Mechanics G J Troup

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Longman introductory physics

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Preface

This book grew out of a series of lectures on mechanics given to first-year science students at Monash University over a number of years. Though the name of only one author appears in the book, many of my colleagues and co-lecturers contributed ideas and treatments of topics over the years. Professor W.A. Rachinger set up an 'ideal' syllabus for the course, and with the passage of time some of this was pruned and some topics were added, but the 'core' remained. My co-lecturers have been Dr A.P. Roberts, Mr R. Turner, Dr K. Thompson, and Dr J. Cashion; and Dr H.S. Perlman and Dr L. Francey have often given aid in valuable discussions. One should not forget the students either; on occasions their penetrating questions and location of errors have indeed been helpful. Finally, I should like to thank my hosts, Professor U.M. Palma and Professor I. Cicarello at the Istituto di Fisica dell' Universita, Palermo, Sicily, where this book was written 'in santa pace' during a period of study leave.

I have tried to stress the importance of the Galilean Transformation in Newtonian mechanics; the relation of conservation of quantities to symmetry; and the use of conservation laws to simplify problems. The centre-of-mass frame is also used a good deal. Some of the approaches should remain useful to the reader even when he begins quantum mechanics, and this has been borne in mind in the writing.

Melbourne 1975 Gordon Troup

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Introductory definitions

1.1 INTRODUCTORY REMARKS

Mechanics is the oldest branch of the exact, experimental sciences, and was founded as such by Galileo Galilei. In exact sciences the laws are expressed by mathematical relationships (e.g. polynomial or differential equations) and the quantities related by the laws must also have precise mathematical definitions. These quantities are, in general, constructs of our own minds; the scientific laws enable us to predict what will happen if we do something to the world, but tell us nothing about what we might call 'ultimate reality'.

The physicist today regards length, time, mass, and charge as fundamental quantities, and derives all other quantities from them. From length, mass, and time the concept of force may be derived, and we know that forces can cause the motion, or changes in the motion, of objects. From this point of view mechanics is a unifying branch of physics: wherever there are forces and motion, we may use mechanics to analyze the situation, irrespective of the origin of the forces. We shall be dealing with so-called 'classical' mechanics, and so ignoring the particular behaviour of matter on the atomic scale. However, a thorough grounding in classical mechanics is a great help in learning quantum mechanics; and indeed many principles, such as the conservation laws, are common to both.

One problem in setting up a university textbook on mechanics is that much of the material has been familiar to the reader for many years, so that he feels he knows it all and thus becomes bored by the subject. However, familiarity does not imply complete knowledge or understanding; and there are sophisticated viewpoints and unifying concepts that the reader will not have met before. In this text we shall be stressing the Galilean Transformation and Galilean invariance, just as the Lorentz Transformation and Lorentz invariance are stressed in special relativity. We shall also draw examples from many branches of physics to illustrate the unifying nature of mechanics.

1.2 TIME AND LENGTH

We choose to define an interval of *time* by saying that we measure it with a 'clock'. There are many kinds of clocks, from the water-clock used by Galilei in his experiments on inclined planes, through familiar alarm clocks and watches, to the very sophisticated and accurate clocks which count the vibrations of atoms. Clocks are based on phenomena which occur regularly, either in fact or 'on the average'. Thus the period between sunrise and sunset is regular 'on the average', as is the decay of radioactive atoms; whereas the period of a pendulum of given length, for example, shows much greater regularity. The unit of time is the second, which was defined in terms of the 'mean solar day' of twenty-four hours (the definition of the mean solar day is complicated in the extreme) and is now defined in terms of the number of vibrations associated with a certain spectral line of the caesium atom. Since we can make all the different kinds of clocks 'agree' on the time-interval between certain repeatable or regular phenomena, we know how to measure time-intervals very well. It is clear that we can take an 'origin' for our time, and measure the number of intervals after this origin as positive, and the number before as negative. In general, the choice of an origin for timing a phenomenon can be made quite arbitrarily, and we may choose it where most convenient.

We choose to define the *length* between two points A and B in a plane as follows. We take a *ruler*, apply it to A and B simultaneously, and note the number of divisions on the ruler lying between A and B. We associate with each (regular) division on the ruler some unit of length, and say that the length of the interval AB is the appropriate number of units observed. Everyone is familiar with a ruler, which must have a straight edge; this we may define by using a beam of light, for example. The fundamental physical unit of length is the *metre*, defined either in terms of the standard metre etalon kept in Paris, or in terms of the required number of wavelengths of light from a certain atomic transition. This latter definition is regularly reviewed.

1.3 POSITION. CO-ORDINATE SYSTEMS

Once we have defined *length*, we can pass to the definition of *area* and *volume*, and to the specification of the position of an object or a particle. We do this by choosing a *frame of reference*, and setting up a co-ordinate system. Our choice of reference frame is in a sense completely arbitrary, but we shall see later on that there are particular reference frames which are much more useful than others. For the moment, we consider this page of the book as a plane, and presumably it is stationary with respect to the reader. Then the most familiar co-ordinate system is the Cartesian one, in which we set up (in this plane) two lines at right angles to each other. The common zero (origin) is taken as the intersection of the lines, and we label these axes x along the page and y up the page; units of length to the right of the origin 0 along 0x are considered positive, to the left, negative; units along 0y above 0 are considered positive, and below, negative. Thus the point P in figure 1.1 is at x = 3, y = 4, and the point Q, at x = −2, y = −3.

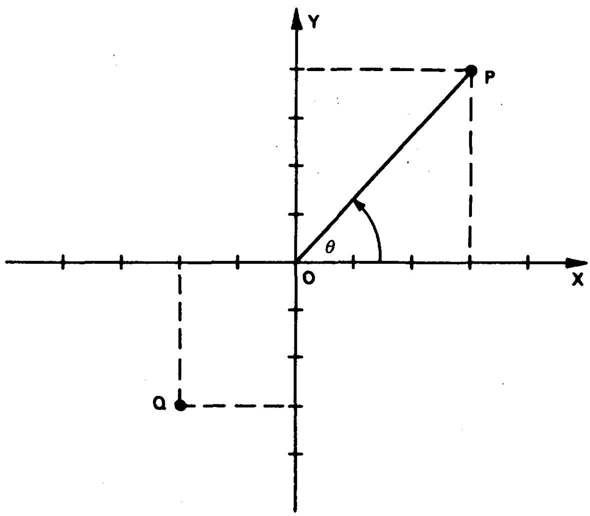

Fig. 1.1 *A plane Cartesian co-ordinate system*

Now Euclidean geometry tells us that the length of OP is 5 units; and if we measured it, we should find that this was so. It is an *experimental fact* that space is Euclidean to a very good approximation, from distances as large as those between stars to distances as small as those between atoms in a solid. This is quite a remarkable experimental fact which the reader may not have contemplated before. We are not talking about distances measured on curved surfaces, such as a sphere or a saddle, but in three dimensions, defined below.

We may also specify the position of the point P by quoting the length OP, taken always as positive, and the angle between OP and Ox, measured anticlockwise from Ox to be positive (angle is defined in terms of lengths). Such co-ordinates are called *polar* co-ordinates, and we shall find them useful later on in problems involving circular motion. If the position of a point is (x, y) in the Cartesian frame, the position in polar co-ordinates is defined from

$$\begin{aligned} OP &= r = [x^2 + y^2]^{\frac{1}{2}} \\ \cos\theta &= \frac{x}{r} \\ \sin\theta &= \frac{y}{r} \end{aligned} \tag{1.1}$$

So are we have confined ourselves to two dimensions. To describe the position of a point in three dimensions, we erect another axis through the origin, perpendicular to both the x and y axes, and call it the z axis. The convention for the positive part of the z axis is as follows: if Ox is rotated towards Oy, the direction of travel of a right-handed screw is the positive direction of the z axis (figure 1.2). We find similar situations if we rotate Oz towards Ox, which gives the positive part of the y axis, and Oy towards Oz, which gives the positive part of the x axis. This is known as a *right-handed* system of Cartesian co-ordinates; we shall be concerned only with right-handed systems in this text.

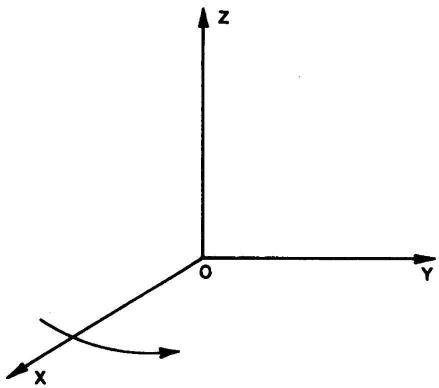

Fig. 1.2 *Right-handed Cartesian co-ordinates. If Ox is rotated towards Oy, the direction of travel of a right-handed screw is along Oz*

The distance r to a general point P(x, y, z) is again given by $r^2 = x^2 + y^2 + z^2$, and it is in this sense of the use of three mutually perpendicular axes that space is Euclidean.

There are two other important things about space which we know from our own experience. If we take a figure, or an object, and *translate* it to a new position, the figure or object does not change its shape. Nor does the figure or object change its shape when we *rotate* it to a new position. In this sense space is invariant under translation and rotation. We believe in this so much that if we perform an experiment with a piece of apparatus and then translate it or rotate it to a new position, and do not get the same results, we start to search for the effects (e.g. force fields) which may have caused this lack of symmetry.

Since all our work in this book will be confined to two dimensions, we shall not need polar co-ordinates in three dimensions. There are many other kinds of co-ordinate systems which are useful in the solution of particular problems – for example, if we wanted to solve the problem of a vibrating elliptical membrane, it is convenient to use ellipsoidal and confocal hyperboloidal co-ordinates, but we shall not be concerned with them.

1.4 SCALARS

We call quantities that are represented by a *number* only (which may be positive, negative, or zero) *scalar* quantities. Thus length is a scalar quantity, and we shall see that mass is also; another example is temperature. Another characteristic of a scalar is that its value at a given point is independent of the co-ordinate system used to calculate it. A scalar having the same algebraic form in all co-ordinate systems is called an *invariant*. The length of some line with end points P(x, y, z) and Q(x′, y′, z′) will have the form

$$\ell = \left\{(x - x')^2 + (y - y')^2 + (z - z')^2\right\}^{\frac{1}{2}} \tag{1.2}$$

in all co-ordinate systems, so it is an invariant. Nevertheless, it is a function of the position of P, say, so there is nothing to stop a scalar being a function of position. The temperature at various points in a vessel containing water that is being heated is another example of a scalar which is a function of position.

1.5 VECTORS. DISPLACEMENT

It is convenient to introduce the idea of a *directed length*. Consider the point P(x, y) and the line OP joining it to the origin O (figure 1.3). We

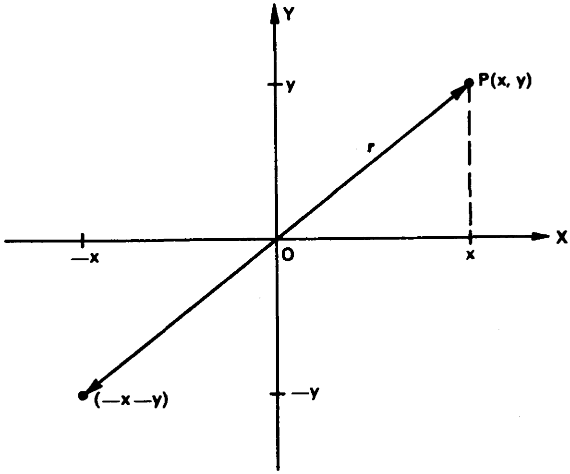

Fig. 1.3 *Displacement*

define a quantity called the *displacement* of **P**, and say that it has the *magnitude* of the length OP, the *direction* of OP in space, and the *sense* of passing from O to P. We observe that the point P′ (−x, −y) would have a displacement exactly similar to OP in magnitude and direction, but the *sense* is now reversed. Quantities requiring the specification of their magnitude, direction, and sense are called vector quantities. They obey the triangle law of addition: the vectors **A**, **B**, and **A** + **B** are shown in figure 1.4. **A** - **B** is defined as **A** + (-**B**), and from our foregoing

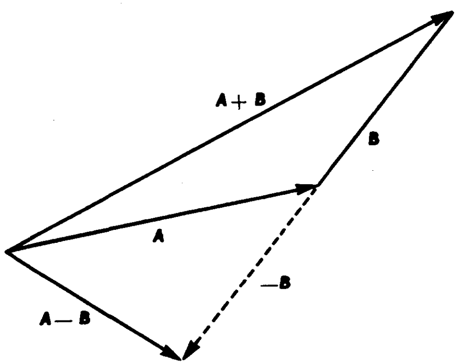

Fig. 1.4 *Vector addition and subtraction*

illustration it is clear that $-\mathbf{B}$ has the magnitude and direction in space of **B**, but the reverse sense. This is illustrated in figure 1.4. Multiplication of the vector **A** by the positive scalar a gives a vector having the same direction and sense as **A**, but having a magnitude a $|\mathbf{A}|$, where $|\mathbf{A}| = A$ is the magnitude of A. A negative scalar will also change the sense of the vector.

Let us call the displacement vector **OP** of figure 1.3 the vector **r**. It is clear that if we take a vector of magnitude x with the direction and sense of the positive x axis, and add to this a vector of magnitude y with the direction and sense of the positive y axis, we obtain **r**. Symbolically,

$$\mathrm{x} + \mathrm{y} = \mathbf{r}$$

But we know that $x = r \cos \theta$, where θ is measured positively anticlockwise from Ox. We say that the vector **r** has the *components* x and y along the directions Ox and Oy respectively, where the magnitude of the component is given by the product of the magnitude of the original vector **r**, and the cosine of the angle between **r** and the particular direction. **r** is said to be 'resolved into its x and y components'.

The concept of a unit vector is often useful. We shall denote unit vectors by a circumflex over the bold-face symbol: thus $\hat{\mathbf{A}}$ is the unit vector (vector of unit magnitude) in the direction and sense of **A**. Clearly, $\mathbf{A} = A\hat{\mathbf{A}}$. The exceptions to this notation will be the unit vectors **i**, **j**, and **k** along Ox, Oy, and Oz respectively; note that the vector **r** of figure 1.3 is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + O\mathbf{k}$$

and

$$r^2 = x^2 + y^2 + O$$

There are other useful vector operations, notably the scalar and vector products of two vectors. The scalar (or 'dot') product **A.B** is defined by

$$\mathbf{A}.\mathbf{B} = AB \cos \theta \tag{1.3}$$

where θ is the angle between **A** and **B**. We note that $\mathbf{A}.\mathbf{A} = A^2$, so that $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$; and that $\mathbf{A}.\mathbf{B} = O$ if **A** is perpendicular to **B**, so that $\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = O$, and therefore

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = a_1b_1 + a_2b_2 + a_3b_3 \tag{1.4}$$

The vector (or 'cross') product is defined by the determinant

$$(a_1 i + a_2 j + a_3 k) \times (b_1 i + b_2 j + b_3 k) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (1.5)$$

i.e. $\mathbf{A} \times \mathbf{B} = \mathbf{AB} \sin \theta \, \hat{\mathbf{n}}$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to **A** and **B**, such that its sense is given by the travel of a right-hand screw when **A** is rotated to **B** by the shortest route. We note that **A** x **A** = O and hence

$$i \times i = j \times j = k \times k = O \quad (1.6a)$$

and

$$i \times j = k, j \times k = i, k \times i = j \quad (1.6b)$$

which last is a vector statement of our right-hand-screw rule for setting up a right-handed Cartesian co-ordinate system. Note that **A** x **B** = −**B** x **A**, i.e. the vector product is non-commutative. The scalar and vector products will be useful to us later on in the mechanics syllabus.

We saw in section 1.4 that we can have a *scalar* which is *invariant*, i.e. its form is always the same in any (Cartesian) frame; an example is the magnitude of a vector. We can also have *vector invariants*; for example, the difference **A** − **B** between two vectors will have the form $\{(x_a - x_b)\, i + (y_a - y_b)\, j + (z_a - z_b)\, k\}$ in any Cartesian frame, although the origins and axis directions may differ.

We have, in this section, chosen to regard a vector quantity as analogous to a directed line element: something requiring the specification of magnitude, direction, and sense. Vectors may also be defined in terms of the way their Euclidean components transform from one stationary Cartesian set of axes to another; this is discussed in Appendix 1. These transformation properties then lead on to more complicated quantities like tensors, with which we shall be very little concerned.

1.6 AVERAGE AND INSTANTANEOUS VELOCITY

Consider a point P whose displacement **r** from the origin O is a function **r**(t) of time t, so that P traces out some *path* or *trajectory* in the x-y plane (figure 1.5). Then if the displacement of P at time t_1 is $\mathbf{r}_1$, and the displacement at $t_2 > t_1$ is $\mathbf{r}_2$, the *average velocity* $\bar{\mathbf{v}}$ of P is defined by

$$\bar{\mathbf{v}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1} = \frac{\text{vector change in displacement}}{\text{time taken}} \tag{1.7}$$

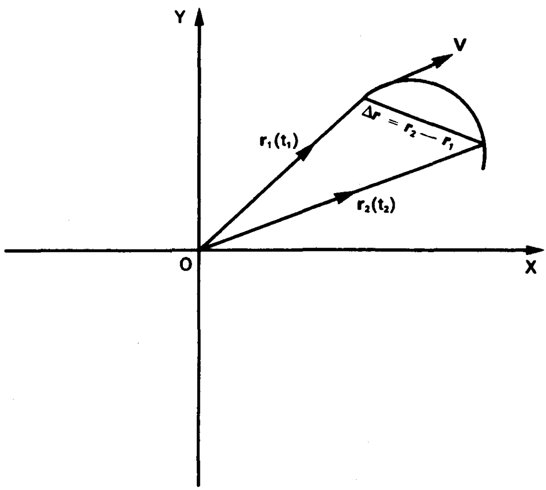

Fig. 1.5 *Average velocity*

It is vital that the reader remembers this definition, and does not become sidetracked by the plausible but incorrect idea of 'average velocity' as the sum of the 'initial' and 'final' velocities divided by 2.

From the definition, average velocity is clearly a vector. If we write $\mathbf{r}_2 - \mathbf{r}_1 = \Delta\mathbf{r}$, $t_2 - t_1 = \Delta t$, we obtain

$$\bar{\mathbf{v}} = \frac{\Delta \mathbf{r}}{\Delta t} \tag{1.8}$$

We now proceed to the limit $\Delta t \to 0$, and thus obtain the *instantaneous velocity* **v**

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}(t)}{\Delta t} \qquad (1.9)$$

which has the magnitude of $d\mathbf{r}/dt$ and the direction and sense of $\Delta\mathbf{r}$ in the limit $\Delta t \to 0$ for the time t. Consideration of figure 1.5 will show that $\mathbf{v}(t)$ is *tangential* to the path at any time t. The magnitude $|d\mathbf{r}/dt|$ of the instantaneous velocity $\mathbf{v} = d\mathbf{r}/dt$ is called the *speed*, and is clearly a scalar. **v** is a vector, and may be resolved into components, for example, as required.

1.7 AVERAGE AND INSTANTANEOUS ACCELERATION

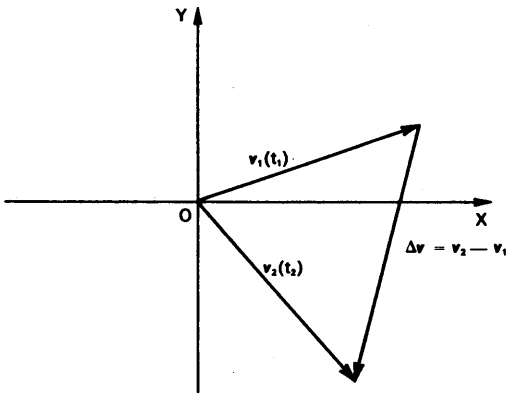

Fig. 1.6 *Average acceleration. This diagram is in 'velocity-space'; the directions of* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are taken from the path of Fig. 1.5*

In a similar fashion we define the *average acceleration* $\bar{\mathbf{a}}$ as

$$\bar{\mathbf{a}} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1} = \frac{\text{change in velocity}}{\text{time taken}} \qquad (1.10)$$

where the point P had velocity $\mathbf{v}_1$ at $t = t_1$ and velocity $\mathbf{v}_2$ at $t = t_2 > t_1$ (figure 1.6). Writing $\mathbf{v}_2 - \mathbf{v}_1 = \Delta\mathbf{v}$, and $t_2 - t_1 = \Delta t$, and proceeding to the limit $\Delta t \to 0$ once more, we obtain the *instantaneous acceleration* **a**,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} \tag{1.11}$$

$$= \frac{d^2\mathbf{r}}{dt^2} \tag{1.12}$$

from equation (1.9).

We note that although the instantaneous velocity is always tangential to the path traced out by the moving point P in the x-y plane, we cannot make any such general statement concerning the acceleration. The relationship of the acceleration to the path depends on the form of the acceleration and the kind of motion. However, if we consider the path traced out in velocity space by the tip of the velocity vector, it is clear that the acceleration vector will always be tangential to this path. Figure 1.6 is plotted in 'velocity-space'.

Having defined displacement, instantaneous velocity, and instantaneous acceleration, we could pass now to the study of *kinematics* – motion *per se* of a particle – without enquiring into the 'cause' (e.g. forces). But it is preferable to arm ourselves with two further principles and definitions before we do, because our study of kinematics will be much more powerful and will have more insight.

CHAPTER 1 PROBLEMS

1.1 A particle moves clockwise in a circle with uniform speed v; the co-ordinate axes are as shown in figure 1.1p.

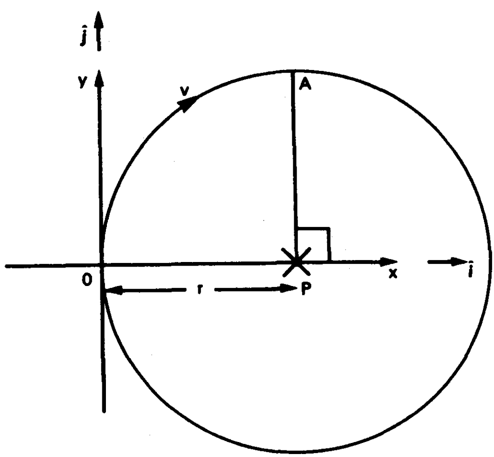

Fig. 1.1p

The particle is initially at 0, and reaches the point A after time T. In terms of r, v, and unit vectors $\hat{i}, \hat{j}$, write down the following quantities:

(a) The displacement from the origin of the particle at A
(b) The average velocity vector of the particle over the path from O to A
(c) The average acceleration vector over the same path

1.2 For general motion of a particle in a plane, which of the following statements is correct?

(a) The displacement vector is tangential to the path:
   (i) always.
   (ii) sometimes.
   (iii) never.
(b) The velocity vector is perpendicular to the path:
   (i) always.
   (ii) sometimes.
   (iii) never.
(c) The acceleration vector is perpendicular to the path:
   (i) always.
   (ii) sometimes.
   (iii) never.

1.3 The equations of motion of a particle moving in a plane are

$$x = -10t + 30t^2$$
$$y = -15t - 20t^2$$

(a) What are the x and y components of the acceleration?
(b) What are the x and y components of the initial velocity?
(c) Plot on the grid provided in figure 1.3p the path of the particle for $t > 1$.
(d) At large values of t the motion is approximately
   (i) circular.
   (ii) elliptic.
   (iii) in a straight line.
   (iv) none of these.

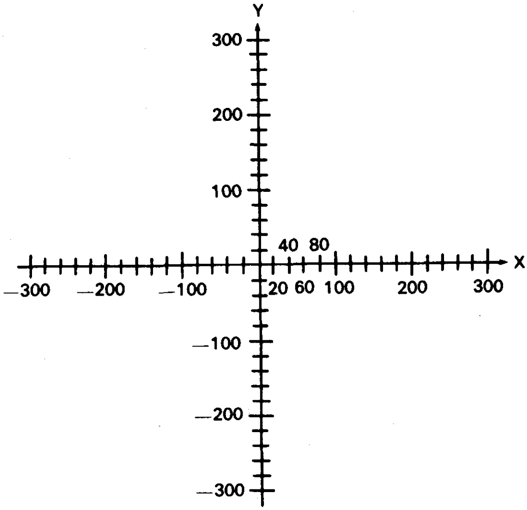

Fig. 1.3p

1.4 A particle of mass m kg moves in the horizontal circle ABCD. The points B and D lie on the north-south diameter of the circle and A and C lie on the east-west diameter (figure 1.4p).

The particle transverses the semi-circle ABC at the speed of 10 m/sec and then traverses the semi-circle CDA at a speed of 20 m/sec. The radius of the circle is 50 m.

(a) What is the magnitude of the average velocity vector of the particle over the path ABC?

(b) What is the direction of the average velocity vector of the particle over the path ABC?

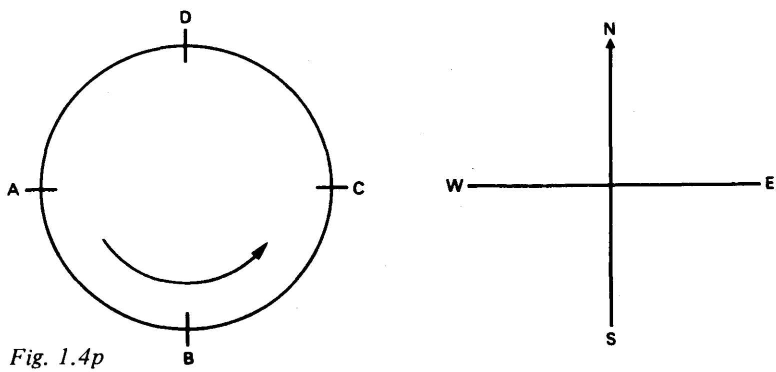

Fig. 1.4p

(c) What is the magnitude of the average velocity vector of the particle over the path ABCDA?
(d) What is the magnitude of the acceleration at point D?
(e) What is the direction of the acceleration at point D?

1.5 A particle moves in the x-y plane. Its position co-ordinates at time t are given by

$$x = 16t$$
$$y = 8t^2$$

where x and y are in metres and t is in seconds.

Answers to the following questions sh ,uld be numerical or in terms of t.

(a) What is the position of the particle at $t = 0$?
(b) What is the position of the particle at $t = 2$ sec?
(c) What is the direction and magnitude of the average velocity vector for the period $t = 0$ to $t = 2$ sec?
(d) What is the value of the x component of velocity at time t?
(e) What is the value of the y component of velocity at time t?
(f) What is the speed of the particle at $t = 2$ sec?
(g) At what time is the kinetic energy of the particle a minimum?
(h) What is the magnitude and direction of acceleration at time t?
(i) What is the radius of curvature of the path of the particle at $t = 0$?

1.6 A particle moves clockwise in the circle shown in figure 1.6p with uniform speed v. At time $t = 0$, the particle is at P. Answer the following questions.

(a) What is the angular speed of the particle about C?
(b) After how long does the particle first reach Q?
(c) What is the displacement vector of the particle at Q?
(d) What is the velocity vector of the particle at Q?
(e) What is the average velocity vector of the particle over the path $P \rightarrow Q$?
(f) What is the average acceleration vector of the particle over the path $P \rightarrow Q$?
(g) What is the instantaneous acceleration vector of the particle at Q?

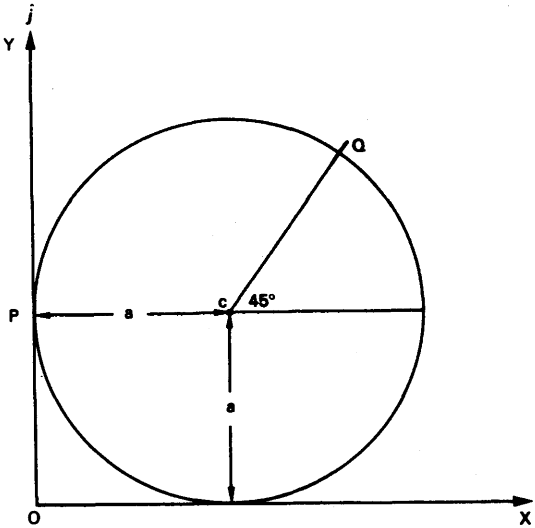

Fig. 1.6p

1.7 A particle starts from 0 at time t = 0 and moves with uniform speed anticlockwise around the circle OBCP shown in figure 1.7p. The angular velocity about the centre of the circle is ω.

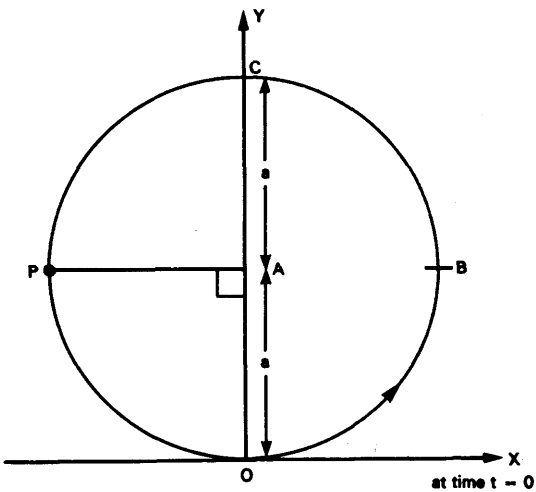

Fig. 1.7p

Answer the following questions *in terms of* ω, a, and the unit vectors i along Ox and j along Oy.

(a) How long does one revolution take?

(b) What is the displacement vector of P from O?

(c) What is the instantaneous velocity vector of the particle at P?

(d) What is the average velocity vector of the particle over the circular path OBCP when the particle first reaches P?

(e) What is the instantaneous acceleration vector of the particle at P?

(f) What is the average acceleration vector of the particle over the circular path OBCP when the particle first reaches P?

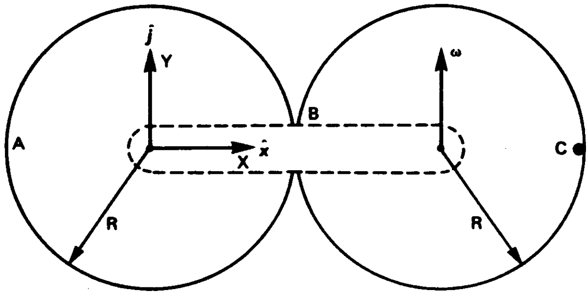

Fig. 1.8p

1.8 Two identical gear wheels A and B, each of radius R, mesh together, as shown in figure 1.8p. Wheel A is fixed in position at the origin of a set of co-ordinates and does not rotate. The centre of wheel B rotates about the origin in an anticlockwise direction with angular velocity ω. C is a spot of paint on the rim of wheel B. At time t = 0, C is at position vector

$$r = (3R\,i + 0\,j) \text{ or } (3R, 0)$$

Write down expressions for the subsequent vectors:

(a) The position vector of C at time t

(b) The velocity vector of C at time t

(c) The average velocity vector of C between the times t = 0 and t = t

1.9 A wheel of radius r_0 rolls without slipping along a horizontal road in the +x direction. The +y direction is vertically upwards, measured from the road surface (figure 1.9p). A nail in the rim of the wheel is at the point $\mathbf{r} = 0$ at time $t = 0$.

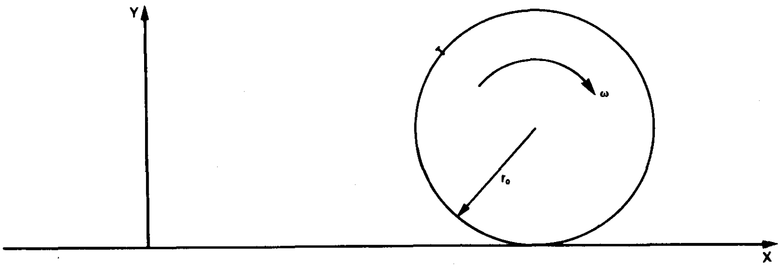

Fig. 1.9p

Using Cartesian co-ordinates write down analytic expressions for the following functions of time.

(a) The position vector $\mathbf{r}(t)$ of the nail

(b) The velocity vector $\mathbf{v}(t)$ of the nail

(c) The acceleration vector $\mathbf{a}(t)$ of the nail

(d) The magnitude of the velocity $|\mathbf{v}(t)|$ of the nail

(e) The magnitude $|\mathbf{a}(t)|$ of the acceleration of the nail

(f) The average velocity vector $\bar{\mathbf{v}}(t)$ of the nail, taken over a long time

(g) The average acceleration vector $\bar{\mathbf{a}}(t)$ of the nail taken over a long time

2
Kinematics

2.1 INERTIAL FRAMES. GALILEAN TRANSFORMATIONS

We now come to the question of which frames of reference are the most useful or suitable in mechanics. The laws of mechanics are in fact related to and formulated in the set of frames known as 'inertial frames'. An inertial frame is one in which the inertial principle of Galilei and Newton holds true, i.e. that a body will remain at rest or continue with uniform velocity in a straight line unless it is compelled to change its state of rest or uniform rectilinear motion by some external influence. This principle was an idealization by Galilei from the results of his experiments in rolling spheres down inclined planes. If a test body which is apparently not acted on by outside influences *accelerates,* the frame is not inertial.

We shall see later on that an inertial frame is also an idealization to some extent, since any frame which is accelerating cannot be truly inertial. But 'approximately inertial' may be good enough for many experiments. Thus a frame attached to a physics laboratory on the earth is only approximately inertial, because the earth is rotating about its axis and about the sun. But we can conceive of a series of frames at rest with respect to the fixed stars, and in which no influences such as gravity act, and these frames should be ideal inertial frames. We then consider frames moving with uniform rectilinear velocity with respect to these, and we find that these too are inertial, because a change of frame does not influence the acceleration of a test body but only changes the constant velocity of the test body by the velocity of the reference frame. We come to see this more precisely when we consider the Galilean Transformation, which enables us to move from a fixed frame to a moving frame. We recall that the postulates of the Galilean Transformation hold good only when the velocity of the moving frame is very much less than the velocity of light – which is in a great many classical mechanical situations!

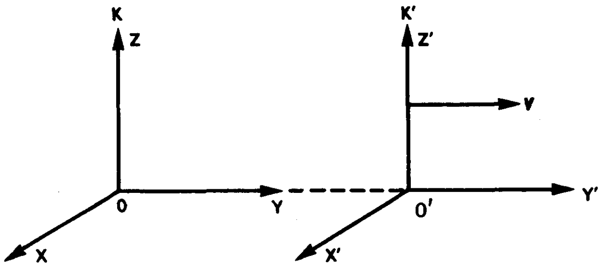

Fig. 2.1 *Galilean frames*

We consider a frame K at rest with respect to the fixed stars, having Cartesian axes x, y, and z, and a frame K′, with Cartesian axes x′, y′, and z′, parallel to x, y, and z respectively; the origins of K and K′ coincide at times t = t′ = O. and K′ is moving with constant velocity V = Vi along Ox of K (figure 2.1). Then the Galilean Transformation which relates x′, y′, and z′ to x, y, and z at times t and t′ is

$$\begin{aligned} x' &= x - Vt && \text{(a)} \\ y' &= y && \text{(b)} \\ z' &= z && \text{(c)} \\ t' &= t && \text{(d)} \end{aligned} \qquad (2.1)$$

and the reverse transformation is clearly

$$\begin{aligned} x &= x' + Vt && \text{(a)} \\ y &= y' && \text{(b)} \\ z &= z' && \text{(c)} \\ t &= t' && \text{(d)} \end{aligned} \qquad (2.2)$$

which we could have obtained by *considering* K′ as stationary, and K as moving with velocity -Vi with respect to K′. The fact that the times are unchanged on changing frames we take from experience when $V \ll c$, where c is the speed of light. We do not expect a good watch to lose or gain time when we travel by train!

Now let us derive the transformation laws relating to velocity and acceleration between K and K′. A velocity component in the x′ direction in K′ will be given by

$$\frac{dx'}{dt'} = \frac{d(x-Vt)}{dt'} = \frac{d(x-Vt)}{dt}$$

since from (2.1d) $t = t'$ and therefore $dt = dt'$, so that

$$\frac{dx'}{dt'} = \frac{dx}{dt} - V \tag{2.3a}$$

and the other equations are clearly

$$\frac{dy'}{dt'} = \frac{dy'}{dt} = \frac{dy}{dt} \tag{2.3b}$$

$$\frac{dz'}{dt'} = \frac{dz'}{dt} = \frac{dz}{dt} \tag{2.3c}$$

Let us consider equation (2.3a) further. dx/dt is the x component of velocity of some object in the frame K; V is the velocity of K′; dx'/dt' is the x′ component of velocity of the object in the frame K′, or *relative to the frame K′*. We therefore see (or define) that the velocity of *A relative to B* is the velocity of A minus the velocity of B. Since this is the difference between two vectors, it is a vector invariant, and must have the same form (and for two particular objects, the same *value*) in all stationary frames, and further, in *all frames related by a Galilean transformation.* This is easily proved from considering equation (2.3a) for two different objects.

We now derive the law of transformation of accelerations. From equations (2.3), since $dt' = dt$, we have

$$\frac{d^2x'}{dt'^2} = \frac{d^2x}{dt^2}$$

$$\frac{d^2y'}{dt'^2} = \frac{d^2y}{dt^2}$$

$$\frac{d^2z'}{dt'^2} = \frac{d^2z}{dt^2} \tag{2.4}$$

i.e. a Galilean Transformation leaves accelerations unchanged. Therefore a body unaccelerated in the initial inertial frame K is unaccelerated in all frames related to K′ by a Galilean Transformation, i.e. all frames related to frames stationary with respect to the fixed stars by Galilean

Transformations are inertial frames. Since we expect the laws of mechanics to be unchanged when we go from one inertial frame to another, we are led to the idea that any law of mechanics, or any expression that we use to relate mechanical quantities, should not change its form when its constituents are subjected to a Galilean Transformation, i.e. the laws should be *covariant* with respect to Galilean Transformations. We shall have occasion to refer back to this principle later on.

We may now collect the expressions of equations (2.1) and (2.3) in a general vector form. That is, we consider a frame K′ defined as previously moving with a general vector velocity **V** with respect to K. Then (2.1) becomes

$$\begin{aligned} \mathbf{r}' &= \mathbf{r} - \mathbf{V}t \\ t' &= t \end{aligned} \qquad (2.5)$$

where $\mathbf{r}'$ is the displacement of an object in K′ and $\mathbf{r}$ is its displacement in K. The velocity transformation becomes

$$\frac{d\mathbf{r}'}{dt'} = \frac{d\mathbf{r}}{dt} - V \qquad (2.6)$$

and the acceleration is the same in both frames.

The simple Galilean Transformation allows us to move with ease between frames moving at constant velocity relative to each other. We shall find this of use in solving problems, as will be illustrated further on in this chapter.

2.2 KINEMATICS

We are now in a position to tackle the study of kinematics, that section of mechanics which studies the motion of a particle or a system of particles without enquiring into the cause. A particle is an idealization from an object whose geometrical dimensions are small in comparison with its motional path. Thus the earth, in its motion about the sun, is a very good approximation to a particle, which ideally has no extension but retains other material properties such as mass and electric charge, if required by the problem.

There is a sense in which the whole of mechanics is contained in the definitions of displacement, instantaneous velocity, and instantaneous acceleration. If we know any one of these quantities as a function of time, we may determine the other two via the appropriate differential or integral equations set out below in vector form.

If **r**, **v**, **a** are the displacement, instantaneous velocity, and instantaneous acceleration of a particle respectively, then the appropriate differential relations are:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \; \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \qquad (2.7)$$

and the appropriate integral relations are

$$\mathbf{v} = \int \mathbf{a}\, dt + \boldsymbol{\alpha} \quad \mathbf{r} = \int \mathbf{v}\, dt + \boldsymbol{\beta} \qquad (2.8)$$

where t is the time and $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are constants of integration.

The vector equations (2.7) and (2.8) conceal certain facts which the component equations reveal. For example, taking components along Ox, Oy, and Oz, we must have

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$$

$$a_y = \frac{dvy}{dt} = \frac{d^2y}{dt^2}$$

$$a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2} \qquad (2.9)$$

where the v_i and a_i are the components of **v** and **a** respectively along the directions Oi, i = x, y, z. Equations (2.9) tell us that the x components of the velocity and displacement, for example, are completely governed by the x component of acceleration. There are many problems in which the components of the acceleration are independent of each other, and have particularly simple forms, so that it is worthwhile to look at the component equations.

It is also worth mentioning here a relationship which may at times be useful in unravelling problems. It is, taking the x component as an example:

$$\frac{d}{dx}\left(\frac{v_x^2}{2}\right) = a_x \qquad (2.10)$$

$$\left(\text{since } \frac{d}{dx}\left(\tfrac{1}{2}v_x^2\right) = v_x\frac{dv_x}{dx} = v_x\frac{dv_x}{dt}\frac{dt}{dx} = v_x\frac{dv_x}{dt}\frac{1}{v_x}\right)$$

which is of use when a_x or v_x are known as functions of x but not perhaps of t. The very important one-dimensional equation

$$\frac{d^2x}{dt^2} = \frac{dv_x}{dt} = -n^2x \qquad (2.11)$$

where n is a constant, may be solved by this method.

From the foregoing we see that kinematics comes down to solving differential or integral equations. Important examples are given in the following sections.

2.3 MOTION UNDER CONSTANT ACCELERATION. PROJECTILES

The first kinematical problem studied experimentally and theoretically in mechanics (by Galilei!) was that of motion under constant acceleration. What took him years to formulate takes us a minute or so because of our knowledge of vector algebra and the integral calculus. We shall first solve the problem in vector form, and then specialize it to various important cases.

Consider a particle which suffers a constant (vector) acceleration **a** throughout its motion. Therefore its velocity **v** is given by

$$\mathbf{v} = \int \mathbf{a}\,dt + \boldsymbol{\alpha} = \mathbf{a}t + \boldsymbol{\alpha} \qquad (2.12)$$

where $\boldsymbol{\alpha}$ is the velocity at t = 0. The displacement **r** is given by

$$\mathbf{r} = \int \mathbf{v}\,dt = \tfrac{1}{2}\,\mathbf{a}t^2 + \boldsymbol{\alpha}t + \mathbf{r}_0 \qquad (2.13)$$

where $\mathbf{r}_0$ is the displacement at t = 0. These two vector equations contain all the solutions to problems in which a particle suffers a constant acceleration. There are two important cases. First, we know that the acceleration due to gravity at the earth's surface is effectively constant – this was the system that Galilei studied. A charged particle in a uniform electric field is also subject to a constant acceleration; this principle was put to work by J.J. Thompson in measuring the charge-to-mass ratio of the electron, and is put to work every day in cathode-ray oscilloscopes. The two problems are identical in terms of the mathematics used to solve them and the concepts of kinematics; but the situations and the origins of the forces are quite different. Here we see mechanics playing its unifying role.

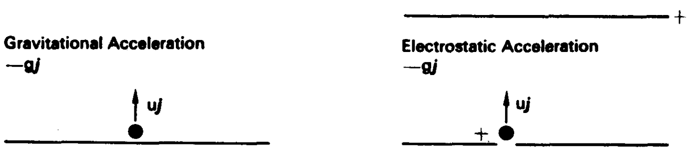

Fig. 2.2 (a) Gravitational acceleration (b) Electrostatic acceleration

Figures 2.2a and 2.2b schematically illustrate the situations of a particle subjected to the acceleration $-g\mathbf{j}$ due to gravity, projected upwards with initial velocity $u\mathbf{j} + O\mathbf{i}$, and a positively-charged particle between two electrodes subject to the same conditions: the *causes* of the accelerations are different. For both systems

$$\frac{d^2y}{dt^2} = \frac{dv_y}{dt} = -g \tag{2.14a}$$

$$v_y = u - gt \tag{2.14b}$$

and if we choose that $y = O$ when $t = O$, we obtain

$$y = ut - \tfrac{1}{2}gt^2 \tag{2.14c}$$

If we use equation (2.10), we obtain

$$v_y^2 - u^2 = -2gy \tag{2.14d}$$

all of which equations are doubtless familiar to the student, together with their interpretations and solutions. To summarize, the maximum height of the particle above $y = O$ is reached in a time $t = u/g$ and has the value $u^2/2g$, the particle reaches $y = O$ at time $2(u/g)$ with velocity $-u$.

If the acceleration exists in the y direction only, this is the solution to the y components of the motion, irrespective of the velocities in the x and z directions.

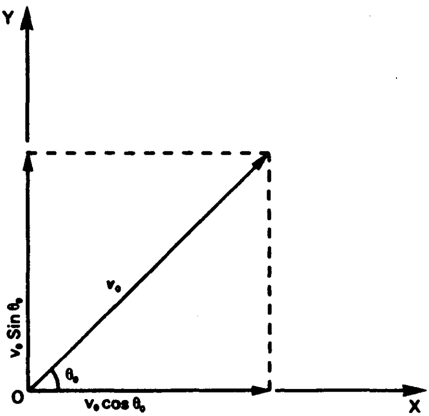

Fig. 2.3 The projectile problem

Consider the two-dimensional problem in which a particle, subject to the constant acceleration –gj, is launched from y = O, x = O at t = O with an initial velocity $v_0 = v_0 \cos\theta_0\, i + v_0 \sin\theta_0\, j$ (figure 2.3). We can tackle this problem in two ways.

(1) We recognize that we have solved the y component of the motion for $uj = v_0 \sin\theta_0\, j$. The x component of displacement is clearly $xi = v_0 \cos\theta_0 t\, i$, and we know that the total displacement **r** is given by xi + yj at any time; the total velocity is given by $v_0 \cos\theta_0 i + v_y \sin\theta_0 j$, where v_y is given by equation (2.14b) for $u = v_0 \sin\theta_0$.

(2) Since the x component of velocity is clearly constant, we solve the problem for all t in the frame moving with this velocity, i.e. *we perform a Galilean Transformation into this frame.* The displacement or velocity at any time t in the original frame will then be obtained from the solutions in the moving frame [i.e. equations (2.14b), (c), (d)] via the reverse transformation.

The work involved in both methods is roughly the same. We might say that method (1) is the 'mathematical' method and method (2) the 'physical' method, because the idea of Galilean Transformations and their relation to the laws of mechanics is an idea imposed by *physics* and not by mathematics.

By either method, we shall obtain the relations

$$x = (v_0 \cos\theta_0)t$$

$$y = (v_0 \sin\theta_0)t - \tfrac{1}{2}gt^2 \qquad (2.15)$$

in the original frame. The elimination of t from these two equations is not difficult, so that we obtain the equation of the path of the particle,

$$y = x(\tan\theta_0) + \dot{x}^2\left(\frac{g\sec^2\theta_0}{2v_0^2}\right) \qquad (2.16)$$

which is clearly a parabola.

We have not only solved the projectile problem: a shell fired with a certain initial velocity and subject to a uniform gravitational acceleration; we have also shown, by method (2), that the path of a particle which moves vertically under constant acceleration in a frame K′ will be a parabola in a frame K where K′ moves with uniform velocity with respect to K, such that $\mathbf{V} = \mathbf{V}_i$.

2.4 UNIFORM CIRCULAR MOTION

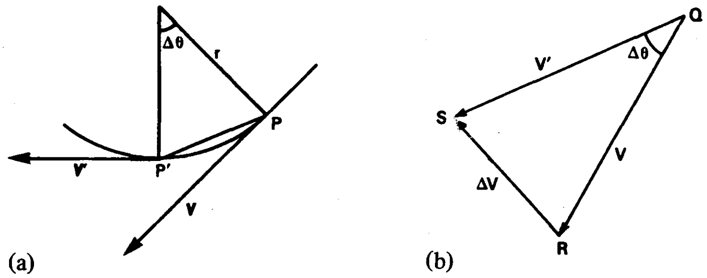

Fig. 2.4 (a) Uniform circular motion (b) Velocity-space diagram

We consider a particle which moves with uniform speed around a circle of radius r. This means that the magnitude of the velocity at any instant is constant, but since the path is a circle the velocity cannot be constant. Therefore the particle is being continuously accelerated. We set out to examine the motion by a vector method.

Let the velocity at some instant t be **V**, and at a later instant t + Δt be **V′**, where |**V**| = |**V′**| = V. In this time the radius from the centre O to the particle will have moved through an angle $\Delta\theta$ (figure 2.4a). Since in the limit $\Delta t \to O$ the arc and the chord become equal, we have

$$r\Delta\theta \doteqdot V\Delta t \tag{2.17}$$

or, in the limit,
$$V = r\frac{d\theta}{dt} = r\omega \tag{2.18}$$

where $\omega = d\theta/dt$ is the *angular speed*. Figure 2.4b illustrates the situation in velocity space: the angle between **V′** and **V** must be $\Delta\theta$, since the velocity is always tangential to the path and therefore perpendicular to the radius in this case. Since |**V′**| = |**V**| = V, the triangles OPP′ of figure 2.4a and QRS of figure 2.4b are similar, and therefore we may write

$$\frac{\Delta V}{V} \doteqdot \frac{V\Delta t}{r} \tag{2.19}$$

or
$$\frac{dV}{dt} = \frac{V^2}{r} = \omega^2 r \tag{2.20}$$

in the limit. The direction of ΔV in the limit is clearly along the radius. Thus for uniform circular motion with speed V in a circle of radius r, the acceleration **a** is given by

$$|\mathbf{a}| = \frac{V^2}{r} = \omega^2 r \text{ where } V = \omega r, \qquad (2.21)$$

ω being the angular speed, and the direction of **a** is always radially inwards towards the centre of the circle from the position of the particle.

We have used a very general vector method to solve this problem. We could also solve it by using a combination of Cartesian and polar co-ordinates. In figure 2.5 we have set the circle of radius r with its centre at the origin of a set of Cartesian co-ordinates, and we show the particle P at time t with the displacement vector **r** such that

$$\mathbf{r} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} \qquad (2.22)$$

where $\theta = \omega t$ and ω is a constant angular speed. Hence

$$\frac{d\mathbf{r}}{dt} = -\omega r\sin\theta\,\mathbf{i} + \omega r\cos\theta\,\mathbf{j} \qquad (2.23)$$

which has magnitude ωr and is at right angles to **r** with the sense shown.

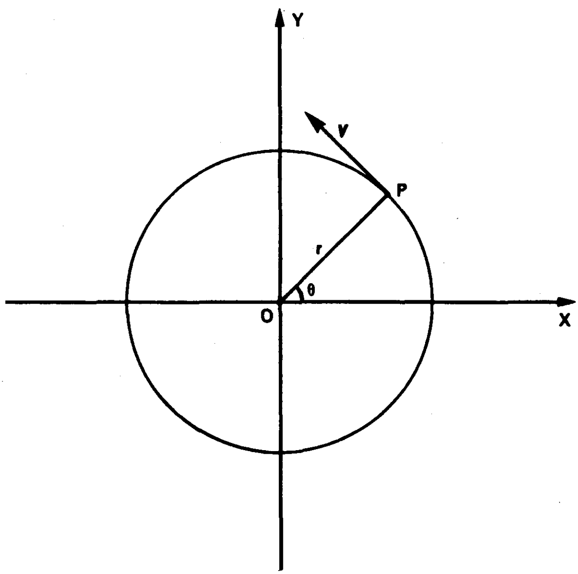

Fig. 2.5 Polar co-ordinates

Finally

$$\frac{d^2 r}{dt^2} = -\omega^2 r \cos\theta\ \mathbf{i} - \omega^2 r \sin\theta\ \mathbf{j} \qquad (2.24)$$

$$= -\omega^2 \mathbf{r} \text{ from } (2.22)$$

and we have the same results as by the general vector method.

2.5 NON-UNIFORM CIRCULAR MOTION

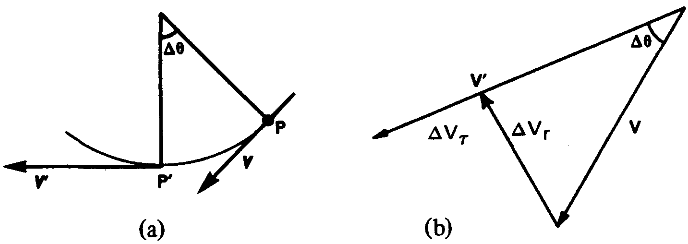

Fig. 2.6 (a) Non-uniform circular motion (b) Velocity-space diagram

The particle now moves in a circle, but the velocity $\mathbf{V}'$ at time $t + \Delta t$ is equal in neither magnitude nor direction to the velocity $\mathbf{V}$ at time t. The situation in the x-y plane and in velocity-space is illustrated in figures 2.6a and 2.6b. We observe from the velocity diagram 2.6b that we may split up the change in velocity $\mathbf{V}' - \mathbf{V} = \Delta\mathbf{V}$ into two components, $\Delta\mathbf{V}_r$, which will be directed inwards along the radius in the limit $\Delta t \to 0$, and $\Delta\mathbf{V}_T$, which will be directed along the tangent in the limit. By the arguments of Section 2.4, the magnitude of the *radial acceleration* a_r is

$$a_r = \lim_{\Delta t \to 0} \frac{\Delta V_r}{\Delta t} = \frac{V^2}{r} = \omega^2 r \qquad (2.25)$$

and the magnitude of the *tangential acceleration* a_T is

$$a_T = \lim_{\Delta t \to 0} \frac{\Delta V_T}{\Delta t} = \frac{d|V|}{dt} \qquad (2.26)$$

The total acceleration $\mathbf{a} = \mathbf{a}_r + \mathbf{a}_T$ at any time. It is reasonably clear that the instantaneous angular speed, ω, must also depend on t, if we define $V(t) = \omega(t)\, r$ at any instant. The kind of analysis involving Cartesian and polar co-ordinates carried out in section 2.4 becomes more tedious and complex, but not difficult. The results are identical with (2.25) and (2.26) above, except perhaps that we gain the insight that

$$a_T = \frac{d|V|}{dt} = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2} \tag{2.27}$$

which we could have obtained from the definition of instantaneous angular velocity.

2.6 GENERAL TWO-DIMENSIONAL MOTION

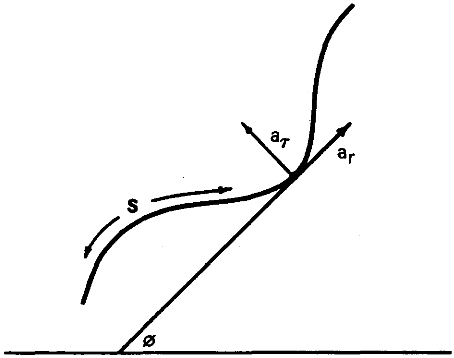

Fig. 2.7 General planar motion

Consider a particle that traces out a curving path in the x-y plane with varying speed (figure 2.7). If we know the expression for the acceleration as a function of time or position, we may in theory calculate the equation of the path; or we may determine the x and y components of acceleration from our knowledge of the path and the instantaneous velocity. The treatment given in the previous section may also sometimes be useful, if extended in the following way. The *radial* component of acceleration (perpendicular to the path) is given by

$$a_r = -\frac{v^2}{r_c} \tag{2.28}$$

where r_c is the *instantaneous radius of curvature* of the path at the particular point; and the *transverse* acceleration (tangential to the path) is given by

$$a_T = \frac{d|v|}{dt} \tag{2.29}$$

where we have stressed the magnitude change only. The expression for r_C is

$$r_C = \frac{ds}{d\phi} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} \left(\frac{d^2y}{dx^2}\right)^{-1} \tag{2.30}$$

(see figure 2.7), where ϕ is the angle the tangent makes with Ox, and s is the distance measured along the path. Hence, in general, this is a somewhat complicated way of looking at things and we shall not be concerned with it.

2.7 MOTION IN A CONSTANTLY ACCELERATED FRAME I

Now that we have solved the equations associated with constant acceleration, we examine the following important problem. Consider a reference frame N, with Cartesian axes x', y', and z', whose origin is initially coincident with that of an inertial frame K at time $t = t' = 0$; the axes x, y, z of K are parallel to the axes x', y', z' of N. Let N be accelerated with uniform acceleration A along Ox. We want to find the transformation of displacement, velocity, and acceleration between K and N. We have:

$$\begin{aligned} x' &= x - \tfrac{1}{2}At^2 \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{2.31}$$

Fixing our attention now on x and x' components, we have

$$\frac{dx'}{dt'} = \frac{dx'}{dt} = \frac{d}{dt}(x - \tfrac{1}{2}At^2) = \frac{dx}{dt} - At \tag{2.32}$$

for the velocity transformation, and

$$\frac{d^2x'}{dt'^2} = \frac{d^2x'}{dt} = \frac{d^2x}{dt^2} - A \tag{2.33}$$

for the acceleration transformation. Thus any constantly (and hence variably!) accelerated frame does *not* keep accelerations constant, and therefore *cannot* be an inertial frame. It is nevertheless sometimes convenient to work in accelerated frames and, of course, necessary to know the consequences of being in an accelerated frame. We shall have more to say on the topic in Chapter 3.

CHAPTER 2 PROBLEMS

2.1 A train travels due north at 10 km/h relative to the ground. A ship travels due west at 5 km/h relative *to a current* flowing south at 2 km/h (relative to the ground). What is the velocity vector of the train relative to the ship?

2.2 A motor boat which travels at 12 m/sec in still water is pointed directly across a running river in which the velocity of the water increases from zero at the banks to a maximum value at the centre. Relative to a stationary observer on the bank of the river, the boat's *speed* varies as shown in figure 2.2p (i).

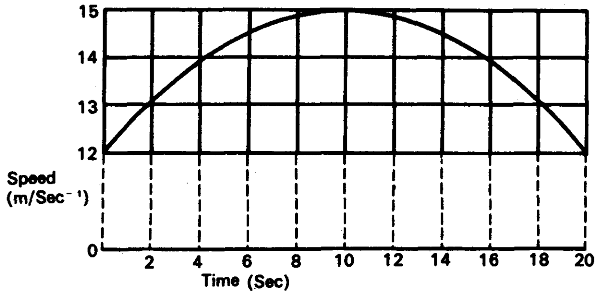

Fig. 2.2p(i)

(a) How wide is the river?
(b) What is the approximate total length of the path followed by the boat?
(c) Which of the paths shown in figure 2.2p (ii) best represents the path taken by the boat?

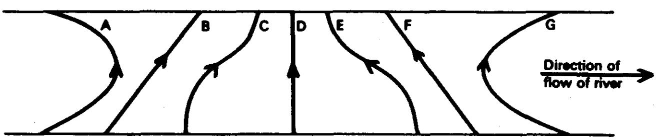

Fig. 2.2p(ii)

2.4 A rocket with a cabin of height 7 m is accelerating vertically upwards with a constant acceleration of 4 m/sec^2. The speed of the rocket at $t = 0$ is zero. At $t = 0$, a particle is released and falls under gravity from the cabin ceiling. Take the acceleration due to gravity as 10 m/sec^2.

(a) At what time T does the particle strike the floor?

(b) What is the velocity of the particle immediately before it strikes the floor

  (i) relative to the rocket?

  (ii) relative to an observer on the ground?

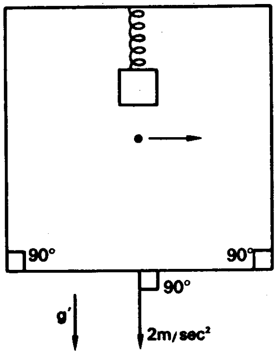

Fig. 2.5p

2.5 A spaceship is accelerating downwards at 2 m/sec^2, as shown in figure 2.5p. The local gravitational acceleration g' is 3 m/sec^2 downwards, parallel to the spaceship's motion. A mass of 1 kg is suspended from the 'ceiling' as shown by a spring whose spring constant is k = 100 newton/metres.

(a) What is the extension of the spring?

An astronaut squirts a water pistol at a wall so that the initial velocity of the water is parallel to the floor. If the initial speed of the water is 2 m/sec, and the path to the wall is 2 m,

(b) how far from the initial line of flight does the water jet strike the wall?

(c) does it strike the wall above or below the initial line of flight?

(d) If the jet initiates at time t = 0, how far does the spaceship move in the time of flight of the jet to the wall, given that the spaceship's velocity at t = 0 is 1 km/sec?

2.6 A merry-go-round at an amusement park is rotating with an anti-clockwise angular velocity $\omega = 0.5$ rad/sec and has a radius of 5 m. The operator of mass 100 kg starts walking at t = 0 from the centre along one of the radial supporting beams with a constant velocity 0·5 m/sec with respect ot the beam.

(a) What is the centrifugal force acting on him at time t ($t < 10$ sec)?

(b) A child riding on the circumference lets a yo-yo dangle loosely. At what angle to the vertical will it lie ($g = 10\ m/sec^2$)?

2.7 A 747 Jumbo Jet is circling at uniform speed V in a horizontal circle of radius R. Take the acceleration due to gravity to be g.

(a) At what angle must the pilot bank the plane in order that the passengers experience local 'gravity' perpendicular to the floor?

(b) What is the magnitude of the local 'gravitational' acceleration in the plane?

(c) A stewardess walking at speed u with respect to the plane crosses the cabin from one side to the other side going 'uphill', i.e. from the inside of the turning circle outwards. At what angle θ must she lean sideways in order to maintain correct balance? In which direction does she lean?

2.8 A stone tied to a piece of string describes a circle with non-uniform speed, the string always remaining taut. Which of the following statements is false?

(a) The velocity vector is always tangential to the circle.

(b) The acceleration vector is always along the radius to the stone.

(c) The angular momentum of the stone about the centre of the circle is not conserved.

(d) The average velocity over one revolution is zero.

2.9 A projectile is fired with speed V from a point 0 at a height h above a horizontal plane. Prove that the greatest distance from 0 at which the projectile can strike the plane is $h + (V^2/g)$.

2.10 An object moves in a circular path with a constant speed v of 50 cm/sec. The velocity vector **v** changes direction by 30° in 2 sec.

(a) Find the magnitude of the change in velocity, Δ**v**.

(b) Find the magnitude of the average acceleration during the interval.

(c) What is the centripetal acceleration of the uniform circular motion?

2.11 Initially two particles are at positions $x_1 = 5$ cm, $y_1 = 0$, and

$x_2 = 0$, $y_2 = 10$ cm with $\mathbf{v}_1 = -4\hat{x}$ cm/sec and $\mathbf{v}_2$ along $-\hat{y}$ as shown in figure 2.11p.

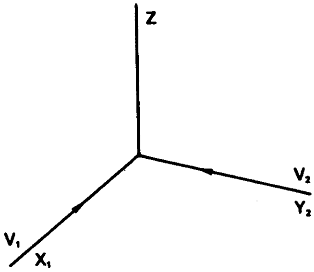

Fig. 2.11p

(a) What must be the value of $\mathbf{v}_2$ if they are to collide?
(b) What is the value of $\mathbf{v}_r$, the relative velocity?
(c) Establish a general criterion for recognizing a collision course for two objects in terms of their positions $\mathbf{r}_1$, $\mathbf{r}_2$ and velocities $\mathbf{v}_1$, $\mathbf{v}_2$.

3

Dynamics of a particle

3.1 OPERATIONAL DEFINITION OF MASS. MOMENTUM, FORCE

We now come to study dynamics, which involves looking at the cause of motion, e.g. forces. We avoid a great many philosophical and logical difficulties associated with Newton's laws of motion and the definition of force if we define *mass* operationally.[1] We also gain insight into the relations between the laws and conservation principles by this method.

We begin by observing the interaction of two bodies in an ideal situation where friction is reduced to a minimum; we can always then extrapolate to the case of zero friction, as Galilei did for his principle of inertia. We might observe two boys on ice-skates tugging at a pole, or two dry-ice pucks connected by a string; the bodies may rotate about each other, or collide, or whatever.

We should find empirically that

(i) the two bodies are always accelerated in opposite directions;

(ii) the ratio of the accelerations is always the same for a particular pair of bodies, irrespective of the motion, although the ratio may be different for different pairs of bodies.

If the co-ordinates of bodies 1 and 2 along the line of the accelerations are x_1, x_2, we should have

$$\frac{\ddot{x}_1}{\ddot{x}_2} = -k_{12} \tag{3.1}$$

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[1] This treatment follows that of K.R. Symon (1961) *Mechanics* (2nd edn, Addison-Weslay, Reading, Mass.)

where $\ddot{x}$ denotes d^2x/dt^2, the minus sign shows that the accelerations are oppositely directed, and k_{12} is a positive constant, characteristic of the particular pair of bodies 1 and 2.

If both bodies were solid and of the same material, we should find that the larger body was always accelerated the less. So some *property* of a body determines how it is accelerated. We call this property *mass,* and give it a number, as follows.

Choose a 'standard mass' – body 1, for example. Then take two different bodies, 2 and 3, and study the accelerations of three pairs in turn, i.e. we determine k_{12}, k_{23}, and k_{31}.

We find that

$$k_{12}\, k_{23}\, k_{31} = 1 \tag{3.2}$$

Let body 1 have *unit* mass. Then we can *define* as follows:

$$\text{mass of body i} = m_i = k_{1i} = -\frac{\ddot{x}_1}{\ddot{x}_i} \tag{3.3}$$

Because of relation (3.2) we can get

$$\begin{aligned}\frac{\ddot{x}_2}{\ddot{x}_3} = -k_{23} &= -\frac{1}{k_{12}\, k_{31}} \\ &= -\frac{k_{13}}{k_{12}} = -\frac{m_3}{m_2}\end{aligned} \tag{3.4}$$

i.e. the *ratio* of two masses is the negative inverse of the ratio of their accelerations when paired, irrespective of the unit of mass. By equations (3.3) and (3.4) we then obtain

$$m_2\ddot{x}_2 = -m_1\ddot{x}_1 \tag{3.5}$$

suggesting that the product $m_i\ddot{x}_i$ will be important. We therefore call this quantity the *force* acting on the body. By its definition it is also a *vector*, like acceleration.

Mass turns out to be a scalar; we find empirically that masses m_1 and m_2 fastened together behave as a single body of mass $(m_1 + m_2)$. The unit of mass is the kilogramme (kg), which is roughly the mass of 10^3 cc of water. The unit of force is the newton (N), which gives 1 kg an acceleration of 1 $msec^{-2}$.

If we *define* a quantity $m\dot{x}$, the *momentum*, we find that, if m does not change,

$$\frac{d}{dt}(mx) = m\ddot{x} = \text{force} \tag{3.6}$$

which is *Newton's Second Law of Motion.*

Further, if we integrate equation (3.5) we obtain

$$m_2 \dot{x}_2 = - m_1 \dot{x}_1 + \text{constant} \tag{3.7}$$

which leads to conservation of momentum.

Note also that equation (3.5), referring to the two interacting masses, contains *Newton's Third Law of Motion,* since the force $m_1 \ddot{x}_1$ exerted by m_2 is equal and *opposite* (the negative sign) to the force $m_2 \ddot{x}_2$ exerted by m_1.

The mass we have defined is *inertial mass.* If we allowed only gravitational, or only electromagnetic, interactions, we should then define 'gravitational mass' or 'electromagnetic mass'.

It is an empirical fact that all 'masses' so defined are equal.

The equality of inertial and gravitational mass has been confirmed to great accuracy in a large series of experiments, beginning with those of Newton, who observed the periods of pendulums of the same length but different bobs, found that the periods were the same, and concluded that gravitational and inertial mass were equal to at least 1 part in 10^3. Further experiments were performed by Eötvos in the nineteenth century, using a torsion balance and the acceleration on a body due to the earth's rotation, and, more recently, by Dicke and others;[2] these last took the equality to 1 part in 10^{10}. The most recent experiments are those of Braginskii and Panov[3] who report equality to $\sim$1 part in 10^{12}.

3.2 NEWTON'S THREE LAWS OF MOTION. SOME CLARIFICATIONS

For convenience, we re-state here Newton's three laws of motion. Note that we have defined mass operationally, and force from mass and acceleration.

(1) *The principle of inertia* Every body will continue in its state of rest or of uniform rectilinear motion unless compelled to change that state by the action of some *force.* This principle has been discussed previously (Section 2.1).

(2) The rate of change of momentum is proportional to the impressed force, and takes place in the direction of the force.

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[2] R.H. Dicke (1961), *Scientific American,* 205, 84.
[3] V.B. Braginskii and V.I. Panov (1972), *Soviet Physics – JETP,* **34**, 463.

We express this mathematically thus:

$$\frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} = \mathbf{F} \tag{3.8}$$

where we have taken the mass to be constant and **F** is the vector sum of *all* the forces acting on the particle.

Why have we insisted that the mass should remain constant? First, our experience tells us that in the macroscopic situations with which classical mechanics deals successfully the total mass is always conserved. Secondly, the naive mathematical treatment of (3.8) leads to trouble, as can be shown immediately. Let us write

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F} = m\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dm}{dt} \tag{3.9}$$

If this equation is to be meaningful, it must be covariant with respect to a Galilean Transformation, as equation (3.8) is. We find that the terms m, $\frac{d\mathbf{v}}{dt}$, and $\frac{dm}{dt}$ are unchanged by a Galilean transformation, but that **v** is changed, and therefore the expression, though correct mathematically, is incorrect *physically*. If **v** could be interpreted as a *relative* velocity, equation (3.9) would then be covariant with respect to Galilean Transformations, since relative velocity is a vector invariant. We shall derive the force equation for systems of variable mass later on, and we shall indeed see that rather than **v**, we must insert the *relative* velocity of the infinitesimal mass dm emitted in time dt. The reader should therefore beware of following mathematics blindly without checking whether the physics is correct.

(3) Action and reaction are equal and opposite.

We can express this mathematically by saying that if $\mathbf{F}_{12}$ is the force due to the action of body 2 on body 1, and $\mathbf{F}_{21}$ is the force due to the action of body 1 on body 2, then

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \tag{3.10}$$

The reader must understand that *in no sense can these action and reaction forces be in equilibrium, since they act on different bodies.* Furthermore, the 'paired' bodies must be specified. If a book is resting on a table, the answer to the question 'What is Newton's Third Law reaction force to the gravitational attraction of the earth for the book?' is *not* 'the reaction force from the table', but 'the gravitational attraction of the book for the earth'.

The analysis of a very simple problem may help to clarify matters further. Figure 3.1a shows a block of mass M being pulled along by a rope of mass m; the rope is pulled by a hand. Figure 3.1b shows the system 'broken' at various points, so that we can see the various action and reaction forces coming into play. The suffix convention is that the body acted *on* comes first: thus F_{Mm} is the force exerted on M by m, and so on.

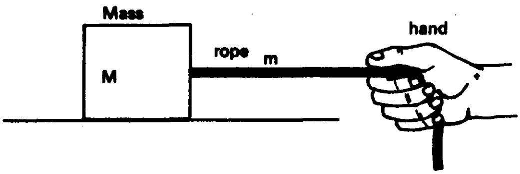

Fig. 3.1a *A simple system*

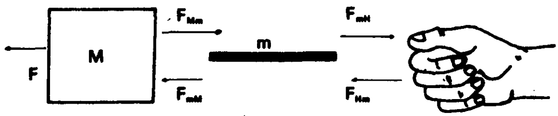

Fig. 3.1b *The action and reaction forces*

Consider the situation when the whole system is at rest; we may suppose that a frictional force **F** acts on the left on M. Then we have the equations:

(i) for M $\qquad \mathbf{F}_{Mm} + \mathbf{F} = 0 \qquad$ (3.11a)

(ii) for m $\qquad \mathbf{F}_{mM} + \mathbf{F}_{mH} = 0 \qquad$ (3.11b)

(iii) Newton's Third Law $\mathbf{F}_{Mm} = -\mathbf{F}_{mM} \qquad$ (3.11c)

from which we conclude that $\mathbf{F}_{mH} = -\mathbf{F}$.

Now let us suppose that the whole system is accelerating to the right with acceleration a, and that there is no friction. The equations now become:

(i) for M $\qquad \mathbf{F}_{Mm} = M\mathbf{a} \qquad$ (3.12a)

(ii) for m $\qquad \mathbf{F}_{mM} + \mathbf{F}_{mH} = m\mathbf{a} \qquad$ (3.12b)

(iii) Newton's Third Law $\quad F_{Mm} = -F_{mM} \quad$ (3.12c)

from which we conclude that $F_{mH} = (m + M)a$. It is clear that, although the action and reaction forces are equal and opposite in this case, there is no equilibrium, because the whole system is accelerating. This is the resolution of the apparent 'horse and cart' paradox, in which the possibility of the horse accelerating the cart is supposed to be *denied* by Newton's Third Law.

We observe from equation (3.12b) that the smaller m is, the more nearly $F_{mM} = F_{mH}$, i.e. strictly speaking only a massless string transmits a tension unchanged. The exercise of calculating the distribution of tension in a string of mass per unit length μ and length ℓ hauling a mass M with acceleration a is not difficult, and is left to the reader.

3.3 THE FUNDAMENTAL FORCES OF NATURE. BEHAVIOUR OF FORCES

It is thought at this stage in the development of physics that there are only four fundamental interactions which may be thought of as giving rise to all the various kinds of forces. For example, a spring force has its origin in the electromagnetic interactions between the atoms of which the spring is composed. In order of their strength, these interactions are (i) the strong interaction, (ii) the electromagnetic interaction, (iii) the weak interaction, and (iv) the gravitational interaction.

The strong interaction holds the nuclei of atoms together. Its dependence is roughly of the form

$$F \propto r^{-2} \exp\left(-\frac{r}{r_0}\right)$$

where r_0 is of the order of 10^{-13} cm. It is thus effectively a very short-range force, but very strong within the 10^{-13} cm range.

Let us compare the fundamental interactions by giving the forces a number in a particular situation; we choose to consider two protons 10^{-13} cm apart. Then the strong interaction has the value $\sim 2 \times 10^3$ newton.

The electromagnetic interaction (Coulomb interaction) holds atoms together. It is responsible for mechanical forces such as friction, spring forces, and inter-body forces in contact collisions. The Coulomb force is a long-range force, having an inverse square dependence on the distance r from the source, i.e.

$$F \propto r^{-2}$$

The electrostatic force between two protons 10^{-13} cm apart is $\sim 2 \times 10^2$ newton.

The weak interaction is an even shorter-range force than the strong interaction; it is responsible for the decay of radioactive nuclei which emit β particles (electrons) and neutrinos. The force between the protons 10^{-13} cm apart from the weak interaction is 2×10^{-11} newton.

The gravitational interaction is also a long-range, inverse square force, which despite its weakness acts over vast distances and holds together the solar system, galaxies, and clusters of galaxies. Gravity is not well understood, despite the fact that it probably affects our lives the most of all the fundamental interactions. The gravitational attraction between two protons 10^{-13} cm apart is $\sim 2 \times 10^{-34}$ newton.

We should recognize that talking about the 'forces' due to the strong and weak interactions is, strictly, going beyond the realm of classical mechanics, because for fundamental particles and small distances we should be using quantum mechanics. From quantum mechanics comes the viewpoint that we may regard the various interactions as arising from the exchange of 'particles' between the interacting objects. The particles involved in the strong interaction are mesons; in the electromagnetic interaction, photons; in the weak interaction, hypothetical intermediate bosons. The 'graviton' has been postulated for gravity. We note that two skaters throwing a heavy ball to each other would eventually separate, so that the 'exchange' of the ball leads to an effective repulsive force; to obtain a similar attractive effective force requires a little more ingenuity!

We often make models of forces, and it is worth considering some of them. For example, a first approximation to the collision force between rigid bodies is shown in figure 3.1c, plotted against separation.

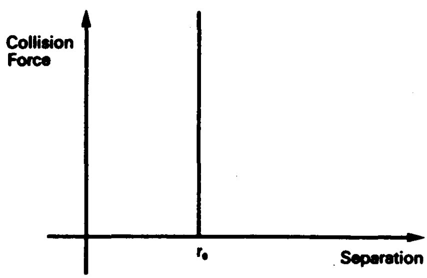

Fig. 3.1c Collision force between two rigid bodies. A first approximation

The rigid bodies affect each other only on impact, and then with an infinite force. A next approximation is the so-called 'hard-core' one, in which the force becomes very large at some distance, r_0 say, and builds up gradually to this, with an initial, almost linear, spring-force region near the contact region. This is shown in figure 3.1d. In a collision the force on a body would then vary with time as shown in figure 3.1e. We shall have occasion to return to this model later on.

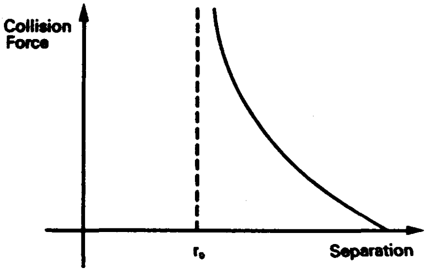

Fig. 3.1d *The 'hard-core' approximation for collision between two rigid bodies*

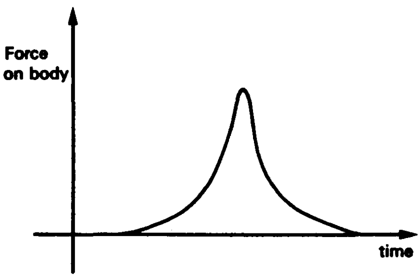

Fig. 3.1e *Actual collision force between two rigid bodies*

The frictional force in sliding friction is often represented as being dependent only on the reaction force **R** of the surface to the sliding body. The dependence of the magnitude of the frictional force is the familiar

$$F = \mu R \tag{3.13}$$

which is clearly velocity-independent. But we may have frictional forces such as damping forces which depend on the velocity and always oppose it,

$$\mathbf{F} = -b\,\mathbf{v} \tag{3.14}$$

or resistance forces which depend on the square of the velocity, e.g.

$$\mathbf{F} = -cv^2\hat{\mathbf{v}} \tag{3.15}$$

Finally, a velocity-dependent force which is of considerable importance is the force exerted by a magnetic field of induction **B** on a particle of charge q having velocity **v**: it is given by

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{3.16}$$

3.4 MOTION IN A CONSTANTLY ACCELERATED FRAME II. PRINCIPLE OF EQUIVALENCE

We return now to the study of motion in a constantly accelerated frame, and its implications in terms of Newton's laws of motion. Figure 3.2 depicts the following situation.

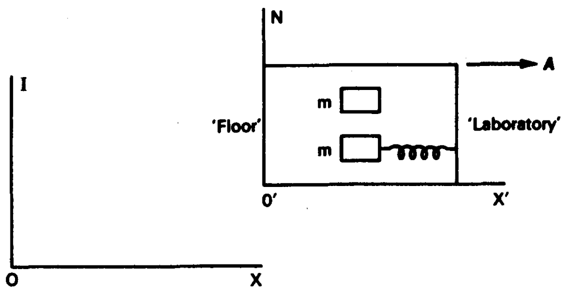

Fig. 3.2 Experiments in a constantly accelerated laboratory

With respect to the fixed inertial frame I, the non-inertial frame N is moving along Ox with a constant acceleration $A\hat{\mathbf{x}}$. Consider a 'laboratory' in N as shown; an observer in it cannot see out of the laboratory, but can communicate with I, which can see into the 'laboratory'.

The observer in N performs the following experiments.

(1) He releases a mass m from 'rest' in N.

(2) He suspends a mass m from a spring balance as shown.
(3) He observes a pulse of light which is emitted from I parallel to Oy.

The observer in I describes the results as follows. On release, the mass m continues moving along parallel to Ox with the instantaneous velocity it had just before release, since no forces now act and the law of inertia holds. However, the laboratory N accelerates past the mass m, which is therefore struck by the 'floor'.

The mass m requires a force **F** to keep it accelerating. This force is given by **F** = mA, and so the spring will be stretched by the amount necessary to make the tension force in it equal in magnitude to mA.

Let the pulse of light enter the laboratory at time t = 0, and travel for a time Δt *parallel to Oy.* The distance travelled parallel to Oy will be $c\Delta t$; meanwhile the laboratory will have moved a distance $\frac{1}{2}A(\Delta t)^2$ parallel to Ox.

The observer in N describes the results as follows. On release, the mass m accelerates towards the floor with acceleration – A.

The spring is stretched by just the amount that it would be stretched if there were a 'weight force' –mA acting towards the floor.

The light beam travels a distance $c\Delta t$ parallel to O′y′, and a distance $-\frac{1}{2}A(\Delta t)^2$ parallel to O′x′. These are the parametric equations of a *parabola.* (Refer to a particle projected horizontally in a uniform vertical gravitational field of acceleration A downwards; see figure 3.3.)

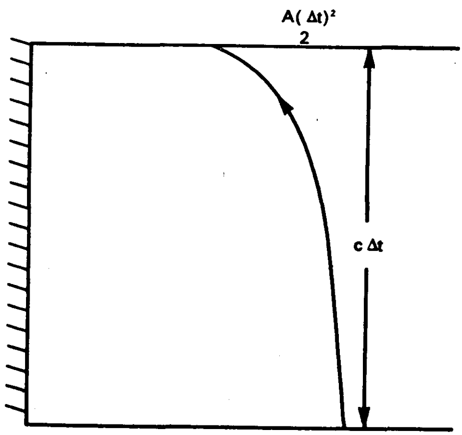

Fig. 3.3 *Apparent light path in the accelerated frame*

Since N cannot see out of the laboratory, he cannot tell

(i) whether the laboratory is being uniformly accelerated parallel to Ox (as I *knows* it is); or
(ii) whether he (N) is *stationary* in a uniform gravitational field of acceleration $-\mathbf{A}$ parallel to $O'x'$.

Thus in every dynamical experiment N performs he must postulate the existence of the force $-m\mathbf{A}$ on a mass m if, for him, the law of inertia is to hold. From the viewpoint of the *inertial frame* I, this force is *non-existent.* It is therefore called a 'pseudo-force', 'fictitious force', or (better) an 'inertial force', since it is very real to anyone experiencing it, as in a rapidly decelerating car. Hence:

(i) in a uniformly accelerated frame the effects of the acceleration are equivalent to those of a uniform gravitational field of the same acceleration, but oppositely directed;
(ii) the acceleration $\mathbf{a}_N$ relative to the non-inertial frame N is equal to the acceleration $\mathbf{a}_I$ relative to the inertial frame I *minus* the acceleration $\mathbf{A}$ of the non-inertial frame:

$$\mathbf{a}_N = \mathbf{a}_I - \mathbf{A} \text{ or } \mathbf{a}_I = \mathbf{a}_N + \mathbf{A} \tag{3.17}$$

Einstein's 'Principle of Equivalence', on which is founded his General Theory of Relativity, states that it is impossible to distinguish between uniform acceleration of a laboratory and an oppositely directed, uniform gravitational acceleration (field) of the same magnitude by experiments performed solely in the laboratory. A consequence of this is (refer to experiment 3) that the path of a light ray should be *curved* in passing through a gravitational field. The results of measurements on the bending of starlight passing near the sun are not inconsistent with Einstein's General Relativistic prediction. Very accurate measurements of the delay of radar echoes from Venus[4] which pass by the sun, and of the radio signals from deep space probes,[5] have agreed very well with Einstein's predictions. A further prediction is that frequencies of electromagnetic radiation emitted by similar sources will differ if the sources are in different gravitational fields, or if they are at different heights in a constant gravitational field (Appendix 2). This has also been verified exactly.[6] Finally, the General Theory of Relativity predicts gravitational waves, whereas the Newtonian theory does not. It *may* be that these waves have been detected.[7]

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[4] I.I. Shapiro *et al.* (1968) *Phys. Rev. Letters,* **20**, 1265.
[5] J.D. Anderson (1971), *NASA-JPL Technical Memorandum,* 33-499.
[6] R.V. Pound and G.A. Rebka Jr (1960), *Phys. Rev. Letters,* **4**, 337.
[7] J. Weber (1970), *Phys. Rev. Letters,* **24**, 276.

It is worth remembering that Einstein's Principle of Equivalence came from thinking about experiments in a constantly accelerating frame, and that the very simple thoughts we have been examining were put forward *for the first time* by Einstein early in this century.

Examples: 'Weightlessness'

We consider first a mass M in a lift which is in a uniform gravitational field of acceleration $-g\mathbf{j}$, and is being accelerated upwards with an acceleration $+a\mathbf{j}$ (figure 3.4a).

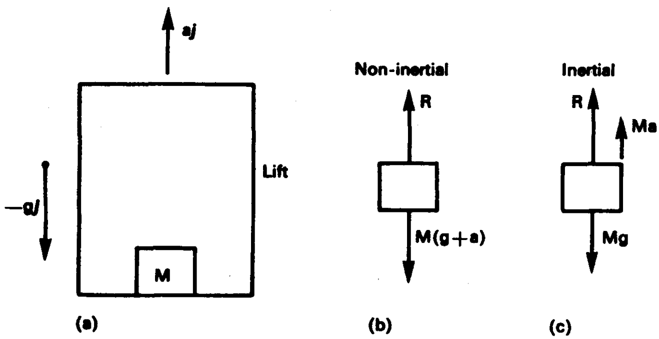


Fig. 3.4 (a) The lift problem (b) Non-inertial frame (c) Inertial frame

In the non-inertial frame of the accelerating lift, the effective gravitational field is therefore the sum of the actual gravitational field and that given by the principle of equivalence, i.e. we may write (figure 3.4b);

$$g_{eff}\,\mathbf{j} = -g\mathbf{j} - a\mathbf{j} \qquad (3.18)$$

and since the mass is stationary in the non-inertial frame, the reaction force **R** from the floor to the lift must be given by

$$\mathbf{R} - M(g\mathbf{j} + a\mathbf{j}) = 0, \text{ i.e. } \mathbf{R} = M(a + g)\,\mathbf{j} \qquad (3.19)$$

In the fixed inertial frame (figure 3.4c) we observe that the only forces acting on M are (i) the actual gravitational force $-Mg\mathbf{j}$ and (ii) the reaction force **R** from the floor. But M has acceleration $+a\mathbf{j}$, and therefore

$$\mathbf{R} - Mg\mathbf{j} = Ma\mathbf{j} \qquad (3.20)$$

giving $\mathbf{R} = M\,(g + a)\,\mathbf{j}$ as before

Consider now a lift in a gravitational field of acceleration $-g\mathbf{j}$, which is in 'free fall', i.e. the actual acceleration is also $-g\mathbf{j}$. *In the non-inertial frame* the actual acceleration $-g\mathbf{j}$ of the frame gives rise to the equivalent 'gravitational acceleration' $+g\mathbf{j}$. The resultant of this and the *actual* gravitational acceleration $-g\mathbf{j}$ is zero: therefore $\mathbf{R} = 0$, and we have a 'weightless' situation.

In the fixed inertial frame the only forces acting on M are (i) the weight force $-Mg\mathbf{j}$ and (ii) the reaction force $\mathbf{R}$. But M has acceleration $-g\mathbf{j}$:

$$\mathbf{R} - Mg\mathbf{j} = -Mg\mathbf{j}, \text{ i.e. } \mathbf{R} = 0 \tag{3.21}$$

and we come to the same conclusion

In the non-inertial frame of a satellite in a circular orbit about the earth the actual centripetal acceleration gives rise to an effective centrifugal gravitational field. But the satellite *obtains* its centripetal acceleration from an *actual* gravitational field, such that if the mass of the satellite is m, $mg(r) = mv^2(r)/r$, where $v(r)$ is the velocity of the circular motion at radius r. Hence, as Newton recognized for the moon, the satellite is in 'free fall' towards the earth, and therefore by exactly similar arguments as for the lift in free fall, we have a weightless situation.

In the fixed inertial frame the only force acting is the gravitational force $mg(r)$, which gives rise to the centripetal acceleration of value $g(r)$. Therefore, by the same argument as for the lift case, an object at rest on the 'floor' of the satellite closest to the earth will experience no reaction force. Since we must go very far away from gravitating matter before the actual gravitational field is very close to zero, man is likely to experience weightlessness in space travel only under 'free fall' conditions!

3.5 MOTION IN A ROTATING FRAME (TWO DIMENSIONS)

We have seen that the law of inertia is obeyed only in certain frames of reference called inertial frames. The law is not obeyed in any reference frame which is accelerating with respect to an inertial frame. A further example is a rotating reference frame. However, as with constantly accelerated frames, we can maintain a fiction that the law of inertia *is* obeyed in such a frame if we determine the inertial forces and allow these to act, as well as any real forces which may be present.

To see what happens in a rotating frame of reference we shall allow a particle unrestricted movement in the x-y plane and from the vector position $\mathbf{r}$ derive its velocity and acceleration vectors. We shall not use the co-ordinates x and y but the more convenient polar

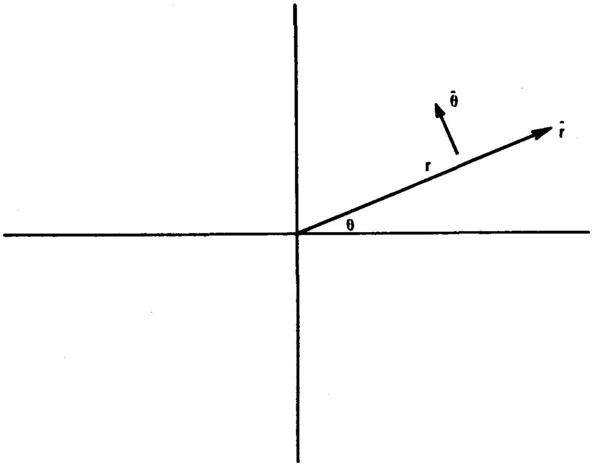


Fig. 3.5 Polar co-ordinates

co-ordinates r and θ (figure 3.5). At any position on the plane we can define two new unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$:

$$\hat{\mathbf{r}} = (\cos\theta, \sin\theta) \qquad (3.22)$$

$$\hat{\boldsymbol{\theta}} = (-\sin\theta, \cos\theta)$$

By forming the product $\hat{\mathbf{r}}.\hat{\boldsymbol{\theta}}$ we see that these are mutually perpendicular and by forming $|\hat{\mathbf{r}}|$ and $|\hat{\boldsymbol{\theta}}|$ we see that they are unit vectors.

We describe the position of the particle in the fixed frame of reference by

$$\mathbf{r} = (x, y) = (r\cos\theta, r\sin\theta) = r(\cos\theta, \sin\theta) \qquad (3.23)$$

In the possible motion both the radial co-ordinate r and the angular co-ordinate θ can vary; hence we shall include both in any differentiation. The velocity is

$$\dot{\mathbf{r}} = \dot{r}(\cos\theta, \sin\theta) + r\dot{\theta}(-\sin\theta, \cos\theta) = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \qquad (3.24)$$

We see that in general the motion is partly in the radial direction $\hat{\mathbf{r}}$ and partly in the tangential direction $\hat{\theta}$. The acceleration is

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{r}(\cos\theta, \sin\theta) + \dot{r}\dot{\theta}(-\sin\theta, \cos\theta) \\ &\quad + \dot{r}\dot{\theta}(-\sin\theta, \cos\theta) + r\ddot{\theta}(-\sin\theta, \cos\theta) \\ &\quad + r\dot{\theta}^2(-\cos\theta, -\sin\theta) \\ \ddot{\mathbf{r}} &= (\ddot{r} - \dot{r}\theta^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \qquad (3.25)\end{aligned}$$

i.e. any acceleration is partly radial and partly tangential.

So far we have described things as viewed in the fixed frame which is an inertial frame; hence we know that any acceleration $\ddot{\mathbf{r}}$ is produced by a real force. By a real force we mean a force produced by the deformation of a restraining spring or the bending of the supports of the particle or an electromagnetic or gravitational force. An observer in the fixed frame sees only real forces producing real accelerations in the fixed frame. For example, a real centripetal force produces a real centripetal acceleration.

In the rotating frame of reference these real forces are still present (the spring is still stretched, the support is still bent). But an observer in the rotating frame sees different accelerations, and hence in order to maintain the fiction that the law of inertia works in his rotating frame, he has to 'invent' extra 'fictitious forces'. These 'fictitious forces' (inertial forces) are unnecessary, and do not exist in the non-rotating, fixed inertial frame, but in the rotating frame they are needed to make the law of inertia work. Let us consider two special examples.

(a) The particle is at rest in a uniformly rotating frame. In our description this means $\dot{r} = 0$, $\ddot{r} = 0$, and $\ddot{\theta} = 0$, and in the fixed frame we have a real centripetal force as the only acting force. In the rotating frame this real force still acts but the particle is at rest. Hence we require an inertial 'centrifugal force' to balance it. This is equal in size and opposite in direction to the real force.

(b) The particle moves uniformly along a radius in the rotating frame. Then $r > 0$, $\ddot{r} = 0$, and $\ddot{\theta} = 0$, and apart from centripetal acceleration in the radial direction there is an acceleration of $2\dot{r}\dot{\theta}\,\hat{\theta}$ in the tangential direction. In the fixed frame this is produced by a real force. In the rotating frame we see the particle moving in a radial straight line. Hence there can be no net tangential force on it, so we require an inertial force equal and opposite to the real tangential force. This inertial force is called the *Coriolis force.*

The surprising thing is that if, in the rotating frame, we remove the real forces (remove the supports on the particle), then the inertial forces still remain in the rotating frame and produce real accelerations as seen in the rotating frame. Of course in the fixed frame of reference the acceleration will be zero in such a case.

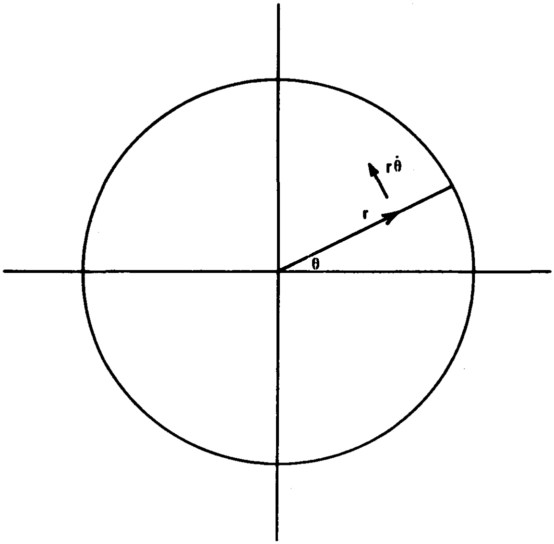


Fig. 3.6 Origin of the Coriolis acceleration (i)

We can visualize the origin of the real tangential force as follows. We find it to come from two sources. To fix our ideas, let $\ddot{r} = 0, \ddot{\theta} = 0$, $\dot{r}$ = constant > 0, and $\dot{\theta}$ = constant > 0, so that the particle is moving outwards along a radius at uniform radial velocity, while the radius turns at angular speed $\dot{\theta}$ (figure 3.6). Then the linear speed of the particle in the tangential direction is just $r\dot{\theta}$. If in time δt the radial position increases to $r + \delta r$, the tangential speed increases to $(r + \delta r)\dot{\theta}$, So the acceleration is

$$\frac{(r + \delta r)\,\dot{\theta} - r\dot{\theta}}{\delta t} \rightarrow \dot{r}\dot{\theta}$$

We see that a term $\dot{r}\dot{\theta}$ arises as Coriolis acceleration as a result of changing the magnitude of the tangential velocity at constant direction.

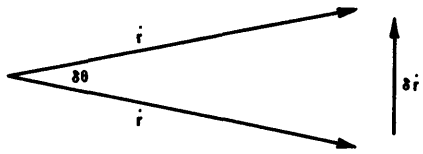


Fig. 3.7 Origin of the Coriolis acceleration (ii)

Now suppose we consider the radial velocity. In time t the direction of the radial velocity changes by $\dot{\theta}\delta t$ (figure 3.7).

$$\delta\dot{r} = \dot{r}\delta\theta$$

Hence the vector change $\qquad = \dot{r}\dot{\theta}\delta t$

So $\qquad \dfrac{\delta\dot{r}}{\delta t} = \dot{r}\dot{\theta}$

is the term which arises in the Coriolis acceleration as a result of change in direction of the radial velocity at constant magnitude. Therefore the total Coriolis acceleration is $2\dot{r}\dot{\theta}\ \hat{\theta}$ The effective 'gravitational acceleration', the Coriolis force per unit mass, is therefore $-2\dot{r}\dot{\theta}\ \hat{\theta}$ when we are *in* the rotating frame.

On the surface of a rotating sphere the forces are just as easy to understand in principle, but the mathematical expression is a little more complex. Coriolis forces cause wind velocity patterns to be whirlpool-shaped. They make corrections necessary in gunnery range-finding, and also cause the strange motion of the Foucault Pendulum. Thus the Coriolis force, like the centrifugal force, is one which arises in a rotating frame of reference. A particle at rest in a rotating frame has a centrifugal force acting on it. Note that an observer in the rotating frame of reference can account for stationary or uniformly moving particles in the rotating frame only by postulating the existence of centrifugal and Coriolis forces. Although 'fictitious', the forces can be treated as real in the rotating frame – they do work on the particle – and are examples of inertial forces which arise in frames that are accelerating relative to inertial frames.

3.6 DYNAMICS OF A PARTICLE. EXAMPLES

The dynamics of a particle are effectively contained in the Second Law of Motion:

$$\mathbf{F} = \sum_i \mathbf{F}_i = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = m\frac{d^2\mathbf{r}}{dt^2} \tag{3.26}$$

where $\sum_i \mathbf{F}_i$ is the sum of the forces acting on the particle of mass m. The fact that m is a constant for the particle effectively reduces the problem to the kinematical one discussed in Chapter 2. Thus if we know the dependence of the components of the resultant force **F** on the co-ordinates and the time, we can, in principle at any rate, determine completely the velocity and displacement components of the particle. This is not to say that we shall find every problem easy to solve!

We now give some simple examples of dynamical situations which reduce to kinematical situations which have already been dealt with.

(i) Particle of mass m and charge q in a constant electric field $\mathbf{E} = E\mathbf{j}$. We have

$$m\ddot{y} = qE \text{ or } \ddot{y} = \frac{q}{m} E \tag{3.27}$$

which is the case of constant acceleration dealt with in Section 2.3.

(ii) Particle of mass m subject to the restoring force $\mathbf{F} = -kx\mathbf{i}$. We have

$$m\ddot{x} = -kx$$

or

$$\ddot{x} = -\frac{k}{m} x \tag{3.28}$$

which we discussed but did not solve in Section 2.2. The solution (check by substitution) is

$$x = a \cos\left(\sqrt{\frac{k}{m}}\, x + \phi\right)$$

where A and ϕ are constants of integration, usually specified by the initial conditions.

(iii) The magnetic force on a charged particle. We consider a particle of mass m and charge q projected into a magnetic field of induction **B** with velocity **v**. Then

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

By virtue of the properties of the vector product, the component of **v** parallel to **B** contributes nothing to the force. We may therefore choose our z axis along **B**, and let the x and y axes be perpendicular to **B**. With this convention we may write

$$\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}$$

$$\mathbf{B} = B\mathbf{k} \qquad (3.29)$$

Since the force is now only in the x-y plane, we move into a Galilean frame x′, y′, z′, moving with velocity v_z along 0_z; we assume that **B** is unchanged by this transformation (and it turns out that, even using special relativity, we are correct). In this frame

$$\mathbf{v}' = v_x\,\mathbf{i}' + v_y\,\mathbf{j}'$$

$$\mathbf{B}' = B\mathbf{k}'$$

and

$$\mathbf{F}' = \mathbf{F} = q\mathbf{v}' \times \mathbf{B} \qquad (3.30)$$

Now since **B** is constant in direction, **F** is always perpendicular to **B**; by the vector-product definition, it is always perpendicular to **v**′. Therefore it changes only the *direction* of **v**′ with time, and not the *magnitude*, just as occurs in circular motion; therefore in the x′, y′, z′ frame we have circular motion such that

$$\frac{m|\mathbf{v}'|^2}{r} = q|\mathbf{v}'|B \qquad (3.31)$$

so that the radius of the circle is given by

$$r = \frac{m|\mathbf{v}'|}{qB} \qquad (3.32)$$

Since we can choose the x′ and y′ axes anywhere, if the velocity **v**′ at the instant of the injection of the particle is along Ox′, say, it is clear that mv′ is actually the magnitude of the momentum of the particle perpendicular to **B**. This can therefore be used as a velocity- or momentum-selecting property for particles of a particular q/m ratio. We further note that the period T of the circular motion is given by

$$T = \frac{2\pi m}{qB} = \frac{2\pi}{\omega} \text{ where } \omega = \frac{qB}{m} \qquad (3.33)$$

and is the angular frequency: i.e. the period depends only on q/m and B, and is independent of r. Transforming back to the Oxyz frame, it is clear that the motion of the charged particle is a *spiral* about the z axis, with *pitch* $v_z T$;

$$v_z T = \frac{2\pi m v_z}{qB} \tag{3.34}$$

(iv) Resistive force. We consider a particle of mass m given an initial velocity v_0 in a medium where it experiences a resistive force $-bv$, where v is the velocity. For simplicity, we reduce the situation to one dimension, and we have

$$m\ddot{x} = -b\dot{x} = -bv = m\frac{dv}{dt} \tag{3.35}$$

whence $$\frac{dv}{v} = -\frac{b}{m}\,dt$$

giving $$v = v_0 \exp\left(-\frac{b}{m}t\right) \tag{3.36}$$

and $$x = \int_0^t v\,dt = v_0\frac{m}{b}\left[1 - \exp\left(-\frac{b}{m}t\right)\right] \tag{3.37}$$

so that the particle actually comes to rest at a *finite distance* from the origin after an infinite time!

(v) Terminal velocity. Consider a particle of mass m which is subject to a constant acceleration, say g, and a resistive force proportional to the velocity. We then have

$$m\ddot{x} = mg - b\ddot{x} \tag{3.38}$$

and we see that $\ddot{x} = 0$ when $\dot{x} = \frac{mg}{b}$, i.e. eventually the particle, accelerated from rest, will reach a *terminal velocity* mg/b. This fact is made use of in Millikan's experiment, where the constant acceleration of the charged oil-drops is provided by a constant electric field. Note that since the particle at its terminal velocity is moving with *constant* velocity, the *resultant* of the constant force and the velocity-dependent force must be zero.

(vi) The reflex klystron. The reflex klystron is an electronic valve used to generate microwaves, i.e. electromagnetic waves, with wavelengths of a few centimetres to a few millimetres.

An electron beam is generated and shaped by a suitable 'electron gun'. It passes through a pair of very closely spaced grids, which can be regarded as the plates of a parallel-plate capacitor; the grids are at a high positive potential with respect to the electron gun. After passing through the grids the beam is subjected to a retarding force by means of an electrode called the 'reflector', which is at a negative potential with respect to the electron gun. The beam is thus turned back on itself, and passes through the grids again.

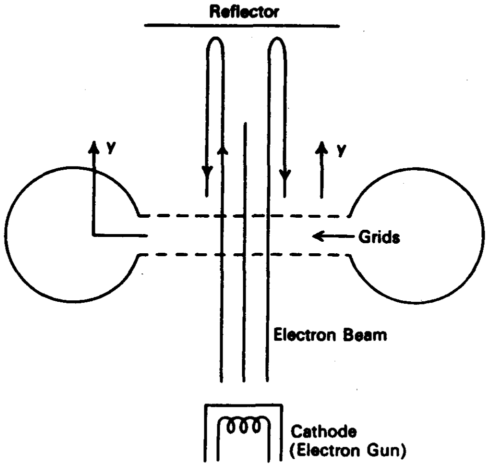


Fig. 3.8 *The reflex klystron (schematic)*

We consider all the electrodes as plane-parallel so that we need consider the motion in one dimension only. We ignore interaction between individual electrons, and the transit-time effects through the grids. Figure 3.8 is a schematic diagram. First, suppose that the electrons in the beam are uniformly spaced in time, say τ seconds apart. Let us examine the motion in the grid-to-reflector space, assuming a uniform retarding field and no alternating voltage on the grids. Let the velocity of the electrons at the grids be v_0, and the field in the grid-reflector space be E. If the electron mass is m, Newton's Law of Motion tells us that

$$m\ddot{y} = -eE$$

or

$$\ddot{y} = -\frac{eE}{m}$$

from which it is not difficult to show that if an electron passes through the grids at time t_1, its time of flight in the grid-reflector region will be

$$t_o = \frac{2m}{eE} v_o \qquad (3.39a)$$

and so it returns to the grids at time

$$t_1 + t_o = t_1 + \frac{2m}{eE} v_o \qquad (3.39b)$$

Hence the electrons will return to the grids still uniformly spaced in time, and there will be no resultant a-c component of current (or no changing charge density) induced on the grids considered as a parallel-plate capacitor.

However, the grids form part of a circuit which is tuned to a frequency, say ω. Noise of thermal origin is always present, so a small alternating field will exist between the grids; if the circuit is sharply tuned, this field must be at frequency ω. Let this field be $E_1 \cos \omega t$; then Newton's Second Law tells us that

$$m\ddot{y} = -eE_1 \cos \omega t \quad \text{(between the grids)}$$

so that

$$\dot{y} = v_o - \frac{eE_1}{\omega m} \sin \omega t$$

i.e. there is now a small periodic velocity component on the electron stream after it passes through the grids. We choose to consider its effect by saying that if an electron has passed through the grids at time t_1, the alternating component velocity is

$$v_1 \sin \omega t_1$$

(Note: $v_1 \neq eE_1/\omega m$, because of transit-time effects.)

The time of flight in the field, from (3.40a), is then

$$t_f = \frac{2m}{eE} (v_o + v_1 \sin \omega t_1)$$

which is *no longer independent* of t_1, i.e. of the time when the electron enters the retarding field. The time of return of this electron (1) to the grids is

$$t_{R1} = t_1 + t_f = t_1 + t_o + \frac{2m}{eE} v_1 \sin \omega t_1 \qquad (3.40)$$

We shall now show that 'bunching' of the beam occurs, i.e. that there are regions in the beam where the spacing of the electrons in time is *less* than the uniform initial spacing τ

(bunches), and therefore also regions where they are spaced farther apart in time than τ (rarefactions). We shall also show that the *bunches* are uniformly spaced in time, so that as the bunches and rarefactions pass back through the grids, an alternating charge density will be induced on the grids. If the returning bunches are *declerated* by the alternating field E_1, they must give up their energies to it, so causing it to increase in magnitude, and increasing the bunching, etc. until the alternating charge density on the grids becomes self-sustaining and an a-c signal at the frequency ω is generated.

Consider an electron (2) passing through the grids at time $(t_1 + \tau)$. Its time of arrival back at the grids will be, by (3.41),

$$t_{R2} = t_1 + \tau + (\text{time of flight}) = t_1 + \tau + t_0 + \frac{2m}{eE} v_1 \sin \omega (t_1 + \tau)$$

The difference in arrival time back at the grids of electrons (1) and (2) is then

$$t_{R2} - t_{R1} \doteqdot \tau + \frac{2m}{eE} v_1 \left[\sin \omega (t_1 + \tau) - \sin \omega t_1\right] \qquad (3.41)$$

and if τ is very small, this becomes

$$t_{R2} - t_{R1} \doteqdot \tau + \frac{2m}{eE} v_1 \cos \omega t_1 . \omega\tau \qquad (3.42)$$

$$< \tau \text{ for } \cos \omega t_1 - \text{ve (bunches)}$$

$$> \tau \text{ for } \cos \omega t_1 + \text{ve (rarefactions)}$$

and $t_{R2} - t_{R1}$ is a *minimum* (i.e. maximum bunching) for $\cos \omega t_1 = 1$, i.e. $\sin \omega t_1 = 0$, and decreasing. Thus the alternating component of velocity is passing from positive through zero to negative, and the alternating field on the grids must have passed from an accelerating effect through zero to a decelerating effect. The returning electrons must see a decelerating field in order to give up their energy to it; since they are travelling in the opposite direction to the electrons leaving the grids for the first time, this corresponds to an accelerating field for the outgoing electron stream. Maximum accelerating effect has been exerted by the field on the grids when $v_1 \sin \omega t_1$ is a *maximum*; this will occur $n + 3/4$ cycles ($n = 0,1,2...$) after the time for maximum bunching effect.

Hence we expect a *series* of mean times of flight t_0 for which oscillations will occur.

Since

$$t_o = \frac{2m}{eE} v_o$$

and

$$|E| = \frac{V_r}{d} \text{ (plane parallel geometry)}$$

where V_r is the grid-reflector potential difference, we expect a series of values of V_r for which oscillations will occur, and regions of V_r where oscillations are impossible. This is indeed the case.

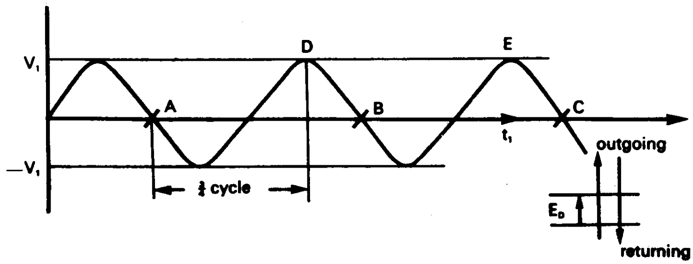


Fig. 3.9 *The alternating velocity A,B,C – maximum bunching; D, E – maximum accelerating effect for outgoing electrons, and therefore maximum decelerating effect for returning electrons*

A plot of power output *versus* reflector voltage from an actual klystron is shown in Fig. 3.10.

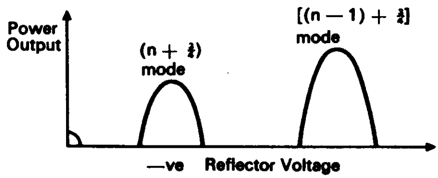


Fig. 3.10 *Output power versus reflector voltage*

Note that to solve this problem, given the alternating component of velocity, we have used only Newton's Second Law of Motion and some mathematics, and we have assumed conservation of energy (the energy of the decelerated electrons is given to the retarding field), although we have not discussed it yet.

CHAPTER 3 PROBLEMS

3.1 A body of mass M moves in outer space with speed V. It is explosively separated into two equal parts such that they move along the same direction in the same sense as before. If the speed of one part is V/3, what is the speed of the other?

3.2 A book rests on a table. The reaction force (in the sense of Newton's Third Law) to the weight of the book is

(a) the force exerted by the table top on the book.

(b) the force exerted by the floor on the table legs.

(c) the gravitational attraction exerted by the book on the earth.

(d) none of these.

3.3 A man stands in a lift which is accelerating *downwards*. The force exerted by the lift floor on the man is

(a) equal to

(b) less than

(c) greater than

his weight.

3.4 A particle of charge q and mass m passes through a constant electric field **E** between two plates of length x. The particle enters the field with a velocity **v** perpendicular to the field.

(a) What is the time of flight through the field?

(b) What is the acceleration of the particle in the direction of the field?

(c) What is the component v_E of final velocity in the direction of the field?

(d) What is the magnitude of the final speed?

3.5 At time t = 0 a train, previously moving at a constant speed of 30 m/sec, starts to decelerate at a constant rate of 1 m/sec^2. At time t = 3 sec a passenger drops a coin from a height of 2 m above the floor. Assume that at t = 3 sec the coin is at rest relative to the train, and take the acceleration of gravity as 10 m/sec^2

(a) At what time does the coin strike the floor?

(b) Does the coin move towards the front or rear of the train as it falls?

(c) How far towards the front or the rear of the train does the coin move during its fall?

(d) What is the magnitude and direction of the effective gravitational field experienced by the coin? Express your answer with respect to the true vertical axis.

3.6 A merry-go-round at an amusement park has a radius of 5 m and is rotating with an angular velocity $\omega = 0{\cdot}5$ rad/sec in an anticlockwise direction as viewed from above.

The operator of mass 100 kg starts walking at $t = 0$ from the centre along one of the radial supporting beams with a constant velocity of 0·5 m/sec with respect to the beam.

(a) At the instant shown in figure 3.6p, which of the four arrows correctly represents the direction of the two pseudo-forces the operator experiences?

(i) Centrifugal force

A B C D

(ii) Coriolis force

A B C D

(b) What is the magnitude of the centrifugal force acting on him at time $t (t < 10$ sec)?

(c) What is the magnitude of the Coriolis force acting on him at time $t (t < 10$ sec)?

(d) What is the magnitude of the operator's speed with respect to the ground at the time $t (t < 10$ sec)?

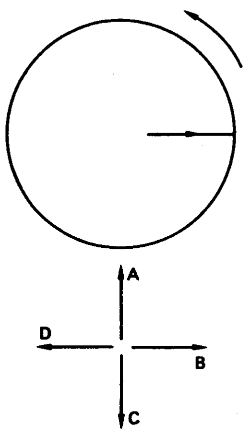


Fig. 3.6p

3.7 A charged particle moves in the x direction through a region in which there is an electric field E_y and a perpendicular magnetic field B_z. What is the condition necessary to ensure that the net force on the particle will be zero? Show the v, E, and B vectors on a diagram. What is the condition on v_x if $B_y = 10$ statvolts/cm and $B_z = 300$ gauss?

3.8 A particle of charge q and mass M with an initial velocity v_{ox} enters an electric field $-E_y$ (figure 3.8p). We assume that E is uniform, i.e. its value is constant at all points in the region between plates of length L (except for small variations near the edges of the plates which we shall neglect).

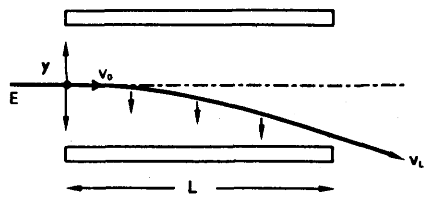


Fig. 3.8p

(a) What forces act in the x and y directions respectively?
(b) Will a force in the y direction influence the x component of the velocity?
(c) Solve for v_x and v_y as functions of time, and write the complete vector equation for **v**(t).
(d) Choose the origin at the point of entry, and write the complete vector equation for the position of the particle as a function of time while the particle is between the plates.

3.9 A very small cube of mass m is placed on the inside of a funnel rotating with its axis vertical at a constant rate of ν revolutions/sec (figure 3.9p). The wall of the funnel makes an angle θ with the horizontal. The coefficient of static friction between the cube and the funnel is μ and the centre of the cube is a distance r from the axis of rotation. What are the largest and smallest values of ν for which the block will not move with respect to the funnel? To answer this you may assume that $m < \tan\theta < (1/\mu)$.

What would happen if either or both of these conditions is not valid?

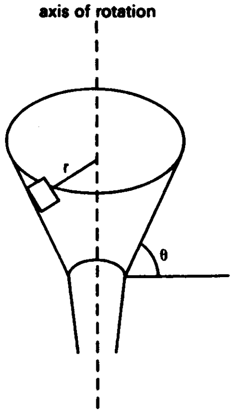


Fig. 3.9p

4

Dynamics of a system of particles

4.1 IMPULSE. CONSERVATION OF LINEAR MOMENTUM

When we come to deal with systems of particles, and interactions between particles such as collisions, we find that momentum is a most useful concept. We return to the idea of two interacting bodies which we considered in Section 2.1, and combine it with the 'hard-core' collision model of Section 3.3.

By Newton's Third Law, the force experienced by body 1 due to body 2 is equal and opposite to that experienced by body 2 due to body 1, i.e.

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \tag{4.1}$$

Let us assume for the moment that these are the only forces acting on the particles. Then, since the momentum $\mathbf{p}_1$ of body 1 is related to $\mathbf{F}_{12}$ by $\mathbf{F}_{12} = d\mathbf{p}_1/dt$, we may write $\Delta\mathbf{p}_1 = \int \mathbf{F}_{12}\,\Delta t$, where the *change* in *momentum* $\Delta\mathbf{p}_1$ due to the action of $\mathbf{F}_{12}$ for time Δt is called the *impulse*. Similarly, $\Delta\mathbf{p}_2 = \int \mathbf{F}_{21}\,\Delta t$, and if we use the 'hard-core' collision model, we shall have a plot of $\mathbf{F}_{12}$, say, against t, like that shown in figure 3.1d. Clearly, since $\mathbf{F}_{12} + \mathbf{F}_{21} = 0$ at every instant, we have

$$\int \mathbf{F}_{12}\,\Delta t + \int \mathbf{F}_{21}\,\Delta t = \Delta\mathbf{p}_1 + \Delta\mathbf{p}_2 = 0 \tag{4.2}$$

i.e. the *total* change in momentum of the two colliding particles is zero. These contact collision forces are called *impulsive forces*, and can be of enormous magnitude.

Now we know that electromagnetic forces, for example, are not transmitted instantaneously, so that during the collision of, say, two positively-charged particles, equation (4.1) cannot hold. But we postulate – and indeed find to be true experimentally – that the total momentum measured a long time before the collision is equal to the total momentum measured a long time after the collision. We also know that mechanical forces cannot be transmitted instantaneously, so that we must postulate the same thing. It should be clear that a collision does not have to imply actual *contact:* it is a situation in which the motion of two bodies is mutually affected by the interaction between them. An explosion is a collision in this sense.

Now suppose that the two contacting bodies are both subject to some external force, which acts during the time Δt of the collision. We see that if the impulsive forces $\mathbf{F}_{12}$, $\mathbf{F}_{21}$ are very much greater than the external force, or Δt is very *small*, or both, momentum will *still* be conserved to a very good approximation during the collision. Well before and well after the collision the external force will have changed the momentum, but we may consider that over the (very short) period Δt of the collision the momentum is conserved. This is extremely important, because it means that *momentum is conserved even in collisions where energy is not conserved,* i.e. in so-called *inelastic* collisions. Thus the presence of external resistive or dissipative forces does not *necessarily* mean that momentum will not be conserved; provided the impulsive forces are large enough, and the collision is short enough in time, momentum will be conserved.

The reader is probably familiar with the idea of kinetic energy (KE), though we have not yet discussed it. We shall here define only the 'kinetic energy' of a mass moving with speed v as $\frac{1}{2}mv^2$, and the kinetic energy of a number of masses m_i moving with respective speeds v_i as $\sum_i \frac{1}{2}m_iv_i^2$. A perfectly elastic collision is one in which the total KE before the collision equals the total KE after the collision.

4.2 CENTRE OF MASS. THE CENTRE-OF-MASS FRAME

A most useful concept in treating the motion of a system of particles is the *centre of mass.* Consider a system of particles, with masses denoted by m_i; suppose a mass m_i has a position vector $\mathbf{r}_i$ in a particular inertial frame. Then the co-ordinates of the centre of mass in this frame are defined by

$$\bar{\mathbf{r}} = \frac{\Sigma\, m_i\mathbf{r}_i}{\Sigma\, m_i} = \frac{\Sigma\, m_i\mathbf{r}_i}{M} \tag{4.3a}$$

or

$$\bar{\mathbf{r}}\, \Sigma\, m_i = \bar{\mathbf{r}}M = \Sigma\, m_i\mathbf{r}_i \tag{4.3b}$$

where $M = \Sigma m_i$ is the total mass of the system of particles. We shall find the form (4.3b) the more useful, although (4.3a) is the strict definition. Let us call $m_i\mathbf{r}_i$ the *moment* of the mass m_i in the particular inertial frame. Then the sum of the moments of the masses in the centre-of-mass frame, i.e. the frame whose origin has co-ordinates $\bar{\mathbf{r}}$, is zero. For, to move into the centre-of-mass frame, we must write the co-ordinate of each m_i as $\mathbf{r}_i - \bar{\mathbf{r}}$. The sum of the moments of all the masses is therefore

$$\sum_i m_i(\mathbf{r}_i - \bar{\mathbf{r}}) = \sum_i m_i\mathbf{r}_i - \bar{\mathbf{r}}\,\Sigma m_i$$

$$= 0 \text{ from (4.3b)}$$

This result is rarely proved, but is sometimes useful.

Let us differentiate equation (4.3b) with respect to time. We obtain

$$\dot{\bar{\mathbf{r}}}\sum_i m_i = \dot{\bar{\mathbf{r}}}M = \sum_i m_i\dot{\mathbf{r}}_i \tag{4.4}$$

which defines the velocity of the centre of mass $\dot{\bar{\mathbf{r}}}$ in terms of the total mass and the sum of the momenta of the particles. In words, the total momentum equals the total mass times the velocity of the centre of mass. If we move into the centre-of-mass frame, i.e. that frame having velocity $\dot{\bar{\mathbf{r}}}$, we find by an argument similar to that above that the *sum of the momenta in the centre-of-mass frame is zero.* This is a most important and useful result. The centre-of-mass frame is therefore often called the *zero-momentum frame.*

Let us now differentiate equation (4.4). We obtain

$$\ddot{\bar{\mathbf{r}}}\,\Sigma m_i = \ddot{\bar{\mathbf{r}}}M = \Sigma m_i\ddot{\mathbf{r}}_i \tag{4.5a}$$

$$= \Sigma\mathbf{F}_i \tag{4.5b}$$

by Newton's Second Law of Motion, where the $\mathbf{F}_i$ are the forces experienced by each of the i particles.

We note that each $\mathbf{F}_i$ is made up of two parts:

$$\mathbf{F}_i = \Sigma\,\mathbf{F}_{ext\,i} + \Sigma\mathbf{F}_{int\,i} \tag{4.6}$$

where the $\mathbf{F}_{ext}$ are forces external to the system of particles, and the $\mathbf{F}_{int}$ are *internal*, or action-reaction forces in the sense of Newton's

Third Law.

Over the *whole system* of *particles*, since the action-reaction forces occur in pairs for each pair of particles, we must have

$$\Sigma \mathbf{F}_{\text{int } i} = 0$$

which leaves us with the equation

$$\ddot{\mathbf{r}}\ \Sigma m_i = \ddot{\mathbf{r}}M = \Sigma\ \mathbf{F}_{\text{ext } i} \tag{4.7}$$

i.e. the total mass times the acceleration of the centre of mass equals the sum of all the *external* forces acting on the particles. Hence if *no* external forces act, the acceleration of the centre of mass is *zero*: the *momentum* of the centre of mass is therefore a constant; since the *total mass M* of the system is constant, the *velocity* of the centre of mass is constant; and therefore the centre-of-mass frame with no external forces acting *is an inertial frame*. Note also that the *kinetic energy* of the centre of mass when no external forces act, defined by

$$KE_{cm} = \tfrac{1}{2}M(\dot{\mathbf{r}})^2 \tag{4.8}$$

must also be a constant, by definition. Therefore, if only *internal* dissipative forces act, only the KE *relative* to the centre of mass can be destroyed, and not the KE of the centre of mass. Note that we can determine the motion of the centre of mass if we know the sum of the external forces. If any collisions take place in the system of particles (including explosions), we can apply conservation of momentum to calculate the positions and motions of the particles, since by our previous discussion (Section 4.1), momentum is conserved during collisions and explosive events even when an external force acts.

Since the centre-of-mass frame is inertial in the absence of external forces, frames located on particles moving relative to the centre of mass will in general be non-inertial if internal forces are present. This can lead to important results. For example, consider the earth-moon system in the particle approximation, and ignore outside influences. Since gravitation is an internal force for the system, the frame of the common centre of mass is inertial in our approximation. The earth and the moon are in 'free fall' towards each other (by Newton's Third Law), so neither is an inertial frame; a little thought shows that they must both rotate about the common centre of mass. It is the 'competition' between the centrifugal force of rotation about the common centre of mass and gravitational force due to the moon that gives rise to the tides on the earth's surface.

We now turn to the topic of one- and two-dimensional collisions in the centre-of-mass frame (CM frame).

4.3 TWO-BODY COLLISIONS AND THE CM FRAME

We now specialize to the important case of *two-body collisions.* These should be of interest to anyone who watches demonstrations on an air-track, or plays billiards, or travels in a motor-car. Moreover, most of what we know about the kinds of particles that exist, and the forces they exert, comes from watching what happens when one particle is bounced off another. This is true whether we are thinking of Rutherford's experiment, scattering alpha-particles off gold-foil, or the more sophisticated work with cyclotrons, synchrotrons, and cosmic-ray showers. In all of these the results when one particle hits another are recorded in some way (Geiger counters, cloud chambers, bubble chambers, photographic emulsion, etc.) and analyzed.

Since only the collision forces (which are internal) are acting, the CM frame is an inertial frame ($\mathbf{v}_{cm}$ = constant). Suppose that, as a result of the collision, the velocities of the particles change from $\mathbf{u}_1$, $\mathbf{u}_2$ to $\mathbf{v}_1$, $\mathbf{v}_2$. Then the scattering angles θ_1, θ_2 are defined as shown:

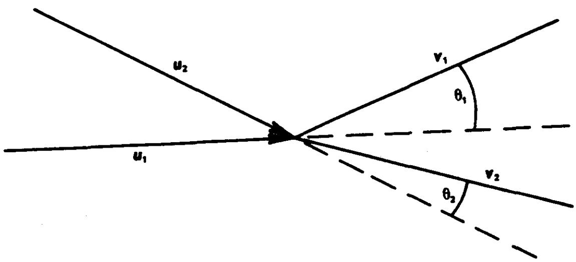


Fig. 4.1 Laboratory frame

If the measurements are done in the CM frame we call the angles θ_1^*, θ_2^*. Since $\mathbf{P}^* = 0$, we have

$$m_1\mathbf{u}_1^* = -m_2\mathbf{u}_2^*, \text{ and } m_1\mathbf{v}_1^* = -m_2\mathbf{v}_2^* \qquad (4.9)$$

showing that, in the CM frame, the particles before the collision travel in opposite directions along the same straight line, while after the collision they again are travelling in opposite directions along a (generally different) straight line. So $\theta_1^* \neq \theta_2^*$

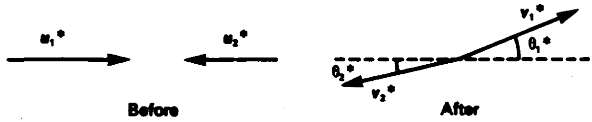


Fig. 4.2 *CM frame: left, before collision; right, after collision*

Now suppose that the collision is *elastic*, i.e. the total kinetic energy K remains unchanged. (In the laboratory frame, or in the CM frame? It does not matter which, for even when the number of particles is arbitrary

$$\begin{aligned} K^* &= \tfrac{1}{2}\,\Sigma\, m_n(\mathbf{v}_n - \mathbf{v}_{cm})^2 \\ &= \tfrac{1}{2}\,\Sigma\, m_n v_n^2 + \tfrac{1}{2} v_{cm}^2\,\Sigma\, m_n - \tfrac{1}{2}\,\Sigma\, 2m_n \mathbf{v}_n\cdot\mathbf{v}_{cm} \\ &= K + \tfrac{1}{2}Mv_{cm}^2 - \mathbf{v}_{cm}\cdot\,\Sigma\, m_n\mathbf{v}_n \\ &= K + \tfrac{1}{2}Mv_{cm}^2 - \mathbf{v}_{cm}\cdot M\,\mathbf{v}_{cm} \\ &= K - \tfrac{1}{2}Mv_{cm}^2 \end{aligned}$$

But $\mathbf{v}_{cm}$ is constant. Thus if K is unchanged, so is K*, and vice versa).

$$(K^*)_{before} = \tfrac{1}{2}m_1 u_1^{*2} + \tfrac{1}{2}m_2 u_2^{*2}$$
$$= \tfrac{1}{2}m_1 u_1^{*2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \quad \text{[from (4.9)]}$$

Similarly

$$(K^*)_{after} = \tfrac{1}{2}m_1 v_1^{*2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \quad \text{[from (4.9)]}$$

i.e. since

$$(K^*)_{before} = (K^*)_{after}$$
$$v_1^* = u_1^*$$

Similarly $v_2^* = u_2^*$ (4.10)

i.e. *a particle's speed in the CM system is unchanged by an elastic collision.*

The *relative speed* u_{12} of the two particles before collision is defined as the length of the vector $(\mathbf{u}_1 - \mathbf{u}_2)$, i.e. it is $|\mathbf{u}_1 - \mathbf{u}_2|$.

Similarly $\quad v_{12} = |\mathbf{v}_1 - \mathbf{v}_2|$

so that relative speed is the same in the CM frame as in the laboratory frame, since

$$u^*_{12} = |u^*_1 - u^*_2| = |(u_1 - v_{cm}) - (u_2 - v_{cm})|$$

$$= |u_1 - u_2| = u_{12}$$

Speeds are unchanged in the CM frame in an elastic collision $|v_1 - v_2| = u^*_{12}$, and so (by this last result)

$$v_{12} = u_{12} \tag{4.11}$$

i.e. *the relative speed in the laboratory frame is unchanged by an elastic collision.* In fact, this result is sometimes used as the *definition* of an elastic collision.

This last result illustrates a common use of the CM frame. We are usually interested in measurements made in the laboratory frame, since this is often the frame in which we (and our detecting instruments) are at rest: i.e. it is *our* frame. But it may be much easier to work out the problem in the CM frame. So we do this, and then deduce from these results the laboratory frame results we are really concerned with. This trick is especially useful when using scattering angles; but we first need to know the relation between θ and θ^*.

Scattering Angles — Laboratory Frame and CM Frame

We look at the most common kind of scattering experiment: a particle of mass m, travelling at velocity **u**, collides elastically with a stationary 'target' of mass M:

Fig. 4.3 Laboratory frame: before collision

We deduce $$v_{cm} = \frac{mu}{m+M} = \alpha u$$

where $$\alpha = \frac{m}{m+M}$$

In the CM frame we have the initial velocity **u*** given by

$$u^* = u - v_{cm} = (1 - \alpha)u$$

since speed is unchanged in the CM frame (figure 4.4).

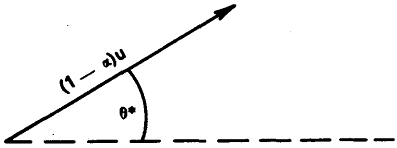


Fig. 4.4 *CM frame: after collision*

To find the velocity in the laboratory frame, we simply add $\mathbf{v}_{cm} = \alpha\mathbf{u}$ (figure 4.5).

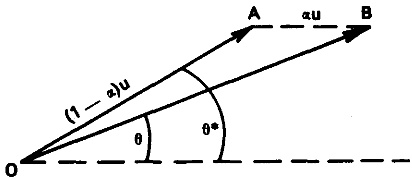


Fig. 4.5 *Transforming from CM to laboratory frame*

Applying the sine rule to the triangle OAB:

$$\frac{\sin\theta}{(1-\alpha)u} = \frac{\sin(\theta^* - \theta)}{\alpha u}$$

i.e.

$$\frac{\alpha}{1-\alpha}\sin\theta = \sin\theta^*\cos\theta - \cos\theta^*\sin\theta$$

giving

$$\tan\theta = \frac{\sin\theta^*}{\cos\theta^* + \dfrac{m}{M}} \qquad (4.12)$$

We can now use this result to draw conclusions about possible scattering angles θ (which are the ones we actually observe). Note first that, so long as equation (4.9) is satisfied, both momentum and energy are conserved; since (4.9) is compatible with *any* scattering angle θ^*, we conclude that all values of θ^* are possible. Is the same true of θ?

Definitely not! For example, if $m/M \geqslant 1$, (4.12) shows that $\tan\theta$ can never be negative, i.e. $\theta \leqslant \pi/2$. (Look at the denominator on the RHS and remember that $\sin\theta$ must be $\geqslant 0$.) Or: a particle cannot be scattered backward unless the target particle has a greater mass than it. For $m/M = 1$, we find from physical considerations (vector momentum conservation!) that $\theta = \pi/2$ is not allowed, but we may get as close to it as we wish.[1]

Can a particle be back-scattered if the target particle is more massive than it ($m/M < 1$)? Certainly. In fact it can then be scattered straight back along the path of approach ($\theta = \pi$). For, when $\theta^* = \pi$, the RHS of (4.12) is zero; so $\tan\theta$ is zero, or $\theta = \pi$. (It is not $\theta = 0$, for when θ^* is *close* to π, we can easily show from (4.12) that $\tan\theta$ is negative and small, i.e. θ is close to π, if $(m/M) < 1$.)

Example 4.3(i): An Air-track Problem

To show how convenient the CM system can be, let us solve the following problem. Two cars of masses m and M are released from a height h on a tilted air-track. (Neglect their lengths: take them both as being initially at height h.) The second car is infinitesimally behind the first, and rebounds elastically from it after the first car has itself rebounded elastically from the lower-end buffer. To what height does the second car rebound? Let us take $M \gg m$, so that m/M can be neglected; then the CM is located at all times to a good approximation in the more massive particle, so that its frame is the CM frame.

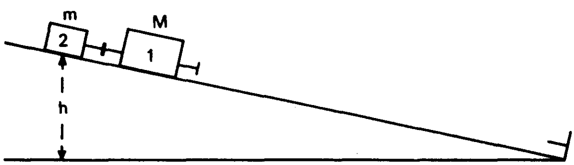


Fig. 4.6 The tilted air-track problem

In this tilted air-track problem, just before the first collision, we get

$$v_2^* = u + v_{cm}$$

but now,

$$v_{cm} = u, \text{ the velocity of M}$$

So

$$v_2^* = v_2^* + v_{cm}$$
$$= 3u$$

giving $h' = 9h$ (see Section 2.3), a result which is surprising when one sees it without the analysis.

Example 4.3(ii)

Large mass M stationary, small mass m moving with velocity v (one-dimensional collision). Since the CM frame is the frame of M, $v_{cm} \doteq 0$.

[1] This is an example of a limit point of a set not being a member of the set. It is also a case where *naive* mathematics and no physics would give the wrong result.

Since the speed is unchanged in an elastic collision, m must rebound with speed v and therefore velocity –v approximately. The KE transfer is negligible, and hence this is no way to slow down particles of small mass: the masses of the colliding particles need to be *comparable* if the stationary mass is to take away much kinetic energy.

Example 4.3(iii)

Small mass m stationary, large mass M moving with velocity **v**. In the CM frame (approximately that of M), the mass m has velocity **–v**. Since the speed is unchanged in an elastic collision, the velocity of m in the CM frame after collision must be **+v**. Therefore in the laboratory frame it is +2**v**.

The General Two-dimensional Collision We shall see later on that nearly all two-body collisions occur in a plane, owing to a conservation law. The situation in which a target particle 2 is stationary, and another, 1, is fired at it with velocity ui is quite general, since we can obtain any velocity of 2 by a suitable Galilean Transformation. Conservation of momentum then gives (see figure 4.7)

$$m_1 u = m_1 v_1 \cos\theta + m_2 v_2 \cos\phi$$

$$0 = m_1 v_1 \sin\theta + m_2 v_2 \sin\phi \qquad (4.13)$$

and conservation of KE for a perfectly elastic collision gives

$$m_1 u^2 = m_1 v_1^2 + m_2 v_2^2 \qquad (4.14)$$

If we know m_1, m_2, and u, we still cannot solve the general problem, since we have only three equations and there are four unknowns (v_1, v_2, θ, ϕ). We have to *specify* one of the unknowns. In a perfectly *inelastic* collision, where 1 and 2 go off joined together, we have $\theta = \phi$, $v_1 = v_2$, and therefore the problem becomes soluble.

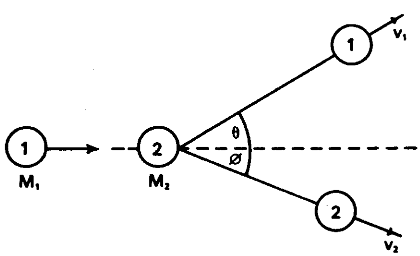


Fig. 4.7 General planar collision (laboratory frame)

4.4 SYSTEMS OF VARIABLE MASS. ROCKETS

Although the *total* momentum of a system of particles remains constant if no *external* forces act, it is clear that, by collisions for example, a transfer of momentum can take place. A stream of particles striking a body will transfer momentum to the body, and we may regard the rate of increase of momentum of this body as an effective force. So we proceed first to calculate the momentum transferred per unit time by a stream of particles each having mass m and velocity **v**. Suppose that the particles strike a wall normally. The transferred momentum per particle will depend on the final momentum of the particle: if the collision is perfectly elastic, the particles will reverse their velocity and so the momentum transfer per particle is 2mv. If the particles are brought to rest (inelastic collision), the momentum transfer per particle is mv.

Let the number of particles per unit volume be n. Then the mass density is $\rho = nm$. A particle at a distance Δx from the wall will reach it in a time $\Delta t = \Delta x/v$; if the stream of particles has a cross-sectional area A normal to **v**, all the particles in the volume $A\Delta x$ will strike the wall in time Δt. This number of particles is therefore

$$nA\Delta x = nAv\Delta t \tag{4.15}$$

and the number of collisions per second is nAv. The momentum transfer per second, for perfectly elastic collisions, i.e. the effective force, is

$$F = Anv(2mv) = 2Anmv^2 = 2A\rho v^2 \tag{4.16}$$

It is also clear that if a stream of similar particles *emerges* from a body with velocity v, the effective force on the body must be half that set out in (4.16), i.e. $Anmv^2 = A\rho v^2$. This is the principle of the rocket and the jet engine. But if we regard the rocket motor *by itself*, for example, its mass is changing, and apart from our discussion in section 3.2 we are as yet unsure how to treat this. Accordingly we set out the problem from first principles, so that we can see how the correct expression for the force arises in a system where the mass of a part varies.

We consider a mass M, moving with velocity **v** at time t in an inertial frame (figure 4.8). At time $t + \Delta t$ it has emitted a portion ΔM with velocity **u** as shown, and the remaining $(M - \Delta M)$ moves with velocity $\mathbf{v} + \Delta\mathbf{v}$.

Suppose that an external force F_{ext} acts. Then the motion of the centre of mass of the *whole system* ($M - \Delta M$, and ΔM) is given by

$$F_{ext} = \frac{dP}{dt} \tag{4.17}$$

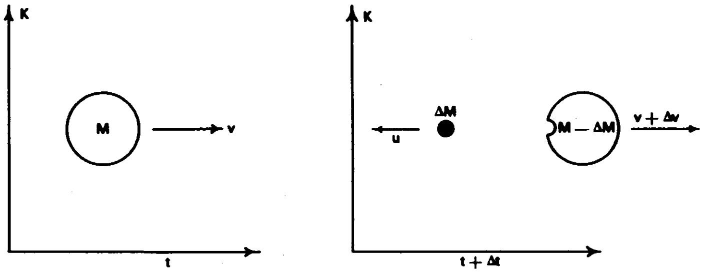


Fig. 4.8 *The rocket problem*

The change in momentum ΔP is given by the difference between the final and the initial momenta:

$$\Delta \mathbf{P} = (\mathrm{M} - \Delta\mathrm{M})(\mathbf{v} + \Delta\mathbf{v}) + \Delta\mathrm{M}\mathbf{u} - \mathrm{M}\mathbf{v} \tag{4.18}$$

$$\therefore \frac{\Delta \mathbf{P}}{\Delta t} = \mathrm{M}\frac{\Delta \mathbf{v}}{\Delta t} + [\mathbf{u} - (\mathbf{v} + \Delta\mathbf{v})]\frac{\Delta \mathrm{M}}{\Delta t} \tag{4.19}$$

Taking $\lim_{\Delta t \to 0}$, we obtain

(i) $(\Delta \mathbf{v}/\Delta t) \to (d\mathbf{v}/dt)$

(ii) $(\Delta \mathrm{M}/\Delta t) \to (d\mathrm{M}/dt)$.

If we consider ΔM as the mass *ejected* by the more massive body, i.e. we focus attention on the more massive body, dM/dt will be a negative quantity, and we may rewrite (4.19) as

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_{ext} = \mathrm{M}\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{d\mathrm{M}}{dt} - \mathbf{u}\frac{d\mathrm{M}}{dt} \tag{4.20a}$$

$$= \mathrm{M}\frac{d\mathbf{v}}{dt} + (\mathbf{v} - \mathbf{u})\frac{d\mathrm{M}}{dt} \tag{4.20b}$$

i.e. the velocity of the ejected mass must be taken into account. We note that $(\mathbf{v} - \mathbf{u})$ is the relative velocity of the 'main' mass to the ejected mass in the limit $\Delta t \to 0$. Hence the expression (4.20b) for variable mass is now Galilean covariant, and the naive expression

$$\frac{d\mathbf{P}}{dt} = \mathrm{M}\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{d\mathrm{M}}{dt} \tag{4.21}$$

is seen to be a special case where the velocity of the emitted mass is zero in the inertial frame, so that the velocity of the main mass relative to the ejected mass is $\mathbf{v}$ in this special case.

Before passing on to the case of the rocket, it is instructive to consider the following problem. Sand is falling vertically into a box at a rate dm/dt; the box is to be kept moving horizontally with constant velocity $\mathbf{v}_0$ (figure 4.9). What is the horizontal force required?

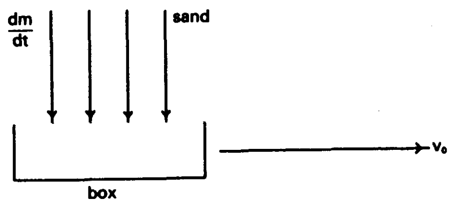


Fig. 4.9 *The sand-box problem*

We have to note a number of things. The velocity of the box is constant, so $(d\mathbf{v}/dt) = 0$. The velocity of the box relative to the sand *before* it collects the sand is $\mathbf{v}_0$; the *final* relative velocity is 0. So we have the situation of equation (4.20b), except that it is effectively *reversed* with respect to the previous problem: dM is now a positive quantity, whereas we treated it as negative in the previous problem. The final result is therefore

$$\mathbf{F} = \mathbf{v}_0 \frac{dm}{dt} \qquad (4.22)$$

If the sand were projected into the box with a horizontal component of velocity $\mathbf{v}_0$, no external force would be required to pull the box along, because the relative velocity of the box to the sand in the horizontal direction would always be zero.

Of course the *vertical* component of velocity of the sand that is destroyed exerts a downward force on the box, but this did not enter the problem. It should be clear from the foregoing discussion that even apparently simple variable-mass problems must be approached with care, and that it is probably safest to go back to first principles.

We return now to the problem of the rocket. In equation (4.20b) $(\mathbf{u} - \mathbf{v})$ is the relative velocity of the exhaust gases to the main body of the rocket; call it $\mathbf{v}_{rel}$. We can then rewrite (4.20b) as

$$M\frac{d\mathbf{v}}{dt} = \mathbf{F}_{ext} + \mathbf{v}_{rel}\frac{dM}{dt} \qquad (4.23)$$

from which it is clear that if $\mathbf{F}_{ext}$ *opposes* the motion (as air resistance would), M(dv/dt) is *reduced*. So a rocket is clearly most efficient in the *absence* of any resistive forces. If we therefore set $\mathbf{F}_{ext} = 0$, we obtain

$$M\frac{dv}{dt} = v_{rel}\frac{dM}{dt} \quad \text{or} \quad dv = v_{rel}\frac{dM}{M} \qquad (4.24)$$

and if we presume that the rocket starts with zero velocity, we obtain

$$\int_o^v dv = v_{rel}\int_{M_o}^{M}\frac{dM}{M}$$

i.e.

$$v = -v_{rel}\,\ell n\,(M_o/M)$$

where M_o is the initial mass of the rocket. The negative sign occurs because $\mathbf{v}_{rel}$ as we have set up the original equation (4.20b) is opposite in sense to $\mathbf{v}$. We see that the larger the ratio of the initial to the final mass, the larger is the final velocity; for a certain $\mathbf{v}_{rel}$ and a certain final velocity $\mathbf{v}_f$, the ratio of the final to the initial mass is

$$(M_f/M_o) = \exp(-v_f/v_{rel})$$

which explains why rockets with high final velocities must have most of their original mass in fuel. We also see that for a given (M_o/M), $\mathbf{v}$ is directly proportional to $\mathbf{v}_{rel}$, which is therefore made as high as possible. For a given temperature T in the combustion chamber of a rocket, the less massive the combustion products, the higher will be the $\mathbf{v}_{rel}$ of the exhaust. This explains why troublesome liquid hydrogen is sometimes used as a rocket fuel.

It should be noted that if a rocket starts from rest in empty space under the influence of *no* external forces, the centre of mass of the rocket remains stationary. The main body of the rocket is a non-inertial frame moving with a variable acceleration dv/dt, but it is possible to solve the rocket problem in this non-inertial frame. Try it!

CHAPTER 4 PROBLEMS

4.1 A particle of mass m executes motion such that its velocity is described by $v = v_o \sin(2\pi t/T)$ where T (a constant) is the period of the motion and v_o is constant.

(a) What is the impulse received by the mass from t = 0 to t = T/4?

(b) What is the average force acting on m from t = 0 to t = T/4?

(c) What is the impulse received by the mass from t = 0 to t = T/2?

(d) What is the average force acting on m from $t = T/4$ to $t = T/2$?

4.2 Two masses m_1 and m_2 have velocities v_1 and v_2 respectively along Ox.

(a) What is the velocity of the centre of mass?

(b) In the centre-of-mass frame, what is the velocity of m_2 relative to m_1?

(c) The two masses collide *perfectly elastically.* What is the speed of m_2 relative to m_1 after the collision, in the centre-of-mass frame?

4.3 Two masses collide *perfectly inelastically.* Which of the following statements are false?

(a) The relative velocity after the collision is zero.

(b) The kinetic energy of the centre of mass is not conserved.

(c) The kinetic energy of the particles with respect to the centre of mass is not conserved.

(d) Momentum is conserved.

4.4 A 10 kg block is at rest on a frictionless table. Three separate experiments are performed.

I The block is struck by a metal ball which rebounds.

II The block is struck by a lump of hard clay which is at rest after collision.

III The block is struck by a lump of soft clay which adheres to the block after collision. The metal ball and lumps of clay have masses of 2 kg. They have the same initial speed and collide head-on with the 10 kg block.

(a) In which experiment (or experiments) does the block acquire the greatest speed?

(b) In which experiment (or experiments) does the block acquire the least speed?

(c) In which experiment will there be the greatest reduction in total kinetic energy?

(d) Which of the projectiles will suffer the greatest impulse during the collision?

(e) Which of the projectiles will suffer the greatest change of kinetic energy?

4.5 Two particles of masses m_1 and m_2 and speeds u_1 and u_2 collide. Their speeds after the collision are v_1 and v_2 respectively. Which one or more of the following statements are true? (Assume that no forces act on the particles save during the collison).

(a) The total momentum in any inertial reference frame is constant.

(b) The total momentum in the centre-of-mass frame is constant.

(c) The total momentum in the centre-of-mass frame is constant and zero.

(d) The total momentum in the centre-of-mass frame is constant and zero if and only if the collision is perfectly elastic.

(e) In the centre-of-mass frame $u_1 = v_1$.

(f) In the centre-of-mass frame $u_1 = v_1$ if and only if the collision is perfectly elastic.

4.6 A wall is struck by 300 particles per second, each of mass 5 g, moving at a speed of 2 m/sec normal to the wall.

(a) What is the average force exerted on the wall when the particles are absorbed?

(b) If, in the last question, the particles rebound perfectly elastically from the wall, what is the average force exerted on the wall?

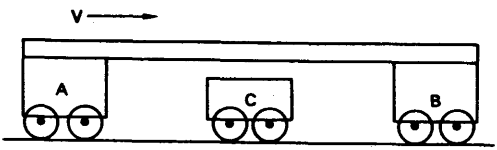


Fig. 4.7p

4.7 Two similar cars A and B each of mass 2 m are connected rigidly together by a light rod (figure 4.7p). Car C has mass m. Initially C is at rest midway between A and B, which are moving with a velocity V to the right.

(a) What is the final velocity of the system if A and C collide perfectly inelastically?

(b) If the collision between A and C is perfectly elastic but that between B and C is perfectly inelastic, what then is the final velocity of the system? Compare with (a).

(c) If both collisions are perfectly elastic, what are the laboratory velocities of C after the first and second collisions? Use the centre-of-mass reference frame.

4.8 A particle of mass m and initial velocity v_i strikes a stationary particle of mass M. The collision is perfectly elastic. It is observed that after the collision the particles have equal and opposite velocities in the laboratory frame.

(a) What is the velocity of the centre of mass?

(b) What is the ratio M/m?

(c) What is the final kinetic energy of m in the laboratory frame?

(d) Newton calculated the resistive force for an object travelling through a fluid by supposing that the particles of the fluid (supposedly initially stationary) rebounded elastically when struck by the object. On this model the resistive force would vary as some power, n, of the speed v of the object. Calculate the value of n.

4.9 Two particles move in the same straight line. One is of mass m kg and moves initially according to the position-time relationship $x_1 = 5t$. The other is of mass 2m kg and moves initially according to the relationship $x_2 = 21 - 2t$. x_1 and x_2 are in metres and the time t is in seconds.

(a) Find the speed of mass m at $t = 0$.

(b) Find the speed of mass 2m at $t = 0$.

(c) A collision will occur at what time?

(d) A collision will occur at what position?

(e) Find the speed of mass m just before collision.

(f) Find the speed of mass 2m just before collision.

(g) Find the momentum of the system before collision.

(h) If the collision is perfectly elastic, what is the momentum of the system after collision?

(i) If the collision is completely inelastic, what is the momentum of the system after collision?

4.10 A truck on an African safari expedition is struck broadside on by a charging rhinoceros. The truck has a mass of 3000 kg and is travelling at 20 m/sec in the +x direction. The rhinoceros has a mass of 2000 kg and is travelling at 10 m/sec in the +y direction. The collision is completely inelastic and the rhinoceros is carried along with the truck after the collision. What are the *components in the x and y directions* of the following?

(a) The velocity of the centre of mass of the truck and the rhinoceros

(b) The velocity of the truck in the centre-of-mass frame before the collision

(c) The velocity of the rhinoceros in the centre-of-mass frame before the collision

(d) The velocity of the rhinoceros-truck combination relative to the ground after the collision

(e) In joules, what is the energy lost in the collision?

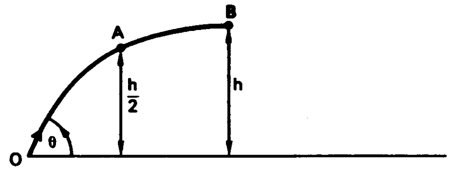


Fig. 4.11p(i)

4.11 Two bodies each of mass 5·0 kg are joined by a spring of negligible mass. The spring is compressed and the assembly projected with an initial speed of 196 m/sec at an angle θ to the horizontal. At point B the projectile reaches its maximum height h, 10 seconds after firing [figure 4.11p (i)]. Take g as constant and equal to 9·8 m/sec^2 and neglect air resistance.

(a) Show that angle $\theta = 30°$.

(b) What is the maximum height h reached?

(c) Find the vector velocity at point B.

(d) What is the kinetic energy at point A?

(e) What is the total energy at the instant of projection?

At point B the spring is released and all three components separate. Body 1 follows the trajectory BCD, which is horizontal at B [figure 4.11p (ii)].

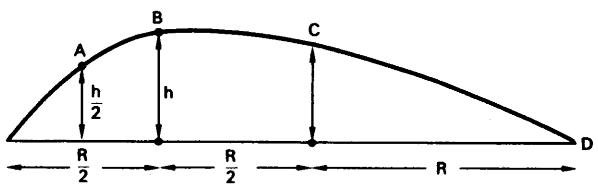


Fig. 4.11p(ii)

(f) How long after projection does body 1 reach D?

(g) What is the horizontal component of velocity at point C?

(h) Find the impulse imparted to body 1 by the spring.

(i) How long after projection does body 1 reach the height h/2 on the way down?

(j) What impulse is given to body 2 by the spring?

(k) How long after projection does body 2 land?

(l) Where does body 2 land?

(m) How much work is done by gravity during the complete motion?

(n) How much work is done by the spring?

(o) If the mass of the spring, although negligible for the preceding questions, is not zero, where and when does the spring land?

5

Energy and its conservation

5.1 WORK, ENERGY, CONSERVATIVE FORCES

If a force is applied to a particle, for example, and a component of the particle's motion is in the same direction and sense as the force, we say that the force does *work*. We define the element of work dW done by the force **F** in displacing the particle by **dS** by

$$dW = \mathbf{F}.\mathbf{dS} \tag{5.1}$$

The unit of work is the newton/metre, or joule. Work is clearly a scalar, and will be *negative* if the component of displacement in the direction of the force has the opposite sense to the force. *Power* is defined as the rate of doing work, and the instantaneous power P is defined as

$$P = \frac{dW}{dt} = \mathbf{F}.\mathbf{v} \tag{5.2}$$

where **v** is the instantaneous velocity. The unit of power is the watt, which equals one joule/sec.

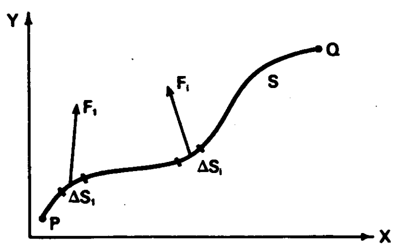


Fig. 5.1 Particle path in a force field

Suppose that a particle moves through a force field along a path S from point P to point Q (figure 5.1). For simplicity, we illustrate the situation in two dimensions. The force may vary in magnitude and direction over the path. To find the total work done, we split up the path into elements, ΔS_i, and approximate the force acting over each ΔS_i by its value $\mathbf{F}_i$ at the beginning of each element, for example. Then the total work W done by the force **F** over the path S from P to Q is

$$W = \lim_{\Delta S_i \to 0} \sum_i \mathbf{F}_i . \Delta \mathbf{S}_i = \int_P^Q \mathbf{F}.d\mathbf{S} \qquad (5.3)$$

where $\int_P^Q$ denotes what we have defined: the *line-integral* of the varying force **F** over the path S from P to Q.

Let us confine our attention to the x component of the motion, say from x_o to x, of the particle of mass m. We will have

$$W = \int_{x_o}^{x} F_x \, dx \qquad (5.3)$$

where F_x is the x component of the force, and we may drop the bar-sign because clearly we now have an ordinary integral. By Newton's Second Law we have

$$W = \int_{x_o}^{x} F_x dx = \int_{x_o}^{x} m\ddot{x} dx = \int_{x_o}^{x} m v_x \frac{dv_x}{dx} dx = \int_{v_{x_o}}^{v_x} m v_x dv_x$$

$$= \tfrac{1}{2} m v_x^2 - \tfrac{1}{2} m v_{x_o}^2 \qquad (5.4)$$

where v_x and v_{x_o} are the velocities of the particle at x and x_o respectively. We note that the *increase* in the quantity $\frac{1}{2}mv_x^2$ is equal to the work done on the particle by the x component of the force. We shall find similar relations between the y and z components of the velocity and the y and z components of the force. Hence we *define* a quantity, the *kinetic energy:*

$$T = \tfrac{1}{2} m |\mathbf{v}|^2 = \tfrac{1}{2} m \mathbf{v}.\mathbf{v}$$

and we note that the increase in the kinetic energy is equal to the work done *on* the particle *by* the resultant force. Clearly, a *decrease* in the kinetic energy must mean that work is done *by* the particle. So we may also consider the kinetic energy of a body as the work it is capable of doing before being brought to rest.

Suppose we take a particle around a *closed path* in some force field; we may enquire what is the total work done by the force on the particle during the round trip. It will be the line-integral around the closed path, denoted by $\oint$:

$$W = \oint \mathbf{F}.\mathbf{dS} \tag{5.5}$$

There are forces for which equation (5.5) gives W = 0 for *any* closed path. We call such forces *conservative* forces. Let us consider a particular round trip, shown in figure 5.2, for a conservative force field. Then, since

$$\oint \mathbf{F}.\mathbf{dS} = \int_A^B \mathbf{F}.\mathbf{dS}_1 + \int_B^A \mathbf{F}.\mathbf{dS}_2 = 0 \tag{5.6}$$

we must have

$$\int_A^B \mathbf{F}.\mathbf{dS}_1 = \int_A^B \mathbf{F}.\mathbf{dS}_2 \tag{5.7}$$

i.e. the work done in going from A to B is *independent* of the path taken, and depends only on the positions of A and B. This is an equivalent definition of a conservative force.

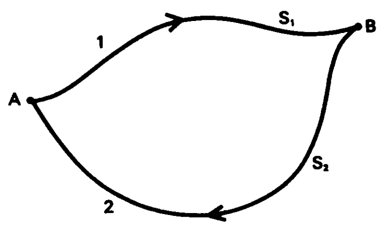


Fig. 5.2 *A closed path*

Some simple examples are now in order. Consider a particle of mass m moving in a uniform gravitational field of acceleration −gj, and for simplicity, consider the paths shown in figure 5.3 to be in the x-y plane, such that P is at height h_1j and Q is at height h_2j. The work done by the force −mgj in taking the particle from Q to P by *either* path is clearly $mg(h_2 - h_1)$, since any x components of the path contribute nothing to the elementary scalar products −mgj.**dS**. Therefore this gravitational force is a conservative force, by the criterion of the previous paragraph.

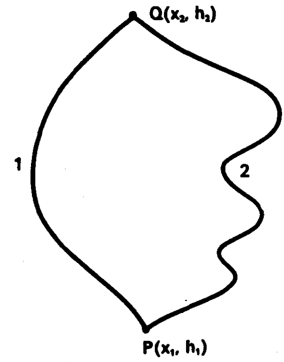


Fig. 5.3 *Closed path in a uniform gravitational field*

Consider now the degenerate closed path of figure 5.4, lying along the x axis from A to B, such that AB = ℓ. Suppose that a constant frictional force acts so that it always opposes the direction of motion. Then the work done by the force in going from A → B (path 1) is $(-F)\ell$. The work done in going from B → A (path 2) is $F(-\ell)$; and the total work around the whole closed path is $-2F\ell \neq 0$. Hence, as we suspected, this frictional force is *not conservative.* Dissipative forces such as friction are not conservative; but we can always choose to ignore them to a first approximation at any rate, and consider any conservative forces in a given situation first. We shall see in the next sections that conservative forces are capable of a very elegant treatment which greatly simplifies many problems in mechanics.

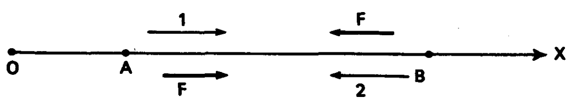


Fig. 5.4 *Closed path with a frictional force*

5.2 POTENTIAL ENERGY. POTENTIAL ENERGY DIAGRAMS IN ONE DIMENSION

Since the work done by a conservative force on a particle moving between two points depends only on the position of the points, and not on the path, there is a sense in which we can regard the particle as being capable of doing work by reason of its position. For example, a particle at a height h in a gravitational field will gain kinetic energy $\frac{1}{2}mv^2 = mgh$ on falling through the height h; this kinetic energy we may then *use* in some way. We are thus led to the idea of potential energy, which we define in the following way. We say that there exists a scalar potential function U(**r**) which has a single value for each **r** throughout the conservative force field. We define the change ΔU in this function between the points **r** and $\mathbf{r}_0$ as

$$\Delta U = U(\mathbf{r}) - U(\mathbf{r}_0) = \text{change in potential energy}$$

and we define

$$\Delta U = -\Delta T \tag{5.8}$$

where ΔT is the change in *kinetic* energy in going from $\mathbf{r}_0$ to **r**.

Now

$$\Delta T = \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}.d\mathbf{S} = -\int_{\mathbf{r}}^{\mathbf{r}_0} \mathbf{F}.d\mathbf{S}$$

$$= -\Delta U, \text{ from (5.8)}$$

Hence

$$\Delta U = U(\mathbf{r}) - U(\mathbf{r}_0) = \int_{\mathbf{r}}^{\mathbf{r}_0} \mathbf{F}.d\mathbf{S} \tag{5.9}$$

i.e. the change in potential energy between **r** and $\mathbf{r}_0$ is the work done *by* the force in moving from **r** to $\mathbf{r}_0$.

From equation (5.8) we see that if the kinetic energy *increases*, the potential energy *decreases*. From (5.8) and (5.4) we have

$$U(\mathbf{r}) - U(\mathbf{r}_0) = -[\tfrac{1}{2}mv^2(\mathbf{r}) - \tfrac{1}{2}mv_0^2(\mathbf{r}_0)]$$

or

$$U(\mathbf{r}) + \tfrac{1}{2}mv^2(\mathbf{r}) = U(\mathbf{r}_0) + \tfrac{1}{2}mv_0^2(\mathbf{r}_0) \tag{5.10}$$

$$= \text{constant}$$

From (5.10) we observe that the sum of the potential and kinetic energies is a *constant*; we call this the total energy, E,

i.e.

$$E = T + U \tag{5.11}$$

We also observe that the addition of an arbitrary constant to each side of (5.10) or (5.11) preserves the equality. Hence the *zero* of total, and therefore potential, energy is arbitrary, and we may take it where it is most convenient. This is because only *differences* in potential energy matter. This refers to the defining equation (5.8).

Let us specialize (5.9) to one dimension: the x axis, say. Then we obtain

$$\Delta U = \int_{x}^{x_o} F dx = -\int_{x_o}^{x} F dx$$

or

$$dU = -F dx, \text{ giving } F = -\frac{dU}{dx} \qquad (5.12)$$

That is, the negative of the slope of the one-dimensional potential-energy function at some point gives the value of the force at that point. Hence a plot of the potential-energy function can be a most useful device for examining the motion of a dynamical system.

Relation (5.12) can be most useful in deciding how to draw a potential-energy diagram: some difficulty is often experienced in knowing where to set the (arbitrary!) zero, for example in the case of electrostatic potential energy. Consider the potential energy of a test *positive* charge at a distance r from a positive charge q, for example. When $r \to \infty$, the force tends to zero, so the potential-energy curve must asymptote to the r axis. As r *decreases*, the test charge experiences a repulsive force; work must be done *against* this force, so the potential energy must increase as r decreases, and we obtain the conventional r^{-1} curve shown in figure 5.5a. If we have a positive test charge, and a charge $-q$, the force still tends to zero as $r \to \infty$; but as r decreases, the force is attractive and therefore does work, and hence the potential energy must *decrease* as r decreases, giving rise to the curve of figure 5.5b, clearly the mirror image of figure 5.5a.

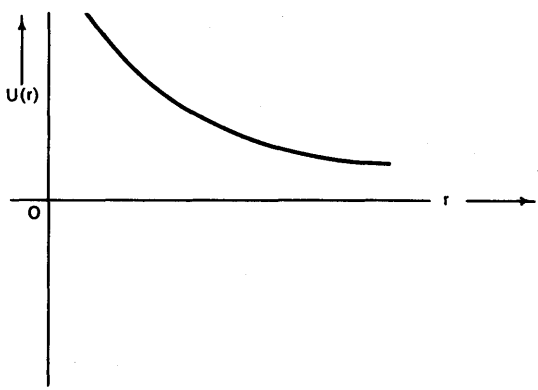


Fig. 5.5a Potential energy of a positive charge in the field of a charge +q

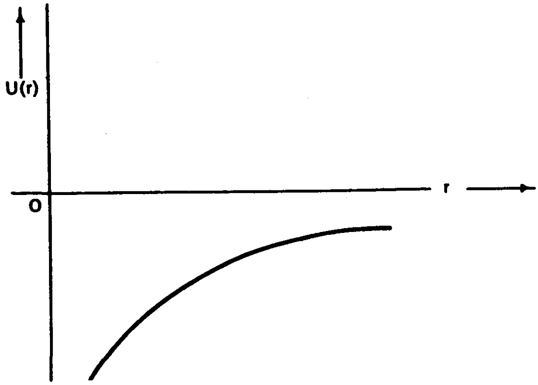


Fig. 5.5b *Potential energy of a positive charge in the field of a charge* $-q$

We may even construct potential-energy diagrams when we do not know much about the origin or nature of the force. Consider an α-particle (positively charged) approaching a positively-charged nucleus. It must experience the r^{-1} potential of figure 5.5a at first. But when it comes sufficiently close to the nucleus, the short-range, 'strong-interaction' force must take over, because we know that α-particles exist inside the nucleus. Therefore we may construct the potential-energy curve of figure 5.6a, where in the region of the attractive unclear force the slope of the curve is determined by the force $F = -\frac{dU}{dr}$. We can now investigate our model by *firing* α-particles at a nucleus. It turns out that the potential-energy diagram is of the right form, but its predictions are at variance with the facts: an α-particle with total energy E, say, *classically* speaking cannot get out of the nucleus, but nuclei do emit

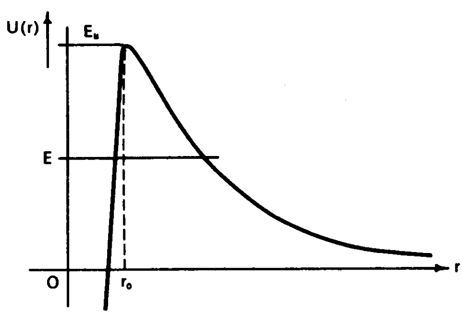


Fig. 5.6a *Potential energy of an α-particle near a nucleus*

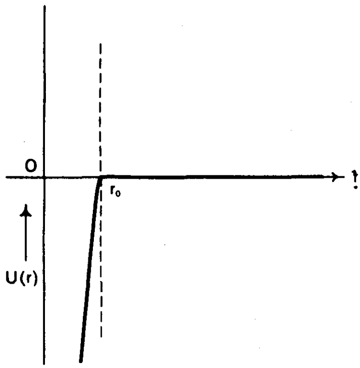


Fig. 5.6b   Potential energy of a neutron near a nucleus

α-particles of energy lower than the height E_B of the 'potential barrier'. This 'tunnelling' through a potential-energy barrier can be explained only by quantum mechanics.

For completeness, consider now a possible potential-energy diagram for a *neutron.* It experiences no Coulomb force, and only the 'strong-interaction' force, so that the potential-energy diagram would look like that of 5.6b; again, the slope is determined by $F = -\frac{dU}{dr}$.

Since the potential energy is a *scalar* , we may add potential energies arising from different kinds of forces. Thus a mass in a gravitational field has a potential energy mgh, where h is the height above a given point (figure 5.7a). If the mass experiences a spring force, the potential energy from this alone is $\frac{1}{2}kx^2$, say, where x is the displacement from the equilibrium point of the spring (figure 5.7b). The potential-energy

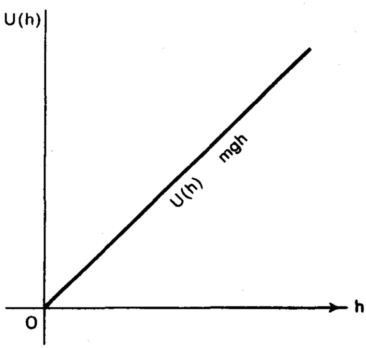


Fig. 5.7a   Potential energy in a uniform gravitational field

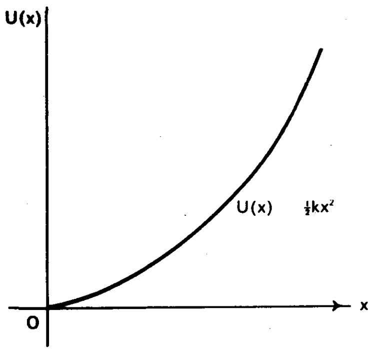


Fig. 5.7b Potential energy of a spring force

diagram of a mass suspended from a spring under gravity is the sum of the two potential-energy diagrams, as shown in figure 5.7c, with suitable redefinitions of x and h. We note that adding a straight line to a parabola still gives us a parabola so that the *character* of the motion under the spring force is unchanged.

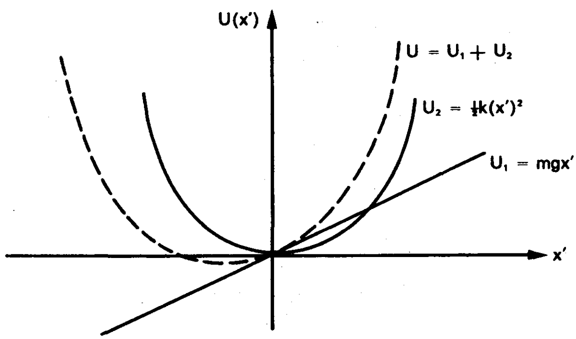


Fig. 5.7c Resultant potential energy from (a) and (b)

5.3 COMPLETE SOLUTION OF ONE-DIMENSIONAL PROBLEMS

In principle, one-dimensional dynamical problems in which we know the form of the potential-energy curves are completely soluble. Let us commence with

$$U(x) + \tfrac{1}{2}mv^2 = U(x_0) + \tfrac{1}{2}mv_0^2 = E$$

and eliminate v, so that we may solve for x as a function of time. At any instant, we have

$$\tfrac{1}{2}mv^2 = T = E - U(x) \qquad (5.13a)$$

so we obtain first

$$v = \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}[E - U(x)]} \qquad (5.13b)$$

where the positive and negative solutions depend on the direction of v, and the total energy E is constant. We may write (5.11) as

$$\left\{\frac{2}{m}[E - U(x)]\right\}^{-\frac{1}{2}} dx = dt \qquad (5.14)$$

or

$$\int_{x_0}^{x} \frac{dx}{\left\{\frac{2}{m}[E - U(x)]\right\}^{\frac{1}{2}}} = \int_{t_0}^{t} dt = t - t_0 \qquad (5.15)$$

an integral equation, which can be solved (numerically if necessary!) for x. As it stands, equation (5.15) does not appear to give us much information about the motion, so we return to the simpler equation (5.13). We note that $v = 0$ for $E = U(x)$, and that $U(x) > E$ yields an 'imaginary' velocity which we reject in classical mechanics as unphysical. Hence the regions for which $U(x) > E$ represent regions where a particle cannot be. The points for which $U(x) = E$ are called 'turning points' of the motion. The relation between U and the force, $F = -\frac{dU}{dx}$, tells us that the slope of the U(x) versus x curve is zero when $F = 0$. At such a point, U(x) may be a *maximum* (d^2U/dx^2 −ve), a minimum (d^2U/dx^2 +ve), or run parallel to the x axis $[(d^2U/dx^2) = 0]$. These correspond to positions of *unstable*, *stable*, and *neutral* equilibrium respectively. Consider a general potential-energy diagram as shown in figure 5.8. When we depart a little way from a local maximum, we see that the force, by equation (5.12), acts in a direction away from the local maximum, and tends to increase the displacement. When we depart a little from a local minimum, the force acts towards the local minimum, and tends to decrease the displacement. When U(x) is parallel to the x axis, there is no force produced by a small displacement. Thus the use of potential-energy diagrams, coupled with equations (5.12) and (5.13), gives us considerable information about the motion of a dynamical system.

To show that equation (5.15), despite its awkward-looking form, is useful, let us solve it for the case $U(x) = \tfrac{1}{2}kx^2$, the potential of a restoring spring force $F = -kx$. We have

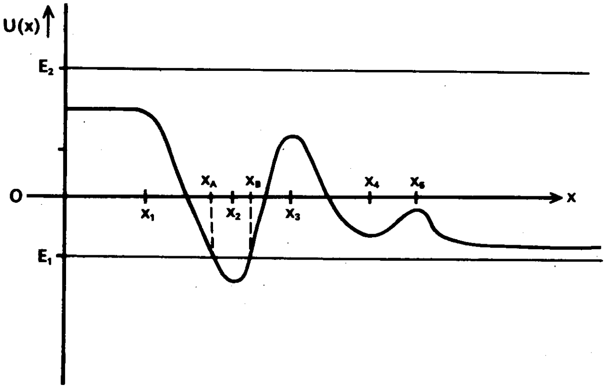


Fig. 5.8 *Some general potential-energy functions: 0-x_1, neutral equilibrium; x_2, x_4, stable equilibrium; x_3, x_5, unstable equilibrium. Note that a particle with total energy E is confined to the region $x_A \leqslant x \leqslant x_B$; x_A, x_B are the* turning points *of the motion. A particle with energy E_2 may be anywhere along the x axis. At any point the kinetic energy $T = E - V$, where E is the total energy*

$$\left(\frac{m}{2}\right)^{\frac{1}{2}} \int_{x_0}^{x} \frac{dx}{(E - \frac{1}{2}kx^2)^{\frac{1}{2}}} = t - t_0 \qquad (5.16)$$

$$\therefore \qquad \left(\frac{m}{k}\right)^{\frac{1}{2}} \int_{0}^{x} \frac{dx}{[(2E/k) - x^2]^{\frac{1}{2}}} = t$$

where we have put $x_0 = t_0 = 0$ for simplicity. The LHS is a standard form:

$$\left(\frac{m}{k}\right)^{\frac{1}{2}} \arcsin\left(\frac{k}{2E}\right)^{\frac{1}{2}} x = t$$

or if x = 0, t = 0, we obtain

$$x = \left(\frac{2E}{k}\right)^{\frac{1}{2}} \sin\left(\frac{k}{m}\right)^{\frac{1}{2}} t \qquad (5.17)$$

which is simple harmonic motion with angular frequency $\left(\frac{k}{m}\right)^{\frac{1}{2}}$ and amplitude $\left(\frac{2E}{k}\right)^{\frac{1}{2}} = x_0$ if $U(x) = E$ at $x = x_0$.

5.4 FIELD THEORY I. POTENTIAL IN THREE DIMENSIONS. THE GRADIENT OPERATOR

In the previous section we dealt with the potential-energy function U(x), a scalar function of the single space co-ordinate x, and we found it useful to have the related quantity (dU/dx), whose value at the point x measures the rate at which U increases from one point to another. We are now going to generalize potential functions to three dimensions; it is convenient to consider *any* general scalar field quantity V(x, y, z) whose value depends on three space co-ordinates, and then specialize to the case of potential. In this way we get a better introduction to what is called *field theory*.

If we are dealing with a scalar field quantity V(x, y, z), whose value depends on *three* space co-ordinates, it turns out to be useful here also to have a quantity which measures the rate of increase of V. But obviously we can move away from a given point (x, y, z) in an infinite number of directions, and generally speaking the rate of increase of V will be different in these different directions. So things are more complicated. But it turns out that we need not introduce more than *one* vector quantity related to V in order to be able to get the rate of increase of V in *any* direction. We do it like this.

Suppose the quantity V has a constant value V_1 at all points on some surface S_1. Take a point P on S_1, and consider the surface S_2, very close to S_1, on which V has the constant value V_2 – which of course will be very close to V_1 (see figure 5.9). The *shortest* way to get to S_2 from P is obviously to go along the *normal* to S_1 at P. (If we take a small enough area around P we can treat it as flat and the normal is just the straight line at right angles to this bit of a plane.) So it is in *this* direction PN away from P that V is increasing fastest. In fact its rate of increase is just $(V_2 - V_1)/PN$. In any other direction PQ, at an angle θ to PN, the rate of increase is $(V_2 - V_1)/PQ = (V_2 - V_1) \cos \theta/PN$. (For $PQ = PN/\cos \theta$.)

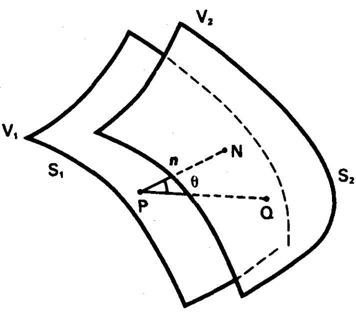


Fig. 5.9 Equipotential surfaces

We now define a vector quantity called the *gradient* of V, written **grad** V, or ∇V, by letting $V_2 - V_1$ become small, and saying: The *magnitude* of ∇V at P is

$$\lim_{V_1 \to V_2} (V_2 - V_1)/PN \tag{5.18}$$

and the *direction* of ∇V at P is the direction of the normal $\vec{PN}$. Then the rate of increase of V, in the direction $\vec{PQ}$ making an angle θ with $\vec{PN}$, is just $\nabla V.PQ$, where PQ is a *unit* vector in the direction $\vec{PQ}$. (Check this from the above result, and the definition of the 'dot' product.)

If we multiply this rate of increase by the small length PQ the result is evidently $V_2 - V_1$. Putting this in a neater form: the increase δV corresponding to a displacement $\delta \mathbf{r}$ is

$$\delta V = \nabla V.\delta \mathbf{r} \tag{5.19}$$

Suppose that $\delta \mathbf{r}$ is in fact the direction of the x axis (the direction of the unit vector **i**). Then obviously the rate of increase of V is (by definition!) the partial derivative $\partial V/\partial x$. But it is also $\nabla V.\mathbf{i}$ from the above, and we know that this is the expression for the x component of ∇V. So we have that the x component of ∇V is just $\partial V/\partial x$, and similarly the y component is $\partial V/\partial y$ and the z component is $\partial V/\partial z$. So we have *another* way of defining ∇V:

$$\nabla V = \frac{\partial V}{\partial x}\,\mathrm{i} + \frac{\partial V}{\partial y}\,\mathrm{j} + \frac{\partial V}{\partial z}\cdot\mathrm{k} \tag{5.20}$$

This form is useful when we have to *calculate* with the gradient; we choose a convenient co-ordinate system (x, y, z) and work out the components. Of course the value of $\partial V/\partial x$ will depend on what we choose as the x axis, and similarly for $\partial V/\partial y$ and $\partial V/\partial z$. But our original definition of ∇V shows that the *whole vector* ∇V is the same no matter what co-ordinate system we choose; only its division into components changes. We say that ∇V is a *vector invariant.*

An important interpretation of the gradient arises when the scalar field is a potential function. This means (by the definition of 'potential') that the work done against the (vector) field **E** in moving a small charge from a point of potential V to a point of potential $(V + \delta V)$ is $q\delta V = \nabla V.\delta \mathbf{r}$, where $\delta \mathbf{r}$ is the displacement of q.

But, since $q\mathbf{E}$ is the force, this work is also (from the definition of 'work') $-q\mathbf{E}.\delta\mathbf{r}$. Since these expressions have to hold for all paths we see that

$$\mathbf{E} = -\nabla V \text{ or } \mathbf{F} = -\nabla U$$

is the three-dimensional version of the relation $E = -\frac{dV}{dx}$ or $F = -\frac{dU}{dx}$ in one dimension.

A further use of the ∇-operator, in determining whether known force fields are conservative, is given in Appendix 3.

5.5 ENERGY AND SYMMETRY I

The total energy function

$$E = \frac{1}{2}\sum_i m_i v_i . v_i + \sum_i U_i(r) = T + U \qquad (5.21)$$

of a system of particles contains a great deal of information about the motion of that system by virtue of its *symmetry* or *invariance* properties. For simplicity we confine ourselves to a single particle for the moment.

First, we note that the expression for the kinetic energy of a single particle is unchanged in form by a Galilean Transformation, since

$$T_{K'} = \tfrac{1}{2}m(v_K - V).(v_K - V) = \tfrac{1}{2}mv_{K'}.v_{K'}$$

where V is the velocity of K. It is also clearly unchanged by any *rotations*, since any rotation of axes cannot alter the sum $v_x^2 + v_y^2 + v_z^2 = |v|^2$; we shall have occasion to refer to this later on. Clearly these statements can be suitably generalized to a multi-particle system.

We see therefore that the symmetry of the total energy is governed by that of the potential function. We now state (without proof) the following theorem (valid for any number of particles).

Theorem If a potential function is invariant under translation in a particular direction, then the *momentum* in that direction is conserved, i.e. is a constant of the motion in all frames.

As an example, consider a particle moving under the gravitational potential mgy. It is clear that this is invariant under a translation along the x axis, and therefore the x component of momentum is conserved. These are just the conditions we set up for the projectile problem in Section 2.3.

Consider now two interacting particles. If the interaction potential energy depends only on $|(\mathbf{r}_1 - \mathbf{r}_2)|$, say, where $\mathbf{r}_1$, $\mathbf{r}_2$ are the position vectors of the particles, it is clear that this will also be invariant under translation, and that again momentum will be conserved. We note that the gravitational and electrostatic forces are of this form. The generalization of this kind of interaction to a system of particles is obvious.

The invariance theorem above is important in higher mechanics, *and also carries over into quantum mechanics.* Constants of the motion usually help to reduce the number of co-ordinates necessary in the solution of a problem. There is, in fact, a technique for solving complicated dynamical problems by finding co-ordinate transformations that give constants of the motion, eventually ending up with an effective one-dimensional problem which we know is completely soluble.

CHAPTER 5 PROBLEMS

5.1 A massless coiled spring AB lies on a horizontal frictionless table, the end B of the spring being held fixed. A body of mass 0·5 kg moving with a constant speed of 4 m/sec in the direction AB strikes the end A of the spring and compresses it by 10 cm. The body and spring remain in contact until the spring regains its original length.

(a) Determine the force constant K of the spring.

(b) Draw sketch graphs of the potential energy and total energy of the system as a function of the body's position.

(c) Draw a sketch graph of the velocity of the body as a function of position before, during, and after the collision.

(d) What was the total impulse delivered to the mass by the spring?

(e) What was the total amount of work done on the body by the spring during their time of contact?

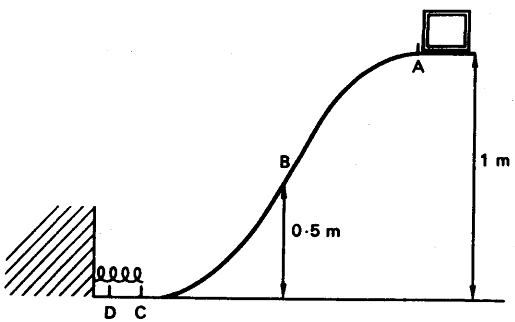


Fig. 5.2p

5.2 A mass of 1 kg passes point A in figure 5.2p with a speed of 1 m/sec and proceeds down the frictionless incline. It collides with a horizontal massless spring at C, remaining in contact while it compresses the spring 10 cm to the point D. The spring then resumes its original length and the body moves back up the incline. The value of g, the gravitational acceleration, is 10 m/sec^2.

(a) What is the kinetic energy of the body at B?

(b) What is the total mechanical energy at C?

(c) What impulse is delivered to the spring while the body moves from C to D?

(d) What total impulse is delivered to the spring by the body during the total time for which the two are in contact?

(e) What is the work done on the spring in compressing it from C to D?

(f) What is the force constant of the spring?

5.3 A mass of 10 kg rests on top of a vertical coil spring of negligible mass, thereby compressing it by 0·1 m. The acceleration due to gravity is 9·8 m/sec.

(a) What is the value of k, the restoring force for unit displacement, in newtons/metre?

(b) What is the elastic potential energy of the spring when the block rests on the spring?

(c) What is then the gravitational energy of the block, taking the top of the uncompressed spring as the zero level?

The block is now removed and later dropped on to the spring from a height of 0·4 m above the uncompressed position.

(d) What is the total mechanical energy E of the block while it is falling towards the spring?

(e) The block strikes the spring and compresses it, no mechanical energy being lost on impact. At the instant of maximum compression what is the total mechanical energy of the block/spring system?

(f) At this instant of maximum compression what is the total mechanical energy of the block alone?

5.4 A projectile of mass m is fired in a uniform gravitational field of intensity g with an initial speed v. It traverses the trajectory shown in figure 5.4p and reaches a maximum height h at the point B. In answering the following questions neglect the effect of air resistance and give your answers in terms of the parameters m, v, g, and h.

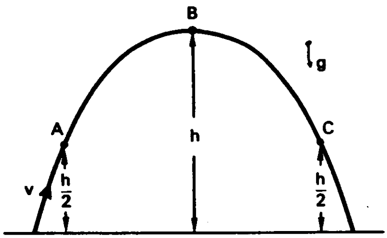


Fig. 5.4p

(a) What is the kinetic energy of the projectile at the point B?
(b) What is the horizontal component of velocity at point B?
(c) What is the horizontal component of velocity at point A?
(d) What is the vector acceleration of the projectile at point B?
(e) What is the vector acceleration of the projectile at point A?
(f) What is the radius of curvature of the trajectory at the highest point B?
(g) Relate θ, the angle of elevation of the gun, to the quantities v, g, h.
(h) What is the total work done on the projectile during its passage from A to C?
(i) What is the change in momentum suffered by the projectile in travelling from A to C?
(j) Describe the conditions under which the radius of curvature at the point B could be zero.
(k) If, at the point A, the projectile exploded into two pieces, one of which was projected vertically downwards, would you expect the maximum height reached by the other piece to be greater than or less than h? State your reasons.

5.5 (a) Distinguish between the mass of a body and its weight.
(b) Two bodies whose masses are m and M are held a distance h apart. Each attracts the other with a force which has a constant value K, independent of their separation. The bodies are released. If the forces of mutual attraction are the only ones acting, how much work has been done by these forces when the bodies collide?

Use momentum and energy considerations to show that, when they collide, the body of mass m has acquired a velocity v given by

$$v^2 = \frac{2Kh}{m\left(1 + \frac{m}{M}\right)}$$

(c) Consider the problem of an object falling from a height h to the earth's surface. Show that this situation approximates to the hypothetical situation described in part (b) and that the result obtained in that part reduces, in this approximation, to the familiar expression for the velocity acquired by a falling object.

5.6 A closed horizontal tube of mass 5 kg rests on a frictionless table. Inside the tube and at its centre are two masses of 1 kg and 4 kg, initially joined together. It is arranged that a small explosion separates these two masses so that they fly towards the opposite ends of the tube (experiencing no retarding forces on the way) and are embedded in the ends. The 1 kg mass starts off with a speed of 12 m/sec.

(a) Calculate the initial speed of the 4 kg mass.
(b) What happens when the 1 kg mass hits the end of the tube?
(c) After this impact, what is the velocity of the 4 kg mass relative to the tube?
(d) Describe the subsequent motion of the system.

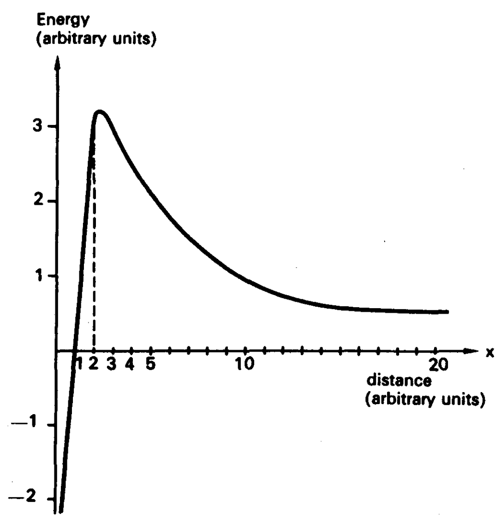


Fig. 5.7p

5.7 The potential energy curve in figure 5.7p represents the force experienced by a positively-charged particle (proton) approaching an atomic nucleus. Answer the following questions.

(a) In which region or regions of the diagram could a proton move if it had a total energy of
   (i) −1 unit?
   (ii) +3 units?

(b) Indicate any equilibrium points on the diagram, and state the nature of the equilibrium.

(c) What is the minimum total energy a proton must have to get into or out of the nucleus?

(d) A proton is originally far to the right, moving to the left with a total energy of 2 units. Describe its subsequent motion.

(e) What is the magnitude and direction of the force on a positively-charged particle in the region $1 < x < 2$?

(f) A neutron experiences the same attractive force as the positively-charged particle inside the nucleus, but no force outside. Taking the nucleus to occupy $0 \leq x \leq 2$, sketch the appropriate potential-energy curve for a neutron.

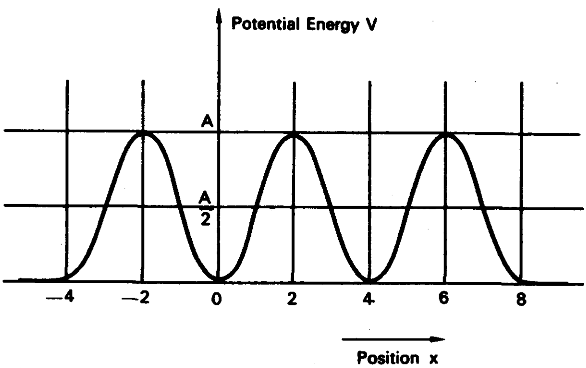


Fig. 5.8p

5.8 A particle moves in the x direction shown in figure 5.8p. It is subjected to a conservative force field. The potential V of the particle is given by

$$V = A \sin^2\left(\frac{\pi x}{4}\right) \text{ in the range } -4 \leq x \leq 8,$$

$$V = 0 \text{ elsewhere.}$$

(a) Which of the following positions are positions of equilibrium? Give your reasons.

$$x = -2, -1, 0, +1, +2, +3, +4, +5, +6$$

(b) Which of these are positions of stable equilibrium? Give your reasons.

(c) Which ranges of x are forbidden to a particle whose total energy is A/2? Give your reasons.

Questions (d) to (i) relate to the motion of a particle which is projected in the +x direction from the point x = 0 with a kinetic energy of 2A, and is subjected to the force field described above.

(d) At which points will the kinetic energy reach a minimum value?

(e) At which points will the total energy of the system reach a maximum value?

(f) At which points will the speed of the particle be a minimum?

(g) At which points will the speed of the particle be a maximum?

(h) What is the magnitude and direction of the force exerted on the particle when it is at the position x = 2/3?

(i) Write down an expression for the kinetic energy of the particle as a function of position within the range $0 \leq x \leq 8$.

The next two questions relate to a particle which is projected in the +x direction from the point x = 0 with a kinetic energy of $\frac{3}{4}$A, and is subjected to the force field described above.

(j) At which point does the particle come to rest?

(k) Describe the subsequent motion of the particle.

5.9 Figure 5.9p represents a potential-energy diagram for a particle which can move along the x direction. The particle has a total energy E, and is *initially* at x = 0, moving in the +x direction.

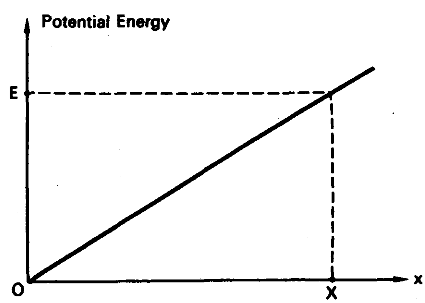


Fig. 5.9p

(a) Which *one* of the following correctly describes the subsequent motion of the particle?
   (i) It continues to move along the positive x direction and never returns to x = 0.
   (ii) It reaches x = X and remains at rest at this point.
   (iii) It moves to x = X and then returns to x = 0.
(b) The energy of the particle at x = 0 is
   (i) all kinetic energy.
   (ii) all potential energy.
   (iii) partly kinetic and partly potential.
(c) The *force* on the particle is
   (i) constant.
   (ii) directly proportional to the displacement from 0.
   (iii) proportional to the square of the displacement from 0.
   (iv) proportional to the displacement from X.
(d) The *direction* of the force is
   (i) in the positive x direction.
   (ii) in the negative x direction.
   (iii) along the direction of the potential-energy curve.
   (iv) at right angles to Ox.

5.10 A particle within a certain nucleus has a potential energy V(x) like that shown in figure 5.10p, where x = 0 at the centre of the nucleus. The particle is said to be 'bound' if it cannot move outside the limits x = ±1, and 'free' otherwise.

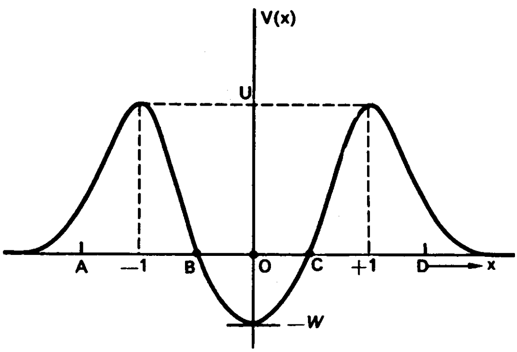


Fig. 5.10p

In the region between A and D,

$$V(x) = (U + W)\, x^2\, (2 - x^2) + K$$

where K is a constant. Outside this region V(x) falls rapidly to zero as shown. Answer the following questions in terms of the quantities V and W.

(a) What is the value of the constant K in the above expression for V(x)?

(b) What is the greatest energy a bound particle can have?

(c) What is the greatest speed a bound particle can have?

(d) What is the least amount of work that would need to be done to free a particle originally at rest at the centre of the nucleus (x = 0)?

(e) How much work would have to be done to shift a stationary particle from C to a point a large distance to the right of D?

(f) What is the magnitude of the force on the particle at the point $x = +\frac{1}{2}$?

(g) The direction of the force at the point $x = +\frac{1}{2}$ is
   (i) towards the centre of the nucleus.
   (ii) away from the centre of the nucleus.
   (iii) along the tangent to the potential-energy curve.
   (iv) out of the paper in a sense given by the right-hand screw rule.

(h) Name a point labelled in the diagram where the force is zero.
   (i) What is the least speed a particle (of mass m) would need to start with in order to pass right through the nucleus from a point far away to the right?

6 Simple harmonic motion-free, damped and driven

6.1 FREE SHM FROM THE ENERGY CONSERVATION VIEWPOINT

We have seen in previous sections, 3.6 and 5.3, that a particle of mass m, subject to a restoring force $\mathbf{F} = -kx\,\mathbf{i}$, for example, moves such that its displacement x from the origin is given by

$$x = A \sin(\omega_0 t + \phi) \tag{6.1}$$

where $\omega_0 = \sqrt{k/m}$ is the angular frequency, t is the time, ϕ is a phase angle, and A is the amplitude (magnitude of the maximum displacement) of the motion. This kind of repetitive, oscillatory motion is known as *simple harmonic motion* (SHM), and we are now going to delve into some aspects of it in considerable detail.

Why? Because of its universality, and its importance in physics and engineering. Let us demonstrate the universality first. Suppose we have some potential function, V(x), which has a *local minimum*, which we may take as being at x = 0 without loss of generality. We recall from section 5.2 that a particle with this minimum energy is in a state of stable equilibrium, i.e. will experience a *restoring force* for small displacements from the location of the actual minimum. Now it is a fact that, if we know the *derivatives* of a function at x = 0, we may expand that function for small displacements x in a series; thus

$$V(x) = V(0) + xV'(0) + \frac{x^2}{2!}V''(0) + \ldots \tag{6.2}$$

This is Maclaurin's theorem. Using $F = -(dV/dx)$, and the fact that $V'(0) = (dV/dx)_{x=0} = 0$ for the minimum at x = 0, we have that the *restoring force* F for small displacements is given by

$$F = -\frac{dV}{dx} \doteq -x\,V''(0) + \text{higher-order terms} \qquad (6.3)$$

Hence all small oscillations about stable equilibrium are (at least to a good approximation) simple harmonic oscillations.

Simple harmonic oscillations occur in electromagnetic circuits, and the quantum-mechanical simple harmonic oscillator forms the basis for the quantization of the electromagnetic field, among other things. So the simple harmonic oscillator deserves a full treatment.

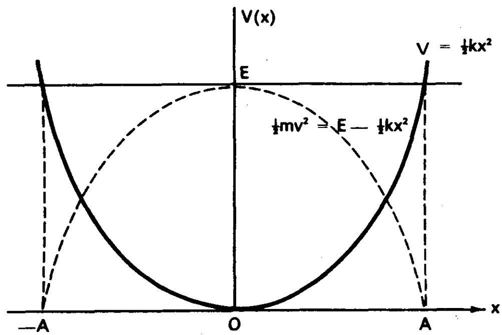


Fig. 6.1 Potential energy of a simple harmonic oscillator

The potential-energy curve V(x) for a simple harmonic oscillator is a *parabola,* $V(x) = \frac{1}{2}kx^2$ (figure 6.1). It is clear from the figure that the greater the value of the total energy, the greater is the amplitude. Since the total energy is all potential at the turning points of the motion, we have immediately:

$$E = \tfrac{1}{2}kA^2 \quad \text{or} \quad A = \left(\frac{2E}{k}\right)^{\frac{1}{2}}$$

And since the kinetic energy is the difference between the total energy and the potential energy,

$$\tfrac{1}{2}mv^2 = E - V(x) = E - \tfrac{1}{2}kx^2 \qquad (6.5)$$

i.e. the kinetic energy follows an inverted parabola with respect to the displacement x. This is also illustrated in figure 6.1. Finally, the maximum kinetic energy is clearly at x = 0, and

$$\tfrac{1}{2}mv^2_{max} = E = \tfrac{1}{2}kA^2 \quad \text{or} \quad v^2_{max} = \frac{kA^2}{m} \qquad (6.6)$$

These results are often obtained by manipulations with x and $\dot{x}$ as functions of t, but are much more simply obtained from the potential-energy curve and energy conservation considerations. Equations (6.1) (6.5), and (6.6) completely categorize the simple harmonic motion: and we recall that equation (6.1) was obtained by the energy method of solution in section 5.3.

6.2 DAMPED SHM – QUALITATIVE SOLUTION

It is clear that, up to now, we have ignored any dissipative (i.e. non-conservative) forces in our treatment of simple harmonic motion. A very commonly encountered force is the *viscous damping force,*

$$F = -bv = -b\dot{x} \tag{6.7}$$

where b is some constant. A line integral over a degenerate closed path along the x axis shows clearly that this is a non-conservative force; even more simply, the motion always does work against this force, and therefore the total energy (unless supplied from outside) *must decrease.* Using this fact, and a potential-energy diagram, we can obtain a qualitative idea of the motion. First, let us suppose that the damping is light: this means that the total energy runs down slowly, i.e. it takes many periods of the oscillation before, say, half of the total energy is dissipated. Suppose that the SHM oscillator starts off from $x = -A$; it will not reach $x = +A$, because the energy will have run down to some value, say E_1. The next excursion in the negative x direction will be even smaller, because the energy will have run down to E_2, say (figure 6.2). So by these qualitative considerations we obtain a plot of displacement versus time like that in figure 6.3.

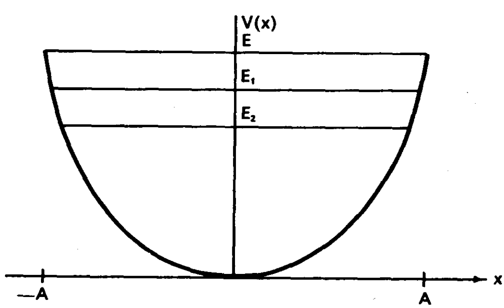


Fig. 6.2 Damped SHM energy diagram

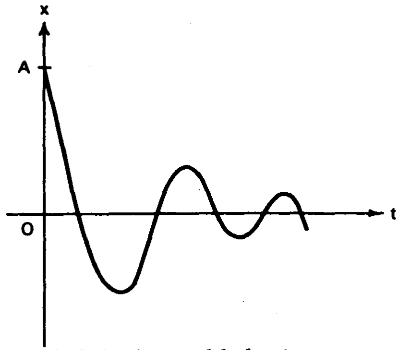


Fig. 6.3 Qualitative lightly damped behaviour

Suppose now that the damping is so heavy that the total energy runs down almost to zero in half a period of oscillation. This must mean that the 'oscillator', if it starts from −A, say, will not even have the 'runningdown' oscillations of figure 6.3, but must behave as shown in figure 6.4.

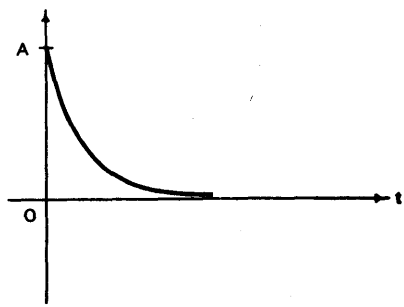


Fig. 6.4 Qualitative heavily damped behaviour

Since the motion is always being opposed, we should expect that the time-interval between two successive zero-crossings in figure 6.3 would be greater than the half-period T/2 of the undamped oscillator. This must be so, since if, as in figure 6.4, the damped oscillator never makes a zero-crossing, there is a sense in which the period is infinite.

We have gone about as far as we can go with our qualitative considerations and the energy diagram. It is time to turn to the exact solution.

6.3 DAMPED SHM – QUANTITATIVE SOLUTION

The equation of motion of the SHM oscillator subject to the viscous damping force of equation (6.7) is

$$m\ddot{x} = -b\dot{x} - k\dot{x} \qquad (6.8a)$$

or

$$m\ddot{x} + b\dot{x} + kx = 0 \qquad (6.8b)$$

We have already solved this equation for k = 0 (section 3.6 (iv)), and we found that the velocity decreased exponentially with respect to time. We solve (6.8) by making an inspired guess at the solution, and checking this in the original differential equation (quite a fruitful way of solving differential equations).
We set

$$x = A\exp(-t/\tau)\cos(\omega' t + \phi) \qquad (6.9)$$

and we find that this is a solution if

$$\tau = \frac{2m}{b}, \quad \omega' = \left[\frac{k}{m} - \left(\frac{b}{2m}\right)^2\right]^{\frac{1}{2}}$$
$$= [\omega_0^2 - \frac{1}{\tau^2}]^{\frac{1}{2}} \qquad (6.10)$$

i.e. we have

$$x = A\exp\left(-\frac{bt}{2m}\right)\cos\left\{\left[\frac{k}{m} - \left(\frac{b}{2m}\right)^2\right]^{\frac{1}{2}} t + \phi\right\}$$
$$= A\exp(-t/\tau)\cos\left\{\left[\omega_0^2 - \frac{1}{\tau^2}\right]^{\frac{1}{2}} t + \phi\right\} \qquad (6.11)$$

Consider the case $\omega' > 0$, i.e. $\omega_0^2 > \tau^{-2}$, or $\tau > \omega_0^{-1}$. We have an *oscillation*, of period $(2\pi/\omega')$, which has a decreasing amplitude, $A\exp(-t/\tau)$. This is shown in figure 6.5; the similarity to the qualitative predictions of figure 6.3 is not surprising. The quantity τ is the *time-constant* of of the decay of the amplitude: the amplitude has fallen to e^{-1} of its original value at t = 0 in time τ. The time-constant for the decay of the *energy* ($\propto A^2$) is therefore 2τ.

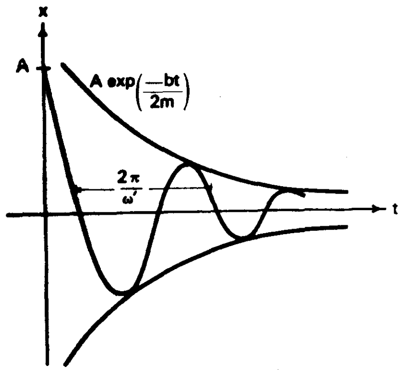


Fig. 6.5 *Lightly damped oscillations*

Consider the case $\omega' = 0$, i.e. $\tau = \tau_c = \omega_0^{-1}$. We now have no oscillations, but a simple exponential approach to zero, if the oscillator began at x = A, with time-constant τ_c (figure 6.6). We note that τ_c is less than any of the τ associated with the previous case $\omega' > 0$; the approach to zero is quicker.[1]

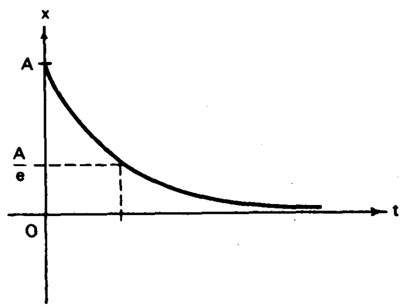


Fig. 6.6 *Critically damped behaviour*

What happens when ω' is imaginary, i.e. $\tau < \omega_0^{-1}$? We must now take courage and make use of the mathematical relationship ($i = \sqrt{-1}$):

$$\cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta})$$

which gives us, setting $\phi = 0$ for convenience,

$$\cos(i\omega' t) = \tfrac{1}{2}(e^{\omega' t} + e^{-\omega' t})$$

[1] This treatment of so-called 'critically damped motion' is not quite correct (see e.g. K.R. Symon, *Mechanics, op. cit.*, p. 47) but it will serve our purpose.

So that equation (6.9) becomes effectively

$$x = A \exp(-t/\tau)\tfrac{1}{2}[\exp(\omega' t) + \exp(-\omega' t)] \qquad (6.12)$$

We note that this expression contains the exponential function $A \exp\left\{-[\tau^{-1} + \omega']t\right\}$: in other words, the effective time-constant has *increased* from τ_C to $(\tau^{-1} + \omega')$, so that the approach to zero [actually a sum of two exponentials from (6.12)] is without oscillations, but *slower* than in the case $\omega' = 0$.

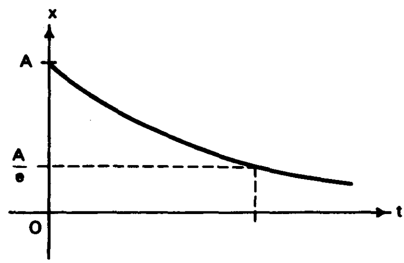


Fig. 6.7 Overdamped behaviour

In the case where $\omega' > 0$, and damped (decaying) oscillations occur, the oscillator is said to be *underdamped*; in the case where $\omega' = 0$, and we have the *minimum* time-constant τ_C, the oscillator is said to be *critically damped*; in the case ω' imaginary, where we have no oscillations but again an effective time-constant greater than τ_C, the oscillator is said to be *overdamped*.

When we have something like a meter which reads electrical current, for example, and the needle works against a restoring spring force, we like to know the value of the current as quickly as possible. So we build in some damping, preferably so that the system is *critically* damped, i.e. it settles down with the minimum possible time-constant. It is possible to do this either mechanically or electrically, since we can have damped electromagnetic oscillations in a circuit which has resistance, capacitance, and inductance present.

6.4 DRIVEN SHM. RESONANCE

Since friction or viscous damping is always present in any practical system, we must supply energy from outside if we expect a practical simple harmonic oscillator to have a constant amplitude. It is quite easy to generate simple harmonic driving forces, either mechanically or electrically, so we choose to investigate the case where a damped harmonic oscillator is subject to the harmonic driving force

$$F = F_0 \cos \omega t \qquad (6.13)$$

say, where ω may differ from the frequency $\omega_0 = \sqrt{k/m}$ of the undamped harmonic oscillator. In fact, we shall investigate what happens as the driving frequency, ω, is varied.

Before solving the equation of motion, we shall give the results of an experiment which is readily set up on an air-track. A carriage of a certain mass and a spring of a certain stiffness form the oscillator; the air-track and the spring supply some friction. The oscillator is driven by means of a piston from a variable speed motor; the piston is coupled to the spring by a suitable connection (figure 6.8). We determine first the approximate frequency ω_0 of the oscillator; we then start the motor at a speed such that the driving frequency ω is considerably greater than ω_0.

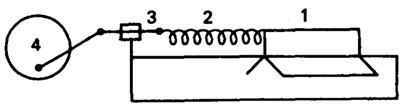


Fig. 6.8 *A driven oscillator: (1) air track carriage; (2) spring; (3) piston; (4) wheel, driven by variable-speed motor*

We observe that the carriage oscillates with the frequency ω of the driving piston, but 180° out of phase with it. We decrease ω, and observe the motion for a series of values of ω as it approaches ω_0. We find that the amplitude of the motion *increases* as $\omega \to \omega_0$, and at $\omega \doteqdot \omega_0$ the amplitude gets very large (sometimes disastrously!), and that here the motion of the carriage is about 90° out of phase with that of the driving piston.

As we decrease ω below ω_0, and observe the motion for various values, we find that the amplitude of the motion decreases once more; when $\omega \ll \omega_0$, the motion of the mass is in phase with the driving piston. A typical set of results is plotted in figures 6.9a and 6.9b.

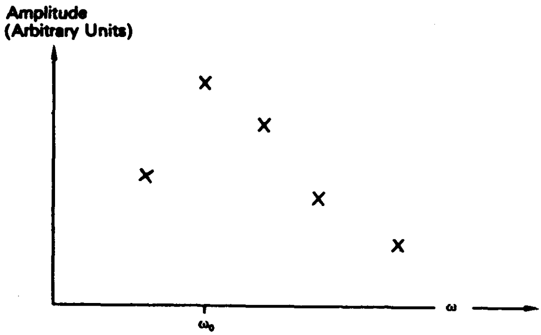


Fig. 6.9a Amplitude versus driving frequency (experimental)

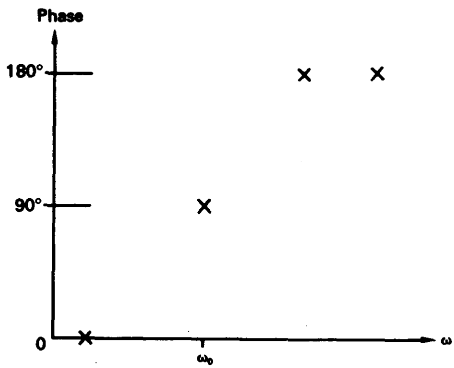


Fig. 6.9b Phase versus driving frequency (experimental)

Let us now table the equation of motion of the driven SHM oscillator, which is

$$m\ddot{x} = -kx - b\dot{x} + F_0 \cos \omega t \tag{6.14a}$$

or

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega t \tag{6.14b}$$

To bring out the significance of the solution, it is now convenient to rewrite (6.14b) in the following form:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos \omega t$$

or

$$\ddot{x} + \tau_e^{-1}\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos \omega t \tag{6.14c}$$

where we have set $(b/m) = \tau_e^{-1}$, the time-constant at which the energy runs down [see equations (6.10) and (6.11) and the following discussion]. Now we expect the *steady-state* solution to be a sinusoidal oscillation at the driving frequency, of constant amplitude (dependent on F_o and ω), with a phase different from that of the driving frequency, i.e. a solution of the form

$$x = \frac{F_o}{m\Omega}\cos(\omega t + \phi) \tag{6.15}$$

say, where the constants Ω and ϕ can be determined by substituting (6.15) back in the original equation. After some tedious but not difficult algebra, we find that

$$\Omega = [(\omega^2 - \omega_o^2)^2 + \omega^2 \tau_e^{-2}]^{\frac{1}{2}} \tag{6.16}$$

$$\tan\phi = \frac{-\omega/\tau_e}{\omega_o^2 - \omega^2}$$

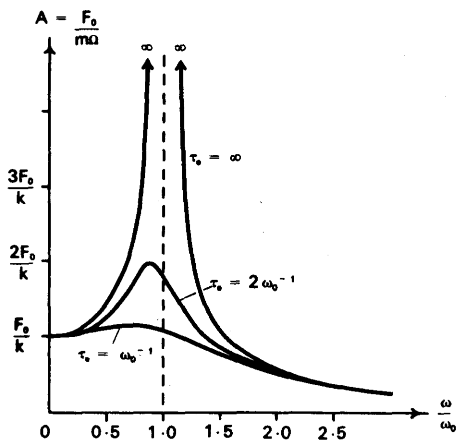


Fig. 6.10a Oscillation amplitude versus driving frequency

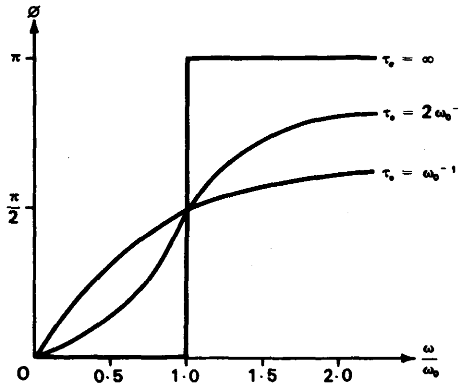


Fig. 6.10b Oscillation phase versus driving frequency

The solutions for the amplitude $A = F_0/m\Omega$ and for the phase angle ϕ are plotted, for various values of τ_e, in figures 6.10a and 6.10b. We observe that when τ_e is large (i.e. the damping is small), the amplitude is a maximum near $\omega \doteqdot \omega_0$, and falls off on either side; the phase variation corresponds to that we found by experiment. The phenomenon of the large amplitude when $\omega \doteqdot \omega_0$, i.e. the driving frequency is approximately equal to the resonant frequency of the undamped oscillator, is known as *resonance.* It can be useful, or disastrous. Soldiers 'break step' when crossing a bridge, for example, to avoid possible disastrous consequences of resonance. We see also that the frequency width of the resonance curve between the points $(A_{max}/\sqrt{2})$, where A_{max} is the maximum amplitude, decreases as the damping decreases; these points are known as the 'half-power points', for a reason which should be obvious [equations (6.4)]. We can therefore use the property of resonance to *select* only certain frequencies; this is just what is done in a radio or television receiver, where the behaviour of a suitable circuit containing inductance, capacitance, and resistance is exactly analogous to that of our mechanically-driven harmonic oscillator. The spectral lines emitted by atoms also have the characteristic bell-shaped curve for power versus frequency. If $\tau_e = 2\tau$ is *large*, we may set $\Delta\omega = \omega - \omega_0$, $\omega \doteqdot \omega_0$, and we find that the *square of the amplitude* falls off as $[(\Delta\omega)^2 + \tau^{-2}]^{-1}$ or $[1 + \tau^2(\Delta\omega)^2]^{-1}$. This is known as a Lorentz line-shape, and is of considerable importance in physics.

6.5 ANHARMONIC OSCILLATORS. AN EXAMPLE

Although the simple harmonic oscillator is very important in physics, we can have periodic phenomena which are not simple harmonic: these are called *anharmonic*. For example, if the higher-order terms in the expansion of the potential function V(x) about a local minimum are *not* negligible, we shall have anharmonic oscillations. Rather than treat one of these cases, we consider a simple but illustrative example: a particle (such as a steel ball) making a perfectly elastic collision with a plane surface (such as a steel plate) under gravity. We assume that the velocity of the ball on striking the plate is reversed in sign, and unchanged in magnitude; and we confine our attention to one dimension, the y direction, say. Let the plate be at y = 0, and the particle commence from height H at t = 0. The potential-energy diagram is shown in figure 6.11. From this we obtain that the amplitude of the motion H is *directly* proportional to the energy:

$$H = \frac{E}{mg} \tag{6.18}$$

and that the maximum kinetic energy

$$\tfrac{1}{2}mv_{max}^2 = E = mgH \text{ or } v_{max}^2 = 2gH \tag{6.19}$$

while the kinetic energy curve is a straight line:

$$\tfrac{1}{2}mv^2 = E - mgy \tag{6.20}$$

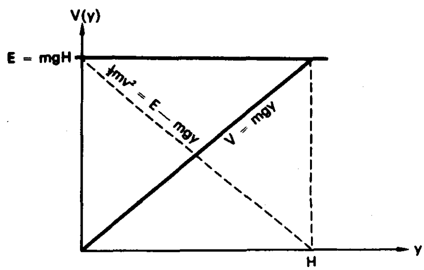


Fig. 6.11 Potential-energy diagram for the 'bouncing particle' oscillator

These equations should be compared with (6.4), (6.5), and (6.6). The period T of the motion is given by

$$T = (2H/g)^{\frac{1}{2}} \tag{6.21}$$

whereas the period of SHM is *independent* of the amplitude. Finally, we have for the displacement versus time plot:

$$y = H - \tfrac{1}{2}g(t - nT)^2 \quad n = 1, 2, 3 \ldots \text{(y decreasing)}$$

$$y = \tfrac{1}{2}g\left(t - \frac{mT}{2}\right)^2 \quad m = 1, 3, 5 \ldots \text{(y increasing)} \tag{6.22}$$

because of the discontinuity in the motion. Each kind of anharmonic oscillator has to be analyzed separately.

CHAPTER 6 PROBLEMS

6.1 (a) A simple pendulum on the earth swings with a period of T sec. If it were transported to the surface of the moon, the time for one oscillation would be

(i) T.
(ii) greater than T.
(iii) less than T.

(b) A body is suspended by a spring and performs vertical oscillations of period T sec on the earth. If the system were transported to the surface of the moon, the time for one oscillation would be

(i) T.
(ii) greater than T.
(iii) less than T.

(c) A body of mass m lies on a horizontal frictionless table and is connected by a light spring to a fixed point on the table. On the earth the period of oscillation is T. If the system were transported to the moon, the time for one oscillation would be

(i) T.
(ii) greater than T.
(iii) less than T.

6.2 A body of mass m oscillates with simple harmonic motion according to the equation

$$x = 3{\cdot}0 \cos\left(3\pi t + \frac{\pi}{3}\right) \text{ metres}$$

Find

(a) the displacement at the time t = 2 sec;
(b) the velocity at the time t = 2 sec;
(c) the acceleration at the time t = 1 sec;
(d) the phase angle;
(e) the period of the motion;
(f) the maximum kinetic energy of the system;
(g) the values of t corresponding to this maximum kinetic energy.
(h) How many times per second does the potential energy of the system pass through its maximum value?
(i) At what values of x does this occur?

6.3 The equation of motion of a simple harmonic oscillator is

$$m\ddot{x} + kx = 0$$

where m is the mass of the system and k is the restoring force per unit displacement.

The solution of this equation is of the form:

$$x = A \cos(\omega t + \delta)$$

(a) What is the significance of the quantities A and δ?

(b) How is ω related to k and how is it related to the period of the motion?

(c) How does the period of the motion depend on the amplitude?

Assume in the following that $\delta = 0$.

(d) What are the possible values of velocity and acceleration when the displacement is equal to $+\frac{1}{2}A$? At what values of t will this occur?

(e) At what values of t will the system attain its maximum value of:

  (i) kinetic energy;

  (ii) potential energy; and

  (iii) total energy?

(f) How many times per second does the kinetic energy of the system pass through its minimum value? At what values of x does this occur?

(g) If the system is subjected to a damping force proportional to the speed, what is the equation of motion?

(h) If, in addition, the damped system is subjected to an external sinusoidal force, what is the equation of motion?

6.4 The equation of motion of an undamped harmonic oscillator subjected to an impressed force F(t) is

$$m\ddot{x} + kx = F(t) \qquad (1)$$

The solution is always the sum of two parts, the first of the form $x = A \sin(\omega t + \delta)$ where $\omega^2 = K/m$, and the second dependent on F(t).

(a) Let $F(t) = F_0$ (a constant independent of time). Show that $x = F_0/m\omega^2$ satisfies equation (1).

(b) The total solution is now

$$x = A \sin(\omega t + \delta) + \frac{F_o}{m\omega^2}$$

What does the second term represent physically? (Hint: Consider the difference between a mass on the end of a spring oscillating either horizontally or vertically.)

(c) Now let $F(t) = k\alpha \cos \Omega t$. Show that $x = X \sin(\Omega t + \varphi)$ satisfies equation (1) for all values of t and hence find the values of X and φ.

(d) The total solution is now $x = A \sin(\omega t + \delta) + X \sin(\Omega t + \varphi)$. If the frequencies Ω and ω are very similar we can write $\Omega = \omega + \epsilon$ where ϵ is small. Show that this leads to an oscillation of frequency ω which has a time-dependent amplitude. [It is convenient to write x in the complex form

$$x = A^* \exp(i\omega t) + X^* \exp(i\Omega t).]$$

(e) As what physical phenomenon is this time-dependence of the amplitude usually referred to?

6.5 One end of a light horizontal spring, with a force constant of 800 newtons/metre, is fixed, and a 2 kg mass is attached to the other end. The mass is pulled along a horizontal table so that the spring is stretched by 5 m and then released from rest at time $t = 0$. The resulting motion is simple harmonic.

(a) Find the force required to give the displacement of 5 m.
(b) What is the period T of oscillation?
(c) What is the amplitude A?
(d) Find the kinetic energy of the oscillator at time $t = T/2$.
(e) Find the potential energy at time $t = T/2$.
(f) Find the total mechanical energy at time $t = 3T/4$.
(g) What is the potential energy when the displacement is A/2?
(h) What is the kinetic energy when the displacement is A/2?
(i) What is the total energy when the displacement is zero?

Sand is dropped on the table to introduce a friction force $F_f = -b(dx/dt)$ where b is constant.

(j) Write down dimensions for b.

6.6 A mass m on a spring of modulus k is made to oscillate along the x direction by application of an oscillatory force $F = F_0 \cos \omega t$. The mass experiences a small viscous drag force proportional to its velocity, $-b\dot{x}$. Hence the equation of motion is

$$m\ddot{x} + b\dot{x} + kx = F_o \cos \omega t$$

The solution is of the form

$$x = A \cos \omega t + B \sin \omega t$$

with $$A = F_o \frac{(k - m\omega^2)}{(k - m\omega^2)^2 + b^2\omega^2}$$

$$B = F_o \frac{b\omega}{(k - m\omega^2)^2 + b^2\omega^2}$$

(a) Write out the expression for the velocity of the mass.
(b) Write out the expression for the rate at which the force F is doing work.
(c) What is the mean rate of working (average over one cycle)?
(d) At what frequency is the mean rate of working a maximum?

6.7 From equations (6.16) and (6.17), write the response of the driven simple harmonic oscillator in terms of an in-phase and an out-of-phase part, i.e. as

$$x = A \cos \omega t + B \sin \omega t$$

Show that the average power $P = \langle F\dot{x} \rangle$ dissipated (average over one cycle) is given by

$$P = \frac{F_o^2}{2m} \frac{\omega^2/\tau_e}{(\omega_o^2 - \omega^2)^2 + \omega^2\tau_e^{-2}}$$

and is a result of the *out-of-phase* component. Show that for $\omega_o\tau_e \gg 1$ (light damping), the 'Q' or 'quality factor' defined by

$$Q = 2\pi \left(\frac{\text{energy stored}}{\text{average energy loss in 1 period}}\right)$$

is given by $Q \doteqdot \omega_o\tau_e$.

Show that the power absorption is reduced to one-half the value at resonance at frequencies given approximately by $\omega = \omega_o \pm (1/2\tau)$ for light damping. Hence show that

$$Q = \frac{\text{resonant frequency}}{\text{full width at half maximum power}}$$

7
Rotational motion 1

7.1 PRIMITIVE CONCEPTS. TORQUE. ANGULAR MOMENTUM

We know a great deal about rotational motion by practical experience, without ever having formalized it. For example, door-handles are set as far away as practically possible from the hinges of a door, and when we go to open a door, we pull at right angles to the plane of the door (initially at least) because we know we shall then need the least effort. We know that the closer the door-handle is set to the hinges, the greater will be the effort required to open the door, and that if we pull at an angle other than perpendicular to the plane of the door, we shall also have to use more effort.

The physicist formalizes and quantifies these ideas into the concept of *torque*, or the *vector moment of a force about a point*. If a force **F** acts at some point P with position vector **r** (figure 7.1), then the torque **J** is defined by

$$\mathbf{J} = \mathbf{r} \times \mathbf{F} = r\,F \sin\theta\,\hat{n} \qquad (7.1)$$

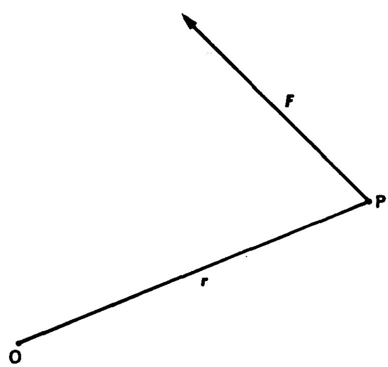


Fig. 7.1 Definition of torque

where $\hat{n}$ is the unit vector perpendicular to **r** and **F**, with its sense given by the right-hand screw rule for the vector product. It is easy to check from figure 7.1 that if the force tends to rotate **r** anticlockwise as shown, then the sense of **J** is out of the paper: i.e. the direction of *rotation* of the right-hand screw is the direction in which **F** tends to turn **r**. We also see that when **r** is perpendicular to **F**, **F** must be large if **r** is small for constant **J** and vice versa; and either **r** or **F** must increase for constant **J**, if $\sin\theta \neq 1$. So if we think of torque as a measure of 'turning ability', our experience with doors is well formalized and quantified.

How can we fit this into our foregoing mechanics framework? Before we do, we must define a new quantity, called the *moment of momentum* or the *angular momentum.* We shall use the latter term, but the former is a good mnemonic for the definition of the angular momentum **L** of a particle of momentum **p** = mv about a point 0 with respect to which the particle has a position vector **r** (figure 7.2):

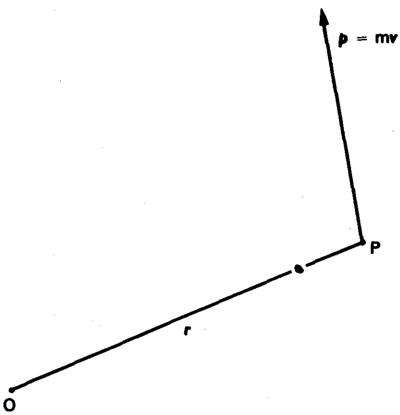


Fig. 7.2 Definition of angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v} \tag{7.2}$$

We observe here also that the direction of *rotation* of the right-hand screw is the direction in which **p** is tending to rotate **r**. We note also that the definitions of torque and of angular momentum depend very much on the specification of the *origin* of **r**, in a way in which *linear* momentum in fixed frames, for example, does not.

Let us differentiate both sides of equation (7.2). We have, by the rules for differentiating a vector product,

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \frac{d\mathbf{r}}{dt} \times m\mathbf{v} + \mathbf{r} \times m\frac{d\mathbf{v}}{dt} \tag{7.3a}$$

$$= 0 + \mathbf{r} \times \mathbf{F} \tag{7.3b}$$

since $(d\mathbf{r}/dt) = \mathbf{v}$ and $\mathbf{v} \times \mathbf{v} = 0$. In other words, *the rate of change of angular momentum* is proportional to the impressed torque, and takes place in the direction of the torque. But we must always remember that the directions of the torque of a force **F**, and of the moment of a momentum mv, are at right angles to **F** and mv respectively, given by the vector-product rule. We also note that if

$$\mathbf{J} = 0, \text{ i.e. } \mathbf{r} = 0 \text{ or } \mathbf{F} = 0 \text{ or } \mathbf{r} \parallel \mathbf{F} \tag{7.4}$$

then $$\frac{d\mathbf{L}}{dt} = 0, \text{ i.e. } \mathbf{L} \text{ constant,}$$

which means that we have a principle of *rotational inertia:* a particle rotating about an origin will continue to do so in the absence of any *torques.* Note that, by (7.4), forces can be present. In fact, let us consider a very simple situation, in which a very light particle of mass m is in a circular gravitational orbit about a very massive particle, which is thus a good approximation of the centre-of-mass inertial frame. We observe that the gravitational force **F** always acts *along* the line joining the light particle to the origin on the massive particle. Hence the torque of the gravitational force is zero, and once the particle is *set* revolving, it must continue to revolve! Ideally, we can build up rigid bodies from assemblies of particles, so if a rigid body is set revolving and there are no torques, then it must continue to revolve. We are rather less familiar with rotational inertia than we are with linear inertia, so the consequences of rotational inertia seem strange to us; thus, for example, we wonder at the behaviour of a gyroscope (to be discussed later) but not at the Galilei-Newton law of inertia.

7.2 ANGULAR VELOCITY AND ANGULAR ACCELERATION AS VECTORS

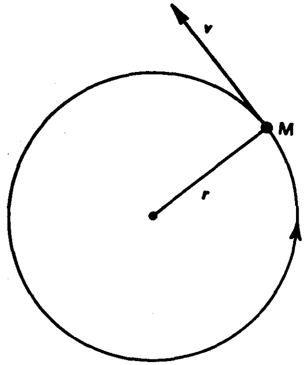


Fig. 7.3 Uniform circular motion

Let us consider a particle of mass m moving in a circle of radius r with uniform speed v. Then its angular momentum is (figure 7.3)

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} = mvr\,\hat{\mathbf{L}} \qquad (7.5)$$

$$= mr^2\omega\,\hat{\mathbf{L}}$$

where ω is the angular speed of the particle. We observe that if we could specify $\boldsymbol{\omega}$ as a *vector*, with magnitude ω, direction perpendicular to the plane of rotation, and sense given by the right-hand-screw rule, so that it lies along the axis of rotation, we should have the convenient relationship, *for this special case,*

$$\mathbf{L} = mr^2\boldsymbol{\omega} = I\boldsymbol{\omega} \qquad (7.6)$$

where $I = mr^2$ is called the *moment of inertia* of m about the origin 0.

Since $\omega = (d\theta/dt)$ and time is a scalar, if $\boldsymbol{\omega}$ is a vector, then $\mathbf{d\theta}$ – an infinitesimal angular displacement – must be a vector. It is easy to verify that large, finite angular rotations about different axes – say, 90° rotation about two axes at right angles – do not *commute,* so that *finite* rotations cannot be represented by vectors, whose addition law is commutative. But as we make the angles of rotation smaller and smaller, we find that, in the limit, infinitesimal rotations, $\mathbf{d\theta}_1$ and $\mathbf{d\theta}_2$ say, about axes 1 and 2 *do* commute, and we may represent them by vectors of magnitude $d\theta_1$ and $d\theta_2$, with directions and senses given by the axes 1 and 2 respectively and the right-hand screw rule: so $\mathbf{d\theta}_1 + \mathbf{d\theta}_2 = \mathbf{d\theta}_2 + \mathbf{d\theta}_1$. So we are justified in representing $(\mathbf{d\theta}/dt) = \boldsymbol{\omega}$ as a vector, with magnitude ω and direction and sense given by the axis of rotation and the right-hand-screw rule.

Differentiating equation (7.6) once more with respect to time, we obtain (since **r** is constant, and therefore **I** is constant)

$$\frac{d\mathbf{L}}{dt} = I\dot{\boldsymbol{\omega}} = I\frac{d^2\theta}{dt^2} = \mathbf{r} \times \mathbf{F} \tag{7.7}$$

showing that the rate of change of angular momentum is given by the product of the moment of inertia and the *angular* acceleration. The case of *uniform* speed gives $\mathbf{r} \times \mathbf{F} = 0$, since the force is always parallel to **r**; but if we have a tangential force changing v (see section 2.5), but **r** remains constant, then $\boldsymbol{\omega}$ would be non-zero, and parallel to $\mathbf{r} \times \mathbf{F}$ *for this special case.* Why do we keep stressing this? The reason will emerge in the next section.

7.3 ANGULAR MOMENTUM OF A SYSTEM OF PARTICLES

The total angular momentum of a system of i particles with masses m_i, position vectors $\mathbf{r}_i$, and velocities $\mathbf{v}_i$ is given by

$$\mathbf{L} = \sum_i \mathbf{L}_i = \sum_i \mathbf{r}_i \times m\mathbf{v}_i \tag{7.8}$$

While we are considering the system, suppose that the centre of mass has velocity **V** and position co-ordinate **R**. Then we may write

$$\mathbf{L} = \sum_i (\mathbf{R} + \mathbf{r}_i') \times m_i(\mathbf{V} + \mathbf{v}_i')$$

where $\mathbf{r}_i'$ and $\mathbf{v}_i'$ are the position vector and velocity respectively of the i^{th} particle with respect to the centre of mass. Hence

$$\mathbf{L} = \mathbf{R} \times \mathbf{V}(\Sigma m_i) + \mathbf{R} \times \Sigma m_i\mathbf{v}_i' + \Sigma \mathbf{r}_i' \times m_i\mathbf{v}_i + (\Sigma m_i\mathbf{r}_i') \times \mathbf{V} \tag{7.9}$$

The second and fourth terms in the expansion are zero, since we know (section 4.2) that $\Sigma m_i\mathbf{v}_i' = \Sigma m_i\mathbf{r}_i' = 0$; hence

$$\mathbf{L} = \mathbf{R} \times M\mathbf{V} + \sum_i \mathbf{r}_i' \times m_i\mathbf{v}_i \tag{7.10}$$

$$= \mathbf{L}_{CM} + \mathbf{S}$$

where $\Sigma m_i = M$, $\mathbf{L}_{CM}$ is the angular momentum of the centre of mass about 0, and **S** is the angular momentum of the system about its centre of mass. This latter is often called the 'spin angular momentum' of the system.

A short digression is in order here. Most of the elementary particles — electrons, protons, neutrons, photons, for example — have an intrinsic spin angular momentum, and so do many nuclei. It turns out that electrons, protons, and neutrons have spins of $\frac{1}{2}\hbar$ units of angular momentum, where $\hbar$ is Planck's constant ($6{\cdot}6 \times 10^{-34}$ joule/sec) divided by 2π. Photons have spin $\hbar$. The *behaviour* of particles with spin that is an odd multiple of $\frac{1}{2}\hbar$ is quite different from those with integral $\hbar$ values for spin; the former particles are called *fermions*, the latter, *bosons*. A classic example is the different behaviour of the He^3 and He^4 nuclei: He^3 behaves like a normal liquid until it freezes, but He^4 becomes a 'superfluid' with quite remarkable properties before it eventually solidifies. So angular momentum is a very fundamental property of matter!

Now we return to the main path of our argument, and differentiate equation (7.10) with respect to time. We will obtain [see equation (7.3)]

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{L}_{CM}}{dt} + \frac{d\mathbf{S}}{dt} = \mathbf{R} \times M\mathbf{A} + \sum_i \mathbf{r}'_i \times \mathbf{F}_i \qquad (7.11)$$

where $\mathbf{A}$ is the acceleration of the centre of mass: we know that $M\mathbf{A} = \Sigma \mathbf{F}_{ext}$, the sum of the external forces (section 4.2). So if $\mathbf{R} \times \mathbf{F}_{ext} = 0$, i.e. no resultant *torques* act about the origin, owing to forces acting on the centre of mass, the angular momentum of the centre of mass, $\mathbf{L}_{CM}$, remains constant. What about the spin S? We write

$$\frac{d\mathbf{S}}{dt} = \sum_i \mathbf{r}'_i \times \mathbf{F}_{ext\,i} + \sum_i \mathbf{r}'_i \times \mathbf{F}_{int\,i} \qquad (7.12)$$

where $\mathbf{F}_{ext\,i}$ and $\mathbf{F}_{int\,i}$ are the sums of the external and internal forces respectively acting on the i^{th} particle. The first term is the sum of the external torques *relative* to the centre of mass. If this is zero, we are left with the last term, which we can show is zero by Newton's Third Law, provided that the interaction forces act along the lines joining the various particles. Consider, for example, the j^{th} and $(j + 1)^{th}$ particle, and denote the force on j due to j + 1 as F_j. Then we have

$$\mathbf{r}_i \times \mathbf{F}_j = (\mathbf{r}_j - \mathbf{r}_{j+1}) \times \mathbf{F}_j + \mathbf{r}_{j+1} \times \mathbf{F}_j$$

$$= \mathbf{r}_{j+1} \times \mathbf{F}_j \text{ (since } (\mathbf{r}_j - \mathbf{r}_{j+1}) \parallel \mathbf{F}_j)$$

$$= \mathbf{r}_{j+1} \times (-\mathbf{F}_{j+1})$$

by Newton's Third Law. So the torques of all such internal forces will vanish in pairs, and we have the result that if no external torques act *about* the centre of mass, the angular momentum **S** *about* the centre of mass is constant. Note that the results obtained for $\mathbf{L}_{CM}$ and **S** are more general than requiring the external forces to vanish. Consider, for example, two equal masses joined by a straight rod of negligible mass which rests on a knife-edge in a uniform gravitational field of acceleration $-g\mathbf{j}$ (figure 7.4). If the rod lies along the x axis, then the sum of the external forces on the system clearly vanishes. However, unless the knife-edge is *at* the centre of mass, there are resultant *torques* about the knife-edge and about the centre of mass, so that the angular momentum of the system must change with time.

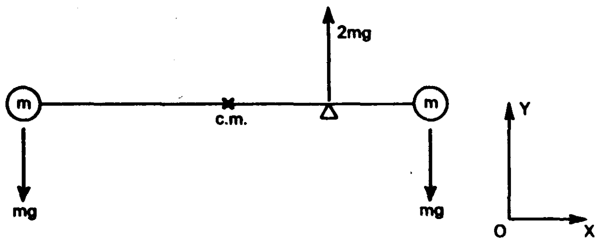


Fig. 7.4 Dumbbell on knife-edge

It is important to note that in two-body *collisions*, where we have only interaction forces of the kind discussed above, *angular momentum is conserved.* Therefore, two-body collisions of this kind *must be confined to a plane,* as we mentioned but did not prove in section 4.3.

Finally, we now take a very simple example to show why equations (7.6) and (7.7), relating angular momentum and its rate of change to angular velocity and angular acceleration respectively, are special cases. In order to do this, we first need to refine our definition of angular velocity and to extend our definition of moment of inertia.

If a rigid body, considered as an assembly of particles, is rotating about an axis, and the rate of change of the angle perpendicular to the axis is ω, then the angular velocity vector $\boldsymbol{\omega}$ lies along the axis of rotation, has the magnitude ω, and sense given by the right-hand-screw rule for rotation.

The *velocity* **v** of any point with position vector **r** of the rotating rigid body is given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{7.13}$$

where the origin of **r** lies somewhere on the rotational axis.

The *moment of inertia* of a rigid body, considered as an assembly of i particles, *about an axis* along Oz is given by

$$I_z = \sum_i m_i(x_i^2 + y_i^2) \qquad (7.14)$$

The moments of inertia I_x, I_y, about O_x, O_y, can be obtained from equation (7.14) by cyclic permutation of the co-ordinates.

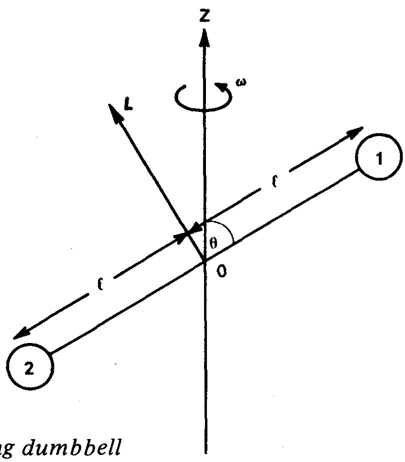


Fig. 7.5 Rotating dumbbell

Consider the 'dumb-bell' shown in figure 7.5: two particles of mass m connected by a rod of length 2ℓ and negligible mass. It is made to rotate about the axis shown, with angular speed ω. The angular velocity vector therefore lies along this axis, and its direction is vertically upwards. The top mass 1 is moving into the paper, the bottom mass 2, out of the paper.

In terms of the quantities shown in the figure, taking the origin at 0, the *velocities* of the masses 1 and 2 are

$$\mathbf{v}_1 = \omega \times \mathbf{r}_1 = \omega\ell \sin\theta \text{ into paper}$$

$$\mathbf{v}_2 = \omega \times \mathbf{r}_2 = \omega\ell \sin\theta \text{ out of paper}$$

The *angular momentum* of the system about 0 is

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 = \mathbf{r}_1 \times m\mathbf{v}_1 + \mathbf{r}_2 \times m\mathbf{v}_2$$

$$= (m\ell.v_1 + m\ell.v_2) \text{ perpendicular to the line joining 1 and 2}$$

$$= 2m\ell.\omega\ell \sin\theta \text{ perpendicular to the line joining 1 and 2}$$

$$= 2m\ell^2\omega \sin\theta\, \hat{L} \qquad (7.15)$$

where $\hat{L}$ is the unit vector perpendicular to the line joining 1 and 2 (see figure 7.5).

It is quite clear that **L** is *not* parallel to **ω**. We therefore *cannot* write

$$L = I_Z\omega \qquad (7.16)$$

where I_Z is the moment of inertia about the axis of rotation, taken as being the z axis. In fact

$$I_Z = 2m\ell^2 \sin^2\theta \qquad (7.17)$$

so that even the *magnitude* is wrong as given by relation equation (7.16). Some thought shows that as the 'dumbbell' rotates about the axis shown, the angular momentum vector *rotates* about the angular velocity vector. Hence the angular momentum is changing, and a *torque* must be supplied to keep it changing. This torque must come from the bearings of the axis of rotation. Hence an equal and opposite torque must be exerted *on* the bearings.

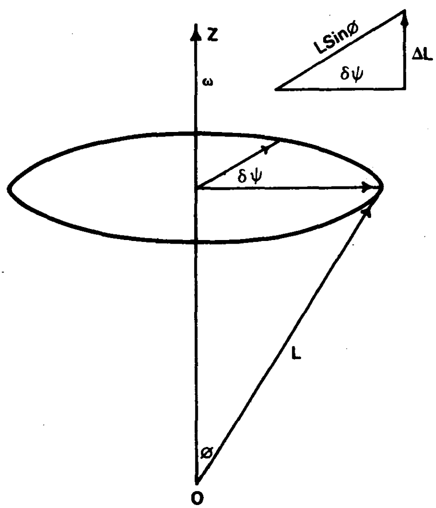


Fig. 7.6 Angular momentum diagram

From figure 7.6 we see that the tip of **L** traces out a circle in a plane perpendicular to Oz. Let the component of **L** in this plane rotate by a small angle $\delta\psi$ in time δt. The change $\Delta \mathbf{L}$ in angular momentum is given by $\mathbf{T}\delta t$, where $\mathbf{T}$ is the torque:

$$\Delta \mathbf{L} = \mathbf{T}\delta t$$

But $\Delta L = L \sin \phi \cdot \delta \psi$ (from the figure).

Hence $$T\delta t = L \sin \phi \cdot \delta \psi$$

or $$T = L \sin \phi \cdot \frac{\delta \psi}{\delta t}$$

$$= L \sin \phi \cdot \omega$$

in the limit since the component of $\mathbf{L}$ perpendicular to Oz clearly rotates with angular speed ω. Both *magnitude* and *direction* of $\mathbf{T}$ are correctly given by the relationship

$$\frac{d\mathbf{L}}{dt} = \mathbf{T} = \boldsymbol{\omega} \times \mathbf{L} \text{(Check this.)} \qquad (7.18)$$

Such a torque must *always* exist if $\boldsymbol{\omega}$ is constant and not parallel to $\mathbf{L}$.

When *is* $\boldsymbol{\omega}$ parallel to $\mathbf{L}$? This question is answered in the next chapter.

CHAPTER 7 PROBLEMS

7.1 Two particles A and B of masses m and 2m respectively are connected by a rigid rod of negligible mass, of length 2r, freely pivoted at its centre 0 (figure 7.1p). These two particles are, at first, stationary. Another particle C of mass m travelling with velocity $\mathbf{v}$ relative to the laboratory and perpendicular to the direction of the rod strikes A, the smaller of the two connected particles.

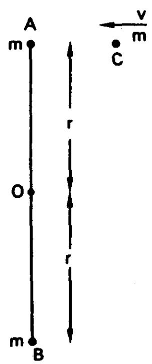


Fig. 7.1p

(a) What is the kinetic energy of the complete system of three masses relative to the laboratory before the collision?
(b) What is the velocity vector of the centre of mass relative to the laboratory before the collision?
(c) What is the velocity vector of mass C relative to the centre of mass before the collision?
(d) What is the velocity vector of masses A and B relative to the centre of mass before the collision?
(e) What is the total kinetic energy relative to the centre of mass before the collision?

On collision particle C sticks to A and the system starts to rotate about point 0 fixed in the laboratory.

(f) What is the velocity vector of the centre of mass relative to the laboratory immediately after the collision?
(g) What is the impulse due to external forces which acted on the system during the collision?
(h) Where was this external impulse applied?
(i) What change occurred in the total angular momentum about 0 during the collision?
(j) What is the total angular momentum about 0 before the collision?
(k) What is the total angular momentum about 0 after the collision?
(l) What is the moment of inertia of the system about 0 after the collision?
(m) What is the angular velocity of the system about 0 after the collision?
(n) What is the kinetic energy relative to the laboratory after the collision?

7.2 A dumbbell consists of two massive particles, A and B, each of mass m, attached to the ends of a bar of length d and negligible mass. The dumbbell is at rest on a horizontal frictionless surface. Another ball, C, also of mass m, moves along a line perpendicular to the dumbbell with speed v_0, and collides with and sticks to B.

(a) Where is the centre of mass of the system ABC located?
(b) What is the total momentum of the system ABC?
(c) What is the total angular momentum of the system ABC about its centre of mass?
(d) What is the total kinetic energy of the system ABC?
(e) What is the velocity of the centre of mass?

The following questions refer to the motion *after* the collision.

(f) What is the total linear momentum of the system after the collision?

(g) What is the total angular momentum of the system about its centre of mass after the collision?

(h) What is the *velocity* of the centre of mass after the collision?

(i) What is the moment of inertia of the system ABC about the centre of mass?

(j) What is the angular velocity of rotation of the system about the centre of mass?

(k) What is the kinetic energy of the centre of mass after the collision?

(l) What is the kinetic energy of rotation about the centre of mass after the collision?

(m) Is kinetic energy conserved in the collision?

7.3 A metre stick lies on a frictionless horizontal table. It has mass M, moment of inertia M/12 about its centre of mass, and is pivoted so that it can rotate about its centre of mass 0, but not move horizontally.

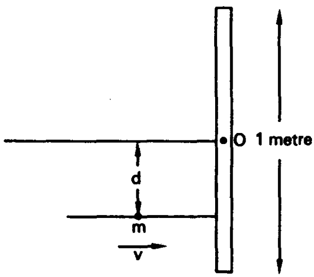


Fig. 7.3p

A particle of mass m moving with velocity v as shown in figure 7.3p collides with the metre stick at a point d from its centre and adheres to the stick. Answer the following questions.

(a) What is the total angular momentum of the system about 0 before the collision?

(b) What is the total angular momentum of the system about 0 after the collision?

(c) What is the moment of inertia about 0 of the system metre stick and particle after the collision?

(d) What is the angular velocity of the system about 0?

(e) What is the rotational kinetic energy of the system about 0?
(f) What is the kinetic energy of the particle before the collision?
(g) Is kinetic energy conserved in the collision?
(h) Does an external force act on the system during the collision? If so, where?

7.4 A body with moment of inertia I about a principal axis, about which it can rotate, is subject to a restoring torque T such that

$$T = \frac{dL}{dt} = k\theta$$

where θ is the angle through which the body has rotated about the principal axis. Show that the body will execute rotational simple harmonic motion.

7.5 A truck travelling at 50 km/h has wheels of 1 m diameter.
(a) What is the angular speed of the wheels about the axle?
(b) If the wheels are brought to a stop uniformly in 30 turns, what is the angular acceleration?
(c) How far does the truck advance during the braking period?

7.6 Assume the earth to be a sphere of uniform density.
(a) What is the rotational kinetic energy? Take the radius of the earth to be $6{\cdot}4 \times 10^3$ km and its mass to be $6{\cdot}0 \times 10^{24}$ kg.
(b) Suppose that this energy could be harnessed for man's use. For how many years could the earth supply 1 kw of power to each of the $3{\cdot}5 \times 10^9$ persons on the earth?
(c) We find from astronomical observations that the earth's rotation is slowing down so that the day is lengthening at the rate of 0·001 seconds per century. This is the result of tidal friction exerting a retarding torque on the earth. What is the magnitude of this torque?
(d) In kilowatts, what is the rate of dissipation of kinetic energy of the earth's rotation as a result of rotation against this torque?

8
Rotational motion 2

8.1 THE ROTATION OF RIGID BODIES

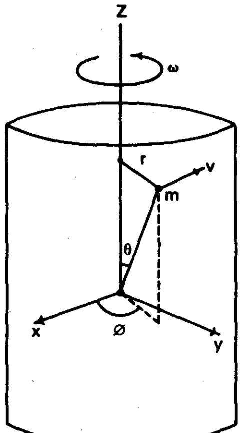


Fig. 8.1 Particle in a rotating rigid body (general)

We are now going to consider a rigid body as an assembly of particles, and consider the contribution of a particular particle to the total angular momentum when the rigid body is rotated about an axis with angular velocity $\boldsymbol{\omega}$. The general situation is shown in figure 8.1 for the element of mass m, where $\boldsymbol{\omega} = \omega\mathbf{k}$; we shall take a new set of axes such that x′ lies along r, the line joining m_i to Oz parallel to the x-y plane, and Oz′ = Oz. The general situation, projected onto the plane of the paper, is shown in figure 8.2, and this is little different from figure 7.5, which we have already considered. From the arguments advanced for figure 7.5 we see that the angular momentum $\mathbf{L_i}$ of $\mathbf{m_i}$ about 0 is given by

$$\mathbf{L_i} = m\ell^2 \omega \sin\theta\, \hat{\mathbf{L}} \tag{8.1}$$

where $\hat{\mathbf{L}}$ is perpendicular to ℓ. We may write this as

$$\begin{aligned} \mathbf{L_i} &= -L_{ix'}\,\mathbf{i} + L_{iz}\,\mathbf{k} \\ &= m\ell^2 \omega \sin\theta . \sin\theta\, \mathbf{k} - m\ell^2 \omega \sin\theta . \cos\theta\, \mathbf{i} \\ &= m\ell^2 \omega \frac{x'}{\ell} . \frac{x'}{\ell} \mathbf{k} - m\ell^2 \omega \frac{x'}{\ell} \frac{z}{\ell} \mathbf{i} \\ &= m\omega x'^2\, \mathbf{k} - m\omega x' z\, \mathbf{i}' \end{aligned} \tag{8.2}$$

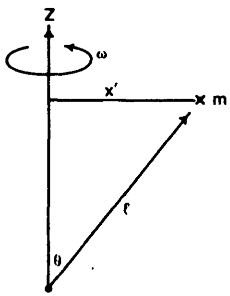


Fig. 8.2 The new axes for the particle

Now we note that mx'^2 is just the *moment* of *inertia* of the element of mass **m** about Oz. Let us transform the term in $\mathbf{i}'$ back to the original **i** and **j** co-ordinates, by writing

$$\mathbf{i}' = \cos\phi\, \mathbf{i} + \sin\phi\, \mathbf{j}$$

$$\cos\phi = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin\phi = \frac{y}{\sqrt{x^2 + y^2}} \tag{8.3}$$

$$x' = \sqrt{x^2 + y^2}$$

whence $\quad -m\omega x' z\mathbf{i} = -m\omega(xz\mathbf{i} + yz\mathbf{k}) \quad$ (8.4)

In other words, an element of mass m of a rigid body rotating with angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$ has *in general* angular momentum components along all three axes. The z component of angular momentum is given by

$$L_z = I_m\omega = m(x^2 + y^2)\omega \qquad (8.5)$$

where I_m is the *moment of inertia* of m about Oz, while the other components depend on the products of inertia −mxz and −myz. The *total* contribution from these will *vanish* if Oz is an *axis of symmetry*, i.e. if to *every* element m at (x, y) there corresponds an equivalent element m at (−x, −y). Another way of defining an axis of symmetry is to say that it is a direction through a body such that if the body is rotated about this axis by all successive 1/n of a revolution, n integral > 1, the body always appears identical; for example, a regular *hexagonal prism* has an axis of sixfold symmetry through the centres of the hexagonal faces (figure 8.3).

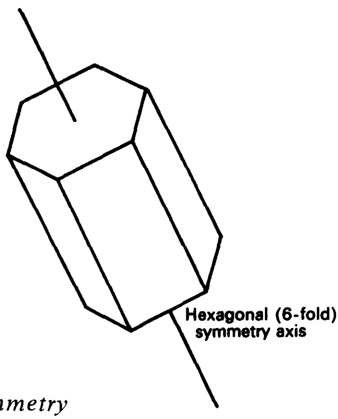


Fig. 8.3 Axis of symmetry

We see, therefore, that if the axis of rotation or a rigid body is an *axis of symmetry*, **ω** is parallel to **L** . In general, therefore, if we rotate a rigid body about some axis, **L** and **ω** are *not* parallel, and a torque must be supplied to keep **L** rotating about **ω** (see figure 7.6). Hence the disastrous effects of unbalanced wheels on car steering!

If a body has *no* axes of symmetry, nevertheless the following theorem can be proved (we shall not prove it):

Theorem In *any* rigid body, there are always three mutually perpendicular axes, called the *principal axes of inertia*, such that rotation about any one of these axes will give $\mathbf{L} \parallel \boldsymbol{\omega}$.

To specify completely what is going on when a rigid body is rotating, we need to know the moments of inertia and products of inertia of the body. These can be collected into a single entity, the so-called *inertia tensor,* which can be represented by a 3 x 3 matrix. If we consider it 'operating' on the angular velocity $\boldsymbol{\omega}$, we notice that, in general, it changes the direction and magnitude of $\boldsymbol{\omega}$. Symbolically,

$$\mathbf{L} = [\mathrm{I}]\ \boldsymbol{\omega} \text{ where } [\mathrm{I}] \text{ is the inertia tensor.} \qquad (8.6)$$

Written out completely, we have the relation

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_z \\ \omega_y \\ \omega_z \end{bmatrix} \qquad (8.7)$$

where

$$I_{zz} = \Sigma\, m_i(x_i^2 + y_i^2)$$

is the moment of inertia about the z axis, for example, and

$$I_{zx} = I_{xz} = \Sigma\, m_i x_i z_i$$

is the appropriate product of inertia, for example.

Sometimes we set $I_{zz} = k_z^2\, \Sigma\, m_i$, where k_z is the appropriate *radius of gyration.* This is set down here for completeness only, and we shall not be using it.

Consider now a different case: a body rotating *freely* in space, i.e. with its axis of rotation *not* fixed by any 'supports' (so that $\boldsymbol{\omega}$ can change its direction), and with no external torque **J** on it. (Think of a satellite spinning about a non-principal axis.) Since **J** = 0, the angular momentum **L** must be constant. But this tells us straightaway that the direction of $\boldsymbol{\omega}$ – i.e. the axis of rotation – not only *can* change, but in fact *must* change, in the general case. (Think of the rotating dumbbell we examined in Chapter 7; we found that, if $\boldsymbol{\omega}$ is fixed, **L** must change. It follows that, when **L** cannot change, $\boldsymbol{\omega}$ cannot be fixed.)

It is not hard to show that now the axis of rotation must whirl (precess) about the (fixed) direction $\hat{\mathbf{L}}$.

So, summing up: in the general case of rotation about a non-principal axis, when $\boldsymbol{\omega}$ is fixed (i.e. there are 'supports'), **L** rotates about the fixed direction, $\boldsymbol{\omega}$. The supports provide the needed torque

$$\mathbf{J} = \frac{d\mathbf{L}}{dt}$$

When there are no supports, and no external torque, then $d\mathbf{L}/dt = 0$ and $\boldsymbol{\omega}$ rotates about the fixed direction $\hat{\mathbf{L}}$.

To calculate the moments and products of inertia for continuous rigid bodies, we must go over to the integral formulations

$$I_{zz} = \int_0^M (x^2 + y^2)dm, \quad I_{zx} = \int_0^M xzdm$$

where $M = \int dm$ is the total mass of the rigid body. A representative table for commonly met solids is given on page . Two useful theorems help us to extend these results: the *parallel axis* theorem (for *any* solid) and the *perpendicular axis* theorem (for laminae). These are (without proof):

Parallel axis theorem If a body has a moment of inertia I_{CM} about some axis through the centre of mass, then the moment of inertia I_O about a parallel axis through a point O distance h from the centre of mass (figure 8.4) is

$$I_O = I_{CM} + Mh^2$$

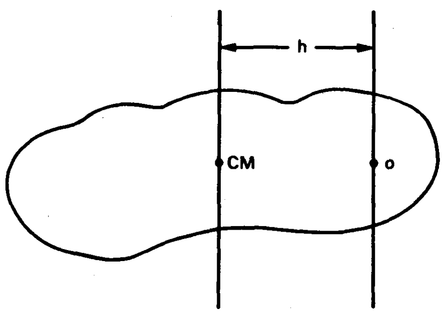


Fig. 8.4 Parallel axis theorem

Perpendicular axis theorem If the moments of inertia about two perpendicular axes Ox, Oy in the plane of the lamina are I_x and I_y respectively, then the moment of inertia about the axis Oz perpendicular to both of them and to the plane of a lamina (figure 8.5) is given by

$$I_z = I_x + I_y$$

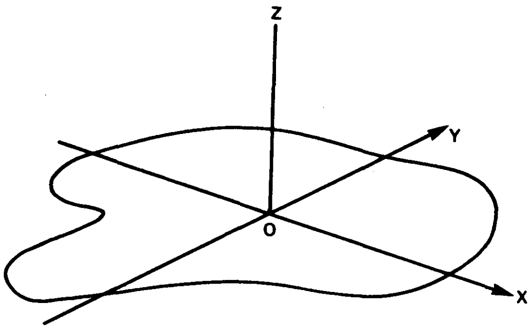


Fig. 8.5 Perpendicular axis theorem

8.2 ROTATIONAL KINETIC ENERGY

Consider once more a particle of mass m rotating in a circle of radius r with uniform angular velocity ω. The kinetic energy of the particle, with speed v, is

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2 r^2 = \tfrac{1}{2}I\omega^2 = L^2/2I \qquad (8.8)$$

where $I = mr^2$ is the moment of inertia and $L = I\omega$ is the angular momentum. These results can clearly be generalized to the case of a rigid body rotating with angular velocity ω_z about a principal axis, say Oz:

$$T = \tfrac{1}{2}I_{zz}\omega_z^2 = \frac{L_z^2}{2I_{zz}} \qquad (8.9)$$

If x and y are the other principal axes, and we have the components ω_x and ω_y of ω along these axes, we must have (since L_x is parallel to ω_x, L_y is parallel to ω_y by definition in this case):

$$\begin{aligned} T &= \tfrac{1}{2}I_{xx}\omega_x^2 + \tfrac{1}{2}I_{yy}\omega_y^2 + \tfrac{1}{2}I_{zz}\omega_z^2 \\ &= \frac{L^2{}_x}{2I_{xx}} + \frac{L^2{}_y}{2I_{yy}} + \frac{L^2{}_z}{2I_{zz}} \end{aligned} \qquad (8.10)$$

If we take axes other than the principal axes of inertia of a body, the expression for the kinetic energy involves the products of inertia (we state it here for completeness):

$$T = \tfrac{1}{2}(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 + 2\omega_x\omega_y I_{xy} + 2\omega_y\omega_z I_{yz} + 2\omega_z\omega_x I_{zx})$$

which can be proved from the general relationship $T = \Sigma m_i v_i^2 = \Sigma m_i(\boldsymbol{\omega} \times \mathbf{r}_i)^2$. For this case x, y, and z are *not* the principal axes of inertia, and $\boldsymbol{\omega} = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$.

It is clear that a rigid body can have only *rotational* energy with respect to the centre of mass. A system of particles which are free to move, however, may have both rotational and translational energy with respect to the centre of mass. Thus two masses attached by a spring and free to move on a frictionless table may vibrate and rotate with respect to the centre of mass.

The importance of taking into account a rigid body's rotation with respect to the centre of mass is brought out by a very simple example. We first consider a uniform sphere of mass M which simply *slips* down a plane of length L and height $h = L \sin\theta$ (figure 8.6). The velocity of the centre of mass would then be given by

$$\tfrac{1}{2}MV^2 = MgL\sin\theta, \text{ i.e. } V^2 = 2gL\sin\theta \qquad (8.11)$$

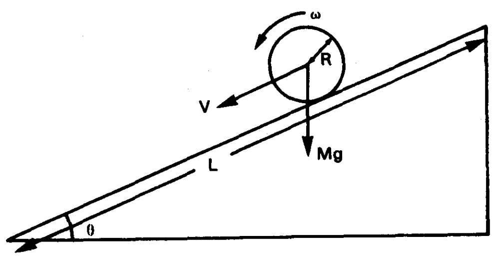


Fig. 8.6 Sphere on inclined plane

If now the sphere *rolls* down the plane *without slipping*, we have the kinetic energy of the centre of mass, and of rotation with respect to the centre of mass. Rolling without slipping means that the point of the sphere in contact with the plane may be considered instantaneously at rest, so that the centre of mass of the sphere has a velocity, relative to this point, of $R\omega$, where ω is angular velocity about the centre of mass. The energy equation is now

$$T = \tfrac{1}{2}MV^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}MV^2 + \tfrac{1}{2}I\left(\frac{V}{R}\right)^2$$
$$= \tfrac{1}{2}MV^2 + \tfrac{1}{2}\,.\,\frac{2}{5}MR^2\left(\frac{V}{R}\right)^2$$
$$= \frac{1}{10}\,MV^2 = MgL\sin\theta \qquad (8.12)$$

so that $V^2 = (10/7)\,gL\sin\theta$ for this case. [compare (8.11).]

8.3 GYROSCOPES AND TOPS. PRECESSION

We are now in a position to understand the apparently strange behaviour of gyroscopes and tops. We can take a gyroscope as being a suitably mounted flywheel that rotates very rapidly about its axis of cylindrical symmetry. Suppose that it is free to rotate in a ball-and-socket support at one end, and that the other end is initially supported so that the rotational axis is horizontal (figure 8.7); this support is then removed. The gyroscope will not drop down significantly towards the vertical, as one might expect, but will rotate in the horizontal plane with a constant angular velocity about a vertical axis through the ball-and-socket support. Let us analyze the motion from the rotational dynamics viewpoint.

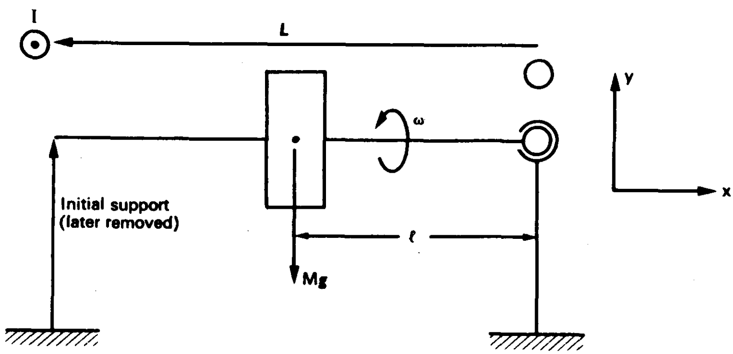


Fig. 8.7 Gyroscope

Let the angular speed of the gyro flywheel be Ω, and the moment of inertia about the rotational axis be I; let the mass of the gyroscope be M, situated effectively at its centre of mass distant ℓ from the support. Then we clearly have a torque $Mg\ell$ acting about O, such that if figure 8.7 represents the initial situation, the direction and sense of the torque are *out* of the paper. The change in angular momentum caused by this torque $\mathbf{T}$ acting for a time δt is $\delta\mathbf{L} = \mathbf{T}\delta t$. The angular momentum $\mathbf{L} = \mathbf{I}\boldsymbol{\Omega}$ of the flywheel we take as being to the *left*. Hence $\mathbf{L} + \delta\mathbf{L}$ is out of the plane of the paper and, viewed from *above*, as in figure 8.8, has rotated *anticlockwise* with respect to $\mathbf{L}$. It should be obvious that if $\mathbf{L}$ were directed to the *right*, i.e. the gyroscope flywheel were spinning in the opposite direction, the addition of $\delta\mathbf{L}$ would still bring the resultant angular momentum vector out of the plane of the paper, but the rotation, viewed from above, of $\mathbf{L} + \delta\mathbf{L}$ would now be *clockwise.*

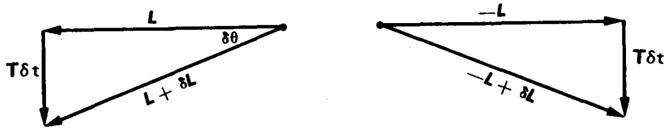


Fig. 8.8 *A view looking vertically down on the gyroscope in the plane of the paper, i.e. down the y axis*

When the gyroscope axis has rotated or *precessed* in the time δt through the angle $\delta\theta$ (figure 8.8), we see that the torque of the force $-Mg$ is still perpendicular to $(\mathbf{L} + \delta\mathbf{L})$. In other words, the magnitude of $\mathbf{L}$ is not changing, but its direction is. Since the magnitude of the force $-Mg$ is constant, the axis of the gyroscope will precess (rotate) with a constant angular velocity about the vertical direction. We can calculate this constant angular speed $\omega = \lim_{\delta t \to 0} (\delta\theta/\delta t)$ in the following way from figure 8.8. We clearly have

$$L\delta\theta = T\delta t \tag{8.13}$$

or

$$L\frac{\delta\theta}{\delta t} = T$$

giving

$$\omega L = T \text{ in the limit.} \tag{8.14}$$

If we represent ω and $\mathbf{L}$ by their appropriate vectors, we find

$$\mathbf{T} = \frac{d\mathbf{L}}{dt} = \boldsymbol{\omega} \times \mathbf{L} \tag{8.15}$$

[Compare equation (7.8), Section 7.3]

Since $\mathbf{T}$ has magnitude $Mg\ell$, we note that the angular velocity of precession has magnitude

$$|\omega| = \frac{Mg\ell}{L} = \frac{Mg\ell}{I\Omega} \tag{8.16}$$

so that ω is directly proportional to M and ℓ and inversely proportional to L, i.e. to I and Ω.

The precessional motion of a gyroscope appears strange to us merely because we are not used to thinking in terms of torques, but are used to thinking in terms of *forces*. Thus we naively expect the gyroscope axis in figure 8.7 to *fall* under the gravitational force $-M\mathbf{g}\mathbf{j}$; but rotational inertia keeps **L** in the original plane, and it is the *torque* of the force $-M\mathbf{g}\mathbf{j}$ which changes the direction of **L** in this plane.

Car wheels and bicycle wheels possess angular momentum, whose direction must be changed if we want to turn corners. By Newton's Third Law, or by a consideration of the conservation of angular momentum, we conclude that to every applied torque there is an equal and opposite reaction torque. If we wish to turn left while travelling forwards on a bicycle, we lean to the left to apply the appropriate torque. If we just wrench the handlebars around, the resulting reaction torque will tilt us to the right. This is also why a motor-car body needs springs!

Spinning tops are familiar objects, and we have all seen the axis of rotation of the top precessing about the vertical direction. Let us try to understand this *free precession* of a top from our knowledge of rotational motion.

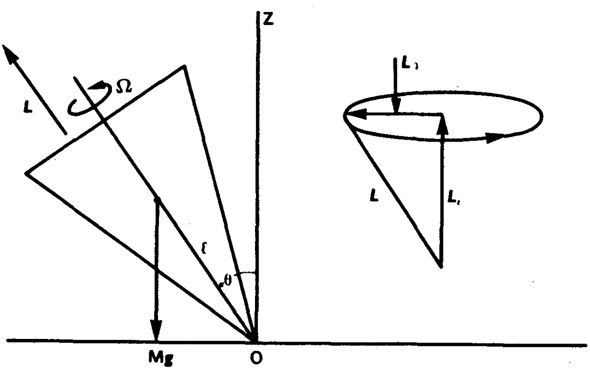


Fig. 8.9 Precessing top

Let the top have mass M, and let the centre of mass be at a distance ℓ along the axis from the tip (figure 8.9). Let the axis of the top make an angle θ with the vertical axis Oz, and we presume that the angular momentum **L** of the top about the axis through its centre of mass is very much greater than the angular momentum of the centre of mass about 0. It is convenient to resolve **L** into a component $L_z\mathbf{k}$ in the z direction, and a component $L_\perp$ in the x-y plane. Since the torque vector due to the force $-Mg\mathbf{k}$ must be confined to the x-y plane, it is only $L_\perp$ which will change. By exactly similar arguments to those used previously for the gyroscope, we see that the *magnitude* of $L_\perp$ will not change, only the direction, since the torque always remains perpendicular to $L_\perp$. From diagrams 8.10a and 8.10b we see that if the torque $\mathbf{T} = Mg\ell \sin\theta \hat{\mathbf{T}}$ acts for a time δt, and changes the angular momentum by $\Delta\mathbf{L}$, we must have

$$T\Delta t = \Delta L = L_\perp \delta\phi \tag{8.17}$$

so that

$$\frac{\delta\phi}{\delta t} = \frac{T}{L_\perp}$$

and in the limit $\delta t \to 0$, the angular velocity of precession has magnitude

$$\omega = \frac{d\phi}{dt} = \frac{T}{L_\perp} = \frac{Mg\ell \sin\theta}{L \sin\theta} = \frac{Mg\ell}{L} \tag{8.18}$$

which is *independent* of the angle θ that the top makes with the vertical.

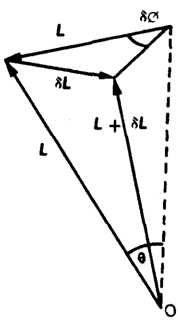


Fig. 8.10a
General angular momentum diagram

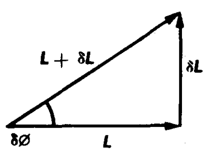


Fig. 8.10b
The changing component of angular momentum

We also have

$$L\omega \sin\theta = T = \frac{dL}{dt} \tag{8.19}$$

and it is easy to check that, in fact

$$\mathbf{T} = \frac{d\mathbf{L}}{dt} = \boldsymbol{\omega} \times \mathbf{L} \tag{8.20}$$

[Compare equations (8.15) and (7.8).]

Why do these three situations (the rotating 'unbalanced' dumbbell, the precessing gyroscope, and the precessing top) all have the equation $(d\mathbf{L}/dt) = \boldsymbol{\omega} \times \mathbf{L}$ in common? It is a consequence of the fact that, in each case, there is a component of **L** which is fixed, and a component of **L** does not change in the rotating frame. If we have a frame rotating with angular velocity $\boldsymbol{\omega}$, we can relate the rate of change d/dt' of quantities in this rotating frame to the rate of change d/dt in a fixed (inertial frame) by

$$\left[\frac{d}{dt}\right]_{\text{inertial}} = \left[\frac{d}{dt}\right]_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{r} \tag{8.21}$$

A very simple application of this was hidden in our definition of $\boldsymbol{\omega}$ for a rotating body, in which we said that $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ for a point with instantaneous position vector **r** with respect to a fixed origin on the axis of rotation; in the rotating frame **r** is seen as *constant*, so in the fixed frame,

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$ [1]

The relation (8.21) allows us to calculate the Coriolis and centrifugal accelerations in a very elegant way. Let us suppose that **r** *can* change in the rotating frame: we then have, using suffixes i for inertial and r for rotating,

$$\left[\frac{d}{dt}\right]_i \mathbf{r} = \left[\frac{d}{dt}\right]_r \mathbf{r} + \boldsymbol{\omega} \times \mathbf{r} \tag{8.22}$$

and we assume that $\boldsymbol{\omega}$ is constant. Now apply $\left[\frac{d}{dt}\right]_i$ to both sides:

$$\left[\frac{d}{dt}\right]_i^2 \mathbf{r} = \left[\frac{d}{dt}\right]_i \left[\frac{d}{dt}\right]_r \mathbf{r} + \left\{\left[\frac{d}{dt}\right]_i \boldsymbol{\omega}\right\} \times \mathbf{r} + \boldsymbol{\omega} \times \left[\frac{d}{dt}\right]_i \mathbf{r} \tag{8.23}$$

[1] The equation $(d\mathbf{L}/dt) = \boldsymbol{\omega} \times \mathbf{L}$, equations (8.21), (8.22), and the associated transformations between fixed and rotating frames are very important in quantum mechanics [R.P. Feynman *et.al.* (1957), *J. Appl. Phys.* **28**, 49] and in the field of magnetic resonance, where they form the basis of the famous 'Bloch equations' [F. Bloch (1946), *Phys. Rev.* **70**, 460]

Now we must convert the expressions with $\left[\frac{d}{dt}\right]_i$ on the RHS by (8.21) in order to obtain a result containing only $\boldsymbol{\omega}$ and things observed in the rotating frame. The second term vanishes because $\boldsymbol{\omega}$ is constant, and we obtain finally

$$\left[\frac{d}{dt}\right]_i^2 \mathbf{r} = \left[\frac{d}{dt}\right]_r^2 \mathbf{r} + 2\boldsymbol{\omega} \times \left[\frac{d}{dt}\right]_r \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (8.24)$$

or

$$\mathbf{a}_i = \mathbf{a}_r + 2\boldsymbol{\omega} \times \mathbf{v}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (8.25)$$

where $\mathbf{a}_i$ is the acceleration observed in the inertial frame, $\mathbf{a}_r$ is the acceleration observed in the rotating frame, and $\mathbf{v}_r$ is the velocity observed in the rotating frame. Hence the terms $2\boldsymbol{\omega} \times \mathbf{v}_r$ and $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ are due to the *acceleration of the frame* (see section 3.4). Hence an observer *in* the rotating frame must postulate the inertial forces giving rise to the inertial accelerations $-2\boldsymbol{\omega} \times \mathbf{v}_r$ (the Coriolis force) and $-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ (the centrifugal force). The two-dimensional elementary derivation of these inertial forces was given in section 3.5.

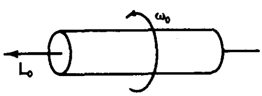


Fig. 8.11a Initial spin axis

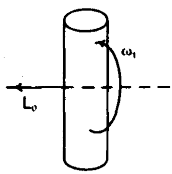


Fig. 8.11c Final spin axis

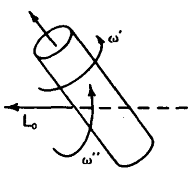


Fig. 8.11b Intermediate axis with precession

As a final example of conservation of angular momentum, we consider a cylindrical spacecraft initially set spinning about the *long* axis of the cylinder (figure 8.11a). We may write the initial angular momentum as $\mathbf{L}_0 = I_0\boldsymbol{\omega}_0$, say, where I_0 is the moment of inertia about the long axis and $\boldsymbol{\omega}_0$ is the initial angular velocity; the rotational kinetic energy will be $T_0 = L_0^2/2I_0$. Even though no external forces act, internal torques and body forces can cause the spacecraft to start precessing about the direction of $\mathbf{L}_0$ (figure 8.11b). Internal strains and dissipative forces will begin to take up some of the rotational kinetic energy. Conservation of angular momentum holds in the absence of external forces: the only way angular momentum can be conserved is for the precession angle to become larger, until eventually the satellite spins about an axis perpendicular to the long axis of the cylinder (figure 8.11c); so that $L_O = I_1\omega_1$, where I_1 is the larger moment of inertia about the axis, and $\omega_1 < \omega_O$, so that $T_1 = (L_O^2/2I_1) < T_O$. This is clearly, therefore, the most *stable* axis of rotation, about which the spacecraft should be set spinning initially.

8.4 ENERGY AND SYMMETRY II: ANGULAR MOMENTUM

We have seen that the angular momentum of a mechanical system is constant when no external torque is acting. However, in many cases of interest the angular momentum, or one of its components, is constant even when forces are acting on the system. This may be deduced from the symmetry of the system with regard to the total energy.

For a free particle of mass m the total energy is given by

$$E = \frac{p^2}{2m} \tag{8.26}$$

where **p** is the linear momentum. This is just the kinetic energy.

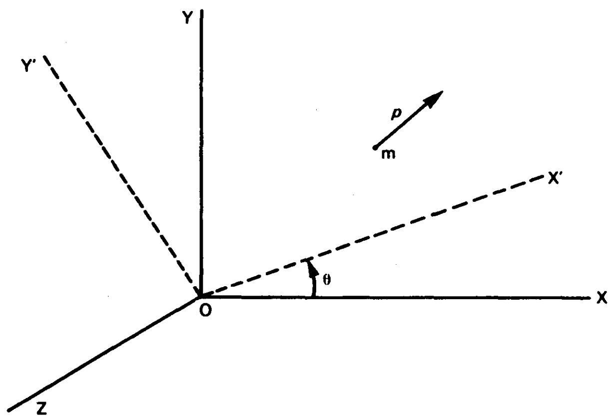


Fig. 8.12 Rotation of axes

Equation 8.26 has complete spherical symmetry. That means that the expression is invariant for co-ordinate transformations involving a rotation about any axis through the origin. Consider the simple example illustrated in figure 8.12. The mass m has momentum $\mathbf{p} = p_x\mathbf{i} + p_y\mathbf{j}$ with $p_z = 0$. In the co-ordinate system x′, y′, z′, which is rotated through the angle θ about the Oz axis relative to xyz the momentum components are

$$p'_x = p_x \cos\theta + p_y \sin\theta$$

$$p'_y = p_y \cos\theta - p_x \sin\theta$$

$$p'_z = 0$$

and $$p'^2_x + p'^2_y = p^2_x + p^2_y$$

Similar results hold for rotation about any axis through 0. For a particle of mass m moving in a force field **F**, the total energy is given by

$$E + \frac{p^2}{2m} + V(\mathbf{r}) \tag{8.27}$$

where p is the linear momentum and $V(\mathbf{r})$ is the potential energy at **r**, the position vector of the particle. $V(\mathbf{r})$ is the work done against the force **F** in bringing the particle from infinity to the point **r**. That is, $V(\mathbf{r}) = \int_{r}^{\infty} \mathbf{F}.d\mathbf{r}$, and V is a scalar quantity.

Since the kinetic energy is invariant for rotations, the symmetry of E depends entirely on that of V and hence on that of **F**.

The angular momentum **L** of a mass m about an axis through 0 is given by the vector product

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

where **r** is the radius vector to m and **p** is the linear momentum of m.

Thus $$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$

$$= \mathbf{r} \times \mathbf{F}(\mathbf{r})$$

where $\mathbf{F}(\mathbf{r})$ is the force acting on m at the position **r**. The angular momentum **L** is constant for a free particle, since **F** is zero. This means that all three components of **L** are constant and that $L^2 = L^2_x + L^2_y + L^2_z$ is constant. If the particle has intrinsic angular momentum or spin, this is maintained throughout the motion. The angular momentum is constant when the force **F** is parallel to **r**. This holds for the important case of a *central force,* which by definition has **F** parallel to **r**. For a central force towards the origin 0 the magnitude of the force depends only on the magnitude of the radius vector $|\mathbf{r}|$. Spherical surfaces concentric with 0 form equipotential surfaces, so that V is a spherically symmetric potential. Thus E is spherically symmetric, and when this is the case **L** is constant.

Note that the force **F** is always perpendicular to the equipotential surfaces (section 5.4). A particle moving on an equipotential surface is doing no work. The general rule is that the components of **L** along the axis of symmetry are constants. For central forces all components of **L** are constants, since **r** is parallel to **F** everywhere. Thus again $\mathbf{L}^2$ is a constant for motion in a central force field.

In a field of force which acts only in one direction, the component of angular momentum about an axis in the direction of the force is constant. For this case the equipotential surfaces are parallel planes perpendicular to the direction of the force. The local gravitational force gives an example. For such a force field in the z direction F_z is the only non-zero component of the force and

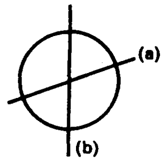
$$\frac{dL_z}{dt} = (\mathbf{r} \times \mathbf{F})_z$$

$$= 0$$

Thus L_z is constant. The only axis of symmetry for this system is the z axis.

The motion of a charged particle in a uniform magnetic field is of considerable interest. The force acting, the Lorentz force, is such that the charged particle moves in a circle whose plane is perpendicular to the direction of the magnetic field (section 3.6). The radius of the circle is determined by the mass and charge of the particle and by the magnitude of the magnetic field. For the particle to move in a circle the Lorentz force must be a centripetal force. There is no component of force in the direction of the magnetic field which we may choose as the z direction.

The equipotential surfaces here are the curved surfaces of cylinders co-axial with the z axis. For this motion it may be seen that L_z is constant. Once again there is symmetry for rotation about the z axis only. Hence we may state the theorem: if V(**r**) has an axis of *rotational* symmetry, the *angular* momentum about this axis is conserved, i.e. is a constant of the motion.



(i)

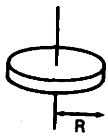
Fig. 8.13 Moments of Inertia

(i) *(a)* *Uniform thin circular ring of mass M, radius R, about an axis through the centre, perpendicular to the plane of the ring:*

$$I = MR^2$$

(b) *about an axis through the centre in the plane of the ring:*

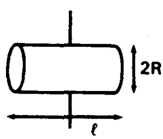
$$I = \tfrac{1}{2} MR^2$$



(ii)

(ii) *Solid circular disc of mass M about an axis perpendicular to the plane of the disc, through its centre:*

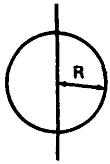
$$I = \tfrac{1}{2} MR^2$$



(iii)

(iii) *Solid circular cylinder of mass M about an axis perpendicular to the cylinder axis through the centre of mass:*

$$I = \frac{MR^2}{4} + \frac{M\ell^2}{12}$$



(iv)

(iv) Solid sphere about any axis through its centre:

$$I = \frac{2}{5} MR^2$$

Other moments of inertia for these solids (e.g. disc about an axis perpendicular to a plane passing through the periphery of the disc) can be calculated simply from the parallel axis theorem

Quantum Mechanics

The link between the symmetry of the energy under rotations and conservation of angular momentum has significance beyond classical mechanics. In quantum mechanics it can be shown that $\mathbf{L}^2$ is a constant of the motion if the total energy function is spherically symmetric. Such a spherically symmetric function has cylindrical symmetry about any chosen axis, so that L_Z is also constant in this case. Similarly it can be shown that an energy function having an axis of symmetry implies that the component of angular momentum along the axis is constant. In quantum mechanics, the relations

$$\mathbf{L}^2 = \ell(\ell + 1)\frac{\hbar^2}{4\pi^2}$$

where $\ell = 0, 1 \ldots (n - 1)$ is the total angular momentum quantum number, n is the principal quantum number, and $\hbar$ is Planck's constant and

$$L_Z = M_\ell \frac{\hbar}{2\pi}$$

where $M_\ell = 0, \pm 1, \ldots, \pm \ell$ is the magnetic quantum number, hold for the hydrogen atom, in which the potential energy of the Coulomb force is spherically symmetric.

CHAPTER 8 PROBLEMS

8.1 Two equivalent particles 1 and 2 of mass m are joined by a straight rod of negligible mass and length 2r with its centre at the origin (figure 8.1p). The system is forced to rotate with constant angular velocity ω about the y axis, so that the angle θ is constant as shown in figure 8.1p. The z axis is pointing out of the page towards you. The questions below refer to the instant in time at which the system lies in the x-y plane. Give your answers in terms of r, θ', ω and the unit vectors **i**, **j**, **k** along the x, y and z axes.

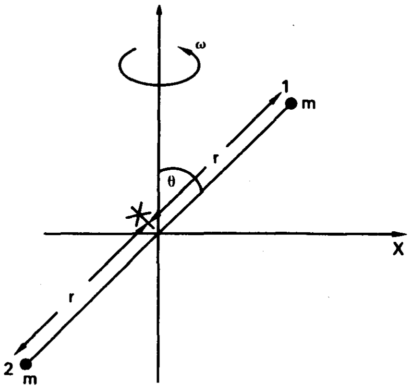


Fig. 8.1p

(a) What is the velocity of mass 1?
(b) What is the moment of inertia of the system about the y axis?
(c) From *first principles*, or otherwise, calculate the instantaneous magnitude and direction of the angular momentum vector.
(d) What is the torque required to keep the angular momentum vector precessing with constant angular velocity about the y axis?
(e) This torque is applied with a lever arm a**j**. In what direction does the necessary force occur?

8.2 A dumbbell consisting of two particles of mass m separated by a rod of length 2ℓ and negligible mass is forced to rotate about the vertical axis Oz shown in figure 8.2p with angular speed ω. The sense of rotation is such that the lower mass is moving out of the paper.

(a) What is the direction of the angular velocity vector?

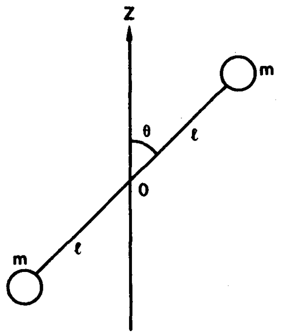


Fig. 8.2p

(b) What are the direction and magnitude of the angular momentum vector at the instant depicted above?
(c) Is there a component of angular momentum which is conserved? If so, which one?
(d) What is the magnitude of the torque required to keep the dumbbell rotating?

8.3 A hoop of radius R and mass M which lies flat on a horizontal frictionless table is set in rotation about its centre with angular velocity ω (figure 8.3p). The moment of inertia about the centre is MR^2. Initially the hoop has no translational motion. A piece of putty of mass m, moving on a straight path at speed v strikes the hoop tangentially and sticks to it.

Fig. 8.3p

(a) What is the linear momentum of the whole system before the collision?
(b) What is the angular momentum about 0 of the whole system before the collision?
(c) What is the angular momentum about 0 of the whole system after the collision?
(d) What is the moment of inertia about 0 of the whole system after the collision?

(e) With respect to a fixed co-ordinate system with origin 0, what is the total kinetic energy of the whole system (translational plus rotational) before the collision?

(f) Is kinetic energy conserved in this collision?

(g) Does the hoop move across the table after collision? Give your reasons.

8.4 A cylinder of radius r and moment of inertia $\frac{1}{2}mr^2$ is allowed to roll without slipping around the inside of a larger fixed cylinder of radius R (figure 8.4p). Initially the small sylinder is at rest with its centre of mass on the horizontal radius of the large cylinder, and at time t its position is given by the angle ϕ.

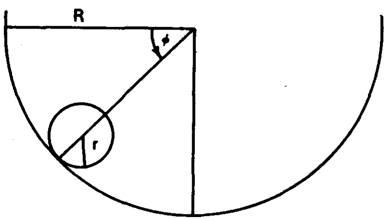


Fig. 8.4p

(a) What is the potential energy at the position ϕ?

(b) Show that its kinetic energy at the position ϕ is $\frac{3}{4}m(R-r)^2\,\dot{\phi}^2$.

8.5 A man of mass m stands on the edge of a freely rotating horizontal wheel of moment of inertia I and radius R (figure 8.5p). The man then walks inwards along a radial arm of the wheel at constant speed u. The initial angular velocity of wheel and man is Ω, and you may neglect the dimensions of the man compared to the radius of the wheel.

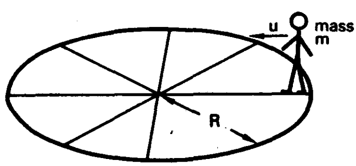


Fig. 8.5p

(a) What is the angular velocity ω of the wheel as a function of time for $0 < t < (R/u)$?

(b) What is the angular velocity of the wheel when the man reaches the centre?

(c) Is kinetic energy conserved in this motion?

(d) If your answer to part (c) is 'yes', explain why; if it is no, explain where the change in energy comes from or goes to.

9
Central forces

9.1 DEFINITION AND PROPERTIES

If a particle experiences a force which is always directed along the position vector $\mathbf{r}$ of the particle with respect to some origin or *centre*, and which is some function f(r) of r only, the force is said to be a *central* force. Thus if the origin is at 0, and the particle is at P (figure 9.1), the central force $\mathbf{F}$ is given by

$$\mathbf{F} = f(r)\hat{\mathbf{r}} \tag{9.1}$$

where $\hat{\mathbf{r}}$ is the unit vector along the position vector $\mathbf{r} = \mathbf{OP}$.

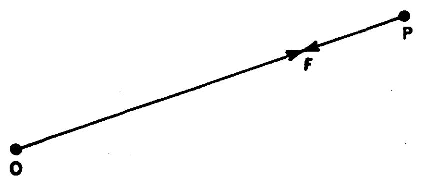


Fig. 9.1 Central force

The Coulomb and gravitational forces are of this type. But as far back as section 3.1, when we studied the interaction of two bodies only, we discovered that in an ideal situation the interaction forces are directed along the line joining the two bodies. In other words, *all* Newtonian action and reaction forces between pairs of particles are central forces. It is conceivable that the action and reaction forces between two bodies could be as is illustrated in figure 9.2, where they are still equal and opposite; but this is not the case in ideal situations.

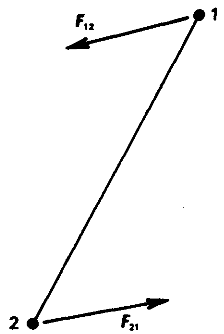


Fig. 9.2 Conceivable action-reaction forces

We have seen that central interaction forces

(i) conserve the total *linear* momentum of a system of particles (the momentum of the centre of mass): section 4.2, when no external forces act;

(ii) conserve the total *angular* momentum of a system of particles about the centre of mass when no external torques act: section 7.3;

(iii) conserve the total angular momentum of a system of particles when no external forces act.

As a consequence of (i) and (ii), momentum is conserved in a two-body collision where the interaction forces are central; and so is angular momentum, as a consequence of (ii) and (iii), which leads to the fact that all two-body central-force collisions and orbits must be *planar.*

We see therefore that central forces are not only a very common and important class of force, but that they also imply certain conservation principles which make problem-solving much simpler.

9.2 THE TWO-BODY PROBLEM. CONSTANCY OF AREAL VELOCITY

A very familiar central-force problem is that where we consider only the interaction of two bodies – for example, the rotation of the earth about the sun, or of an electron about a nucleus. We show below that the two-body problem can always be reduced to the problem of a *single* body moving about an origin of the particular central force. Thus the two-body problem is always exactly soluble, even though numerical methods may be necessary for peculiar cases. Three-body or many-body problems cannot be solved exactly; approximations must always be made.

Consider two particles, of mass m_1, m_2, with position vectors $\mathbf{r}_1$, $\mathbf{r}_2$

respectively with respect to an origin 0 in some inertial frame. The central interaction force between them lies always along the direction of $\mathbf{r}_2 - \mathbf{r}_1$, and depends only on $|\mathbf{r}_2 - \mathbf{r}_1|$. Suppose that the force is *attractive:* if we write $\mathbf{r}_{21} = \mathbf{r}_2 - \mathbf{r}_1$, we have

$$\text{(a)} \quad \ddot{m}_1 \mathbf{r}_1 = -f(r_{21})\hat{\mathbf{r}}_{21} \qquad \text{(b)} \quad m_2 \ddot{\mathbf{r}}_2 = f(r_{21})\hat{\mathbf{r}}_{21} \tag{9.2}$$

where $\hat{\mathbf{r}}_{21}$ is unit vector along $\mathbf{r}_2 - \mathbf{r}_1$, and $f(r_{21})$ is some function of $r_{21} = |\mathbf{r}_2 - \mathbf{r}_1|$ only (figure 9.3). Now we know that the centre of mass must lie along $\mathbf{r}_{21}$; let us *move into the centre-of-mass frame,* and consider the motion of m_2, say, with respect to the centre of mass. We assume that there are no external forces acting, so the centre-of-mass frame is inertial.

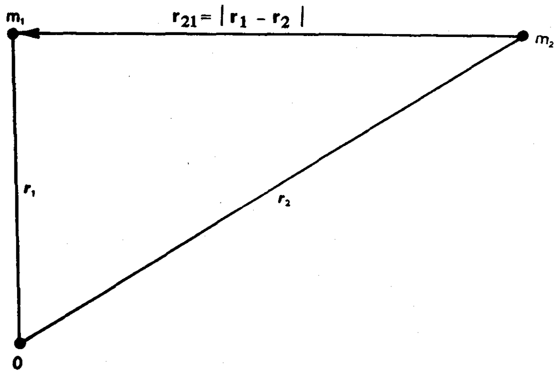


Fig. 9.3 Two-body problem

The position-vector of m_2 with respect to the centre of mass is

$$\mathbf{r}_2' = \frac{m_1}{m_1 + m_2}\mathbf{r}_{21}$$

by definition of the centre of mass, so we may write

$$m_2 \ddot{\mathbf{r}}_2' = \frac{m_2 m_1}{m_1 + m_2}\ddot{\mathbf{r}}_{21} = f(r_{21})\hat{\mathbf{r}}_{21} \tag{9.3}$$

since the force in unchanged by a Galilean Transformation. It is clear from equation (9.3) that we now have only a single vector $\mathbf{r}_{21}$ to deal with, so we have reduced the problem to a *one*-body one; in this case we have 'fixed' m_1, and the result is that the 'effective' mass of m_2 is 'reduced' in the ratio $m_1/(m_1 + m_2)$. We may therefore rewrite equation (9.3), without loss of generality, as

$$m\ddot{\mathbf{r}} = -f(r)\hat{\mathbf{r}} = \mathbf{F}(r) \tag{9.4}$$

for an attractive force, where we understand that m may have to be suitably 'reduced' as in equation (9.3). We shall now prove a property common to all central-force orbits, *whatever the form of f(r)*, which derives from the conservation of angular momentum.

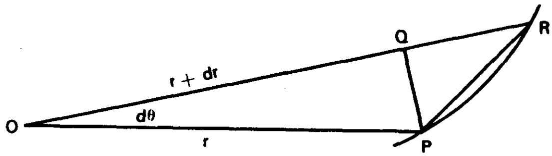


Fig. 9.4 'Areal velocity'

The angular momentum is $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$. Let us resolve $\mathbf{v}$ into a component $\mathbf{v}_r$ along $\mathbf{r}$ and a component $\mathbf{v}_\theta$ which is perpendicular to $\mathbf{r}$. Then, by the properties of the vector product, $\mathbf{L} = \mathbf{r} \times m\mathbf{v}_\theta$. But, instantaneously,

$$v_\theta = r\frac{d\theta}{dt} = r\omega \tag{9.5}$$

where ω is the angular velocity, and consideration of figure 9.4 shows that the area of the elementary triangle OPQ, which must tend to that of OPR in the limit, is just $\frac{1}{2}r.rd\theta$, swept out in time dt. We therefore have

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v}_\theta = mr.r\frac{d\theta}{dt}\hat{\mathbf{L}} = 2mA_V\hat{\mathbf{L}} \tag{9.6}$$

where $A_V = \frac{1}{2}r^2(d\theta/dt)$ is the 'areal velocity', or area swept out per unit time by $\mathbf{r}$. Since $\mathbf{L}$ is constant, we see, from (9.6), that A_V is constant. Thus the law of equal areas being swept out in equal times by the line joining the 'moving' body to the 'fixed' body applies to *all* central-force orbits, whatever the form of f(r), and whether the force is attractive or repulsive. As applied to the motion of the planets about the sun under gravity, it is doubtless familiar as Kepler's second law.

9.3 SOLUTION OF THE TWO-BODY PROBLEM USING 'EFFECTIVE POTENTIAL'

We now show how, in principle at any rate, we can use the method of section 5.3 to solve completely the two-body central-force problem for any f(r). In doing so, we reduce the problem from a two-dimensional problem to a one-dimensional one, making use of a constant of the motion – in this case, the angular momentum.

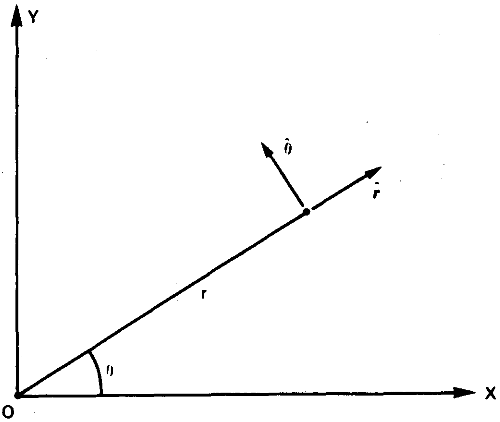


Fig. 9.5 Polar co-ordinate system

We know that the motion is planar because of angular momentum conservation. Taking polar co-ordinates r and θ in the (x, y) plane, we write the velocity **v** as

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \tag{9.7}$$

where $\hat{\boldsymbol{\theta}}$ is normal to $\hat{\mathbf{r}}$ (figure 9.5). Hence the total energy (assumed *conserved*) is

$$E = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 + V(r) \tag{9.8}$$

where V(r) is the potential energy corresponding to f(r); if we absorb the *sign* of the force in f(r), we have of course (section 5.2) $f(r) = -[dV(r)]/dr$, since V(r) does not depend on θ. The angular momentum is

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} = mr^2\dot{\theta}\mathbf{k} = \text{constant} \tag{9.9}$$

and we now substitute the value of $\dot{\theta}$ obtained from (9.9) into (9.8), thus reducing the two-dimensional problem to a one-dimensional problem, by eliminating the co-ordinate $\dot{\theta}$ associated with the constant of the motion **L**.[1] We thus obtain

[1] This kind of co-ordinate is called 'ignorable' in advanced mechanics because it can always be eliminated by an appropriate substitution.

$$E = \tfrac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) \tag{9.10}$$

This is equivalent to a one-dimensional energy equation

$$\tfrac{1}{2}m\dot{r}^2 + V'(r) = E$$

where

$$V'(r) = \frac{L^2}{2mr^2} + V(r) \tag{9.11}$$

is an 'effective potential energy'. When we substitute the value of $\dot{\theta}$ from equation (9.9) and (9.8), we have *moved into the non-inertial frame* rotating with angular velocity $\dot{\theta}$. We expect to see an inertial force appearing, and it turns up in the energy equation as the effective *centrifugal potential* $L^2/2mr^2$ due to the angular momentum L. The effective centrifugal force due to L is L^2/mr^3 (check this). There remains now nothing more to do but solve for r as a function of t in equation (9.11) by the method of section 5.3. This can of course be a very difficult problem, and for certain situations numerical methods may be necessary.

9.4 INVERSE SQUARE LAW FORCES

Two very important forces, the Coulomb force and the gravitational force, are of the inverse square type; they may be written

$$F(r) = Kr^{-2} \tag{9.12}$$

where the constant K contains the sign of the force (attractive or repulsive). The resulting potential is

$$V(r) = Kr^{-1} \tag{9.13}$$

so that equation (9.10) becomes

$$E = \tfrac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + \frac{K}{r}$$

$$V'(r) = \frac{L^2}{2mr^2} + \frac{K}{r} \tag{9.14}$$

In order to obtain information about the motion, we plot the effective potential $V'(r)$ of equation (9.14) in figure 9.6 for the following cases:

(i) $K > 0$ (repulsive force)

(ii) $K = 0$ (angular momentum contribution alone)

(iii) $K < 0, L \neq 0$ (attractive force)
(iv) $K < 0, L = 0$ (inverse square attractive force alone)

We note that for $K > 0, K = 0$, the particle can never be 'bound' to the centre of force.

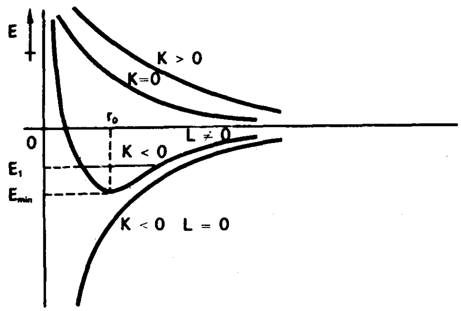


Fig. 9.6 'Effective potentials' for inverse square law force

For $K > 0, L \neq 0$, particles with energies less than O can be 'trapped' in the effective-potential well, with minimum at r_0. A particle with energy E_{min} is in *stable equilibrium* at r_0 – and clearly this must be a circular orbit. So the circular orbit is the most stable orbit, for earth satellites, for example.

For a total energy > 0, particles are 'reflected' by the contribution from the angular momentum, i.e. owing to angular momentum the effective force on the particle close to the force centre becomes *repulsive*. The derivation of the orbits other than the circular one is somewhat laborious, and requires some knowledge of conic sections. Without proof, the orbits are:

$$K < 0 \quad \begin{cases} E < 0 & \text{ellipse} \\ E = 0 & \text{parabola} \\ E > 0 & \text{hyperbola} \end{cases} \qquad \begin{matrix} L = 0, K < 0 \\ \text{straight line} \end{matrix}$$

$$K > 0, \qquad \text{hyperbola unless } L = 0$$

Thus, if two attracting particles approach each other from infinity, they can never contact each other unless they approach along the same straight line, i.e. $L = 0$ for system, a result true for Newtonian mechanics only. The foregoing material illustrates the usefulness of the 'effective potential' and potential-energy diagram technique in solving problems. When we consider that Newton worked out all the planetary orbits from first principles, notwithstanding the labour and the difficulty, we should have an even greater respect for his genius.

Finally, a note about Kepler's laws of planetary motion. The first law (that the planets move about the sun in ellipses, with the sun at one focus) emerges from the orbit calculation which we have omitted; it is trivial for the circular orbit case. We have already proved the second law (constancy of areal velocity) for *any* central force. The proof of the third law ($T^2 \alpha R^3$, where R is the mean distance from the sun and T is the period for the orbit) is almost trivial for the circular orbit, and is left as an exercise; for the elliptical orbit the proof is more difficult, but it illustrates the use of the conservation laws.

Let us therefore sketch a proof of Kepler's third law, using certain properties of the ellipse, conservation of energy, and conservation of angular momentum. First, let us integrate the constant areal velocity, A_V, which is equal to the total angular-momentum L divided by twice the reduced mass, μ, [see equations (9.3), (9.6)] over one period, T. We obtain the area of the ellipse

$$\pi ab = \frac{LT}{2\mu}$$

or

$$T = \frac{2\pi ab\mu}{L} \qquad (9.15)$$

where a and b are the semi-major and semi-minor axes of the ellipse respectively. When the radius vector lies along the semi-major axis of the ellipse, the velocity vector is perpendicular to it: hence the kinetic energy K may be written in the form

$$K = \frac{1}{2\mu}\left(\frac{L}{\mu d}\right)^2 \qquad (9.16)$$

as for motion in a circle, where d is now the larger or smaller distance from the origin (focus) along the semi-major axis. Let us call these r_1 and r_2 respectively. Then the total energy becomes

$$\begin{aligned} E &= K + U \\ &= \frac{L^2}{2\mu}\left(\frac{1}{r_1{}^2}\right) - \frac{GM_1M_2}{r_1} = \frac{L^2}{2\mu}\left(\frac{1}{r_2{}^2}\right) - \frac{GM_1M_2}{r_2} \end{aligned} \qquad (9.17)$$

Now, for an ellipse, the quantity e, defined by $b^2 = a^2 (1 - e^2)$, is known as the ellipticity; and also the distances r_1 and r_2 defined above are given by $r_1 = a(1 + e)$, $r_2 = a(1 - e)$. If we substitute these into the energy equation (9.16) and eliminate E, we shall obtain the relation

$$L^2 = a(1 - e^2)\, GM_1M_2\, \mu \qquad (9.18)$$

Squaring equation (9.15), substituting for L^2, and using the relation between b and a involving the ellipticity, we finally obtain

$$T^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)} \tag{9.19}$$

i.e. T^2 is proportional to the cube of the semi-major axis of the ellipse. Since a is related to b via the ellipticity, the relation is equivalent to, but more general than, Kepler's original statement involving the mean distance from the sun, because $T^2 \propto R^3$, from equation (9.19), will be strictly true only when the mass of a 'planet' can be ignored in comparison with that of its 'sun'.

CHAPTER 9 PROBLEMS

9.1 Find the components of force for the following potential-energy functions.
   (a) $V = axy^2z^3$, where a is a constant
   (b) $V = \frac{1}{2}Kr^2$, where K is a constant
   (c) $V = \frac{1}{2}K_x x^2 + K_y y^2 + \frac{1}{2}K_z z^2$, where K_x, K_y, and K_z are different constants

9.2 Find the force on a particle which moves in a region of potential

$$V = -\frac{a^2}{r_1} - \frac{a^2}{r_2}$$

where a is a constant and r_1 is the distance of the particle to the point $x = -b$, $y = 0$, $z = 0$, and r_2 is its distance to the point $x = b$, $y = 0$, $z = 0$.

9.3 Plot against r the functions V and $V' = V + (L^2/2mr^2)$ for a particle of mass m and angular momentum **L** which moves under the action of a central force **F**(r) for each of the following potential functions.
   (a) $V = \alpha r$, where α is a positive constant
   (b) $V = \frac{\alpha}{r}$
   (c) $V = \frac{-\alpha}{r^2}$, where $\alpha < \frac{L^2}{2m}$
   (d) $V = \frac{-\alpha}{r^2}$, where $\alpha > \frac{L^2}{2m}$

9.4 Show that $\mathbf{F} = \mathbf{n}F(r)$, where $\mathbf{n}$ is a unit vector directed away from the origin, is a conservative force by showing that

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}.d\mathbf{r}$$

along any path between $\mathbf{r}_1$ and $\mathbf{r}_2$ depends only on r_1 and r_2. (Hint: Express $\mathbf{F}$ and $d\mathbf{r}$ in spherical co-ordinates.)

9.5 The attractive force between a neutron and a proton has the potential (in Yukawa's theory) $V(r) = ke^{-\alpha r}/r$, where k is a negative constant.

(a) Find the force.

(b) Compare this with an inverse square law of force.

(c) What types of motion can occur if a particle of mass m moves under the action of such a force?

(d) How do the motions differ from the corresponding motions for an inverse square law of force?

(e) Find L and E for motion in a circle of radius a.

10
Gravitation

10.1 INTRODUCTION TO FIELD THEORY II. GAUSS'S THEOREM

We know that we can have distributions of charge or of matter which will give rise to electric or gravitational fields respectively whose behaviour throughout space may not be inverse square, although the basic *particle* interaction of these fields is inverse square. A very simple example is the field of the electric dipole illustrated in figure 10.1a. The *resultant* electric field along the y axis, for example, is clearly not proportional to y^{-2}, because of the vector sum involved; nor is it any longer a central force. But there are certain symmetrical and continuous distributions of charge or of matter for which the resultant force fields are calculated very simply, by the use of a theorem called Gauss's theorem, which we shall now prove. Before we do so, we need some further definitions of terms used in field theory.

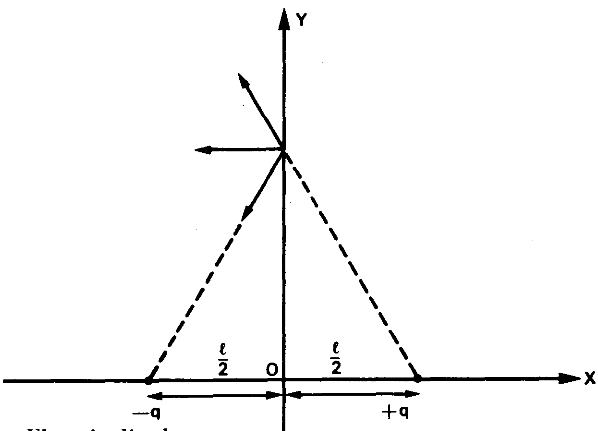


Fig. 10.1a Electric dipole

Vector Field: Flux and Divergence

The gradient ∇V is a *vector invariant* associated with a *scalar* field $V(x, y, z)$. There are also useful *scalar invariants* associated with a *vector* field $\mathbf{E}(x, y, z)$.

Consider a small portion, of area δS and unit normal $\mathbf{n}$, of a closed surface S. The direction of $\mathbf{n}$ is conventionally taken *outward* from the surface. Consider the quantity

$$\mathbf{E}.\mathbf{n}\delta S$$

Its physical significance is most easily seen in the case where $\mathbf{E}$ stands for the quantity of something – water, perhaps – flowing through unit area perpendicular to $\mathbf{E}$ per second. $\mathbf{E}.\mathbf{n}\delta S$ is then simply the quantity of water flowing through this bit of surface per second. (The dot product correctly allows for the 'slant' of the surface – only its area normal to $\mathbf{E}$ counts.)

However, this quantity is useful even when $\mathbf{E}$ does not represent the flow of anything physical. We add up all the products $\mathbf{E}.\mathbf{n}\delta S$ for all portions of the surface S, in the limit where δS goes to zero, and write the result

$$\int_S \mathbf{E}.\mathbf{n}dS \text{ or } \int_S \mathbf{E}.\mathbf{dS}$$

(The vector $\mathbf{dS}$ conventionally denotes $\mathbf{n}dS$.) We call this integral the *flux* of $\mathbf{E}$ through S. Thus the flux is the 'surface integral of the normal component'. We shall now point out three useful properties of the flux.

Flux of a Conserved Quantity

Suppose that $\mathbf{E}$ is really a flow of some quantity whose total amount inside S is Q. Q may be water (in grams), heat (in calories), neutrons (their number), etc. Suppose further that this quantity cannot be created or destroyed inside S, i.e. it is *conserved*. Then change in Q with time can occur only by flow into or out of S. Since the total flow *out* of S is simply the flux as we defined it (*out* because we took the normal outwards), we have immediately:

$$\int_S \mathbf{E}.\mathbf{dS} = -\frac{dQ}{dt} \tag{10.1}$$

This apparently trivial result is quite basic in relating variation in space to variation in time, and can later on give us important 'equations of continuity'.

Slicing up a Surface

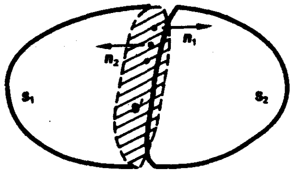


Fig. 10.1b Slicing up a surface

Suppose we cut through the surface S so that it divides into two pieces S_1 and S_2, with S' the surface of separation (figure 10.1b). Consider now the *sum* of the fluxes through S_1 and S_2. How is it related to the flux through the original surface S? The difference

$$\int_{S_1} + \int_{S_2} - \int_{S}$$

will be seen to arise only from the contributions from S', i.e. the difference is

$$\int_{S'} \mathbf{E}.\mathbf{n}_1 \, dS' + \int_{S'} \mathbf{E}.\mathbf{n}_2 \, dS'$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit normals at a point to S_1 and S_2 respectively. But $\mathbf{n}_1$ and $\mathbf{n}_2$ differ only because of the convention that we take the normal in the *outwards* direction – which makes $\mathbf{n}_2 = -\mathbf{n}_1$. Thus these last two integrands cancel each other at every point. Conclusion: the sum of the fluxes through S_1 and S_2 is *equal* to the flux through S.

If we now consider S_1 and S_2 each further divided in two, the same result applies to the sum of the *four* fluxes. Continuing thus, we conclude: if we dissect a volume into any number of pieces, the sum of the fluxes through the surfaces of these pieces is equal to the flux through the original bounding surface. Note that the constituent pieces can even be *infinitesimal.*

Flux Related to Enclosed Volume

In some physical problems we are given the value of a *vector* field on a *surface;* in others the data naturally occurs as the values of a *scalar* field inside a *volume*. It is very handy to be able to jump from one to the other and this we can do by means of Gauss's theorem, which we specialize to the case of the inverse square law.

Suppose that our field **E** is due to a point source at 0, that its direction at a point P is along the line OP, and that its magnitude falls off as $1/(OP)^2$. For the sake of being definite we shall talk in terms of this electrostatic law, for which the field **E**(r) at a distance r from the source (of charge q) has the magnitude

$$E(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \qquad (10.2)$$

where ϵ_0 is a universal constant. The direction of **E** is from the source to the field point when q is positive. We are after a result about the normal surface integral of **E**. We shall begin with a special surface and then work our way up to an arbitrary one.

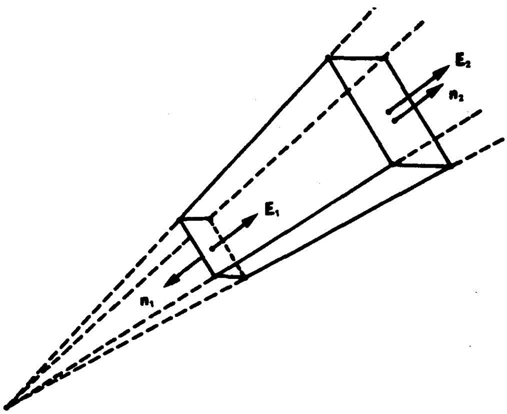


Fig. 10.2 Surface for proving Gauss's theorem

The 'box' in figure 10.2 has its long faces (not infinitesimal) parallel to **E**, and so their contribution to the flux integral is zero. The end faces are 'caps' of spheres centred on the charge q, of area δA_1 and δA_2, small enough for us to neglect their curvature when we want to. Then the flux integral is

$$(\mathbf{E}_1 . \mathbf{n}_1)dA_1 + (\mathbf{E}_2 . \mathbf{n}_2)dA_2$$

$$= -E_1 dA_1 + E_2 dA_2$$

Now, E *decreases* as $1/r^2$, while dA *increases* as $1/r^2$; so the product EdA is constant, and the total flux is thus zero.

Now we make the 'box' a little more complicated by allowing the end faces to 'tilt'. If the 'tilt' is through an angle θ, the area increases by a factor $1/\cos\theta$. But the normal component of **E** is no longer E, but $E\cos\theta$. (Check these last two sentences.) Thus the product EdA is unchanged; so the flux remains zero.

But now we have a thin 'splinter' of arbitrary length, with end faces of arbitrary slope. By putting splinters like this together, when the end faces are small enough we can make any surface we like. Thus the flux integral is zero for any surface. But there is an important proviso: our proof holds good only if the surface (like the box above) does not enclose the charge q. What if it *does* enclose it?

Here we use a very effective trick. Take a very special 'enclosing' surface: a small sphere of radius r centred on the charge q. Then **E** on the surface is always normal to it (it is along a radius) and is of magnitude $(1/4\pi\epsilon_0)(q/r^2)$. Since the total area is $4\pi r^2$, the flux is $(4\pi r^2) \times (1/4\pi\epsilon_0)(q/r^2) = q/\epsilon_0$. Now given an arbitrary surface S enclosing the charge, we make a sphere s like this, small enough to fit inside S, and then we consider the surface consisting of s + S. This may seem a queer surface, but if you think about it you will see that there is no problem in building it up out of the end faces of our thin splinters. They fill the space between s and S, *which contains no charge.* So the flux through these splinters, and hence through s + S, is zero.

Thus flux through S = $-$(flux through s)

$$= -\left(-\frac{q}{\epsilon_0}\right)$$

(We have to put in the extra minus sign because the normal to s has to be **outward** from the space between s and S, and so runs inward now along the radius.)

$$\int_S \mathbf{E.dS} = 0, \text{ when q is outside S}$$

$$= q/\epsilon_0, \text{ when q is inside S}$$

When more than one charge is involved, we simply add the results for each charge separately; fluxes due to different charges obviously add to give the total flux. We then get

$$\int_S \mathbf{E.dS} = (\text{total charge inside S})/\epsilon_0 \qquad (10.3)$$

This is Gauss's Law for electrostatics. For gravitation, it becomes

$$\int_S \mathbf{g.dS} = 4\pi GM \qquad (10.4)$$

where **g** is the acceleration due to gravity over the surface S, G is the universal gravitational constant, and M is the mass enclosed by the surface S. *Note that the surface S is perfectly general, so we may imagine it anywhere convenient.* Such surfaces are called 'Gaussian surfaces'. We now give some examples of the use of Gauss's theorem as applied to gravitation.

10.2 GRAVITATIONAL FIELDS OF SOME UNIFORM, SYMMETRIC OBJECTS

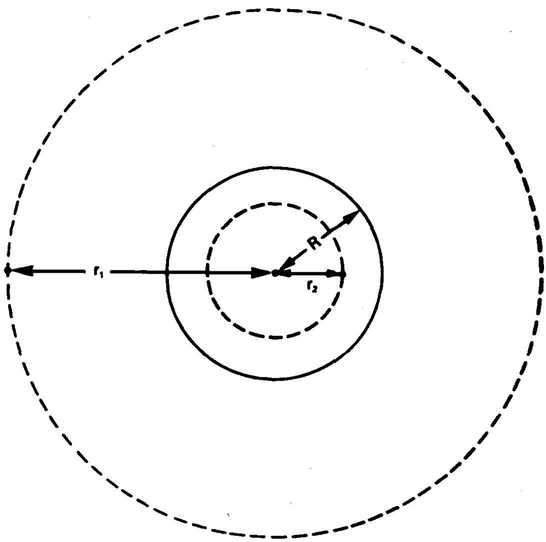


Fig. 10.3 Uniform sphere. The Gaussian surfaces are dotted

Consider a uniform sphere of radius R, centre at 0, which has a total mass M. We want to determine the gravitational acceleration (i) at some radius $r_1 > R$; (ii) at some radius $r_2 < R$ (figure 10.3). By symmetry, **g** is always perpendicular to spherical surfaces concentric with the uniform sphere, and its value is the same at all points on any one surface. The surface area of a sphere of radius $r_1 > R$ centred on 0 is $4\pi r_1{}^2$, and so the flux of **g** through this Gaussian surface is

$$-g_1 . 4\pi r_1{}^2 = 4\pi GM \qquad (10.5)$$

by Gauss's theorem, since the total mass M is enclosed by this surface: g_1 is the value of g on the surface. Hence

$$g_1 = -\frac{GM}{r_1{}^2} \qquad (10.6)$$

which is a familiar result: the negative sign occurs because g is directed *towards* the centre of the uniform sphere.

Now we consider case (ii), i.e. we consider the flux of $g = g_1$ everywhere on the surface of a sphere of radius $r_2 > R_1$ centred on 0. Again, by symmetry, g is everywhere perpendicular to this Gaussian surface, and its flux is

$$-g_2 4\pi r_2^2 = 4\pi G M_2 \tag{10.7}$$

by Gauss's theorem, where M_2 is the mass enclosed by the surface of radius r_2. It should be clear that M_2/M is given by the ratio of the volume of the sphere of radius r_2 to that of radius R, so that

$$M_2 = \frac{r_2^3}{R^3} M \tag{10.8}$$

and equation (10.7) becomes

$$-g_2 r_2^2 = G\frac{r_2^3}{R^3} M$$

giving

$$g_2 = -r_2 GM \tag{10.9}$$

The results of these calculations are shown plotted in figure 10.4.

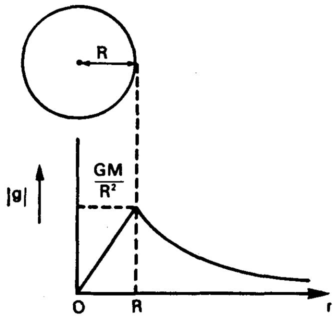


Fig. 10.4 Gravitational field of uniform sphere

Next, consider an infinitely long cylinder of mass m per unit length and radius R (figure 10.5); this could be an approximation to a gas-cloud in space, or to the slowly-curving arm of a spiral nebula. Because the cylinder is infinitely long, g must everywhere be perpendicular to the surfaces of concentric cylinders, and by symmetry must have the same value at any point of any one cylinder. We again calculate g at some radius $r_1 > R$, and some radius $r_2 < R$, using Gauss's theorem.

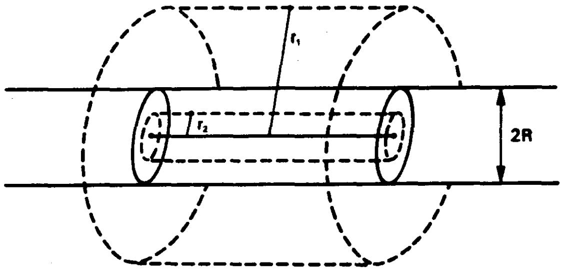


Fig. 10.5 Infinite uniform cylinder: Gaussian surfaces are dotted

For $r_1 > R$, consider the *unit length* of a cylinder of radius $r_1 > R$ concentric with the cylinder of radius R. The flux of g through the surface is

$$-g_1 2\pi r_1$$

since g is everywhere perpendicular to the cylinder surface, and the end faces of the unit length, chosen normal to the cylinder axis, make no contribution. The enclosed mass is clearly m, so we have, by Gauss's theorem,

$$-g_1 2\pi r_1 = 4\pi G m \qquad (10.10)$$

which gives

$$g_1 = -\frac{2Gm}{r_1} \qquad (10.11)$$

For $r_2 < R$, the flux of g presents no problems. The mass enclosed by the unit length of the cylinder of radius $r_2 < R$ is

$$m_2 = \frac{r_2^2 m}{R^2} \qquad (10.12)$$

so that eventually we obtain by Gauss's theorem

$$g = -\frac{2r_2 Gm}{R^2} \qquad (10.13)$$

a result analogous to that for the sphere. Note that for both the sphere and the cylinder we could have treated a non-uniform *radial* distribution of density quite successfully.

10.3 A 'MODERN' USE OF GAUSS'S THEOREM

We shall now briefly describe a use of Gauss's theorem in a very up-to date problem: that of 'mascons', or concentrations of mass, on the moon. These were first discovered by anomalies (accelerations) in the orbits of lunar artificial satellites. We shall grossly simplify the problem to bring out the principles.

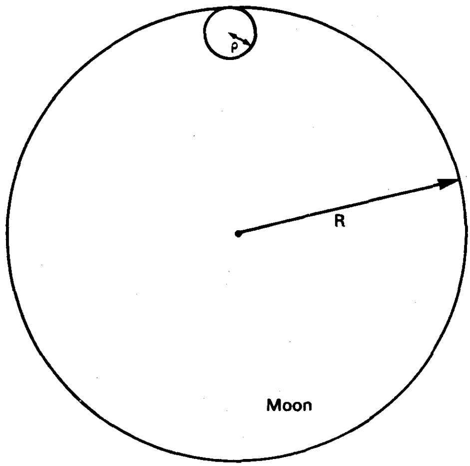


Fig. 10.6 A 'mascon' on the moon (idealized)

Suppose that the moon is a uniform sphere of mass M, except for a small, more dense sphere of mass $m \ll M$ and radius ρ which touches the surface as shown in figure 10.6. If the radius of the moon is $R \gg \rho$, the *average* gravitational acceleration at its surface will be given by $-(GM/R^2)$ to a very good approximation, since the density difference to the 'mascon' is only a small anomaly. At the point where the 'mascon' touches the surface, we may consider the acceleration due *only to the mascon* by drawing our spherical 'Gaussian surface' to coincide with the spherical mascon. The gravitational acceleration due to the mascon is then given by $-(Gm/\rho^2)$ *exactly*, by our model, and this must be the value of the gravitational acceleration at the point where the mascon touches the surface. We can then proceed, from the known average lunar density and supposed densities for the mascon for example, to calculate ρ. This use of Gauss's theorem to calculate mascon 'radii' or densities has given estimates which are in reasonable agreement with those determined by other methods.

10.4 ESCAPE VELOCITIES

Knowing how the gravitational force falls off from the surface of some object to infinity, we may use conservation of energy to determine the minimum velocity necessary to escape from the gravitational field of the object, i.e. to arrive at infinity with zero velocity. We have the the total energy of a mass m:

$$E = \tfrac{1}{2}mv^2 + V(r) = \tfrac{1}{2}mv^2 - \frac{GMm}{R} \qquad (10.14)$$

for a sphere of mass M and radius R. Since the object is to arrive at infinity with zero velocity, and the potential energy will also be zero there, we have E = 0, giving us

$$v_e^2 = \frac{2GM}{R} \qquad (10.15)$$

where v_e is the escape velocity. The result for the uniform cylinder of mass m per unit length and radius R is that v_e^2 is infinite, i.e. it is impossible to escape from such a system. This can be seen more easily from the fact that the work done to go from **R** to infinity must be

$$W = -\int_R^{\infty} \frac{2GM}{r}\, dr$$

which is an infinite integral, than from the form of the potential energy.

CHAPTER 10 PROBLEMS

10.1 Gauss's theorem for gravitation is

$$\int_S \mathbf{g}.d\mathbf{S} = -4\pi G \int_V \rho dV$$

(a) (i) In the above equation what is g?
(ii) What is the meaning of the left-hand side of the equation?
(b) (i) What is the meaning of the right-hand side?
(ii) What is the significance of the negative sign?
(c) Use the above equation ot calculate the magnitude and direction of the gravitational field at a point inside a very long cylinder of uniform density.

10.2 The gravitational force of attraction between two bodies of masses m_1 and m_2 at a separation r is

$$F(r) = -\frac{Gm_1m_2}{r^2}$$

Assume that the moon (mass m) moves in a circular orbit of radius r_0 about the earth (mass M). Assume also that $m \ll M$.

(a) Find an expression for the moon's speed in its orbit.

(b) Find an expression for the time for one orbit.
Suppose that the moon were to be arrested in its orbit at separation r_0 and then released at rest with respect to the earth.

(c) Write down an equation relating the kinetic energy and the potential energy of the moon in the earth's gravitational field when the moon is at separation r.

(d) How long will it take the moon to crash into the earth? (ignore the earth's motion about the sun, consider the earth's radius as negligibly small, and take

$$\int_0^1 \frac{\epsilon^{\frac{1}{2}}\,d\epsilon}{(1-\epsilon)^{\frac{1}{2}}} = \frac{\pi}{2})$$

10.3 Gauss's theorem for gravitation is

$$\int_S \mathbf{g}.\mathbf{dS} = -4\pi G \int_V \rho \, dV$$

where **g** is the gravitational field intensity; the integral is taken over a closed surface S; and **dS** is an element of area of this surface. The surface S encloses volume V over which the density ρ is integrated.

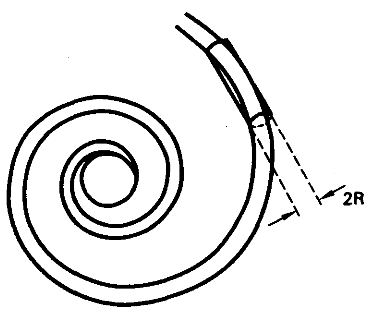


Fig. 10.3p

The outer arm of a spiral galaxy can be approximated by a cylinder of density ρ, radius R, and infinite length (figure 10.3p).

(a) Using Gauss's theorem, calculate:

(i) the gravitational field at a distance $r(>R)$ from the centre of the cylinder;

(ii) the gravitational field at a distance $r(<R)$ from the centre of the cylinder.

(b) Calculate the period of revolution of a small satellite moving in a circular orbit of radius r very close to the surface of the cylinder.

10.4 Gauss's Law for gravitation is

$$\int_S \mathbf{g}.d\mathbf{S} = -4\pi GM$$

where G is the universal gravitational constant, **g** is the acceleration due to gravity, and the surface integral of **g** is over a surface S which contains a total mass M.

Derive an expression for the acceleration due to gravity at a point situated a distance r from the centre of a solid sphere of mass density ρ and radius R

(a) for $r > R$;

(b) for $r < R$.

10.5 A mass M is moving in a uniform circular orbit of radius R about a centre of gravitational attraction 0. The magnitude of the acceleration due to gravity at distance R from 0 is g. Answer the following questions.

(a) What is the torque about 0 of the gravitational force on M?

(b) What is the amount of inertia of the mass M about 0?

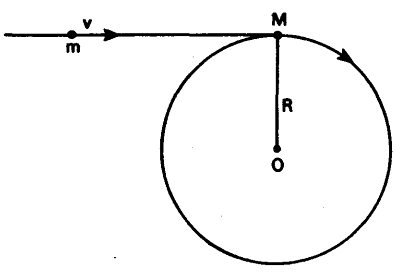


Fig. 10.5p

A small mass m is moving with speed v tangentially to the orbit of M, as shown in figure 10.5p. It strikes M when their velocities are parallel, and sticks to M. Answer the following questions.

(c) What is the total angular momentum of m and M about 0 before the collision?

(d) What is the total angular momentum of m and M about 0 after the collision?

(e) What is the instantaneous new angular *velocity* of m and M about 0?

(f) What is the total rotational kinetic energy before the collision?

(g) What is the total rotational kinetic energy instantaneously after the collision?

(h) What is the instantaneous new velocity, $\mathbf{v}'$, of M and m?

(i) The resulting combined mass takes on an *elliptical* orbit about 0. Give *brief* arguments to show that R is now the *closest* approach of this orbit to 0.

(j) The furthest distance of the new orbit from 0 is R' Using Newton's law of gravitation, express the gravitational acceleration at R' in terms of g, R, and R' *only*.

(k) Using the above expression, and conservation, of angular momentum, derive an expression for R' in terms of R, g, m, M, and v' [see (h)].

10.6 Suppose that the mass m in problem 10.5 is moving in the direction opposite to that shown in figure 10.5p. Answer parts (c) – (h) of problem 10.5 under this new condition.

(i) Give brief arguments to show that R is now the most distant part of the new elliptical orbit from 0.

(j) The closest approach of the new orbit to 0 is R'. Using Newton's law of gravitation, express the gravitational acceleration at R' in terms of g, R, and R' *only*.

(k) As for (k) of problem 10.5, using (h) of 10.6.

10.7 A small spaceship, of mass 2m (which can be considered as a particle), is in a circular orbit of radius r about a uniform spherical planet of radius R. The acceleration due to gravity at the surface of the planet is g. The spaceship separates into two equal halves of mass m: the relative velocity of separation is initially 2v, and the (small) impulse for separation is tangential to, and in the plane of, the orbit.

(a) Give physical reasons to show that the decelerated mass will now go into an elliptical orbit about the planet, with the point of separation being the furthest point on the new orbit from the planet.

(b) What is the orbit of the spaceship's centre of mass after separation?

(c) Find the maximum value of v for which the decelerated mass just fails to graze the planet on the side opposite to the spaceship's point of separation.

Appendix 1: Vector analysis

Frames of Reference or Co-ordinate Systems in Three-dimensional Euclidean[1] Space

Rectangular Cartesian Co-ordinate System K

Consists of three mutually perpendicular straight lines meeting at the origin, as indicated in figure A1.1. (It is right-handed, i.e. rotation from the positive x axis to the positive y axis of a right-hand screw will produce an advance in the direction of the positive z axis.)

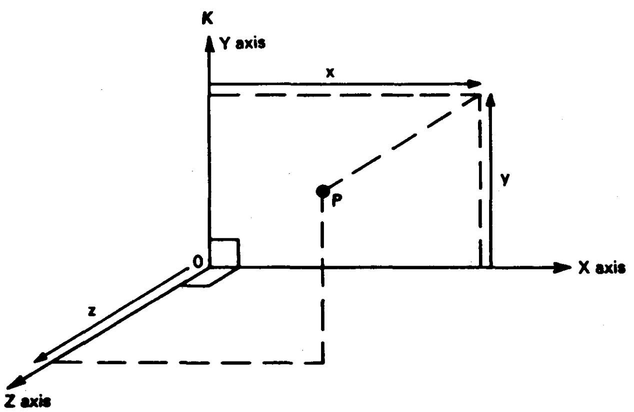


Fig. A1.1 Cartesian co-ordinates

[1] *Comment on Euclidicity* Rectangular Cartesian systems can be used only in Euclidean or flat space. One cannot use them in 'curved' spaces – consider a two-dimensional curved space such as the surface of a sphere.

It is an experimental fact that physical space is approximately Euclidean in our region.

Position of a Point with Respect to K

Definition: The position of P with respect to a frame K is specified by the ordered triple of numbers x, y, z measured as indicated.

Other Co-ordinate Systems

For example: spherical polar co-ordinates as in figure A1.2. The position of P is specified by r, θ, ϕ.

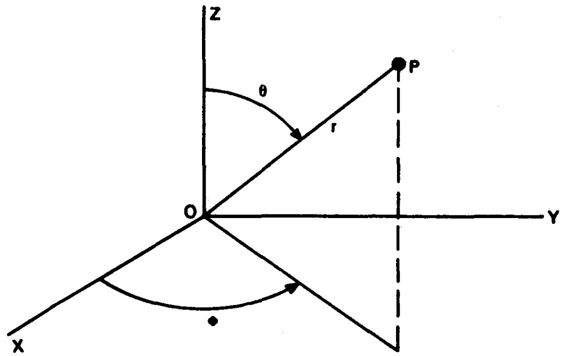


Fig. A1.2 Spherical polar co-ordinates

Note: We shall use only rectangular Cartesian systems below. Hence, by frame of reference or co-ordinate system we shall mean rectangular Cartesian system.

Co-ordinate Transformations between Rectangular Cartesian Frames (Stationary with Respect to Each Other)

Definition (see figure A1.3):

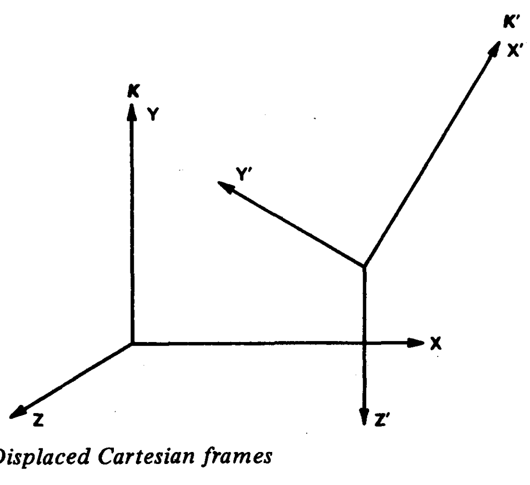


Fig. A1.3 Displaced Cartesian frames

Call x, x_1

y, x_2

z, x_3

then $x'_i = \sum_{j=1}^{3} a_{ij}x_j + b_i \qquad i = 1, 2, 3$ (A1.1)

(rotation constants: a_{ij}; translation constants: b_i)

where the a_{ij} and b_i are constants[2] which depend on the relative orientation of K′ with respect to K and the Euclidicity of space. Note that it is a linear co-ordinate transformation.
Example: If there is no displacement of origins and if the x_3 and x'_3 axis of K and K′ coincide, we have the situation in figure A1.4.

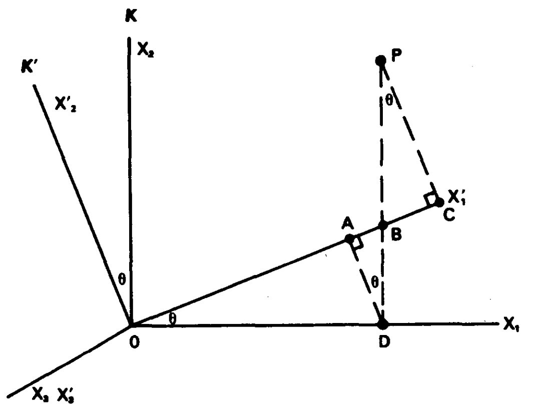


Fig. A1.4 Rotated Cartesian frames

Consider point P with co-ordinates $(x_1, x_2, 0)$ with respect to K
$(x'_1, x'_2, 0)$ with respect to K′

[2] We can write out the nine transformation constants a_{ij} explicitly in a square array called a matrix thus:

$$\begin{Bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Bmatrix}$$

Clearly

$$x_1' = OA + AB + BC$$
$$= OD \cos\theta + DB \sin\theta + BP \sin\theta$$
$$= OD \cos\theta + DP \sin\theta$$

But $OD = x_1$ and $DP = x_2$

$$\therefore \quad x_1' = x_1 \cos\theta + x_2 \sin\theta \qquad \text{(A1.2a)}$$

Similarly, we can show

$$x_2' = -x_1 \sin\theta + x_2 \cos\theta \qquad \text{(A1.2b)}$$

It is also clear that even if P has a non-zero x_3 component

$$x_3' = x_3 \text{ here.} \qquad \text{(A1.2c)}$$

Thus we see that the transformation constants, if K′ is oriented with respect to K as in the diagram, are, by comparing each of (A1.2a), (b), and (c) with (A1.1)

$$\begin{aligned} a_{11} &= \cos\theta, & a_{12} &= \sin\theta, & a_{13} &= 0 \\ a_{21} &= -\sin\theta, & a_{22} &= \cos\theta, & a_{23} &= 0 \\ a_{31} &= 0, & a_{32} &= 0, & a_{33} &= 1 \end{aligned} \qquad \text{(A1.3)}$$

or in matrix form,

$$\begin{Bmatrix} \cos\theta, & \sin\theta, & 0 \\ -\sin\theta, & \cos\theta, & 0 \\ 0, & 0, & 1 \end{Bmatrix}$$

Note again that we got (A1.2) from the orientation of K′ with respect to K, using properties of plane trigonometry and the fact that the coordinate systems were rectangular Cartesian. These last two features depend on the Euclidicity of space.

Scalar Quantity (or an Invariant)

Definition: Any quantity (it can be a function of position) whose numerical magnitude is the same in all frames (stationary), e.g. mass, time, distance. Clearly position is *not* a scalar.

A Defining Property of Euclidean Space

Consider any two points Q $(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3)$ and P (x_1, x_2, x_3) with respect to *any* K (rectangular Cartesian of course) (see figure A1.5). Then the distance $(\Delta S)^2$ between Q and P is a scalar* defined by

$$(\Delta S)^2 = (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2$$

$$= \sum_i \Delta x_i \Delta x_i \text{ (this is simply Pythagoras' theorem)}$$

$$\left(= \sum_j \Delta x'_j \Delta x'_j \right), \text{ etc.} \qquad \text{(A1.4)}$$

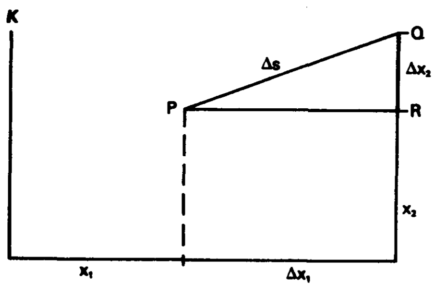


Fig. A1.5 Euclidean space

Note: (A1.4) is not true in curved space. Consider

$$\Delta S = QR = PR \text{ (figure A1.6)}$$

$$\therefore \quad (\Delta S)^2 \neq (QR)^2 + (PR)^2$$

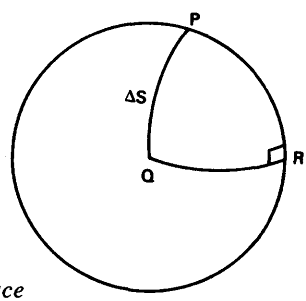


Fig. A1.6 Curved space

*That it is indeed a scalar will be evident after magnitudes of vectors have been discussed (page 189).

Transformation Law for Co-ordinate Differences Δx_i

From (A1.1) we deduce that

$$\Delta x'_i = \sum_j a_{ij} \Delta x_j \qquad \text{(A1.5)}$$

Relation between the a_{ij}:

From the fact that distance is a scalar given by (A1.4), i.e. from

$$\sum_i \Delta x'_i \Delta x'_i = \sum_k \Delta x_k \Delta x_k$$

we have, using (A1.5), that

$$\sum_i (\sum_j a_{ij} \Delta x_j \sum_k a_{ik} \Delta x_k) = \sum_k \Delta x_k \Delta x_k$$

i.e.

$$\sum_{j,k} (\sum_i a_{ij}\, a_{ik}) \Delta x_j \Delta x_k = \sum_k \Delta x_k \Delta x_k$$

This means that

$$\sum_i a_{ij}\, a_{ik} = 1 \text{ when } j = k$$

$$= 0 \text{ when } j \neq k$$

We write this:

$$\sum_i a_{ij}\, a_{ik} = \delta_{jk} \qquad \text{(A1.6)}$$

where δ_{jk}, called the Kronecker delta, is defined by

$$\delta_{jk} = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ if } j \neq k \end{cases}$$

Vectors

Definition: Consider the co-ordinate transformation $x'_i = \sum_j a_{ij}\, x_j$ between K′ and K.

Let A'_i and A_i ($i = 1, 2, 3$) be ordered sets of quantities with respect to K′ and K respectively. We say that A_i and A'_i are the *components* along the x_i' and x_i axes with respect to K and K′ respectively of the one *vector* **A** if and only if A_i' and A_i are related by

$$A_i' = \sum_j a_{ij} A_j \tag{A1.7}$$

i.e. if they transform like co-ordinate differences.

Note: Although A_1, A_2, A_3 specify **A** with respect to K, we have not yet defined precisely how **A** and A_1, A_2, A_3 are related.

Examples: Co-ordinates of a point are the components of the position vector (if we consider only rotation).

Co-ordinate differences are the components of the relative position vector.

Velocity components $v_i \equiv dx_i/dt$ (dt scalar) are the components of the velocity vector.

Acceleration components $a_i \equiv dv_i/dt$ are the components of the acceleration vector.

Force components $f_i \equiv ma_i$ (m scalar) are the components of the force vector.

Equality of Vectors

Definition: **A** = **B** if and only if $A_i = B_i$ $(i = 1, 2, 3)$ with respect to any one co-ordinate system K.

Significance of Vectors

If two vectors are equal in one co-ordinate system, they are equal in all co-ordinate systems.

Proof: Since $A_i' = \sum_i a_{ij} A_j$, $B_i' = \sum_i a_{ij} B_j$

then $A_j = B_j$ implies $A_i' = B_i'$.

Hence **A** = **B** in all co-ordinate systems if **A** = **B** in one.

Corollary: If all the components of a vector **A** vanish in one co-ordinate system (we then write **A** = 0), they vanish in all, i.e. **A** = 0 in all co-ordinate systems.

Multiplication of Vector by Scalar m

Definition: mA is defined as that mathematical object whose components are mA_i.

Property: mA is a vector.

Proof: Trivial.

Addition and Subtraction of Vectors

Definition: $\mathbf{A} \pm \mathbf{B}$ is defined as that mathematical object whose components are $A_i \pm B_i$.

Properties of Addition and Subtraction (Proofs Trivial)

(1) $\mathbf{A} \pm \mathbf{B}$ is a vector.
(2) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (addition commutative)
(3) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (addition associative)

Scalar Product of Two Vectors (or Dot Product)

Definition: The scalar product of **A** and **B**, written **A.B**, is defined by

$$\mathbf{A.B} = \sum_i A_i B_i \qquad \text{(A1.8)}$$

Property: The scalar product is a scalar, i.e. invariant.

Proof:

$$\begin{aligned} \sum_i A_i' B_i' &= \sum_i \left(\sum_j a_{ij} A_j \sum_k a_{ik} B_k\right) \\ &= \sum_{jk} \left(\sum_i a_{ij} a_{ik}\right) A_j B_k \\ &= \sum_{jk} \delta_{jk} A_j B_k, \text{ using (A1.6)} \\ &= \sum_j A_j B_j \end{aligned}$$

But $\sum_i A_i' B_i'$ is the scalar product of **A** and **B** with respect to K′

and $\sum_i A_i B_i$ is the scalar product of **A** and **B** with respect to K.

QED

Magnitude or Norm of a Vector

Definition: Magnitude of **A**, written A, is defined by

$$A = (\mathbf{A}.\mathbf{A})^{\frac{1}{2}}$$

$$= \sqrt{\sum_i A_i A_i}$$

Property: The magnitude of a vector is a scalar, i.e. invariant with respect to the frame of reference (from invariance of scalar product).

Unit Vector

Definition: Any vector, written $\hat{\mathbf{e}}$, say, whose magnitude is 1:

$$\text{i.e. } \hat{\mathbf{e}}.\hat{\mathbf{e}} = 1$$

Orthogonal Vectors

Definition: Any two non-zero vectors **A** and **B** (whose components are not all zero) such that **A.B** = 0 are said to be mutually orthogonal.

Note: This 'orthogonality' will shortly be seen to mean perpendicularity.

Basic Vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ (or $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$)

Definition: Consider a co-ordinate system K.

Let $\hat{\mathbf{e}}_1$ have components (1, 0, 0)
$\hat{\mathbf{e}}_2$ have components (0, 1, 0)
$\hat{\mathbf{e}}_3$ have components (0, 0, 1) with respect to K (A1.9)

Clearly $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ are mutually orthogonal unit vectors (i.e. $\hat{\mathbf{e}}_i.\hat{\mathbf{e}}_j = \delta_{ij}$) and we can regard them as unit vectors along the directions of the x, y, and z axes of K.

Expansion of any Vector in Terms of its Components with Respect to a Basis Set of Vectors

Any vector **A** can be written as a linear combination of the basis vectors $\hat{\mathbf{e}}_i$ of a frame of reference K, the coefficients being the components A_i of **A** with respect to K.

i.e. $$\mathbf{A} = \sum_i A_i \hat{\mathbf{e}}_i$$

i.e. $$A = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \tag{A1.10}$$

Proof: The only vectors which could not be written as linear combinations of the form $A = \alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2 + \alpha_3 \hat{e}_3$ would be those which are orthogonal to each of $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ (just as $\hat{e}_3$ cannot be written as a linear combination of $\hat{e}_1$ and $\hat{e}_2$).

By the definition of the three-dimensionality of space, there can be no vectors orthogonal to $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$. Hence every vector A can be written

$$A = \sum_i \alpha_1 \hat{e}_i$$

To show that the $\alpha_i = A_i$, scalar multiply $A = \sum_i \alpha_i \hat{e}_i$ by $\hat{e}_1$.

We get $$A.\hat{e}_1 = \sum_i \alpha_i \hat{e}_i . \hat{e}_1$$

i.e. $$A_1 \times 1 + A_2 \times 0 + A_3 \times 0 = \sum_i \alpha_i \delta_{i1}$$

i.e. $\alpha_1 = A_1$. Similarly $\alpha_2 = A_2$, $\alpha_3 = A_3$

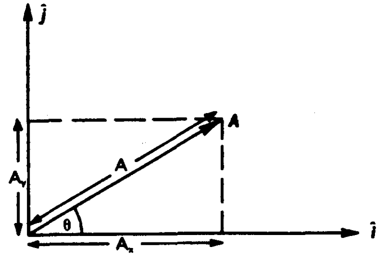


Fig. A1.7 Vector components

Comment: This expansion is very useful for the geometric interpretation of vectors. Thus (in two dimensions, to simplify drawing) we can depict $A = A_x \hat{i} + A_y \hat{j}$ as in figure A1.7, and we see that $A_x = A \cos\theta$, $A_y = A \sin\theta$. Also, for instance, since the addition of **A** and **B** can now be expressed by $A + B = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$ we have, in two dimensions, the situation shown in figure A1.8. This shows that our definition of the sum of two vectors is equivalent to the parallelogram rule for addition (see figure A1.8).

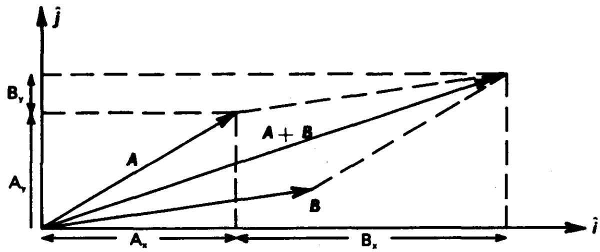


Fig. A1.8 Vector addition

Alternative Expression for Scalar Product
For any two vectors **A** and **B**,

$$\mathbf{A.B} = \mathrm{AB} \cos \theta \text{ (figure A1.9)} \qquad \text{(A1.11)}$$

where A and B are the magnitudes of **A** and **B** and θ is the angle between **A** and **B**.

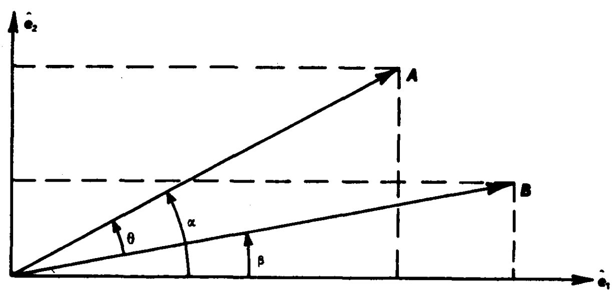


Fig. A1.9 Scalar product

Proof: Choose a co-ordinate system so that the x_1 and x_2 axes are in the plane of **A** and **B**.

Then $$\mathbf{A.B} = A_1B_1 + A_2B_2$$

$$= \mathrm{A} \cos \alpha \, \mathrm{B} \cos \beta + \mathrm{A} \sin \alpha \, \mathrm{B} \sin \mathrm{B}$$

$$= \mathrm{AB} (\cos \alpha \cos \beta + \sin \alpha \sin \beta)$$

$$= \mathrm{AB} \cos (\alpha - \beta)$$

i.e. $$\mathbf{A.B} = \mathrm{AB} \cos \theta$$

And since A.B is a scalar and the right-hand side involves only scalars, it is true in all co-ordinate systems and not just the special one we chose.

Vector Product (or Cross Product)

Definition: The vector product, written $\mathbf{C} \equiv \mathbf{A} \times \mathbf{B}$, of any two vectors **A** and **B** with components A_i and B_i with respect to a frame K, is defined as that mathematical object whose components C_i with respect to K are

$$C_1 = A_2 B_3 - A_3 B_2$$

$$C_2 = A_3 B_1 - A_1 B_3 \text{ (Note cyclic } 1 \to 2 \to 3 \to 1.)$$

$$C_3 = A_1 B_2 - A_2 B_1 \qquad \text{(A1.12)}$$

The Vector Product of Two Vectors Is a Vector

Proof: We show that the components of **C** defined above transform like the components of a vector. Consider a co-ordinate system K so that **A** and **B** lie in the $(x_1\ x_2)$ plane. This is no restriction. We then have $A_3 = B_3 = 0$ and therefore, from the above definition, $C_1 = C_2 = 0$. Consider now a rotation of a co-ordinate system $K \to K'$ around the x_3 axis of K. This is a restriction, but the proof is too hard without it. The coefficients of the co-ordinate transformation have already been derived [see equation (A1.3)], namely

$$\begin{Bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Bmatrix} = \begin{Bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

Hence
$$A_1' = A_1 \cos\theta + A_2 \sin\theta + 0$$
$$A_2' = -A_1 \sin\theta + A_2 \cos\theta + 0$$
$$A_3' = 0 \times 0 + 1 \times 0 = 0$$

and similarly for B_i'.

Hence
$$C_1' \equiv A_2' B_3' - A_3' B_2' = 0$$

because
$$B_3' = B_3 = 0$$

and
$$A_3' = A_3 = 0$$

Likewise $$C_2' = 0$$

Hence we can write

$$
\begin{array}{llll}
C_1' = a_{11} & C_1 + a_{12} & C_2 + a_{13} & C_3 \\
 & \searrow & \searrow & \searrow \\
 & 0 & 0 & 0 \\
 & \uparrow & \uparrow & \nearrow \\
C_2' = a_{21} & C_1 + a_{22} & C_2 + a_{23} & C_3
\end{array}
\qquad
\begin{array}{r}
(i) \\ \\ \\ \\ (ii)
\end{array}
$$

Finally,

$$C_3' = A_1' B_2' - A_2' B_1'$$

$$= (A_1 \cos\theta + A_2 \sin\theta)(-B_1 \sin\theta + B_2 \cos\theta)$$

$$-(-A_1 \sin\theta + A_2 \cos\theta)(B_1 \cos\theta + B_2 \sin\theta)$$

$$= A_1 B_2 \cos^2\theta - A_2 B_1 \sin^2\theta - A_1 B_1 \sin\theta \cos\theta + A_2 B_2 \sin\theta \cos\theta$$

$$+ A_1 B_1 \sin\theta \cos\theta - A_2 B_2 \sin\theta \cos\theta + A_1 B_2 \sin^2\theta - A_2 B_1 \cos^2\theta$$

$$= A_1 B_2 - A_2 B_1$$

$$= C_3$$

Hence we can write

$$
\begin{array}{llll}
C_3' = a_{31} & C_1 + a_{32} & C_2 + a_{33} & C_3 \\
 & \downarrow & \downarrow & \searrow \\
 & 0 & 0 & 1
\end{array}
\qquad (iii)
$$

Thus from (i), (ii), and (iii) we see that the C_i transform like the components of a vector. Hence the vector product is indeed a vector.

Expansion of Vector Product in Terms of Components and Basis Vectors

From the definition of **A** x **B** and expansion (A1.10) we see that

$$\mathbf{A} \times \mathbf{B} = (A_2 B_3 - A_3 B_2)\hat{e}_1 + (A_3 B_1 - A_1 B_3)\hat{e}_2 + (A_1 B_2 - A_2 B_1)\hat{e}_3 \qquad (A1.13)$$

or more conveniently

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Alternative Expression for Vector Product

$$\mathbf{A} \times \mathbf{B} = AB \sin\theta\, \hat{\mathbf{n}} \qquad \text{(A1.14)}$$

where θ is the angle between **A** and **B** and $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane of **A** and **B** and the direction of advance of a right-hand screw being rotated from **A** to **B**.

Proof: Choose a co-ordinate system so that **A** and **B** are in the $(x_1\ x_2)$ plane, i.e. so that $A_3 = B_3 = 0$ (figure A1.10).

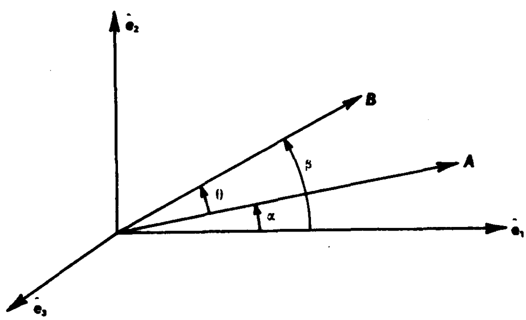


Fig. A1.10 Vector product

Then, from (A1.13),

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \hat{e}_3(A_1B_2 - A_2B_1)\\ &= \hat{e}_3\,(A\cos\alpha\, B\sin\beta - A\sin\alpha\, B\cos\beta)\\ &= \hat{e}_3\, AB\sin(\beta-\alpha)\end{aligned}$$

$$\therefore \qquad \mathbf{A} \times \mathbf{B} = \hat{e}_3\, AB\sin\theta$$

$$\text{i.e.} \qquad \mathbf{A} \times \mathbf{B} = AB\sin\theta\, \hat{\mathbf{n}}$$

Since LHS is a vector and RHS is a vector multiplied by scalars, the relation is true in all co-ordinate systems.

Example of Vector Product

If a charge q moves with velocity v in a magnetic field **B** then it experiences a force **F** given by **F** = qv x **B**.

Properties of Vector Product

From (A1.14) we see that

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

and

$$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$$

$$\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$$

$$\hat{e}_3 \times \hat{e}_1 = \hat{e}_2$$

$$\hat{e}_1 \times \hat{e}_1 = \hat{e}_2 \times \hat{e}_2 = \hat{e}_3 \times \hat{e}_3 = 0$$

Covariance

Definition: An equation is said to be covariant with respect to given co-ordinate transformations if its algebraic form is unchanged by those transformations.

Ultimate Significance of Vectors

Vector equations are covariant with respect to co-ordinate transformations between co-ordinate systems which are stationary with respect to each other. This is clear whether we write the equation in component form or in abstract vector form.

For example, if **A**, **B**, and **C** are vectors,

and if $A_i + B_i = C_i$ with respect to a given K,

then $A_i' + B_i' = C_i'$ with respect to all K′,

and we can write **A** + **B** = **C** irrespective of the co-ordinate system.

Back to Physics

Since the basic laws of physics are vector equations, we can formulate what we will call a *Principle of Relativity,* albeit a *trivial one,* namely: The laws of physics are covariant with respect to co-ordinate transformations between frames of reference which are at rest with respect to each other.

The starting point of the *Theory of Relativity* is the question: Is this still true if the frames are moving with respect to each other? We *require* this to be true for the Newtonian laws of mechanics under Galilean Transformations.

Some Useful Vector Identities

$$A.B \times C = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$A \times (B \times C) = (A.C)B - (A.B)C$$

$$(A \times B) \times C = (A.C)B - (B.C)A$$

$$\frac{d}{dt}(fA) = \frac{df}{dt}A + f\frac{dA}{dt} \text{ (f a scalar)}$$

$$\frac{d}{dt}(A.B) = \frac{dA}{dt}.B + A.\frac{dB}{dt}$$

$$\frac{d}{dt}(A \times B) = \frac{dA}{dt} \times B + A \times \frac{dB}{dt}$$

Appendix 2: The principle of equivalence

Clock Rates in a Uniform Gravitational Field

We consider two clocks, A and B, separated by a distance h, and subject to a uniform *upward* acceleration g (figure A2.1). These clocks pass close to a third clock C in an inertial frame. All three clocks have previously been synchronized to make sure that they keep the same time when alone in an inertial frame and at rest with respect to each other.

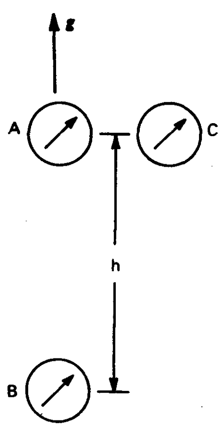


Fig. A2.1 Accelerating clocks

The clocks A and B have upward velocities v_A and v_B respectively as they pass C in succession. We assume that these velocities are very much less than c, the velocity of light, and proceed to apply the special theory of relativity to the situation. We want to find the relation between a fixed interval of time T_C as measured by C, and the corresponding intervals T_A and T_B as measured by A and B as they each pass by C.

We have
$$T_A = T_C\left(1 - \frac{V_A^2}{c^2}\right)^{-\frac{1}{2}} \doteq T_C\left(1 + \frac{V_A^2}{2c^2}\right) \tag{A2.1}$$

and similarly
$$T_B = T_C\left(1 - \frac{v_B^2}{c^2}\right)^{-\frac{1}{2}} \doteq T_C\left(1 + \frac{v_B^2}{2c^2}\right) \tag{A2.2}$$

Hence in the approximation v_A, v_B, both $\ll c^2$,

$$T_B \doteq T_A\left(1 + \frac{v_B^2 - v_A^2}{2c^2}\right) = T_A\left(1 + \frac{gh}{c^2}\right) \tag{A2.3}$$

since
$$v_B^2 = v_A^2 + 2gh$$

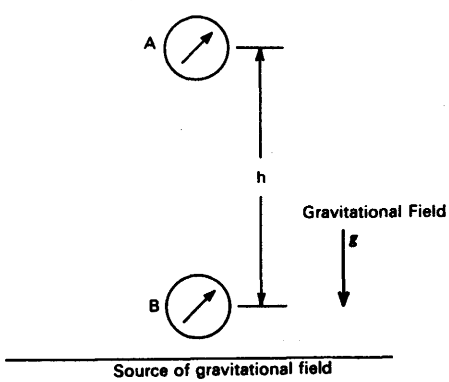


Fig. A2.2 Clocks in a gravitational field

By the principle of equivalence, the result (A2.3) should apply if the clocks A and B are separated and at rest in a uniform gravitational field directed oppositely to the original upward acceleration. That is, for the situation of figure A2.2, we must also have

$$T_B = T_A\left(1 + \frac{gh}{c^2}\right) = T_A\left(1 - \frac{V_A - V_B}{c^2}\right) \tag{A2.4}$$

where V_A and V_B are the gravitational potentials at A and B respectively [see S. Chandrasekhar, *An. J. Phys.* **40**, 224 (1972); L.I. Schiff, *ibid.* 28, 340 (1960)].

Appendix 3: The 'curl' (∇x) operator and conservative forces

We saw in section 5.4 that if we had a scalar potential function V(x, y, z), then the force **F**(x, y, z) was given by

$$\mathbf{F} = -\nabla V = -\mathbf{grad}\ V \qquad \text{(A3.1)}$$

where $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ is the *vector operator,* 'del'. Consider now the operation ∇x, which we define as

$$\mathbf{curl}\ \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \qquad \text{(A3.2)}$$

for some vector function of position F(x, y, z). If we took seriously the the way ∇ 'looks like' a vector, we should guess that '$\nabla \times \nabla$' is a *zero* operator, from our knowledge of the vector product. It is in fact a very simple exercise to show that, for any scalar function V(x, y, z),

$$\nabla \times (\nabla V) = (\nabla \times \nabla)V = \mathbf{curl}\ \mathbf{grad}\ V = 0 \qquad \text{(A3.3)}$$

From this, and from the relation (3.1), we conclude that for any vector *(force)* field **F** with a scalar potential,

$$\mathbf{curl}\ \mathbf{F} = 0$$

This gives us an immediate method of determining whether a force field **F**(x, y, z) is conservative or not; if **curl F** = 0, then **F** is the gradient of some scalar function V, i.e. we have a potential function and **F** is *conservative.*

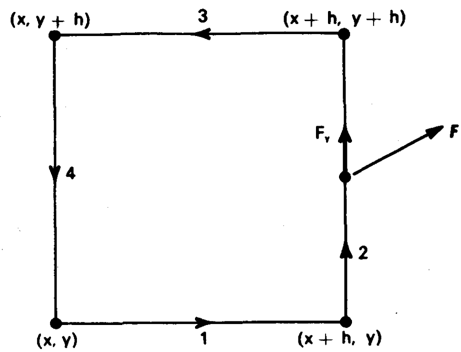


Fig. A3.1 Elementary closed path

Let us see why this is so *physically*. Remember that a conservative field is one in which the line-integral $\oint \mathbf{F}.d\boldsymbol{\ell}$ around *any* closed path is zero. Consider a little square of side h, with its side parallel to the x and y axes (figure A3.1). Along side 1 the tangential component of **F** is the x component. Taking its value at the middle of side 1 as F_X, the line-integral along side 1 is approximately $F_X h$. For side 3 the x component of **F** has increased to $[F_X + h\,(\partial F_X/\partial y)]$. Noticing the opposite direction of integration on side 3 as compared to 1, we see that the line-integral along 3 is

$$-\left(F_X + h\,\frac{\partial F_X}{\partial y}\right)h$$

so the pair of sides 1 and 3 contributes a net amount of

$$-h^2\,\frac{\partial F_X}{\partial y}$$

Similarly, the pair of sides 2 and 4 contributes an amount

$$h^2\,\frac{\partial F_Y}{\partial x}$$

to the line-integral. The total value of the line-integral around the path 1234 anticlockwise is

$$h^2\left(\frac{\partial F_Y}{\partial x} - \frac{\partial F_X}{\partial y}\right)$$

which we can rewrite as

$$(\nabla \times \mathbf{F})_z \, \delta A$$

where $\delta A = h^2$ is the area of the square, and $(\nabla \times \mathbf{F})_z$ is the z component of $\nabla \times \mathbf{F}$ as defined in (A3.2). If we associate the direction **k** with the normal to the area enclosed by the path in the anticlockwise direction (the right-hand-screw rule), we may write

$$(\nabla \times \mathbf{F})_z . \delta A = \oint \mathbf{F}.d\boldsymbol{\ell}$$

for this path 1234 around the elementary square. So that, in the limit $h \to 0$, the components of $\nabla \times \mathbf{F}$ measure the value of the line-integral of **F** around infinitesimal closed paths in planes normal to the co-ordinate axes. So taking the curl of a force **F** effectively samples the line-integrals around *all* infinitesimal closed paths. If the result is everywhere zero, **F** must therefore be conservative by definition.

The final total result $\oint_L \mathbf{F}.d\boldsymbol{\ell} = \int_S (\text{curl } \mathbf{F}).d\mathbf{A}$ is known as *Stokes's theorem;* S is the surface surrounded by the closed path L.

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