horizontal position by vertical springs 1 and 2, of stiffnesses k_1 and k_2, at distances a and b on either side of the centre of gravity G (Fig. 465). The mass of the bar is M and its moment of inertia about an axis through G perpendicular to the paper is $I = \frac{1}{3}Ml^2$. It is required to find the natural frequencies of vibration of the bar in the vertical plane.

Let x be the vertical displacement of the centre of gravity G at time t and θ be the inclination of the bar to the horizontal, as shown exaggerated. Let the corresponding amplitudes be H and A. The positive directions are downward for x and anti-clockwise for θ.

Spring	Deflection	Additional upward force on bar
1	$x+a\theta$	$k_1(x+a\theta)$
2	$x-b\theta$	$k_2(x-b\theta)$

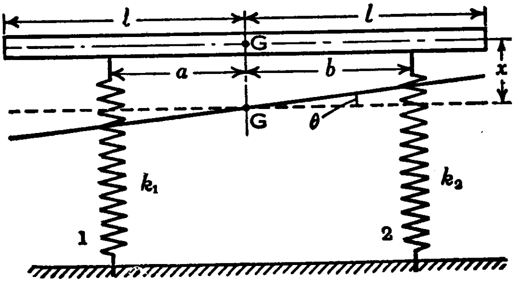

FIG. 465.

The equations of motion are

$$M\frac{d^2x}{dt^2} = -k_1(x+a\theta) - k_2(x-b\theta) \quad . \quad . \quad (1),$$

$$I\frac{d^2\theta}{dt^2} = -k_1(x+a\theta)a + k_2(x-b\theta)b \quad . \quad . \quad (2),$$

which may be written

$$M\frac{d^2x}{dt^2} + (k_1+k_2)x + (k_1a-k_2b)\theta = 0 \quad . \quad (3),$$

$$I\frac{d^2\theta}{dt^2} + (k_1a-k_2b)x + (k_1a^2+k_2b^2)\theta = 0 \quad . \quad (4).$$

Assume $x = H\cos\omega t, \quad \theta = A\cos\omega t$

then $\dfrac{d^2x}{dt^2} = -\omega^2 H\cos\omega t, \quad \dfrac{d^2\theta}{dt^2} = -\omega^2 A\cos\omega t$

and equations (3) and (4) become

$$M\omega^2 H - (k_1+k_2)H - (k_1a-k_2b)A = 0 \quad . \quad (5),$$

$$I\omega^2 A - (k_1a-k_2b)H - (k_1a^2+k_2b^2)A = 0 \quad . \quad (6).$$

From (5) $$\frac{H}{A} = \frac{k_1a-k_2b}{M\omega^2-(k_1+k_2)} \quad . \quad . \quad . \quad (7),$$

and from (6) $$\frac{H}{A} = \frac{I\omega^2-(k_1a^2+k_2b^2)}{k_1a-k_2b} \quad . \quad . \quad (8).$$

Equating the values of H/A gives

$$\{M\omega^2-(k_1+k_2)\}\{I\omega^2-(k_1a^2+k_2b^2)\}=(k_1a-k_2b)^2$$

which reduces to

$$MI\omega^4-\{M(k_1a^2+k_2b^2)+I(k_1+k_2)\}\omega^2+k_1k_2(a+b)^2=0 \quad (9),$$

or, dividing by MI and substituting $I=\frac{1}{3}Ml^2$,

$$\omega^4-\left\{\frac{3(k_1a^2+k_2b^2)}{Ml^2}+\frac{k_1+k_2}{M}\right\}\omega^2+\frac{3k_1k_2(a+b)^2}{M^2l^2}=0 \quad (10).$$

The roots of this quadratic equation in ω^2 are the required values of ω^2 and their square roots are the natural angular frequencies in radians per second, provided the various units are suitably chosen.

Example.—Assume $a=b=0{\cdot}8l$, $k_1=1{\cdot}5k_2$, and use the symbol k for k_2. Making these substitutions, equation (10) becomes

$$\omega^4-(7{\cdot}5\times0{\cdot}64+2{\cdot}5)\frac{k}{M}\omega^2+18\times0{\cdot}64\frac{k^2}{M^2}=0$$

or

$$\omega^4-7{\cdot}3\frac{k}{M}\omega^2+11{\cdot}52\frac{k^2}{M^2}=0$$

and the roots of this equation are

$$\omega^2=2{\cdot}307k/M \quad \text{or} \quad 4{\cdot}993k/M.$$

The natural angular frequencies are

$$\omega_1=\sqrt{2{\cdot}307k/M} \quad \text{and} \quad \omega_2=\sqrt{4{\cdot}993k/M}.$$

The relations between the amplitudes H and A are obtained by substituting the values of ω^2 in either (7) or (8).

When $\omega^2=2{\cdot}307k/M$ it can be shown that

$$\frac{H}{A}=-2{\cdot}07l \quad \text{or} \quad A=-\frac{H}{2{\cdot}07l}.$$

When $\omega^2=4{\cdot}993k/M$,

$$\frac{H}{A}=0{\cdot}16l \quad \text{or} \quad A=\frac{H}{0{\cdot}16l}.$$

These values are illustrated in Figs. 466 and 467 which are diagrammatic only, since the actual values of H are

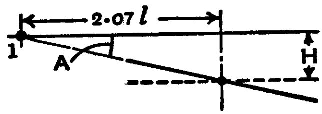

FIG. 466.

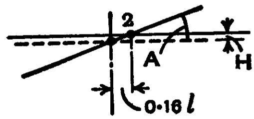

FIG. 467.

unknown. In the first case the bar oscillates about the point 1 and in the second case about the point 2.

BIBLIOGRAPHY (see also p. 452)

Mechanical Vibrations. By J. P. Den Hartog. McGraw-Hill.
Examples in Mechanical Vibrations. By John Hannah and R. C. Stephens. Edward Arnold. This book includes worked examples and many questions for solution.

Exercises XX

1. A mass weighing 20 lb. is suspended by a helical spring weighing 3 lb. and the system is vibrating along the spring axis. Find the ratio of the frequency when the mass of the spring is neglected to the frequency when this mass is taken into account.

2. A steel cantilever 1 in. diameter and $l = 40$ in. carries a mass of 5 lb. at the free end. The mass of the bar is equivalent to two concentrated masses each of 4 lb. at 10 in. and 30 in. respectively from the fixed end. Assuming that, when vibrating freely, the deflection curve is the same as the statical deflection curve due to the end load W only, i.e. $y = \frac{Wl^3}{3EI}\left\{\frac{3}{2}\left(\frac{x}{l}\right)^2 - \frac{1}{2}\left(\frac{x}{l}\right)^3\right\}$, where x is the distance from the fixed end, find by using an energy method the frequency of transverse vibration. $E = 30 \times 10^6$ lb./in.^{2}.
[I.Mech.E.]

(*Note.*—The answer was obtained by using the method given in Art. 210.)

3. (*a*). Solve the preceding Ex. by the method explained in Art. 222, para. 2, but allowing for the intermediate concentrated masses by making two additions to the mass at the free end.

(*b*). What is the value of the transverse frequency if it is assumed that the uniformly distributed mass of the bar is 8 lb. and 33/140 of it is added to the 5 lb. mass?

4. The natural frequency of torsional vibration of an unloaded shaft fixed at one end is (see top of p. 437)

$$f=\frac{1}{4l}\sqrt{\frac{Cg}{w}}$$

where l is the length, w is the weight per unit volume, and C is the modulus of rigidity.

It is shown in Art. 222, para. 4, that when the shaft carries a flywheel at the free end and torsional vibrations are set up, an allowance for the mass of the shaft can be made by adding one-third of its moment of inertia to that of the flywheel. Show that if there is no flywheel and this approximation is used to find the natural frequency of the shaft,

$$f=\frac{\sqrt{3}}{2\pi l}\sqrt{\frac{Cg}{w}}$$

and that the error is $+10{\cdot}27$ per cent.

5. A mass of 50 lb. hangs from a spring of stiffness 10 lb. per inch. The upper end of the spring is given simple harmonic motion with a total vertical movement of 1 in. and frequency of 50 cycles per minute.

Find (*a*) the amplitude of the motion of the mass; (*b*) the amplitude if the frequency is 100 cycles per minute. Point out the difference between the two types of response. [U.L.]

6. An electric motor weighing 700 lb. and resting on a flexible support without damping is set into resonant vertical vibration as the speed passes through the value of 800 r.p.m. Calculate the stiffness of the support in lb. per in. of deflection. If at the final running speed of 1200 r.p.m. the total movement arising from the forced vibration is 0·02 in., estimate the out of balance in the motor rotor in lb. at 1 in. radius. [I.Mech.E.]

7. A beam section is formed of two channels with flange plates, the moment of inertia of the section being 55 in.^{4}. The beam, of total weight 600 lb., is freely supported at the ends of a span of 15 ft. and carries at the centre of the span a motor of total weight 3000 lb. and rotor weight 1000 lb. The rotor is slightly out of balance and at a speed of 370 r.p.m. a forced vibration of 0·1 in. is measured at the motor position. Neglecting damping, estimate the amount by which the rotor is out of balance. To allow for the effect of the mass of the beam half its weight may be added to the motor weight. $E=30\times10^6$ lb./in.^{2}. [U.L.]

(It is assumed forced amplitude $=\frac{1}{2}\times0{\cdot}1$ in.; $k=48EI/l^3$.)

8. Two masses M_1 and M_2 (or W_1/g and W_2/g) are suspended, with M_1 above M_2, by springs of stiffnesses k_1 and k_2 respectively, as in Fig. 458.

Show that when the system is vibrating along the common axis of the springs the natural angular frequencies are given by the roots of the equation

$$\omega^4 - \left(\frac{k_1+k_2}{M_1}+\frac{k_2}{M_2}\right)\omega^2+\frac{k_1k_2}{M_1M_2}=0.$$

If $W_1=40$ lb., $W_2=20$ lb., $k_1=80$ lb./in., and $k_2=60$ lb./in., find the two natural frequencies in vibrations per minute and the values of a_1/a_2, the ratio of the amplitudes.

9. Masses M_1 and M_2 are suspended between fixed supports by springs of stiffnesses k_1, k_2, and k_3, as in Fig. 468.

Show that if the system is vibrating along the axis in a natural mode the ratio of the amplitudes of the masses is

$$\frac{a_1}{a_2}=\frac{k_2}{k_1+k_2-M_1\omega^2}=\frac{k_2+k_3-M_2\omega^2}{k_2}$$

and the natural angular frequencies are obtained from the roots of the equation

$$\omega^4-\left(\frac{k_1+k_2}{M_1}+\frac{k_2+k_3}{M_2}\right)\omega^2+\frac{k_1k_2+k_2k_3+k_3k_1}{M_1M_2}=0.$$

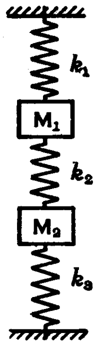

FIG. 468.

10. In Art. 226 on absorbing vibrations, equations (3) and (4), pp. 496 and 497, give the amplitudes a_1 and a_2 of the main mass M_1 and the mass M_2 of the absorber system respectively. In each case the amplitude tends to become infinite when the denominator is zero. Assuming $\omega_1^2=\omega_2^2$ and $k_2/k_1=M_2/M_1=\mu$, show that resonance occurs when

$$\frac{p^2}{\omega_2^2}=1+\tfrac{1}{2}\mu\pm\sqrt{\mu+\tfrac{1}{4}\mu^2}.$$

Find the values of p/ω_2 when (*a*) $M_2=0{\cdot}1M_1$ and (*b*) $M_2=0{\cdot}4M_1$.

11. Two simple pendulums are suspended one below the other as in Fig. 462. The mass of each bob weighs 2 lb., the upper pendulum is 20 in. long and the lower is 40 in. long. Find the two natural frequencies of small oscillations (per minute) and the values of the ratio of the amplitudes of the bobs.

ANSWERS

Exercises II, pp. 28–31.

1. 21·3 ft./sec. at 59° 3′ to OX. 2. 59·5 ft./sec. at 40° 1′ to OX.
3. 6·09, 8·34, 9·74, and 10 ft./sec.
4. (*a*) At A; (*b*) At infinity perpendicular to AC.
5. 30·1 ft./sec. 6. Crankshaft 13·1 ft./sec., crank pin 10·8 ft./sec.
7. (*a*) At point of contact with rail; top point moves at 160 ft./sec.; (*b*) At centre of wheel; (*c*) On vertical diameter and $1\frac{3}{4}$ ft. above rail; sliding velocity 7 ft./sec.; velocity of top point 17 ft./sec.; (*d*) 13 ft./sec. perpendicular to line joining foremost point to instantaneous centre.
8. (*a*) 37° 53′; (*b*) 55·71 ft./sec.; (*c*) 15·01 ft./sec.; (*d*) 73° 48′.
9. Vel. of P = 1·73 ft./sec.; vel. of D = 3·19 ft./sec.
10. $\omega_{BC} = 0{\cdot}36$ rad./sec.; $\omega_{CD} = 1{\cdot}27$ rad./sec.; $v_C = 1{\cdot}49$ ft./sec.; vel. of mid point of BC = 1·56 ft./sec.
11. $v_A = 7{\cdot}72$ ft./sec.; $v_F = 10{\cdot}1$ ft./sec.; $v_H = 5{\cdot}20$ ft./sec.
12. 2·06 ft./sec. 13. 10·2 ft./sec.

Exercises III, pp. 43–46.

1. 55 ft./sec.; 26 ft./sec.². 2. 20·6 ft./sec.; 241·2 ft.
3. 0·966 sec.; −37·1 ft./sec.². 5. 13° 18′ to OX; 680·6 ft.
6. 10 min. 7. 2·75 ft./sec.²; 3·14 ft./sec.²; 49·09 sec.
8. 72 ft./sec.; 3240 ft. 9. 18·20 ft./sec.; 22·36 ft./sec.
10. 33·7 m.p.h.; 72·2 m.p.h.; 0·416 ft./sec.².
12. 0·702 ft./sec.²; 63·51 ft./sec. 13. 0, 29·6, 36·1, 21·5, 0 ft./sec.
14. 6958, 4013, −1283, −4013, −4392 ft./sec.². 15. 79°.
19. $\dfrac{\sqrt{2}\sin\theta}{(3-2\sqrt{2}\cos\theta)^{\frac{1}{2}}}\mathrm{AC}.\omega;\quad \dfrac{-\sqrt{2}\sin\theta}{(3-2\sqrt{2}\cos\theta)^{2}}\omega^{2}.$

Exercises IV, pp. 60–67.

An acceleration is given here as negative when the velocity is decreasing.

1. 6480 ft./sec.². 2. (*a*) $\omega^2 r\left(1+\dfrac{r}{l}\right)$ ft./sec.²; (*b*) $-\omega^2 r\left(1-\dfrac{r}{l}\right)$ ft./sec.².
3. 124 ft./sec.². 4. (*a*) 371 ft./sec.²; (*b*) −215 ft./sec.².
8. $\omega = 3{\cdot}76$ rad./sec.; $\dot{\omega} = 24{\cdot}14$ rad./sec.²; 3·36 ft./sec. at 121° measured anticlockwise from BA.
10. Vel. of C = 44·2 ft./sec.; vel. of E = 25·6 ft./sec.; acc. of C = 11,300 ft./sec.²; acc. of E = −15,600 ft./sec.²; ang. vel. of BC = 79·6 rad./sec.; ang. vel. of DE = 102 rad./sec.; ang. acc. of BC = −15,100 rad./sec.²; ang. acc. of DE = 12,800 rad./sec.².
11. 7·16 ft./sec.; −86·1 ft./sec.².
12. 1·20 rad./sec.; −8·34 rad./sec.².
14. Vel. of E = 18·8 in./sec.; acc. of E = −80 in./sec.².

Exercises V, pp. 83–88.

1. 1·61 ft./sec.^{2}; 24·15 ft./sec.; 5 sec.
2. 7·61 ft./sec.; 0·0328 sec.; (*a*) 18,000 lb.; (*b*) 20,500 lb.
4. $16\frac{2}{3}$ lb.; $33\frac{1}{3}$ lb.
5. 50 lb.; 75 lb.
6. 50·1
7. 63·5 m.p.h.
8. 9·1 ft.
9. 27·7.
10. 129·4 ft.; 0·706; 7·33 ft./sec.
11. 0·715.
12. 3·66 inches.
13. 11·6 inches.
14. 8·3 sec.
15. 38° 43′; 452 lb.
16. 3·64 inches.
17. 44,800; 273·3 sec.
18. 14·2.
19. 1·6 m.p.h.; 2·57 ft.-tons; 8·51.
20. (*a*) 15·2; (*b*) 51·8 lb.
21. 60 m.p.h.
22. $Wc \sin\theta + \mu W(a - c\cos\theta)$, taking $\theta = 0$ when weight is in lowest position.
23. $(Z - H_0)/W$.
24. $\{n_1 - wk(\lambda + \sin\alpha)/h\}k$; 12·5 m.p.h.
26. Per wheel: front +13·1 lb., back −13·1 lb.
27. Per wheel: outer +5·34 lb., inner −5·34 lb.

Exercises VII, pp. 114–116.

1. 0·327 sec.; 3·059 cycles/sec.
2. (i) $x = \frac{1}{8}\cos 4{\cdot}7\pi t$; (ii) $x = \frac{1}{8}\sin 4{\cdot}7\pi t$;
(iii) $x = \frac{1}{16}\cos 4{\cdot}7\pi t - \frac{1}{16}\sqrt{3}\sin 4{\cdot}7\pi t = \frac{1}{8}\cos(4{\cdot}7\pi t + \frac{1}{3}\pi)$.
3. 1·63 ft.; 5·06 ft./sec.; 2·03 sec.
4. (i) 21·65 ft./sec.; (ii) 17·68 ft./sec.
5. 0·091 ft.; 0·955 ft./sec.
6. 89·0 ft./sec.; 22,370 ft./sec.^{2}; 7920 lb.
7. 0·452 sec.; 0·869 ft./sec.
8. 20·94 ft./sec.; 18·14 ft./sec.; 1316 ft./sec.^{2}; 133.
10. $4\pi^2$ ft./sec.^{2}.
11. 1·918 sec.; 0·013 sec.
12. 3·26 ft.; 86·3 sec.
13. 3·32 sec.; 6·64 sec.
14. 1·0174.

Exercises VIII, pp. 129–131.

1. $r = 3{\cdot}95$ inch; $\theta_1 = 13°\ 10'$; acceleration at A is 2924 ft./sec.^{2}; at B, 2847 ft./sec.^{2} and −625 ft./sec.^{2}; at D, −914 ft./sec.^{2}.
2. 0·822 inch.
3. (*a*) 1582 ft./sec.^{2}; (*b*) 0·0117, 0·0469, 0·1055, 0·1875, 0·2695, 0·3281, and 0·3633 inch; (*c*) see Fig. 469. Acceleration is shown here as positive

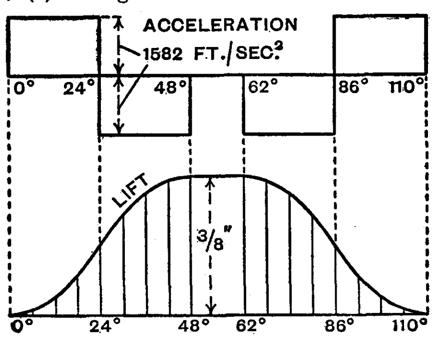

FIG. 469.

or negative according as its direction is that of valve opening or of valve closing.

4. (a) $\frac{1}{24}\cos\theta\left(\frac{d\theta}{dt}\right)^2$ ft./sec.^{2}; (b) 530 r.p.m.
5. (a) 594 r.p.m.; (b) 460 r.p.m.
6. 13° 4′; $\frac{299}{7200}$ or $\frac{1}{24}$ approx.
10. 23·4 lb.
11. 3·70 mm.
12. -990 ft./sec.^{2}; 23·1 lb.

Exercises IX, pp. 156–161.

1. (a) 1·51 ft./sec.^{2}; (b) 7·27 sec.; (c) 1·57 tons; 3·75 ton-ft.
2. 1·32 ton-ft.; 24·6.
3. (1) $\frac{\mu g b}{a+b-\mu h}$; (2) $\frac{\mu g a}{a+b+\mu h}$.
4. (a) 10·7 ft./sec.^{2}; (b) 0·192.
5. 13·7 sec.
6. 6·52 in.
7. (1) $\frac{3}{2}r$; (2) $\frac{17}{12}r$; (3) $\frac{3}{2}r$.
8. (1) $\frac{4\sqrt{2}}{3}a$; (2) $\frac{7\sqrt{2}}{6}a$; (3) $\frac{5}{3}a$.
9. $a=4·07$ in.; $b=9·68$ in.; $k=4·82$ in.
10. $a=6·16$ in.; $b=11·84$ in.; $k=6·63$ in.
11. $R_1=5·43$ lb.; $R_2=101·0$ lb.; 0·90 rad./sec.
13. $2\pi\left\{\frac{\frac{2}{5}r^2+l^2+x^2}{gx}\right\}^{\frac{1}{2}}$.
15. 2·51 ft.; 24·1 ft.-tons.
16. 14·4 ft./sec.
19. $\dot{\theta}=\left\{\frac{2mgb(\cos\theta-\cos a)}{\frac{1}{2}M(k^2+a^2)+m(a^2-2ab\cos\theta+b^2)}\right\}^{\frac{1}{2}}$.
20. $N_1'=43·5$ r.p.m.; $N_2'=32·7$ r.p.m.; 0·28 in.-lb.
21. $N_1'=35·4$ r.p.m.; $N_2'=26·5$ r.p.m.; 31·7 in.-lb.
22. (i) 41·7 rad./sec.^{2}; 23·2 rad./sec.^{2}; (ii) 193 r.p.m.; 0·48 sec.
24. $\omega=\frac{\sqrt{70gh}}{7a-5h}$; $\frac{\sqrt{\omega^2(7a-5h)^2-70gh}}{7a}$.
27. $\frac{1}{2}ma^2\omega\sin^2\theta$.

Exercises X, pp. 196–201.

1. 28·74 lb.; 16° 42′.
2. (a) 209·9 lb.; (b) 223·0 lb.
4. 459 lb. in.
7. 0·0997 W lb. in. and 0·0595 W lb. in., both torques in same direction.
8. 85·2 lb.
9. 78·0 lb.
10. (a) 2·94 in. tons; (b) 5·88 in. tons.
11. (a) 116·7 lb. in.; (b) 112·5 lb. in.
12. 3527 lb. in.
13. 0·80 H.P.
14. 207·5 lb.; 18·9 lb./in.^{2}; 12·0 lb./in.^{2}.
15. 24·6 lb.; 22·7 per cent.
16. 8·61 in.; 459 lb.
17. (a) 220 lb.; (b) 2·46 in.
18. (a) 42·1 H.P.; (b) 15·9 lb./in.^{2}.
19. (a) 1·76 in.; (b) 652 r.p.m.
20. 12·0 lb.
21. 69·3 H.P.
23. 283·7 lb. in.
24. 14.

Exercises XI, pp. 218–221.

1. 20,420 lb. ft.
2. 10·6 ft./sec.; 906 ft./sec.^{2}; 1127 lb. ft.
3. 145·7 lb. ft.
4. 222 lb. ft.
6. 99·9 lb. ft.^{2}.
7. 0·56 per cent.
8. 7267 lb. ft.^{2}; with radial thickness 3 in., width is 7·89 in., say 8 in.;

with radial thickness 6 in., width is 3·91 in., say 4 in. It is sufficiently accurate to assume $k = 2{\cdot}5$ ft., then the calculated widths are respectively 7·91 in. and 3·955 in.

9. 17·2 tons. 10. 59·5; 6·92 per cent.
11. $23{\cdot}1 \times 10^4$ ft. lb.; 1400 lb. ft. 12. $1697\frac{1}{7}$; 23·2 r.p.m.
13. (*a*) 2·55 in.; 1·42 lb.; 5·08 lb.; (*b*) 1·77 lb.; 4·73 lb.; 28·1 lb. ft.
14. 50·6 lb. ft. by calculation; the other solutions gave 51·0 lb. ft. and 50·1 lb. ft., the differences being due to errors in drawing and measurement.

Exercises XII, pp. 237–241.

1. (*a*) 28 ft. 5·06 in.; (*b*) 29 ft. 1·12 in. 2. 73·5 lb.
3. 1·25 tons. 4. 2635 lb.; 9·22 lb.
5. 22·5 in. 6. 551 lb.
7. 0·39. 8. 2·91; creep reduces speed.
10. 8·52; 373 r.p.m. 11. 4 in.
12. 5·58, that is 6 ropes. 13. 6·10, that is 7 ropes.
14. 2·01 sq. in. 15. 5·40 in.; 283 lb.
16. When $v = 4500$ ft./min., max. tension = 1385 lb., bearing load = 1277 lb.; when $v = 3000$ ft./min., max. tension = 1571 lb., bearing load = 1915 lb.

Note.—Bearing load $= (T_1 + T_2 - 2T_c) \cos \alpha$, neglecting belt weight. $\alpha = 10°$.

17. 13·4; 226 lb.; 25·4 in., 50·8 in.
18. 9·86, that is 10 ropes. 19. 37·0.
20. (*a*) 34° 30′; (*b*) Small pulley, 69° 49′; large pulley, 104° 19′.
21. 8 ft. 11·4 in.; 3·09, that is 4 ropes.
22. 4·11, that is 5 ropes. 23. 2·83 in., say 3 in.
24. 0·95 per cent.
25. Mean tension $= \frac{1}{2}(T_1 + T_2) = \frac{1}{2}(146 + 57) = 101{\cdot}5$ lb.; increase in horse-power $= \dfrac{89 - 80}{80} \times 100 = 11{\cdot}3$ per cent.
26. $T_2 = 27$ lb.; mean tension $= \frac{1}{2}(350 + 27) = 188{\cdot}5$ lb.

Exercises XIII, pp. 272–275.

1. 6 in.; 0·6283 in.; 0·2 in. 2. 13·24 ft./sec.; 0; 12·03 ft./sec.
3. 3·31 ft./sec. 4. 0·8578 in.; 0·745 ft./sec.
5. Addenda: pinion 0·812 in., wheel 0·324 in.; path of contact 2·565 in. (*a*) pinion 7·46 ft./sec.; wheel 1·87 ft./sec. (*b*) 5·60 ft./sec.
6. $D = 16{\cdot}5$ in., $d = 7{\cdot}5$ in.; blank diameters, 17 in., 8 in.; $T = 66$, $t = 30$; depth, 0·5393 in.
7. 18° 26′; 0·806 in. 10. 12.
11. $F_t = 500$ lb.; $F_a = 288{\cdot}7$ lb.; $F_r = 210{\cdot}1$ lb.
12. Teeth, 24 and 36; $d = D = 5{\cdot}126$ in.; α (pinion) $= 53° 24'$, $\beta = 26° 36'$; $\eta = 0{\cdot}8554$.
14. 4·462 in.; 70°; (*a*) 0·8715; (*b*) 0·8797. 15. 60°; 4·381 in.
16. Wheel 1: 27° 28′, 9·016 in.; wheel 2: 47° 32′, 18·958 in.
18. Pinion: 4·141 in., 14° 59′; wheel: 13·859 in., 30° 1′; 3·54 ft./sec.

Exercises XIV, pp. 295–302.

1. 47·12 r.p.m. 2. Drivers, 25, 30, 20; driven wheels, 50, 60, 100.
3. 19; include an idle wheel in the train.
4. A, 36; B, 49; 293·9 r.p.m.
5. A, 33; B, 24; C, 44; D, 53; 452·8 r.p.m.
6. 13 and 57 teeth; 526·2 r.p.m.; if 520 r.p.m. with 12 and 52 teeth, decrease is 1 inch; with 15 and 65 teeth, increase is $1\frac{2}{3}$ inches.
7. 21·11 inches. 8. 360 r.p.m. clockwise.
9. 74·375 r.p.m. clockwise. 10. 3·542 r.p.m. anticlockwise.

11. 7·55 N_A and in same direction. 12. $5\frac{1}{3}$.

13. $\dfrac{\text{Hauling speed}}{\text{Lifting speed}}=33.$ 15. G, 21; C, 28; H, 70; 8 r.p.m.

16. 60 r.p.m.; $T_R=175{\cdot}1$ lb. ft.; $T_A=131{\cdot}3$ lb. ft.

17. 16·3 r.p.m.; 1433 lb. ft.; −18·3 r.p.m.

19. 240 r.p.m.; 15·76 lb. ft.; 3·6 H.P.

20. 1·47 lb. ft.

21. (a) 46; (b) $\dfrac{N_X}{N_Y}=\dfrac{4}{27}$; (c) 8·59 lb. in.

22. Shaft turns 2 revs. while cam disc turns $-\frac{1}{4}$ rev.

Exercises XV, pp. 324–327.

1. (a) 123·7 lb.; (b) 27·8 lb. 2. (a) 137·5 lb.; (b) 59·5 lb.
3. 13·3 lb. 4. 10·4 lb.
5. 3·45. 7. Left, 23·1 lb./in.^{2}; right, 6·7 lb./in.^{2}.
10. 14° 2′; 18·4 lb./in.^{2}; 15·8 and 9·5 lb./in.^{2}; 0·674.
15. 1061 lb. in.; 233 lb.

Exercises XVI, pp. 364–369.

1. (b) 5·51 in.; 3·46 in.; (c) 11·78 in.; (d) 56·6 r.p.m.
2. 65·6 r.p.m., 67·1 r.p.m.; 31·8 lb.
4. (a) 6·91 in.; (b) 9·6 r.p.m. 5. (a) 7·61 in.; (b) 6·20 in.
6. 210 r.p.m. 7. 2·0 lb.; 51° 14′. 8. 30 lb.
9. 178 lb./in. 10. 221 lb.; 167 lb./in.
11. 122 lb./in.; 3·04 in. 12. 107 lb./in.; 2·29 in.; 4·05 in.
14. (a) 0·243 lb.; (b) 0·213 lb.; (c) 0·197 lb.
15. 30·9 lb.; 27·2 lb./in.; 824 r.p.m.
16. 707 lb.; 250 lb./in. 17. 910 lb.; 653 r.p.m.
18. 0·53 in.; 219 lb./in. 19. 1·57 lb.; 0·021 in.; 0·033 in. lb.
20. (a) 0·0254 in.; (b) 261 r.p.m. 21. (a) 1·61 lb.; (b) 375 r.p.m.
22. (b) 3·26 lb. in. which is independent of the length OC.

Exercises XVII, pp. 408–413.

1. 18° 26′ between vertical and radius to larger hole; 0·889 inch.
2. 30 lb., 30 lb.; 19·0 lb., 79·0 lb.; 93·9 r.p.m.
3. 244 lb. each; 1341 lb.
4. 22·6 lb., 36·2 lb.; 9 lb., 8 inches.
5. Each weight = 77·9 lb.; in axial plane perpendicular to centre crank, and the smaller angle between radius to each weight and the nearer outside crank is 150°.
6. 7·56 lb. Angular positions from mass A: mass B, 204° 53′; mass C, 96° 8′; mass D, 310° 45′.
7. 6·6 lb. each.
8. 0·63 lb.; relative angular positions of mass centres, measured in one direction: B from A, 157° 49′; C from B, 76° 14′; D from C, 155° 15′; M from D, 180°.
9. Initial unbalance: B, 11·72 lb.-in. at 44° 3′; D, 0·71 lb.-in. at 9° 33′. Final unbalance: B, 8·08 lb.-in. at 201° 48′; D, 3·91 lb.-in. at 309° 48′.
10. 237 lb. each at 156° 48′ from adjacent crank.
12. (a) 124·5 lb. each, at 94° on each side of centre crank; (b) 10,990 lb.
13. 324·9 lb. opposite crank pin; when crank angle is θ the residual unbalanced force is 1512 lb. at an angle $-\theta$.
14. 5628 lb.
15. Left-hand bearing, 434·6 lb. at 30° relative to left-hand crank; right-hand bearing, 434·6 lb. at 210°; balance weight in plane of left-hand crank, 13·3 lb. at 210°; other balance weight, 10·4 lb. at 30°.

16. Angles measured from crank A in direction of rotation. Bearing F, 279 lb. at 321° 31′; bearing G, 884 lb. at 51° 9′. When flywheel is balanced: bearing F, 297 lb. at 215° 39′; bearing G, 695 lb. at 82° 11′.

17. Balance weight, 1·2 lb.; out-of-balance primary force, 109 lb.

18. Primary and secondary forces balanced; max. primary couple 5288 lb.-ft., central crank inclined at 90° and 270° to line of stroke; max. secondary couple 2139 lb.-ft., central crank inclined at 45°, 135°, 225°, and 315° to line of stroke.

Exercises XVIII, pp. 453–457.

1. 0·815 sec. **2.** 187·7; 159; 4·61 inches when using first spring, or 5·71 inches when using both springs.

3. 1·92 lb. **4.** See Fig. 470.

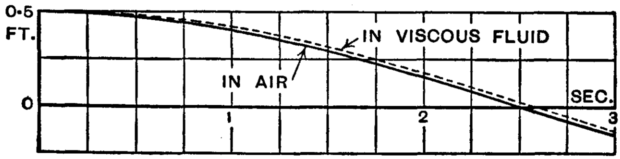

Fig. 470.

5. 1·78 sec.; 1·82 sec.; log. dec. $=0{\cdot}682$; $R=6{\cdot}75$ lb.

7. 6·29 in. **8.** 0·0664 sec. **9.** 26·3 oscillations/sec.

10. $$T=2\pi\sqrt{\frac{32I}{C\pi}\int_0^l \frac{dx}{d^4}}=2\pi\sqrt{\frac{32Il}{3C\pi d_1^3 d_2^3}(d_1^2+d_1d_2+d_2^2)}.$$

11. 132 oscillations/sec. **12.** 5·37 lb.-ft.; (i) $0{\cdot}313m$ lb.; (ii) $0{\cdot}199m$ lb.

13. (*a*) 1622 r.p.m.; (*b*) Total deflection = dynamic + static deflections = 0·000206 in. + 0·013393 in.

14. 0·627 sec.; (1) No damping, $y=3{\cdot}85\cos 10{\cdot}02t$, see Fig. 471;

(2) Damping, $y=e^{-7\cdot51t}(3{\cdot}85\cos\sqrt{44}t+4{\cdot}36\sin\sqrt{44}t)$, see Fig. 471;

$$\text{Amplitude}=\frac{\frac{1}{8}p^2\times 12}{\sqrt{(100{\cdot}4-p^2)^2+225{\cdot}6p^2}}\text{ in.,}$$ see Fig. 472;

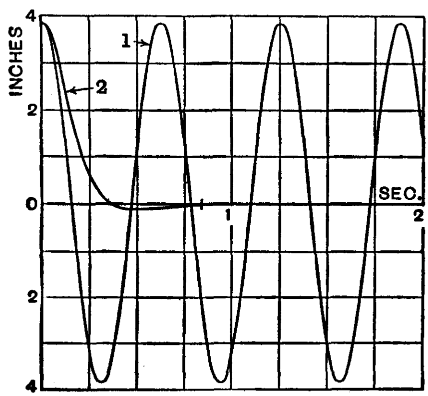

Fig. 471.

Angle of lag $\epsilon = \tan^{-1}\dfrac{15{\cdot}03p}{100{\cdot}4 - p^2}$, see Fig. 473. Since p is in rad./sec., it is convenient to replace p by $\pi N/30$ when drawing the last two graphs, N being in r.p.m.

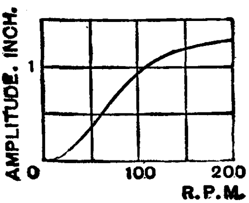

FIG. 472.

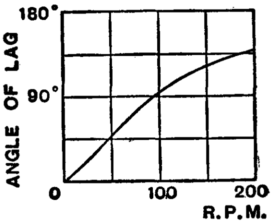

FIG. 473.

15. 16·0 oscillations/sec.
16. 481 r.p.m.
17. *First Method* (using deflections obtained when both loads are on shaft) 920 r.p.m.; *Second Method* 861 r.p.m.
18. 1950 r.p.m.

Exercises XIX, pp. 481–484.

1. 3·73 vib./sec.
2. 22·08 in.; $1{\cdot}34 \times 10^6$ lb. in./rad.
3. 8·21 vib./sec.; flywheel, 0·0325°; engine, 0·0375°.
   24·80 vib./sec.; flywheel, 1·108°; engine, 5·332°.
4. 22·4 vib./sec.; B, 0·482°; C, 1·525°.
   38·7 vib./sec.; B, 0·546°; C, 0·187°.
5. 39·3 vib./sec.; 68·9 vib./sec.
9. 7·16 in. to right of flywheel.
10. 69·9 vib./sec.
11. 1·75 in.; 6·46 vib./sec.; 1·298°.

Exercises XX, pp. 505–507.

1. 1·025.
2. 12·7 vib./sec.
3. (*a*) 10·09 vib./sec.; (*b*) 9·91 vib./sec.
5. (*a*) 0·775 in. (in phase); (*b*) −1·193 in. (180° out of phase).
6. 12,714 lb./in.; 3·89 lb.
7. 0·0098 in.
8. 198·1 vib./min.; 435·6 vib./min.; 0·629; −0·795.
10. (*a*) 0·85 or 1·17; (*b*) 0·73 or 1·37.
11. 25·9 vib./min.; 67·9 vib./min.; 0·236; −4·24.

INDEX

Numbers refer to pages

THEORY
OF
MACHINES

B. B. LOW

LONGMANS