Module 1 : Dynamics of Rigid Bodies in Plane Motion; Dynamic Force Analysis of Machines

Lecture 1 : Dynamics of Rigid Bodies in Plane Motion; Dynamic Force Analysis of Machines.

Objectives

In this lecture you will learn the following

- Introduction to dynamics of machines
- Equations of motion for a planar body
- Equations for a mechanism
- Joint reactions
- Different types of forces

Introduction

The main problems addressed in dynamics of machines can be summarized as:

- Determine forces required to generate given accelerations of a mechanism.
- Determine the acceleration and motion resulting from given forces on a mechanism.

In both cases, internal joint reactions and frictional forces are also determined. We focus mainly on rigid body planar mechanisms.

In the first lecture, we derive equations of motion for planar mechanisms with different types of joints.

In the second and third lectures of this module, we use the above equations to solve the problems of determining forces, accelerations and motions in various types of mechanisms.

Equations of Motion of a Planar Rigid Body

We can use Newton 's equations for particles to derive the equations of motion of a rigid body. A rigid body, which is constrained to move in a plane, has three degrees of freedom, and hence it has three independent equations of motion which relate the forces in the plane to accelerations. Consider the body shown in Fig. 1.1 (a). A set of forces $\vec{F}_k$ and couples $\vec{C}_k$ act on it, with the force $\vec{F}_k$ acting at the point (x_k, y_k). The position coordinates are with respect to an inertial frame. The rigid body can be considered to be made of elemental masses which can be regarded as particles. Newton 's equation for the i^{th} elemental mass Δm_i (Fig. 1.1 (b)) at the location (x_i, y_i) is

$$\Delta m_i \vec{a}_i = \vec{F}i + \sum_j \vec{R}_{ij} \qquad (1)$$

Where a i is the acceleration of the elemental mass and $\vec{F}_i$ is the external force acting on it. Note that every elemental mass may not have an external force. R_{ij} is the force on the i^{th} elemental mass from the j^{th} elemental mass. This arises due to the constraint that the two masses have to remain at a fixed distance from each other.

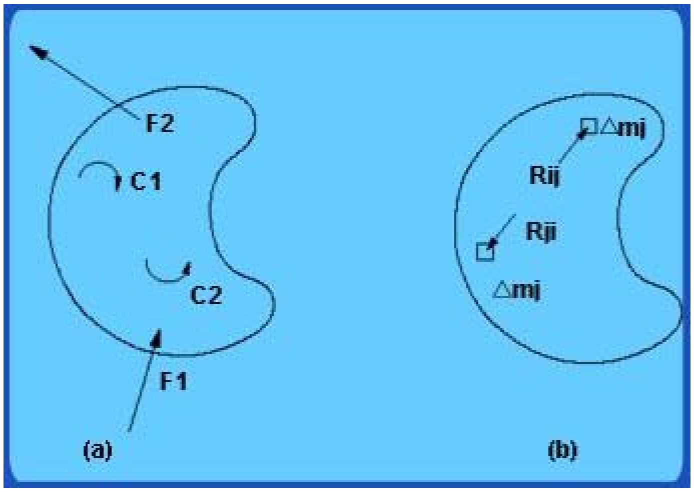


Fig. 1.1.1 A rigid body under planar forces and couples

Adding up the equations of all the elemental masses (keeping in mind that the internal reaction forces occur in equal but opposite pairs), we get the equations of motion for the body, relating linear accelerations to linear forces

$$m\vec{a}_c = \sum_k \vec{F}_k \quad (2)$$

Where a c is the acceleration of the centre of mass. Note that the force and acceleration vectors, though three dimensional, lie in the plane of motion of the rigid body. Hence, resolving the forces along the x and y directions in the plane, we get two scalar equations of motion.

$$m\ddot{x}_c = \sum_k F_{xk}$$
$$m\ddot{y}_c = \sum_k F_{yk} \quad (3)$$

The third equation in planar motion, for the rotation of the body is obtained as:

$$I_c\alpha = \sum_k \vec{r}_{ck} \times \vec{F}_k + \sum_k \vec{C}_k \quad (4)$$

where I_c , the moment of inertia about the center of mass is given by

$$I_c = \sum_i \Delta m_i r_{ic}^2 \quad (5)$$

Equations (3) and (4) are the fundamental equations of motion of a rigid body in planar motion.

Joint Reactions

A mechanism consists of bodies which are connected together by kinematic pairs. One approach to derive the equations of motion for mechanisms (called Newton-Euler approach) considers each body as a free body, along with the forces due to the constraints, called joint reactions. We use this approach here. First, let us consider the nature of joint reactions for different types of joints.

Revolute Joint:

Two bodies connected by a revolute joint are shown in Fig. 1.2(a) and the free body diagrams of the two bodies are shown in Fig. 1.2(b). The revolute joint prevents the two bodies from undergoing relative translational motion along say x and y. Hence, the reaction force is represented by two components, R_x and R_y. The reaction forces on the two bodies are equal and opposite, as required by Newton 's third law.

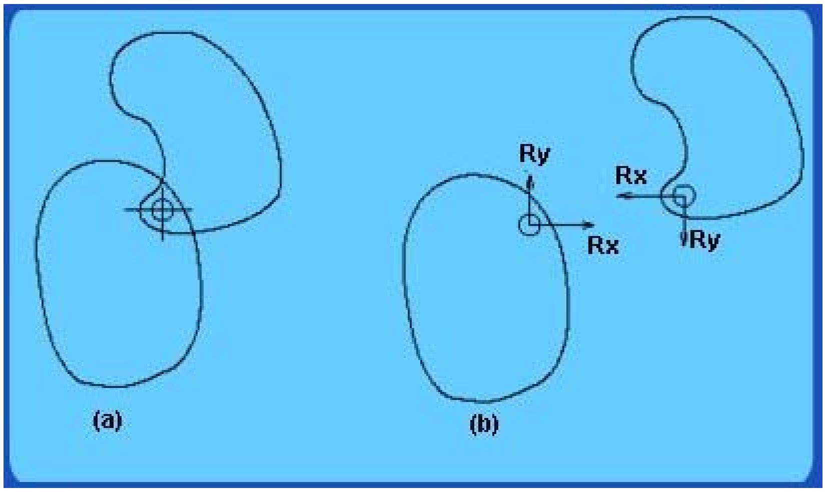


Fig. 1.1.2 Revolute Joint

Prismatic Joint:

Two bodies connected by a prismatic joint are shown in Fig. 1.3(a) and the free body diagrams of the two bodies are shown in Fig. 1.3(b). The prismatic joint prevents relative translation in a direction normal to the line of the joint and also relative rotation. Hence, the reaction forces are represented by normal reaction force R_n and couple R_c. The point at which R_n acts is usually fixed to the piston and moves with the piston. The position of R_n is not important, however, the direction of R_n has to be normal to the direction of relative translation between the two bodies. The reaction forces and couples on the two bodies are equal and opposite, as required by Newton 's third law.

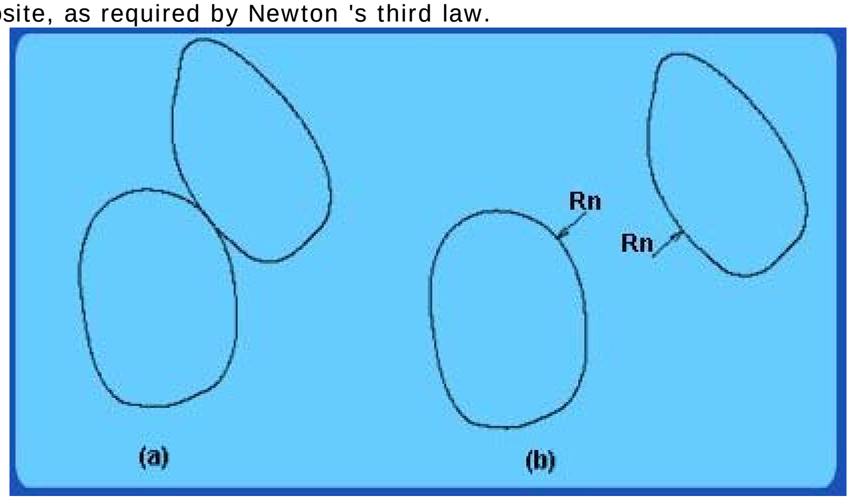


Fig. 1.1.3 Prismatic Joint

Cam-Follower Joint with Sliding:

Two bodies with a cam follower joint which allows sliding are shown in Fig. 1.4(a) and the free body diagrams of the two bodies are shown in Fig. 1.4(b). There is only one reaction force R_n which acts at the point of contact, and is normal to the surfaces at that point. The reaction force is a pushing force as shown and it cannot be negative. Negative value indicates separation of the two bodies. The reaction forces on the two bodies are equal and opposite, as required by Newton 's third law.

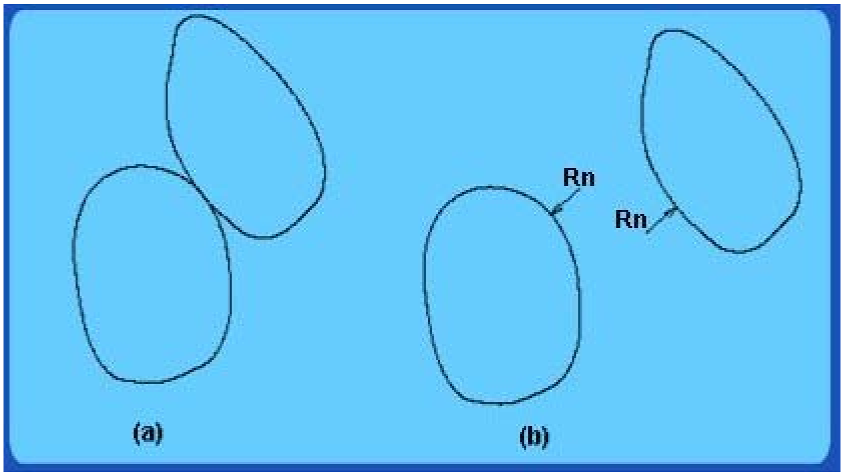


Fig. 1.1.4 Cam-follower-with-sliding

Cam-Follower Joint with Pure Rolling:

Two bodies with a cam follower joint which allows only pure rolling are shown in Fig. 1.5(a) and the free body diagrams of the two bodies are shown in Fig. 1.5(b). There are two reaction forces, both acting at the point of contact. R_n is normal to the surfaces, while R_t is tangential. R_n has the same characteristics (only pushing) as the corresponding force in the case of cam follower joints with sliding. R_t can be in any one of the two tangential directions. The reaction forces on the two bodies are equal and opposite, as required by Newton 's third law.

The joints considered above are those which commonly occur in planar mechanisms. We now show how the equations of motion of the mechanism can be derived using the free body (Newton-Euler) approach.

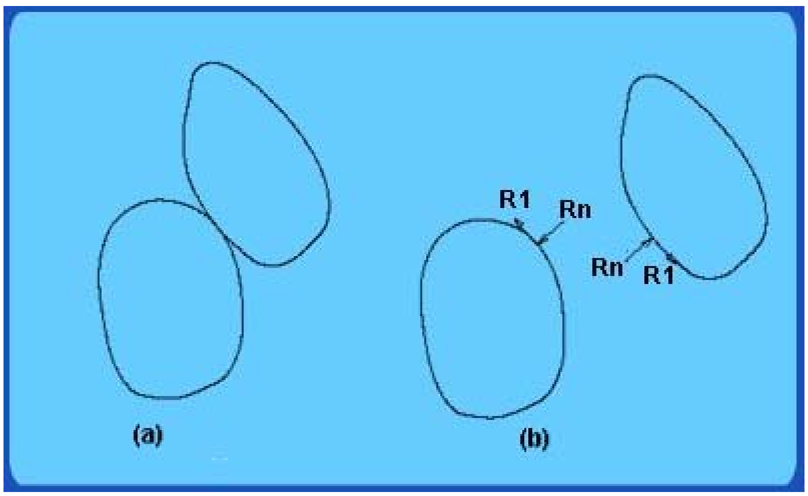


Fig. 1.1.5 Cam-follower-pure-rolling

Equations of Motion of Mechanisms

Consider the mechanism shown in fig. 1.6. It is a 4 link, 2 degree of freedom mechanism with two revolute joints, one prismatic joint, and one cam-follower joint which allows sliding. The link numbered 1 is the frame. It has a revolute joint A with link 2 and a cam-follower joint with link 4, with the contact being at E . Links 2 and 3 are connected by a prismatic joint, while links 3 and 4 are connected by a revolute joint D . The point B is on link 3. The centers of mass of link i is denoted by C i . Note that the center of mass of link 3 is outside the physical bounds of the link, a possibility with links with concave portions in their profile. A torque _ 2 acts on link 2, while a force F 4 acts on link 4 at point G as shown. The direction of gravity is as shown in figure.

The free body diagrams of the moving links are shown in Fig. 1.7, along with the reaction and applied forces on them.

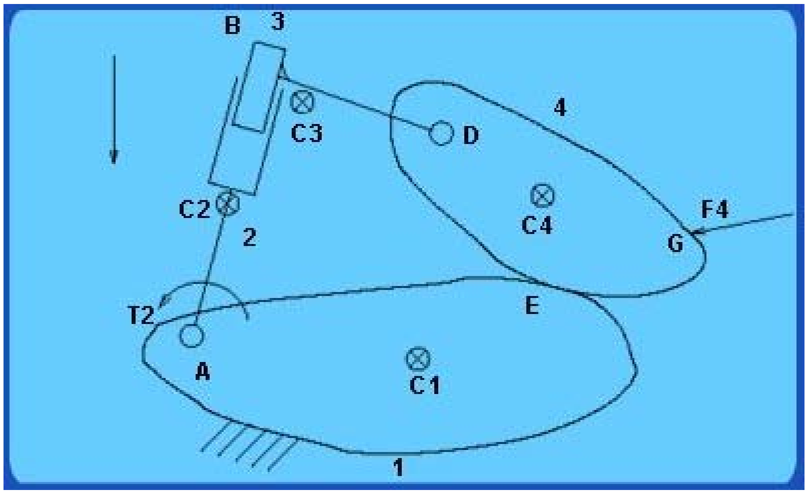


Fig. 1.1.6 Typical planar mechanism

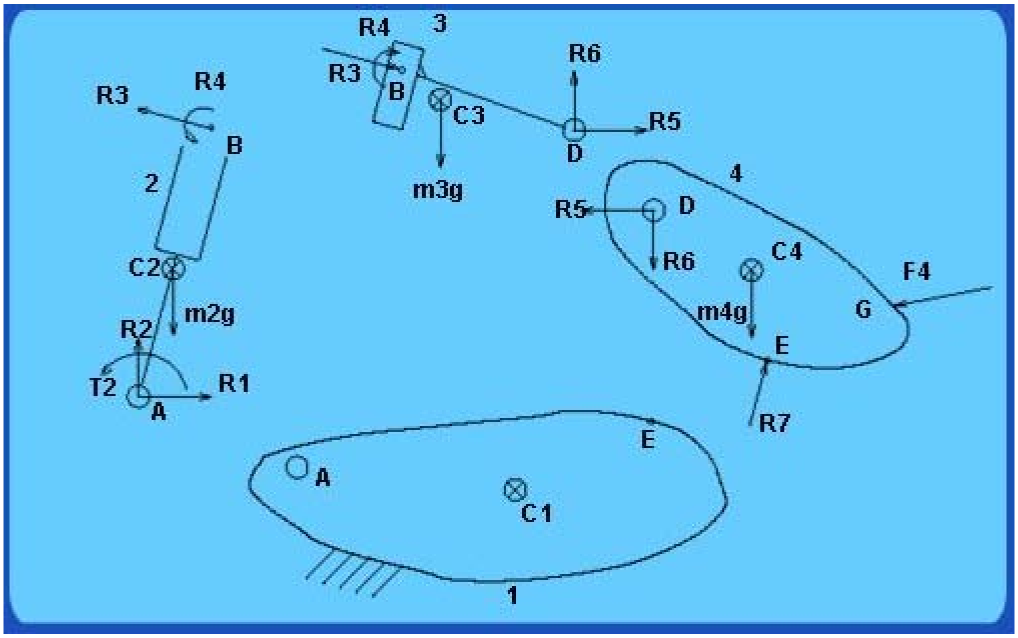


Fig. 1.1.7 Free Body Diagrams for Links of mechanism in Fig. 1.1.6

Recap

In this lecture you have learnt the following

- Developing equations of motion for a planar body/mechanism
- Different types of joints and their reaction forces

Congratulations, you have finished Lecture 1. To view the next lecture select it from the left hand side menu of the page

Module 1 : Dynamics of Rigid Bodies in Plane Motion; Dynamic Force Analysis of Machines

Lecture 2 : Dynamics of Rigid Bodies in Plane Motion; Dynamic Force Analysis of Machines.

Objectives

In this lecture you will learn the following

- Inverse dynamics -- Determination of actuating forces
- Forward dynamics – determination of accelerations given the actuating forces

Introduction

In this lecture, we take up the following two problems.

- Determine forces required to generate given accelerations of a mechanism.
- Determine the acceleration and motion resulting from given forces on a mechanism.

The equations of motion were derived in the previous lecture. Here we use those equations to solve the above two types of problems.

Force Determination

In many situations, it is necessary to determine the forces to be applied on a mechanism to keep it in equilibrium or to accelerate it. Both are part of the same general problem. However, we treat them separately here. First we look at how the force is specified.

Specification of Unknown Force

For an F degrees of freedom mechanism, we need to specify F number of scalar force elements as the unknowns to be determined. To give an example, in the mechanism shown in the previous lecture, as the degree of freedom is 2, we need to specify two force elements. In that case, we can say that the unknowns to be determined are the torque t_2 and the magnitude of the force F_4. It is assumed that the direction of F_4 is known.

Equilibrium

In many situations we are interested in determining the forces that will keep a given mechanism stationary at a position. When the mechanism is at a position with no velocity, and the forces on the mechanism do not cause any acceleration, the mechanism is said to be in equilibrium in that position. The problem of finding forces causing equilibrium can be stated formally as follows.

> *Given a mechanism and its fixed kinematic and inertia parameters (i.e., link lengths, CG locations, masses etc) and the forces already on the mechanism, determine the additional forces to be applied on the mechanism to prevent it from accelerating.*

The solution can be obtained using the equations of motion of the mechanism. As the accelerations and velocities are zero, all terms of the equations involving acceleration and velocity terms disappear. The unknowns in the equation are the reaction forces and the unknown applied forces.

For the mechanism shown in Fig. 1.2.1, considering the magnitudes of F_4 and $\tau 2$ as the unknowns along with the reaction forces, we can write the nine equations of motion as

$$= R_1 - R_3 u_y$$
$$= -m_2 g + R_2 - R_3 u_x$$
$$= \tau_2 + R_4 - R_1(y_A - y_{C_2}) + R_2(x_A - x_{C_2})$$
$$+ R_3 u_y(y_B - y_{C_2}) + R_3 u_x(x_B - x_{C_2})$$
$$= R_3 u_y + R_5$$
$$= -m_3 g - R_3 u_x + R_6$$
$$= -R_4 - R_3 u_y(y_B - y_{C_3}) - R_3 u_x(x_B - x_{C_3})$$
$$- R_5(y_D - y_{C_3}) + R_6(x_D - x_{C_3})$$
$$= -R_5 + R_7 v_x$$
$$= -m_4 g - R_6 + R_7 v_y$$
$$= R_5(y_D - y_{C_4}) - R_6(x_D - x_{C_4}) - R_7 v_x(y_E - y_{C_4})$$
$$+ R_7 v_y(x_E - x_{C_4}) - F_{4_x}(y_G - y_{C_4}) + F_{4_y}(x_G - x_{C_4})$$

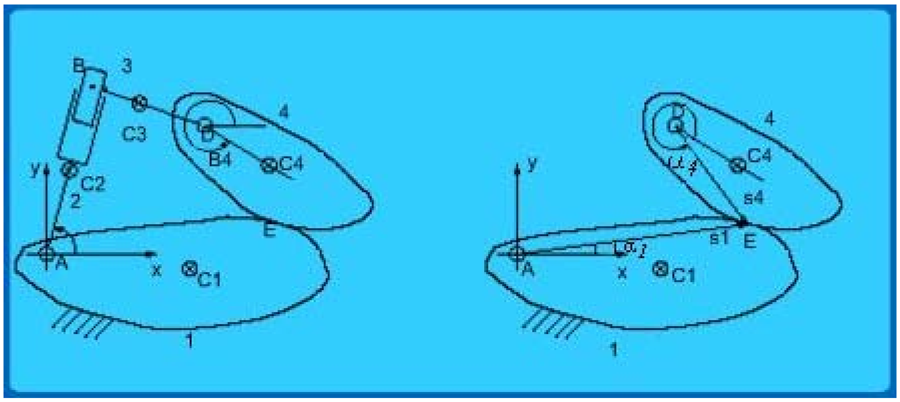


Figure 1.2.1 Typical planar mechanism

Since all the kinematic information about position of various links, their C.Gs is given, the unknowns are only reaction and applied forces namely R_1, R_2, R_3, R_4, R_5, R_6, R_7, F_{4x} or F_{4y} (since direction of F_4 is assumed to be given) and τ_2. It can be seen that the number of unknowns is the same as the number of equations. Note that the unknowns appear in linear form in the equations. Hence the unknowns can be obtained by solving the system of nine linear algebraic equations, using any standard available techniques. Thus the problem of determining the forces required to maintain static equilibrium for a given position of the mechanism is fairly straight-forward.

Solution of the static equilibrium problem (also known as “static force analysis”) is also useful when viewed from another context. Normally, from a specified input-output motion requirement, a mechanism (such as a four-bar mechanism) is synthesized i.e., its link lengths are found. However, purely from this kinematics, link cross-sections and material cannot be decided since these have to be chosen such that the link can withstand the forces being transmitted. Since link cross-sections and material are as yet unknown, link masses and mass moments of inertia required in the general dynamic equations of motion are as yet unknown. Thus, from synthesis, one cannot jump to general dynamic analysis. Static force analysis helps to get some estimate of the forces on the links and the joints, when the accelerations are negligibly small. Using these estimates, we can decide tentative dimensions for link cross-sections and choose an appropriate material. Using these numbers, we can perform dynamic analysis to verify, if under dynamic conditions, the mechanism performs as desired.

Nonzero Accelerations

When the acceleration demanded is nonzero, the problem of determining forces can be stated as follows.

Given a mechanism and its fixed kinematic and inertia parameters, the forces already on the mechanism, and the velocity and acceleration of the mechanism, determine the additional forces to be applied on the mechanism to generate the required acceleration.

Here, the terms "velocity of the mechanism" and "acceleration of the mechanism" means, velocities and accelerations respectively of all links of the mechanism. However since the mechanism has only F degrees of freedom, there are only F independent velocities and accelerations. Thus one approach is to give a set of F independent velocities and accelerations, from which the velocities and acceleration of all links can be determined using standard kinematic velocity and acceleration analysis.

The problem of solving for the forces which generate the given accelerations is solved by substituting the accelerations (and velocities) in the equations of motion and then solving for the unknown reaction forces and additional unknown forces. If reaction dependent friction forces are absent, the unknowns occur in the linear form and hence solution is easy.

Acceleration Determination

Determination of acceleration when the forces are given is necessary to determine the motion of mechanisms subjected to forces. This problem can be stated as follows.

> *Given a mechanism and its fixed kinematic and inertia parameters, all the applied forces on the mechanism, and its position and velocity, determine the acceleration of the mechanism.*

Once the accelerations at this instant of time (t = 0) are found, we can integrate forward in time and determine the possible position and velocity of the mechanism at the next instant of time $(t = \Delta t)$. Again, knowing the position and velocity and the forces, the acceleration can be determined at Δt. Repeating these steps, we can find the complete motion history of the mechanism, from the forces (as a function of time) and initial position/velocity. This problem can also be solved using the equations of motion of the mechanism. However, if we count the unknowns and the number of equations of motion, we find that the unknowns are more than the number of equations. If there are n links, there are n - 1 moving links and hence there are 3(n - 1) equations. The unknowns in the equations of motion are the accelerations which are 3(n - 1) in number and the reaction forces which are equal in number to the number of constraints in the mechanism.

In the particular mechanism example we have been discussing, the unknowns are -- $R_1, R_2, R_3, R_4, R_5, R_6, R_7$, $\ddot{x}_{c2}, \ddot{y}_{c2}, \ddot{\theta}_{c2}, \ddot{x}_{c3}, \ddot{y}_{c3}, \ddot{\theta}_{c3}, \ddot{x}_{c4}, \ddot{y}_{c4}, \ddot{\theta}_{c4}$ and there are nine equations of motion. Thus we need to find more equations to be able to solve the system. We do know that this is a two degree of freedom mechanism and therefore there are only two independent accelerations and seven constraint equations that tie together all others. Therefore we can use the kinematic constraints to generate the necessary additional equations. The number of kinematic constraints and the number of reaction forces are the same – each constraint prevents some motion and hence sets up some reaction force or moment.

Consider the two degrees of freedom example mechanism we have been discussing. To simplify the equations, let us make several simplifications in the mechanism. The global reference frame attached to the fixed link has its origin at A (see Fig.1.2.1). The line on link 2, on which the point B of link 3 is constrained to move, passes through A . The center of mass C_2 is on this line, at a distance r_2 from A . The line BD on link 3 is perpendicular to the line AC_2 on link 2. The center of mass C_3 is on line BD , at a distance r_3 from B . The center of mass C_4 is at a distance r_4 from D . Let the fixed link length BD be called l_3.

Consider as position variables, the angles θ_2 and θ_4 and the distance l_2 from A to B . Here θ 4 is defined as the absolute angle made by the vector DC_4. In addition to the above position variables, to formulate the contact constraint of the cam-follower joint, we introduce the variables (s_1, α_1) which locate the point of contact E on link 1, and (s_4, α_4) which locate the point of contact E on link 4. Note that s_1 is dependent on α_1 and s_4 is dependent on α_4 as the cam profile is given . We can now write the following constraint equations relating the above variables to the Cartesian coordinates of the links (note that some of the former are identical to some of the latter).

$$
\begin{aligned}
x_{C_2} &= r_2 \cos\theta_2 \\
y_{C_2} &= r_2 \sin\theta_2 \\
x_{C_3} &= l_2 \cos\theta_2 + r_3 \cos\theta_3 \\
y_{C_3} &= l_2 \sin\theta_2 + r_3 \sin\theta_3 \\
\theta_3 &= \theta_2 - \frac{\pi}{2} \\
x_{C_4} &= l_2 \cos\theta_2 + l_3 \cos\theta_3 + r_4 \cos\theta_4 \\
y_{C_4} &= l_2 \sin\theta_2 + l_3 \sin\theta_3 + r_4 \sin\theta_4
\end{aligned}
\qquad (2)
$$

We can write three constraints related to the cam-follower joint as follows. The first two constraints (i.e., eq (3)) state that the global location of the point of contact E on the two links be identical.

$$
\begin{aligned}
s_1 \cos\alpha_1 &= l_2 \cos\theta_2 + l_3 \cos\theta_3 + s_4 \cos(\theta_4 + \alpha_4) \\
s_1 \sin\alpha_1 &= l_2 \sin\theta_2 + l_3 \sin\theta_3 + s_4 \sin(\theta_4 + \alpha_4)
\end{aligned}
\qquad (3)
$$

The last constraint states that the two curves are tangential to each other. For this, we use the normal $\hat{n}_1$ to the profile of link 1 at E and the tangent $\hat{n}_4$ to the profile of link 4 at E . The tangency condition is

$$
n_1^T \hat{t}_4 = 0 \qquad (4)
$$

The expressions for $\hat{n}_1 = \begin{Bmatrix} -s_1^t \sin\alpha_1 - s_1 \cos\alpha_1 \\ s_1^t \cos\alpha_1 - s_1 \sin\alpha_1 \end{Bmatrix}$, $\hat{t}_4 = \begin{Bmatrix} s_4^t \cos(\theta_4 + \alpha_4) - s_4 \sin(\theta_4 + \alpha_4) \\ s_4^t \sin(\theta_4 + \alpha_4) + s_4 \cos(\theta_4 + \alpha_4) \end{Bmatrix}$ $\hat{n}_1$ and $\hat{t}_4$ are (5)

Where $s_1' = ds_1/d\alpha_1$ and $s_4' = ds_4/d\alpha_4$

Now equation (4) can be written as

$$
\begin{aligned}
&\left(s_1' \sin\alpha_1 + s_1 \cos\alpha_1\right)\left(s_4' \cos\left(\theta_4 + \alpha_4\right) - s_4 \sin\left(\theta_4 + \alpha_4\right)\right) = \\
&\left(s_1' \cos\alpha_1 - s_1 \sin\alpha_1\right)\left(s_4' \sin\left(\theta_4 + \alpha_4\right) + s_4 \cos\left(\theta_4 + \alpha_4\right)\right)
\end{aligned}
\qquad (6)
$$

Recapitulating our discussion thus far, we have nine position variables x_{C2}, y_{C2} etc and we have introduced additional variables l_2, α_1, α_4. We also have seven reactions as unknowns. On the other hand, we have nine equations of motion and the above ten constraint equations (seven contained in eq (2); two contained in eq (3) and one in eq (6)).

The constraint equations in position coordinates are differentiated twice with respect to time to get constraint equations in velocities and accelerations. These are actually the equations used in any standard kinematic position/velocity/acceleration analysis. It is to be observed that the original equations of motion are differential equations. The constraint equations in position coordinates are algebraic equations. So also the velocity and acceleration equations. Thus, for forward dynamics problems, we need to solve a set of differential – algebraic equations. Recall that inverse dynamic analysis (i.e., given all the kinematic variables the problem of finding the actuating forces) involved only algebraic equations. Thus forward dynamics problem of simulating the mechanism motion is far more involved than the inverse dynamic problem.

We will briefly illustrate the forward dynamics problem on a simple example problem.

Example: 2-R planar manipulator

Consider the two link planar manipulator moving in horizontal plane as shown in Fig. 1.2.2. The free body diagrams are shown in Fig. 1.2.3. The equations of motion for the two moving bodies are given by:

$$m_1 \ddot{x}_{G1} = R_{Ox} + R_{Ax} \qquad \text{(E1)}$$

$$m_1 \ddot{y}_{G1} = R_{Oy} + R_{Ay} \qquad \text{(E2)}$$

$$I_{G1} \ddot{\theta}_1 = T_1 + \left(R_{Ox} - R_{Ax}\right)\frac{l_1}{2}\sin\theta_1 + \left(R_{Ay} - R_{Oy}\right)\frac{l_1}{2}\cos\theta_1 \qquad \text{(E3)}$$

$$m_2 \ddot{x}_{G2} = -R_{Ax} \qquad \text{(E4)}$$

$$m_2 \ddot{y}_{G2} = -R_{Ay} \qquad \text{(E5)}$$

$$I_{G2} \ddot{\theta}_2 = T_2 + \left(-R_{Ax}\right)\frac{l_2}{2}\sin\theta_2 + \left(R_{Ay}\right)\frac{l_2}{2}\cos\theta_2 \qquad \text{(E6)}$$

The unknowns for a forward dynamics simulation problem are

$$\ddot{x}_{G1}, \ddot{y}_{G1}, \ddot{\theta}_1, \ddot{x}_{G2}, \ddot{y}_{G2}, \ddot{\theta}_2, R_{Ox}, R_{Oy}, R_{Ax}, R_{Ay}$$

It is observed that there are 10 unknowns in six equations of motion. It is a two degree of freedom mechanism. Thus even though we have taken six coordinates for the two moving links, only two of these are independent. Let us choose them to be θ_1 *and* θ_2. Other coordinates can be expressed in terms of these two independent coordinates through constraint equations. There are four constraint equations and correspondingly four reaction force unknowns. The kinematic constraint equations are given as:

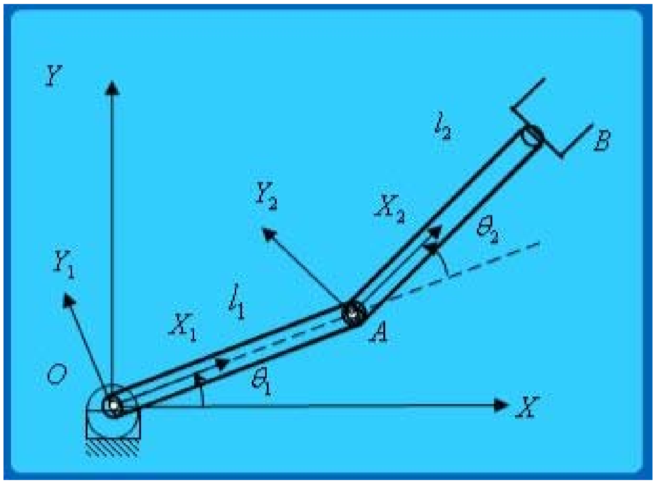
$$x_{G1} = \left(\frac{l_1}{2}\right)\cos\theta_1 \qquad \text{(E7)}$$

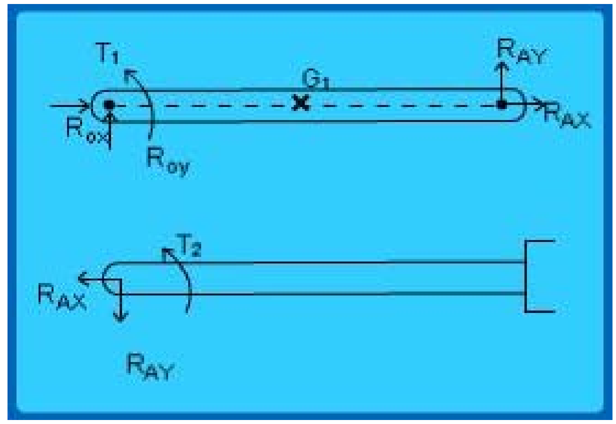
$$y_{G1} = \left(\frac{l_1}{2}\right)\sin\theta_1 \qquad \text{(E8)}$$

$$x_{G2} = \left(l_1\right)\cos\theta_1 + \left(\frac{l_2}{2}\right)\cos\theta_2 \qquad \text{(E9)}$$

(E10)

$$y_{G2}=(l_1)\sin\theta_1+\left(\frac{l_2}{2}\right)\sin\theta_2$$





Let us explicitly work out, how these constraint equations can be used in the solution process. One strategy is to keep them as algebraic equations and solve a set of Differential (equations of motion) and Algebraic (constraint) equations. Other approach is to differentiate the constraint equations once for velocity and a second time for acceleration equations. We illustrate this second approach here. Differentiating equation (E7-E10) once, we get the velocity equations as:

$$\dot{x}_{G1}=-\frac{l_1}{2}(\sin\theta_1)\dot{\theta}_1 \qquad \text{(E11)}$$

$$\dot{y}_{G1}=\frac{l_1}{2}(\cos\theta_1)\dot{\theta}_1 \qquad \text{(E12)}$$

$$\dot{x}_{G2}=-l_1(\sin\theta_1)\dot{\theta}_1-\frac{l_2}{2}(\sin\theta_2)\dot{\theta}_2 \qquad \text{(E13)}$$

$$\dot{y}_{G2}=l_1(\cos\theta_1)\dot{\theta}_1+\frac{l_2}{2}(\cos\theta_2)\dot{\theta}_2 \qquad \text{(E14)}$$

Further differentiation yields the acceleration equations as follows:

$$\ddot{x}_{G1}=-\frac{l_1}{2}(\cos\theta_1)\dot{\theta}_1^2-\frac{l_1}{2}(\sin\theta_1)\ddot{\theta}_1 \qquad \text{(E15)}$$

$$\ddot{y}_{G1}=-\frac{l_1}{2}(\sin\theta_1)\dot{\theta}_1^2+\frac{l_1}{2}(\cos\theta_1)\ddot{\theta}_1 \qquad \text{(E16)}$$

$$\ddot{x}_{G2}=-l_1(\cos\theta_1)\dot{\theta}_1^2-l_1(\sin\theta_1)\ddot{\theta}_1-\frac{l_2}{2}(\cos\theta_2)\dot{\theta}_2^2-\frac{l_2}{2}(\sin\theta_2)\ddot{\theta}_2 \qquad \text{(E17)}$$

$$\ddot{y}_{G2}=-l_1(\sin\theta_1)\dot{\theta}_1^2+l_1(\cos\theta_1)\ddot{\theta}_1-\frac{l_2}{2}(\sin\theta_2)\dot{\theta}_2^2+\frac{l_2}{2}(\cos\theta_2)\ddot{\theta}_2 \qquad \text{(E18)}$$

Substituting for R_{Ax} *and* R_{Ay} from equations (E4) and (E5) into (E6) and using equations (E17) and (E18) in equation (E6) we get the following:

$$\left[\frac{m_2 l_1 l_2}{2}\cos(\theta_2-\theta_1)\right]\ddot{\theta}_1+\left[I_{G2}+\frac{m_2 l_2^2}{4}\right]\ddot{\theta}_2=T_2-\left[\frac{m_2 l_1 l_2}{2}\sin(\theta_2-\theta_1)\right]\dot{\theta}_1^2 \qquad \text{(E19)}$$

Similarly, substituting for R_{Ax} *and* R_{Ay} from equations (E4) and (E5) into (E3) and R_{Ox} and R_{Oy} from equations (E1) and (E2) into (E3) and using equations (E15)-(E18) in

$$\left[I_{G1}+\frac{m_1 l_1^2}{4}+m_2 l_1^2\right]\ddot{\theta}_1+\left[\frac{m_2 l_1 l_2}{2}\cos(\theta_2-\theta_1)\right]\ddot{\theta}_2=T_1+\left[\frac{m_2 l_1 l_2}{2}\sin(\theta_2-\theta_1)\right]\dot{\theta}_2^2 \qquad \text{(E20)}$$

equation (E3) we get the following:

These are the two independent equations of motion in the two independent degrees of freedom namely θ_1 *and* θ_2. All the substitutions etc. that have been carried out, may also be done automatically in a formal computer program. Now the solution process proceeds as follows. To begin with we are given the parameters of link lengths, masses etc; we are also given the position and velocity on both θ_1 *and* θ_2; we are also given the torques $T_1(t)$ *and* $T_2(t)$. Using equations (E19 – E20), we can find the accelerations $\ddot{\theta}_1$ *and* $\ddot{\theta}_2$ at this instant of time. Over a sufficiently small time interval Δt, changes in velocity $\dot{\theta}_1$ *and* $\dot{\theta}_2$ and consequently positions θ_1 and θ_2 can be estimated. Using these new position and velocity variables at time $(t+\Delta t)$, and appropriate values of torques, we can find the accelerations again using equations (E19-E20). This process is repeated for the entire time duration of simulation. At each instant of time, the constraint equations (E7-E10); (E11-E14) and (E15-E18) can be used to estimate other position, velocity and acceleration variables. Using these in equations (E1-E6), we can find the reaction forces at each instant of time. That completes the solution process.

Recap

In this lecture you have learnt the following

- Statement of forward and inverse dynamic problems
- Inverse dynamics problems (in the absence of complications such as friction) are posed as linear algebraic equations and hence readily solved
- Forward dynamics problems where the forces are specified and resulting accelerations for a given position, velocity of the mechanism are to be determined get posed as differential equations which need to be solved for finding the accelerations. Integrating these acceleration will yield the position and velocity information at the next instant of time.

Congratulations, you have finished Lecture 2. To view the next lecture select it from the left hand side menu of the page

Module 2 : Dynamics of Rotating Bodies; Unbalance Effects and Balancing of Inertia Forces

Lecture 3 : Concept of unbalance; effect of unbalance

Objectives

In this lecture you will learn the following

- Unbalance in rotating machinery
- Causes and effects of unbalance
- Response of a simple rotor

Consider a shaft mounted in its bearings as shown in Fig. 2.1.1

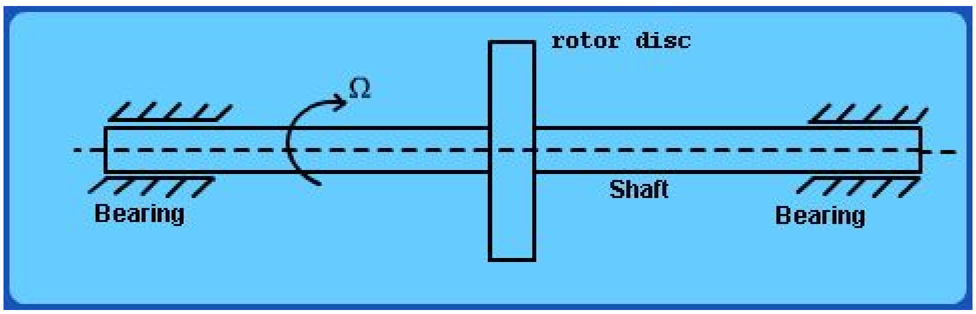


Fig 2.1.1 A simple Rotor

In the idealized situation shown in the figure, the shaft may be assumed mass-less and the rotor is representative of a thin disc with several blades attached around its circumference (such as in a single stage of turbo-machine). The armature of a typical motor can also be represented in this fashion.

It is easy to see that the center of mass of the rotor disc may or may not lie on the geometric axis of the shaft in its bearings. Consider the side-view of the system as shown in Fig. 2.1.2 below. If the center of mass is NOT on the axis of rotation (i.e., the system is “eccentric”), then as the shaft rotates, a centrifugal force will be set-up. The shaft will then be bent away from the line of bearings.

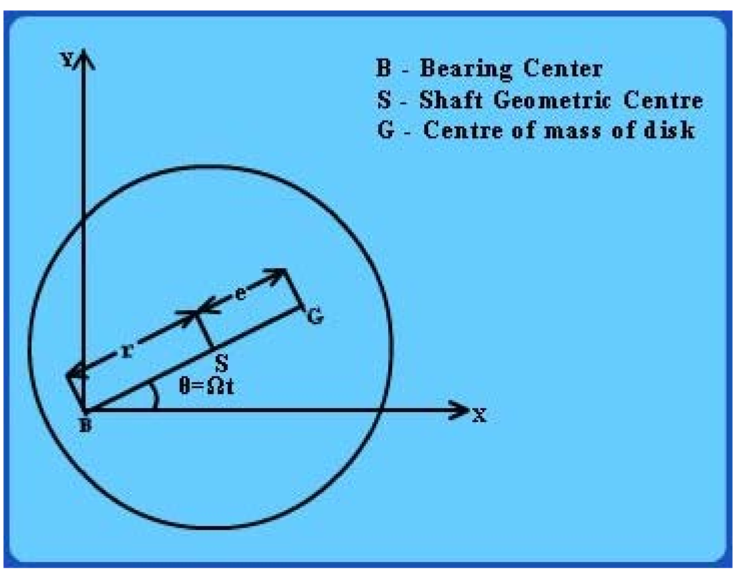


Fig 2.1.2 Side View of The Rotor

Such an unbalance could be normally caused due to manufacturing limitations. For a turbo-machinery bladed disc assembly, there could be as many as 50-100 blades around the disc. Even small fluctuations in the masses of individual blades attached around the periphery of the disk could cause eccentricity of the effective center of mass. Similarly uneven windings in an armature could lead to eccentricity.

For steady state rotation (whirling), from the equilibrium of forces we have,

$$k\,r = m\,\Omega^2\,(r+e) \tag{2.1.1}$$

where,

“k” stands for the equivalent stiffness offered by the shaft to deflection at the location of the disc; If the shaft were simply supported in its bearings at either end and the disc were centrally located, the stiffness of the shaft “k” would be given by $\left(\frac{48EI}{L^3}\right)$.

“r” is the deflection of the shaft's geometric center at the location of the disc;

“m” is the mass of the disc;

“ Ω ” is the speed of rotation;

“e” is the eccentricity i.e., constant distance between the geometric center (S) and center of mass (G).

Thus we have:

$$r = e\,\frac{m\Omega^2}{\left(k m\Omega^2\right)} = e\,\frac{\left(\frac{\Omega}{\omega_n}\right)}{1-\left(\frac{\Omega}{\omega_n}\right)^2} \tag{2.1.2}$$

where w_n is the natural frequency of the system given by

$$\omega_n = \sqrt{\frac{k}{m}} \tag{2.1.3}$$

Typical variation of the deflection of the shaft is given in Fig. 2.1.3 below. It is deserved that for very low speeds of operation, (i.e $\Omega \ll w_n$) $r \approx 0$. The unbalanced forces do not cause significant deflection in the shaft. Towards the high speed end(i.e $\Omega \ll w_n$), $r \approx e$. Thus S and G interchange their relative locations and G tends to coincide with the line of bearings.

In between, when $\Omega \ll w_n$, shaft deflection r becomes excessively large

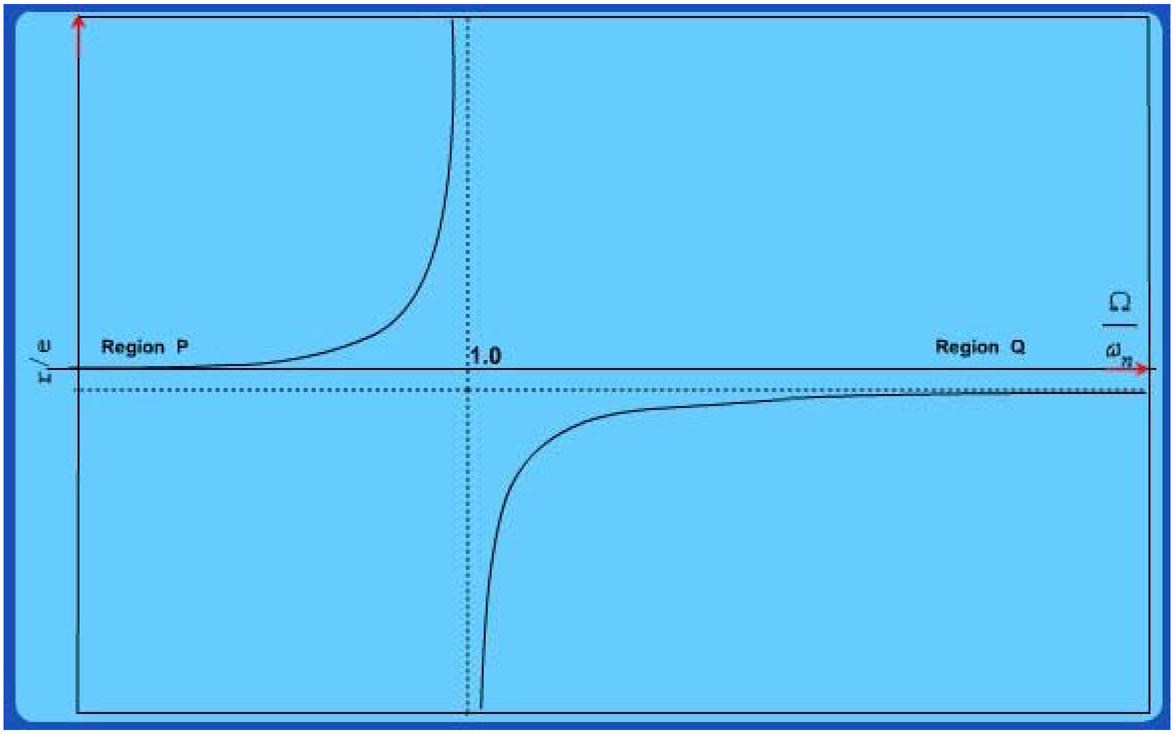


Fig 2.1.3a Variation of the Deflection of the Shaft

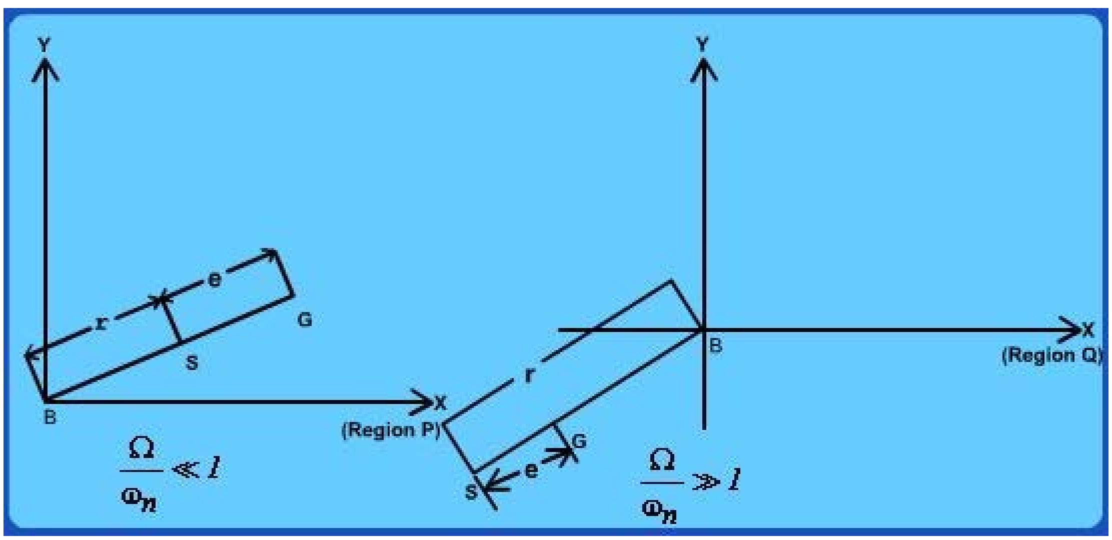


Fig 2.1.3b

From Fig. 2.1.3 it is observed that any eccentricity in the system leads to unbalanced forces and when frequencies match $\left(i.e \dfrac{\Omega}{w_n} \approx 1 \right)$, this could lead to excessive deflections. It is to be observed that because of these forces, the shaft centerline is pulled away from line of bearings and the axis of rotation is thus bent. Please see Fig. 2.1.4. The system continues to rotate with a certain fixed deformed shape, the amount of deflection being given by equation(2.1.2).

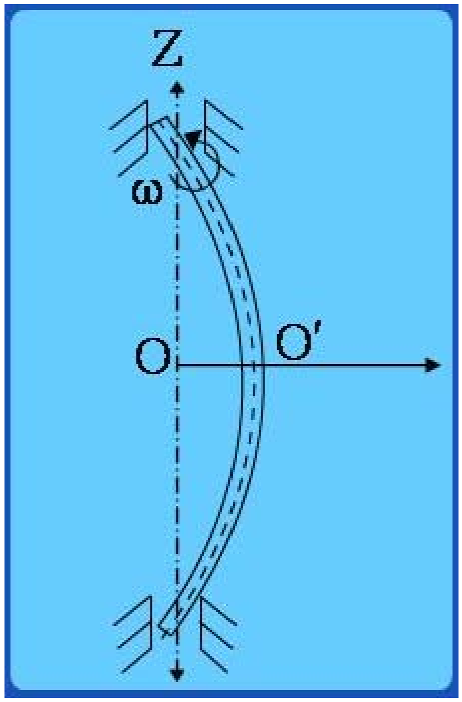


Figure 2.1.4

When the speed of rotation of the shaft about its axis and the speed of rotation of the plane containing the bent shaft about the line of bearings is the same, it is known as "synchronous whirl". In a synchronous whirl, Fig. 2.1.5 shows the side-view of the deformed rotor-shaft system at four different positions in one rotation. Thus the "heavier" fibers (indicated in the figure by the shaded portion) always remain outermost and thus in tension and the inner fibers are always in compression. Thus there is NO cyclic variation of stress from tension-compression for any given fiber. Thus synchronous whirl could lead to excessive amplitudes and possible failure but NOT to fatigue due to alternating stresses.

Fig 2.1.5 Position of Rotor-Disk at 0°, 90°, 180°, 270° in one rotation

In order to reduce the amplitudes of deflection, one could try and reduce the eccentricity or the unbalanced forces (known as the problem of "balancing") or design the system such that the frequencies w_n and Ω do not match. Our present discussion will be focusing on the techniques of balancing out the unbalanced forces.

The "balance grade" suitable for an application is typically prescribed in Indian or International Standards (for example ISO-1940). The balancing grade is usually specified as "G n", where "n" is a number representing the permissible peripheral velocity in mm/s. Thus G100 implies, for a rotational speed of 1000 RPM, a permissible eccentricity of the center of mass from the axis of rotation within 0.1mm. This can be readily obtained from the following formula:

(Eccentricity permitted “e” mm)(speed of rotation ω rad/s) < 100 mm/s (in general "n" in Gn)

Typical balancing grades for a few applications are listed in the Table below to give you an idea of the task of balancing.

Balance Quality Grade “G”	**Rotor Type (application)**
G 4000	Crankshaft-drives of rigidly mounted slow marine diesel engines
G 100 - 250	Crankshaft drives of rigidly mounted fast diesel or gasoline engines for cars, trucks, locomotives
G 16	Rotating parts of agricultural or crushing machinery
G 2.5	Gas and steam turbine rotors
G 0.4	Spindles of precision grinders and gyroscopes

Recap

In this lecture you have learnt the following

- Concept of unbalance in rotating machinery
- Causes and effects of unbalance
- Response of a simple rotor to unbalance
- Typical balancing grades as per standards

Module 2 : Dynamics of Rotating Bodies; Unbalance Effects and Balancing of Inertia Forces; Field Balancing and Balancing Machines.

Lecture 4 : Rotor dynamics problems; Static and dynamic unbalance

Objectives

In this course you will learn the following

- Some interesting issues in rotor dynamics
- Types of unbalance viz., static and dynamic
- Balancing technique for achieving static balance

In the previous lecture we looked at a simple rotor wherein the disc was centrally located on the shaft and we studied the effect of unbalance. Now we will try and gain an appreciation of some more interesting problem situations in the field of rotor dynamics.

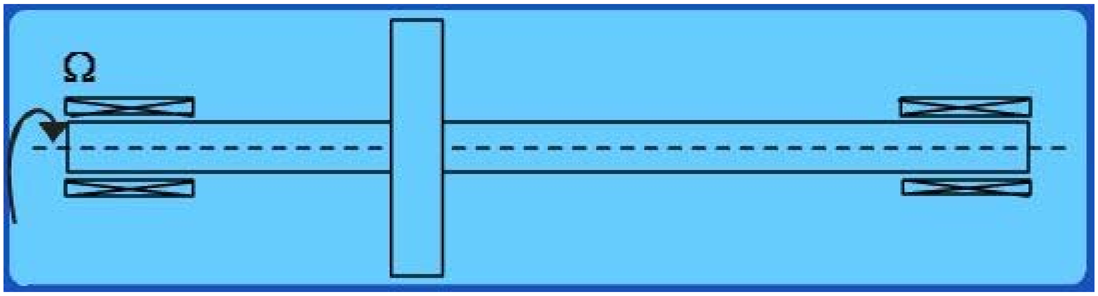


Fig 2.2.1 An offcenter Rotor

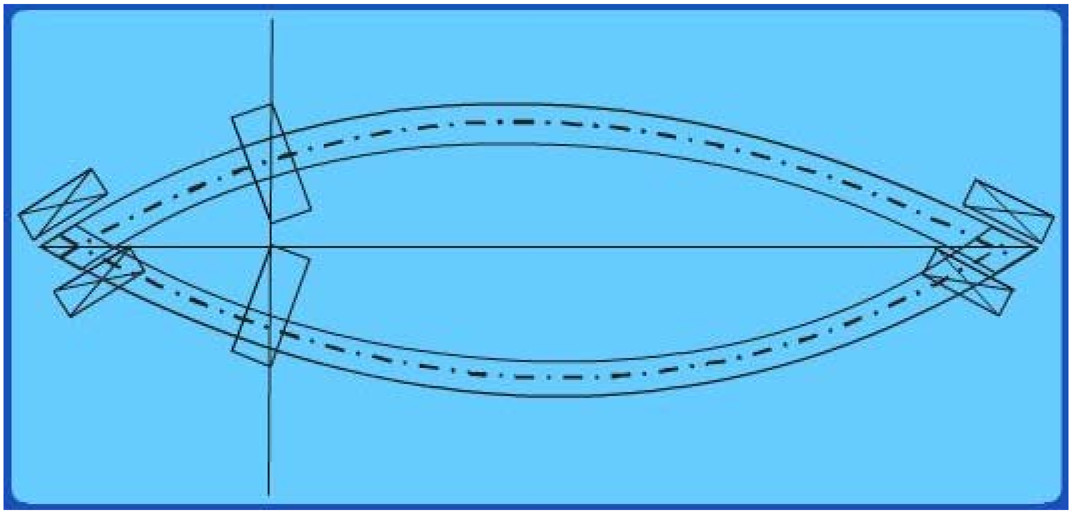


Fig 2.2.2 Two Position of the rotor

Consider the shaft system shown in Fig 2.2.1, wherein the disc is located at a point away from the mid-length of the shaft. Fig. 2.2.2 shows two positions and the point of interest is the orientation of the disc. As per Euler-Bernoulli beam bending theory, a plane cross-section remains plane and normal to the neutral axis even after bending. Thus we observe that the disc has actually rotated about an axis perpendicular to the plane of the figure. The moment of inertia of the disc therefore plays a significant role in this rotation of the disc and has an affect on the dynamics of the rotor-shaft system.

The location of the disc on the shaft and the particular shape of deformation of the shaft determine the extent of the influence. As shown in Fig 2.2.3, the steeper the variations in the slope of the shaft in the deformed shape, the larger the rotations of the disc and hence the larger the influence. Thus an accurate analysis of the dynamics of the rotor will need to take into account this influence.

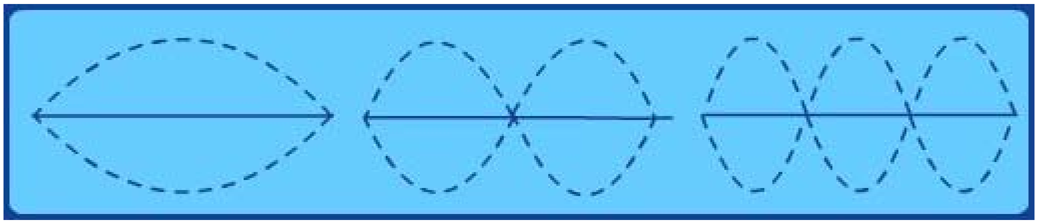


Fig 2.2.3 Some typicalmode shapes f deformation

We have so far assumed that the bearings are stiff. A normal hydrodynamic bearing, however, has finite stiffness and damping. Also, these may not be the same in all directions. A typical model of a rotor in journal bearings is shown in Fig. 2.2.4. We need to take into account the general non-linear and anisotropic bearing stiffness/damping properties. Thus an accurate model of the entire system would become quite complex and will need advanced modeling techniques.

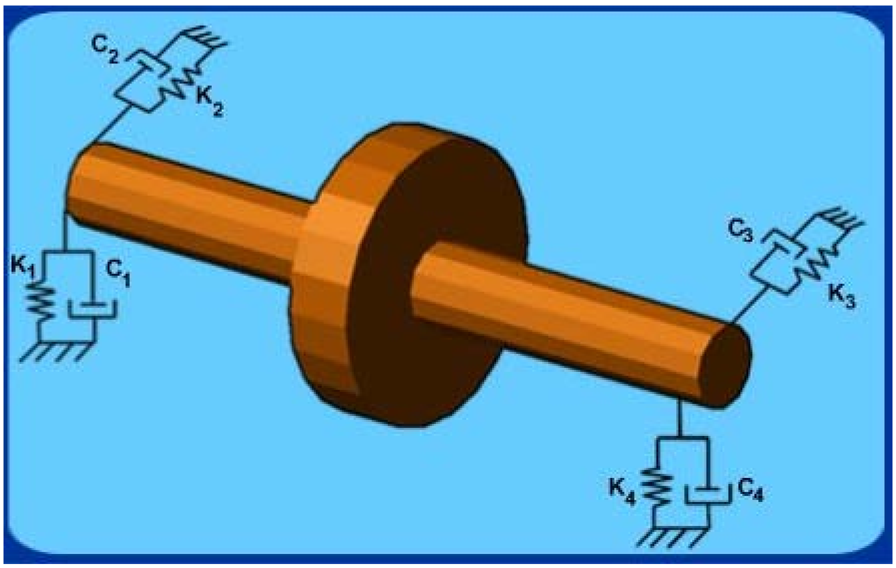


Fig 2.2.4 Model of a Rotor-Bearing system

When we consider a typical rotor of a turbine or compressor it is likely to have multiple stages and in each stage several blades attached around the circumference of the disk. A single stage is schematically shown in Fig. 3.2.5. Several blades are sometimes grouped together using “lacing wires”. Also, not all the blades in a stage will be exactly identical due to various manufacturing process variations etc. The individual blades are of aero-foil cross-section and are twisted / tapered and attached at a stagger angle to the disk. Thus the dynamics of the complete bladed-disk unit is quite a challenging problem.

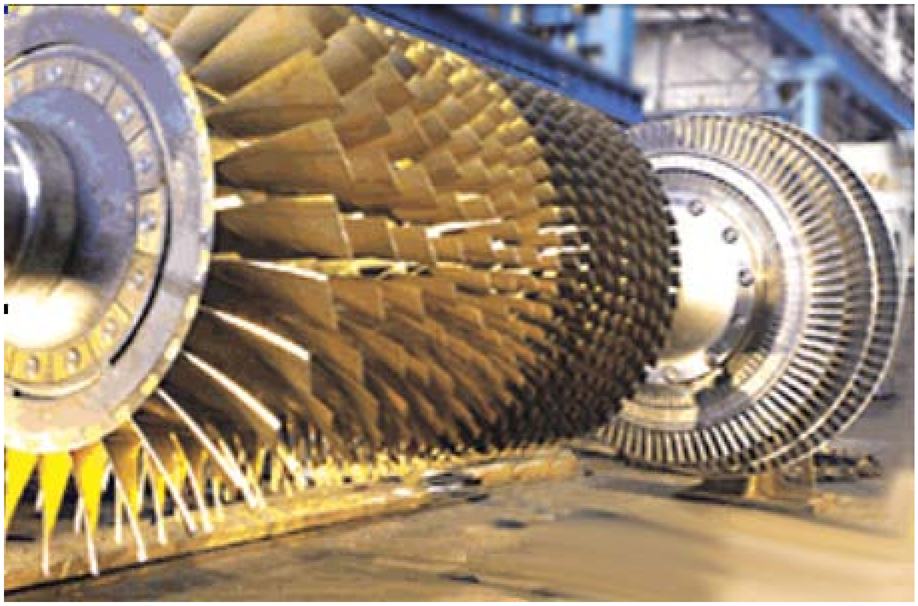


Fig 3.2.5

Thus you will appreciate that the field of rotor-dynamics offers many practically significant and theoretically challenging research problems. We will however restrict our discussion to just the problem of achieving “balance”.

A rotor can in general have two types of unbalance viz., “static” and “dynamic”. It is of course to be appreciated that practical systems will all have dynamic unbalance only and considering it as static unbalance is a “good-enough” approximation for some cases.

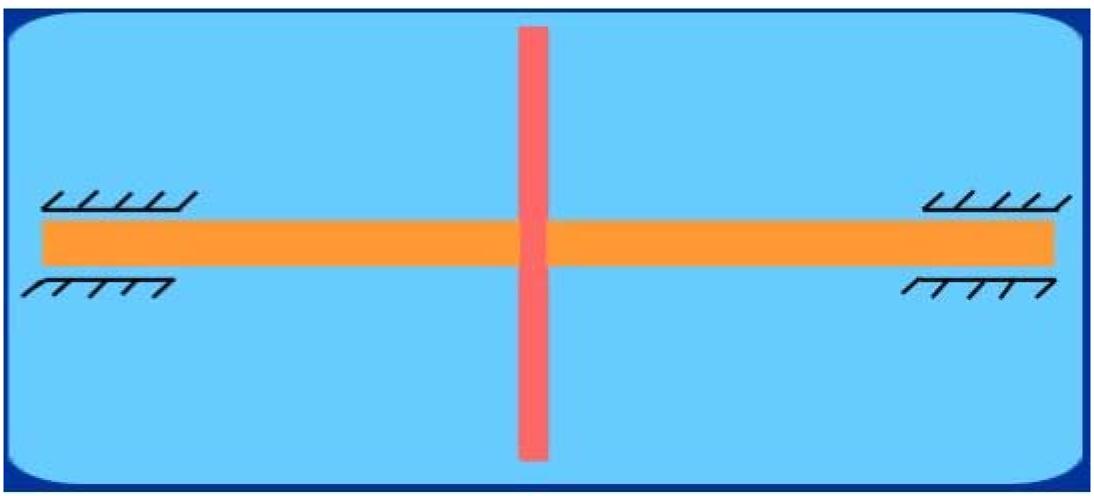


Fig 2.2.6 A thin Rotor Disc - Illustration of Static Unbalance

If the rotor is thin enough (longitudinally) as shown in Fig. 2.2.6, the unbalance force can be assumed to be confined to one plane (the plane of the disc). Such a case is known as “static” unbalance. Such a system when mounted on a knife-edge as shown in Fig. 2.2.7, will always come to rest in one position only – where the centre of gravity comes vertically below the knife-edge point. Thus in order to “balance out”, all we need to do is to attach an appropriate “balancing mass” exactly 180^0 opposite to this position.

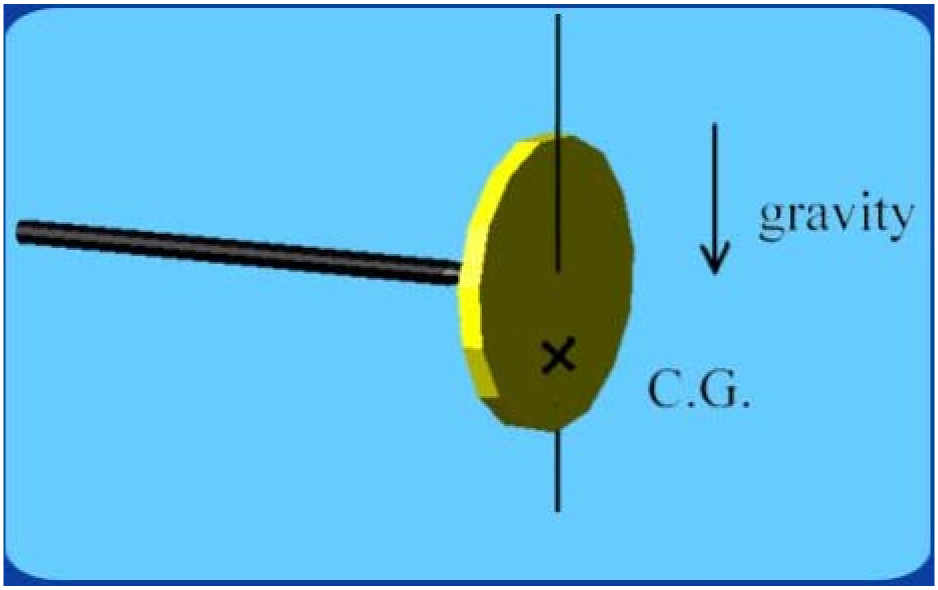


Fig 2.2.7 Thin Rotor on a knife edge - Illustration of Static Unbalance

Thus we first mount the disc on a knife edge and allow it to freely oscillate. Mark the position when it comes to rest. Choose a radial location (180^0 opposite to this position) where we can conveniently attach a balancing mass. By trial and error the balancing mass can be found out. When perfectly balanced, the disc will exhibit no particular preferred position of rest. Also when the disc is driven to rotate by a motor etc., there will be no centrifugal forces felt on the system (for example, at the bearings). Thus the condition for static balance is simply that the effective centre of gravity lie on the axis.


Fig 3.2.8 A Case of Dynamic Unbalance

Consider the rotor shown in Fig. 3.2.8. It is easily observed that mass distribution cannot be approximately confined to just one plane. So unbalance masses and hence unbalance forces are in general present all along the length of the rotor. Such a case is known as “dynamic unbalance”.

The fundamental difference between static and dynamic unbalance needs to be clearly appreciated.

When a rotor as shown in Fig. 3.2.8 is mounted on a knife edge and allowed to oscillate freely, it too may come to rest in one particular position all the time – the position corresponding to the resultant unbalance mass (centre of gravity) vertically below the knife edge. We could, like earlier, mount an appropriate balance mass exactly 180^0 opposite to this position. It would then have no preferred position of rest when mounted on a knife-edge. Thus effective center of gravity lies on the axis.

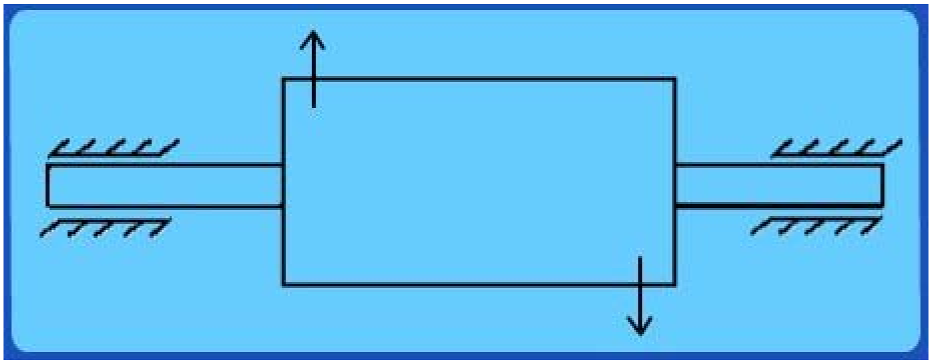


Fig 3.2.9 Example of unbalance masses leading to unbalance force that for a resultant couple because of axial.

However, when mounted in bearings and driven by a motor etc., it could still wobble due to the unbalanced moments of these forces as shown in Fig. 3.2.9. This becomes apparent only when the rotor is driven to rotate and hence the name “dynamic unbalance”. Thus it is not, in general, sufficient to do just static balance but achieving good dynamic balance is more difficult. We will discuss one important method of achieving dynamic balance in the next lecture.

Recap

In this lecture you have learnt the following

- Some interesting issues in rotor dynamics

- Static vs dynamic balance

Module 2 : Dynamics of Rotating Bodies; Unbalance Effects and Balancing of Inertia Forces

Lecture 5 : Two-plane balancing technique

Objectives

In this lecture you will learn the following

- Balancing of rigid rotors
- Two-plane method for balancing

Consider the turbo-machine rotor that was discussed earlier wherein each stage contains several blades around the circumference of a disk. Eventhough typically each stage is balanced in itself to the extent possible, it has a likely net unbalance. When the rotor is set to spin, it will cause dynamic forces and moments on the bearings that support the shaft. Therefore it is of interest to achieve "good balance" of this shaft so that the fluctuating forces on the bearings are reduced. Conceptually our strategy can be simply stated as follows:

Step 1: Consider the shaft supported on its bearings. For each unbalance mass, there will be a centrifugal force set-up when the rotor spins at some speed Ω. This would cause some reactions at the supports. Estimate these support reactions that would come onto the bearings.

Step 2: Estimate the balancing mass that needs to be placed in the plane of bearings, to counter this reaction force due to unbalance mass.

Repeat steps 1 and 2 for each unbalance mass in the system and each time add the balancing masses obtained in step 2 vectorially to determine the resultant balancing mass required.

Let us now understand the details of the technique mentioned earlier. Firstly we choose to place "balancing or correcting" masses on the shaft (rotating along with the shaft) to counter-act the unbalance forces. We understand that this is to be done on the rotor on site, perhaps during a maintenance period. From the point of view of accessibility, we therefore choose the balancing masses to be kept near the bearings.

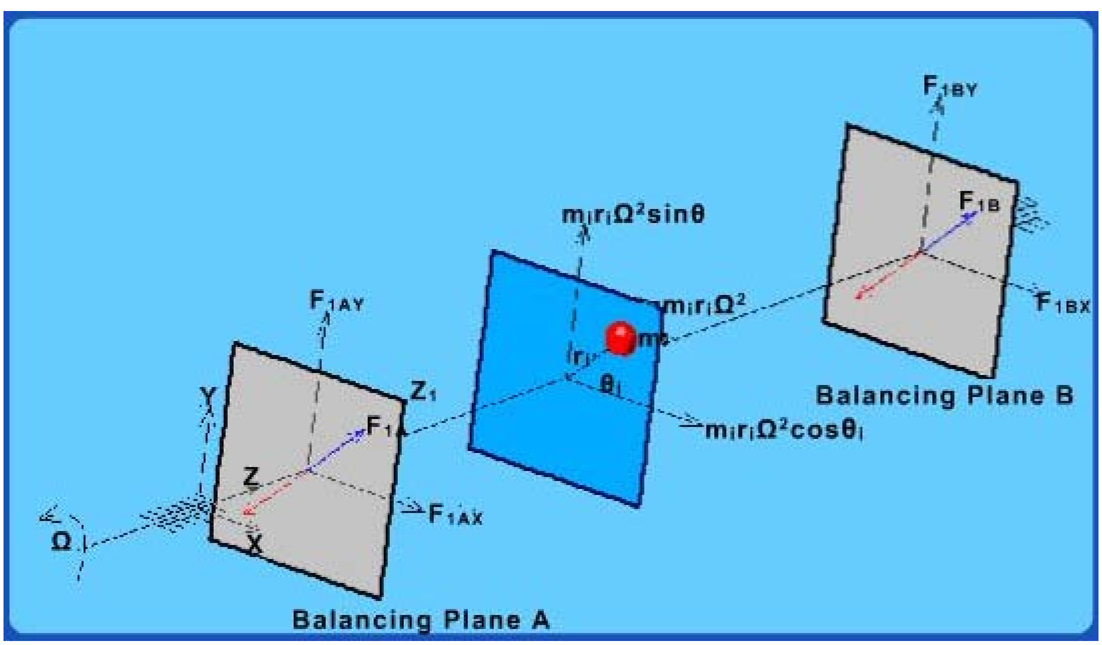


Figure 2.3.1 Two plane balancing technique

The calculations proceed as shown in Fig. 2.3.1. For an unbalance mass m_i situated at an angular location θ_i

in a plane at an axial distance z_i from the left end bearing and rotating at a radius r_i as shown in the figure, the unbalance force is $m_i r_i \Omega^2$. It is resolved into X and Y components as shown in the figure. These forces are represented by EQUIVALENT FORCES in the balancing planes (shown in blue F_{IA}, F_{IB}). These forces can be readily calculated (based on calculations similar to those involved in finding support reactions for a simply supported beam). In order to counterbalance this force, we need to place a balancing mass m_b at a radius r_b in the balancing plane such that it creates an equal and opposite force (shown in red).

Now we need to repeat the calculations for ALL the unbalance masses m_i (i = 1,2,3,.....) and find the resultant equivalent force in the balancing plane as shown in blue in Fig. 2.3.1. This resultant force is balanced out by placing a suitable balancing mass creating an equal and opposite force (shown in red). Since all the masses are rotating at the same speed Ω along with the shaft, we can drop Ω in our calculations – i.e., a rotor balanced at one speed will remain balanced at all speeds or in other words, our technique of balancing is independent of speed. We will review this towards the end of the lecture.

While these calculations can be done in any manner perceived to be convenient, a tabular form (see Table 2.3.1) is commonly employed to organize the computations. While doing this, it is also common practice to include the two balancing masses in the balancing planes as indicated in the table.

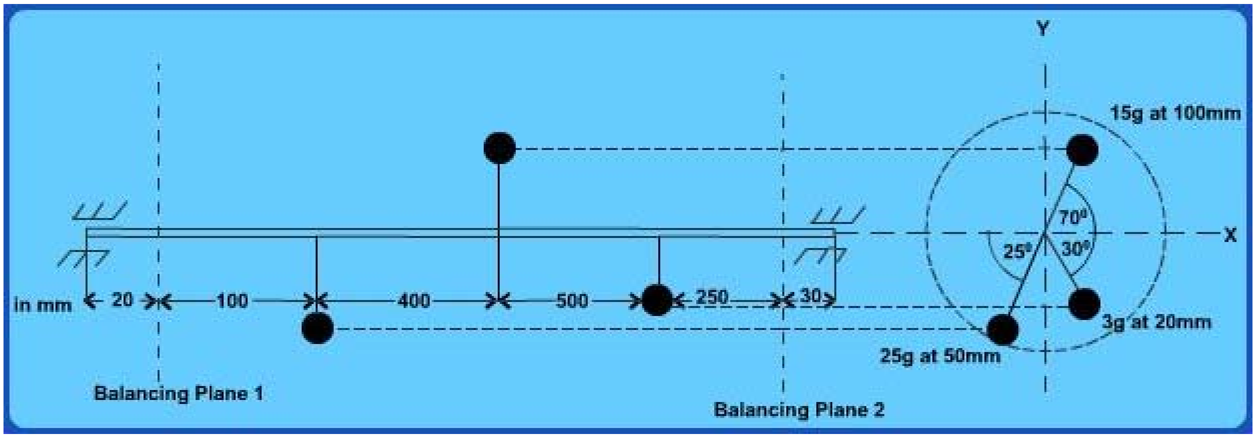
Table 2.3.1 Tabular form of organizing the computations for two-plane balancing technique

Sr. No	Z_i	M_i	R_i	θ_i	$M_i R_i$ Cos(θ_i)	$M_i R_i$ Sin(θ_i)	$M_i R_i Z_i$ Cos(θ_i)	$M_i R_i Z_i$ Sin(θ_i)
1								
2								
3								
….								
…..								
Balancing Plane 1	0	M_{b1}	R_{b1}	θ_{b1}				
Balancing Plane 2	L	M_{b2}	R_{b2}	θ_{b2}				
TOTAL FORCES					**0**	**0**	**0**	**0**

It is observed in Table 2.3.1 that the balancing masses and their locations (radial as well as angular) are unknowns while the location of the balancing plane itself is treated as a known (any accessible location near the bearings etc). The resultant total forces and moments must sum up to ZERO and therefore we have four equations but six unknowns. Thus any two of the six unknowns can be freely chosen and the other four determined from the computations given in the table. This method of balancing is known as the “two-plane balancing technique” since balancing masses are kept in two planes.

We will now work out an example, to illustrate the procedure.

Ex. 2.3.1 For the rotor shown in **Fig. 2.3.2**, find the magnitude and the angular location of the balancing masses.



The calculations are shown in the table below.

Sr. No.	Z_i	M_i	R_i	θ_i	$M_i R_i \cos\theta_i$	$M_i R_i Sin\theta_i$	$M_i R_i Z_i \cos\theta_i$	$M_i R_i Z_i Sin\theta_i$
1	120	25	50	205^0	-1132.88	-528.27	135945.6	-63392
2	520	15	100	70^0	513.03	1409.54	266775.71	732960.24
3	1020	3	20	330^0	51.96	-30.00	53000.75	-30600.00
Balancing Plane1	20	m_{b1}	r_{b1}	θ_{b1}	$m_{b1} r_{b1} \cos\theta_{b1}$	$m_{b1} r_{b1} \sin\theta_{b1}$	$20 m_{b1} r_{b1} \cos\theta_{b1}$	$20 m_{b1} r_{b1} \sin\theta_{b1}$
Balancing Plane2	1270	m_{b2}	r_{b2}	θ_{b2}	$m_{b2} r_2 \cos\theta_{b2}$	$m_{b2} r_2 \sin\theta_{b2}$	$1270 m_{b2} r_2 \cos\theta_{b2}$	$1270 m_{b2} r_2 \sin\theta_{b2}$
Total					**0**	**0**	**0**	**0**

From the table, the final equations are obtained as follows:

$m_{b1} r_{b1} \cos\theta_{b1} + m_{b2} r_2 \cos\theta_{b2} = 567.89$

$m_{b1} r_{b1} \sin\theta_{b1} + m_{b2} r_2 \sin\theta_{b2} = -851.27$

$20 m_{b1} r_{b1} \cos\theta_{b1} + 1270 m_{b2} r_2 \cos\theta_{b2} = -455722.06$

$20 m_{b1} r_{b1} \sin\theta_{b1} + 1270 m_{b2} r_2 \sin\theta_{b2} = -638967.5$

Upon solution, we get

$\theta_{b1} = 340.53^0$;

$m_{b1} r_{b1} = 979.5$ kg-mm

$\theta_{b2} = 236.0^0$

$m_{b2} r_{b2}$ = 634 kg-mm

We have said that the balancing achieved by our method was independent of speed. This is true only up to a certain limit, as we have implicitly assumed the shaft to be perfectly rigid. As the speed increases, the unbalance forces increase and the shaft tends to deform elastically under the action of these forces. Also, as the speed becomes closer to the fundamental transverse bending natural frequency of the shaft in its bearings, chances of resonance are also there. Thus the assumption of shaft being perfectly rigid is no longer valid. A rotor can be assumed to be "rigid" if the speed of rotation is less than about 1/3 rd the fundamental natural frequency. Two plane balancing technique is essentially a rigid rotor balancing method and is useful only within the speed limit mentioned above.

Recap

In this lecture you have learnt the following

- The two plane method of balancing rigid rotors
- The tabular form of organizing the computations

Module 3 : Dynamics of Reciprocating Machines with Single Slider;Unbalance in Single Cylinder Engine Mechanisms

Lecture 6 : Dynamics of Reciprocating Machines with Single Slider;Unbalance in Single Cylinder Engine Mechanisms

Objectives

In this lecture you will learn the following

- Approximate acceleration analysis of an IC Engine mechanism
- Equivalent Link model of a connecting rod
- Estimation of Inertia forces in a crank-slider mechanism

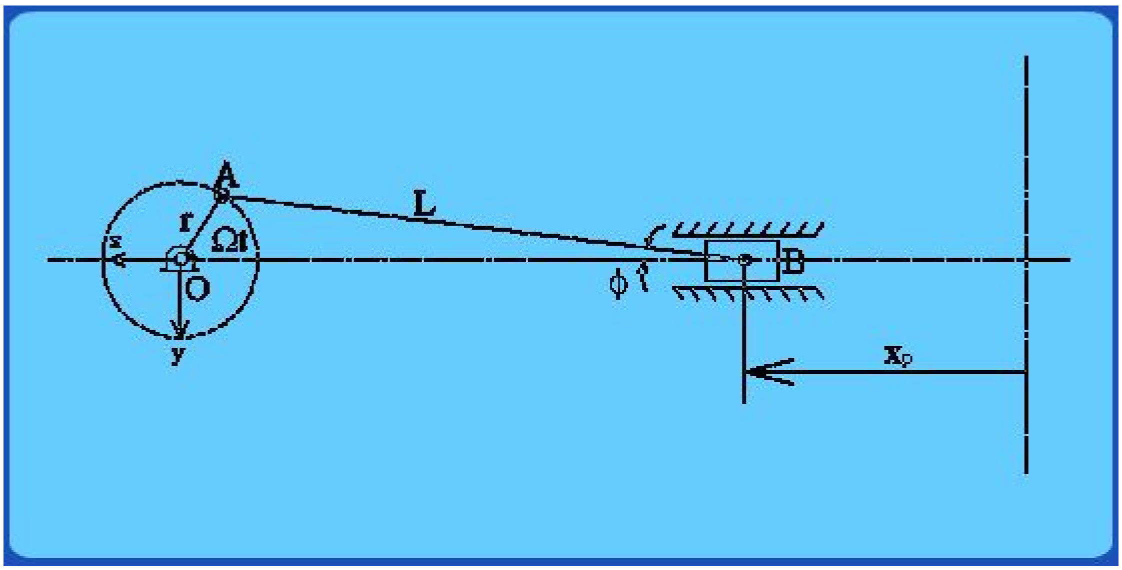


Figure 3.1 Slider-crank Mechanism of IC Engine

A typical crank-slider mechanism as used in an IC Engine is shown in Fig. 3.1. It essentially consists of four different parts viz., frame(i.e., cylinder) ,crank ,connecting rod and reciprocating piston. The frame is supposedly stationary; crank is undergoing purely rotary motion while the piston undergoes to-and-fro rectilinear motion. The connecting rod undergoes complex motion – its one end is connected to the crank (undergoing pure rotation) and the other end is connected to the piston (undergoing pure translation).

We know that the inertia forces are given by mass times acceleration and we shall now estimate the inertia forces (shaking forces and moments) due to the moving parts on the frame (cylinder block).

PISTON

From the geometrical construction shown in Fig 3.1, we have:

$$x_p = r\,(1 - \cos(\Omega t)) + L\,(1 - \cos(\varphi)) \qquad (3.1)$$

$$\sin(\varphi) = (r/L)\,\sin(\Omega t) \qquad (3.2)$$

(3.3)

$$x_p = r\left(1-\cos(\Omega t)\right) + L\left[1-\sqrt{1-\left(\frac{r}{L}\right)^2 \sin^2(\Omega t)}\right]$$

It is to be observed that if r > L, the crank wont rotate fully. Even if r = 0.5 L, the transmission angle $(\pi/2 - \phi)$ will still be poor. Therefore, in practical crank-slider mechanisms used in IC Engines, crank radius "r" is less than one-fourth of the length of the connecting rod. Thus

$$\left(\frac{r}{L}\right)^2 \ll 1 \qquad (3.4)$$

We can therefore approximately write: $\sqrt{1-\left(\frac{r}{t}\right)^2 \sin^2 \Omega t} \approx 1-\frac{1}{2}\left(\frac{r}{t}\right)^2 \sin^2 \Omega t$

$$But\ Sin^2 \Omega t = \frac{1-\cos^2 \Omega t}{2}$$

$$\therefore \sqrt{1-\left(\frac{r}{t}\right)^2 \sin^2 \Omega t} \approx 1-\left(\frac{r}{t}\right)^2 \frac{\left(1-\cos^2 \Omega t\right)}{4}$$

$$\therefore x_p \approx \left(r+\frac{r^2}{4L}\right) - r\left(\cos(\Omega t)+\frac{r}{4L}\cos(2\Omega t)\right) \qquad (3.5)$$

Note: For r/L = 0.25, the error involved in the above approximation is just 0.05% and thus is negligible for all practical purposes. However, for multi-cylinder engines(high Ω possible because of better balance), the higher harmonics ($\cos^4 \Omega t$ etc.) become significant and are typically included in the dynamic analysis.

Differentiating twice with respect to time yields an approximate expression for the acceleration of the piston as follows,

$$\ddot{x}_p \approx r\Omega^2\left(\cos(\Omega t)+\frac{r}{L}\cos(2\Omega t)\right) \qquad (3.6)$$

When multiplied by the mass of the piston, this gives the inertia force due to the piston.

$$(I.F)_p \approx m_p r\Omega^2\left(\cos(\Omega t)+\frac{r}{L}\cos(2\Omega t)\right) \qquad (3.7)$$

The first term signifies variation at the same frequency as the speed of rotation and hence is known as the PRIMARY force term and the second term is known as the SECONDARY force term. For a typical case, these two terms are pictorially shown in Fig 3.2

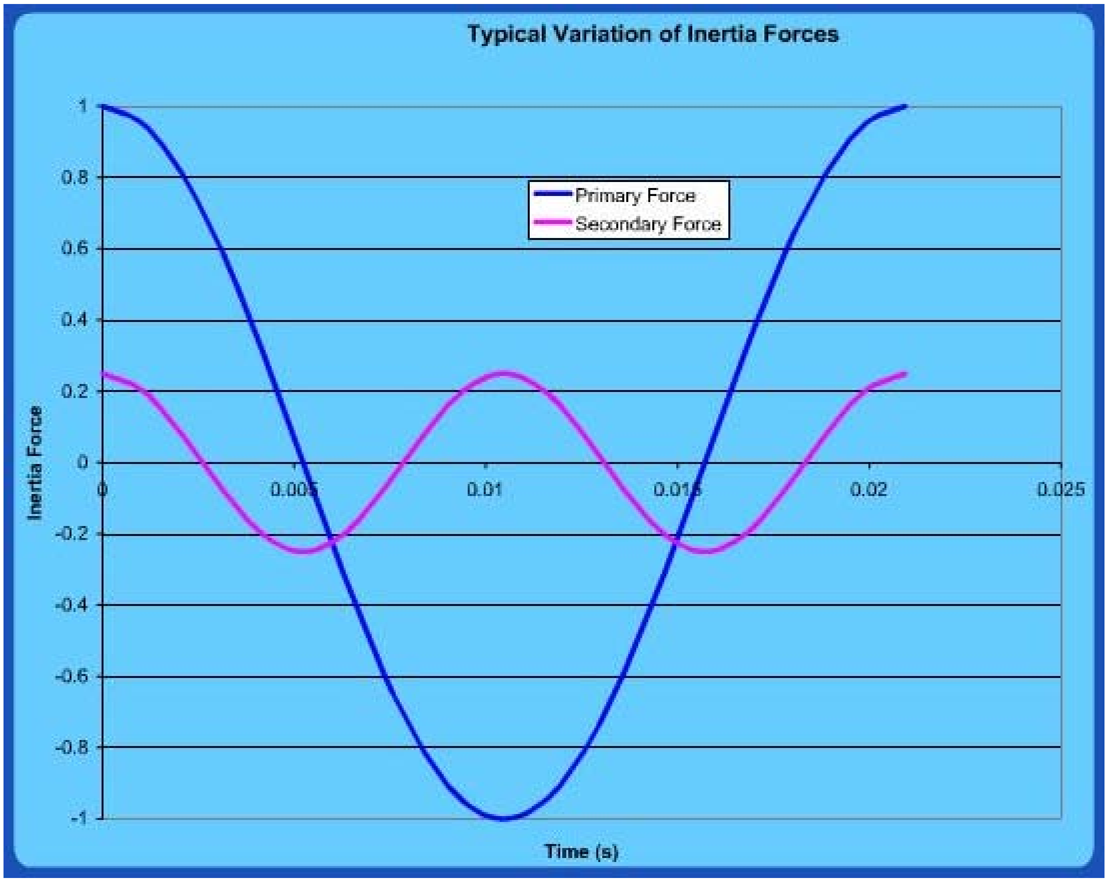


Figure 3.2

CRANK

Crank undergoes pure rotary motion and let us assume that it is rotating at a constant speed Ω. Let G be the center of mass of the crank as shown in Fig 3.3 and so we can write:

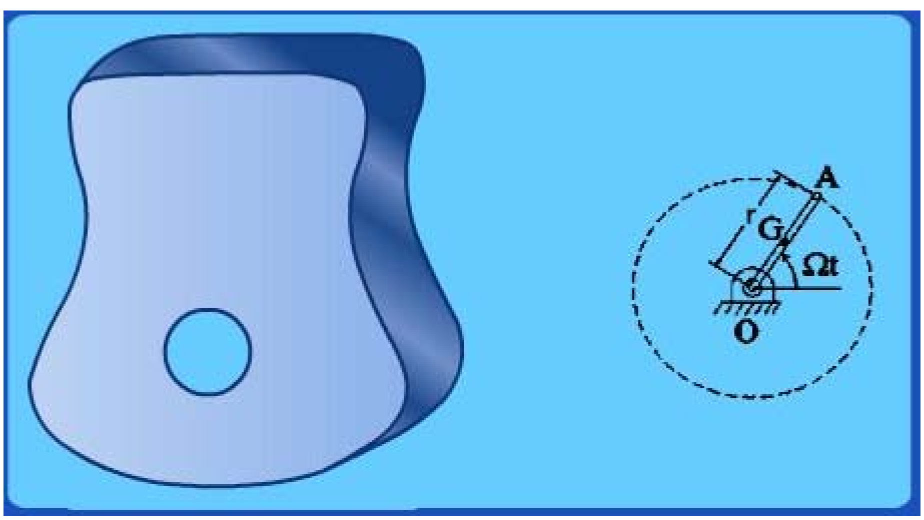


Figure 3. 3 Typical Crank

Centripetal acceleration = (OG) Ω^2 (3.8)

The inertial force (radial) due to crank rotation is given by:

Inertia force = m_c (OG) Ω^2 (3.9)

where m_c is the mass of the crank

This inertia force can be balanced out by the techniques learnt in the previous module and we could make sure that G coincides with O. Then the inertia force would be reduced ideally to zero. If need be, we could always use the above formula to estimate the inertia forces due to crank.

For reasons that will become clear when we discuss the dynamics of connecting rod, it is common to assume that the entire mass of the crank is actually concentrated at the pin A. Thus we can write,

Effective crank mass at A = m_c (OG)/(OA) (3.10)

Comparison of inertia forces due to rotating and reciprocating masses

Now we have seen the inertia forces due to crank which undergoes pure rotation and those due to the piston that undergoes pure translation. We observe that the inertia force due to crank is always of fixed magnitude but ever directed radially i.e., continually changing direction. The inertia force due to a reciprocating mass is always directed along the line of motion but its magnitude is ever changing. This fundamental difference between the two types of inertia forces is captured in Fig. 3.4.

Figure. 3.4

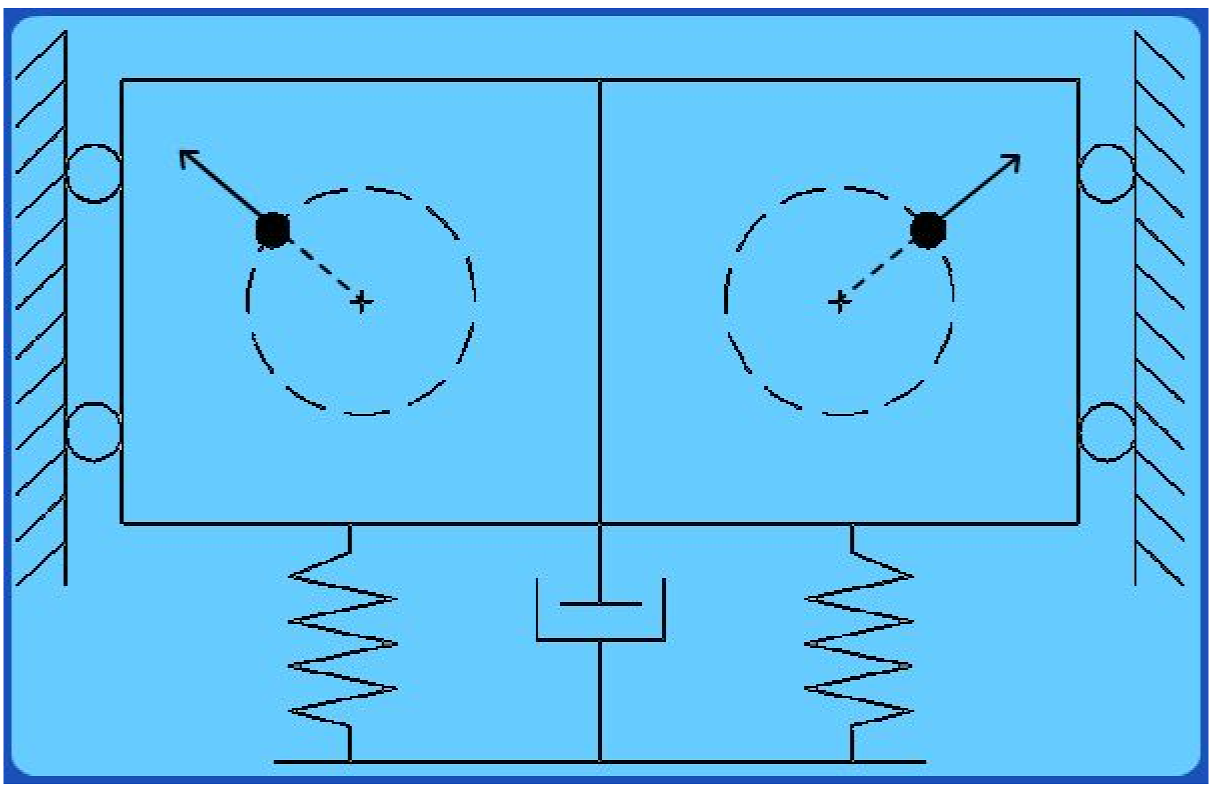


Figure 3.5 Mechanical Shaker

An interesting configuration with two rotating masses (in phase opposition) is shown in Fig. 3.5 which results in a force only along one line/axis. This is effectively used as a mechanical shaker for dynamic testing of structures.

CONNECTING ROD

One end of the connecting rod is circling while the other end is reciprocating and any point in between moves in an ellipse. It is conceivable that we derive a general expression for the acceleration of any point on the connecting rod and hence estimate the inertia forces due to an elemental mass associated with that point. Integration over the whole length of the connecting rod yields the total inertia force due to the entire connecting rod. Instead we try to arrive at a simplified model of the connecting rod by replacing it with a “dynamically equivalent link” as shown in Fig 3.6.

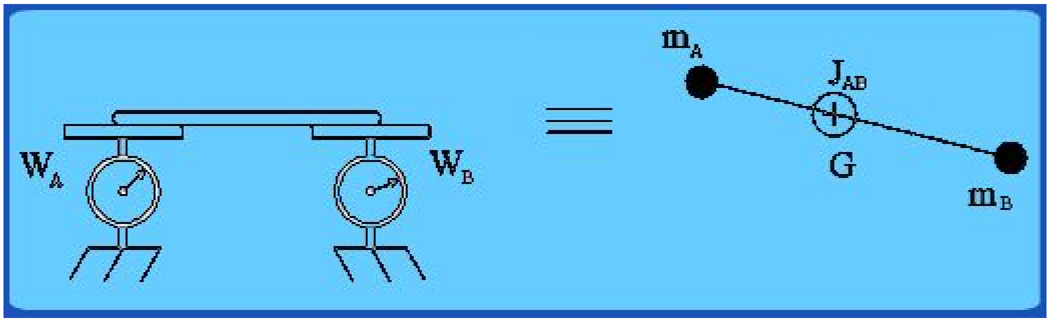


Figure 3.6 Dynamically Equivalent link for a connecting rod .

In order that the two links are dynamically equivalent, it is necessary that:

- Total mass be the same for both the links
- Distribution of the mass be also same i.e., location of CG must be same and the mass moment of inertia also must be same.

Thus we can write three conditions:

$$m = m_A + m_B$$
$$m_A\,(AG) = m_B\,(GB) \qquad (3.11)$$
$$J_G = m_A\,(AG)^2 + m_B\,(GB)^2 + J_{AB}$$

For convenience we would like the equivalent link lumped masses to be located at the big and small end of the original connecting rod and if its center of mass (G) location is to remain same as that of original rod, distances AG and GB are fixed. Given the mass m and mass moment of inertia J_G of the original connecting rod, the problem of finding dynamically equivalent link is to determine m_A, m_B and J_{AB}.

An approximate equivalent link can be found by simply ignoring J_{AB} and treating just the two lumped masses m_A and m_B connected by a mass-less link as the equivalent of original connecting rod. In such a case we take:

$m_A = m\,(GB)/L$

$m_B = m\,(AG)/L$ (3.12)

Thus the connecting rod is replaced by two masses at either end (pin joints A and B) of the original rod. m_A rotates along with the crank while m_B purely translates along with the piston. It is for this reason that we proposed use of crank's effective rotating mass located at pin A, which can now be simply added up to part of connecting rod's mass.

On the shop floor m_A, m_B can be immediately determined by mounting the existing connecting rod on two weighing balances located at A and B respectively. The readings of the two balance give m_A and m_B directly

Dynamic Model of a single cylinder IC Engine Mechanism

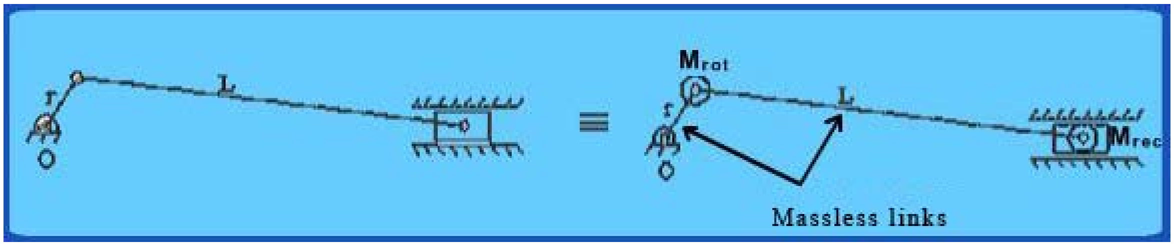


Figure 3.7 Dynamic Model of slider-crank Mechanism

Based on our discussion thus far, we can arrive at a simplified model of the crank-slider mechanism for the purpose of our dynamic analysis as shown in Fig. 3.7. Thus we have either purely rotating masses or purely translating masses and these are given by:

$$M_{rot} = m_c\big|_{at\,A} + m_A$$
$$M_{rec} = m_p + m_B \qquad (3.13)$$

where the first term in the rotating masses is due to the effective crank mass at pin A and the second term is due to the part of equivalent connecting rod mass located at pin A. Similarly the first term in reciprocating masses is due to the mass of the piston and the second is due to the part of equivalent connecting rod mass located at pin B. There are inertia forces due to M_{rot} and M_{rec}.

The inertia forces due to M_{rot} can be nullified by placing appropriate balancing masses as indicated in Fig. 3.9. Thus the effective force transmitted to the frame due to rotating masses can ideally be made zero.

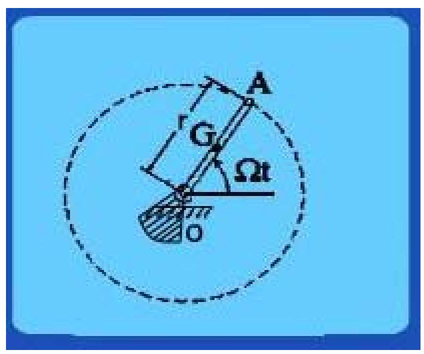


Figure 3.8 Counter balancing rotating masses

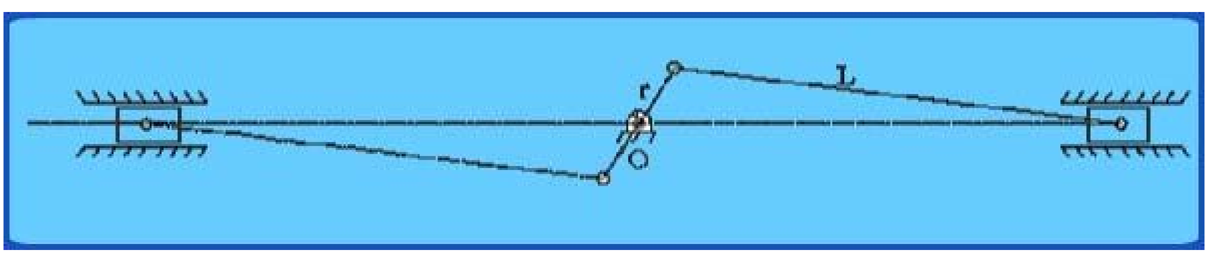


Figure 3.9 Opposed position configuration

However it is not so straight forward to make the unbalanced forces due to reciprocating masses vanish completely. As given in Eq. (3.7) and depicted in Fig. 3.2, there are components of the force which are at the rotational speed and those at twice this speed. It is conceivable to use a configuration as shown in Fig. 3.9 to completely balance out these forces but the mechanism becomes too bulky. Thus a single cylinder engine is inherently unbalanced.

DRIVING TORQUE and INERTIA TORQUE

Gas forces due to the internal combustion of fuel, drive the piston's motion which is transmitted through the connecting rod to the crank. as shown in Fig. 3.10. An expression for the "driving torque" or "gas torque" is given by:

$$T_g = (\rho_g A)\, r\, Sin\Omega t\left[1 + \frac{r}{L}\cos\Omega t\right]$$

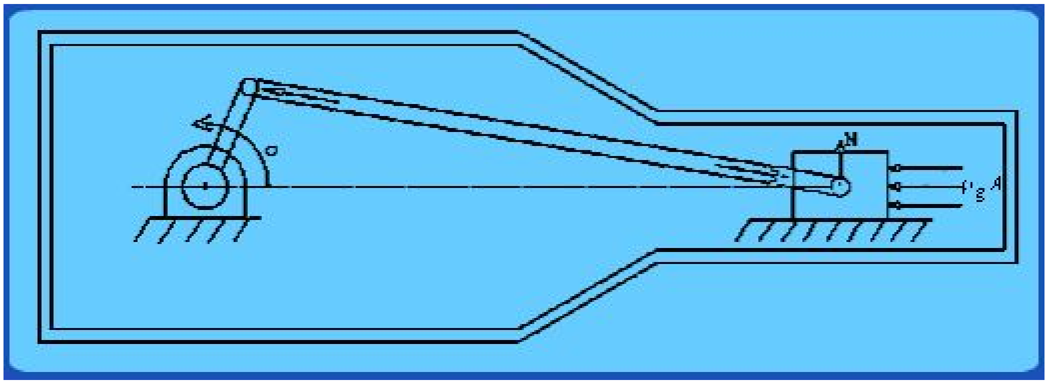


Figure 3.10

For a four-stroke cycle, power is delivered only in one stroke. Thus the gas torque varies with time. When such a source is used as a prime mover, instantaneous speed of the shaft fluctuates from time to time within a cycle of rotation, depending on the “load torque” requirement. If the load torque ideally matches the driving torque at every instant, the speed will be theoretically uniform. Thus in general we will need a device to “iron out” these fluctuations in shaft speed within a cycle. Such a device is known as “flywheel” and we shall discuss it in greater detail in a later module.

By replacing the gas pressure by inertia force due to M_{rec} (i.e., $m_{rec}\ddot{x}_p$), we can estimate the “shaking moment” or “inertia torque” as follows:

$$T_{inertia} = \frac{m_{rec}}{2} r^2 \Omega^2 \left[\frac{r}{2L} Sin\Omega t - Sin^2\Omega t - \frac{3r}{2L} Sin^3\Omega t\right]$$

Recap

In this module you have learnt the following

- A single cylinder IC engine mechanism is inherently unbalanced
- Approximate analysis of the dynamics of single cylinder IC engine
- Estimation of the forces and moments felt on the “frame” i.e., cylinder block during the operation of the IC Engine

Module 4 : Unbalance in Multicylinder Engines – In-line, V-twin and Radial Engines; Balancing Techniques.

Lecture 7 : Unbalance in Multicylinder Engines – In-line, V-twin and Radial Engines; Balancing Techniques.

Objectives

In this lecture you will learn the following

- Typical arrangements of multiple cylinders
- State of balance of typical multi-cylinder engines

Consider a single cylinder engine (Fig 4.1) such as is used in a typical scooter or motor bike. We have analyzed the inertia forces in this case based on our approximate dynamic equivalent link model of connecting rod – replacing it by two masses one at crank pin and another at the piston end. Thus effectively there are reciprocating masses which included the mass of the piston and that part of the connecting rod which was lumped on the piston pin; rotating parts which included the effective mass of crank at its pin and part of connecting rod lumped at crank pin.

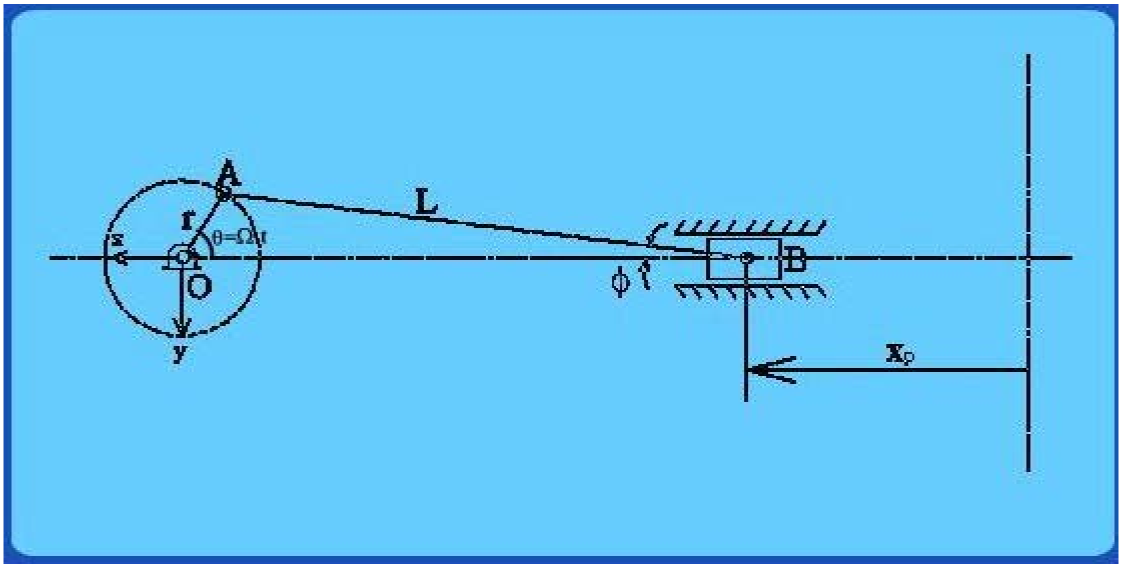


Figure 4.1 Typical Single Cylinder Engine.

From our earlier discussion on balancing it is easily conceived that the rotating parts can be readily balanced out by keeping a balancing mass at an appropriate radial location diametrically opposite m_{rot}. Thus we are left with only the inertia forces due to reciprocating masses. These forces are given by:

$$I.F. = m_{rec}\ddot{x}_p \quad (4.1)$$

Recall that, with reference to Fig. 4.1, and eq(3.7), an approximate expression for the acceleration of the reciprocating parts is given by:

$$\ddot{x}_p = r\Omega^2\left(Cos\theta + \frac{r}{\ell}Cos2\theta\right) \quad (4.2)$$

Thus an approximate expression for the net unbalanced inertia force in a single cylinder IC engine is given by:

$$I.F. = m_{rec} r \Omega^2 \left(Cos\theta + \frac{r}{\ell} Cos2\theta \right) \quad (4.3)$$

It is important to note that this force is directed ALONG the line of reciprocation of the masses m_{rec} and θ is the orientation of the crank measured from this line of reciprocation.

For example, as shown in Fig. 4.2, if the line of reciprocating motion (axis of cylinder) is arbitrarily directed, we can resolve this force into components along Cartesian coordinate axes X- and Y. In the subsequent discussion of the state of balance of multi-cylinder engines, with multiple cylinders arranged in a particular configuration, it is therefore only required to find the X- and Y- components of inertia forces of each cylinder and perform a vector addition of all the component forces and their moments to get the resultant unbalance force/moment for the whole engine.

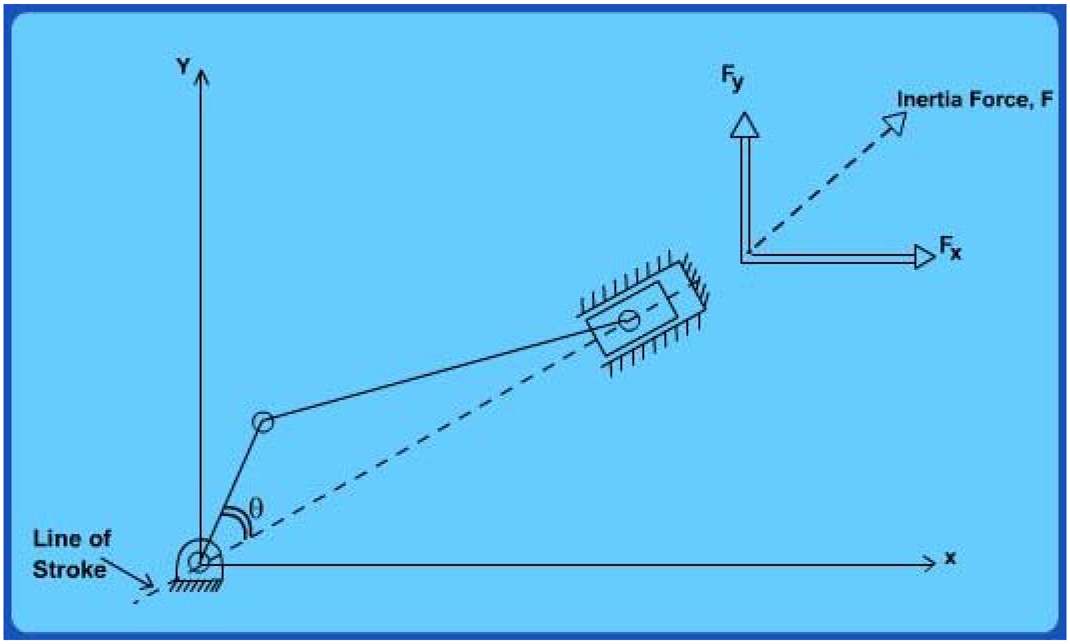


Figure 4.2 Arbitrarily oriented single cylinder engine

INERTIA FORCES

Let us designate the inertia forces due to the reciprocating parts of cylinder “i” as $I.F.|_i$. Thus,

$$I.F.|_1 = m_{rec}\, r\, \Omega^2 \left(Cos\theta_1 + \frac{r}{\ell} Cos2\theta_1 \right)$$

$$I.F.|_2 = m_{rec}\, r\, \Omega^2 \left(Cos\theta_2 + \frac{r}{\ell} Cos2\theta_2 \right)$$

$$= m_{rec}\, r\, \Omega^2 \left(Cos(\theta_1 + \beta_2) + \frac{r}{\ell} Cos2(\theta_1 + \beta_2) \right) \quad (4.5)$$

$$I.F.|_i = m_{rec}\, r\, \Omega^2 \left(Cos(\theta_1 + \beta_i) + \frac{r}{\ell} Cos2(\theta_1 + \beta_i) \right)$$

These forces are all acting along their respective cylinder axes and are therefore axially separated as indicated in Fig. 4.3. Thus the total inertia force due to the reciprocating parts is given by:

(4.6)

$$I.F. = \sum_{i=1}^{n} I.F.\Big|_i$$
$$= m_{rec}\, r\, \Omega^2 \left[\sum_{i=1}^{n} Cos(\theta_1 + \beta_i) + \frac{r}{\ell} \sum_{i=1}^{n} Cos2(\theta_1 + \beta_i) \right]$$

where "n" stands for the number of cylinders. From standard trigonometric relations, we can write,

$$\sum_{i=1}^{n} Cos(\theta_1 + \beta_i) = Cos\theta_1 \sum_{i=1}^{n} Cos\,\beta_i \quad - \quad Sin\theta_1 \sum_{i=1}^{n} Sin\,\beta_i$$
$$\sum_{i=1}^{n} Cos2(\theta_1 + \beta_i) = Cos2\theta_1 \sum_{i=1}^{n} Cos2\,\beta_i \quad - \quad Sin2\theta_1 \sum_{i=1}^{n} Sin2\,\beta_i \qquad (4.7)$$

In order that the total inertia force be zero for all positions of cranks (i.e., for all values of θ_1), we therefore have the following conditions:

$$\sum_{i=1}^{n} Cos\,\beta_i = 0$$
$$\sum_{i=1}^{n} Sin\,\beta_i = 0$$
$$\sum_{i=1}^{n} Cos2\,\beta_i = 0 \qquad (4.8)$$
$$\sum_{i=1}^{n} Sin2\,\beta_i = 0$$

PITCHING MOMENTS

Due to the axial separation of the cylinders, there is a pitching moment developed about a lateral axis as given by $\sum_{i=1}^{n} I.F.\Big|_i z_i$ where z_i stands for the axial distance of i^{th} cylinder from a common reference frame. When the cylinders are arranged along the length direction of the vehicle, such a moment about a lateral axis of the vehicle tends to induce pitching motion of the vehicle and hence this is known as pitching moment. For the pitching moment to be zero for all crank positions, we can similarly derive the following conditions:

$$\sum_{i=1}^{n} z_i\, Cos\,\beta_i = 0$$
$$\sum_{i=1}^{n} z_i\, Sin\,\beta_i = 0$$
$$\sum_{i=1}^{n} z_i\, Cos2\,\beta_i = 0 \qquad (4.9)$$
$$\sum_{i=1}^{n} z_i\, Sin2\,\beta_i = 0$$

Thus for a given crank shaft profile (i.e., z and β_s) we could determine the resultant force/moment on the frame.

We will illustrate the calculations through an example.

Example: Inline Four Cylinder Four Stroke Engine

Consider an inline four cylinder four stroke marine engine arrangement as shown in Fig. 5.4. Let the reciprocating masses in each cylinder be 500kg. Let the crank length be 200mm and the connecting rod length 800mm. Engine speed is 100 RPM. Investigate its state of balance.

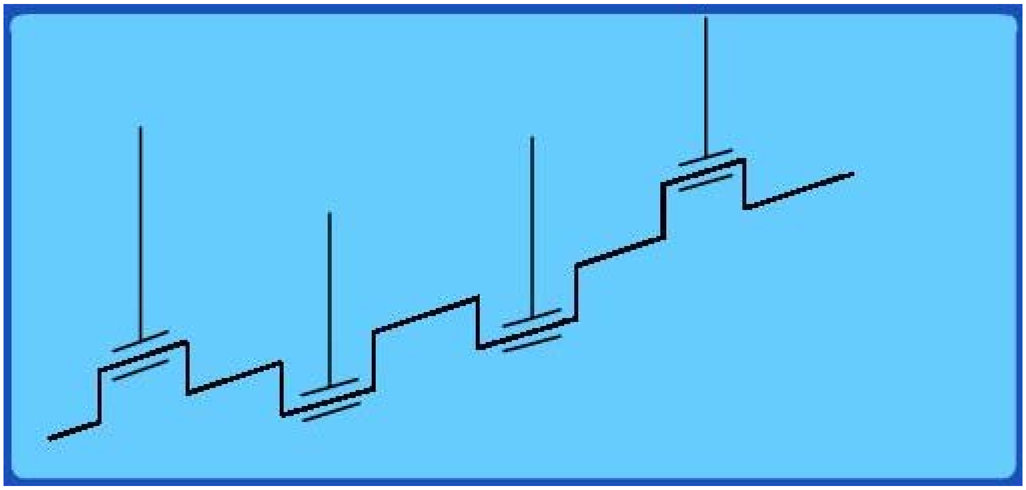


Figure 4.4 Four cylinder In-line engine

Solution:

Let the crank position of the first crank be θ. Then the remaining cylinders' cranks are at (θ+180), (θ +180) and θ respectively. In other words, $\beta_1 = 0;\ \beta_2 = 180^0;\ \beta_3 = 180^0;\ \beta_4 = 0$. Thus,

$$\sum Cos\ \beta_i = 0$$
$$\sum Sin\ \beta_i = 0$$
$$\sum Cos\ 2\beta_i = 4$$
$$\sum Sin\ 2\beta_i = 0$$

Thus the primary forces are completely balanced but not the secondary forces. The magnitude of the unbalanced secondary forces can be estimated as follows:

I.F. = (4) (500) (0.2) (2* π*100/60) 2 (0.2/0.8) = 10966 N

Let us take the mid-plane as the reference plane. Then,

z_1 = - 1.5m

z_2 = - 0.5m

z_3 = 0.5m

z_4 = 1.5m

Thus, we have,

$$\sum z_i\ Cos\ \beta_i = 0$$
$$\sum z_i\ Sin\ \beta_i = 0$$
$$\sum z_i\ Cos\ 2\beta_i = 0$$
$$\sum z_i\ Sin\ 2\beta_i = 0$$

Therefore the primary and secondary moments are completely balanced.

STATE OF BALANCE OF A RADIAL ENGINE

A radial engine is one in which all the cylinders are arranged circumferentially as shown in Fig. 4.5. These engines were quite popularly used in aircrafts during World War II. Subsequent developments in steam/gas turbines led to the near extinction of these engines. However it is still interesting to study their state of balance in view of some elegant results we shall discuss shortly. Our method of analysis remains identical to the previous case i.e., we proceed with the assumption that all cylinders are identical and the cylinders are spaced at uniform interval $\left(\frac{2\pi}{n}\right)$ around the circumference. Thus the crank angle for an i^{th} cylinder will be given by:

$$\theta_i = \theta_1 + (i-1)\left(\frac{2\pi}{n}\right) \tag{4.10}$$

Inertia force due to reciprocating parts of the i^{th} cylinder is given by:

$$I.F.\big|_i = m_{rec} r \Omega^2 \left(Cos\theta_i + \frac{r}{\ell} Cos 2\theta_i\right) \tag{4.11}$$

Resolving this along global Cartesian X-Y axes, we have,

$$I.F.\big|_{iX} = m_{rec} r \Omega^2 \left(Cos\theta_i + \frac{r}{\ell} Cos 2\theta_i\right) Cos\left[(i-1)\frac{2\pi}{n}\right] \tag{4.12}$$

$$I.F.\big|_{iY} = m_{rec} r \Omega^2 \left(Cos\theta_i + \frac{r}{\ell} Cos 2\theta_i\right) Sin\left[(i-1)\frac{2\pi}{n}\right] \tag{4.13}$$

When summed up for all the cylinders, we get the resultant inertia forces for the whole radial engine.

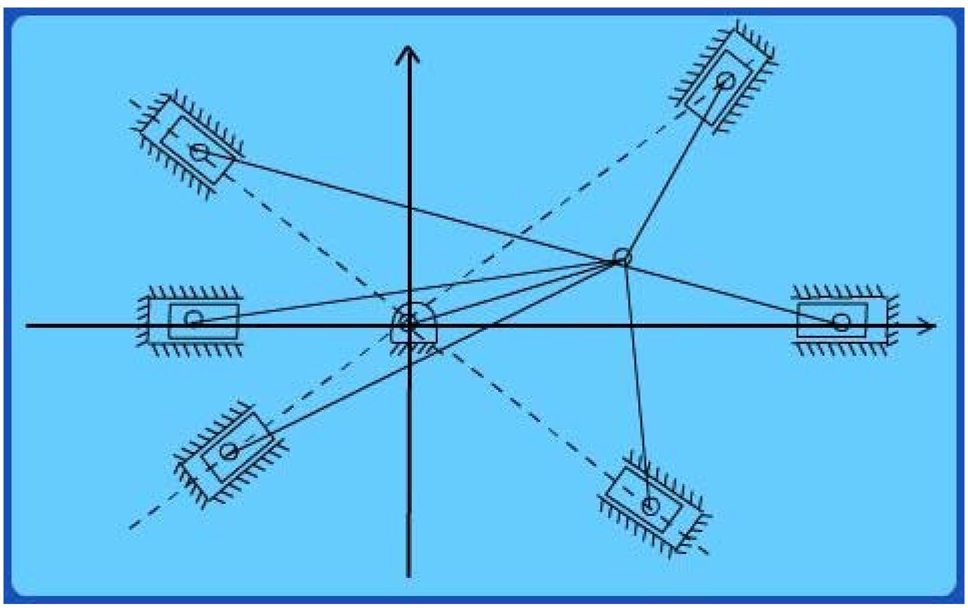


Fig. 4.5 Typical Radial Engine (Not to scale)

Interesting result on the primary forces

The primary component of the forces above sum upto:

$$\sum_{i=1}^{n} I.F.\big|_{XP} = \sum_{i=1}^{n} m_{rec} r\Omega^2 \left(Cos\theta_i\right) Cos\left[(i-1)\frac{2\pi}{n}\right]$$

(4.14)

$$\sum_{i=1}^{n} I.F.\big|_{YP} = \sum_{i=1}^{n} m_{rec} r\Omega^2 \left(Cos\theta_i\right) Sin\left[(i-1)\frac{2\pi}{n}\right]$$

Using standard trigonometric relations, it can be shown that this summation simply reduces to (for n > 2):

$$\sum_{i=1}^{n} I.F.\big|_{XP} = \frac{n}{2} m_{rec} r\Omega^2 \left(Cos\theta_1\right)$$

$$\sum_{i=1}^{n} I.F.\big|_{YP} = \frac{n}{2} m_{rec} r\Omega^2 \left(Sin\theta_1\right)$$

(4.15)

Thus, the arrangement of the radial engine has resulted in a total resultant primary force of fixed magnitude viz., $\frac{n}{2} m_{rec} r\Omega^2$ and directed along the first crank. Thus this fixed magnitude force "rotates" along with the first crank. Such a resultant force can therefore readily be balanced out by an appropriate mass kept on the crank. Therefore it is **possible to get complete balance of the primary forces**.

Further analysis of the inertia forces reveals that for even number of cylinders (n > 2) i.e., for four, six, eight etc. cylinders the secondary forces are also completely balanced out.

Thus we can have an engine where in all the inertia forces can be completely balanced. Since the reciprocating engines can NOT be normally operated at high speeds because of the inertia forces due to the reciprocating masses, this gives an exciting possibility of operating the radial engine at high speeds. However when we think of high speeds, we should revisit our analysis of the inertia forces. Recall that eq. (5.3) involved an approximation as given in eq. (4.2) wherein we neglected the higher order terms in the series. Including these terms also, we get the more accurate expression for the acceleration of the reciprocating parts as:

$$\ddot{x}_p = r\Omega^2 \left(Cos\theta + c_2 Cos2\theta + c_4 Cos4\theta + c_6 Cos6\theta + ...\right) \quad (4.16)$$

where,

$$c_2 = \left[\frac{1}{4}\left(\frac{r}{\ell}\right) + \frac{1}{16}\left(\frac{r}{\ell}\right)^3 + \frac{15}{512}\left(\frac{r}{\ell}\right)^5 + ...\right]$$

$$c_4 = \left[\frac{1}{64}\left(\frac{r}{\ell}\right)^3 + \frac{3}{256}\left(\frac{r}{\ell}\right)^5 + ...\right] \quad (4.17)$$

$$c_6 = \left[\frac{5}{512}\left(\frac{r}{\ell}\right)^5 + ...\right]$$

Since crank radius "r" is usually less than one-fourth the length of the connecting rod, it is common practice to ignore the higher order terms. However in view of the possibility of higher speeds (i.e. large Ω^2), higher order terms may become significant for a radial engine. Fortunately for even number of cylinders (n > 2) i.e., for four, six, eight etc. cylinders it can be shown that all the higher order forces are also completely balanced out .

However as mentioned earlier, radial engines are normally not in use now-a-days. In view of the power-to-weight ratio advantage, gas turbine engines are common in modern aircrafts. However, V-engine configurations wherein the cylinders are arranged in two banks forming a "V" are commonly employed. For example, a six-cylinder engine may have two sets of (three, in-line) cylinders on a common crankshaft arranged as a "V". Similarly, two sets of the in-line four cylinders of previous example could be arranged on the arm of a "V" to give an eight-cylinder engine. Of course, the larger number of cylinders results in more power. The V-angle introduces additional phase shifts. Typical marine engines use the V-8 configuration.

Analysis of V-engines proceeds exactly same as that for radial engines.

Recap

In this modules, you have learnt the following

- Common arrangements of multiple cylinders
- Methods of analyzing the shaking forces/moments
- In particular you have studied the inline four cylinder engine configuration as this is the most common configuration used in the passenger cars you see on the road in India such as Maruti 800, Zen, Santro, Honda City, Toyoto Corolla etc.

Congratulations, you have finished Module 4. To view the next module select it from the left hand side menu of the page

Module 5 : Turning Moment Diagram for Engines and Speed Fluctuation; Power Smoothening by Flywheels.
Lecture 8 : Turning Moment Diagram for Engines and Speed Fluctuation; Power Smoothening by Flywheels.

Objectives

In this module, you will learn the following

- The driving torque generated in an IC Engine due to gas forces
- Issues in Matching of driving and load torques
- Use of flywheels to smoothen the fluctuations in speed within a cycle

Consider a typical four stroke IC Engine. The internal pressure as a function of crank angle is as shown in Fig. 5.1. In a four stroke engine, the four strokes are identified as suction, compression, power and exhaust. Power due to combustion is actually generated ONLY in one of the four strokes and hence the resulting torque on the crank shaft will also fluctuate in a similar manner. We shall now attempt to derive an expression for the gas torque.

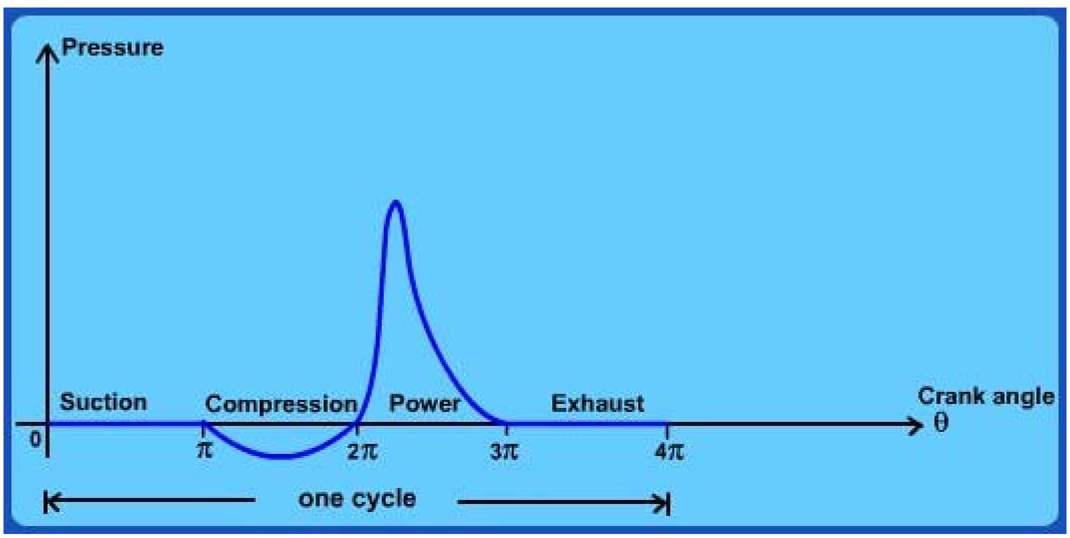


Figure 5.1. Typical Pressure Vs Crank angle for a four stroke engine

For the purpose of gas torque analysis, let us assume for the time being that all the links have negligible mass and ignore any inertia force affects. Of course some of the gas torque will have to be used to overcome the inertia of the moving parts. From the free body diagram shown in Fig. 5.2 and the considerations of static equilibrium, we find that the normal force on the cylinder frame at the piston is given by:

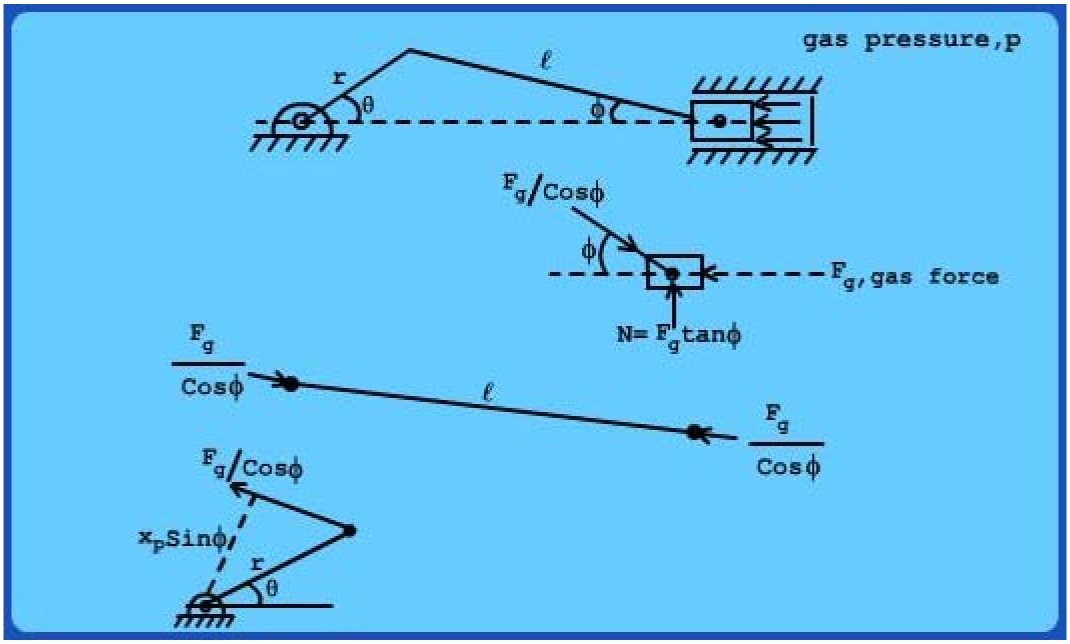


Figure 5.2 Free Body Diagram for gas Pressure loading

$$N = F_g \tan \phi \qquad (5.1)$$

Similarly the force transmitted through the connecting rod is given by:

$$F_{CR} = \frac{F_g}{Cos\,\phi} \qquad (5.2)$$

Taking moment of this force on the crankshaft pivot, we get the gas torque as:

$$T_g = \left(\frac{F_g}{Cos\,\phi}\right)\left(x_p\ Sin\,\phi\right) = F_g \tan \phi \qquad (5.3)$$

From the geometry of the IC Engine mechanism, we can write

$$\tan \phi = \frac{r\ Sin\theta}{\ell\left(\sqrt{1-\left(\frac{r}{\ell}\right)^2 Sin^2\theta}\right)} \qquad (5.4)$$

Thus the gas torque varies as a complex function of time (since $\theta = \Omega t$, where Ω is the speed of rotation).

When an IC engine is used as a prime mover, the driving torque available at the crank shaft may not in general match the load torque demanded from instant to instant. For example, if the load torque is uniform with respect to time, the load and driving torque will have same value in an average (or mean) sense as shown in Fig. 5.3 but won't match at every instant of time. For steady state operation, it is required that over a cycle average torques match. In other words we can say:

$$\oint T_g\, d\theta = \oint T_L\, d\theta \qquad (5.6)$$

If on an average, the load torque over each cycle is more than the supply torque, system will soon come to a halt.

If on an average, the load torque over each cycle is less than the supply torque, system speed will keep increasing indefinitely.

If on an average, the load torque over each cycle is equal to the supply torque, system will operate in a steady state.

However even if the average torques match, in the portion of the cycle when the load torque is instantaneously smaller than driving torque, there will be instantaneous acceleration of the crank shaft. When the load torque momentarily exceeds the driving torque available, crank shaft will decelerate. Thus even though the crankshaft has a “steady” speed of rotation, there will be small fluctuations in speed within each cycle as shown in Fig. 5.4. At steady state, the same cycle will of course repeat itself. We wish to minimize the speed fluctuations, to the extent possible. This is achieved through the addition of a flywheel to the system shaft as shown in Fig. 5.5.

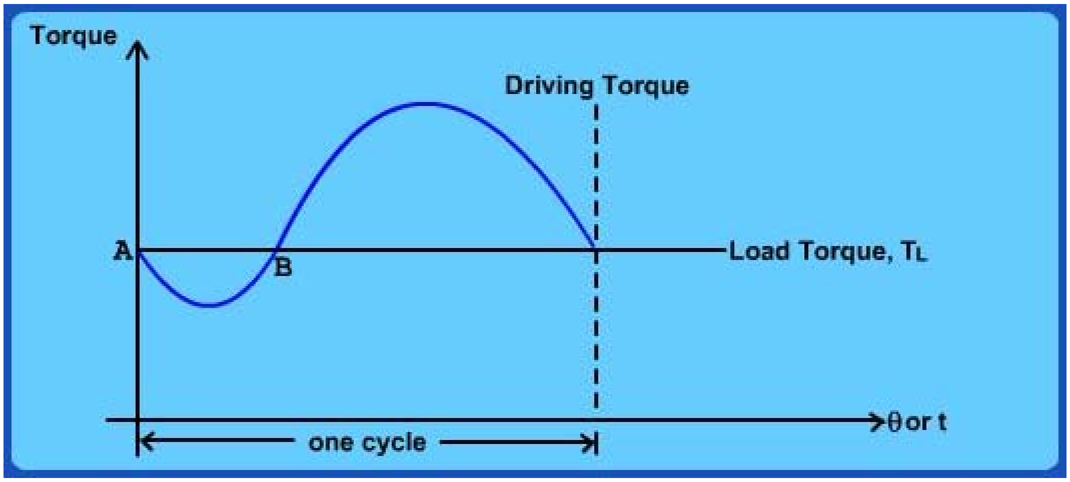


Figure 5.3 Typical torque of prime mover & load

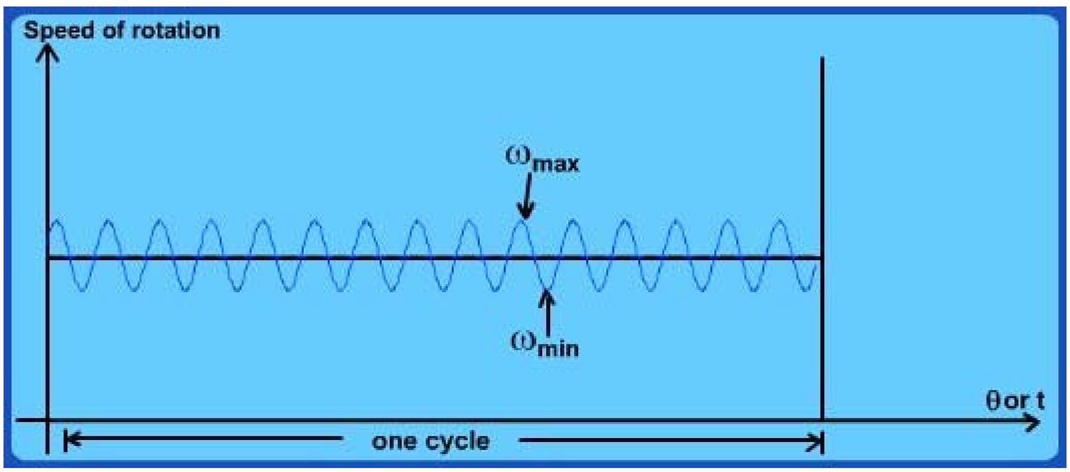


Figure 5.4. Fluctuation in Speed of Rotation within a cycle

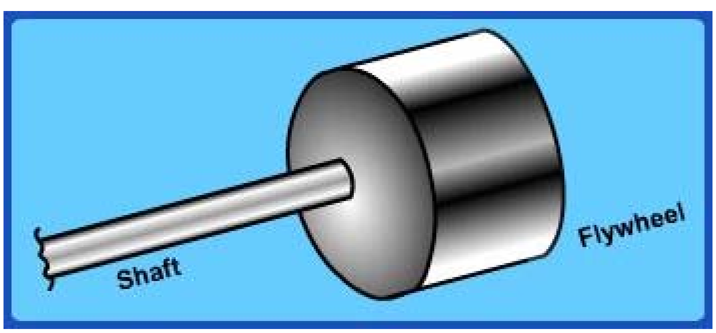


Fig 5.5 Typical shaft-flywheel

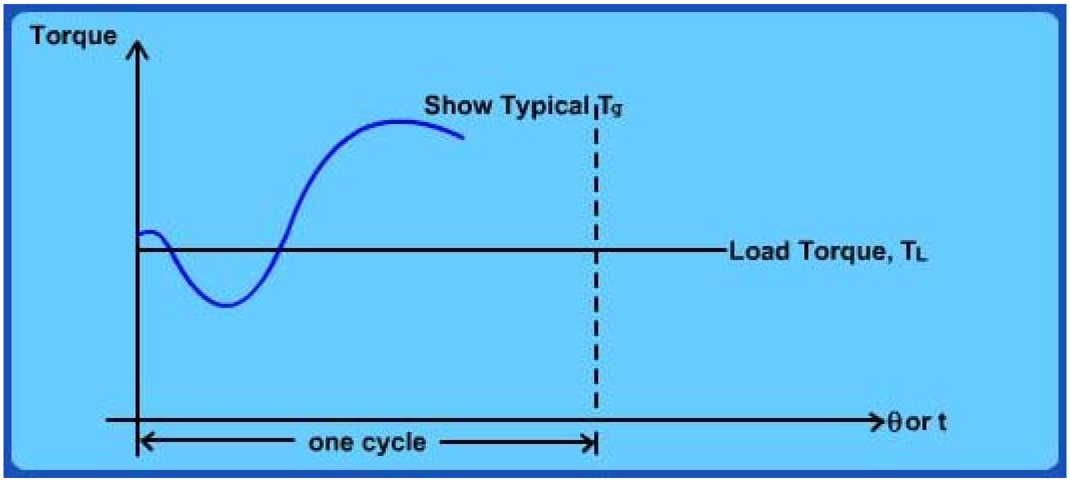


Fig 5.6 Supply and load torque Representative

Let us look at Fig. 5.6 which is a general representation of load and supply torques over a cycle and investigate the dynamics of the system. From point A to B in the cycle, the load torque exceeds the available torque. So the shaft speed falls from A to B. During B-C, the load torque will further slow down the shaft making point C perhaps the minimum speed point. During C-D, the system gains in speed as the supply torque is much more than the load torque. During D-E, the system again slows down gradually coming back to the speed at A (beginning of cycle). Thus we could identify the points in the cycle that correspond to the maximum and minimum speed and write the following energy balance equation:

$$\int_{\theta_1}^{\theta_2} \left(T_g - T_L\right) d\theta = \frac{1}{2} J \left(\omega_{max}^2 - \omega_{min}^2\right) \qquad (5.6)$$

where θ_1, θ_2 correspond to the points of maximum and minimum speeds and J is the mass moment of inertia of the system including crank shaft, other rotating parts along with flywheel. Assuming that the speed fluctuation is actually quite small after the installation of a flywheel, we could approximately write:

$$\omega_{avg} = \frac{\omega_{max} + \omega_{min}}{2} \qquad (5.7)$$

If we define a coefficient of fluctuation of speed k_s as:

$$k_s = \frac{\omega_{max} - \omega_{min}}{\omega_{avg}} \qquad (5.8)$$

We can re-write eq. (5.6) as:

$$\int_{\theta_1}^{\theta_2} \left(T_g - T_L\right) d\theta = J\, k_s \left(\omega_{avg}^2\right) \qquad (5.9)$$

In order to simplify our calculations, we could further assume that the mass moment of inertia of the flywheel (J_f) is perhaps much larger than that of the other rotating parts and hence simply take $J \approx J_f$.

Therefore given the supply torque from the prime mover (say an IC Engine, electric motor etc) and the load torque, we can size the flywheel required to ensure that the fluctuation in speed will remain within the desired limits (as prescribed by k_s). It must be noted that by taking $J \approx J_f$ in the above calculation, we would have overestimated the flywheel size somewhat or in other words, the actual fluctuation in speed will be even smaller than demanded. Thus we see that a flywheel acts similar to a reservoir in a fluid line which tries to maintain pressure fluctuations within limits.

A typical flywheel would appear as shown in Fig. 5.7 wherein more material is kept towards the outer radius so as to get maximum moment of inertia. For designing the cross-section of the spokes etc. it is important to realize that the maximum stress would occur towards the root – at any section the internal resisting force developed is the force required to keep the material from flying away. Thus as we go from tip to the root, more and more material tries to fly away at great speed and therefore greater and greater force is required to be developed internally between the sections towards the root.

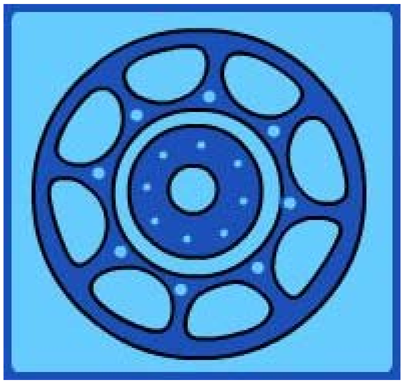


Fig. 5.7 Typical flywheel

A few issues are left for you to think further about viz.,

Is there an upper limit on the inertia of the flywheel – obviously the bigger the wheel, the lesser will be the fluctuations in speed?

Typically we have a transmission unit between the prime mover and the load, in the form of a speed reducer gear box. Which side of the gear box will you keep the flywheel and why?

Recap

In this module you have learnt the following

- Issues in Matching driving and load torques
- Role of a flywheel in a drive
- Estimation of size of flywheel required to keep the speed fluctuations within a specified limit

Module 6 : Vibration of Mechanical Systems; Types of Vibration; Lumped Parameter Models; Linearization of System

Elements; Degrees of Freedom; Types of Restoration and Dissipation Mechanisms; Types of Excitation

Lecture 9 : Overview of Mechanical Vibrations.

Objectives

In this module, you will learn the following

- Examples of Vibrations
- Types of Vibration
  - Free and Forced Vibrations
  - Linear and Non-linear Vibrations
  - Deterministic and Random Vibrations

Most machinery are subject to time varying forces which cause time dependent motions of the system. For example any mechanism such as used in a shaper machine, lathe, grinder etc involves moving masses that accelerate and decelerate in addition to time dependent cutting forces. A building or a nuclear reactor structure subjected to an earthquake undergoes time dependent deformations and stresses. A crankshaft of a multi-cylinder IC engines used in most cars is continuously subjected to time dependent torques. On the other hand, an aircraft at the time of landing is subject to tremendous impact forces that act for an extremely short duration of time. An automobile is subjected to crushing forces over a few milli-seconds whenever an accident takes place.

Acceleration and deceleration of masses causes inertia forces as we have discussed at length when we looked at the problem of balancing. If we are dealing with only rigid bodies, it is essentially a problem of balancing alone i.e., we aim to minimize unbalanced forces. However most systems are not rigid – for example, for all practical purposes, a building appears quite sturdy and rigid but when an earthquake hits, the whole building shakes as a deformable body and may actually break down. On the other hand automobiles have specially-built-in springy elements in the form of suspension/shock absorbers and when negotiating rough roads, the automobile undergoes vibratory oscillations. While the suspension spring is an example of a concentrated dose of springiness in a system lumped at one place in one device, springiness (elastic or plastic) is actually present throughout the body of the vehicle. This becomes apparent when a car hits an obstacle and the body of the vehicle gets dented (plastically deformed).

Whenever a deformable body is subjected to time dependent forces, it undergoes oscillatory motion as follows. Since it is deformable, it undergoes deformation under the load. In the process, the mass of the body has been accelerated and work has been done against the springiness of the body, leading to restoring forces. Thus, under the action of the external disturbances, inertia and restoring internal forces the body continues to move to and fro. Such a motion is called vibration . In the next few modules we will focus on such vibratory motion of typical mechanical systems.

By and large vibration is considered undesirable. A machine component subjected to vibration of sufficiently high magnitude could eventually fail in fatigue. A passenger in a car may feel extremely uncomfortable, when riding over rough roads. An instrument mounted for example, on the surface of an aircraft, may give erroneous results due to vibrations. Thus our aim, in many instances, is to minimize the vibrations. However there are situations such as vibratory bowl feeder in a material handling system in a factory actually works on utilizing vibrations.

We will begin our discussion with an aim to understand the nature and type of vibrations so that you gain a broader perspective of the subject matter. However within the scope of this course we will not be able to discuss the entire range of vibration problems. We will focus our attention, in subsequent modules, on simple harmonic vibrations of linear systems.

We will first need to distinguish between two types of vibration – viz., free vibration and forced vibration .

When a system is subjected to an initial disturbance and then left free to vibrate on its own, the resulting vibrations are referred to as free vibrations .

When a system is subjected continuously to time varying disturbances, the vibrations resulting under the presence of the external disturbance are referred to as forced vibrations .

When a vehicle moves on a rough road, it is continuously subjected to road undulations causing the system to vibrate (pitch, bounce, roll etc). Thus the automobile is said to undergo forced vibrations. Similarly whenever the engine is turned on, there is a resultant residual unbalance force that is transmitted to the chassis of the vehicle through the engine mounts, causing again forced vibrations of the vehicle on its chassis. A building when subjected to time varying ground motion (earthquake) or wind loads, undergoes forced vibrations. Thus most of the practical examples of vibrations are indeed forced vibrations.

Free vibrations arise once the external excitation dies down. Free vibrations are thus the transient vibrations after the external disturbance is removed but before the system comes to a halt. In the laboratory, many a time, free vibrations are intentionally introduced in the system (for example, by using an impact hammer) to study the inherent dynamics of the system. Free or natural vibration behaviour of the system enables us to gain significant insight into the system behaviour. We will learn about the system natural frequencies and corresponding mode shapes. From the transient response, we will actually be able to measure the system damping. Knowledge of these actually helps us to predict how the system will respond to an external disturbance. Thus it is equally important to study free or natural vibrations.

We can also distinguish between vibrating systems based on the nature of their response to external excitation. Some systems are called Linear systems and hence their vibrations are referred to as Linear vibrations. Some other systems, on the other hand, are known as non-linear systems and their vibrations are therefore referred to as non-linear vibrations.

Linear systems exhibit a linear relationship between the magnitude of excitation force and the response. Thus if the excitation force amplitude (at a given frequency) doubles, we expect the system response also to double. Care must be taken at this stage NOT to confuse this with variation in system response for varying excitation frequency . Even when the amplitude of the force remains constant, as the frequency varies the system response will vary and at certain frequencies, the response may become very large (called resonant frequencies). Even near a resonant frequency, for a linear system, when the force amplitude becomes double or triple, the response will become double or triple respectively.

Non-linearities in most systems can be approximated by equivalent linear systems. For example, nonlinearity in a simple pendulum is approximated by a linear system by considering only small oscillations (i.e., $\sin\theta \approx \tan\theta \approx \theta, \text{ for small } \theta$). Linear systems are described mathematically by linear differential equations, which can be more easily solved than non-linear differential equations. Thus, in engineering, we try and simplify. However when the non-linear behaviour of a spring in an automobile suspension needs to be modeled accurately, we need to include its full non-linear force-deflection relationship rather than approximating by a straight linear relationship. Machine tool chatter, flutter of airfoils, wheel shimmy, vibrations with Coulombic friction are instances of non-linear vibrations

We now come to a third way of classifying vibrations namely, deterministic and random vibrations.

When the rotor of a fan or compressor/turbine has an unbalance, as the system rotates there will be a net dynamic force on the system but the amplitude and time variation of this excitation is essentially precisely known. For example, we know that the unbalance excitation will be at the running speed of the system. Thus, in general, when the external disturbance varies with time in a known manner and its amplitude/time variation can be estimated accurately at each instant, it is said to be a case of deterministic vibrations .

On the other hand, road excitation to a car due to road roughness can not be precisely known and leads to what are known as random vibrations . Even though random, these are not arbitrary – in the sense that we can use certain average properties of the excitation. Supposing we look at a few meters of the road at one place and determine the average roughness parameters of the profile. A few kilometers further down the road, we expect the average roughness parameters, again measured over a few meters of the road, would be essentially the same. We can use such “statistical regularity” and study the resulting random vibrations. Random vibrations occur in most natural phenomena such as vibrations due to earthquakes, gusty winds etc. Thus, while the theory of random vibrations is mathematically involved, its applications are many.

Recap

In this module you have learnt the following

- Examples and importance of vibrations
- Different types of vibrations with examples such as free/forced; linear/non-linear; deterministic/random.

Module 6 : Vibration of Mechanical Systems; Types of Vibration; Lumped Parameter Models; Linearization of System

Elements; Degrees of Freedom; Types of Restoration and Dissipation Mechanisms; Types of Excitation

Lecture 10 : Modeling of Mechanical Vibrations.

Objectives

In this module, you will learn the following

- Elements in a Vibrating System
- Degrees of Freedom
- Linearization and Lumping of Elements

Vibration is essentially a to-and-fro motion. Thus there is a force (excitation/disturbance) that initiates the motion. We will learn about different types of excitation forces in the next lecture. Under the influence of the external disturbance excitation, the system masses move i.e., they accelerate and decelerate setting up inertia forces. If external excitation were the only type of force on the system, the system would exhibit a rigid body motion. However since most systems are elastic, the movement of the masses invariably causes stretching or compression of springy elements setting up elastic restoring forces.

For example when an automobile passes on a road, the road roughness is the external excitation. The mass of the vehicle moves up-down (pitch and bounce), left-right (roll) setting up inertia forces. The suspension spring gets stretched and compressed as the vehicle mass moves up and down. When a spring is stretched or compressed from its free length position, it exerts a restoring force on the mass trying to bring it back to its free length position. In the process of course the mass would have gained momentum and continues to travel farther than the static equilibrium, free length position. Once again the spring tries to pull the mass back to its free length position and the cycle repeats.

The cycle of to-and-fro motions would however not repeat forever due to the dissipation present in most systems. For example, an automobile suspension always has a shock absorber i.e., a damper that dissipates the energy of vibration into friction against a moving fluid.

Thus the four fundamental elements of a vibrating system are:

- Mass or Inertia
- Springiness or Restoring element
- Dissipative element (often called damper)
- External excitation

When we refer to modeling a vibrating system, we need to distinguish between two types of models viz., physical models and mathematical models. Physical model of a system is a representation of the physics of the system that we would like to include in our study. For example, we may consider dissipation negligible in a system for a given study and may say that the physical model of the system consists of purely spring-mass systems. Now a mathematical model refers to a mathematical relation that defines the input-output relation of the elements. For example, we could have a simple linear relation between force and deflection for a spring or a more complicated non-linear relation (for example softening or hardening types) as shown in Fig. 6.2.1. For a given problem of engineering, we typically develop first the physical model i.e., decide which physics is important to include in the model and then build the mathematical description of the various elements to develop the mathematical model.

Modeling of mass or inertia of a system seems fairly straight forward, at a first glance. We just need to worry about the total mass/inertia of the system and use it in system models. For example in an automobile, there is a certain mass of the vehicle – chassis, body, engine etc and we can include these masses in the model of the system. Of course the vehicle mass changes with the number of passengers and the luggage but that can also be determined and taken into account.

However as we study the issue deeper we realize some difficulties. For example, we want to develop a mathematical model of the vehicle and study its dynamics (to ensure its satisfactory performance) before the vehicle prototype is actually built. How do we ascertain the system mass moment of inertia at this stage? But we need the inertia numbers for accurate modeling of the roll, yaw, pitch etc. How do we precisely locate the centre of gravity of the vehicle that is yet to be built?! Similarly consider the example of sloshing of a fluid inside a tanker being transported from place to place. Due to road undulations and driving characteristics, the liquid is bound to slosh inside the tanker. When we are modeling the dynamics of the vehicle body subjected to road undulation, how do we exactly model the mass of the fluid? Another interesting example arises in the case of marine structures for examples, ships, submarines etc. When a ship hull vibrates, how much mass of the water adjacent to it participates in the vibration?

Thus we see that while for simple cases it is easy to model the mass of a vibrating system, for complex problems, this could become a big issue and demands deeper study.

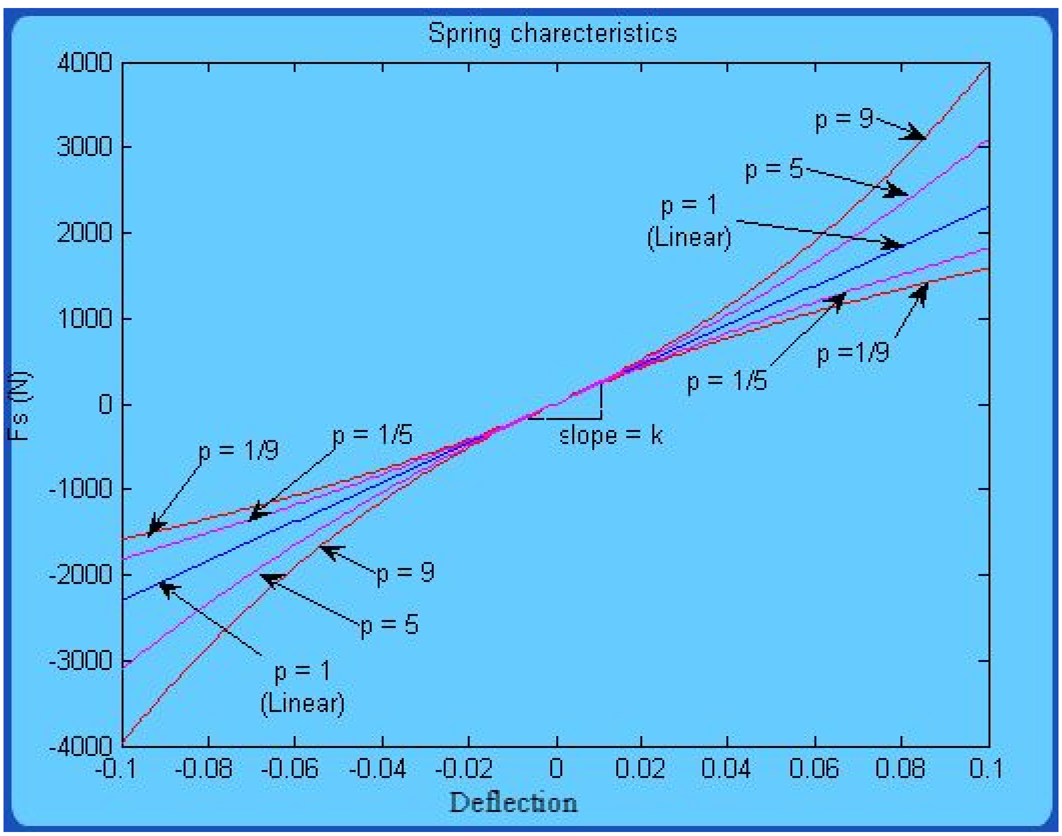


Fig 6.2.1: Nonlinear spring characteristics

The restoring force in a vibrating system may be provided by a real physical spring (say a close coiled helical spring or a leaf spring in an automobile suspension) or by an "effective" spring action. For example in the case of the to-and-fro oscillations of a simple pendulum, the restoring force is actually provided by gravity. It is the weight of the bob that tends to bring it back to its original equilibrium position and thus gravity acts as the effective spring here. There could also be electro-magnetic or such other forces that provide the restoring action. We will consider all of these under the head, springiness in the system.

Dissipation in a system is perhaps the hardest to model mathematically in accurate manner, simply because there can be many sources of damping involving complex physical phenomena. For example, automobile shock absorbers have piston-cylinder kind of fluid friction based dampers normally referred to as viscous dampers. There can also be contact friction between two mating surfaces (such as piston-cylinder interface; cam-follower etc) which is normally considered to be of Coulombic type. Additionally there can be material internal friction (hysteresis) damping. Considering all such various sources of damping present in a system and developing accurate mathematical models for the same is a challenging task. Often, total energy being dissipated is approximately modeled using a simple linear viscous damper.

We will discuss the fourth element of a typical vibrating system namely the external disturbance in the next lecture.

Various elements of a typical vibrating system and the standard configuration for assembly of the vibrating system are shown pictorially in Fig. 6.2.2. The system shown in this figure is what is known as a Single Degree of Freedom system. We use the term degree of freedom to refer to the number of coordinates that are required to specify completely the configuration of the system. Here, if the position of the mass of the system is specified then accordingly the position of the spring and damper are also identified. Thus we need just one coordinate (that of the mass) to specify the system completely and hence it is known as a single degree of freedom system.

What is shown in Fig. 6.2.3 is a two degree of freedom system. With reference to automobile applications, this is referred as "quarter car" model. The bottom mass refers to mass of axle, wheel etc components which are below the suspension spring and the top mass refers to the mass of the portion of the car and passenger. Since we need to specify both the top and bottom mass positions to completely specify the system, this becomes a two degree of freedom system.

A half-car model shown in Fig. 6.2.4 is a four degree of freedom model where G is the centre of mass. A full car model, on the other hand (Fig. 6.2.5) has several degrees of freedom.

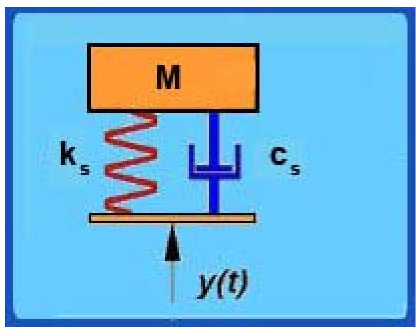


Figure 6.2.2

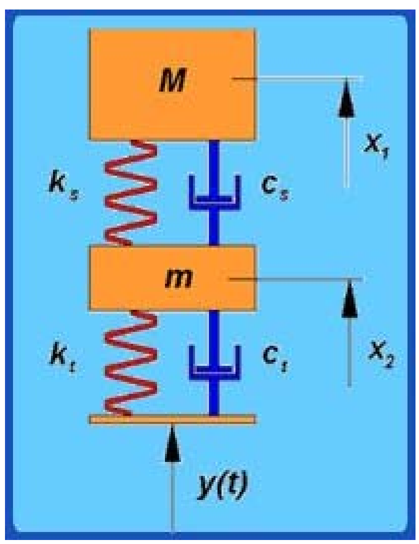


Figure 6.2.3

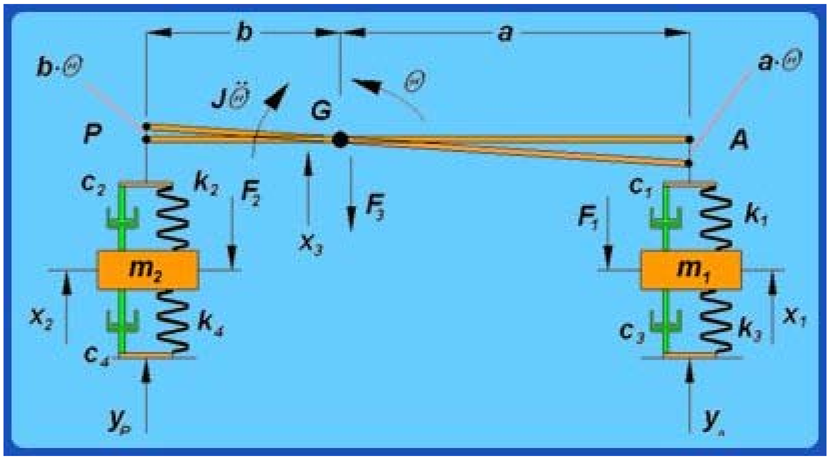


Figure 6.2.4

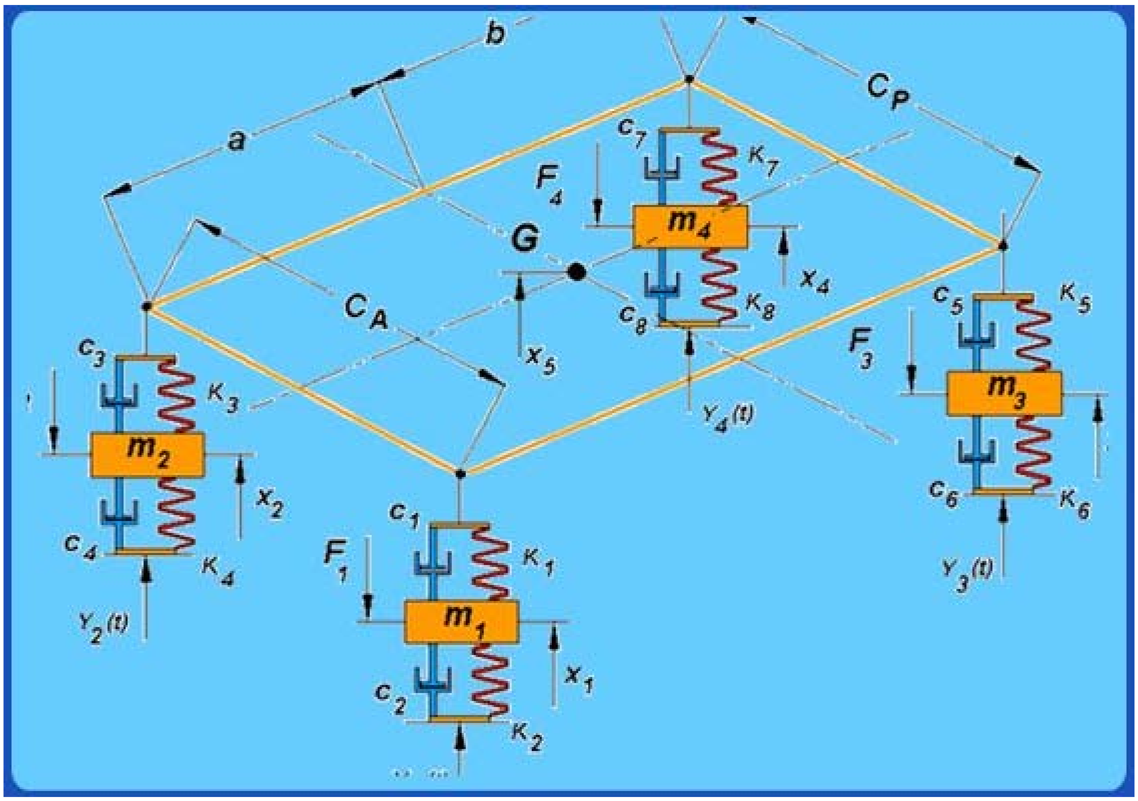


Figure 6.2.5

In all these systems considered as examples so far, the properties of mass/inertia; springiness and dissipation are modeled as "lumped" into certain physical devices. For example the mass does not have any springiness nor dissipation. The spring does not have any mass and does not dissipate either. The mass-less dashpot is a pure viscous damper without any springiness. Thus these are idealized models of typical physical phenomena that are observed in vibrating system namely mass/inertia; restoring force, dissipative force etc. In a building subjected an earthquake, the whole building shakes and it is not possible to isolate which part of the building is a pure mass; which is a pure spring element and which is pure dashpot. These elements are all “distributed” within the complete system and need to be modeled as such.

However modeling as lumped elements eases up the modeling effort because the resulting equations are ordinary differential equations (ODEs), while the distributed parameter models are described by partial differential equations (PDEs). Since ODEs are lot easier to solve than PDEs, it has been common practice to develop simplified lumped parameter models first. Latest computational techniques such as finite element method convert given partial differential equations of distributed parameter models (such as a building etc) into ODEs before solving them on the computer.

A spring or a dashpot is mathematically represented by a force – deformation or force – velocity curve/equation. In general this curve can be straight line or a general curve as shown in Fig. 6.2.1. When the relationship is a straight line i.e., a linear spring or damper model is used, there is direct proportionality between the cause and the effect and the resulting governing equations of motion are also linear. Linear models are extensively used in most engineering analyses, at least as a first approximation in design. The linear equations (differential or algebraic) are lot easier to solve than non-linear equations and hence these models are commonly used eventhrough many times practical systems do exhibit non-linear behaviour. Nonlinear systems exhibit very interesting and sometime unexpected behavior (for example chaos) and do become important in special situations. In our course we will be limiting our attention to only linear systems.

Recap

In this module you have learnt the following

- Spring, damper, inertia and external excitation constitute four primary elements of a Vibrating System;
- Concept of Degrees of Freedom
- Lumped models of springiness, inertia and damping

Module 7 : Free Undamped Vibration of Single Degree of Freedom Systems; Determination of Natural Frequency ; Equivalent Inertia and Stiffness; Energy Method; Phase Plane Representation.

Lecture 11 : Free Undamped Vibration of SDOF systems

Objectives

In this lecture you will learn the following

- Physical Model / Mathematical model
- Lumped and Distributed parameter systems
- Derivation of Equation of motion

  Free Body Diagram

  Use of Newton's Laws of motion

Fig. 7.1_2 Simple Spring Mass System

In the previous module, we learnt of how typical engineering systems experience vibrations and how these can be modeled. We will now begin our detailed study of vibration problems – beginning with a simple single degree of freedom system as shown in Fig. 7.1_2. Our simple system has just a spring and a mass and intentionally we have not included any form of dissipation. We call this a single d.o.f system because it requires just one coordinate (viz., position of mass) to completely specify the configuration of the system.

This is an example of what is known as a Lumped Parameter System i.e., the spring is assumed not to have any mass and the mass has no springiness. The system springiness is lumped into the spring and the inertia is lumped into the mass. Of course this is an idealization – real springs do have mass and no system is perfectly rigid. However this idealization renders the system easy to analyze and we also get very useful insights into the characteristics of typical vibration problems and therefore, we begin our study of vibrations with such lumped parameter models. When we have gained enough insight into the physics and mathematics of such vibrating systems, we will then be ready to study distributed parameter systems in a later module.

The spring mass system can be referred to as a physical model and we need to develop the corresponding mathematical model (governing equations) for our further analysis. The physical model of an engineering system encapsulates the various physical phenomena we wish to model in our study. The physical model, in this example, has provided for some source of springiness and inertia in the system, where as the specific load-deflection characteristics of the spring for example are as yet unspecified. It could be a simple linear function or a complex non-linear relation and will be reflected in the mathematical model that we develop. However, the physical model does not have any element of damping and hence automatically the mathematical model (governing equations) will also not have any damping terms. Thus development of physical model of the complex, real-life engineering systems represents the first source of approximations and will contribute to the deviations between the behavior predicted by the model and the real system behavior. Considering the complexity of the governing equations in the mathematical model, we may further introduce approximations (read errors!) in the solution of these governing equations. Finally every numerical computation involves truncation and round-off errors. Keeping in mind these sources of error, the role of the vibration analyst is to build a model that is easy to handle, yet accurate!

To develop the mathematical model (in the form of governing equation of motion) of a physical model, we rely on the application of laws of physics such as:

Newton 's laws of motion

Energy principles such as conservation of energy.

Mathematical Model by application of Newton's 2nd Law

You may recall Newton 's 2nd Law of motion which can be loosely stated as follows

"The net force on a body in a given direction is equal to the product of the body mass and its acceleration along that direction."

In order to be able to derive the governing equation of motion for the spring-mass system using Newton 's law, we need to draw a free body diagram for the system. We begin our study of vibration with what is known as free vibrations i.e., the body is given an initial disturbance (such as pulling the mass through a certain displacement) and released to vibrate with no external force acting on it. This is also referred to as Natural vibration as it depicts how the system vibrates when left to itself with no external force. After having understood the system behavior under free vibration, we will then take up the study of forced vibration i.e. response of the system to an external, time varying force. Fig. 9.1.2 shows the forces acting on the mass for an assumed displacement x, Δ corresponds to static deflection i.e., deflection of the spring under the weight.

Fig. 7.1.2 Free-body diagram of a spring-mass system

We next equate the inertia force to the sum of all the external force.

$$\text{Inertia Force} = \sum \text{External Forces}$$

$$m\ddot{x} = mg - k(x + \Delta) \quad 7.1.1$$

$$\text{But } mg = k\Delta$$

$$\therefore m\ddot{x} = -kx$$

$$\boxed{\therefore \ddot{x} + \frac{k}{m}x = 0} \quad 7.1.2$$

This is the governing equation of motion.

It is the solution to this differential equation of motion along with its appropriate interpretation which constitutes a major part of vibration analysis. The governing equation as represented by eqn (7.1.2) is a second order, ordinary differential equation. This is characteristic of lumped parameter models. Distributed parameter system models result in partial differential equations , which are definitely more difficult to solve.

DIFFERENTIAL EQUATIONS

We have already studied the algebric equations in our basic mathematics courses.

$$y = mx + c$$

Above equation represents an equation of a line in a two dimensional space. Differential equations in general represent the equation of the tangent for a given system represented by a curve.

$$\frac{dy}{dx} = 2x$$

represents the equation of the tangent of a parabola $y = x^2 + c$ in general.

Above equation could be termed as first order linear differential equation.

Lets first refresh the understanding the concept of order of differential equation.

$\frac{d^2y}{dx^2} = x$ --2nd order differential equation.

$\frac{d^3y}{dx^3} = x$ --3rd order differential equation.

All the above equations are linear differential equations.

All the above mentioned differential equations represent one single dependent and one independent variable.

But there is every possibility of a system to be dependent on two variables.

For example Heat flux propagation through cooling fins depends on distance from source as well as time. Such systems are represented by differential equations termed as partial differential equations.

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = c$ --1st Order partial differential equation

where f is a function of x and y

$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$ --2nd order partial differential equation

Recap

In this lecture you have learnt the following

- Concept of degree of freedom of a system

Different types of systems in general

- Lumped parameter system (finite degrees of freedom represented by ordinary differential equation)
- Distributed parameter system (infinte degrees of freedom, represented by partial differential equations)

- Laws used in mathematical modelling of physical systems viz

- - Newton's laws of motion

- Free body diagram and its role in mathematical modelling

- Derivation of equation of motion for vibration viz

$$\therefore \ddot{x}+\frac{k}{m}x=0$$

Congratulations, you have finished Lecture 7.1. To view the next lecture select it from the left hand side menu of the page

Lecture 12 : Determination of Natural Frequency

Objectives

In this lecture you will learn the following

- Solution of equqtion of motion for undamped-single degree of freedom system.
- Concept of natural frequency.
- Translational and torsional spring mass systems.
- The Equation of motion found earlier for the spring mass system is reproduced below.

$$\ddot{x} = -\frac{k}{m}x \qquad 7.2.1$$

Equation (7.2.1) is called the equation of motion.

It represents a motion wherein the acceleration (second derivative) is proportional to the negative of displacement. This is characteristic of a simple harmonic motion and the most general solution to this equation can be represented as follows:

$$x = A\sin(\sqrt{\frac{k}{m}}t) + B\cos(\sqrt{\frac{k}{m}}t) \qquad 7.2.2$$

wherein the coefficients A and B can be found using the initial conditions – it is a second order system and hence needs two initial conditions viz., initial position and velocity .Let these be [?] and [?] respectively. Thus we have:

$$x(t) = \frac{\dot{x}_0}{\sqrt{\frac{k}{m}}}\sin\left(\sqrt{\frac{k}{m}}t\right) + x_0\cos\left(\sqrt{\frac{k}{m}}t\right) \qquad 7.2.3$$

The above equation forms the complete soultion to the equation of motion of a simple spring mass system. For the initial conditions as follows:

$$x = X_0 \quad \text{at t=0}$$

$$\dot{x} = 0 \quad \text{at t=0}$$

we get the solution as:

$$x(t) = x_0\cos\left(\sqrt{\frac{k}{m}}t\right) \qquad 7.2.4$$

This is a non-decaying sinusoidal vibration of frequency $\sqrt{k/m}$, so that,

$$\omega_n = \sqrt{\frac{k}{m}} \qquad 7.2.5$$

This frequency is referred to as the natural frequency of the system since it represents the natural motion of the system viz., the way the system vibrates when given an initial disturbance [?] and left free to vibrate on its own.

Module 7 : Free Undamped Vibration of Single Degree of Freedom Systems; Determination of Natural Frequency ; Equivalent Inertia and Stiffness; Energy Method; Phase Plane Representation.

Lecture 13 : Equivalent Stiffness and Inertia

Objectives

In this lecture you will learn the following

- Equivalent spring stiffness.
    - Serial combination
    - Parallel combination
- Equivalent mass / moment of inertia including geared shafts

While we have discussed so far the vibration behavior of a spring-mass system, in many practical situations we don't readily find such simple spring-mass systems. Many a time, we may find several springs and masses vibrating together and then we will have several second order differential equations to be solved simultaneously. In some special situations however, we will be able to simplify the system by considering equivalent stiffness and inertia. We may then still be able to model the system as a simple single d.o.f spring-mass case. We will now discuss some of those situations.

When multiple springs are used in an application, they are mainly found in two basic combinations.

- Series Combination
- Parallel Combination

Equivalent Spring Stiffness

Figure 7.3.1

Series Combination

A typical spring mass system having springs in series combination is shown above. The two springs can be replaced by a equivalent spring having equivalent stiffness equal to k as shown in the figure 7.3.2

Fig 7.3.2 Spring in series

When springs are in series, they experience the same force but under go different deflections.

For the two systems to be equivalent, the total static deflection of the original and the equivalent system must be the same.

$$Deflection \Delta = \frac{mg}{k_{eq}} = \frac{mg}{k_1} + \frac{mg}{k_2}$$

$$\therefore \qquad \frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \qquad 7.3.1$$

Therefore if the springs are in series combination, the equivalent stiffness is equal to the reciprocal of sum of the reciprocal stiffnesses of individual springs.As an application example, consider the vertical bounce (up-down) motion of a passenger car on a road. Considering one wheel assembly we can develop what is known as a quarter car model as shown in fig 7.3.3a. For typical passenger cars, the tyre stiffness is of the order of 200,000N/m while the suspension stiffness is of the order of 20,000N/m. Also, the vehicle mass per wheel (sprung mass) can be taken to be of the order of 250kg while the un-sprung mass (i.e., mass of wheel, axle etc not supported by suspension springs) is less than 50kg. Reciprocal of tyre stiffness is negligible compared to reciprocal of suspension stiffness or in other words, the tyre is very rigid compared to the soft suspension. Hence an equivalent one d.o.f. model can be taken to be as shown in Fig. 7.3.3b

Fig 7.3.3a Quater Car model

Fig 7.3.3b Spring mass system

Fig 9.3.4 Spring in parallel

For the springs in parallel combination, the equivalent spring stiffness can be found out as:

Each of the individual spring supports part of the load attached to it but both the springs undergo same deflection.

Therefore the static deflection Δ of the mass is,

$$
\begin{aligned}
&Deflection\,\Delta = \frac{mg}{k_{eq}} = \frac{f_1}{k_1} = \frac{f_2}{k_2} \\
&And \qquad mg = f_1 + f_2 \\
&\Rightarrow \qquad k_{eq}\Delta = k_1\Delta + k_2\Delta \\
&\qquad \therefore\ k_{eq} = k_1 + k_2
\end{aligned}
\qquad 9.3.2
$$

Therefore if the springs are in parallel combination, the equivalent spring stiffness is sum of individual stiffnesses of each spring.

Fig 7.3.6 Springs Connected by lever

Figure 7.3.6 shows a typical arrangement in which springs are connected with a lever.

We herein intend to replace the two springs by one equivalent spring. Let the displacement of the equivalent spring be, x_{eq} and the equivalent spring stiffness be k_{eq}

We select then the equivalent spring displacement as equal to either of the spring displacement. Lets say that its displacement is equal to x_1. Then as the strain energy stored in the orignal system and the system with equivalent spring has to be the same.

7.3.3

$$\text{Strain energy } U = \frac{1}{2}\left\{k_1 x_1^2 + k_2 x_2^2\right\} = \frac{1}{2} k_{eq} x_{eq}^2$$

$$\text{Letting } x_{eq} = x_1 \text{ and since } x_2 = x_1 \frac{l_2}{l_1}$$

$$\text{we get } k_{eq} = k_1 + k_2 \left(\frac{l_2}{l_1}\right)^2 \qquad 7.3.4$$

Equivalent Inertia

We have discussed finding the equivalent stiffness. This equivalence can be termed "statically equivalent system" since the spring controls the static response of the system. However for complete dynamic response to a time varying load to be equivalent, we need to develop a dynamically equivalent model. Ideally, this dynamic equivalent system has analogous equations of motion and produces the same output response for a given input. For example consider the connecting rod of an IC Engine. It has complex shape and undergoes general planar motion. Thus computing the kinetic energy exactly will be a tedious task. Instead we can try and represent the connecting rod with a much simpler system as shown in Fig 7.3.5. Let us now study how to find such equivalent mass/inertia systems.

Fig 7.3.5 Connecting rod

Rigidly Connected Masses

Sometimes the members connecting the various masses could be considered rigid compared to other springy elements in the system . For example consider two masses connected together by a stiff rod as shown in the fig7.3.7

- Both the masses have identical velocities since they are connected by a rigid link.

- Therefore the equivalent mass is the sum of individual masses

$$v_1 = v_2 \qquad 7.3.5$$

$$\therefore\ m_{eq} = m_1 + m_2 \qquad 7.3.6$$

Fig 7.3.7 Rigid connected mass

Masses connected by Lever

For the configuration shown in fig7.3.8 with small amplitude of angular motion we take the velocity of the equivalent mass as equal to velocity of either mass and considering the lever ratio and assuming that the lever used is rigid but massless we get,

$$v_{eq} = v_1 \quad 7.3.7$$

$$v_2 = v_1\left(\frac{L_2}{L_1}\right) \quad 7.3.8$$

We then equate the kinetic energy of the existing system to the equivalent system and find out the equivalent mass from the equation below

$$T = \frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2\right) = \frac{1}{2} m_{eq} v_{eq}^2 \quad 7.3.9$$

$$\therefore m_{eq} = \left(m_1 + m_2\left(\frac{L_2}{L_1}\right)^2\right) \quad 7.3.10$$

Fig 7.3.8 Masses connected by lever

Inertia on geared shafts (rotation)

To find the equivalent inertia on a geared system, consider two mass moments of inertias, J_1 and J_2, each rotating on shafts connected by massless gears. Let the number of teeth on each gear be N_1 and N_2, respectively.

Fig 7.3.9

We select the angular speed of the equivalent shaft as equal to the angular speed of either of the shaft, Say

$$\dot{\theta}_{eq} = \dot{\theta}_1 \qquad 7.3.11$$

Then to determine the polar moment of inertia of the equivalent shaft we evoke the equivalence of kinetic energy,

$$J_{eq} = \left(J_1 + J_2 \left(\frac{N_1}{N_2} \right)^2 \right) \qquad 7.3.12$$

Equivalence of kinetic energy

The basis of conversion is that the kinetic energy and potential energy for the equivalent system should be same as that for the original system.Then if θ_1 and θ_2 are the angular displacement of rotors J_1 and J_2 respectively, then neglecting the inertia's of the gears, the energy equations are given by:

$$K.E. = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 \dot{\theta}_2^2$$

$$P.E. = \frac{1}{2} k_{t1} \theta_1^2 + \frac{1}{2} k_{t2} \theta_2^2$$

Where K_{t1}, K_{t2}, are the torsional stiffnesses of the shafts

since $\theta_2 = n\,\theta_1$ (gear ratio 'n') the expressions become:

$$K.E. = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2(n\dot{\theta}_1)^2$$

$$P.E. = \frac{1}{2}k_{t1}\theta_1^2 + \frac{1}{2}k_{t2}(n\theta_1)^2$$

$$K.E. = \frac{1}{2}\left(J_1 + n^2 J_2\right)\dot{\theta}_1^2$$

$$P.E. = \frac{1}{2}\left(k_{t1} + n^2 k_{t2}\right)\theta_1^2$$

finally

$$J_{eq} = J_1 + n^2 J_2 \; ; \; K_{eq} = K_{t1} + n^2 k_{t2}$$

where

$$n = \frac{N_1}{N_2}$$

wherein N_1 and N_2 are the number of teeth on the two meshing gears.

Recap

In this lecture you have learnt the following

- Types of combinations in springs(series or parallel)
- Concept of equivalent stiffness
  - Equivalent stiffness for springs in series $\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$
  - Equivalent stiffness for springs in parallel $\boxed{k_{eq} = k_1 + k_2}$
- Quarter car model and its simplified model
- Concept of equivalent inertia
- Derivation of equivalent inertia of rigidly connected masses, masses connected by lever and geared shafts

Congratulations, you have finished Lecture 3. To view the next lecture select it from the left hand side menu of the page

Module 7 : Free Undamped Vibration of Single Degree of Freedom Systems; Determination of Natural Frequency ; Equivalent Inertia and Stiffness; Energy Method; Phase Plane Representation.

Lecture 14 : Energy Method

Objectives

In this lecture you will learn the following

- Derivation of equation of motion using energy methods.

Energy Method

Energy methods can also be used to analyse the vibration behaviour of a system, just as we used the Newton's laws of motion.

Law of conservation of energy states that energy can neither be created nor destroyed, but can be only converted from one form to another. In a simple vibrating spring – mass system, the two forms of energy are the kinetic energy and the potential energy. Thus this principle means that the relative contribution of the kinetic and potential energies to the total energy of the system can change from time to time, but the total energy of the system itself has to remain constant.

Rayleigh's method is based on this principle. Let,

- T=Kinetic Energy
- V=Potential Energy
- U=Total Energy

The system shown in fig 7.4.1 has the kinetic energy of the form

$$T = \frac{1}{2} m\dot{x}^2 \qquad 7.4.1$$

And the Potential Energy of the form

$$V = \frac{1}{2}kx^2 \quad 7.4.2$$

The total Energy is

$$U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \quad 7.4.3$$

Since there is no dissipative element in the model, the total energy in this conservative system must remain constant all the time. Differentiating total energy with respect to time and equating the resultant to zero we get,

$$\frac{dU}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = 0 \quad 7.4.4$$

So the Equation of motion is,

$$\ddot{x} + \frac{k}{m}x = 0 \quad 7.4.5$$

which is identical to the equation obtained from Newton 's Laws.(see section 9.1_5)

For the same physical model, the governing equations obtained by any method of formulation must be identical. Application of Newton 's laws will involve drawing free body diagrams and dealing with vector quantities such as forces, accelerations. One has to be extra careful when dealing with moving frames of reference. Energy methods involving quantities such as kinetic and potential energies, on the other hand, deal with scalar quantities. One can choose any method of formulation but the final governing equations will be identical.

Recap

In this lecture you have learnt the following

- Derivation of free undamped vibration system equation using the energy conservation principle.
- Determine the total energy of the system ie sum of kinetic and potential energy of vibrating system.
- Similarity in equations derived using Newton's laws and using energy conservation principle.

Congratulations, you have finished Lecture 4. To view the next lecture select it from the left hand side menu of the page

Module 7 : Free Undamped Vibration of Single Degree of Freedom Systems; Determination of Natural Frequency ; Equivalent Inertia and Stiffness; Energy Method; Phase Plane Representation.

Lecture 15 : Representation of Vibratory Behaviour

Objectives

In this lecture you will learn the following

Representation of Vibratory motion

- Phase plane plots

Phase Plane Method

The vibratory motion of a spring mass system with initial conditions x_0 and $\dot{x}_0$ has been obtained earlier give reference to that section and is reproduced here

$$x(t) = \frac{\dot{x}_0}{\sqrt{\frac{k}{m}}}\sin\left(\sqrt{\frac{k}{m}}t\right) + x_0\cos\left(\sqrt{\frac{k}{m}}t\right) \qquad 7.5.1$$

We depict the vibratory motion in the form of a chart showing displacement, x vs time, t. While this is one common way of plotting the vibration response, we will now discuss another very useful method of depicting the response viz., the phase-plane plot.

Eq. 7.5.1 may be re-written as

$$x = A\sin\left(\omega_n t + \phi\right) \qquad 7.5.2$$

where $A = \sqrt{x_0^2 + \frac{\dot{x}_0^2}{\omega_n^2}}$

and

$$\phi = \tan^{-1}\left(\frac{\omega_n x_0}{\dot{x}_0}\right)$$

Differentiating the equation for displacement, we have

$$\dot{x} = A\omega_n\cos\left(\omega_n t + \phi\right) \qquad 7.5.3$$

or

$$\frac{\dot{x}}{\omega_n} = A\cos\left(\omega_n t + \phi\right) \qquad 7.5.4$$

Squaring and adding 7.5.3 and 7.5.4 , we have

$$x^2 + \left(\frac{\dot{x}}{\omega_n}\right)^2 = A^2 \qquad 7.5.5$$

- Eqn 7.5.5 represents a circle with coordinate axes x and $\dot{x}/\omega_n$.
- Radius of the circle is the amplitude of oscillations and centre is at the origin. This is shown in Fig. 7.5.1 below.

Fig 7.5.1 Phase-Plane Plot

Time is implicit in this plot and from this diagram, displacement and velocity of motion are available from single point which corresponds to a particular time instant. This is called the phase-plane plot. The horizontal projection of the phase trajectory on a time base gives the displacement-time plot of the motion and similarly the vertical projection on time base gives velocity-time plot of the motion. Let us see the correlation of phase plane and regular displacement – time plots .

Fig 7.5.2 Phase and Displacement time plots

As can be seen in Fig. 7.5.2 above, the starting point (with finite displacement and velocity at time t=0) is marked P_1 . After t_1 seconds, we reach P_2 where $\angle P_1OP_2 = \omega_n t_1$ radians.There are many other interesting forms of graphical representation of dynamic response of a system. Since it is an undamped system, when started with some intial conditions, if continues to move forever. Staring point P_1 is reached after every cycle (time period).

If the system is damped, then the mass gratually dissipates away energy and comes to rest. Please see section for the motion of such systems and their phase plane plots.

Recap

In this lecture you have learnt the following

- Phase plane representation of the simple harmonic motion with axes as velocity and displacement.
- Essentially the phase plot of a simple harmonic motion is a circle.

Congratulations, you have finished Lecture 5. To view the next lecture select it from the left hand side menu of the page

Module 8 : Free Vibration with Viscous Damping; Critical Damping and Aperiodic Motion; Logarithmic Decrement;Systems with Coulomb Damping.

Lecture 17 : Free Vibration with Viscous Damping

Objectives

In this lecture you will learn the following

- Solution of equation of motion for damped-single degree of freedom system.
- Concept of damping and damping factor.
- Different types of damped systems (underdamped, overdamped and critically damped systems).

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Free Vibrations with Viscous Damping

Fig 8.1.1 Free Damped Vibrations **Fig 8.1.2 Free body Diagram**

A single degree of freedom damped system and its free body diagram are shown in Fig.8.1.1 and 8.1.2. Applying Newton 's second Law,

$$\text{Inertia Force} = \sum \text{External Forces}$$

$$m\ddot{x} = -kx - c\dot{x} \qquad 8.1.1$$

$$\Rightarrow \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

This is an ordinary linear differential equation of second order.

Assuming the solution to the above equation of the form, $x = e^{St}$ we get

$$s^2 e^{st} + \frac{c}{m} s e^{st} + \frac{k}{m} e^{st} = 0$$

$$s^2 + \frac{c}{m} s + \frac{k}{m} = 0 \qquad 8.1.2$$

This is called the characteristic equation of the system which has two roots,

$$S_{1,2} = \frac{c}{2m}\left\{-1 \pm \sqrt{1 - \frac{4km}{c^2}}\right\} \qquad 8.1.3$$

Therefore the genral solution to the equation of motion is of the form,

$$\boxed{x = Ae^{s_1 t} + Be^{s_2 t}}$$

where A and B are constants to be determined from initial conditions on position and velocity.

We define Damping factor ξ as

$$\xi = \frac{c}{2\sqrt{km}}$$

$$\text{such that } \frac{4km}{c^2} = \frac{1}{\xi^2} \qquad 8.1.4$$

$$\text{And} \qquad \xi = \frac{c}{2m\omega_n}$$

$$\therefore \frac{c}{2m} = \xi\omega_n \qquad 8.1.5$$

From equations 8.1.4 and 8.1.5 , we can write the two roots s_1 and s_2 as follows:

$$s_{1,2} = \omega_n \left\{ -\xi \pm \sqrt{\xi^2 - 1} \right\}$$

Thus mainly three cases arise depending on the value of ξ

$\xi > 1 \Leftrightarrow$ *Overdamped System*

$\xi = 1 \Leftrightarrow$ *Critically damped System*

$\xi < 1 \Leftrightarrow$ *Underdamped System*

When $\xi \geq 1$ the system undergoes aperiodically decaying motion and hence such systems are said to be Overdamped Systems. An example of such a system is a door damper – when we open a door and enter a room, we want the door to gradually close rather than exhibit oscillatory motion and bang into the person entering the room behind us! So the damper is designed such that $\xi \geq 1$

Critically damped motion ($\xi = 1$ a hypothetical borderline case separating oscillatory decay from aperiodic decay) is the fastest decaying aperiodic motion.

When " ξ < 1", x(t) is a damped sinusoid and the system exhibits a vibratory motion whose amplitude keeps diminishing. This is the most common vibration case and we will spend most of our time studying such systems. These are referred to as **Underdamped systems .**

Fig 8.1.3 Damped vibration at various damping factors

In Fig. 8.1.3 we capture typical motions for overdamped, critically damped and underdamped systems.The initial conditions for the system are:

$$t = 0, \quad x = 1.3cm$$

$$\dot{x} = 0$$

Phase plot for underdamped oscillations showing the system coming to rest at its equilibrium position is given in Fig. 8.1.4

Fig 8.1.4 Phase plot of damped oscillations for ξ =0.1

Recap

In this lecture you have learnt the following

- Damping and damping factor
- Free vibration response with damping
- Overdamped, critical and underdamped systems

- Governing differential equations and typical solution plots for these systems

Congratulations, you have finished Lecture 1. To view the next lecture select it from the left hand side menu of the page

Module 8 : Free Vibration with Viscous Damping; Critical Damping and Aperiodic Motion; Logarithmic Decrement; Systems with Coulomb Damping.

Lecture 18 : Critical Damping and Aperiodic Motion

Objectives

In this lecture you will learn the following

- Solution of equqtion of motion for critically damped and overdamped systems

- Motion characteristics when ξ=1 and ξ>1

CRITICAL DAMPING

Critical damping is when a system has to undergo motion with the damping factor $\xi = 1$. The two roots of the characteristic equation are given by the general equation:

$$s_{1,2} = \omega_n \left\{ -\xi \pm \sqrt{\xi^2 - 1} \right\} \qquad 8.2.1$$

For $\xi = 1$ the roots become:

$$s_1 = s_2 = -\omega_n \qquad 8.2.2$$

The resulting motion of the system is given by

$$x = \left(A + Bt\right) e^{-\omega_n t} \qquad 8.2.3$$

The above equation is solution to the system having critical damping, where A and B are to be obtained from initial conditions.

For finding the values of constants A and B we use the initial conditions:, let

$$x = X_0 \quad \text{at t=0}$$
$$\dot{x} = 0 \quad \text{at t=0}$$

Differentiating the displacement equation (8.2.3), we get

$$\dot{x} = Be^{-\omega_n t} - \left(A + Bt\right) \omega_n e^{-\omega_n t} \qquad 8.2.2$$

Substituting the initial conditions, we have

$$X_0 = A$$
$$B - \omega_n A = 0$$
$$\text{which gives} \quad A = X_0$$
$$\text{and} \quad B = \omega_n X_0$$

Substituting the values of A and B in the displacement equation

$$x = X_0 (1 + \omega_n t) e^{-\omega_n t} \qquad 8.2.3$$

The value x in the above equation can be shown to decrease as t increases and ultimately tend to zero as t tends to infinity see fig. 8.1.3.

APERIODIC MOTION

Aperiodic motion or over-damped system has the value of damping $\xi \geq 1$. The system has a very large value of damping. In this case roots of the diferential equation are:

$$s_{1,2} = \omega_n \left\{ -\xi \pm \sqrt{\xi^2 - 1} \right\}$$

Both the roots are real and negative numbers. Therefore the general solution to the equation is

$$x = Ae^{S_1 t} + Be^{S_2 t}$$

which represents a decaying response.

We will now illustrate the determination of A and B from given initial conditions.

Consider the following initial conditions :

$$x = X_0 \quad \text{at t=0}$$
$$\dot{x} = 0 \quad \text{at t=0}$$

Differentiating the equation we get:

$$\dot{x} = AS_1 e^{S_1 t} + BS_2 e^{S_2 t} \qquad 8.2.5$$

Substituting the initial conditions, we get

$$X_0 = A + B$$
$$AS_1 + BS_2 = 0$$
$$\text{giving} \qquad A = \left[\frac{\xi + \sqrt{\xi^2 - 1}}{2\sqrt{\xi^2 - 1}} \right] X_0$$
$$B = \left[\frac{-\xi + \sqrt{\xi^2 - 1}}{2\sqrt{\xi^2 - 1}} \right] X_0$$

Substituting the above values in the displacement equation :

$$x = \frac{X_0}{2\sqrt{\xi^2 - 1}} \left[\left[\xi + \sqrt{\xi^2 - 1} \right] e^{\left[-\xi + \sqrt{\xi^2 - 1} \right] \omega_n t} + \left[-\xi + \sqrt{\xi^2 - 1} \right] e^{\left[-\xi - \sqrt{\xi^2 - 1} \right] \omega_n t} \right] \qquad 8.2.6$$

Since the exponent is negative in both the terms in the above equation, they both decrease exponenetially with t. Higher the damping , more sluggish is the respose of the system. Theoritically the system will take infinite time to come back to its equilibrium position. This type of motion is called aperiodic motion see fig. 8.1.4.

Underdamped system

Underdamped systems have the damping factor $\xi < 1$

The general equation of motion is as follows

$$x = Ae^{S_1 t} + Be^{S_2 t}$$

where $s_{1,2} = \omega_n \left\{ -\xi \pm \sqrt{\xi^2 - 1} \right\}$

With damping factor $\xi < 1$, the values of S_1 and S_2 will be as follows :

$$s_1 = \omega_n \left\{ -\xi + i\sqrt{\xi^2 - 1} \right\}$$
$$s_2 = \omega_n \left\{ -\xi - i\sqrt{\xi^2 - 1} \right\}$$

The general solution to the equation of motion is as follows:

$$x = e^{-\xi\omega_n t}\left[A\ Sin(\sqrt{1-\xi^2}\,\omega_n t) + B\ \mathrm{Cos}(\sqrt{1-\xi^2}\,\omega_n t) \right] \qquad 8.2.8$$

subjected to initial conditions as follows

$x = X_0$ at t=0
$\dot{x} = 0$ at t=0

solving for A and B we get

$$x = X_0 e^{-\xi\omega_n t}\left[\frac{\xi}{\sqrt{1-\xi^2}} Sin(\sqrt{1-\xi^2}\,\omega_n t) + \mathrm{Cos}(\sqrt{1-\xi^2}\,\omega_n t) \right] \qquad 8.2.9$$

where $\omega_n \sqrt{1-\xi^2}$ is called as damped natural frequency.

Recap

In this lecture you have learnt the following

- Concept of critical damping
- Solution of equation of motion for underdamped, critically damped and over damped systems.

Congratulations, you have finished Lecture 2. To view the next lecture select it from the left hand side menu of the page.

Module 8 : Free Vibration with Viscous Damping; Critical Damping and Aperiodic Motion; Logarithmic Decrement; Systems with Coulomb Damping.

Lecture 19 : Logarithmic Decrement

Objectives

In this lecture you will learn the following.

- Concept of logarithmic decrement
- Formulation of equation for logarthmic decrement.
- Application of the concept.

When "$\xi < 1$", x(t) is a damped sinusoid and the system exhibits a vibratory motion whose amplitude keeps diminishing as shown in fig 10.3.1. This is the most common vibration case and we will spend most of our time studying such systems. These are referred to as Underdamped systems

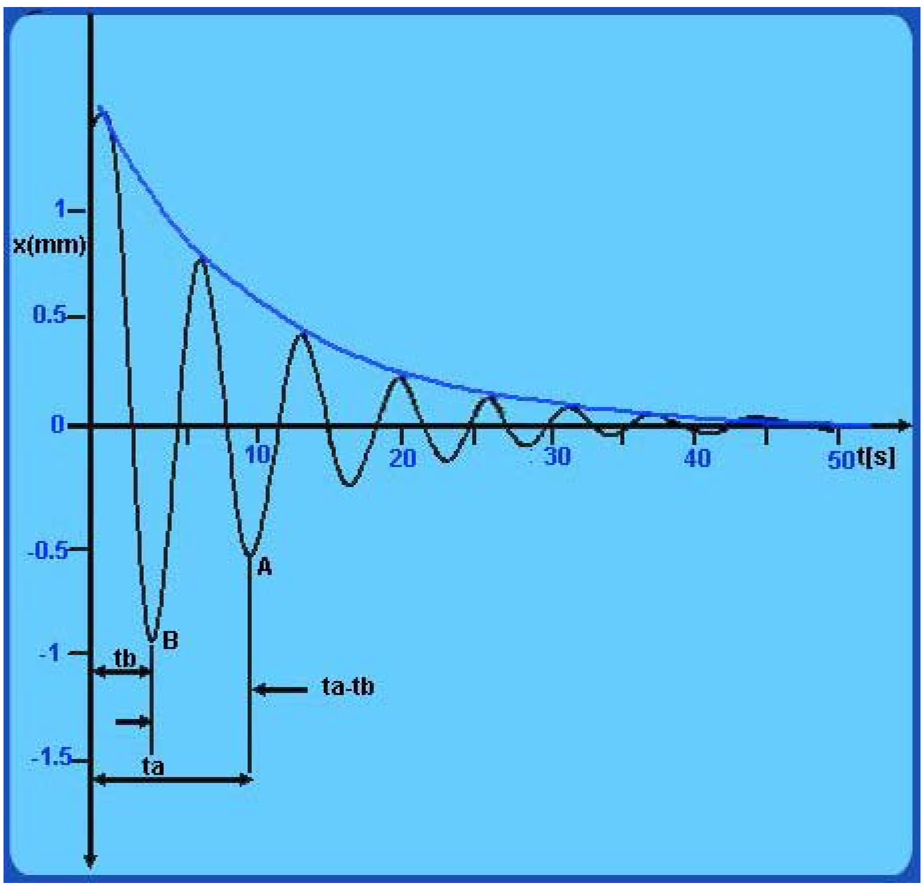

Fig 8.3.1 Underdamped Oscillations

The decrease in amplitude from one cycle to the next depends on the extent of damping in the system. The successive peak amplitudes bear a certain specific relationship involving the damping of the system, leading us to the concept of "logarithmic decrement" which we will now discuss.

It is often necessary to estimate the extent of damping present in a given system. Essentially the experimental techniques to determine damping in a system fall into two categories -- those based on free vibration tests and secondly those based on forced vibration tests. The latter require more sophisticated equipment/instruments, while the former is a relatively simple test. In a free vibration test, based on the measured peak amplitudes over several cycles (and thus estimating the "logarithmic decrement"), one can readily find the damping factor for the given system. We will now discuss these aspects.

Lograthmic decrement comes as an accurate and practically feasible tool to determine the damping in the system.

Logarithmic Decrement:

Consider the two peaks A and B as shown in fig 8.3.1. The amplitude at A and B are x_A and x_B at time t_A and t_B respectively.The periodic displacement from x_A to x_B represents a cycle. The time period for this complete cycle is given by:

$$t_A - t_B = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1-\xi^2}\,\omega_n} \qquad 8.3.1$$

This is the time period of damped oscillations and ω_d is damped natural frequency.

The amplitude of damped oscillations is given by the expression:

$$x = \frac{X_0}{\sqrt{1-\xi^2}\,\omega_n} e^{-\xi\omega_n t} \qquad 8.3.2$$

which is the envelope of maximum of displacement -time curve. Therefore at t= t_A and t_B the amplitudes are given by:

$$x_A = \frac{X_0}{\sqrt{1-\xi^2}\,\omega_n} e^{-\xi\omega_n t_A} \qquad 8.3.3$$

$$x_B = \frac{X_0}{\sqrt{1-\xi^2}\,\omega_n} e^{-\xi\omega_n t_B} \qquad 8.3.4$$

Therefore

$$\frac{x_B}{x_A} = e^{\xi\omega_n(t_A - t_B)}$$

$$\text{But from equation } 8.3.1 \quad t_A - t_B = \frac{2\pi}{\sqrt{1-\xi^2}\,\omega_n} \qquad 8.3.5$$

Therefore

$$\frac{x_B}{x_A} = e^{2\pi\xi / \sqrt{1-\xi^2}} \qquad 8.3.6$$

$$\log_e \frac{x_B}{x_A} = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

This is called the Logarithmic Decrement denoted by

$$\delta = \log_e \frac{x_B}{x_A} = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \qquad 8.3.7$$

if $\xi << 1$ $\delta \approx 2\pi\xi$

This shows that the ratio of any two succesive amplitudes for an underdamped system, vibrating freely , is constant and is a function of the damping only.

Sometimes, in experiments, it is more convenient/accurate to measure the amplitudes after say "n" peaks rather than two succesive peaks (because if the damping is very small, the difference between the succesive peaks may not be significant). The lograthmic decrement can then be given by the equation

$$\delta = \frac{1}{n}\log_e\frac{x_0}{x_n} \qquad 8.3.8$$

Recap

In this lecture you have learnt the following

- Concept of logarithmic decrement and its application as a tool to determine the system damping
- Derivation of formula for lograthmic decrement $\log_e\frac{x_B}{x_A} = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$

Congratulations, you have finished Lecture 3. To view the next lecture select it from the left hand side menu of the page

Module 8 : Free Vibration with Viscous Damping; Critical Damping and Aperiodic Motion; Logarithmic Decrement;Systems with Coulomb Damping.

Lecture 20 : Systems with Coulomb Damping.

Objectives

In this lecture you will learn the following

- Damping effect due to friction and its characteristics.
- Solution of equation of motion for frictional damping.
- Frictionally damped natural frequency.

Dry friction or Coloumb Damping:This type of damping occurs when two machine parts rub against each other, dry or unlubricated. The dry friction is a complex phenomenon and there are several theories about the mechanism of dry friction. A typical dry friction characteristic is as shown in Fig 8.4.1. This form of damping brings in non-linearities and the mathematical model of the system becomes non-linear . Non-linear dynamical systems can exhibit very complex behavior and detailed analysis of such systems is beyond the scope of our present discussion. Interested readers can refer to advanced reference material .

Fig 8.4.1 Coulomb Friction

Consider two dry sliding surfaces with a normal reaction R between them. Then the force of friction acting on each of the two mating surfaces is given by:

$$F = \mu R$$

where μ is defined as the coefficient of friction between the two mating surfaces. For ideally smooth surfaces, dry friction coefficient μ is independent of velocity. For rough surfaces, dry friction cofficient decreases somewhat initially with increase in velocity and finally is constant through out. For lubricated surfaces, μ is approximately proportional to velocity giving approxiamtely viscous damping.

The aspects we are intrested here regarding free vibration with Coulomb damping are:

- The frequency of damped oscillations.
- The rate of decay of these oscillations.

FREQUENCY OF DAMPED OSCILLATIONS

Consider a spring mass system shown in fig 8.4.2 with mass sliding on the dry surface , μ being the coefficient of friction between the two surfaces. At equillibrium position, spring is unstretched and no friction force acts on the mass.

Fig 8.4.2 Equilibrium Position

Mass displaced to right and moving towards right:

Fig 8.4.2 Mass displaced and moving to right

Frictional force as well as the spring force in this case are acting towards left.The equation of motion of mass for this part of motion is as follows:

$$m\ddot{x} = -kx - F \qquad 8.4.1$$

$$or \qquad \ddot{x} + \frac{k}{m}\left(x + \frac{F}{k}\right) = 0 \qquad 8.4.2$$

Let us now introduce a new variable "y" as follows:

$$x + \frac{F}{k} = y$$

$$\therefore \quad \ddot{x} = \ddot{y}$$

Substituting for x and $\ddot{x}$ we have

$$\ddot{y} = -\frac{k}{m}y$$

This is simple harmonic motion about y=0 and is true for that quater of cycle when mass is displaced to right and moving to the right.

Natural frequency for this part of the motion is :

$$\omega_n = \sqrt{\frac{k}{m}} \qquad 8.4.5$$

Mass displaced to right and moving towards left:

Fig 8.4.4

When the mass is under the condition as in fig 8.4.4 displaced to right and moving towards left, equation of motion changes as follows:

$$m\ddot{x} = -kx + F \qquad 8.4.6$$

$$or \qquad \ddot{x} + \frac{k}{m}\left(x - \frac{F}{k}\right) = 0 \qquad 8.4.7$$

$$x - \frac{F}{k} = y \qquad 8.4.8$$

$$\therefore \quad \ddot{x} = \ddot{y}$$

Substituting for x and $\ddot{x}$ we have

$$\ddot{y} + \frac{k}{m}y = 0 \qquad 8.4.9$$

The natural frequency of this part of motion is also:

$$\omega_n = \sqrt{\frac{k}{m}} \qquad 8.4.10$$

The displacement time plot of such system is shown in Fig. 8.4.5:

Fig. 8.4.5

It is to be observed that the amplitude envelope is exponentially decaying for viscous damped system studies earlier, while in this case, it falls off linearly.

Recap

In this lecture you have learnt the following

- Damping effect due to friction
- Coloumb damped natural frequency
- Rate of decay due to friction

Congratulations, you have finished Lecture 4. To view the next lecture select it from the left hand side menu of the page

Module 9 : Forced Vibration with Harmonic Excitation; Undamped Systems and resonance; Viscously Damped

Systems; Frequency Response Characteristics and Phase Lag; Systems with Base Excitation;

Transmissibility and Vibration Isolation; Whirling of Shafts and Critical Speed.

Lecture 21 : Harmonic vibration of single DOF system

Objectives

In this lecture you will learn the following

- Equation of motion for forced vibration
- Response to Harmonic Force
- Dynamic magnification factor
- Concept of resonance
- Role of stiffness, mass and damping in response

Mechanical systems are subject to various excitations and it is of interest to determine the response of the system. It is by no means a simple problem to determine the response of a complex real life engineering system to a general excitation. Consider for example, the response of an automobile when moving on a rough road. The excitation to the system comes from the road roughness which is random in nature and needs to be modeled using statistical techniques. Similarly the tire and the suspension of the vehicle itself could exhibit non-linear behavior and hence the mathematic model and its solution will turn out to be quite complex. As another example, consider the dynamic response of an aircraft (such as the indigenous Light Combat Aircraft) during take-off, flight maneuvers and landing. Accurate modeling and response estimation involves complex calculations often on a digital computer.

While we aim to perform complex vibration analysis on a computer on such practical systems, in order to get insight into the methods of analysis as well as the nature of the response, we begin our discussion (as we did for free vibration case) with a simple single degree of freedom spring – mass-damper system subjected to a sinusoidal excitation of single frequency. As mentioned earlier, it is not necessarily true that damping in every physical system is viscous in nature (i.e., damping force proportional to velocity) however we consider viscous damping as it is perhaps the easiest way of accounting for damping in the system as it leads to linear differential equations with constant coefficients.

Why Sinusoidal Excitation?

The reason for using sinusoidal excitation is simple - any general time varying forcing function can in principle be expressed as a summation of several sinusoids (Fourier series). Assuming linear system behavior, we will be able to superpose the response to individual sinusoids to get the total system response. Moreover, many mechanical systems involve a fundamental operating speed and the excitation source is often predominantly at this speed or its harmonics – for example, unbalance in a shaft causes an excitation at the running speed.

Why single d.o.f. spring-mass -damper system?

The simple spring-mass system exhibits several characteristics (such as resonance) similar to those of real life, complex engineering systems and hence we begin our discussion with this system.

Forced Vibration Analysis of an undamped single d.o.f system

Fig 9.1.1 Single degree of freedom system under harmonic excitation

Fig 9.1.2 Free body diagram of SDOF system

Consider a spring-mass system subjected to a sinusoidal force as shown in Fig. 9.1.1. A practical situation representing this is also shown in the figure. The figure shows a machine on its foundation, where the machine is operating at a rotational speed Ω . If the rotor has some unbalance and if ‘k' denotes the stiffness of the foundation, we can represent the system as an equivalent single d.o.f system as shown in the figure. The sinusoidal excitation corresponds to the unbalance force as indicated in the figure. We will begin our discussion with the simple spring-mass system and later, return to this practical example to study it in greater detail.

The governing equation of motion can be readily obtained from the free body diagram shown in Fig. 9.1.2 as follows:

$$F_o \sin\Omega t - kx - c\dot{x} = m\ddot{x} \qquad (9.1.1)$$

The response of the system as a solution to the differential equation of motion has two parts viz., complementary solution and particular integral. For this system, the complementary solution represents the response of the system to given initial conditions and the particular integral is the response to the particular forcing function (sinusoid in this case). The complementary solution, as obtained in eqn. () is given by:

$$x_c(t) = e^{-\xi w_n t}\left(A \sin\omega_d t + B\cos\omega_d t\right) \qquad (9.1.2)$$

$$\text{where} \quad \omega_d = \omega_n\sqrt{1-\xi^2} \quad \text{and} \quad \omega_n = \sqrt{k/m}$$

The particular integral can be assumed to be of the form:

$$x_p(t) = X_0 \sin(\Omega t - \phi) \qquad (9.1.3)$$

Thus the total response of the system is given by:

$$x(t) = x_c(t) + x_p(t) \qquad (9.1.4)$$

The first part of the response, corresponding to the complementary solution representing the free vibration response, is at the damped natural frequency of the system and decays very fast exponentially !. In practice every system has some

form of dissipation or other and hence this will eventually die down (as we discussed in module 8.) leaving only the "steady state" response pertaining to the particular integral. The steady state response is a sinusoid at the frequency of excitation (called forcing frequency or driving frequency) and has some phase difference with the excitation as depicted in Fig. 9.1.3 of special interest is the amplitude of steady state which can be obtained as follows.

Fig 9.1.3 Phase relation of excitation and response

Substituting eq 9.1.3 in eq 9.1.1 we get

$$\begin{aligned} &-mX_0\Omega^2 \sin(\Omega t-\phi) \\ &+cX_0\Omega\cos(\Omega t-\phi) \\ &+kX_0 \sin(\Omega t-\phi)=F_o \sin\Omega t \end{aligned} \qquad (9.1.6)$$

i,e.,

$$\begin{aligned} &(k-m\Omega^2)X_0[\sin\Omega t\cos\phi-\cos\Omega t\sin\phi] \\ &+cX_0\Omega[\cos\Omega t\cos\phi+\sin\Omega t\sin\phi]=F_o\sin\Omega t \end{aligned} \qquad (9.1.7)$$

Comparing terms of $\sin\Omega t$ and $\cos\Omega t$ on either side we have

$$X_0(k-m\Omega^2)\cos\phi+cX_0\Omega\sin\phi=F_0 \qquad (9.1.8)$$

$$-X_0(k-m\Omega^2)\sin\phi+cX_0\Omega\cos\phi=0 \qquad (9.1.9)$$

which yield

$$X_0=\frac{F_0}{\sqrt{(k-m\Omega^2)^2+(c\Omega)^2}} \qquad (9.1.10)$$

$$\tan\phi=\frac{c\Omega}{(k-m\Omega^2)} \qquad (9.1.9)$$

Now we introduce the notation

undamped natural frequency $\omega_n=\sqrt{\frac{k}{m}}$

frequency ratio $\eta=\frac{\Omega}{\omega_n}$

critical damping $c_c=2m\omega_n$

damping factor $\xi=\frac{c}{c_c}$ i.e. $c=2m\omega_n\xi$

we can rewrite equations 9.1.9 and 9.1.10 as

$$X_0 = \frac{F_0 / k}{\sqrt{[(1-\eta^2)]^2 + [2\xi\eta]^2}} \qquad (9.1.11)$$

and $$\tan\phi = \frac{2\xi\eta}{(1-\eta^2)} \qquad (9.1.12)$$

If we consider that the force of magnitude F_0 were applied statically, the response would then be

$$X_{st} = \frac{F_0}{k} \qquad (9.1.13)$$

Thus re-writing eq (9.1.11), we get,

$$\frac{X_0}{X_{st}} = \frac{1}{\sqrt{(1-\eta^2)^2 + (2\xi\eta)^2}} \qquad (9.1.14)$$

Fig 9.1.4 Variation of dynamic magnification factor with frequency ratio

Since the R.H.S. indicates by how much the dynamic response is more than the static response, it is usually termed the Magnification Factor. If we now plot the variation of the magnification factor with frequency of excitation, we get a curve as shown in Fig. 9.1.4. There are primarily three regions of interest in such a graph.

When $\Omega \ll \omega_n$ ($\eta \ll 1$) [stiffness controlled regime]

In this case, the forcing frequency is much lower than the natural frequency of the system and the magnification factor is nearly equal to unity i.e., the dynamic response is almost same as static response. In other words, since the forcing frequency is so slow, the system responds almost as if the load were static. Thus the response in this regime is governed by the stiffness of the system.

For $\Omega = 0.33\omega_n$, the dynamic magnification factor equals 1.125 i.e., the dynamic response is just about 12.5% more than the static response. In such cases, one needs to critically assess whether it is indeed necessary to perform a full fledged dynamic analysis or it is sufficient to just perform a simple static analysis with the knowledge that the actual response could be slightly higher.

When $\Omega \gg \omega_n$ ($\eta \gg 1$) [inertia controlled regime]

In this case the excitation frequency is much higher than the natural frequency of the system and the magnification factor approaches zero i.e., the mass vibrates very little about the mean equilibrium position. In other words, the disturbance varies so fast that the inertia of the system cannot cope with it and kind of 'gives up'. To be sure the motion is still sinusoidal with the forcing frequency but the amplitude is extremely small. The dynamic response is just about 4% of the static response! Again, in this regime, one needs to critically examine whether it is necessary to perform a full fledged dynamic analysis. It is of course to be noted that high frequency vibration could result in eventual fatigue failure and secondly that even though vibratory displacements are small, the velocity need not be small since $\dot{X}_0 = \Omega X_0$ for a sinusoid.

When $\Omega \approx \omega_n$ ($\eta \approx 1$) [damping controlled]

In this region the excitation frequency is close to the natural frequency of the system and we see a huge build-up of dynamic response. It is this region that is critical and could cause potential damage to many structures and machine elements and needs to be avoided as far as possible. You now appreciate why it is important to accurately determine the natural frequencies of a system and make sure that the operating frequencies are not in the same range. When the excitation and the natural frequencies match, we call this phenomenon "resonance".

$$\frac{X_0}{X_{st}} \approx \frac{1}{2\xi} \qquad (9.1.15)$$

Practical Implication

A real-life complex engineering system may have several natural frequencies and may be experiencing excitation at various frequencies. When any of these excitation frequencies comes close to any of the natural frequencies of the system, we can expect large amplitudes of vibration and this condition should be avoided as far as possible.

Thus we see that the peak amplitude near resonanceis limited purely by the damping present in the system. The lower the damping in the system, the higher the peak amplitude and the sharper the peak. Thus one can guess the extent of damping in a system by merely looking at the dynamic response curves as depicted in Fig. 9.1.4. From the value of vibration amplitude as given by equation (9.1.15) one can think of a method of experimentally determining the damping coefficient viz., measure the response at this condition and use eqn (9.1.15) to estimate the damping factor. However, another easy method of determining damping is commonly employed, which is known as the half-power-point method. When the magnification factor is 0.707 times the peak magnification factor i.e., $\frac{1}{\sqrt{2}}$ times , the corresponding frequency values on either side of resonance are given by (see Fig.9.1.5)

$$\frac{1}{\sqrt{(1-\eta^2)^2 + (2\xi\eta)^2}} = \frac{1}{\sqrt{2}}\left(\frac{1}{2\xi}\right) \qquad (11.1.17)$$

$$\eta^4 + (4\xi^2 - 2)\eta^2 + (1 - \xi^2) = 0 \quad (11.1.18)$$

$$\eta_{1,2}^2 = 1 - 2\xi^2 \pm 2\xi\sqrt{1 + \xi^2} \quad (11.1.19)$$

Usually, $\xi << 1, \quad \xi^2 << 1 \quad \text{and so,} \quad \eta_{1,2}^2 \approx 1 \pm 2\xi$

$$\therefore \quad 4\xi \approx \frac{\Omega_2^2 - \Omega_1^2}{\omega_n^2} = \left(\frac{\Omega_2 + \Omega_1}{\omega_n}\right)\cdot\left(\frac{\Omega_2 - \Omega_1}{\omega_n}\right) \quad (11.1.20)$$

$$\frac{\Omega_2 + \Omega_1}{2} \approx \omega_n, \quad \xi = \frac{\Omega_2 - \Omega_1}{2\omega_n} = \cdot\frac{\eta_2 - \eta_1}{2} \quad (11.1.21)$$

This is an effective method of determining the damping coefficient in a system using a forced vibration test.Logarithmic decrement is another effective technoque and is based on free vibration test.

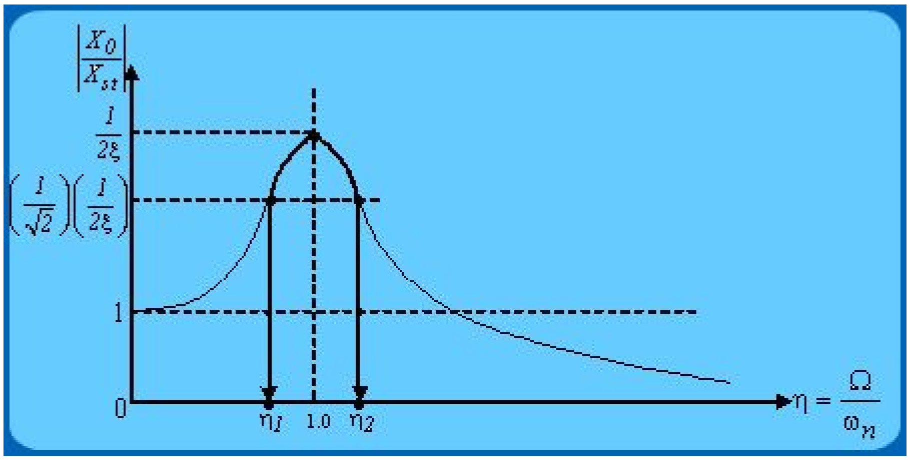

Figure 9.1.5

Recap

In this lecture you have learnt the following

- Solution to the equation for the forced oscillations
- Magnification factor ie ratio between the deflection due to force when applied as sinusoidal vs static $$X_0 = \frac{F_0/k}{\sqrt{[(1-\eta^2)]^2 + [2\xi\eta]^2}}$$
- Effect on vibrations due to change in ratio of the forced frequency to natural frequency of the system $\eta = \dfrac{\Omega}{\omega_n}$

Congratulations, you have finished Lecture 1. To view the next lecture select it from the left hand side menu of the page

Module 9 : Forced Vibration with Harmonic Excitation; Undamped Systems and resonance; Viscously Damped

Systems; Frequency Response Characteristics and Phase Lag; Systems with Base Excitation;

Transmissibility and Vibration Isolation; Whirling of Shafts and Critical Speed.

Lecture 22 : Phase and Phasor representation

Objectives

In this lecture you will learn the following

- Phase between excitation and response.
- Phasor analysis of resonance behaviour.

So far we focussed our attention on the amplitude of vibration and let us now look at the phase as given below:

$$\tan\phi = \frac{2\xi\eta}{1-\eta^2} \qquad (9.2.1)$$

Fig 9.2.1 Variation of Phase angle with frequency ratio

The variation of the phase with the frequency of excitation is shown in Fig. 9.2.1

For an undamped system, as seen in the figure,

Phase = 0 degree for $\Omega < \omega_n$

Phase = 90 degree for $\Omega = \omega_n$

Phase = 180 degree for $\Omega > \omega_n$

even if there is damping in the system, the phase will be 90 degree for $\Omega = \omega_n$. This condition is therefore often used to identify the occurrence of resonance in an experiment.

Let us now discuss and understand the physical meaning of phase between the excitation and the response. The excitation and response of undamped system have zero phase difference for all values of forcing frequency less than the natural frequency of the system i.e., response is said to be in phase with the excitation. When the forcing frequency is greater than the natural frequency of the system, the phase difference becomes exactly 180 degree i.e., the response is in phase-opposition to the excitation. In a phasor diagram, the excitation and response can be plotted as shown in Fig. 9.2.2. In an undamped system as we are discussing here, there are essentially three forces viz., externally applied excitation force, force representing spring resistance, inertia force. These three force vectors are also plotted in this figure for all the cases. Of particular interest is the case of resonance. At resonance($\Omega = \omega_n$), if $x = X_0$ (Sin ω_n t) then the acceleration =- ω_n^2 Sin ω_n t. Hence the spring resistance force and the inertia force magnitudes are given by:

$$\text{Spring force} = kx = kX_0 \sin \omega_n t \qquad (9.2.2)$$

$$\text{Interia force} = m\ddot{x} = -m\omega_n^2 X_0 \sin \omega_n t \qquad (9.2.3)$$

Fig 9.2.2 Phasor representation of forces in an undamped system

Since the natural frequency of the system is given by ω_n^2 = k/m, we see that the spring and inertia forces are exactly equal and opposite as indicated in the figure. In view of the 90 degree phase difference between the excitation and response, the excitation goes unbalanced and keeps pumping energy into the system in each cycle thus leading to a build-up of the amplitudes .Fig. 9.2.1 shows the variation of the phase with respect to the forcing frequency. Due to the presence of viscous damping, the response always lags the excitation. Irrespective of the value of damping, the phase is always 90 degree when $\Omega = \omega_n$ i.e., $\eta = 1$. This is in fact used to identify the occurrence of resonance condition in an experiment. Fig. 9.2.3 (similar to Fig. 9.2.2) shows the phasor representation of spring resistance force, viscous damping force, inertia force and the externally applied excitation force for the cases when $\eta < 1$, $\eta = 1$ and $\eta > 1$. It is observed that at resonance the spring force and inertia force match each other exactly (like in the case of undamped system) while the damping force balances the external excitation force. This is another way of interpreting how the damping limits the vibratory response at resonance, while the vibration amplitudes build-up to infinitely large values for undamped systems.

In the case of a damped system, we observe that the viscous damping force (being proportional to velocity and hence 90 degree phase to the displacement response) balances the external disturbance force and limits the displacement to a finite value.

A pertinent question to ask at this stage is as follows – what happens if the system is excited at resonance, will the amplitude immediately shoot up to infinity and the system fails? Fortunately the answer is no – in all practical systems, there is damping present (however small) and hence the response will not immediately shoot up to infinity. Secondly, it can be readily shown that at resonance the amplitude of vibration keeps building up with time approximately in a linear fashion i.e., it surely increases with time but it does take finite time to build up to a dangerously huge value. So what does this mean – it implies that when operating a machine, if a resonance condition needs to be crossed, we should “rush through” the critical speed without letting the system build up enough vibration. This is followed in many practical system e.g., steam turbines where the operating rotational speeds are beyond the fundamental critical speed of the shaft.

Fig 9.2.3 Phasor representation of forces in a viscously damped system

Practical Implication

Spring and Mass (Inertia) control the value of the natural frequency but have no role to play in deciding the amplitude of vibration near resonance. Design of the damper critically affects the dynamics of the system near resonance and could potentially determine whether or not the system fails.

As the mass is subjected to the excitation force and undergoes vibratory motion, some of the force is transmitted to the ground through the spring and the damper. This force can be represented in a graphical manner as shown in Fig. 11.2.4 and the magnitude can be readily computed as follows:

Fig 11.2.4 Force transmitted to foundation

$$F_T = \sqrt{(kX_0)^2 + (c\Omega X_0)^2} \qquad (11.2.4)$$

While mounting many machines, we wish to ensure that F_T is as small as possible.

Recap

In this lecture you have learnt the following

- Relation between phase angle and frequency ratio $\eta = \frac{\Omega}{\omega_n}$
- Phasor representation of vibration systems in damping.
- Role of damping in supressing the vibrations.
- Concept of force transmissibility.

Congratulations, you have finished Lecture 2 To view the next lecture select it from the left hand side menu of the page.

Module 9 : Forced Vibration with Harmonic Excitation; Undamped Systems and resonance; Viscously Damped Systems; Frequency Response Characteristics and Phase Lag; Systems with Base Excitation; Transmissibility and Vibration Isolation; Whirling of Shafts and Critical Speed.

Lecture 23 : Response of base excitation systems

Objectives

In this lecture you will learn the following

- Examples of system with base excitation
- Equation of motion of such systems
- Nature of solution

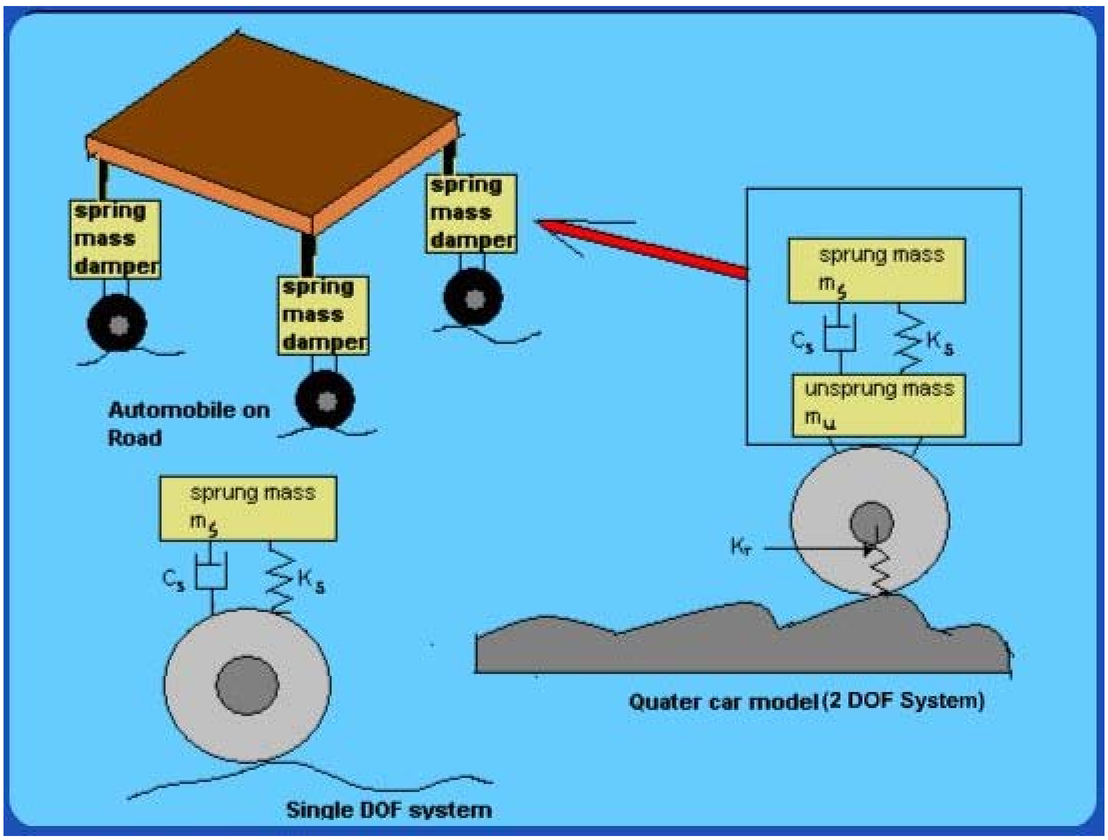

Fig 11.3.1Quater car model

In the previous lecture we discussed the response of the single d.o.f system to a sinusoidal force when the force is acting on the mass, such as in the case of an unbalanced machine operating on its springy foundation. Another important practical situation arises for example in the case of an automobile on the road as shown in Fig. 11.3.1. Typical tire stiffness is in the range of 200 N/mm while that of the suspension is of the order of 20 N/mm. Thus assuming the tire to be practically rigid, we can come up with a simplified single d.o.f model of one-quarter of the car as indicated in the figure. In this case, the excitation of interest for the present discussion is the road undulation. Excitations of course arise from the operation of the engine and hence proper design of the engine mounts becomes important but this will be taken up later. The road undulation is represented as a ground motion disturbance input given to the bottom end of the spring. Let us now investigate the response of such “base-excitation” systems.

Fig 9.3.2

The physical system under consideration and the corresponding free body diagram are given in Fig. 9.3.2. The governing equation of motion can be readily obtained as:

$$\begin{aligned} m\ddot{x} &= -k\left(x - xg\right) \\ &= -c\left(\dot{x} - \dot{x}g\right) \end{aligned} \tag{9.3.1}$$

$$\therefore m\ddot{x} = +c\dot{x} + kx = kx_g + c\dot{x}_g \tag{9.3.2}$$

We could choose two sets of coordinates in this problem viz., the absolute motion of the ground and the mass (or) the motion of the mass relative to the ground. Both have practical utility in different contexts and we shall derive the necessary expressions for both at this stage. For example, if we are interested in the ride comfort of a passenger seated in a car, we will be worried about the absolute acceleration felt by the passenger due to road unevenness. On the other hand if we mounted any measuring instrument on a shaking ground, it can measure only the relative motion of the mass with respect to the ground. If we introduce z = x – y as the relative motion coordinate, the governing equation in terms of z can be given as:

$$m\ddot{z} + c\ddot{z} + kx_z = -m\ddot{x}_g \tag{9.3.3}$$

Assuming that the free vibration transients would have died down due to the damping present in the system, the steady state absolute response of the mass can be shown to be:

$$\text{Given } x_g = X_g \sin \Omega t \tag{9.3.4}$$

$$\text{Assume } x = X_0 \sin(\Omega t - \phi) \tag{9.3.5}$$

$$X_0 = X_g \sqrt{\frac{k^2 + (c\Omega)^2}{(k - (m\Omega)^2)^2 + (c\Omega)^2}} \tag{9.3.6}$$

Variation of the response with forcing frequency is plotted in Fig. 9.3.3 and can be observed to be very similar to that harmonic force excitation acting directly on the mass itself. If we wish the motion of the mass to be less than that of the ground, then the system natural frequency has to be designed in such a manner that

Fig 9.3.3

$$\eta = \frac{\Omega}{\omega_n} \tag{9.3.7}$$

This is a very important observation and has several practical implications as indicated below:

Practical Implication

When the suspension of a car is designed, it has to be designed such that the natural frequency of the car is less than about ? or ¼ the road disturbance frequency, which itself depends on the waviness profile of the road surface as well as the forward speed of the vehicle. Typical values for suspension stiffness are in the range of 20 N/mm for passenger cars such that the natural frequencies for vertical oscillation are about 1 Hz. Thus we need to design a "soft" suspension for better ride comfort. Of course there are other constraints (such as vehicle handling) that determine the optimal design of the suspension.

Similarly we can obtain the steady state RELATIVE motion from eqn (9.3.3) as:

$$z = Z_0 \sin(\Omega t - \phi) \quad (9.3.9)$$

where

$$Z_0 = \frac{m\Omega^2 X_9}{\sqrt{(k - m\Omega^2)^2 + (c\Omega)^2}} \quad (9.3.10)$$

$$\phi = \tan^{-1}\left(\frac{c\Omega}{k - m\Omega^2}\right) \quad (9.3.9)$$

Variation of z (i.e. motion of the mass relative to the base) with respect to the forcing frequency is plotted in Fig. 9.3.4. It is observed that the asymptotes of the response have interchanged when we compare the response to Fig. 9.3.3 i.e., at lower frequencies the relative motion of the mass tends to zero while at high frequencies the magnification factor tends to unity. We will return to this plot at a later stage when we discuss vibration measuring instruments etc.

Fig 9.3.4

Recap

In this lecture you have learnt the following

- Concept of base excitations and their governing equations
- Conept of relative dispacement plot and variation of the relative displacement with base excitations
- Concept of absolute displacement and relevance to

Module 9 : Forced Vibration with Harmonic Excitation; Undamped Systems and resonance; Viscously Damped

Systems; Frequency Response Characteristics and Phase Lag; Systems with Base Excitation;

Transmissibility and Vibration Isolation; Whirling of Shafts and Critical Speed.

Lecture 24 : Transmissibility and Isolation

Objectives

In this lecture you will learn the following

- Transmissibility of force and motion
- Considerations in isolation

Transmissibility and Vibration Isolation

When a machine is operating, it is subjected to several time varying forces because of which it tends to exhibit vibrations. In the process, some of these forces are transmitted to the foundation – which could undermine the life of the foundation and also affect the operation of any other machine on the same foundation. Hence it is of interest to minimize this force transmission. Similarly when a system is subjected to ground motion, part of the ground motion is transmitted to the system as we just discussed e.g., an automobile going on an uneven road; an instrument mounted on the vibrating surface of an aircraft etc. In these cases, we wish to minimize the motion transmitted from the ground to the system. Such considerations are used in the design of machine foundations and in order to understand some of the basic issues involved, we will study this problem based on the single d.o.f model discussed so far.

From eqn (9.2.4), we get the expression for force transmitted to the base as follows:

$$F_T = \sqrt{(kX_0)^2 + (c\Omega X_0)^2} \qquad (9.4.1)$$

where

$$X_0 = X_g\sqrt{\frac{k^2 + (c\Omega)^2}{(k - (m\Omega)^2)^2 + (c\Omega)^2}} \qquad (9.4.2)$$

"Transmissibility Ratio (TR)" is the term commonly used and a plot of the variation of the TR is shown in Fig. 9.4.1. It is observed that when the system is designed such that $\eta <$ (1.414), the TR actually exceeds unity. In order that the TR is sufficiently small, it is desirable to keep the frequency ratio about 5 i.e., the natural frequency of the system must be about 5 times lower than the operating frequency. However, if the springs are made very soft so as to keep the natural frequency low, it could result in excessive static deflection (under the weight of the machine itself). Thus the mountings need to be carefully designed.

Often the mounting system is made up of rubber mounts (e.g., engine mounts in a car), or cork pads apart from the conventional helical coil springs, shock absorbers etc.It is also observed that as $\eta \rangle \sqrt{2}$, damping actually increases the T.R. Thus it would appear necessary to have as low damping as possible. However, to reach an operating point corresponding to $\eta > \sqrt{2}$ (i.e., $\Omega > \sqrt{2}\omega_n$) one needs to pass through $\Omega = \omega_n$, resonance. To limit resonance amplitudes. We need to have sufficient damping in the system.

Fig 9.4.1

Recap

In this lecture you have learnt the following

- Concept of force transmissibility to the base.
- Variation of transmissibility ratio with frequency ratio

Congratulations, you have finished Lecture 4. To view the next lecture select it from the left hand side menu of the page

Module 9 : Forced Vibration with Harmonic Excitation; Undamped Systems and resonance; Viscously Damped

Systems; Frequency Response Characteristics and Phase Lag; Systems with Base Excitation;

Transmissibility and Vibration Isolation; Whirling of Shafts and Critical Speed.

Lecture 25 : Whirling and critical speed

Objectives

In this lecture you will learn the following

- Whirl of shaft with a rotor having some eccentricity
- Synchronous whirl
- Critical speed
- Rayleigh's and Dunkerley's formulae

WHIRLING OF SHAFTS – CRITICAL SPEED

Fig 9.5.1

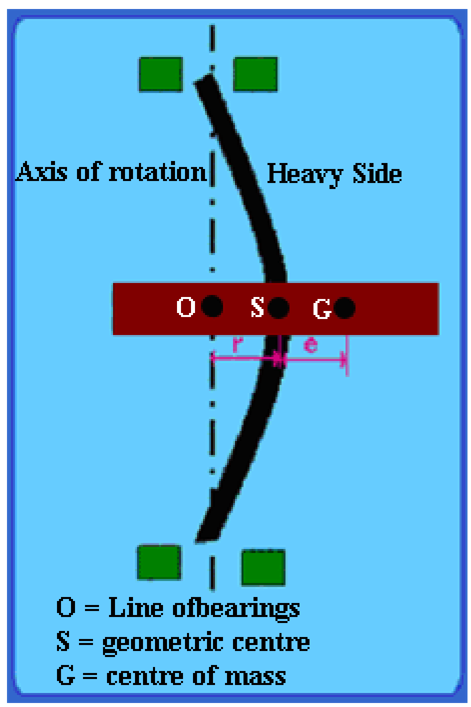

Fig 9.5.2

Consider a typical shaft, carrying a rotor (disk) mounted between two bearings as shown in Fig. 9. 5.1. Let us assume that the overall mass of the shaft is negligible compared to that of the rotor (disk) and hence we can consider it as a simple torsional spring. The rotor (disk) section has a geometric centre i.e., the centre of the circular cross-section and the mass centre due to the material distribution. These two may or may not coincide in general, leading to eccentricity as indicated in Fig. 9.5.2. The eccentricity could be due to internal material defects, manufacturing errors etc. As the shaft rotates, the eccentricity implies that the mass of the rotor rotating with some eccentricity will cause in-plane centrifugal force. Due to the flexibility of the shaft, the shaft will be pulled away from its central line as indicated in the figure. Let us assume that the air-friction damping force is negligible. The centrifugal force for a given speed is thus balanced by the internal resistance force in the shaft-spring and the system comes to an equilibrium position with the shaft in a bent configuration as indicated in the figure. Thus the shaft is rotating about its own axis and the plane containing the bent shaft and the line of bearings rotates about an axis coinciding with the line of bearings. We consider here only the case, wherein these two rotational speeds are identical, called the synchronous whirl.

Fig 9.5.3

Fig. 9.5.3 shows the shaft and rotor system undergoing synchronous whirl in four different positions in a single revolution. Let us write down the force equilibrium equation as follows, following the notation shown in Fig. 9.5.2:

$$\text{Centrifugal force } = m\Omega^2(r+e) \qquad (9.5.1)$$

$$\text{shaft resistance force } = Kr \qquad (9.5.2)$$

wherein, the shaft stiffness *k* is the lateral stiffness of a shaft in its bearings i.e., considering the rotor at mid-span, this is the force required to cause a unit lateral displacement at mid-span of the simply supported shaft. Thus

$$K = 48EI/L^3 \qquad (9.5.3)$$

Where E is the Young's modulus, I is the second moment of area, and L is the length between the supports.

Thus, for equilibrium,

$$Kr = m\Omega^2(r+e) \qquad (9.5.4)$$

$$r = \left(\frac{m\Omega^2 e}{k - m\Omega^2}\right)e$$

ie

$$= \left(\frac{\Omega^2}{\omega_n^2 - \Omega^2}\right)e \qquad (9.5.5)$$

where we have used $\omega_n = \sqrt{\frac{k}{m}}$ to represent the natural frequency of the lateral vibration of the springy-shaft-rotor system. Thus when the rotational speed of the system coincides with the natural frequency of

Ω

lateral vibrations, the shaft tends to bow out with a large amplitude. This speed is known as the critical speed and it is necessary that such a resonance situation is avoided in actual practice. As discussed earlier in the case of resonance, it takes some time for the amplitude to build up to a large value. Some of the turbine rotors whose operating speeds go beyond the critical speed are able to use this fact and rush-through the critical speed. It is necessary to observe from Fig.9.5.3 that, in synchronous whirl. the heavier side remains all the time on the outer side. Thus when the shaft bends, an inner fibre is under compressive stress and outer fibre is under tensible stress but there is NO REVERSAL of stress.

Rayleighs Method

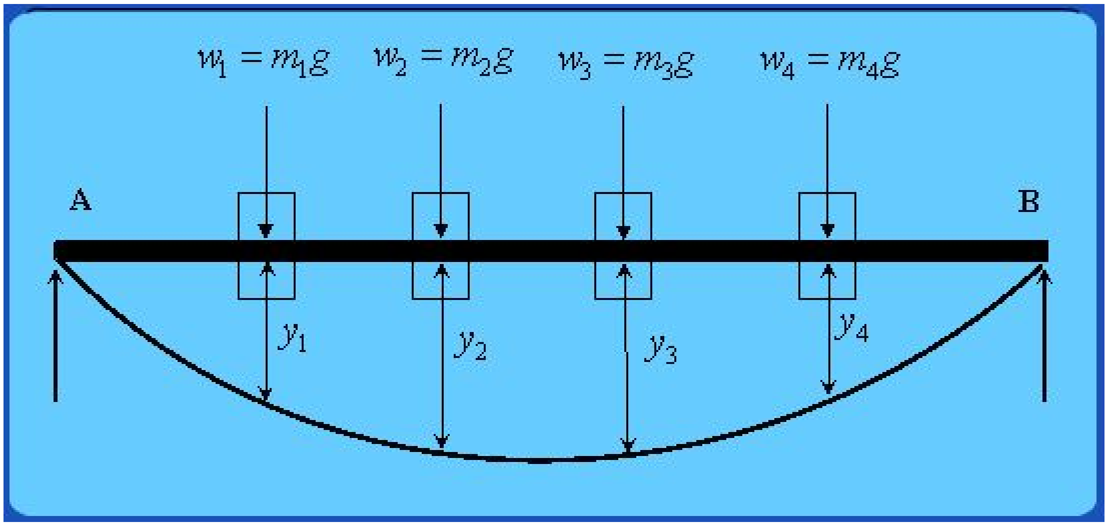

Fig 9.5.3 Multiple disks on a shaft

Rayleigh's method is based on the principle of conservation of energy. The energy in an undamped system consists of the kinetic energy and the potential energy. The kinetic energy T is stored in the mass and is proportional to the square of the velocity. The potential energy U includes strain energy that is proportional to elastic deformations and the potential of the applied forces. For a conservative system, the total energy must remain constant. That is

$$T + U = \text{constant} \qquad (9.5.6)$$

Differentiating this expression, we get the equation of motion as follows.

$$\frac{d}{dx}(T+U) = 0 \qquad (9.5.7)$$

Note that the amounts of kinetic and potential energy in the system may change with time but their sum must remain constant. Thus if T_1 and U_1 are energies at time t_1 and T_2 and U_2 are energies at time t_2 , then

$$T_1 + U_1 = T_2 + U_2 \qquad (9.5.8)$$

For a shaftas shown in Fig.9.5.4, the potential energy is zero at the specific instant of time when the mass is passing through its static equilibrium position and kinetic energy is at its maximum T_{max}. Similarly at the instant when the mass is at its extreme position the kinetic energy is zero and the potential energy is at its maximum U_{max} . Thus we have the following relationship.

$$T_{max} = U_{max}$$
$$T_{max} = \frac{1}{2}mv^2 = \frac{1}{2}m(y\omega_n)^2 \qquad (9.5.9)$$
$$U_{max} = \frac{1}{2}Wy = \frac{1}{2}mgy$$

Therefore we have, considering all the disks on the shaft,

$$\omega_n^2 \sum_{i=1}^{n} m_i y_i^2 = g \sum_{i=1}^{n} m_i y_i \qquad (9.5.10)$$

Where i=1, n represents summation over all the "n" disks.

So we get the frequency of natural vibration as,

$$\omega_n = \sqrt{\frac{g\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i y_i^2}} \qquad (9.5.9)$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i y_i^2}} \qquad (9.5.12)$$

Dunkerley's Empirical Method

When a shaft carries multiple disks it is always efficient to use this method.

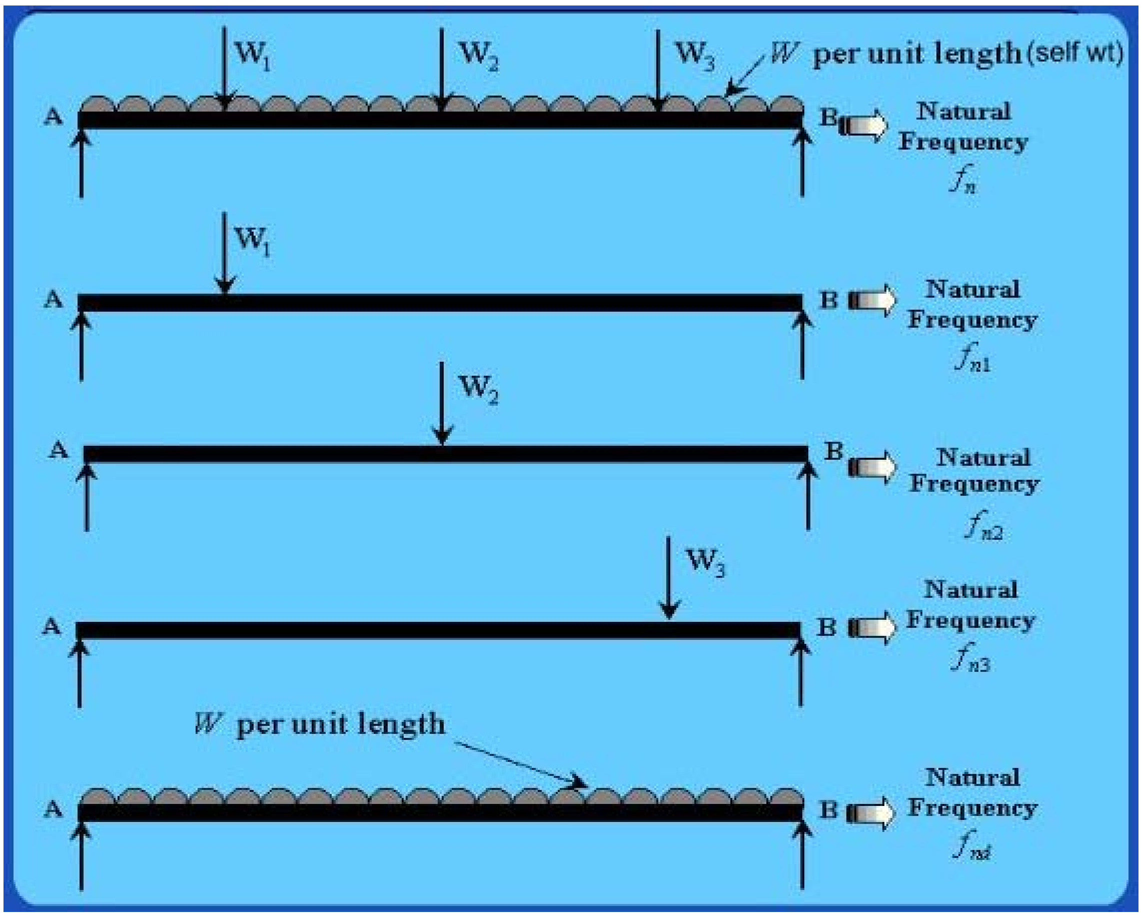

Fig 9.5.4 Dunkerly's approximation for shaft

We consider only one force (wt of disk) acting on the shaft at a time. For each disk, we find the corresponding natural frequency as $f_{n1}, f_{n2}, f_{n3} and\ f_{n4}$. The Nutural frequency of the shaft f_n when all the loads(disks) act on the shaft simultaneously can be found out by the using the formulae:

$$\frac{1}{f_n^2} = \frac{1}{f_{n1}^2} + \frac{1}{f_{n2}^2} + \frac{1}{f_{n3}^2} + \frac{1}{f_{nd}^2} \qquad (9.5.13)$$

For each of the dub-systems ie shaft with only one disk, natural frequency is obtained as $\sqrt{k/m}$ where k is the lateral stiffness of the shaft in its bearings and m is the mass of the disk.

To understand the basis of this method, we need to appreciate multi-d.o.f system vibrations.(Please refer lecture 4, module 10)

Recap

In this lecture you have learnt the following

- Critical speed of rotating shafts

- Use of energy method (Rayliegh's method)and Dunkerly's method for finding the critical speed of shafts.

Congratulations, you have finished Lecture 5. To view the next lecture select it from the left hand side menu of the page.

Module 10 : Vibration of Two and Multidegree of freedom systems; Concept of Normal Mode; Free Vibration Problems and Determination of Ntural Frequencies; Forced Vibration Analysis; Vibration Absorbers; Approximate Methods - Dunkerley's Method and Holzer Method

Lecture 27 : Free Vibration of Two d.o.f. systems

Objectives

In this lecture you will learn the following

- Equation of motion for free vibration of two degree of freedom systems
- Solution of the equations of motion; Natural frequencies
- Concept of Normal Modes

Free Vibration of Two d.o.f. Systems

In the previous modules we discussed the free and forced vibration behavior of single d.o.f. systems. In this lecture, we will extend the discussion to two d.o.f systems. We will begin with free vibrations. We will be interested in determining how the system vibrates given some initial disturbance and left free to vibrate on its own. We will thus determine the natural frequencies of vibration etc. We had observed earlier that damping has marginal effect on natural frequency of single d.o.f system but primarily controls the resonance amplitudes. Thus we will consider undamped two d.o.f. system for our discussion.

Fig 10.1.1 A two d.o.f. spring mass sysytem

Consider a two d.o.f. spring-mass system as shown in Fig. 10.1.1 with two masses and three springs. To specify the configuration of the system, we need to specify the position of both the masses and hence we refer to this as a two d.o.f system.

Fig 10.1.2 Free body diagrams for a two d.o.f. system

The free body diagrams are shown in Fig. 10.1.2. Using Newton 's second Law, we write the equations of motion of each mass as follows:

$$\begin{aligned} m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) &= 0 \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3 x_2 &= 0 \end{aligned} \qquad 10.1.1$$

Using matrix notation, we can re-write as follows:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad 10.1.2$$

Thus we have two coupled, second order ordinary differential equations of motion with constant coefficients.

Observing the derivation of the equation of motion for the single d.o.f and two d.o.f system we notice that the procedure remains essentially the same viz.

- Step 1 : Consider the system in a displaced Configuration
- Step 2 : Draw Free Body diagrams
- Step 3 : Use Newton 's second Law to write the equation of motion.

We use the same procedure to develop the equations of motion even for multi-d.o.f systems.

Solution of equations of motion

Considering harmonic vibrations, let us assume

$$\begin{aligned} x_1(t) &= x_1 \sin \omega t \\ x_2(t) &= x_2 \sin \omega t \end{aligned} \qquad 10.1.3$$

Substituting in Eq (10.1.2), we get,

$$-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad 10.1.4$$

Rewriting this, we get,

$$\begin{bmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 + k_3 - m_1\omega^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad 10.1.5$$

For a non-trivial solution, we can write,

$$\text{determinant} \begin{bmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 + k_3 - m_2\omega^2 \end{bmatrix} = 0 \qquad 10.1.6$$

Therefore we have the frequency equation / Characteristic equation:

$$(k_1 + k_2 - m_1\omega^2)(k_2 + k_3 - m_2\omega^2) - (-k_2)(-k_2) = 0 \qquad 10.1.7$$

$$i.e.\ m_1 m_2 \left(\omega^2\right)^2 - [m_1(k_2 + k_3) + m_2(k_1 + k_2)]\,\omega^2 + [k_1k_2 + k_1k_3 + k_2k_3] = 0 \qquad 10.1.8$$

The “natural frequencies” of the system are obtained as the solutions of the characteristic frequency equation as follows:

$$\left(\omega_n^2\right)_{1,2} = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \quad 10.1.9$$

where $\alpha = m_1 m_2$

$$\beta = m_1(k_2 + k_3) + m_2(k_1 + k_2) \quad 10.1.10$$

$$\gamma = k_1 k_2 + k_1 k_3 + k_2 k_3$$

when $m_1 = m_2 = m$

$k_1 = k_2 = k_3 = k$

we get

$$\omega_{n1} = \sqrt{\frac{k}{m}}$$

$$\omega_{n2} = \sqrt{\frac{3k}{m}}$$

Concept of Normal Modes

Consider $\omega_{n_1} = \sqrt{\frac{k}{m}}$ and $\omega_{n_2} = \sqrt{\frac{3k}{m}}$ 10.1.11

If we substitute $\omega_n = \omega_{n1}$ in eq. (10.1.5), we get,

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad 10.1.10$$

Thus the ratio of the amplitudes of the two masses is:

$$\frac{X_1}{X_2} = 1 \quad 10.1.13$$

Similarly, if we substitute $\omega_n = \omega_{n2}$ in eq. (10.1.5), we get,

$$\begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad 10.1.14$$

Thus the ratio of the amplitudes of the two masses for this case is:

$$\frac{X_1}{X_2} = -1 \quad 10.1.15$$

Such synchronous motion of the masses with constant ratio of amplitudes is known as Normal Mode Vibration.

Physical Meaning of Normal Modes

We can attribute the following physical meaning to the mathematical solution obtained above – if we give initial conditions such that the two amplitudes are in the ratio given above (either eq. (10.1.13) or (10.1.15)) and leave the system free to vibrate on its own, it will continue to vibrate forever maintaining this ratio of amplitudes all the time. This free vibratory motion will be at the frequency ω_{n1} or ω_{n2} respectively. These are known as the natural modes of the system – two for a two d.o.f system and in general “n” for an “n” d.o.f. system. For the present case, these are depicted in Fig. 10.1.3

Fig 10.1.3 Depiction of Mode Shapes (Normal modes)

It must be appreciated that these are only ratios of amplitudes of the two masses and not absolute magnitudes of vibratory displacement. Thus they indicate a certain shape of vibrating system rather than any particular amplitudes and hence they are also known as Mode Shapes.

Physical Meaning of Normal Modes (contd….)

In the first mode of vibration, both the masses have the same amplitude and hence the middle spring is undeformed. Hence we can ignore the presence of this spring and each spring-mass system is operating independently. Hence the natural frequency is same as a simple spring-mass SDOF system i.e.

$$\omega_{n_1} = \sqrt{\frac{k}{m}} \qquad 10.1.16$$

In the second mode of vibration, the two masses are exactly out of phase i.e. the two ends of the intermediate spring move by the same amount in opposite directions. Thus the mid-point of the intermediate spring will be at rest and hence we can consider the intermediate spring to be cut into two and arrested at the middle as shown in Fig. 10.1.4 below. We know that when a spring is cut into two, its stiffness is doubled and when twosprings are in parallel, their stiffnesses add up. Thus the equivalent system is as shown in the figure. Thus the natural frequency is given by:

Fig 10.1.4 Interpretation of Second Normel Mode

$$\omega_{n_2} = \sqrt{\frac{k + 2k}{m}} = \sqrt{\frac{3k}{m}} \qquad 10.1.17$$

Which is same as what we got by solving the two d.o.f system equations.

Recap

In this lecture you have learnt the following.

- Procedure for developing the equations of motion is same for single or two d.o.f systems.
- A two d.o.f system is represented by two second order ordinary differential equations of motion.
- The two d.o.f system has two natural modes of vibration – these are the shapes of vibration with constant amplitude ratios for the two masses under synchronous motion at the system natural frequency. If the system is set to vibrate initially in either of these two modes, it will continue to vibrate forever in that mode at that natural frequency.

Congratulations, you have finished Lecture 1. To view the next lecture select it from the left hand side menu of the page.

Module 10 : Vibration of Two and Multidegree of freedom systems; Concept of Normal Mode; Free Vibration Problems and Determination of Ntural Frequencies; Forced Vibration Analysis; Vibration Absorbers; Approximate Methods - Dunkerley's Method and Holzer Method

Lecture 28 : Forced Vibration of two d.o.f. systems

Objectives

In this lecture you will learn the following

- Dynamic response of two degree of freedom system to sinusoidal exciation.
- Multiple Resonances.
- Concept of Vibration Absorber.

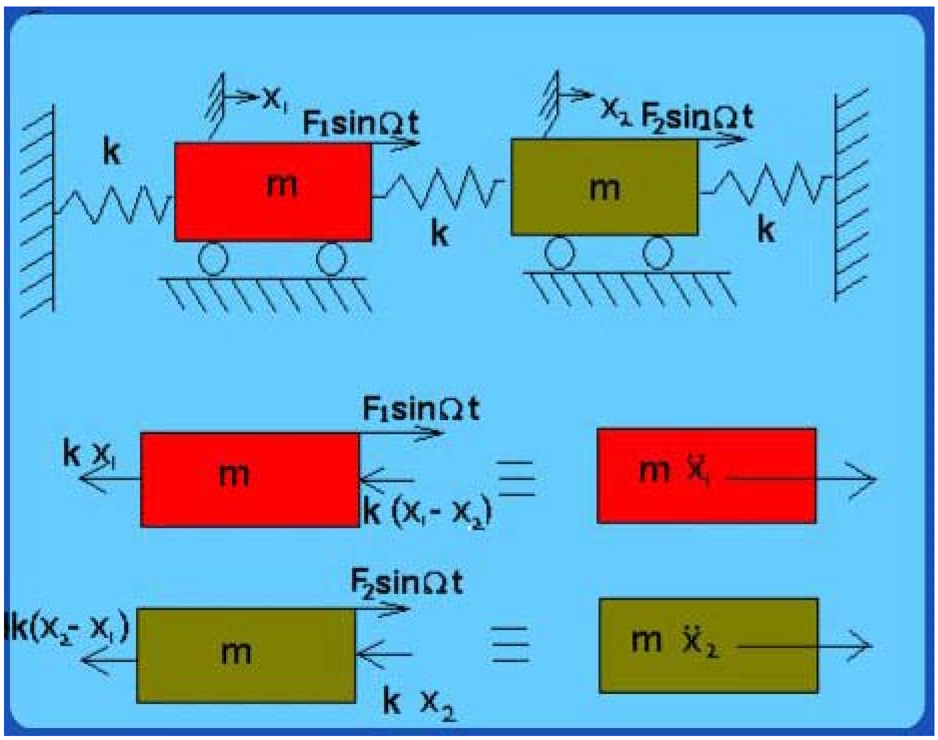

Figure 10.2.1

Consider the undamped two d.o.f system as we discussed in the previous lecture, shown in Fig. 10.2.1. We now study its response to harmonic excitation. Accordingly we have shown two sinusoidal forces acting on the masses. The free body diagrams are shown in the figure and the equations of motion can be readily written down based on Newton 's second Law as follows:

$$\begin{aligned} m\ddot{x}_1 + kx_1 + k(x_1 - x_2) &= F_1 \sin \Omega t \\ m\ddot{x}_2 + k(x_2 - x_1) + kx_2 &= F_1 \sin \Omega t \end{aligned} \qquad 10.2.1$$

Re-writing in matrix notation, we get,

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k_2 \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \sin \Omega t \qquad 10.2.2$$

Under sinusoidal excitation, the response is also sinusoidal at the same frequency as the excitation (as we discussed for single d.o.f case) and so we assume the vibratory movements of the two masses to be as follows:

$$x_1 = X_1 \sin \Omega t$$
$$x_2 = x_2 \sin \Omega t \qquad 10.2.3$$

Substituting in eqn 10.2.2 , we get,

$$\begin{bmatrix} 2k - m\Omega^2 & -k \\ -k & 2k - m\Omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad 10.2.4$$

The solution is readily obtained as follows:

$$x = X_0 \sin(\Omega t - \phi) \qquad 10.2.5$$

Typical response of a two d.o.f system to a sinusoidal excitation is shown in Fig.10.2.2

Fig 10.2.2 Typical Forced vibration response of a two d.o.f. system

As can be seen from eqn 10.2.5 as well as from Fig. 10.2.2 , we notice two resonance peaks i.e. when the forcing (i.e., driving or excitation) frequency matches with either of the natural frequencies, the amplitude of vibration of both the masses shoots upto infinit large values. Thus a two d.o.f. system exhibits two resonant frequencies and our design should ensure that the operating frequency is not near either of the resonant frequencies.

Undamped Vibration Absorber

An interesting practical application situation of a two d.o.f system is when one spring-mass sub-system is designed and used so as to absorb (i.e. suppress) the vibration of the main spring-mass system. Such a system is known as "Vibration Absorber" and we will now discuss this interesting example.

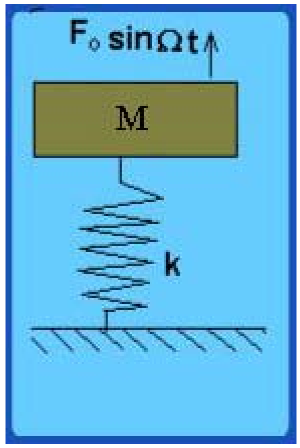

Fig 10.2.3 Undamped Main Spring-mass System

Consider a main spring-mass (K, M respectively) single d.o.f system (such as a machine on its foundation etc) as shown in Fig. 10.2.3 subjected to a forcing function F_0 Sin Ω . We have already studied such a system extensively and we know that it exhibits large amplitude vibrations near its resonance . We now attach another spring-mass system (k_a and m_a respectively) as shown in Fig. 10.2.4. The primary issue of interest is the following:

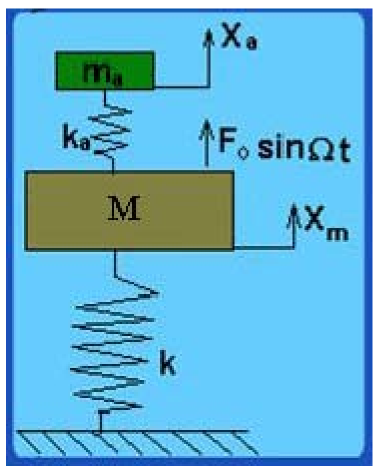

Fig 10.2.4 Main system with an undamped vibration absorber

Can we choose appropriate values for k_a and m_a such that the amplitude of vibration of the main mass, under this excitation, can be made zero?

The equation of motion is given by:

$$\begin{bmatrix} k+k_a & -k_a \\ -k_a & k_a \end{bmatrix} \begin{Bmatrix} x_M \\ x_a \end{Bmatrix} + \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \ddot{x}_M \\ \ddot{x}_a \end{Bmatrix} = \begin{Bmatrix} F_0 \sin \Omega t \\ 0 \end{Bmatrix} \qquad 10.2.6$$

$$\text{Let} \quad x_M = X_M \sin \Omega t$$
$$x_a = X_a \sin \Omega t \qquad 10.2.7$$

We get the vibratory response of the two masses to be:

$$\begin{Bmatrix} X_M \\ X_a \end{Bmatrix} = \frac{1}{[(k+k_a-\Omega^2 M)(k_a-\Omega^2 m_a)-k_a^2]} \begin{bmatrix} k_a-\Omega^2 m_a & k_a \\ k_a & k+k_a-\Omega^2 M \end{bmatrix} \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} \quad 10.2.9$$

In order to make $X_M = 0$ for all time, we have,

$$k_a - \Omega^2 m_a = 0$$

$$ie \quad \sqrt{\frac{k_a}{m_a}} = \Omega$$

Thus k_a and m_a are to be chosen such that the natural frequency of the absorber system is same as the excitation frequency i.e.

$$\omega_a = \sqrt{\frac{k_a}{m_a}} = \Omega$$

It is important to realize that the absorber system will work **ONLY WHEN** $\sqrt{\frac{k_a}{m_2}} = \Omega$

Since k_a and m_a are fixed values for a given absorber, it is therefore imperative that the excitation frequency does not change. So this type of absorber can work only for one frequency. Since the amplitude of vibration for the parent system is likely to be large only within the vicinity of its own natural frequency, we are usually interested in using the absorber when $\omega_n \approx \Omega$. This should not lead to the misconception that this is the governing equation for an absorber design. It must be clearly be remembered that, for the absorber to work, the only condition is that the absirber natual frequency must be equal to the excitation frequency.

If we have put an absorber and the excitation frequency changes even slightly, we could be in real big trouble as evident from the response of the system given in Fig. 10.2.5 below. Instead of the earlier one resonance peak we now have two frequencies (since it has now become a two d.o.f system) when the amplitude will be very large. Therefore it is imperative to provide sufficient amount of damping to limit the peak amplitudes.

Fig 10.2.5 Forced Vibration response of a system with vibration absorber

Recap

In this lecture you have learnt the following

- Developement of equation of motion for forced excitation of two degree of freedom system.
- Solution for the dynamic response.
- Criterion for the design of vibration absorber. $\omega_a = \sqrt{\frac{k_a}{m_a}} = \Omega$

Congratulations, you have finished Lecture 2. To view the next lecture select it from the left hand side menu of the page

Module 10 : Vibration of Two and Multidegree of freedom systems; Concept of Normal Mode; Free Vibration Problems and Determination of Ntural Frequencies; Forced Vibration Analysis; Vibration Absorbers; Approximate Methods - Dunkerley's Method and Holzer Method

Lecture 29 : Free Vibration of Multi-d.o.f systems.

Objectives

In this lecture you will learn the following

- Equation of motion of multi degree of freedom system.
- Matrix representation of the equation of motion.
- Appreciation of Solution methods.

It must be appreciated that any real life system is actually a continuous or distributed parameter system (i.e. infinitely many d.o.f). Hence to derive its equation of motion we need to consider a small (i.e., differential) element and draw the free body diagram and apply Newton 's second Law. The resulting equations of motion are partial differential equations and more complex than the simple ordinary differential equations we have been dealing with so far. Thus we are interested in modeling the real life system using lumped parameter models and ordinary differential equations. The accuracy of such models (i.e. how well they can model the behavior of the infinitely many d.o.f. real life system) improves as we increase the number of d.o.f. Thus we would like to develop mult-d.o.f lumped parameter models which still yield ordinary differential equations of motion – as many equations as the d.o.f . We would discuss these aspects in this lecture.

Derivation of Equations of Motion

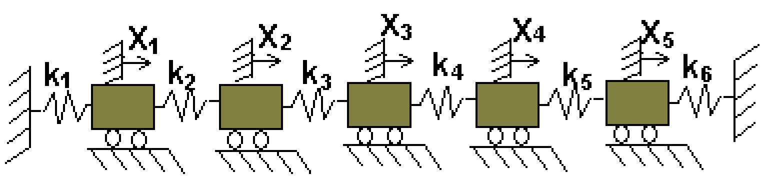

Fig 10.3.1 A Typical multi-d.o.f. system

Consider a typical multi-d.o.f system as shown in Fig. 10.3.1. As mentioned earlier our procedure to determine the equations of motion remains the same irrespective of the number of d.o.f of the system and is recalled to be:

Step 1 : Consider the system in a displaced Configuration

Step 2 : Draw Free Body diagrams

and Step 3 : Use Newton 's second Law to write the equation of motion

From the free body diagrams shown in Fig. 10.3.1, we get the equations of motion as follows:

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = F_1(t)$$

$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_1 x_1 = F_2(t) \qquad 10.3.1$$

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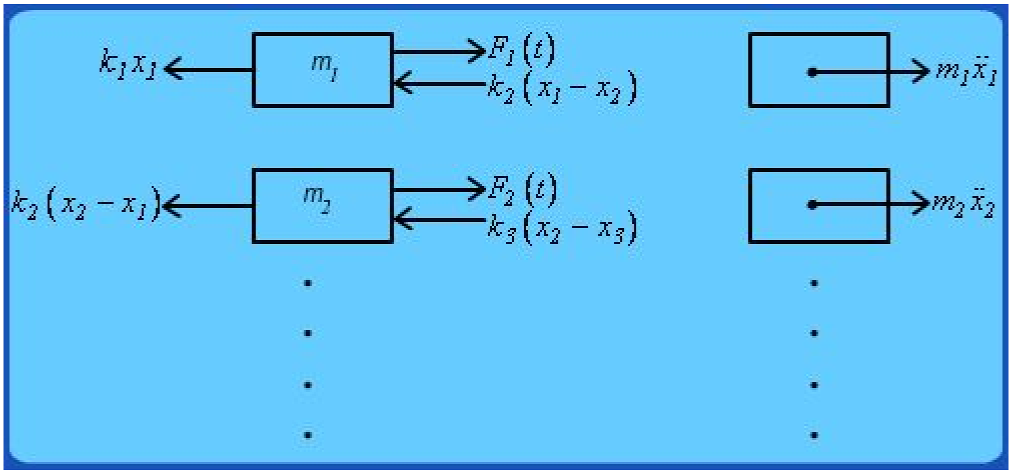

Rewriting the equations of motion in matrix notation, we get:

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 \\ 0 & 0 & 0 & 0 & m_5 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \\ \ddot{x}_5 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 & 0 \\ 0 & 0 & -k_4 & k_4+k_5 & -k_5 \\ 0 & 0 & 0 & -k_5 & k_5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \\ F_4(t) \\ F_5(t) \end{Bmatrix} \qquad 10.3.2$$

Or in compact form,

$$[M]\{\ddot{X}\} + [K]\{X\} = \{F\} \qquad 10.3.3$$

There are "n" equations of motion for an "n" d.o.f system. Correspondingly the mass and stiffness matrices ([M] and [K] respectively) are square matrices of size (n x n).

If we consider free vibrations and search for harmonic oscillations, $\{F\} = \{0\}$ and $\{X\} = \{X_0\}\ Sin\ \Omega t$. Substituting these in eqn 10.3.3 , we get,

$$-\omega^2[M]\{X\} + [K]\{X_0\} = \{0\} \qquad 10.3.4$$

ie

$$[[K] - \omega^2[M]]\ \{X_0\} = \{0\} \qquad 10.3.5$$

For a non-trivial solution to exist, we have the condition that the determinant of the coefficient matrix must vanish. Thus, we can write,

$$\left| [K] - \omega^2 [M] \right| = 0 \qquad 10.3.6$$

In principle this (n x n) determinant can be expanded by row or column method and we can write the characteristic equation (or frequency equation) in terms of ω^2 , solution of which yields the "n" natural frequencies of the "n" d.o.f. system just as we did for the two d.o.f system case.

We can substitute the values of ω_n in eqn 10.3. and derive a relation between the amplitudes of various masses yielding us the corresponding normal mode shape. Typical mode shapes are schematically depicted in

Fig. 10.3.2 for a d.o.f system.

Recap

In this lecture you have learnt the following

- Development of equation of motion for forced excitation of multi degree of freedom system.

- Bandedness of the stiffness matrix $\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 \\ 0 & 0 & 0 & 0 & m_5 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \\ \ddot{x}_5 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & k_4 & 0 \\ 0 & 0 & -k_4 & k_4+k_5 & -k_5 \\ 0 & 0 & 0 & -k_5 & k_5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \\ F_4(t) \\ F_5(t) \end{Bmatrix}$

- Solution of $\left| [K] - \omega^2 [M] \right| = 0$ gives us the eigen values of the system which are nothing but the natural frequencies of the system and finding the eigen vecors gives us the mode shapes of the system.

Congratulations, you have finished Lecture 3 To view the next lecture select it from the left hand side menu of the page.

Module 10 : Vibration of Two and Multidegree of freedom systems; Concept of Normal Mode; Free Vibration Problems and Determination of Ntural Frequencies; Forced Vibration Analysis; Vibration Absorbers; Approximate Methods - Dunkerley's Method and Holzer Method

Lecture 30 : Approximate Methods (Dunkerly's Method)

Objectives

In this lecture you will learn the following

- Dunkerly's method of finding natural frequency of multi- degree of freedom system

We observed in the previous lecture that determination of all the natural frequencies of a typical multi d.o.f. system is quite complex. Several approximate methods such as Dunkerly's method enable us to get a reasonably good estimate of the fundamental frequency of a multi d.o.f. system.

Basic idea of Dunkerly's method

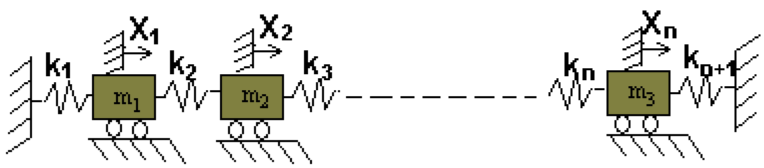

Fig 10.4.1 A typical multi d.o.f. system

Consider a typical multi d.o.f. system as shown in fig 10.4.1

Dunkerly's approximation to the fundamental frequency of this system can be obatined in two steps:

Step1: Calculate natural frequency of all the modified systems shown in Fig 10.4.2. These modified systems are obatined by considering one mass/inertia at a time. Let these frequencies be $\omega_{11}, \omega_{12}, \omega_{13}, \omega_{14}, \ldots\ldots$

Step2: Dunkerly's estimate of fundamental frequency is now given as:

$$\frac{1}{\omega_{1d}^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{12}^2} + \frac{1}{\omega_{13}^2} + \frac{1}{\omega_{14}^2} + \cdots\cdots$$

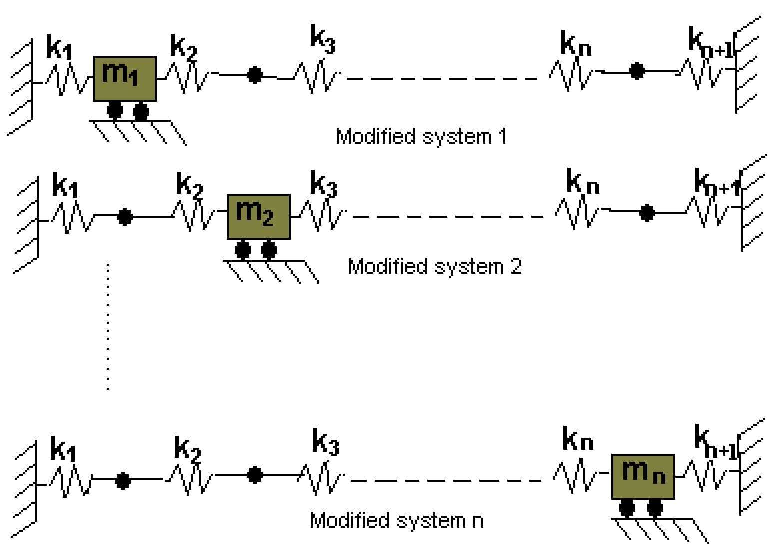

Fig 10.4.2 Modified system considered in Dunkerly's Method

Explanation of Basic Idea

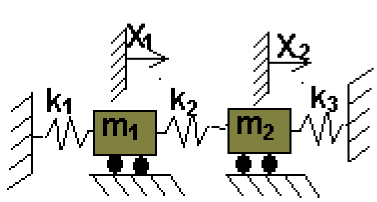

Fig 10.4.3 A Typical two d.o.f. example

Consider a typical two d.o.f. system as shown in Fig 10.4.3 and the equations of motion are given as:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad 10.4.2$$

For harmonic vibration, we can write:

$$\begin{aligned} x_1 &= X_1 \sin \omega t \\ x_2 &= X_2 \sin \omega t \end{aligned} \qquad 10.4.3$$

Thus,

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}=\omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \qquad 10.4.4$$

Inverting the stiffness matrix and re-writing the equations

$$\frac{1}{\omega^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}=\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \qquad 10.4.5$$

$$\text{i.e.,} \quad \begin{bmatrix} \frac{1}{\omega^2}-c_{11}m_1 & -c_{12}m_2 \\ -c_{21}m_1 & \frac{1}{\omega^2}-c_{22}m_2 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad 10.4.6$$

The equation characteristic can be readily obtained by expanding the determinant as follows:

$$\left(\frac{1}{\omega^2}\right)^2-\left(\frac{1}{\omega^2}\right)(c_{11}m_1+c_{22}m_2)+(m_1m_2)(c_{11}c_{22}-c_{12}\ c_{21}) = 0 \qquad 10.4.7$$

As this is a two d.o.f. system, it is expected to have two natural frequencies viz ω_1 and ω_2 ., Thus we can write Eq. 10.4.7 as:

$$\left(\frac{1}{\omega^2}-\frac{1}{\omega_1^2}\right)-\left(\frac{1}{\omega^2}-\frac{1}{\omega_2^2}\right)=\left(\frac{1}{\omega^2}\right)^2-\left(\frac{1}{\omega^2}\right)(c_{11}m_1+c_{22}m_2)+(m_1m_2)(c_{11}c_{22}-c_{12}m_{21}) \qquad 10.4.8$$

Comparing coefficients of like terms on both sides, we have:

$$\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}=(c_{11}m_1+c_{22}m_2) \qquad 10.4.9$$

$$\left(\frac{1}{\omega_1^2}\right)\left(\frac{1}{\omega_2^2}\right) = m_1m_2(c_{11}+c_{22}-c_{12}m_{21}) \qquad 10.4.10$$

It would appear that these two equations (10.4.9-10) can be solved exactly for and . While this is true for this simpleexample, we can't practically implement such a scheme for an n-d.o.f system, as it would mean similar computational effort as solving the original problem itself. However, we could get an approximate estimate for the fundamental frequency.

If $\omega_2 >> \omega_1$, then we can approximately write from Eq. (10.4.9),

$$\frac{1}{\omega_1^2} \approx c_{11}m_1+c_{22}m_2 \qquad 10.4.11$$

Let us now study the meaning of and . It is easily verified that

$c_{11}m_1 \qquad c_{22}m_2$

$$c_{11} = \frac{k_2 + k_3}{k_1k_2 + k_2k_3 + k_3k_1} \qquad 10.4.10$$

$$c_{22} = \frac{k_1 + k_2}{k_1k_2 + k_2k_3 + k_3k_1} \qquad 10.4.13$$

These can be readily verified to be the reciprocal of the equivalent stiffness values for the modified systems shown in Fig. 10.4.4.

Thus, we can write:

$$\frac{1}{\omega_1^2} \approx \frac{m_1}{k_{eq1}} + \frac{m_2}{k_{eq2}} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{12}^2} \qquad 10.4.14$$

Practical Implication

The most crucial assumption in Dunkerley's method is when we go from Eq. (10.4.9) to Eq.(10.4.11).

We are assuming that the second frequency and hence other higher frequencies are far higher than the fundamental frequency. In other words, when the system modes are well separated, Dunkerley's method works well.

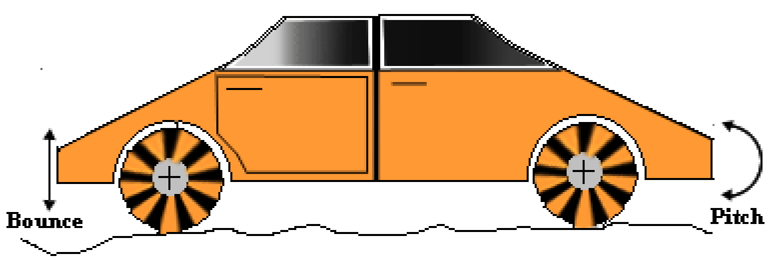

Fig 10.4.5 Bounce and Pitch modes of an automobile

For a typical automobile, for example, the frequencies of bounce and pitch are both in the range of 1-2Hz and the system has several frequencies within the 0-15 Hz range. Hence Dunkerley's approximation is unlikely to give good estimates.

Recap

In this lecture you have learnt the following

Dunkerly's method of determining approximate by the fundamental natural frequencies of system

- $\frac{1}{\omega_{1d}^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{12}^2} + \frac{1}{\omega_{13}^2} + \frac{1}{\omega_{14}^2} + \cdots\cdots$
- Concept of Dunkerly's method.
- Practical implication of Dunkerly's method.

Congratulations, you have finished Lecture 4. To view the next lecture select it from the left hand side menu of the page

Module 10 : Vibration of Two and Multidegree of freedom systems; Concept of Normal Mode; Free Vibration Problems and Determination of Ntural Frequencies; Forced Vibration Analysis; Vibration Absorbers; Approximate Methods - Dunkerley's Method and Holzer Method

Lecture 31 : Approximate Methods (Holzer's Method)

Objectives

In this lecture you will learn the following

- Holzer's method of finding natural frequency of a multi-degree of freedom system

Holzer's Method

This method is an iterative method and can be used to determine any number of frequencies for a multi-d.o.f system. Consider a typical multi-rotor system as shown in Fig. 12.5.1.

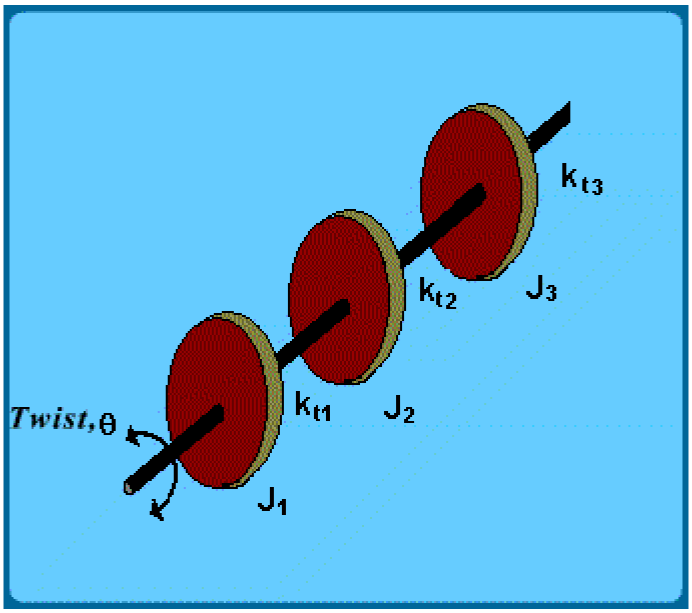

Fig 12.5.1 Typical multi-rotor system

The equations of motion for free vibration can be readily written as follows:

$$J_1\ddot{\theta}_1 + k_{t_1}(\theta_1 - \theta_2) = 0$$
$$J_2\ddot{\theta}_2 + k_{t_1}(\theta_2 - \theta_1) + k_{t_2}(\theta_2 - \theta_3) = 0 \qquad 12.5.1$$
$$J_3\ddot{\theta}_3 + k_{t_2}(\theta_3 - \theta_2) + k_{t_3}(\theta_3 - \theta_4) = 0$$

For harmonic vibration, we assume

$$\theta_i = \Theta_i \sin \omega t \qquad 12.5.2$$

Thus:

$$-\omega^2 J_1 \Theta_1 + k_{t_1}(\Theta_1 - \Theta_2) = 0$$
$$-\omega^2 J_2 \Theta_2 + k_{t_1}(\Theta_2 - \Theta_1) + k_{t_2}(\Theta_2 - \Theta_3) = 0 \qquad 12.5.3$$
$$-\omega^2 J_3 \Theta_3 + k_{t_2}(\Theta_3 - \Theta_2) + k_{t_3}(\Theta_3 - \Theta_4) = 0$$

Summing up all the equations of motion, we get:

12.5.4

$$\sum_{i=1}^{n} J_i \Theta_i \omega^2 = 0$$

This is a condition to be satisfied by the natural frequency of the freely vibrating system.

Holzer's method consists of the following iterative steps:

Step 1: Assume a trial frequency $\omega_n \approx \omega_{try}$

Step 2: Assume the first generalized coordinate $\Theta_1 = 1$ say

Step 3: Compute the other d.o.f. using the equations of motion as follows:

$$\Theta_2 = \Theta_1 - \frac{\omega_{try}^2 J_1 \Theta_1}{k_{t_1}}$$

$$\Theta_3 = \Theta_2 - \frac{\omega_{try}^2 (J_1 \Theta_1 + J_2 \Theta_2)}{k_{t_2}} \quad \mathbf{10.5.5}$$

Step 4: Sum up and verify if Eq. (10.5.4) is satisfied to the prescribed degree of accuracy.

If Yes, the trial frequency is a natural frequency of the system. If not, redo the steps with a different trial frequency.

In order to reduce the computations, therefore one needs to start with a good trial frequency and have a good method of choosing the next trial frequency to converge fast.

Two trial frequencies are found by trial and error such that $\sum J_i \Theta_i \omega_{try}^2$ is a small positive and negative number respectively than the mean of these two trial frequencies(i.e. bisection method) will give a good estimate of for which $\sum J_i \Theta_i \omega_{try}^2 \approx 0$.

Holzer's method can be readily programmed for computer based calculations

Recap

In this lecture you have learnt the following

- Holzer method of determining natural frequencies based on $\sum_{i=1}^{n} J_i \Theta_i \omega^2 = 0$

Module 11 : Free Vibration of Elastic Bodies; Longitudinal Vibration of Bars; Transverse Vibration of Beams;
Torsional Vibration of Shaft; Approximate Methods – Rayleigh's Method and Rayleigh-Ritz Method.

Lecture 33 : Longitudinal vibration of bars

Objectives

In this lecture you will learn the following

- Derivation of the governing partial differential equation for longitudinal vibration of bars
- Solution of the governing equations in terms of the natural frequencies and mode shapes

Consider a long, slender bar as shown in Fig. 11.2.1. We aim to study its vibration behavior in the longitudinal (i.e., axial) direction. Recall that when it undergoes axial deformation, we assume that the whole cross-section moves together by the same displacement. Thus the axial deformation "u" could vary from point to point along the length of the bar (i.e., u is a function of x) but all points in the cross-section at a given axial location (i.e., x) have the same displacement. Of course, the axial displacement at any given point varies with time as the system vibrates. Thus we write u(x,t).

When a rod undergoes deformation u, the strain at any point is given by:

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad (11.2.1)$$

Assuming linear elastic homogeneous material obeying Hooke's Law, we have:

$$\sigma_x = E\,\varepsilon_x = E\frac{\partial u}{\partial x} \qquad (11.2.2)$$

When a cross-section of area A is subjected to this stress, the axial force is given by:

$$F = A\sigma_x = AE\frac{\partial u}{\partial x} \qquad (11.2.3)$$

Since axial displacement "u" is a function of x, all these quantities (the strain, the stress and the internal force in the cross-section) are all dependent on x and vary from point to point along the length of the bar.

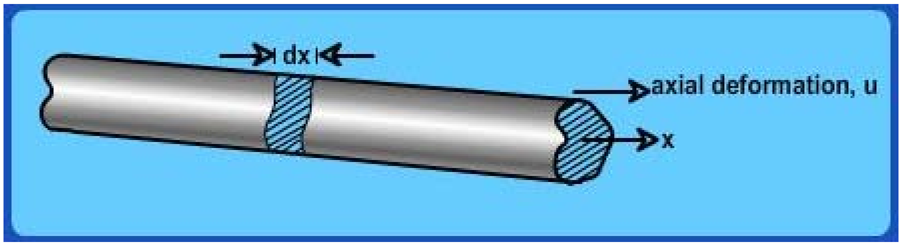

Figure 11.2.1

Contd...

Recall that the first order approximation to the Taylor 's series expansion of a function f(x) in the neighborhood of x is given by:

$$f(x+h) \approx f(x) + h\frac{df}{dx} \qquad (11.2.4)$$

With this brief background, consider the free body diagram of a differential element of length (dx) shown in Fig. 11.2.2. From Newton 's second law, the algebraic sum of all the forces in the axial direction must equal mass time acceleration. Thus we can write:

$$F_x + \left(\frac{\partial F_x}{\partial x}\right)(dx) - F_x = \left(\frac{\partial F_x}{\partial x}\right)(dx) = \rho A(dx)\frac{\partial^2 u}{\partial t^2} \qquad (11.2.5)$$

Substituting from Eq. (11.2.3) in Eq. (11.2.5), and assuming that the area of cross-section and Young's modulus are constant, we get:

$$AE\frac{\partial^2 u}{\partial x^2} = \rho A\frac{\partial^2 u}{\partial t^2} \qquad (11.2.6)$$

Eq. (11.2.6) can be re-written as:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad (11.2.7)$$

where $c = \sqrt{E/\rho}$, the wave speed i.e. the speed of sound wave (acoustic wave) propagation in that medium.

Particular solutions can be obtained for this wave equation when the boundary conditions are specified, for example the left end of the rod may be fixed (i.e. u(0,t)=0) etc. We will illustrate this on one set of boundary conditions here.

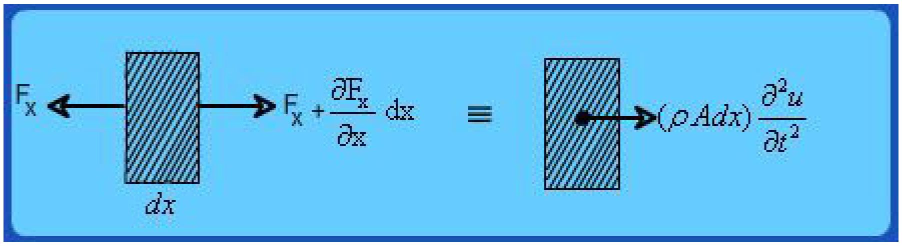

Figure 11.2.2

Example:

Let the left end of the rod be fixed and the right end be free i.e., no force on right end. Thus we get:

$$u(0,t) = 0; \quad \frac{\partial u}{\partial x}(l,t) = 0 \qquad (11.2.8)$$

We can use the method of separation of variables i.e.,

$$u(x,t) = U(x)\,T(t) \qquad (11.2.9)$$

Substituting in eq. (11.2.7), and re-arranging the terms, we get:

$$\frac{1}{T(t)}\frac{d^2T}{dt^2} = c^2\frac{1}{U(x)}\frac{d^2U}{dx^2} \qquad (11.2.10)$$

Contd...

Since the left hand side is only a function of time and right hand side is only a function of spatial coordinate "x", each of them must be equal to a constant. Let this constant be $-\omega^2$. Thus we can write:

$$\frac{1}{T(t)}\frac{d^2T}{dt^2} = -\omega^2 \qquad (11.2.11)$$

$$c^2\frac{1}{U(x)}\frac{d^2U}{dx^2} = -\omega^2 \qquad (11.2.12)$$

Thus, we get,

$$T(t) = A\ Sin\ (\omega t) + B\ Cos\ (\omega t) \qquad (11.2.11)$$

$$U(x) = C\ Sin\ (\omega x/c) + D\ Cos\ (\omega x/c) \qquad (11.2.14)$$

Boundary condition that U(0) = 0 at all times requires that D = 0. The second boundary condition requires that:

$$Cos\ (\omega L/c) = 0 \qquad (11.2.15)$$

i.e., $\omega L/c = \frac{n\pi}{2}$, for n = 1, 3, 5, ...

Thus harmonic vibration takes place at discrete frequencies called the natural frequencies of the system. The natural frequencies of the clamped-free bar under axial vibration are:

$$\boxed{\omega^2 = n^2\frac{\pi^2 E}{4\rho L^2}} \quad n = 1, 3, 5, ... \qquad (11.2.16)$$

The corresponding deformation shapes are given by:

$$u(x,t) = C\ Sin\ (\omega x/c)\ [A\ Sin\ (\omega t) + B\ Cos\ (\omega t)] \qquad (11.2.17)$$

The constants A and B are determined using the prescribed initial conditions. The "shape" of vibratory displacement varies sinusoidally along the length of the bar. These are called the mode shapes.

Recap

In this lecture you have learnt the following.

- Developing governing partial differential equations for longitudinal vibration of rods
- Obtaining the solutions to the free vibration problem
- Natural frequencies and mode shapes of axially vibrating rods

Module 11 : Free Vibration of Elastic Bodies; Longitudinal Vibration of Bars; Transverse Vibration of Beams;
Torsional Vibration of Shaft; Approximate Methods – Rayleigh's Method and Rayleigh-Ritz Method.

Lecture 34 : Torsional Vibration of Shafts

Objectives

In this lecture you will learn the following

- Significance of torsional vibrations in shafts
- Derivation of governing partial differential equation
- Natural frequencies and mode shapes for torsional vibrations

Shafts transmit power and in the process, are subjected to time-varying torques. For example an IC Engine crankshaft is subjected to pulsating torques as we have discussed in an earlier module. Such fluctuating torques set-up vibratory motion. However the shaft is already undergoing rotation. Thus the torsional (elastic twisting and untwisting) vibration is superposed on this rigid body rotation, making it somewhat complex to visualize. When subjected to excessive vibrations for a sufficiently long time, the shafts may fail and thus they need to be analyzed carefully for torsional vibrations. While most of the real-life shafts may be quite complex in shape, we begin our discussion with a simple, uniform cross-section circular shaft as shown in Figure 11.3.1.

Consider a small elemental length " Δx ". Under the action of the torque, let the left end rotate by ψ and the right end by $\psi + \Delta\psi$. For small deformations, the shear strain is given by:

$$\gamma = \lim_{\Delta x \to 0} \frac{CC'}{CD} = \lim_{\Delta x \to 0} \frac{r\,\Delta\psi}{\Delta x} = r\frac{d\psi}{dx} \qquad (11.3.1)$$

The shear stress is given by:

$$\tau = G\gamma = Gr\frac{d\psi}{dx} \qquad (11.3.2)$$

wherein $d\psi/dx$ represents rate of twist or angle of twist per unit length.

The shear stress acting on an elemental area "dA" at a radial distance "r" causes an elemental torque "dT" as shown in Fig. 11.3.2:

$$dT = r\,(\tau\, dA) \qquad (11.3.3)$$

Integrating over the whole area, the total twisting moment is obtained as:

$$T = \iint r\,\tau\, dA = \iint G\frac{d\psi}{dx} r^2 dA = G\frac{d\psi}{dx}\iint r^2 dA = GI_p\frac{d\psi}{dx} \qquad (11.3.4)$$

Considering a slice of length "dx" as shown in Fig. 11.3.3, from equilibrium considerations (Newton 's second law), we can write:

$$\left(T+\frac{\partial T}{\partial x}dx\right)-T=\left(\rho I_p\, dx\right)\frac{\partial^2\psi}{\partial t^2} \tag{11.3.5}$$

Substituting from eq. (11.3.4), we get:

$$\frac{\partial^2\psi}{\partial t^2}=\frac{G}{\rho}\frac{\partial^2\psi}{\partial x^2}=c^2\frac{\partial^2\psi}{\partial x^2} \tag{11.3.6}$$

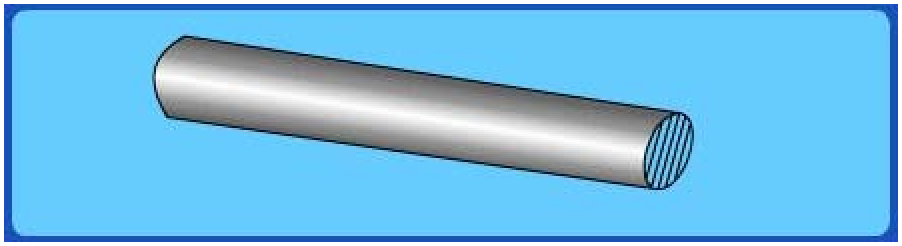

Figure 11.3.1

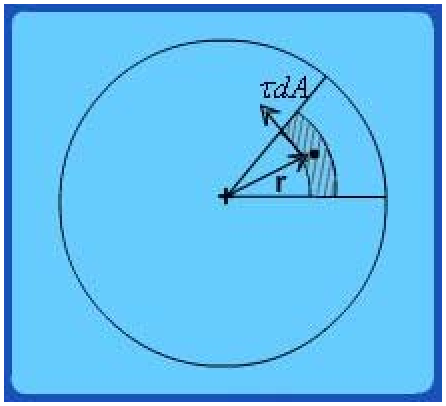

Figure 11.3.2

This is essentially similar to eq. (11.2.7), the wave equation for axial vibrations of rods. In both the cases, it is important to note that the governing equation does not contain any terms pertaining to area of cross-section or polar moment of area of cross-section etc. As long as the cross-section is uniform (and circular for torsion case), these equations can be used. The solution procedure also proceeds in a very similar fashion.

A typical multi-cylinder IC engine crankshaft is shown in Fig. 11.3.3 and it is readily observed that it deviates significantly from the idea uniform cross-section shaft we assumed so far. An equivalent uniform diameter shaft can be drawn-up as shown in Fig. 11.3.4 and the above equations can be used for its torsional dynamics. Such an analysis will help us determine the natural frequencies approximately. A more detailed analysis will require a full three dimensional finite element model as shown in Fig. 11.3.5. Finite Element Method, P.Seshu, Prentice Hall of India,2006.

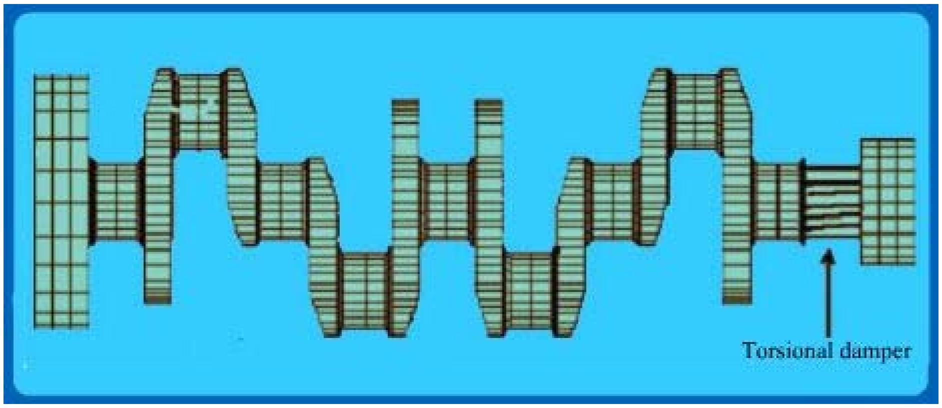

Fig. 11.3.3 A typical multi-cylinder engine crankshaft

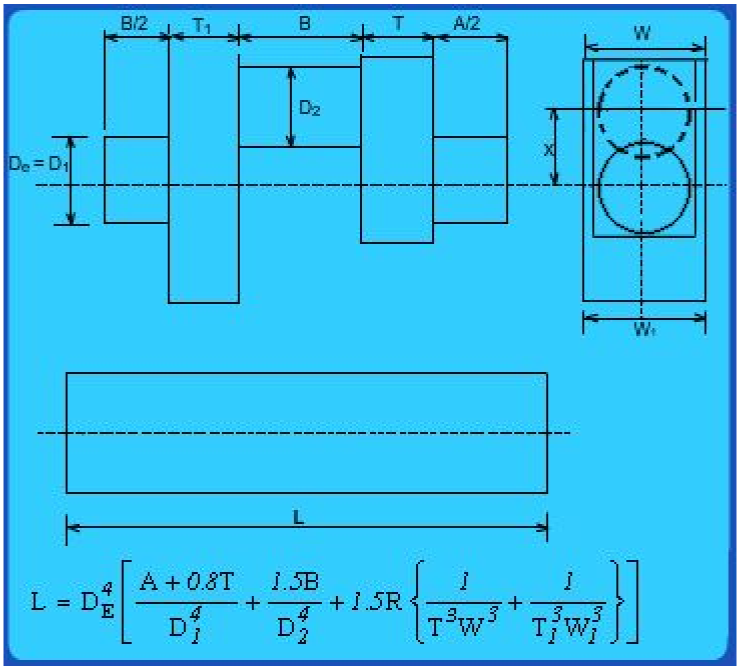

Fig. 11.3.4 Equivalent Shaft for Crankt

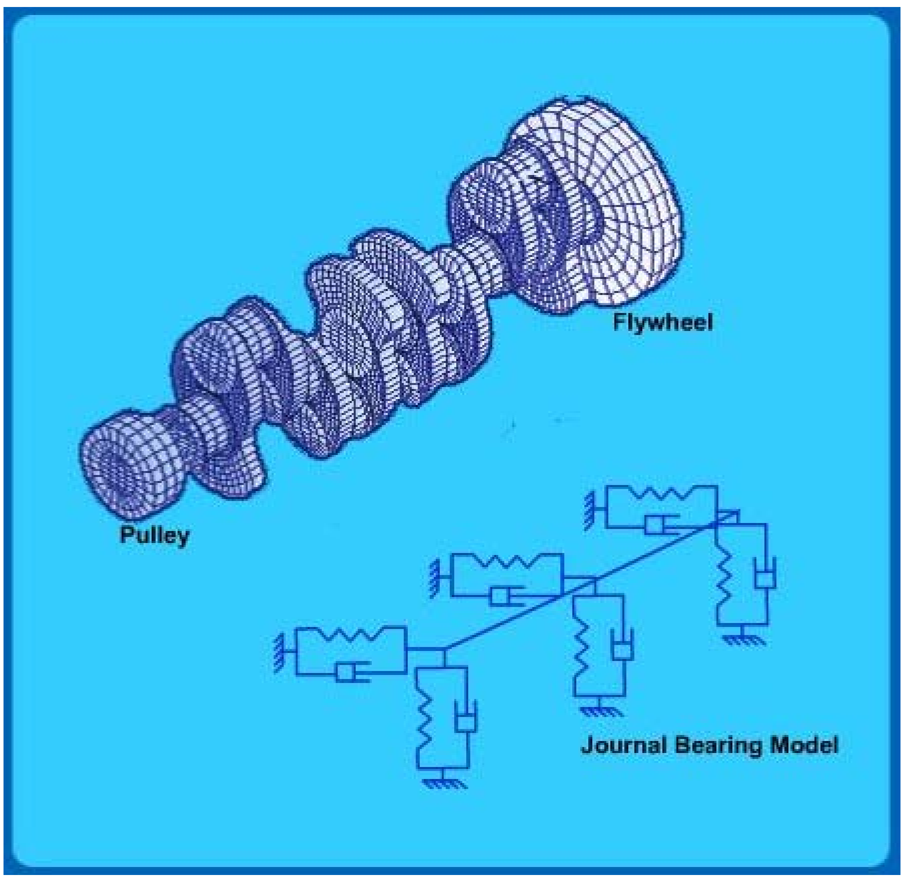

Fig. 11.3.5 Representative 3-D finite element model of crank shaft

Recap

In this lecture you have learnt the following.

- Torsional vibrations of uniform circular cross-section shafting

- Derivation of the governing equation

- Simplifications required to model real-life crankshafts

Module 11 : Free Vibration of Elastic Bodies; Longitudinal Vibration of Bars; Transverse Vibration of Beams;
Torsional Vibration of Shaft; Approximate Methods – Rayleigh's Method and Rayleigh-Ritz Method.

Lecture 35 : Transverse Vibration of Beams

Objectives

In this lecture you will learn the following

- Significance of transverse vibrations of beams.
- Derivation of governing partial differential.
- Natural frequencies and mode shapes.

So far we studied the axial and torsional vibrations of long, prismatic members. We will now study their transverse vibrations. We will assume that we could use the conventional Euler-Bernoulli beam model. This would imply that we invoke several assumptions. We assume that the beam is originally straight and uniform (in terms of both cross-section and material properties) along its length. We assume that the beam cross-section is symmetrical about the plane of loading as shown in Fig. 11.4.1 and the deformation is restricted to this plane of symmetry. We ignore any contribution of the shear force in the cross-section to the transverse deformation. The geometry of deformation is assumed to be such that plane cross-sections which are originally straight and normal to the “neutral axis” remain so even after bending. We assume that the longitudinal fibers are free to expand or contract in the lateral direction.

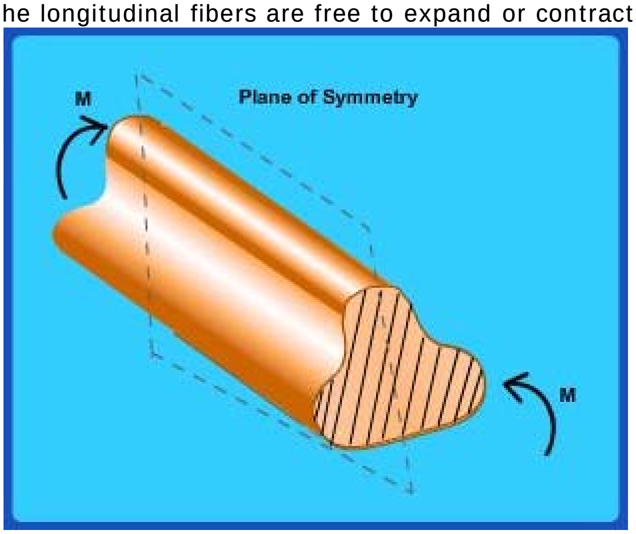

Fig. 11.4.1

When such a beam undergoes time dependent transverse deformations, we can draw a free body diagram of an elemental length as shown in Fig. 11.4.2 from which we can write:

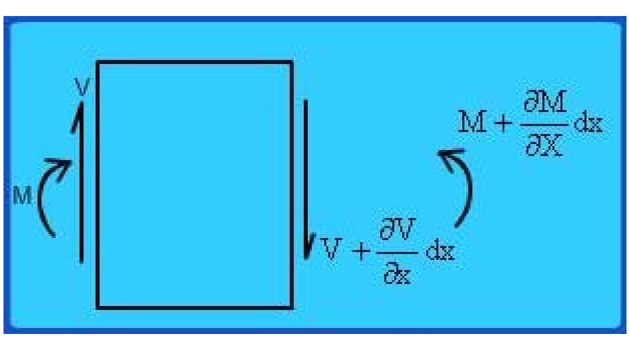

Fig. 11.4.2

$$-\left(V+\frac{\partial V}{\partial x}dx\right)+V=(\rho A dx)\frac{\partial^2 w}{\partial t^2} \qquad (11.4.1)$$

Where V is the shear force, ρ is the density, A is the cross-sectional area and w is the transverse displacement. However the shear force is related to the deformations by:

$$V = EI\,\frac{\partial^3 w}{\partial x^3} \qquad (11.4.2)$$

Thus we have:

$$EI\,\frac{\partial^4 w}{\partial x^4}+\rho A\,\frac{\partial^2 w}{\partial t^2}=0 \qquad (11.4.3)$$

Let $c^2 = {EI}/{\rho A}$. Then we can write:

$$\frac{\partial^2 w}{\partial t^2} = -c^2\,\frac{\partial^4 w}{\partial x^4} \qquad (11.4.4)$$

Contd....

This is the governing equation for free transverse vibration of a beam. The solution proceeds on lines similar to the case of axial vibrations of rods. Using the variables separable method, we can write:

$$w(x,t) = X(x)\,T(t) \qquad (11.4.5)$$

Substituting in (11.4.4), we get:

$$\frac{1}{T}\frac{d^2T}{dt^2} = -c^2\,\frac{1}{X}\frac{d^4X}{dx^4} \qquad (11.4.6)$$

Since the LHS is only a function of time and the RHS is only a function of spatial coordinate, then for (11.4.6) to hold good, each side should be equal to a constant, say $-\omega^2$. We get:

$$\frac{d^2T}{dt^2} = -\omega^2 T$$
$$\frac{d^4X}{dx^4} = \beta^4\,X \qquad (11.4.7)$$

where $\beta^4 = {\omega^2}/{c^2} = \rho A\,{\omega^2}/{EI}$. Thus:

$$T(t) = \alpha_1\cos\omega t + \alpha_2\sin\omega t$$
$$X(x) = \alpha_3\cos\beta x + \alpha_4\sin\beta x + \alpha_5\cosh\beta x + \alpha_6\sinh\beta x \qquad (11.4.8)$$

The coefficients in the above equation depend on the boundary/initial conditions. Let us consider a simple boundary condition viz., a beam simply supported at both ends. Thus we have:

$$w(0,t)=w(l,t)=0$$
$$\frac{\partial^2 w(0,t)}{\partial x^2}=\frac{\partial^2 w(l,t)}{\partial x^2}=0 \qquad (11.4.9)$$

We observe that since this is a fourth order differential equation in x, we have four boundary conditions. From these boundary conditions, we can write:

$$\alpha_3+\alpha_5=0$$
$$-\alpha_3+\alpha_5=0 \qquad (11.4.10)$$

From which we can write that $\alpha_3=\alpha_5=0$. Similarly we can write:

$$\alpha_4\,Sin\,\beta l+\alpha_6\,Sinh\,\beta l=0$$
$$-\alpha_4\,Sin\,\beta l+\alpha_6\,Sinh\,\beta l=0 \qquad (11.4.11)$$

From (11.4.11), we can write ($\alpha_6=0$ and since α_4 can not also be zero):

Sin (βl) = 0 (11.4.12)

i.e., βl = n π, *n = 1,2,3...*

Contd....

Therefore, the natural frequencies of free vibration viz., ω are given by:

$$\omega=n^2\pi^2\sqrt{\frac{EI}{\rho A l^4}} \qquad (11.4.11)$$

Thus, for a given beam, the successive natural frequencies are in the ratio 1, 4, 9, 16, ….

Corresponding mode shapes of vibration are given by:

$$X(x)=\alpha_4\,Sin\,\frac{i\pi x}{l} \qquad (11.4.14)$$

Typical mode shapes are shown in Fig. 11.4.3.

Fig. 11.4.3 Typical mode shapes of vibration of a simply supported beam

It is observed that the constant α_4 remains undetermined and X(x) remains the shape of vibration rather than any particular amplitude.

Beams with different boundary conditions will have different natural frequencies and mode shapes.

Contd....

Recap

In this lecture you have learnt the following.

- Transverse vibrations of beams
- Derivation of governing partial differential equation
- Solution for Natural frequencies and mode shapes for simple beams

Module 12 : Instruments for Dynamic Measurements

Lecture 36 : Vibration Measurements

Objectives

In this lecture you will learn the following

- Review of response of systems with Base Excitation
- Working principle of seismograph and accelerometers
- Selection and use of accelerometers
- Time vs Frequency Domain
- Fiber-optic and Laser based non-contact vibration instruments

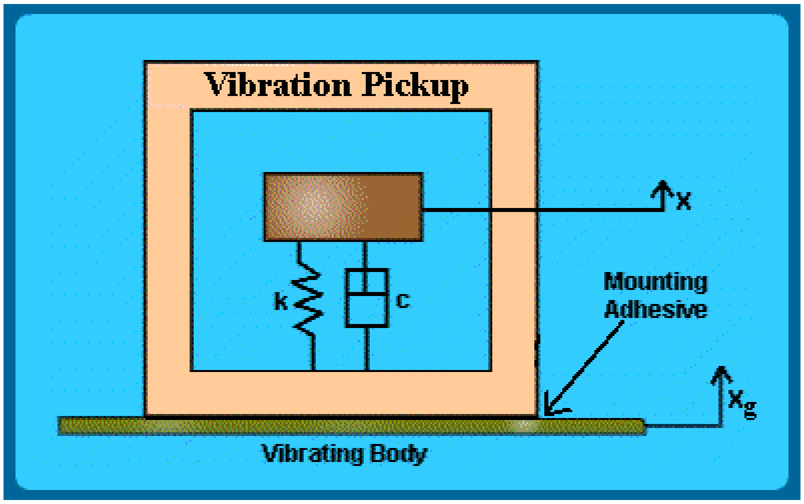

Fig 12.1.1Contact Type Vibration Instrument

A 'contact' type vibration measuring instrument is typically mounted on the system whose vibration is to be measured (as shown in Fig. 12.1.1) and thus the vibrating structure transmits the excitation to the base of the instrument. In order to understand the operating principle of vibration instruments therefore we will first review the analysis of systems subjected to base excitation.

It must be clear that any contact-type instrument acting as a point mass, implicitly affects the vibration of the system onto which it is mounted. Thus non-contact type instruments are also of significant interest to the vibration analyst. These are typically optics based instruments such as a fiber optic probe or a laser Doppler vibrometer etc. We will discuss these later on.

Review of systems with base excitation

The physical system under consideration and the corresponding free body diagram are given in Fig. 12.1.2.

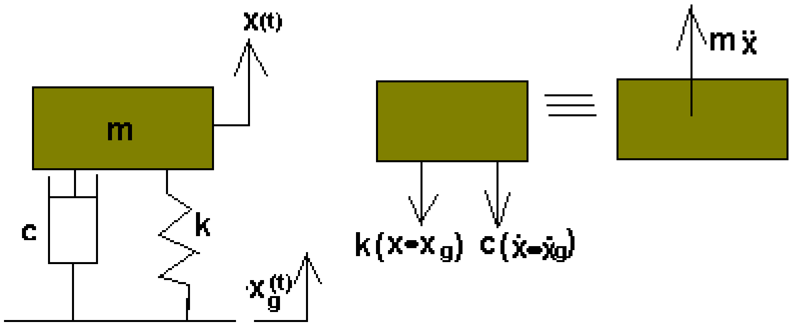

Fig 12.1.2 Free Body diagram of Physical system

Any measuring instrument on a shaking ground can measure only the relative motion of the mass with respect to the ground. If we introduce z = x – xg as the relative motion coordinate, the governing equation in terms of z can be given as:

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{x}_g \qquad 12.1.1$$

We have derived an expression for the steady state response of the mass when xg = Xg sin(Ω t) as:

$$Z_0 = \frac{\left(\frac{\Omega}{\omega_n}\right)^2 X_g}{\sqrt{\left[1-\left(\frac{\Omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\Omega}{\omega_n}\right)^2}} \qquad 12.1.2$$

Variation of z (i.e. motion of the mass relative to the base) with respect to the forcing frequency is plotted in Fig. 12.1.3. It is observed that at lower frequencies the relative motion of the mass tends to zero while at high frequencies the magnification factor tends to unity.

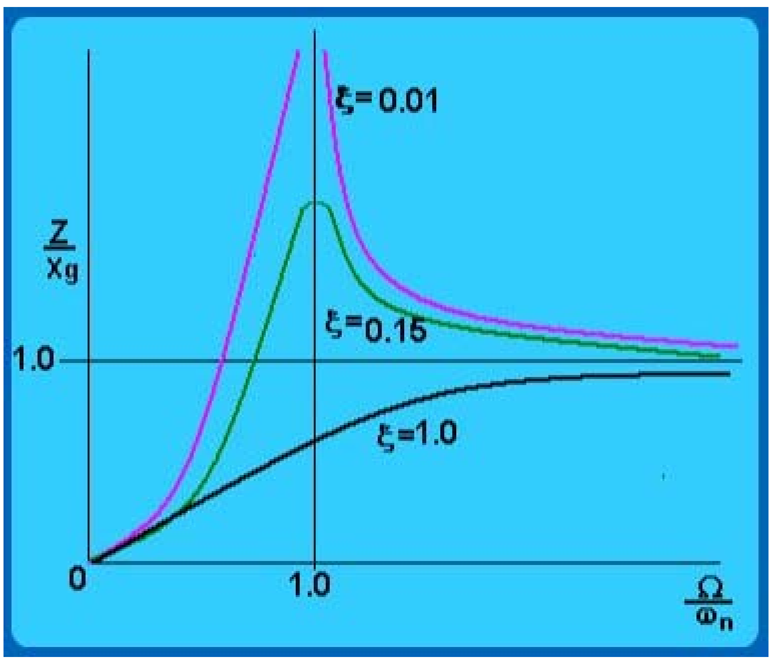

Fig 12.1.3

Acceleration Measurement (Accelerometer)

When the spring-mass of the instrument are so designed that its natural frequency is very large compared to the frequency of vibration of the base i.e. $\omega_n >>> \Omega$, then Ω / ω_n tends to 0 and we can write:

tends to approach unity

$$\sqrt{\left[1-\left(\frac{\Omega}{\omega_n}\right)^2\right]^2+\left(2\xi\frac{\Omega}{\omega_n}\right)^2}$$

hence we get

$$Z_0 = \left(\frac{\Omega}{\omega_n}\right)^2 X_g \qquad 12.1.3$$

$\Omega^2 X_g$ represents the acceleration of the vibrating body. It is observed that the read-out by the instrument are this case (Zo) will be proportional to the acceleration of the base and thus this instrument will be useful for acceleration measurement.

Displacement Measurement (Seismograph)

When the spring-mass of the instrument is so designed that its natural frequency is very small compared to the frequency of vibration of the base, i.e. $\omega_n <<< \Omega$, $\frac{\Omega}{\omega_n}$ approaches high values.

$$Z_0 = \frac{\left(\frac{\Omega}{\omega_n}\right)^2 X_g}{\sqrt{\left[1-\left(\frac{\Omega}{\omega_n}\right)^2\right]^2+\left(2\xi\frac{\Omega}{\omega_n}\right)^2}}$$

then Zo approaches values close to Xg hence indicating the displacement of the base and this is known as a siesmograph.

It is observed that the read-out by the instrument in this case will be proportional to the displacement of the base and thus this instrument will be useful for displacement measurement.

Piezoelectric accelerometers converting vibratory motion into an electric signal,followed by electronic signal conditioning, have dramatically revolutionized vibration measurements over the past couple of decades.
Some of the advantages of a piezoelectric accelerometer are:

- Very wide frequency range
- Linearity
- Characteristics remain stable over a long period of time
- Self-generating i.e. no external power supply needed
- No moving parts so no wear/tear

Typical accelerometers are shown in Fig. 12.1.4 below.

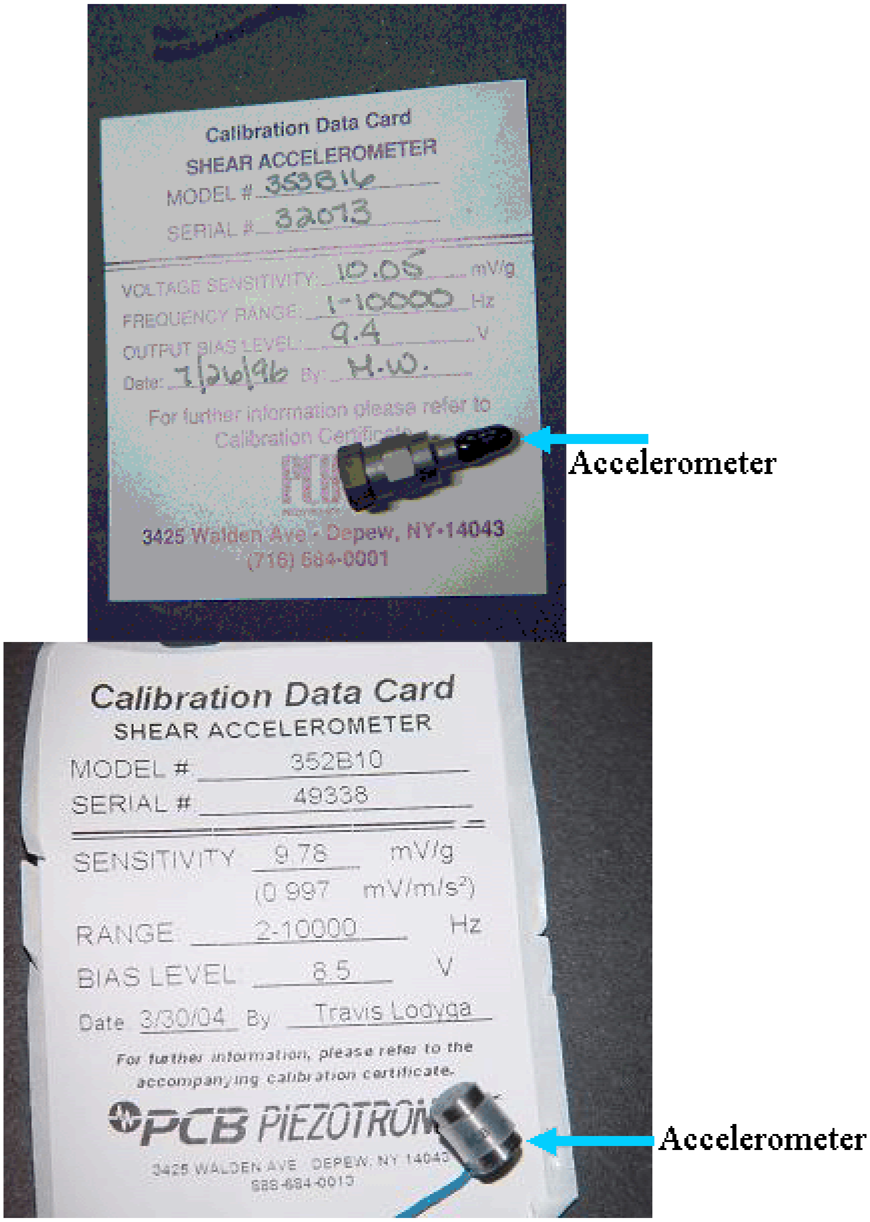

Figure 12.1.4

Selection and use of accelerometers

Several issues are important for proper application of accelerometers in a measurement. We will summarise some of these here:

- Typical measurement frequencies are up to about 10kHz. An accelerometer has two frequency cut-off points viz., the low frequency cut-off and the high frequency cut-off. The high frequency cut-off is due to the resonant frequency of the spring-mass system of the accelerometer.
- As a rule of thumb, **the highest frequency to be measured must be less than one third this upper cut-off frequency** . General purpose accelerometers may have resonant frequency in the range of 20kHz or so while the upper cut-off frequency for miniature accelerometers may be more likely in the range of 200kHz.
- At the lower end, near-DC frequency measurement is generally not very good with the conventional compression type accelerometer but the shear type accelerometer permits accurate measurements even at about 1 Hz.
- Typical general purpose accelerometers tolerate temp eratures up to about 250^0C but special care needs to be taken for high temperature applications.
- Errors (noise related) could creep into the measurements due to the cable itself viz., the cable's mechanical motion (best to tape or glue down the cable as close to the accelerometer as possible);

electromagnetic noise when the cable runs in the vicinity of a running machine.

- Accelerometers may be sensitive to transverse vibrations also i.e., other than its main axis of measurement but this is typically of the order of 1%.
- When dropped from a height, an accelerometer can be subjected to severe shock and cause permanent damage.
- It is generally necessary to make periodic calibration of the accelerometer.

Displacement or Velocity or Acceleration?

As integration is electronically performed, vibratory velocity or displacement can be readily obtained from the measured acceleration.

For a sinusoid, velocity amplitude is frequency times the displacement amplitude and acceleration amplitude is frequency times the velocity amplitude. Thus in general, acceleration measurement will be weighted more towards higher frequencies and displacement is weighted more towards lower frequencies.

In general vibratory **velocity** has been found to be the **best indicator of severity of vibration** . Many standards indicate the permissible vibration levels in terms of velocity limits. However, it must be pointed out that ride comfort of a passenger traveling in an automobile is best quantified in terms of frequency weighted RMS acceleration; similarly displacement can be used as an indicator of unbalance in a rotating machine.

Typical vibration charts are shown Fig.12.1.5

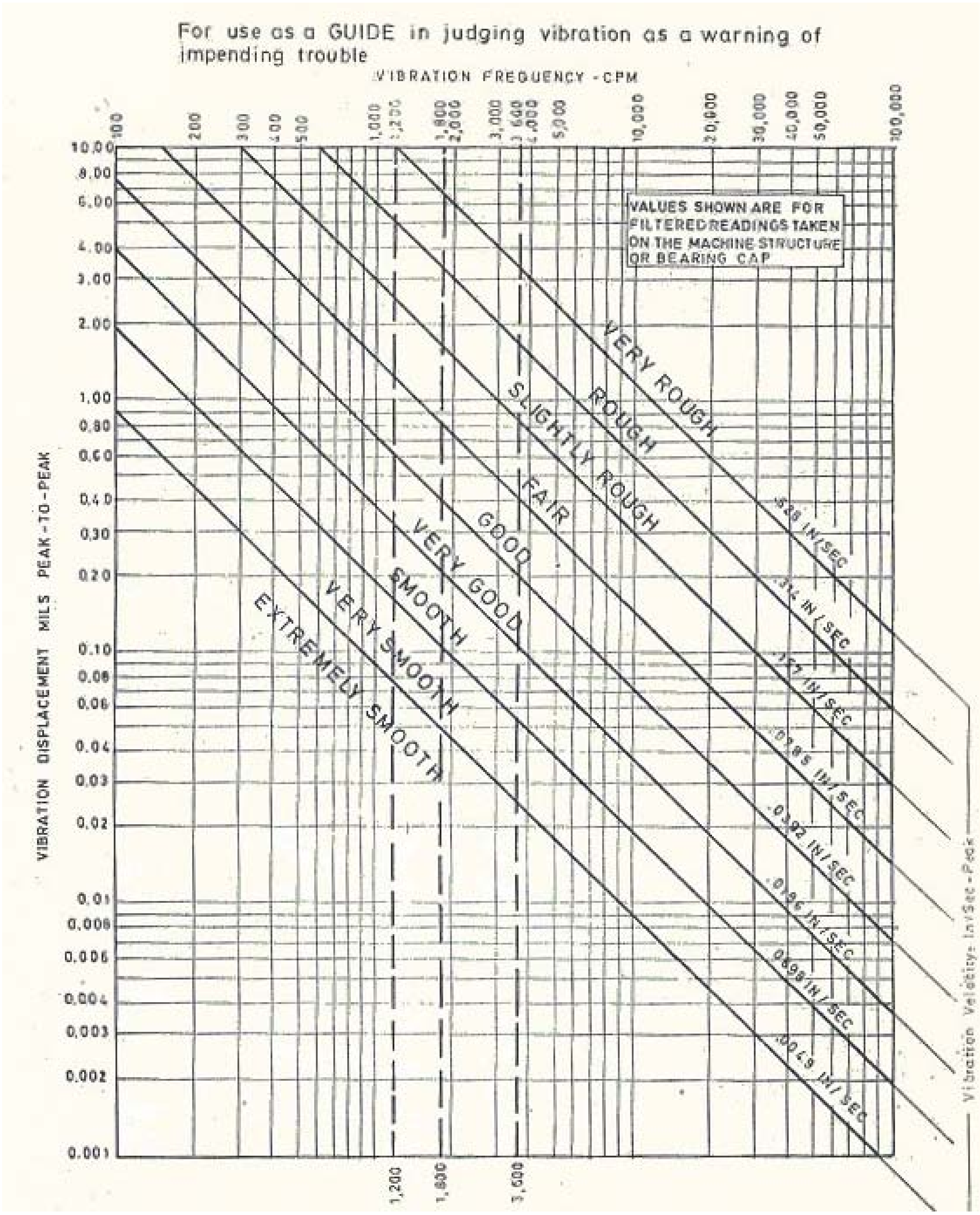

Figure 12.1.5

Constant velocity lines (units inch/sec) are shown in the figure with X-axis as frequency and Y-axis as displacement amplitude. As the velocity of vibration decreases, the machine operates more and more smoothly.

Time vs Frequency Domain

Vibration measurement records usually contain several frequency components – so just looking at the amplitude-time record is not enough.

Fig 12.1.6 Impulse response of a beam

We therefore need to perform an FFT (Fast Fourier Transform) and study the frequency components contained in the signal

Fig 12.1.7 FFT of signal above

Once we understand the frequency content of the measured vibration, we can correlate that to the operating speeds of the machine (or its harmonics) thus leading to an understanding of the source and possible remedial measures.

Measurement of Force, Impedance etc

A force transducer is used to measure the dynamic forces on a structure and this too has a piezoelectric element which gives an electrical output proportional to the force transmitted through it. Therefore the same amplifier and other instrumentation as used for accelerometer, can be used for the force transducer also.

When the force measured from the force transducer and the vibrational velocity obtained using say integration of the accelerometer signal are put together, we get the mechanical impedance of the system.

Use of non-contact measuring instruments

We mentioned two popular non-contact type measuring instruments viz., fiber-optic probe and laser Doppler vibrometer. These are shown below:

Fig 12.1.8 Fibre Optic Probe

Fig 12.1.9 Laser Doppler Vibrometer Schematic

The fiber-optic probe essentially contains several bundles of optical fibers transmitting light which will fall on the vibrating structure and the intensity of reflected light as collected by the receiving fibers is a measure of the displacement.
The laser Doppler vibrometer, as the name implies, relies on the Doppler shift caused by the vibrating body (The **Doppler effect** can be described as the effect produced by a moving source of waves in which there is an apparent upward shift in frequency for observers towards whom the source is approaching and an apparent downward shift in frequency for observers from whom the source is receding. It is important to note that the effect does not result because of an actual change in the frequency of the source.)

Associated Equipments

A typical vibration test will involve use of several other equipment as shown in Figure 12.1.10:

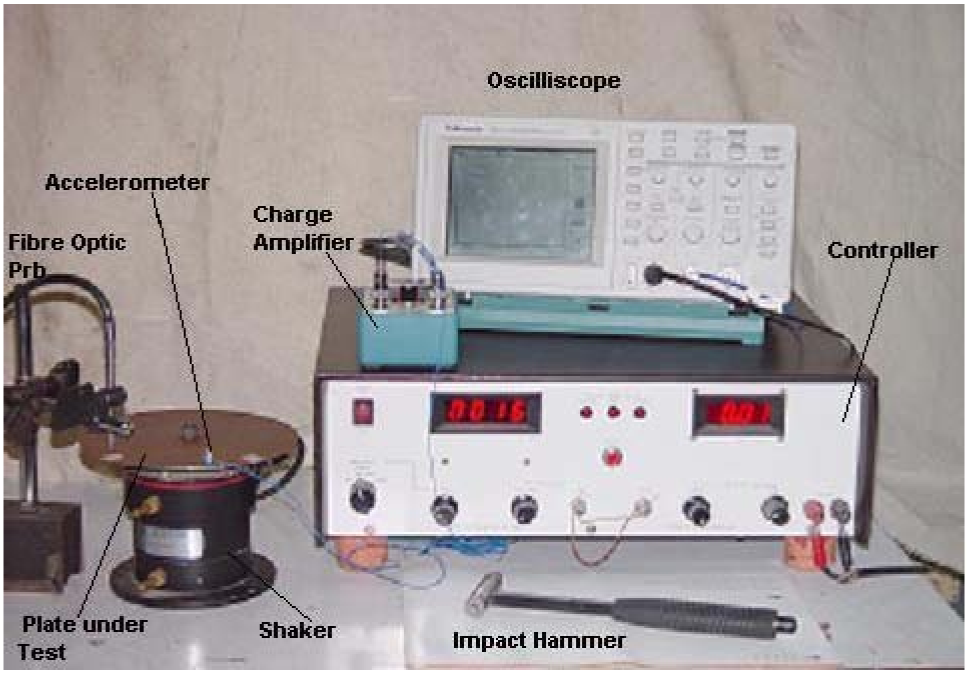

Figure 12.1.10

The experimental setup above shows both free as well as forced vibration instruments. The shaker shown here , induces the forced vibrationinto the plate. The vibrations are picked up by the piezo accelerometer , sent to charge amplifier and could be viewed on oscilliscope. The non-conatct type of instrument fibre optic probe is used to measure displacement of plate at various points.

For free vibrations , impact hammer is used.

Recap

In this lecture you have learnt the following

- The concept of measuring instruments based on fundamentals of dynamics of systems base excitation.
- Types of measuring instruments in vibration analysis.
- Range of operation of instruments and limitations of their application.

Congratulations, you have finished Lecture 1. To view the next lecture select it from the left hand side menu of the page