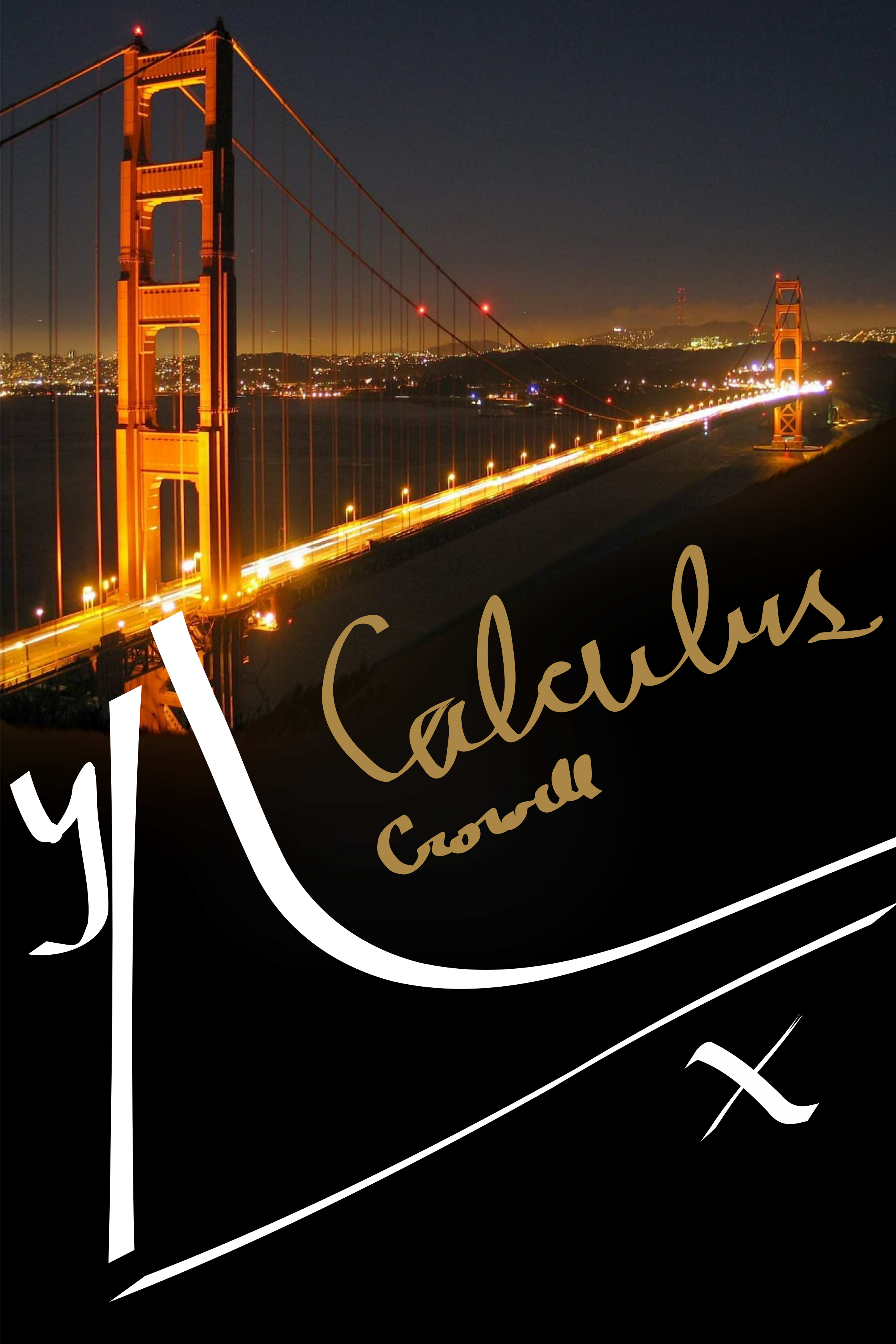
Calculus
Crowell
y
x

Light and Matter
Fullerton, California
www.lightandmatter.com

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rev. December 31, 2005

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Calculus isn't a hard subject.

Algebra is hard. I still remember my encounter with algebra. It was my first taste of abstraction in mathematics, and it gave me quite a few black eyes and bloody noses.

Geometry is hard. For most people, geometry is the first time they have to do proofs using formal, axiomatic reasoning.

I teach physics for a living. Physics is hard. There's a reason that people believed Aristotle's bogus version of physics for centuries: it's because the real laws of physics are counterintuitive.

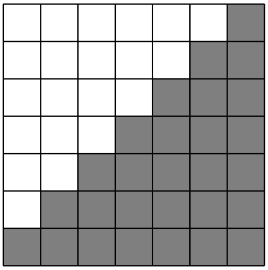
Calculus, on the other hand, is a very straightforward subject that rewards intuition, and can be easily visualized. Silvanus Thompson, author of one of the most popular calculus texts ever written, opined that "considering how many fools can calculate, it is surprising that it should be thought either a difficult or a tedious task for any other fool to master the same tricks."

Since I don't teach calculus, I can't require anyone to read this book. For that reason, I've written it so that you can go through it and get to the dessert course without having to eat too many Brussels sprouts and Lima beans along the way. The development of any mathematical subject involves a large number of boring details that have little to do with the main thrust of the topic. These details I've relegated to a chapter in the back of the book, and the reader who has an interest in mathematics as a career — or who enjoys a nice heavy pot roast before moving on to dessert — will want to read those details when the main text suggests the possibility of a detour.

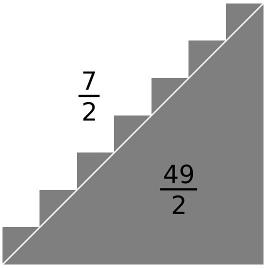
1 Rates of Change

1.1 Change in discrete steps

Toward the end of the eighteenth century, a German elementary school teacher decided to keep his pupils busy by assigning them a long, boring arithmetic problem. To oversimplify a little bit (which is what textbook authors always do when they tell you about history), I'll say that the assignment was to add up all the numbers from one to a hundred. The children set to work on their slates, and the teacher lit his pipe, confident of a long break. But almost immediately, a boy named Carl Friedrich Gauss brought up his answer: 5,050.



a / Adding the numbers from 1 to 7.



b / A trick for finding the sum.

Figure a suggests one way of solving this type of problem. The filled-in columns of the graph represent the numbers from 1 to 7, and adding them up means finding the area of the shaded region. Roughly half the square is shaded in, so if we want only an approximate solution, we can simply calculate $7^2/2 = 24.5$.

But, as suggested in figure b, it's not much more work to get an exact result. There are seven sawteeth sticking out out above the diagonal, with a total area of 7/2, so the total shaded area is $(7^2 + 7)/2 = 28$. In general, the sum of the first n numbers will be $(n^2 + n)/2$, which explains Gauss's result: $(100^2 + 100)/2 = 5,050$.

Two sides of the same coin

Problems like this come up frequently. Imagine that each household in a certain small town sends a total of one ton of garbage to the dump every year. Over time, the garbage accumulates in the dump, taking up more and more space.

c / Carl Friedrich Gauss (1777-1855), a long time after graduating from elementary school.

Let's label the years as $n = 1, 2, 3, \ldots$, and let the function[1] $x(n)$ represent the amount of garbage that has accumulated by the end of year n. If the population is constant, say 13 households, then garbage accumulates at a constant rate, and we have $x(n) = 13n$.

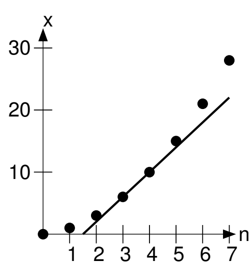
But maybe the town's population is growing. If the population starts out as 1 household in year 1, and then grows to 2 in year 2, and so on, then we have the same kind of problem that the young Gauss solved. After 100 years, the accumulated amount of garbage will be 5,050 tons. The pile of refuse grows more and more every year; the rate of change of x is not constant. Tabulating the examples we've done so far, we have this:

rate of change	*accumulated result*
13	$13n$
n	$(n^2+n)/2$

The rate of change of the function x can be notated as $\dot{x}$. Given the function $\dot{x}$, we can always determine the function x for any value of n by doing a running sum.

Likewise, if we know x, we can determine $\dot{x}$ by subtraction. In the example where $x = 13n$, we can find $\dot{x} = x(n) - x(n-1) = 13n - 13(n-1) = 13$. Or if we knew that the accumulated amount of garbage was given by $(n^2+n)/2$, we could calculate the town's population like this:

$$\frac{n^2+n}{2} - \frac{(n-1)^2+(n-1)}{2}$$
$$= \frac{n^2+n-\left(n^2+2n-1-n+1\right)}{2}$$
$$= n$$



d / $\dot{x}$ is the slope of x.

The graphical interpretation of this is shown in figure d: on a

[1] Recall that when x is a function, the notation $x(n)$ means the output of the function when the input is n. It doesn't represent multiplication of a number x by a number n.

graph of $x = (n^2 + n)/2$, the slope of the line connecting two successive points is the value of the function $\dot{x}$.

In other words, the functions x and $\dot{x}$ are like different sides of the same coin. If you know one, you can find the other — with two caveats.

First, we've been assuming implicitly that the function x starts out at $x(0) = 0$. That might not be true in general. For instance, if we're adding water to a reservoir over a certain period of time, the reservoir probably didn't start out completely empty. Thus, if we know $\dot{x}$, we can't find out everything about x without some further information: the starting value of x. If someone tells you $\dot{x} = 13$, you can't conclude $x = 13n$, but only $x = 13n + c$, where c is some constant. There's no such ambiguity if you're going the opposite way, from x to $\dot{x}$. Even if $x \neq 0$, we still have $\dot{x} = 13n + c - [13(n-1) + c] = 13$.

Second, it may be difficult, or even impossible, to find a *formula* for the answer when we want to determine the running sum x given a formula for the rate of change $\dot{x}$. Gauss had a flash of insight that led him to the result $(n^2 + n)/2$, but in general we might only be able to use a computer spreadsheet to calculate a number for the running sum, rather than an equation that would be valid for all values of n.

Some guesses

Even though we lack Gauss's genius, we can recognize certain patterns. One pattern is that if $\dot{x}$ is a function that gets bigger and bigger, it seems like x will be a function that grows even faster than $\dot{x}$. In the example of $\dot{x} = n$ and $x = (n^2+n)/2$, consider what happens for a large value of n, like 100. At this value of n, $\dot{x} = 100$, which is pretty big, but even without pawing around for a calculator, we know that x is going to turn out really really big. Since n is large, n^2 is quite a bit bigger than n, so roughly speaking, we can approximate $x \approx n^2/2 = 5,000$. 100 may be a big number, but 5,000 is a lot bigger. Continuing in this way, for $n = 1000$ we have $\dot{x} = 1000$, but $x \approx 500,000$ — now x has far outstripped $\dot{x}$. This can be a fun game to play with a calculator: look at which functions grow the fastest. For instance, your calculator might have an x^2 button, an e^x button, and a button for $x!$ (the factorial function, defined as $x! = 1 \cdot 2 \cdot \ldots \cdot x$, e.g., $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$). You'll find that 50^2 is pretty big, but e^{50} is incomparably greater, and 50! is so big that it causes an error.

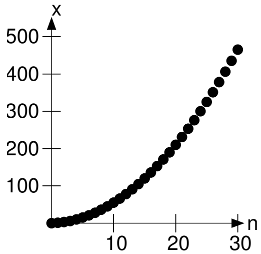
All the x and $\dot{x}$ functions we've seen so far have been polynomials. If x is a polynomial, then of course we can find a polynomial for $\dot{x}$ as well, because if x is a polynomial, then $x(n) - x(n-1)$ will be one too. It also looks like every polynomial we could choose for $\dot{x}$ might also correspond to an x that's a polynomial. And not only that, but it looks as though there's a pattern in the power of n. Suppose x is a polynomial, and the highest power of n it contains is a certain number — the "order" of the polynomial. Then $\dot{x}$ is a polynomial of that order minus one. Again, it's fairly easy to prove this going one way, passing from x to $\dot{x}$, but more difficult to prove the opposite relationship: that if $\dot{x}$ is a polynomial of a certain order, then x must be a polynomial with an order that's greater by one.

We'd imagine, then, that the running sum of $\dot{x} = n^2$ would be a polynomial of order 3. If we calculate $x(100) = 1^2 + 2^2 + \ldots + 100^2$ on a computer spreadsheet, we get 338,350, which looks suspiciously close to $1,000,000/3$. It looks like $x(n) = n^3/3 + \ldots$, where the dots represent terms involving lower powers of n such as n^2.

1.2 Continuous change

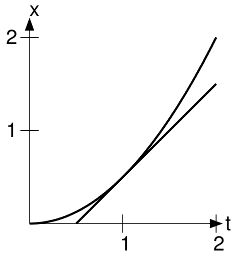
Did you notice that I sneaked something past you in the example of water filling up a reservoir? The x and $\dot{x}$ functions I've been using as examples have all been functions defined on the integers, so they represent change that happens in discrete steps, but the flow of water into a reservoir is smooth or continuous. Or is it? Water is made out of molecules, after all. It's just that water molecules are so small that we don't notice them as individuals. Figure f shows a graph that is discrete, but almost appears continous because the scale has been chosen so that the points blend together visually.

e / Isaac Newton (1643-1727)



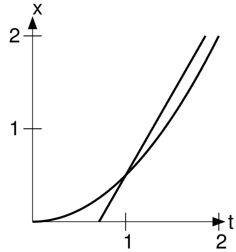
f / On this scale, the graph of $(n^2 + n)/2$ appears almost continuous.

The physicist Isaac Newton started thinking along these lines in the 1660's, and figured out ways of analyzing x and $\dot{x}$ functions that were truly continuous. The notation $\dot{x}$ is due to him (and he only used it for continuous functions). Because he was dealing with the continous *flow* of change, he called his new set of mathematical techniques the method of *fluxions*, but nowadays it's known as the calculus.



g / The function $x(t) = t^2/2$, and its tangent line at the point $(1, 1/2)$.

Newton was a physicist, and he needed to invent the calculus as part of his study of how objects move. If an object is moving in one dimension, we can specify its position with a variable x, and x will then be a function of time, t. The rate of change of its position, $\dot{x}$, is its speed, or velocity. Earlier experiments by Galileo had established that when a ball rolled down a slope, its position was proportional to t^2, so Newton inferred that a graph like figure g would be typical for any object moving under the influence of a constant force. (It could be $7t^2$, or $t^2/42$, or anything else proportional to t^2, depending on the force acting on the object and the object's mass.)



h / This line isn't a tangent line: it crosses the graph.

Because the functions are continuous, not discrete, we can no longer define the relationship between x and $\dot{x}$ by saying x is a running sum of $\dot{x}$'s, or that $\dot{x}$ is the difference between two successive x's. But we already found a geometrical relationship between the two functions in the discrete case, and that can serve as our definition for the continuous case: x is the area under the graph of $\dot{x}$, or, if you like, $\dot{x}$ is the slope of the tangent line on the graph of x. For now we'll concentrate on the slope idea.

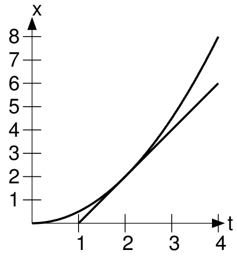
The tangent line is defined as the line that passes through the graph at a certain point, but, unlike the one in figure h, doesn't cut across the graph.[2] By measuring with a ruler on figure g, we find that the slope is very close to 1, so evidently $\dot{x}(1) = 1$. To prove this, we construct the function representing the line: $\ell(t) = t - 1/2$. We want to prove that this line doesn't cross the graph of $x(t) = t^2/2$. The dif-

[2] For a more formal definition, see page 81.

ference between the two functions, $x - \ell$, is the polynomial $t^2/2 - t + 1/2$, and this polynomial will be zero for any value of t where the line touches or crosses the curve. We can use the quadratic formula to find these points, and the result is that there is only one of them, which is $t = 1$. Since $x - \ell$ is positive for at least some points to the left and right of $t = 1$, and it only equals zero at $t = 1$, it must never be negative, which means that the line always lies below the curve, never crossing it.

A derivative

That proves that $\dot{x}(1) = 1$, but it was a lot of work, and we still don't want to that amount of work to evaluate $\dot{x}$ at every value of t. There's a way to avoid all that, and find a formula for $\dot{x}$. Compare figures g and i. They're both graphs of the same function, and they both look the same. What's different? The only difference is the scales: in figure i, the t axis has been shrunk by a factor of 2, and the x axis by a factor of 4. The graph looks the same, because doubling t quadruples $t^2/2$. The tangent line here is the tangent line at $t = 2$, not $t = 1$, and although it looks like the same line as the one in figure g, it isn't, because the scales are different. The line in figure g had a slope of rise/run $= 1/1 = 1$, but this one's slope is $4/2 = 2$. That means $\dot{x}(2) = 2$. In general, this scaling argument shows that $\dot{x}(t) = t$ for any t.



i / The function $t^2/2$ again. How is this different from figure g?

This is called *differenting*: finding a formula for the function $\dot{x}$, given a formula for the function x. The term comes from the idea that for a discrete function, the slope is the difference between two successive values of the function. The function $\dot{x}$ is referred to as the *derivative* of the function x, and the art of differentiating is differential calculus. The opposite process, computing a formula for x when given $\dot{x}$, is called integrating, and makes up the field of integral calculus; this terminology is based on the idea that computing a running sum is like putting together (integrating) many little pieces.

Note the similarity between this result for continuous functions,

$$x = t^2/2 \qquad \dot{x} = t \qquad ,$$

and our earlier result for discrete ones,

$$x = (n^2 + n)/2 \qquad \dot{x} = n \qquad .$$

The similarity is no coincidence. A continuous function is just a smoothed-out version of a discrete one. For instance, the continuous version of the staircase function shown in figure b on page 7 would simply be a triangle without the saw teeth sticking out; the area of those ugly sawteeth is what's represented by the $n/2$ term in the discrete result $x = (n^2 + n)/2$, which is the only thing that makes it different from the continuous result $x = t^2/2$.

Properties of the derivative

It follows immediately from the definition of the derivative that multiplying a function by a constant multiplies its derivative by the same constant, so for example since we know that the derivative of $t^2/2$ is t, we can immediately tell that the derivative of t^2 is $2t$, and the derivative of $t^2/17$ is $2t/17$.

Also, if we add two functions, their derivatives add. To give a good example of this, we need to have another function that we can differentiate, one that isn't just some multiple of t^2. An easy one is t: the derivative of t is 1, since the slope of the graph of $x = t$ is a line with a slope of 1, and the tangent line lies right on top of the original line.

The derivative of a constant is zero, since a constant function's graph is a horizontal line, with a slope of zero.

Example 1

The derivative of $5t^2 + 2t$ is the derivative of $5t^2$ plus the derivative of $2t$, since derivatives add. The derivative of $5t^2$ is 5 times the derivative of t^2, and the derivative of $2t$ is 2 times the derivative of t, so putting everything together, we find that the derivative of $5t^2 + 2t$ is $(5)(2t) + (2)(1) = 10t + 2$.

Example 2

▷ An insect pest from the United States is inadvertently released in a village in rural China. The pests spread outward at a rate of s kilometers per year, forming a widening circle of contagion. Find the number of square kilometers per year that become newly infested. Check that the units of the result make sense. Interpret the result.

▷ Let t be the time, in years, since the pest was introduced. The radius of the circle is $r = st$, and its area is $a = \pi r^2 = \pi(st)^2$. To make this look like a polynomial, we have to rewrite this as $a = (\pi s^2)t^2$. The derivative is

$$\dot{a} = (\pi s^2)(2t)$$
$$\dot{a} = (2\pi s^2)t$$

The units of s are km/year, so squaring it gives $\text{km}^2/\text{year}^2$. The 2 and the π are unitless, and multiplying by t gives units of km^2/year, which is what we expect for $\dot{a}$, since it represents the number of square kilometers per year that become infested.

Interpreting the result, we notice a couple of things. First, the rate of infestation isn't constant; it's proportional to t, so people might not pay so much attention at first, but later on the effort required to combat the problem will grow more and more quickly. Second, we notice that the result is proportional to s^2. This suggests that anything that could be done to reduce s would be very helpful. For instance, a measure that cut s in half would reduce $\dot{a}$ by a factor of four.

Higher-order polynomials

So far, we have the following results for polynomials up to order 2:

function	*derivative*
1	0
t	1
t^2	$2t$

Interpreting 1 as t^0, we detect what seems to be a general rule, which is that the derivative of t^k is kt^{k-1}. The proof is straightforward but not very illuminating if carried out with the methods developed in this chapter, so I've relegated it to page 81. It can be proved much more easily using the methods of chapter 2.

Example 3

▷ If $x = 2t^7 - 4t + 1$, find $\dot{x}$.

▷ This is similar to example 1, the only difference being that we can now handle higher powers of t. The derivative of t^7 is $7t^6$, so we have

$$\begin{aligned}\dot{x} &= (2)(7t^6) + (-4)(1) + 0\\ &= 14t^6 + -4\end{aligned}$$

The second derivative

I described how Galileo and Newton found that an object subject to an external force, starting from rest, would have a velocity $\dot{x}$ that was proportional to t, and a position x that varied like t^2. The proportionality constant for the velocity is called the acceleration, a, so that $\dot{x} = at$ and $x = at^2/2$. For example, a sports car accelerating from a stop sign would have a large acceleration, and its velocity at at a given time would therefore be a large number. The acceleration can be thought of as the derivative of the derivative of x, written $\ddot{x}$, with two dots. In our example, $\ddot{x} = a$. In general, the acceleration doesn't need to be constant. For example, the sports car will eventually have to stop accelerating, perhaps because the backward force of air friction becomes as great as the force pushing it forward. The total force acting on the car would then be zero, and the car would continue in motion at a constant speed.

Example 4

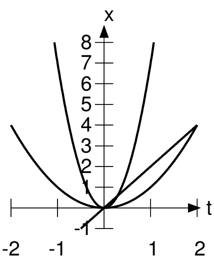
Suppose the pilot of a blimp has just turned on the motor that runs its propeller, and the propeller is spinning up. The resulting force on the blimp is therefore increasing steadily, and let's say that this causes the blimp to have an acceleration $\ddot{x} = 3t$, which increases steadily with time. We want to find the blimp's velocity and position as functions of time.

For the velocity, we need a polynomial

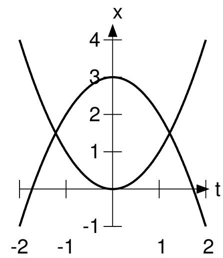
whose derivative is $3t$. We know that the derivative of t^2 is $2t$, so we need to use a function that's bigger by a factor of $3/2$: $\dot{x} = (3/2)t^2$. In fact, we could add any constant to this, and make it $\dot{x} = (3/2)t^2 + 14$, for example, where the 14 would represent the blimp's initial velocity. But since the blimp has been sitting dead in the air until the motor started working, we can assume the initial velocity was zero. Remember, any time you're working backwards like this to find a function whose derivative is some other function (integrating, in other words), there is the possibility of adding on a constant like this.

Finally, for the position, we need something whose derivative is $(3/2)t^2$. The derivative of t^3 would be $3t^2$, so we need something half as big as this: $x = t^3/2$.

The second derivative can be interpreted as a measure of the curvature of the graph, as shown in figure j. The graph of the function $x = 2t$ is a line, with no curvature. Its first derivative is 2, and its second derivative is zero. The function t^2 has a second derivative of 2, and the more tightly curved function $7t^2$ has a bigger second derivative, 14.



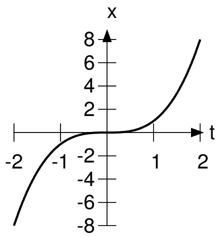
j / The functions $2t$, t^2 and $7t^2$.



k / The functions t^2 and $3 - t^2$.

Positive and negative signs of the second derivative indicate concavity. In figure k, the function t^2 is like a cup with its mouth pointing up. We say that it's "concave up," and this corresponds to its positive second derivative. The function $3-t^2$, with a second derivative less than zero, is concave down. Another way of saying it is that if you're driving along a road shaped like t^2, going in the direction of increasing t, then your steering wheel is turned to the left, whereas on a road shaped like $3 - t^2$ it's turned to the right.

Figure l shows a third possibility. The function t^3 has a derivative $3t^2$, which equals zero at $t = 0$. This called a point of inflection. The concavity of the graph is down on the left, up on the right. The



l / The functions t^3 has an inflection point at $t = 0$.

inflection point is where it switches from one concavity to the other. In the alternative description in terms of the steering wheel, the inflection point is where your steering wheel is crossing from left to right.

Maxima and minima

When a function goes up and then smoothly turns around and comes back down again, it has zero slope at the top. A place where $\dot{x} = 0$, then, could represent a place where x was at a maximum. On the other hand, it could be concave up, in which case we'd have a minimum.

Example 5

▷ Fred receives a mysterious e-mail tip telling him that his investment in a certain stock will have a value given by $x = -2t^4 + (6.4577 \times 10^{10})t$, where $t \geq 2005$ is the year. Should he sell at some point? If so, when?

▷ If the value reaches a maximum at some time, then the derivative should be zero then. Taking the derivative and setting it equal to zero, we have

$$0 = -8t^3 + 6.4577 \times 10^{10}$$

$$t = \left(\frac{6.4577 \times 10^{10}}{8}\right)^{1/3}$$

$$t = \pm 2006.0 \qquad .$$

Obviously the solution at $t = -2006.0$ is bogus, since the stock market didn't exist four thousand years ago, and the tip only claimed the function would be valid for $t \geq 2005$.

Should Fred sell on New Year's eve of 2006?

But this could be a maximum, a minimum, or an inflection point. Fred definitely does *not* want to sell at $t = 2006$ if it's a minimum! To check which of the three possibilities hold, Fred takes the second derivative:

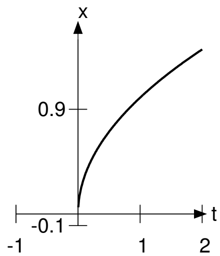
$$\ddot{x} = -24t^2 \qquad .$$

Plugging in $t = 2006.0$, we find that the second derivative is negative at that time, so it is indeed a maximum.

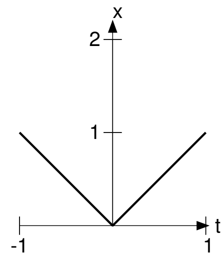
Implicit in this whole discussion was the assumption that the maximum or minimum where the function was smooth. There are some other possibilities.

In figure m, the function's minimum occurs at an end-point of its domain.

Another possibility is that the function can have a minimum or maximum at some point where its derivative isn't well defined. Figrue n shows such a situation. There is a kink in the function at $t = 0$, so a wide variety of lines could be placed through the graph



m / The function $x = \sqrt{t}$ has a minimum at $t = 0$, which is not a place where $\dot{x} = 0$. This point is the edge of the function's domain.



n / The function $x = |t|$ has a minimum at $t = 0$, which is not a place where $\dot{x} = 0$. This is a point where the function isn't differentiable.

there, all with different slopes and all staying on one side of the graph. There is no uniquely defined tangent line, so the derivative is undefined.

Example 6

▷ Rancher Rick has a length of cyclone fence L with which to enclose a rectangular pasture. Show that he can enclose the greatest possible area by forming a square with sides of length $L/4$.

▷ If the width and length of the rectangle are t and u, and Rick is going to use up all his fencing material, then the perimeter of the rectangle, $2t + 2u$, equals L, so for a given width, t, the length is $u = L/2 - t$. The area is $a = tu = t(L/2 - t)$. The function only means anything realistic for $0 \le t \le L/2$, since for values of t outside this region either the width or the height of the rectangle would be negative. The function $a(t)$ could therefore have a maximum either at a place where $\dot{a} = 0$, or at the endpoints of the function's domain. We can eliminate the latter possibility, because the area is zero at the endpoints.

To evaluate the derivative, we first need to reexpress a as a polynomial:

$$a = -t^2 + \frac{L}{2}t \qquad .$$

The derivative is

$$\dot{a} = -2t + \frac{L}{2} \qquad .$$

Setting this equal to zero, we find $t = L/4$, as claimed. This is a maximum, not a minimum or an inflection point, because the second derivative is the constant $\ddot{a} = -2$, which is negative for all t, including $t = L/4$.

Problems

1 Graph the function t^2 in the neighborhood of $t = 3$, draw a tangent line, and use its slope to verify that the derivative equals $2t$ at this point.

2 Graph the function $\sin e^t$ in the neighborhood of $t = 0$, draw a tangent line, and use its slope to estimate the derivative. Answer: 0.5403023058. (You will of course not get an answer this precise using this technique.)

3 Differentiate the following functions with respect to t: 1, 7, t, $7t$, t^2, $7t^2$, t^3, $7t^3$.

4 Differentiate $3t^7 - 4t^2 + 6$ with respect to t.

5 Differentiate $at^2 + bt + c$ with respect to t. [Thompson, 1919]

6 Find two different functions whose derivatives are the constant 3, and give a geometrical interpretation.

7 Find a function x whose derivative is $\dot{x} = 3t^7 - 4t^2 + 6$. In other words, integrate the given function.

8 Let t be the time that has elapsed since the Big Bang. In that time, light, traveling at speed c, has been able to travel a maximum distance ct. The portion of the universe that we can observe is therefore a sphere of radius ct, with volume $v = (4/3)\pi r^3 = (4/3)\pi(ct)^3$. Compute the rate $\dot{v}$ at which the observable universe is expanding, and check that your answer has the right units, as in example 2 on page 13.

9 Kinetic energy is a measure of an object's quantity of motion; when you buy gasoline, the energy you're paying for will be converted into the car's kinetic energy (actually only some of it, since the engine isn't perfectly efficient). The kinetic energy of an object with mass m and velocity v is given by $K = (1/2)mv^2$. For a car accelerating at a steady rate, with $v = at$, find the rate $\dot{K}$ at which the engine is required to put out kinetic energy. $\dot{K}$, with units of energy over time, is known as the *power*. Check that your answer has the right units, as in example 2 on page 13.

10 A metal square expands and contracts with temperature, the lengths of its sides varying according to the equation $\ell = (1+\alpha T)\ell_o$. Find the rate of change of its surface area with respect to temperature. That is, find $\dot{\ell}$, where the variable with respect to which you're differentiating is the temperature, T. Check that your answer has the right units, as in example 2 on page 13.

11 Find the second derivative of $2t^3 - t$.

12 Locate any poins of inflection of the function $t^3 + t^2$. Verify by graphing that the concavity of the function reverses itself at this point.

13 Two atoms will interact via electrical forces between their protons and electrons. To put them at a distance r from one another (measured from nucleus to nucleus), a certain amount of energy E is required, and the minimum energy occurs when the atoms are in equilibrium, forming a molecule. Often a fairly good approximation to the energy is the Lennard-Jones expression

$$E(r) = k\left[\left(\frac{a}{r}\right)^{12} - 2\left(\frac{a}{r}\right)^{6}\right] \quad ,$$

where k and a are constants. Show that there is an equilibrium at $r = a$. Verify (either by graphing or by testing the second derivative) that this is a minimum, not a maximum or a point of inflection.

14 Prove that the total number of maxima and minima possessed by a third-order polynomial is at most two.

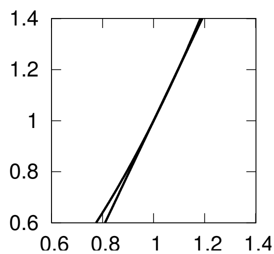
2 To infinity — and beyond!

a / Gottfried Leibniz (1646-1716)

Little kids readily pick up the idea of infinity. "When I grow up, I'm gonna have a million Barbies." "Oh yeah? Well I'm gonna have a billion in my house." "Well I'm gonna have infinity Barbies." "So what? I'll have two infinity of them." Adults laugh, convinced that infinity, ∞, is the biggest number, so 2∞ can't be any bigger. This is the idea behind the joke in the movie Toy Story: Buzz Lightyear's slogan is "To infinity — and beyond!" We assume there *isn't* any beyond. Infinity is supposed to be the biggest there is, so by definition there can't be anything bigger, right?

2.1 Infinitesimals

Actually mathematicians have invented several many different logical systems for working with infinity, and in most of them infinity does come in different sizes and flavors. Newton, as well as the German mathematician Leibniz who invented calculus independently,[1] had a strong intuitive idea that calculus was really about numbers that were infinitely small: infinitesimals, the opposite of infinities. For instance, consider the number $1.1^2 = 1.21$. That 2 in the first decimal place is the same 2 that appears in the expression $2t$ for the derivative of t^2.



b / A close-up view of the function $x = t^2$, showing the line that connects the points $(1, 1)$ and $(1.1, 1.21)$.

[1] There is some dispute over this point. Newton and his supporters claimed that Leibniz plagiarized Newton's ideas, and merely invented a new notation for them.

Figure b shows the idea visually. The line connecting the points $(1,1)$ and $(1.1,1.21)$ is almost indistinguishable from the tangent line on this scale. Its slope is $(1.21-1)/(1.1-1) = 2.1$, which is very close to the tangent line's slope of 2. It was a good approximation because the points were close together, separated by only 0.1 on the t axis.

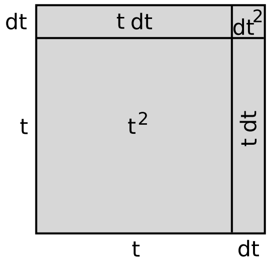
If we needed a better approximation, we could try calculating $1.01^2 = 1.0201$. The slope of the line connecting the points $(1,1)$ and $(1.01,1.0201)$ is 2.01, which is even closer to the slope of the tangent line.

Another method of visualizing the idea is that we can interpret $x = t^2$ as the area of a square with sides of length t, as suggested in figure c. We increase t by an infinitesimally small number $\mathrm{d}t$. The d is Leibniz's notation for a very small difference, and $\mathrm{d}t$ is to be read is a single symbol, "dee-tee," not as a number d multiplied by a number t. The idea is that $\mathrm{d}t$ is smaller than any ordinary number you could imagine, but it's not zero. The area of the square is increased by $\mathrm{d}x = 2t\mathrm{d}t + \mathrm{d}t^2$, which is analogous to the finite numbers 0.21 and 0.0201 we calculated earlier. Where before we divided by a finite change in t such as 0.1 or 0.01, now we divide by $\mathrm{d}t$, producing

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2t\,\mathrm{d}t + \mathrm{d}t^2}{\mathrm{d}t} = 2t + \mathrm{d}t$$

for the derivative. On a graph like figure b, $\mathrm{d}x/\mathrm{d}t$ is the slope of the tangent line: the change in x divided by the changed in t.

But adding an infinitesimal number $\mathrm{d}t$ onto $2t$ doesn't really change it by any amount that's even theoretically measurable in the real world, so the answer is really $2t$. Evaluating it at $t = 1$ gives the exact result, 2, that the earlier approximate results, 2.1 and 2.01, were getting closer and closer to.



c / A geometrical interpretation of the derivative of t^2.

Example 7

To show the power of infinitesimals and the Leibniz notation, let's prove that the derivative of t^3 is $3t^2$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{(t+\mathrm{d}t)^3 - t^3}{\mathrm{d}t} = \frac{3t^2\,\mathrm{d}t + 3t\,\mathrm{d}t + \mathrm{d}t^3}{\mathrm{d}t} = 3t^2 + \ldots \quad ,$$

where the dots indicate infinitesimal terms that we can neglect.

This result required significant sweat and ingenuity when proved on page 81 by the methods of chapter 1, and not only that but the old method would have required a completely different method of proof for a function that wasn't a polynomial, whereas the new one can be applied more generally, as shown in the following example.

Example 8

The derivative of $x = \sin t$, with t in units of radians, is

$$\frac{dx}{dt} = \frac{\sin(t + dt) - \sin t}{dt} \quad ,$$

and with the trig identity $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, this becomes

$$= \frac{\sin t\ \cos dt + \cos t\ \sin dt - \sin t}{dt} \quad .$$

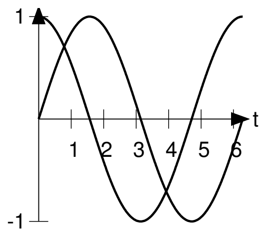
Applying the small-angle approximations $\sin u \approx u$ and $\cos u \approx 1$, we have

$$\frac{dx}{dt} = \frac{\cos t\, dt}{dt} = \cos t \quad .$$

But are the approximations good enough? The situation is similar to the one we encountered earlier, in which we computed $(t + dt)^2$, and neglected the dt^2 term represented by the small square in figure c. Being a little less cavalier, I should demonstrate explicitly that the error introduced by the small-angle approximations is really of the same order of magnitude as dt^2, i.e., a number that is infinitesimally small compared even to the infinitesimal size of dt; I've done this on page 82.

Figure d shows the graphs of the function and its derivative. Note how the two graphs correspond. At $t = 0$, the slope of $\sin t$ is at its largest, and is positive; this is where the derivative, $\cos t$, attains its maximum positive value of 1. At $t = \pi/2$, $\sin t$ has reached a maximum, and has a slope of zero; $\cos t$ is zero here. At $t = \pi$, in the middle of the graph, $\sin t$ has its maximum negative slope, and $\cos t$ is at its most negative extreme of -1.

Physically, $\sin t$ could represent the position of a pendulum as it moved back and forth from left to right, and $\cos t$ would then be the pendulum's velocity.



d / Graphs of $\sin t$, and its derivative $\cos t$.

Example 9

What about the derivative of the cosine? The cosine and the sine are really the same function, shifted to the left or right by $\pi/4$. If the derivative of the sine is the same as itself, but shifted to the left by $\pi/4$, then the derivative of the cosine must be a cosine shifted to the left by $\pi/4$:

$$\frac{d\cos t}{dt} = \cos(t + \pi/4) = -\sin t \quad .$$

e / Bishop George Berkeley (1685-1753)

2.2 Safe use of infinitesimals

The idea of infinitesimally small numbers has always irked purists. One prominent critic of the calculus was Newton's contemporary George Berkeley, the Bishop of Cloyne. Although some of his complaints are clearly wrong (he denied the possibility of the second derivative), there was clearly something to his criticism of the infinitesimals. He wrote sarcastically, "They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them ghosts of departed quantities?"

Infinitesimals seemed scary, because if you mishandled them, you could prove absurd things. For example, let $\mathrm{d}u$ be an infinitesimal. Then $2\mathrm{d}u$ is also infinitesimal. Therefore both $1/\mathrm{d}u$ and $1/(2\mathrm{d}u)$ equal infinity, so $1/\mathrm{d}u = 1/(2\mathrm{d}u)$. Multiplying by $\mathrm{d}u$ on both sides, we have a proof that $1 = 1/2$.

In the eighteenth century, the use of infinitesimals became like adultery: commonly practiced, but shameful to admit to in polite circles. Those who used them learned certain rules of thumb for handling them correctly. For instance, they would identify the flaw in my proof of $1 = 1/2$ as my assumption that there was only one size of infinity, when actually $1/\mathrm{d}u$ should be interpreted as an infinity twice as big as $1/(2\mathrm{d}u)$. The use of the symbol ∞ played into this trap, because the use of a single symbol for infinity implied that infinites only came in one size. However, the practitioners of infinitesimals had trouble articulating a clear set of principles for their proper use, and couldn't prove that a self-consistent system could be built around them.

By the twentieth century, when I learned calculus, a clear consensus had existed that infinite and infinitesimal numbers weren't numbers at all. A notation like $\mathrm{d}x/\mathrm{d}t$, my calculus teacher told me, wasn't really one number divided by another, it was merely a symbol for the limit

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \qquad ,$$

where Δx and Δt represented finite changes. That satisfied me until we got to a certain topic (implicit differentiation) in which we were encouraged to break the $\mathrm{d}x$ away from the $\mathrm{d}t$, leaving them on opposite sides of the equation. I buttonholed my teacher after class and asked why he was now doing what he'd told me you couldn't really do, and his response was that dx and dt weren't really numbers, but most of the time you could get away with treating them as if they were, and you would get the right answer in the end. *Most of the time!?* That bothered me. How was I supposed to know when it *wasn't* "most of the time?"

f / Abraham Robinson (1918-1974)

But unknown to me and my teacher, mathematician Abraham Robinson had already shown in the 1960's that it was possible to construct a self-consistent number system that included infinite and infinitesimal numbers. He called it the hyperreal number system, and it included the real numbers as a subset.[2]

Moreover, the rules for what you can and can't do with the hyperreals turn out to be extremely simple. Take any true statement about the real numbers. Suppose it's possible to translate it into a statement about the hyperreals in the most obvious way, simply by replacing the word "real" with the word "hyperreal." Then the translated statement is also true. This is known as the *transfer principle*.

Let's look back at my bogus proof of $1 = 1/2$ in light of this simple principle. The final step of the proof, for example, is perfectly valid: multiplying both sides of the equation by the same thing. The following statement about the real numbers is true:

> For any real numbers a, b, and c, if $a = b$, then $ac = bc$.

This can be translated in an obvious way into a statement about the hyperreals:

> For any hyperreal numbers a, b, and c, if $a = b$, then $ac = bc$.

However, what about the statement that both $1/\mathrm{d}u$ and $1/(2\mathrm{d}u)$ equal infinity, so they're equal to each other? This isn't the translation of a statement that's true about the reals, so there's no reason to believe it's true — and in

[2] The reader who wants to learn more about the hyperreal system might want to start by skimming the Wikipedia article *Non-standard analysis* for general background, and then read the relevant parts of Keisler's *Elementary Calculus: An Approach Using Infinitesimals*, an out-of-print calculus text that uses infinitesimals, available for free from the author's web site. The standard (difficult) book on the subject is Robinson's *Non-Standard Analysis*.

fact it's false.

What the transfer principle tells us is that the real numbers as we normally think of them are not unique in obeying the ordinary rules of algebra. There are completely different systems of numbers, such as the hyperreals, that also obey them.

How, then, are the hyperreals even different from the reals, if everything that's true of one is true of the other? But recall that the transfer principle doesn't guarantee that every statement about the reals is also true of the hyperreals. It only works if the statement about the reals can be translated into a statement about the hyperreals in the most simple, straightforward way imaginable, simply by replacing the word "real" with the word "hyperreal." Here's an example of a true statement about the reals that can't be translated in this way:

> For any real number a, there is an integer n that is greater than a.

This one can't be translated so simplemindedly, because it refers to a subset of the reals called the integers. It might be possible to translate it somehow, but it would require some insight into the correct way to translate that word "integer." The transfer principle doesn't apply to this statement, which indeed is false for the hyperreals, because the hyperreals contain infinite numbers that are greater than all the integers. In fact, the contradiction of this statement can be taken as a definition of what makes the hyperreals special, and different from the reals: we assume that there is at least one hyperreal number, H, which is greater than all the integers.

As an analogy from everyday life, consider the following statements about the student body of the high school I attended:

1. Every student at my high school had two eyes and a face.
2. Every student at my high school who was on the football team was a jerk.

Let's try to translate these into statements about the population of California in general. The student body of my high school is like the set of real numbers, and the present-day population of California is like the hyperreals. Statement 1 can be translated mindlessly into a statement that every Californian has two eyes and a face; we simply substitute "every Californian" for "every student at my high school." But statement 2 isn't so easy, because it refers to the subset of students who were on the football team, and it's not obvious what the corresponding subset of Californians would be. Would it include everybody who played high school, college, or pro football? Maybe it shouldn't include the pros, because they belong to an organization covering a region bigger than California. Statement 2 is the kind of statement that the transfer principle doesn't apply to.

Example 10

As a nontrivial example of how to apply the transfer principle, let's consider how to handle expressions like the one that occurred when we wanted to differentiate t^2 using infinitesimals:

$$\frac{\mathrm{d}\left(t^2\right)}{\mathrm{d}t} = 2t + \mathrm{d}t \qquad .$$

I argued earlier than $2t+\mathrm{d}t$ is so close to $2t$ that for all practical purposes, the answer is really $2t$. But is it really valid in general to say that $2t+\mathrm{d}t$ is the same hyperreal number as $2t$? No. We can apply the transfer principle to the following statement about the reals:

> For any real numbers a and b, with $b \neq 0$, $a+b \neq a$.

Since $\mathrm{d}t$ isn't zero, $2t+\mathrm{d}t \neq 2t$.

More generally, example 10 leads us to visualize every number as being surrounded by a "halo" of numbers that don't equal it, but differ from it by only an infinitesimal amount. Just as a magnifying glass would allow you to see the fleas on a dog, you would need an infinitely strong microscope to see this halo. This is similar to the idea that every integer is surrounded by a bunch of fractions that would round off to that integer. We can, however, define the *standard part* of a finite hyperreal number, which means the unique real number that differs from it infinitesimally. For instance, the standard part of $2t + \mathrm{d}t$, notated $\mathrm{st}(2t + \mathrm{d}t)$, equals $2t$. The derivative of a function should actually be defined as the standard part of $\mathrm{d}x/\mathrm{d}t$, but we often write $\mathrm{d}x/\mathrm{d}t$ to mean the derivative, and don't worry about the distinction.

One of the things Bishop Berkeley disliked about infinitesimals was the idea that they existed in a kind of hierarchy, with $\mathrm{d}t^2$ being not just infinitesimally small, but infinitesimally small compared to the infinitesimal $\mathrm{d}t$. If $\mathrm{d}t$ is the flea on a dog, then $\mathrm{d}t^2$ is a submicroscopic flea that lives on the flea, as in Swift's doggerel: "Big fleas have little fleas/ On their backs to ride 'em,/ and little fleas have lesser fleas,/And so, ad infinitum." Berkeley's criticism was off the mark here: there is such a hierarchy. Our basic assumption about the hyperreals was that they contain at least one infinite number, H, which is bigger than all the integers. If this is true, then $1/H$ must be less than $1/2$, less than $1/100$, less then $1/1,000,000$ — less than $1/n$ for any integer n. Therefore the hyperreals are guaranteed to include infinitesimals as well, and so we have at least three levels to the hierarchy: infinities comparable to H, finite numbers, and infinitesimals comparable to $1/H$. If you can swallow that, then it's not too much of a leap to add more rungs to the ladder, like extra-small infinitesimals that are comparable to $1/H^2$. If this seems a little crazy, it may comfort you to think of statements about the hyperreals as descriptions of limiting processes involving real numbers. For instance, in the sequence of numbers $1.1^2 = 1.21$, $1.01^2 = 1.0201$, $1.001^2 = 1.002001$, ..., it's clear that the number represented by the digit 1 in the final decimal place is getting smaller faster than the contribution due to the digit 2 in the middle.

2.3 The product rule

When I first learned calculus, it seemed to me that if the derivative of $3t$ was 3, and the derivative of $7t$ was 7, then the derivative of t multiplied by t ought to be just plain old t, not $2t$. The reason there's a factor of 2 in the correct answer is that t^2 has two reasons to grow as t gets bigger: it grows because the first factor of t is increasing, but also because the second one is. In general, it's possible to find the derivative of the product of two functions any time we know the derivatives of the individual functions.

The product rule
If x and y are both functions of t, then the derivative of their product is

$$\frac{\mathrm{d}(xy)}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} \cdot y + x \cdot \frac{\mathrm{d}y}{\mathrm{d}t} \quad .$$

The proof is easy. Changing t by an infinitesimal amount $\mathrm{d}t$ changes the product xy by an amount

$$\begin{aligned}(x + \mathrm{d}x)(y + \mathrm{d}y) - xy \\ = y\mathrm{d}x + x\mathrm{d}y + \mathrm{d}x\mathrm{d}y \quad ,\end{aligned}$$

and dividing by dt gives,

$$= \frac{\mathrm{d}x}{\mathrm{d}t} \cdot y + x \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}x\mathrm{d}y}{\mathrm{d}t} \quad ,$$

whose standard part is the result to be proved.

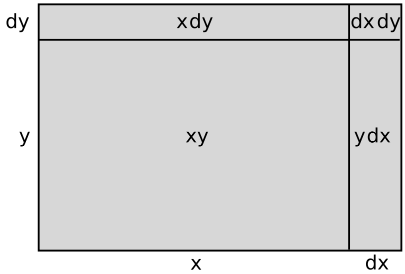
Example 11

▷ Find the derivative of the function $t \sin t$.

▷

$$\begin{aligned}\frac{\mathrm{d}(t \sin t)}{\mathrm{d}t} &= t \cdot \frac{\mathrm{d}(\sin t)}{\mathrm{d}t} + \frac{\mathrm{d}t}{\mathrm{d}t} \cdot \sin t \\ &= t \cos t + \sin t\end{aligned}$$

Figure g gives the geometrical interpretation of the product rule. Imagine that the king, in his castle at the wouthwest corner of his rectangular kingdom, sends out a line of infantry to expand his territory to the north, and a line of cavalry to take over more land to the east. In a time interval $\mathrm{d}t$, the cavalry, which moves faster, covers a distance $\mathrm{d}x$ greater than that covered by the infantry, $\mathrm{d}y$. However, the strip of territory conquered by the cavalry, $y\mathrm{d}x$, isn't as great as it could have been, because in our example y isn't as big as x.



g / A geometrical interpretation of the product rule.

A helpful feature of the Leibniz notation is that one can easily use it to check whether the units of an answer make sense. If we measure distances in meters and time in seconds, then xy has units of square meters (area), and so does the change in the area, $\mathrm{d}(xy)$. Dividing by $\mathrm{d}t$ gives the number of square meters per second being conquered. On the right-hand side of the product rule, $\mathrm{d}x/\mathrm{d}t$ has units of meters per second (velocity), and multiplying it by y makes the units square meters per second, which is consistent with the left-hand side. The units of the second term on the right likewise check out. Some beginners might be tempted to guess that the product rule would be $\mathrm{d}(xy)/\mathrm{d}t = (\mathrm{d}x/\mathrm{d}t)(\mathrm{d}y/\mathrm{d}t)$, but the Leibniz notation instantly reveals that this can't be the case, because then the units on the left, m^2/s, wouldn't match the ones on the right, $\mathrm{m}^2/\mathrm{s}^2$.

Because this unit-checking feature is so helpful, there is a special way of writing a second derivative in the Leibniz notation. What Newton called $\ddot{x}$, Leibniz wrote as

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \quad .$$

Although the different placement of the 2's on top and bottom seems strange and inconsistent to many beginners, it actually works out nicely. If x is a distance, measured in meters, and t is a time, in units of seconds, then the second derivative is supposed to have units of acceleration, in units of meters per second per second, also written (m/s)/s, or $\mathrm{m/s}^2$. (The acceleration of falling objects on Earth is 9.8 $\mathrm{m/s}^2$ in these units.) The Leibniz notation is meant to suggest exactly this: the top of the fraction looks like it has units of meters, because we're not squaring x, while the bottom of the fraction looks like it has units of seconds, because it looks like we're squaring $\mathrm{d}t$. Therefore the units come out right. It's important to realize, however, that the symbol d isn't a number (not a real one, and not a hyperreal one, either), so we can't really square it; the notation is not to be taken as a literal statement about infinitesimals.

Example 12

A tricky use of the product rule is to find the derivative of $\sqrt{t}$. Since $\sqrt{t}$ can be written as $t^{1/2}$, we might suspect that the rule $\mathrm{d}(t^k)/\mathrm{d}t = kt^{k-1}$ would work, giving a derivative $\frac{1}{2}t^{-1/2} = 1/(2\sqrt{t})$. However, the methods used to prove that rule in chapter 1 only work if k is an integer, so the best we could do would be to confirm our conjecture approximately by graphing.

Using the product rule, we can write $f(t) = \mathrm{d}\sqrt{t}/\mathrm{d}t$ for our unknown derivative, and back into the result using the product rule:

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}t} &= \frac{\mathrm{d}(\sqrt{t}\sqrt{t})}{\mathrm{d}t}\\ &= f(t)\sqrt{t} + \sqrt{t}f(t)\\ &= 2f(t)\sqrt{t}\end{aligned}$$

But $\mathrm{d}t/\mathrm{d}t = 1$, so $f(t) = 1/(2\sqrt{t})$ as claimed.

The trick used in example 12 can also be used to prove that the power rule $\mathrm{d}(x^n)/\mathrm{d}x = nx^{n-1}$ applies to cases where n is an integer less than 0, but I'll instead prove this on page 34 by a technique that doesn't depend on a trick, and also applies to values of n that aren't integers.

2.4 The chain rule

Figure h shows three clowns on seesaws. If the leftmost clown moves down by a distance $\mathrm{d}x$, the middle one will come up by $\mathrm{d}y$, but this will also cause the one on the right to move down by $\mathrm{d}z$. If we want to predict how much the rightmost clown will move in response to a certain amount of motion by the leftmost one, we have

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \quad .$$

This relation, called the chain rule, allows us to calculate a derivative of a function defined by one function inside another.

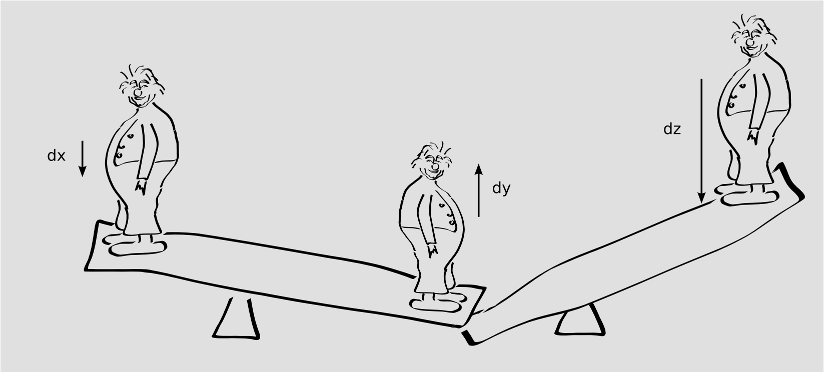
Example 13

▷ Find the derivative of the function $z(x) = \sin(x^2)$.

▷ Let $y(x) = x^2$, so that $z(x) = \sin(y(x))$. Then

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{\mathrm{d}z}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}\\ &= \cos(y) \cdot 2x\\ &= 2x\cos(x^2)\end{aligned}$$

The way people usually say it is that the chain rule tells you to take the derivative of the outside function, the sine in this case, and then multiply by the derivative of "the inside stuff," which here is the square. Once you get used to doing it, you don't need to invent a third, intermediate variable, as we did here with y.



h / Three clowns on seesaws demonstrate the chain rule.

2.5 Exponentials and logarithms

The exponential

An important application of the chain rule comes up when we want to differentiate the omnipresent function e^x, where $e = 2.71828\ldots$ is the base of natural logarithms. We have

$$\begin{aligned}\frac{\mathrm{d}e^x}{\mathrm{d}x} &= \frac{e^{x+\mathrm{d}x} - e^x}{\mathrm{d}x} \\ &= \frac{e^x e^{\mathrm{d}x} - e^x}{\mathrm{d}x} \\ &= e^x \, \frac{e^{\mathrm{d}x} - 1}{\mathrm{d}x}\end{aligned}$$

The second factor, $\left(e^{\mathrm{d}x} - 1\right)/\mathrm{d}x$, doesn't have x in it, so it must just be a constant. Therefore we know that the derivative of e^x is simply e^x, multiplied by some unknown constant,

$$\frac{\mathrm{d}e^x}{\mathrm{d}x} = c\, e^x .$$

A rough check by graphing at, say $x = 0$, shows that the slope is close to 1, so c is close to 1. But how do we know it's exactly one? The proof is given on page 83.

Example 14

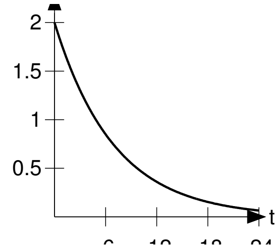
▷ The concentration of a foreign substance in the bloodstream generally falls off exponentially with time as $c = c_o e^{-t/a}$, where c_o is the initial concentration, and a is a constant. For caffeine in adults, a is typically about 7 hours. An example is shown in figure i. Differentiate the concentration with respect to time, and interpret the result. Check that the units of the result make sense.

▷ Using the chain rule,

$$\begin{aligned}\frac{\mathrm{d}c}{\mathrm{d}t} &= c_o e^{-t/a} \cdot \left(-\frac{1}{a}\right) \\ &= -\frac{c_o}{a} e^{-t/a}\end{aligned}$$

This can be interpreted as the rate at which caffeine is being removed from the blood and put into the person's urine. It's negative because the concentration is decreasing. According to the original expression for x, a substance with a large a will take a long time to reduce its concentration, since t/a won't be very big unless we have large t on top to compensate for the large a on the bottom. In other words, larger values of a represent substances that the body has a harder time getting rid of efficiently. The derivative has a on the bottom, and the interpretation of this is that for a drug that is hard to eliminate, the rate at which it is removed from the blood is low.

It makes sense that a has units of time, because the exponential function has to have a unitless argument, so the units of t/a have to cancel out. The units of the result come from the factor of c_o/a, and it makes sense that the units are concentration divided by time, because the result represents the rate at which the concentration is changing.



i / Example 14. A typical graph of the concentration of caffeine in the blood, in units of milligrams per liter, as a function of time, in hours.

Example 15

▷ Find the derivative of the function $y = 10^x$.

▷ In general, one of the tricks to doing calculus is to rewrite functions in forms that you know how to handle. This one can be rewritten as a base-10 logarithm:

$$\begin{aligned} y &= 10^x \\ \ln y &= \ln\left(10^x\right) \\ \ln y &= x \ln 10 \\ y &= e^{x \ln 10} \end{aligned}$$

Applying the chain rule, we have the derivative of the exponential, which is just the same exponential, multiplied by the derivative of the inside stuff:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x \ln 10} \cdot \ln 10 \qquad .$$

In other words, the "c" referred to in the discussion of the derivative of e^x

becomes $c = \ln 10$ in the case of the base-10 exponential.

The logarithm

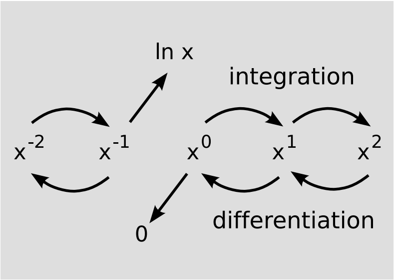
The natural logarithm is the function that undoes the exponential. In a situation like this, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y} \qquad ,$$

where on the left we're thinking of y as a function of x, and on the right we consider x to be a function of y. Applying this to the natural logarithm,

$$\begin{aligned}
y &= \ln x \\
x &= e^y \\
\frac{\mathrm{d}x}{\mathrm{d}y} &= e^y \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{e^y} \\
&= \frac{1}{x} \\
\frac{\mathrm{d}\ln x}{\mathrm{d}x} &= \frac{1}{x} \qquad .
\end{aligned}$$

This is noteworthy because it shows that there must be an exception to the rule that the derivative of x^n is nx^{n-1}, and the integral of x^{n-1} is x^n/n. (On page 30 I remarked that this rule could be proved using the product rule for negative integer values of k, but that I would give a simpler, less tricky, and more general proof later. The proof is example 16 below.) The integral of x^{-1} is not $x^0/0$, which wouldn't make sense anyway because it involves division by zero.[3] Likewise the derivative of $x^0 = 1$ is $0x^{-1}$, which is zero. Figure j shows the idea. The functions x^n form a kind of ladder, with differetiation taking us down one rung, and integration taking us up. However, there are two special cases where differentiation takes us off the ladder entirely.



j / Differentiation and integration of functions of the form x^n. Constants out in front of the functions are not shown, so keep in mind that, for example, the derivative of x^2 isn't x, it's $2x$.

[3]Speaking casually, one can say that division by zero gives infinity. This is often a good way to think when trying to connect mathematics to reality. However, it doesn't really work that way according to our rigorous treatment of the hyperreals. Consider this statement: "For a nonzero real number a, there is no real number b such that $a = 0b$." This means that we can't divide a by 0 and get b. Applying the transfer principle to this statement, we see that the same is true for the hyperreals: division by zero is undefined. However, we can divide a finite number by an infinitesimal, and get an infinite result, which is almost the same thing.

Example 16

▷ Prove $d(x^n)/dx = nx^{n-1}$ for any real value of n.

▷

$$\begin{aligned} y &= x^n \\ &= e^{n \ln x} \end{aligned}$$

By the chain rule,

$$\begin{aligned} \frac{dy}{dx} &= e^{n \ln x} \cdot \frac{n}{x} \\ &= x^n \cdot \frac{n}{x} \\ &= nx^{n-1} \quad . \end{aligned}$$

(For $n = 0$, the result is zero.)

2.6 Quotients

So far we've been successful with a divide-and-conquer approach to differentiation: the product rule and the chain rule offer methods of breaking a function down into simpler parts, and finding the derivative of the whole thing based on knowledge of the derivatives of the parts. We know how to find the derivatives of sums, differences, and products, so the obvious next step is to look for a way of handling division. This is straightforward, since we know that the derivative of the function function $1/u = u^{-1}$ is $-u^{-2}$. Let u and v be functions of x. Then by the product rule,

$$\frac{d(v/u)}{dx} = \frac{dv}{dx} \cdot \frac{1}{u} + v \cdot \frac{d(1/u)}{dx}$$

and by the chain rule,

$$\frac{d(v/u)}{dx} = \frac{dv}{dx} \cdot \frac{1}{u} - v \cdot \frac{1}{u^2} \frac{du}{dx}$$

This is so easy to rederive on demand that I suggest not memorizing it.

By the way, notice how the notation becomes a little awkward when we want to write a derivative like $d(v/u)/dx$. When we're differentiating a complicated function, it can be uncomfortable trying to cram the expression into the top of the $d \ldots / d \ldots$ fraction. Therefore it would be more common to write such an expression like this:

$$\frac{d}{dx}\left(\frac{v}{u}\right)$$

This could be considered an abuse of notation, making d look like a number being divided by another number dx, when actually d is meaningless on its own. On the other hand, we can consider the symbol d/dx to represent the operation of differentiation with respect to x; such an interpretation will seem more natural to those who have been inculcated with the taboo against considering infinitesimals as numbers in the first place.

Using the new notation, the quotient rule becomes

$$\frac{d}{dx}\left(\frac{v}{u}\right) = \frac{1}{u} \cdot \frac{dv}{dx} - \frac{v}{u^2} \cdot \frac{du}{dx} \quad .$$

The interpretation of the minus sign is that if u increases, v/u decreases.

Example 17

▷ Differentiate $y = x/(1 + 3x)$, and check that the result makes sense.

▷ We identify v with x and u with $1+x$. The result is

$$\frac{d}{dx}\left(\frac{v}{u}\right) = \frac{1}{u}\cdot\frac{dv}{dx} - \frac{v}{u^2}\cdot\frac{du}{dx}$$
$$= \frac{1}{1+x} - \frac{3x}{(1+x)^2}$$

One way to check that the result makes sense it to consider extreme values of x. For very large values of x, the 1 on the bottom of $x/(1+x)$ becomes negligible compared to the $3x$, and the function y approaches $x/3x = 1/3$ as a limit. Therefore we expect that the derivative dy/dx should approach zero, since the derivative of a constant is zero. It works: plugging in bigger and bigger numbers for x in the expression for the derivative does give smaller and smaller results. (In the second term, the denominator gets bigger faster than the numerator, because it has a square in it.)

Another way to check the result is to verify that the units work out. Suppose arbitrarily that x has units of gallons. (If the 3 on the bottom is unitless, then the 1 would have to represent 1 gallon, since you can't add things that have different units.) The function y is defined by an expression with units of gallons divided by gallons, so y is unitless. Therefore the derivative dy/dx should have units of inverse gallons. Both terms in the expression for the derivative do have those units, so the units of the answer check out.

2.7 Differentiation on a computer

In this chapter you've learned a set of rules for evaluating derivatives: derivatives of products, quotions, functions inside other functions, etc. Because these rules exist, it's always possible to find a formula for a function's derivative, given the formula for the original function. Not only that, but there is no real creativity required, so a computer can be programmed to do all the drudgery. For example, you can download a free, open-source program called Yacas from `yacas.sourceforge.net` and install it on a Windows or Linux machine. A typical session with Yacas looks like this:

Example 18

```
D(x) x^2
   2*x
D(x) Exp(x^2)
   2*x*Exp(x^2)
D(x) Sin(Cos(Sin(x)))
   -Cos(x)*Sin(Sin(x))
      *Cos(Cos(Sin(x)))
```

Upright type represents your input, and italicized type is the program's output.

First I asked it to differentiate x^2 with respect to x, and it told me the result was $2x$. Then I did the derivative of e^{x^2}, which I also could have done fairly easily by hand. (If you're trying this out on a computer as you real along, makre sure to capitalize functions like Exp, Sin, and Cos.) Finally I tried an example where I didn't know the answer off the top of my head, and that would have been a little tedious to calculate by hand.

Unfortunately things are a little less rosy in the world of integrals. There are a few rules that can help you do integrals, e.g., that the integral of a sum equals the sum of the integrals, but the rules don't cover all the possible cases. Using Yacas to evaluate the integrals of the same functions, here's what happens.[4]

Example 19

```
Integrate(x) x^2
   x^3/3
Integrate(x) Exp(x^2)
   Integrate(x)Exp(x^2)
Integrate(x)
     Sin(Cos(Sin(x)))
   Integrate(x)
       Sin(Cos(Sin(x)))
```

The first one works fine, and I can easily verify that the answer is correct, by taking the derivative of $x^3/3$, which is x^2. (The answer could have been $x^3/3+7$, or $x^3/3+c$, where c was any constant, but Yacas doesn't bother to tell us that.) The second and third ones don't work, however; Yacas just spits back the input at us without making any progress on it. And it may not be because Yacas isn't smart enough to figure out these integrals. The function e^{x^2} can't be integrated at all in terms of a formula containing ordinary operations and functions such as addition, multiplication, exponentiation, trig functions, exponentials, and so on.

That's not to say that a program like this is useless. For example, here's an integral that I wouldn't have known how to do, but that Yacas handles easily:

Example 20

```
Integrate(x) Sin(Ln(x))
   (x*Sin(Ln(x)))/2
       -(x*Cos(Ln(x)))/2
```

This one is easy to check by differentiating, but I could have been marooned on a desert island for a decade before I could have figured it out in the first place. There are various rules, then, for integration, but they don't cover all possible cases as the rules for differentiation do, and sometimes it isn't obvious which rule to apply. Yacas's ability to integrate $\sin \ln x$ shows that it had a rule in its bag of tricks that I don't know, or didn't remember, or didn't realize applied to this integral.

Back in the 17th century, when Newton and Leibniz invented calculus, there were no computers, so it was a big deal to be able to find a simple formula for your result. Nowadays, however, it may not be such a big deal. Suppose I want to find the derivative of $\sin \cos \sin x$, evaluated at $x = 1$. I can do something like this on a calculator:

Example 21

```
sin cos sin 1 =
   0.61813407
sin cos sin 1.0001 =
```

[4] If you're trying these on your own computer, note that the long input line for the function $\sin \cos \sin x$ shouldn't be broken up into two lines as shown in the listing.

```
    0.61810240
 (0.61810240-0.61813407)
       /.0001 =
    -0.3167
```

I have the right answer, with plenty of precision for most realistic applications, although I might have never guessed that the mysterious number -0.3167 was actually $-(\cos 1)(\sin\sin 1)(\cos\cos\sin 1)$. This could get a little tedious if I wanted to graph the function, for instance, but then I could just use a computer spreadsheet, or write a little computer program. In this chapter, I'm going to show you how to do derivatives and integrals using simple computer programs, using Yacas. The following little Yacas program does the same thing as the set of calculator operations shown above:

Example 22

```
1  f(x):=Sin(Cos(Sin(x)))
2  x:=1
3  dx:=.0001
4  N( (f(x+dx)-f(x))/dx )
     -0.3166671628
```

(I've omitted all of Yacas's output except for the final result.) Line 1 defines the function we want to differentiate. Lines 2 and 3 give values to the variables x and dx. Line 4 computes the derivative; the `N( )` surrounding the whole thing is our way of telling Yacas that we want an approximate numerical result, rather than an exact symbolic one.

An interesting thing to try now is to make dx smaller and smaller, and see if we get better and better accuracy in our approximation to the derivative.

Example 23

```
5  g(x,dx):=
        N( (f(x+dx)-f(x))/dx )
6  g(x,.1)
     -0.3022356406
7  g(x,.0001)
     -0.3166671628
8  g(x,.0000001)
     -0.3160458019
9  g(x,.00000000000000001)
     0
```

Line 5 defines the derivative function. It needs to know both x and dx. Line 6 computes the derivative using $\mathrm{d}x = 0.1$, which we expect to be a lousy approximation, since $\mathrm{d}x$ is really supposed to be infinitesimal, and 0.1 isn't even that small. Line 7 does it with the same value of $\mathrm{d}x$ we used earlier. The two results agree exactly in the first decimal place, and approximately in the second, so we can be pretty sure that the derivative is -0.32 to two figures of precision. Line 8 ups the ante, and produces a result that looks accurate to at least 3 decimal places. Line 9 attempts to produce fantastic precision by using an extremely small value of $\mathrm{d}x$. Oops — the result isn't better, it's worse! What's happened here is that Yacas computed $f(x)$ and $f(x + \mathrm{d}x)$, but they were the same to within the precision it was using, so $f(x+\mathrm{d}x) - f(x)$ rounded off to zero.[5]

[5]Yacas can do arithmetic to any precision you like, although you may

Example 23 demonstrates the concept of how a derivative can be defined in terms of a limit:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

The idea of the limit is that we can theoretically make $\Delta y/\Delta x$ approach as close as we like to $\mathrm{d}y/\mathrm{d}x$, provided we make Δx sufficiently small. In reality, of course, we eventually run into the limits of our ability to do the computation, as in the bogus result generated on line 9 of the example.

2.8 Continuity

Intuitively, a continuous function is one whose graph has no sudden jumps in it; the graph is all a single connected piece. Formally, $f(x)$ is defined to be continuous if for any real x and any infinitesimal $\mathrm{d}x$, $f(x + \mathrm{d}x) - f(x)$ is infinitesimal.

Example 24

Let the function f be defined by $f(x) = 0$ for $x \leq 0$, and $f(x) = 1$ for $x > 0$. Then $f(x)$ is discontinuous, since for $\mathrm{d}x > 0$, $f(0+\mathrm{d}x) - f(0) = 1$, which isn't infinitesimal.

If a function is discontinuous at a given point, then it is not differentiable at that point. On the other hand, a function like $y = |x|$ shows that a function can be continuous without being differentiable.

Another way of thinking about continuous functions is given by the *intermediate value theorem*. Intuitively, it says that if you are moving continuously along a road, and you get from point A to point B, then you must also visit every other point along the road; only by teleporting (by moving discontinuously) could you avoid doing so. More formally, the theorem states that if y is a continuous function on the interval from a to b, and if y takes on values y_1 and y_2 at certain points within this interval, then for any y_3 between y_1 and y_2, there is some x in the interval for which $y(x) = y_3$.[6]

run into practical limits due to the amount of memory your computer has and the speed of its CPU. For fun, try `N(Pi,1000)`, which tells Yacas to compute π numerically to 1000 decimal places.

[6] For a proof of the intermediate value theorem starting from our definition of continuity, see Keisler's *Elementary Calculus: An Approach Using Infinitesimals*, p. 162, available online at `http://www.math.wisc.edu/~keisler/calc.html`.

Problems

1 Carry out a calculation like the one in example 7 on page 22 to show that the derivative of t^4 equals $4t^3$.

2 Example 9 on page 23 gave a tricky argument to show that the derivative of $\cos t$ is $-\sin t$. Prove the same result using the method of example 8 instead.

3 Suppose H is a big number. Experiment on a calculator to figure out whether $\sqrt{H+1}-\sqrt{H-1}$ comes out big, normal, or tiny. Try making H bigger and bigger, and see if you observe a trend. Based on these numerical examples, form a conjecture about what happens to this expression when H is infinite.

4 Suppose $\mathrm{d}x$ is a small but finite number. Experiment on a calculator to figure out how $\sqrt{\mathrm{d}x}$ compares in size to $\mathrm{d}x$. Try making $\mathrm{d}x$ smaller and smaller, and see if you observe a trend. Based on these numerical examples, form a conjecture about what happens to this expression when $\mathrm{d}x$ is infinitesimal.

5 To which of the following statements can the transfer principle be applied? If you think it can't be applied to a certain statement, try to prove that the statement is false for the hyperreals, e.g., by giving a counterexample.

(a) For any real numbers x and y, $x+y=y+x$.
(b) The sine of any real number is between -1 and 1.
(c) For any real number x, there exists another real number y that is greater than x.
(d) For any real numbers $x \neq y$, there exists another real number z such that $x<z<y$.
(e) For any real numbers $x \neq y$, there exists a rational number z such that $x<z<y$. (A rational number is one that can be expressed as an integer divided by another integer.)
(f) For any real numbers x, y, and z, $(x+y)+z=x+(y+z)$.
(g) For any real numbers x and y, either $x<y$ or $x=y$ or $x>y$.
(h) For any real number x, $x+1 \neq x$.

6 Differentiate $(2x+3)^{100}$ with respect to x.

7 Differentiate $(x+1)^{100}(x+2)^{200}$ with respect to x.

8 Differentiate the following with respect to x: e^{7x}, e^{e^x}.

9 Differentiate $a\sin(bx+c)$ with respect to x.

10 Find a function whose derivative with respect to x equals $a\sin(bx+c)$. That is, find an integral of the given function.

11 The range of a gun, when elevated to an angle θ, is given by

$$R=\frac{2v^2}{g}\sin\theta\,\cos\theta \qquad .$$

Find the angle that will produce the maximum range.

12 The hyperbolic cosine function is defined by

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad .$$

Find any minima and maxima of this function.

13 Differentiate $\tan\theta$ with respect to θ.

14 Differentiate $\sqrt[3]{x}$ with respect to x.

15 Differentiate the following with respect to x:
(a) $y = \sqrt{x^2+1}$
(b) $y = \sqrt{x^2+a^2}$
(c) $y = 1/\sqrt{a+x}$
(d) $y = a/\sqrt{(}a - x^2)$

[Thompson, 1919]

16 Differentiate $\ln(2t+1)$ with respect to t.

17 If you know the derivative of $\sin x$, it's not necessary to use the product rule in order to differentiate $3\sin x$, but show that using the product rule gives the right result anyway.

18 The Γ function (capital Greek letter gamma) is a continuous mathematical function that has the property $\Gamma(n) = 1\cdot 2\cdot \ldots \cdot (n-1)$ for n an integer. $\Gamma(x)$ is also well defined for values of x that are not integers, e.g., $\Gamma(1/2)$ happens to be $\sqrt{\pi}$. Use computer software that is capable of evaluating the Γ function to determine numerically the derivative of $\Gamma(x)$ with respect to x, at $x = 2$. (In Yacas, the function is called Gamma.)

19 For a cylinder of fixed surface area, what proportion of length to radius will give the maximum volume?

20 Use a trick similar to the one used in example 12 to prove that the power rule $\mathrm{d}(x^k)/\mathrm{d}x = kx^{k-1}$ applies to cases where k is an integer less than 0. ⋆

3 Integration

3.1 Definite and indefinite integrals

Because any formula can be differentiated symbolically to find another formula, the main motivation for doing derivatives numerically would be if the function to be differentiated wasn't known in symbolic form. A typical example might be a two-person network computer game, in which player A's computer needs to figure out player B's velocity based on knowledge of how her position changes over time. But in most cases, it's numerical integration that's interesting, not numerical differentiation.

As a warm-up, let's see how to do a running sum of a discrete function using Yacas. The following program computers the sum $1+2+\ldots+100$ discussed to on page 7. Now that we're writing real computer programs with Yacas, it would be a good idea to enter each program into a file before trying to run it. In fact, some of these examples won't run properly if you just start up Yacas and type them in one line at a time. If you're using Adobe Reader to read this book, you can do `Tools>Basic>Select`, select the program, copy it into a file, and then edit out the line numbers.

Example 25

```
1 n := 1;
2 sum := 0;
3 While (n<=100) [
4   sum := sum+n;
5   n := n+1;
6 ];
7 Echo(sum);
```

The semicolons are to separate one instruction from the next, and they become necessary now that we're doing real programming. Line 1 of this program defines the variable `n`, which will take on all the values from 1 to 100. Line 2 says that we haven't added anything up yet, so our running sum is zero do far. Line 3 says to keep on repeating the instructions inside the square brackets until `n` goes past 100. Line 4 updates the running sum, and line 5 updates the value of `n`. If you've never done any programming before, a statement like `n:=n+1` might seem like nonsense — how can a number equal itself plus one? But that's why we use the `:=` symbol; it says that we're redefining `n`, not stating an equation. If `n` was previously 37, then after this statement is executed, `n` will be redefined as 38. To run the program on a Linux computer, do this (assuming you saved the program in a file named `sum.yacas`):

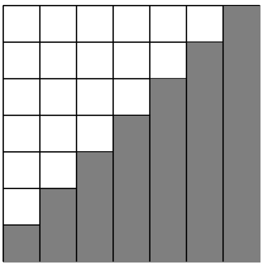
```
% yacas -pc sum.yacas
```

5050

Here the % symbol is the computer's prompt. The result is 5,050, as expected. One way of stating this result is

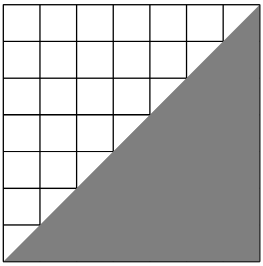
$$\sum_{n=1}^{100} n = 5050 \qquad .$$

The capital Greek letter Σ, sigma, is used because it makes the "s" sound, and that's the first sound in the word "sum." The $n = 1$ below the sigma says the sum starts at 1, and the 100 on top says it ends at 100. The n is what's known as a dummy variable: it has no meaning outside the context of the sum. Figure a shows the graphical interpretation of the sum: we're adding up the areas of a series of rectangular strips. (For clarity, the figure only shows the sum going up to 7, rather than 100.)



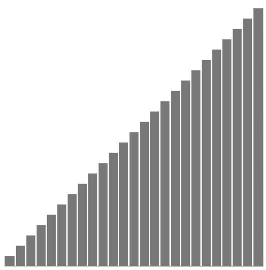
a / Graphical interpretation of the sum 1+2+...+7.

Now how about an integral? Figure b shows the graphical interpretation of what we're trying to do: find the area of the shaded triangle. This is an example we know how to do symbolically, so we can do it numerically as well, and check the answers against each other. Symbolically, the area is given by the integral. To integrate the function $\dot{x}(t) = t$, we know we need some function with a t^2 in it, since we want something whose derivative is t, and differentiation reduces the power by one. The derivative of t^2 would be $2t$ rather than t, so what we want is $x(t) = t^2/2$. Let's compute the area of the triangle that stretches along the t axis from 0 to 100: $x(100) = 100^/2 = 5000$.



b / Graphical interpretation of the integral of the function $\dot{x}(t) = t$.

Figure c shows how to accomplish the same thing numerically. We break up the area into a whole bunch of very skinny rectangles. Ideally, we'd like to make the width of each rectange be an infinitesimal number dx, so that we'd be adding up an infinite number of infinitesimal areas. In reality, a computer can't do that, so we divide up the interval from $t = 0$ to $t = 100$ into H rectangles, each with finite width $\mathrm{d}t = 100/H$. Instead of making H infinite, we make it the largest number we can without making the computer take too long to add up the areas of the rectangles.



c / Approxmating the integral numerically.

Example 26

```
tmax := 100;
H := 1000;
dt := tmax/H;
sum := 0;
t := 0;
While (t<=tmax) [
  sum := N(sum+t*dt);
  t := N(t+dt);
];
Echo(sum);
```

In example 26, we split the interval from $t = 0$ to 100 into $H = 1000$ small intervals, each with width $\mathrm{d}t = 0.1$. The result is 5,005, which agrees with the symbolic result to three digits of precision. Changing H to 10,000 gives $5,000.5$, which is one more digit. Clearly as we make the number of rectangles greater and greater, we're converging to the correct result of 5,000.

In the Leibniz notation, the thing we've just calculated, by two different techniques, is written like this:

$$\int_0^{100} t\,\mathrm{d}t = 5,000$$

It looks a lot like the Σ notation, with the Σ replaces by a flattened-out letter "S." The t is a dummy variable. What I've been casually referring to as an integral is really two different but closely related things, known as the definite integral and the indefinite integral.

Definition of the indefinite integral
If $\dot{x}$ is a function, then a function x is an indefinite integral of $\dot{x}$ if, as implied by the notation, $\mathrm{d}x/\mathrm{d}t = \dot{x}$.
Interpretation: Doing an indefinite integral means doing the opposite of differentiation. All the possible indefinite integrals are the same function except for an additive constant.

Example 27

▷ Find the indefinite integral of the function $\dot{x}(t) = t$.

▷ Any function of the form

$$x(t) = t^2/2 + c \qquad ,$$

where c is a constant, is an indefinite integral of this function, since its derivative is t.

Definition of the definite integral
If $\dot{x}$ is a function, then the definite integral of $\dot{x}$ from a to b is defined as

$$\int_a^b \dot{x}(t)\mathrm{d}t = \lim_{H\to\infty}\sum_{i=0}^{H} \dot{x}\left(a+i\Delta t\right) \quad ,$$

where $\Delta t = (b-a)/H$.
Interpretation: What we're calculating is the area under the graph of $\dot{x}$, from a to b. (If the graph dips below the t axis, we interpret the area between it and the axis as a negative area.) The thing inside the limit is a calculation like the one done in example 26, but generalized to $a \neq 0$. If H was infinite, then Δt would be an infinitesimal number $\mathrm{d}t$.

3.2 The fundamental theorem of calculus

The fundamental theorem of calculus
Let x be an indefinite integral of $\dot{x}$, and let $\dot{x}$ be a continuous function (one whose graph is a single connected curve). Then

$$\int_a^b \dot{x}(t)\mathrm{d}t = x(b) - x(a) \quad .$$

Interpretation: In the simple examples we've been doing so far, we were able to choose an indefinite integral such that $x(0) = 0$. In that case, $x(t)$ is interpreted as the area from 0 to t, so in the expression $x(b) - x(a)$, we're taking the area from 0 to a, but subtracting out the area from 0 to b, which gives the area from a to b. If we choose an indefinite integral with a different c, the c's will just cancel out anyway in the difference $x(b) - x(a)$.

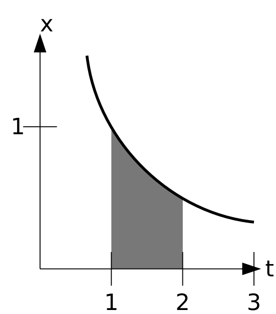
The fundamental theorem is proved on page 83.

Example 28

▷ Interpret the indefinite integral

$$\int_1^2 \frac{1}{t}\,\mathrm{d}t \quad .$$

graphically; then evaluate it it both symbolically and numerically, and check that the two results are consistent.



d / The indefinite integral $\int_1^2 (1/t)\mathrm{d}t$.

▷ Figure d shows the graphical interpretation. The numerical calculation requires a trivial variation on the program from example 26:

```
a := 1;
b := 2;
H := 1000;
dt := (b-a)/H;
sum := 0;
t := a;
While (t<=b) [
  sum := N(sum+(1/t)*dt);
  t := N(t+dt);
];
Echo(sum);
```

The result is 0.693897243, and increasing H to 10,000 gives 0.6932221811, so we can be fairly confident that the result equals 0.693, to 3 decimal places.

Symbolically, the indefinite integral is $x = \ln t$. Using the fundamental theorem of calculus, the area is $\ln 2 - \ln 1 \approx 0.693147180559945$.

Judging from the graph, it looks plausible that the shaded area is about 0.7.

This is an interesting example, because the natural log blows up to negative infinity as t approaches 0, so it's not possible to add a constant onto the indefinite integral and force it to be equal to 0 at $t = 0$. Nevertheless, the fundamental theorem of calculus still works.

3.3 Properties of the integral

Let f and g be two functions of x, and let c be a constant. We already know that for derivatives,

$$\frac{\mathrm{d}}{\mathrm{d}x}(f+g) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}(cf) = c\frac{\mathrm{d}f}{\mathrm{d}x} \qquad .$$

But since the indefinite integral is just the operation of undoing a derivative, the same kind of rules must hold true for indefinite integrals as well:

$$\int (f+g)\mathrm{d}x = \int f\mathrm{d}x + \int g\mathrm{d}x$$

and

$$\int (cf)\mathrm{d}x = c\int f\mathrm{d}x \qquad .$$

And since a definite integral can be found by plugging in the upper and lower limits of integration into the indefinite integral, the same properties must be true of definite integrals as well.

Example 29

▷ Evaluate the indefinite integral

$$\int (x + 2\sin x)\mathrm{d}x \qquad .$$

▷ Using the additive property, the integral becomes

$$\int x\mathrm{d}x + \int 2\sin x\mathrm{d}x \qquad .$$

Then the property of scaling by a constant lets us change this to

$$\int x\mathrm{d}x + 2\int \sin x\mathrm{d}x \qquad .$$

We need a function whose derivative is x, which would be $x^2/2$, and one whose derivative is $\sin x$, which must be $-\cos x$, so the result is

$$\frac{1}{2}x^2 - 2\cos x + c \quad .$$

3.4 Applications

Averages

In the story of Gauss's problem of adding up the numbers from 1 to 100, one interpretation of the result, 5,050, is that the average of all the numbers from 1 to 100 is 50.5. This is the ordinary definition of an average: add up all the things you have, and divide by the number of things. (The result in this example makes sense, because half the numbers are from 1 to 50, and half are from 51 to 100, so the average is half-way between 50 and 51.)

Similarly, a definite integral can also be thought of as a kind of average. In general, if y is a function of x, then the average, or mean, value of y on the interval from $x = a$ to b can be defined as

$$\bar{y} = \frac{1}{b-a}\int_a^b y\,dx \quad .$$

In the continuous case, dividing by $b - a$ accomplishes the same thing as dividing by the number of things in the discrete case.

Example 30

▷ Show that the definition of the average makes sense in the case where the function is a constant.

▷ If y is a constant, then we can take it outside of the integral, so

$$\begin{aligned}\bar{y} &= \frac{1}{b-a}y\int_a^b 1\,dx \\ &= \frac{1}{b-a}y\,x\big|_a^b \\ &= \frac{1}{b-a}y(b-a) \\ &= y\end{aligned}$$

Example 31

▷ Find the average value of the function $y = x^2$ for values of x ranging from 0 to 1.

$$\begin{aligned}\bar{y} &= \frac{1}{1-0}\int_0^1 x^2\,dx \\ &= \frac{1}{3}x^3\Big|_0^1 \\ &= \frac{1}{3}\end{aligned}$$

The mean value theorem
If the continuous function $y(x)$ has the average value $\bar{y}$ on the interval from $x = a$ to b, then y attains its average value at least once in that interval, i.e., there exists ξ with $a < \xi < b$ such that $y(\xi) = \bar{y}$.

The mean value theorem is proved on page 84.

Example 32

▷ Verify the mean value theorem for $y = x^2$ on the interval from 0 to 1.

▷ The mean value is 1/3, as shown in example 31. This value is achieved at $x = \sqrt{1/3} = 1/\sqrt{3}$, which lies between 0 and 1.

Work

In physics, work is a measure of the amount of energy transferred by a force; for example, if a horse sets a wagon in motion, the horse's force on the wagon is putting some energy of motion into the wagon. When a force F acts on an object that moves in the direction of the force by an infinitesimal distance $\mathrm{d}x$, the infinitesimal work done is $\mathrm{d}W = F\mathrm{d}x$. Integrating both sides, we have $W = \int_a^b F\mathrm{d}x$, where the force may depend on x, and a and b represent the initial and final positions of the object.

Example 33

▷ A spring compressed by an amount x relative to its relaxed length provides a force $F = kx$. Find the amount of work that must be done in order to compress the spring from $x = 0$ to $x = a$. (This is the amount of energy stored in the spring, and that energy will later be released into the toy bullet.)

▷

$$
\begin{aligned}
W &= \int_0^a F\mathrm{d}x \\
&= \int_0^a kx\mathrm{d}x \\
&= \frac{1}{2}kx^2\Big|_0^a \\
&= \frac{1}{2}ka^2
\end{aligned}
$$

The reason W grows like a^2, not just like a, is that as the spring is compressed more, more and more effort is required in order to compress it.

Problems

1 Write a computer program similar to the one in example 28 on page 44 to evaluate the definite integral

$$\int_0^1 e^{x^2} \qquad .$$

2 Evaluate the integral

$$\int_0^{2\pi} \sin x \, dx \qquad ,$$

and draw a sketch to explain why your result comes out the way it does.

3 Sketch the graph that represents the definite integral

$$\int_0^2 -x^2 + 2x \qquad ,$$

and estimate the result roughly from the graph. Then evaluate the integral exactly, and check against your estimate.

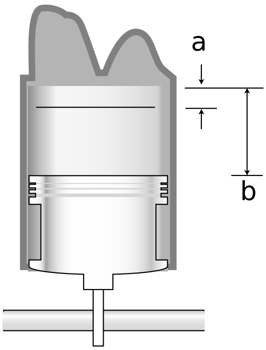
4 Show that the mean value theorem's assumption of continuity is necessary, by exhibiting a discontinuous function for which the theorem fails.

5 Show that the fundamental theorem of calculus's assumption of continuity for $\dot{x}$ is necessary, by exhibiting a discontinuous function for which the theorem fails.

6 Find the average value of $\sin x$ for $0 < x < \pi$.

7 Sketch the graphs of $y = x^2$ and $y = \sqrt{x}$ for $0 \le x \le 1$. Graphically, what relationship should exist between the integrals $\int_0^1 x^2 \, dx$ and $\int_0^1 \sqrt{x} \, dx$? Compute both integrals, and verify that the results are related in the expected way.

8 In a gasoline-burning car engine, the exploding air-gas mixture makes a force on the piston, and the force tapers off as the piston expands, allowing the gas to expand. (a) In the approximation $F = k/x$, where x is the position of the piston, find the work done on the piston as it travels from $x = a$ to $x = b$, and show that the result only depends on the ratio b/a. This ratio is known as the compression ratio of the engine. (b) A better approximation, which takes into account the cooling of the air-gas mixture as it expands, is $F = kx^{-1.4}$. Compute the work done in this case.



Problem 8.

9 A perfectly elastic ball bounces up and down forever, always coming back up to the same height h. Find its average height. ⋆

4 Techniques and applications

4.1 Newton's method

In the 1958 science fiction novel **Have Space Suit — Will Travel**, by Robert Heinlein, Kip is a high school student who wants to be an engineer, and his father is trying to convince him to stretch himself more if he wants to get anything out of his education:

"Why did Van Buren fail of re-election? How do you extract the cube root of eighty-seven?"

Van Buren had been a president; that was all I remembered. But I could answer the other one. "If you want a cube root, you look in a table in the back of the book."

Dad sighed. "Kip, do you think that table was brought down from on high by an archangel?"

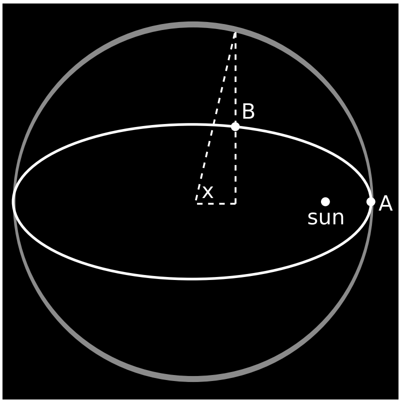
We no longer use tables to compute roots, but how does a pocket calculator do it? A technique called Newton's method allows us to calculate the inverse of any function efficiently, including cases that aren't preprogrammed into a calculator. In the example from the novel, we know how to calculate the function $y = x^3$ fairly accurately and quickly for any given value of x, but we want to turn the equation around and find x when $y = 87$. We start with a rough mental guess: since $4^3 = 64$ is a little too small, and $5^3 = 125$ is much too big, we guess $x \approx 4.3$. Testing our guess, we have $4.3^3 = 79.5$. We want y to get bigger by 7.5, and we can use calculus to find approximately how much bigger x needs to get in order to accomplish that:

$$\begin{aligned}
\Delta x &= \frac{\Delta x}{\Delta y}\Delta y \\
&\approx \frac{\mathrm{d}x}{\mathrm{d}y}\Delta y \\
&= \frac{\Delta y}{\mathrm{d}y/\mathrm{d}x} \\
&= \frac{\Delta y}{3x^2} \\
&= \frac{\Delta y}{3x^2} \\
&= 0.14
\end{aligned}$$

Increasing our value of x to $4.3 + 0.14 = 4.44$, we find that $4.44^3 = 87.5$ is a pretty good approximation to 87. If we need higher precision, we can go through the process again with $\Delta y = -0.5$, giving

$$\begin{aligned}
\Delta x &\approx \frac{\Delta y}{3x^2} \\
&= 0.14 \\
x &= 4.43 \\
x^3 &= 86.9 \qquad .
\end{aligned}$$

This second iteration gives an excellent approximation.



a / Example 34.

Example 34

▷ Figure 34 shows the astronomer Johannes Kepler's analysis of the motion of the planets. The ellipse is the orbit of the planet around the sun. At $t = 0$, the planet is at its closest approach to the sun, A. At some later time, the planet is at point B. The angle x (measured in radians) is defined with reference to the imaginary circle encompassing the orbit. Kepler found the equation

$$2\pi\frac{t}{T} = x - e\sin x \qquad ,$$

where the period, T, is the time required for the planet to complete a full orbit, and the eccentricity of the ellipse, e, is a number that measures how much it differs from a circle. The relationship is complicated because the planet speeds up as it falls inward toward the sun, and slows down again as it swings back away from it.

The planet Mercury has $e = 0.206$. Find the angle x when Mercury has completed 1/4 of a period.

▷ We have

$$y = x - (0.206)\sin x \qquad ,$$

and we want to find x when $y = 2\pi/4 = 1.57$. As a first guess, we try $x = \pi/2$ (90 degrees), since the eccentricity of Mercury's orbit is actually much smaller than the example shown in the figure, and therefore the planet's speed doesn't vary all that much as it goes around the sun. For this value of x we have $y = 1.36$, which is too small by 0.21.

$$\begin{aligned}\Delta x &\approx \frac{\Delta y}{\mathrm{d}y/\mathrm{d}x}\\ &= \frac{0.21}{1-(0.206)\cos x}\\ &= 0.21\end{aligned}$$

(The derivative $\mathrm{d}y/\mathrm{d}x$ happens to be 1 at $x = \pi/2$.) This gives a new value of x, 1.57+.21=1.78. Testing it, we have $y = 1.58$, which is correct to within rounding errors after only one iteration. (We were only supplied with a value of e accurate to three significant figures, so we can't get a result with precision better than about that level.)

4.2 Implicit differentiation

We can differentiate any function that is written as a formula, and find a result in terms of a formula. However, sometimes the original problem can't be written in any nice way as a formula. For example, suppose we want to find $\mathrm{d}y/\mathrm{d}x$

in a case where the relationship between x and y is given by the following equation:

$$y^7 + y = x^7 + x^2 \quad .$$

There is no equivalent of the quadratic formula for seventh-order polynomials, so we have no way to solve for one variable in terms of the other in order to differentiate it. However, we can still find $\mathrm{d}y/\mathrm{d}x$ in terms of x and y. Suppose we let x grow to $x + \mathrm{d}x$. Then for example the x^2 term will grow to $(x+\mathrm{d}x)^2 = x+2\mathrm{d}x+\mathrm{d}x^2$. The squared infinitesimal is negligible, so the increase in x^2 was really just $2\mathrm{d}x$, and we've really just computer the derivative of x^2 with respect to x and multiplied it by $\mathrm{d}x$. In symbols,

$$\begin{aligned} \mathrm{d}(x^2) &= \frac{\mathrm{d}(x^2)}{\mathrm{d}x} \cdot \mathrm{d}x \\ &= 2x\,\mathrm{d}x \quad . \end{aligned}$$

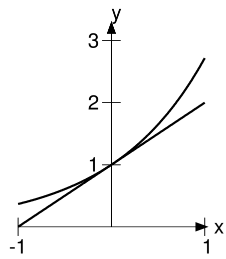
That is, the change in x^2 is $2x$ times the change in x. Doing this to both sides of the original equation, we have

$$\begin{aligned} \mathrm{d}(y^7 + y) &= \mathrm{d}(x^7 + x^2) \\ 7y^6\,\mathrm{d}y + 1\,\mathrm{d}y &= 7x^6\,\mathrm{d}x + 2x\,\mathrm{d}x \\ (7y^6 + 1)\mathrm{d}y &= (7x^6 + 2x)\mathrm{d}x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{7y^6+1}{7x^6+2x} \quad . \end{aligned}$$

This still doesn't give us a formula for the derivative in terms of x alone, but it's not entirely useless. For instance, if we're given a numerical value of x, we can always use Newton's method to find y, and then evaluate the derivative.

4.3 Taylor series

If you calculate $e^{0.1}$ on your calculator, you'll find that it's very close to 1.1. This is because the tangent line at $x = 0$ on the graph of e^x has a slope of 1 ($\mathrm{d}e^x/\mathrm{d}x = e^x = 1$ at $x = 0$), and the tangent line is a good approximation to the exponential curve as long as we don't get too far away from the point of tangency.

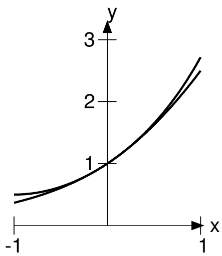


b / The function e^x, and the tangent line at $x = 0$.

How big is the error? The actual value of $e^{0.1}$ is $1.10517091807565\ldots$, which differs from 1.1 by about 0.005. If we go farther from the point of tangency, the approximation gets worse. At $x = 0.2$, the error is about 0.021, which is about four times bigger. In other words, doubling x seems to roughly quadruple the error, so the error is proportional to x^2; it seems to be about $x^2/2$. Well, if we want a handy-dandy, super-accurate estimate of e^x for small values of x, why not just account for this error. Our new and improved estimate is

$$e^x \approx 1 + x + \frac{1}{2}x^2$$

for small values of x.

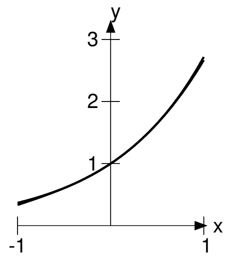


c / The function e^x, and the approximation $1 + x + x^2/2$.

Figure c shows that the approximation is now extremely good for sufficiently small values of x. The difference is that whereas $1 + x$ matched both the y-intercept and the slope of the curve, $1+x+x^2/2$ matches the curvature as well. Recall that the second derivative is a measure of curvature. The second derivatives of the function and its approximation are

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(1 + x + \frac{1}{2}x^2\right) = 1$$



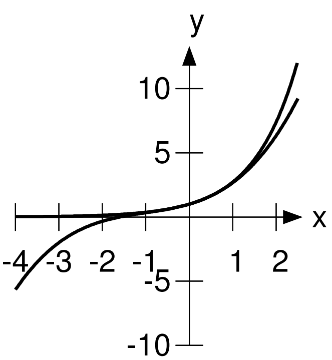
d / The function e^x, and the approximation $1 + x + x^2/2 + x^3/6$.

We can do even better. Suppose we want to match the third derivatives. All the derivatives of e^x, evaluated at $x = 0$, are 1, so we just need to add on a term proportional to x^3 whose third derivative is one. Taking the first derivative will bring down a factor of 3 in front, and taking and the second derivative will give a 2, so to cancel these out we need the third-order term to be (1/2)(1/3):

$$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{2\cdot 3}x^3$$

Figure d shows the result. For a significant range of x values close to zero, the approximation is now so good that we can't even see the difference between the two functions on the graph.

On the other hand, figure e shows that the cubic approximation for somewhat larger negative and positive values of x is poor — worse, in fact, than the linear approximation, or even the constant approximation $e^x = 1$. This is to be expected, because any polynomial will blow up to either positive or negative infinity as x approaches negative infinity, whereas the function e^x is supposed to get very close to zero for large negative x. The idea here is that derivatives are *local* things: they only measure the properties of a function very close to the point at which they're evaluated, and they don't necessarily tell us anything about points far away.



e / The function e^x, and the approximation $1 + x + x^2/2 + x^3/6$, on a wider scale.

It's a remarkable fact, then, that by taking enough terms in a polynomial approximation, we can always get as good an approximation to e^x as necessary — it's just that a large number of terms may be required for large values of x. In other words, the *infinite series*

$$1 + x + \frac{1}{2}x^2 + \frac{1}{2 \cdot 3}x^3 + \ldots$$

always gives exactly e^x. But what is the pattern here that would allows us to figure out, say, the fourth-order and fifth-order terms that were swapt under the rug with the symbol "..."? Let's do the fifth-order term as an example. The point of adding in a fifth-order term is to make the fifth derivative of the approximation equal to the fifth derivative of e^x, which is 1. The first, second, ... derivatives of x^5 are

$$\frac{\mathrm{d}}{\mathrm{d}x}x^5 = 5x^4$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}x^5 = 5 \cdot 4x^3$$
$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}x^5 = 5 \cdot 4 \cdot 3x^2$$
$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}x^5 = 5 \cdot 4 \cdot 3 \cdot 2x$$
$$\frac{\mathrm{d}^5}{\mathrm{d}x^5}x^5 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

The notation for a product like $1 \cdot 2 \cdot \ldots \cdot n$ is $n!$, read "n factorial." So to get a term for our polynomial whose fifth derivative is 1, we need $x^5/5!$. The result for the infinite series is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad ,$$

where the special case of $0! = 1$ is assumed.[1] This is called the *Taylor series* for e^x, evaluated around $x = 0$, and it's true, although I haven't proved it, that this particular Taylor series always converges

[1] This makes sense, because, for example, 4!=5!/5, 3!=4!/4, etc., so we should have 0!=1!/1.

to e^x, no matter how far x is from zero.

A Taylor series can be used to approximate other functions besides e^x, and when you ask your calculator to evaluate a function such as a sine or a cosine, it may actually be using a Taylor series to do it. In general, the Taylor series around $x = 0$ for a function y is

$$T_0(x) = \sum_{n=0}^{\infty} a_n x^n \qquad ,$$

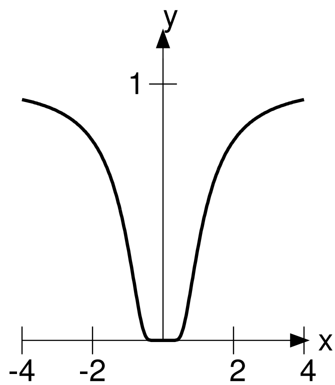
where the condition for equality of the nth order derivative is

$$a_n = \frac{1}{n!} \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\bigg|_{x=0} \qquad .$$

Here the notation $|_{x=0}$ means that the derivative is to be evaluated at $x = 0$.

Example 35

The function $y = e^{-1/x^2}$, shown in figure f, never converges to its Taylor series, except at $x = 0$. This is because the Taylor series for this function, evaluated around $x = 0$ is exactly zero! At $x = 0$, we have $y = 0$, $\mathrm{d}y/\mathrm{d}x = 0$, $\mathrm{d}^2y/\mathrm{d}x^2 = 0$, and so on for every derivative. The zero function matches the function $y(x)$ and all its derivatives to all orders, and yet is useless as an approximation to $y(x)$.



f / The function e^{-1/x^2} never converges to its Taylor series.

In general, every function's Taylor series around $x = 0$ converges to the function for all values of x in the range defined by $|x| < r$, where r is some number, known as the radius of convergence. For the function e^x, the radius of convergence happens to be infinite, whereas for e^{-1/x^2} it's zero.

A function's Taylor series doesn't have to be evaluated around $x = 0$. The Taylor series around some other center $x = c$ is given by

$$T_c(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \qquad ,$$

where

$$\frac{a_n}{n!} = \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\bigg|_{x=c} \qquad .$$

Example 36

▷ Find the Taylor series of $y = \sin x$, evaluated around $x = 0$.

▷ The first few derivatives are

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d^2}{dx^2}\sin x = -\sin x$$
$$\frac{d^3}{dx^3}\sin x = -\cos x$$
$$\frac{d^4}{dx^4}\sin x = \sin x$$
$$\frac{d^5}{dx^5}\sin x = \cos x$$

We can see that there will be a cycle of sin, cos, − sin, and − cos, repeating indefinitely. Evaluating these derivatives at $x = 0$, we have 0, 1, 0, −1, ... All the even-order terms of the series are zero, and all the odd-order terms are $\pm 1/n!$. The result is

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots \quad .$$

The linear term is the familiar small-angle approximation $\sin x \approx x$.

The radius of convergence of this series turns out to be infinite. Intuitively the reason for this is that the factorials grow extremely rapidly, so that the successive terms in the series eventually start diminish quickly, even for large values of x.

Example 37

▷ Find the Taylor series of $y = 1/(1-x)$ around $x = 0$, and see what you can say about its radius of convergence.

▷ Rewriting the function as $y = (1-x)^{-1}$ and applying the chain rule, we have

$$y|_{x=0} = 1$$
$$\left.\frac{dy}{dx}\right|_{x=0} = \left.(1-x)^{-2}\right|_{x=0} = 1$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = \left.2(1-x)^{-3}\right|_{x=0} = 2$$
$$\left.\frac{d^3y}{dx^3}\right|_{x=0} = \left.2\cdot 3(1-x)^{-4}\right|_{x=0} = 2\cdot 3$$
$$\ldots$$

The pattern is that the nth derivative is $n!$. The Taylor series therefore has $a_n = n!/n! = 1$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

The radius of convergence of this series definitely can't be greater than 1, since for $x = 1$ the series is $1 + 1 + 1 + \ldots$, which grows indefinitely without ever converging to a specific number. Likewise for $x = -1$ the series becomes $1 - 1 + 1 - 1 + \ldots$, which oscillates back and forth rather than converging. Intuitively we can see a couple of hints as to why this happens: (1) The function $1/(1-x)$ itself misbehaves at $x = 1$, blowing up to infinity; (2) The only way a series can get closer and closer to a finite value is if the absolute values of the terms decrease, and decrease sufficiently rapidly. But the coefficients a_n of this Taylor series don't decrease with n, so so the only way the absolute values of the terms can decrease is for $|x| < 1$.

4.4 Methods of integration

Change of variable

Sometimes and unfamiliar-looking integral can be made into a familiar one by substituting a a new variable for an old one. For example, we know how to integrate $1/x$ — the answer is $\ln x$ — but what about

$$\int \frac{\mathrm{d}x}{2x+1} \qquad ?$$

Let $u = 2x + 1$. Differentiating both sides, we have $\mathrm{d}u = 2\mathrm{d}x$, or $\mathrm{d}x = \mathrm{d}u/2$, so

$$\begin{aligned}\int \frac{\mathrm{d}x}{2x+1} &= \int \frac{\mathrm{d}u/2}{u} \\ &= \frac{1}{2}\ln u + c \\ &= \frac{1}{2}\ln(2x+1) + c \qquad .\end{aligned}$$

In the case of a definite integral, we have to remember to change the limits of integration to reflect the new variable.

Example 38

▷ Evaluate $\int_3^4 \mathrm{d}x/(2x+1)$.

▷ As before, let $u = 2x + 1$.

$$\begin{aligned}\int_{x=3}^{x=4} \frac{\mathrm{d}x}{2x+1} &= \int_{u=7}^{u=9} \frac{\mathrm{d}u/2}{u} \\ &= \frac{1}{2}\ln u \Big|_{u=7}^{u=9}\end{aligned}$$

Here the notation $|_{u=7}^{u=9}$ means to evaluate the function at 7 and 9, and subtract the former from the latter. The result is

$$\begin{aligned}\int_{x=3}^{x=4} \frac{\mathrm{d}x}{2x+1} &= \frac{1}{2}(\ln 9 - \ln 7) \\ &= \frac{1}{2}\ln\frac{9}{7} \qquad .\end{aligned}$$

Sometimes, as in the next example, a clever substitution is the secret to doing a seemingly impossible integral.

Example 39

▷ Evaluate

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\mathrm{d}x \qquad .$$

▷ The only hope for reducing this to a form we can do is to let $u = \sqrt{x}$. Then $\mathrm{d}x = \mathrm{d}(u^2) = 2u\mathrm{d}u$, so

$$\begin{aligned}\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\mathrm{d}x &= \int \frac{e^u}{u} \cdot 2u\,\mathrm{d}u \\ &= 2\int e^u \mathrm{d}u \\ &= 2e^u \\ &= 2e^{\sqrt{x}} \qquad .\end{aligned}$$

Example 39 really isn't so tricky, since there was only one logical choice for the substitution that had any hope of working. The following is a little more dastardly.

Example 40

▷ Evaluate

$$\int \frac{\mathrm{d}x}{1+x^2} \qquad .$$

▷ The substitution that works is $x = \tan u$. First let's see what this does to the expression $1 + x^2$. The familiar identity

$$\sin^2 u + \cos^2 u = 1 \quad ,$$

when divided by $\cos^2 u$, gives

$$\tan^2 u + 1 = \sec^2 u \quad ,$$

so $1 + x^2$ becomes $\sec^2 u$. But differentiating both sides of $x = \tan u$ gives

$$\begin{aligned} dx &= d\left[\sin u(\cos u)^{-1}\right] \\ &= (d \sin u)(\cos u)^{-1} \\ &\quad + (\sin u) d\left[(\cos u)^{-1}\right] \\ &= \left(1 + \tan^2 u\right) du \\ &= \sec^2 u\, du \quad , \end{aligned}$$

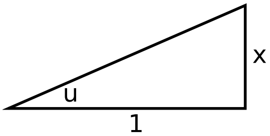
so the integral becomes

$$\begin{aligned} \int \frac{dx}{1+x^2} &= \int \frac{\sec^2 u\, du}{\sec^2 u} \\ &= u + c \\ &= \tan^{-1} x + c \quad . \end{aligned}$$

What mere mortal would ever have suspected that the substitution $x = \tan u$ was the one that was needed in example 40? One possible answer is to give up and do the integral on a computer:

```
Integrate(x) 1/(1+x^2)
  ArcTan(x)
```

Another possible answer is that you can usually smell the possibility of this type of substitution, involving a trig function, when the thing to be integrated contains something reminiscent of the Pythagoran theorem, as suggested by figure g. The $1 + x^2$ looks like what you'd get if you had a right triangle with legs 1 and x, and were using the Pythagorean theorem to find its hypotenuse.



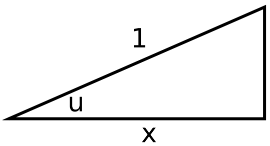
g / The substitution $x = \tan u$.

Example 41

▷ Evaluate $\int dx/\sqrt{1-x^2}$.

▷ The $\sqrt{1-x^2}$ looks like what you'd get if you had a right triangle with hypotenuse 1 and a leg of length x, and were using the Pythagorean theorem to find the other leg, as in figure h. This motivates us to try the substitution $x = \cos u$, which gives $dx = -\sin u\ du$ and $\sqrt{1-x^2} = \sqrt{1-\cos^2 u} = \sin u$. The result is

$$\begin{aligned} \int \frac{dx}{\sqrt{1-x^2}} &= \int \frac{-\sin u\, du}{\sin u} \\ &= u + c \\ &= \cos^{-1} x \quad . \end{aligned}$$

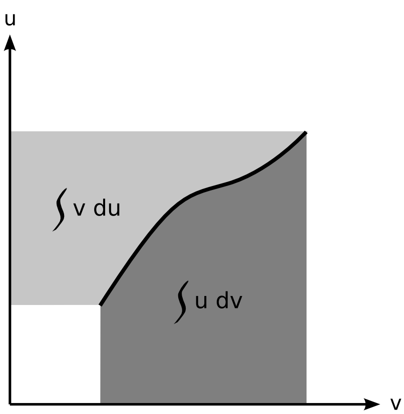


h / The substitution $x = \cos u$.

4.5 Integration by parts

Figure i shows a technique called integration by parts. If the integral $\int v\mathrm{d}u$ is easier than the integral $\int u\mathrm{d}v$, then we can calculate the easier one, and then by simple geometry determine the one we wanted. Identifying the large rectangle that surrounds both shaded areas, and the small white rectangle on the lower left, we have

$$\int u\,\mathrm{d}v = \text{(area of large rectangle)} - \text{(area of small rectangle)} \int v\,\mathrm{d}u \quad .$$



i / Integration by parts.

In the case of an indefinite integral, we have a similar relationship derived from the product rule:

$$\mathrm{d}(uv) = u\,\mathrm{d}v + v\,\mathrm{d}u$$
$$u\,\mathrm{d}v = \mathrm{d}(uv) - v\,\mathrm{d}u$$

Integrating both sides, we have the following relation.

Integration by parts

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u \quad .$$

Since a definite integral can always be done by evaluating an indefinite integral at its upper and lower limits, one usually uses this form. Integrals don't usually come prepackaged in a form that makes it obvious that you should use integration by parts. What the equation for integration by parts tells us is that if we can split up the integrand into two factors, one of which (the dv) we know how to integrate, we have the option of changing the integral into a new form in which that factor becomes its integral, and the other factor becomes its derivative. If we choose the right way of splitting up the integrand into parts, the result can be a simplification.

Example 42

▷ Evaluate

$$\int x \cos x \,\mathrm{d}x$$

▷ There are two obvious possibilities for splitting up the integrand into factors,

$$u\,dv = (x)(\cos x\,dx)$$

or

$$u\,dv = (\cos x)(x\,dx) \qquad .$$

The first one is the one that lets us make progress. If $u = x$, then $du = dx$, and if $dv = \cos x\,dx$, then integration gives $v = \sin x$.

$$\begin{aligned}\int x\cos x\,dx &= \int u\,dv \\ &= uv - \int v\,du \\ &= x\sin x - \int \sin x\,dx \\ &= x\sin x + \cos x\end{aligned}$$

Of the two possibilities we considered for u and dv, the reason this one helped was that differentiating x gave dx, which was simpler, and integrating $\cos x dx$ gave $\sin x$, which was no more complicated than before. The second possibility would have made things worse rather than better, because integrating $x dx$ would have given $x^2/2$, which would have been more complicated rather than less.

4.6 Partial fractions

Given a function like

$$\frac{-1}{x-1} + \frac{1}{x+1} \qquad ,$$

we can rewrite it over a common denominator like this:

$$\begin{aligned}&\left(\frac{-1}{x-1}\right)\left(\frac{x+1}{x+1}\right) \\ &+ \left(\frac{1}{x+1}\right)\left(\frac{x-1}{x-1}\right) \\ &= \frac{-x-1+x-1}{(x-1)(x+1)} \\ &= \frac{-2}{x^2-1} \qquad .\end{aligned}$$

But note that the original form is easily integrated to give

$$\int\left(\frac{-1}{x-1}+\frac{1}{x+1}\right)\mathrm{d}x$$
$$= -\ln(x-1)+\ln(x+1)+c \qquad ,$$

while faced with the form $-2/(x^2-1)$, we wouldn't have known how to integrate it.

The idea of the method of partial fractions is that if we want to do an integral of the form

$$\int\frac{\mathrm{d}x}{P(x)} \qquad ,$$

where $P(x)$ is an nth order polynomial, we can always rewrite $1/P$ as

$$\frac{1}{P(x)} = \frac{A_1}{x-r_1} + \ldots \frac{A_n}{x-r_n} \qquad ,$$

where $r_1 \ldots r_n$ are the roots of the polynomial, i.e., the solutions of the equation $P(r) = 0$. If the polynomial is second-order, you can find the roots r_1 and r_2 using the quadratic formula; I'll assume for the time being that they're real. For higher-order polynomials, there is no surefire, easy way of finding the roots by hand, and you'd be smart simply to use computer software to do it. In Yacas, you can find the real roots of a polynomial like this:

```
FindRealRoots(x^4-5*x^3
   -25*x^2+65*x+84)
  {3.,7.,-4.,-1.}
```

(I assume it uses Newton's method to find them.) The constants A_i can then be determined by algebra, or by the trick of evaluating $1/P(x)$ for a value of x very close to one of the roots. In the example of the polynomial $x^4 - 5x^3 - 25x^2 + 65x + 84$, let $r_1 \ldots r_4$ be the roots in the order in which they were returned by Yacas. Then A_1 can be found by evaluating $1/P(x)$ at $x = 3.000001$:

```
P(x):=x^4-5*x^3-25*x^2
  +65*x+84
N(1/P(3.000001))
  -8928.5702094768
```

We know that for x very close to 3, the expression

$$\frac{1}{P} = \frac{A_1}{x-3} + \frac{A_2}{x-7} + \frac{A_3}{x+4} + \frac{A_4}{x+1}$$

will be dominated by the A_1 term, so

$$-8930 \approx \frac{A_1}{3.000001-3}$$
$$A_1 \approx (-8930)(10^{-6}) \qquad .$$

By the same method we can find the other four constants:

```
dx:=.000001
N(1/P(7+dx),30)*dx
  0.2840908276e-2
N(1/P(-4+dx),30)*dx
  -0.4329006192e-2
N(1/P(-1+dx),30)*dx
  0.1041666664e-1
```

(The `N( ,30)` construct is to tell Yacas to do a numerical calculation rather than an exact symbolic one, and to use 30 digits of precision, in order to avoid problems with rounding errors.) Thus,

$$\begin{aligned} \frac{1}{P} = & \frac{-8.93 \times 10^{-3}}{x-3} \\ & + \frac{2.84 \times 10^{-3}}{x-7} \\ & - \frac{4.33 \times 10^{-3}}{x+4} \\ & + \frac{1.04 \times 10^{-2}}{x+1} \qquad . \end{aligned}$$

The desired integral is

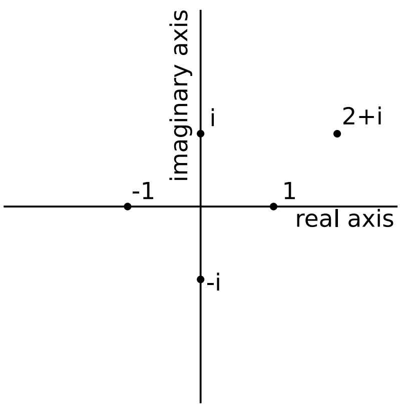
$$\begin{aligned} \int \frac{dx}{P(x)} = & -8.93 \times 10^{-3} \ln(x-3) \\ & + 2.84 \times 10^{-3} \ln(x-7) \\ & - 4.33 \times 10^{-3} \ln(x+4) \\ & + 1.04 \times 10^{-2} \ln(x+1) \\ & + c \qquad . \end{aligned}$$

There are some possible complications: (1) The same factor may occur more than once, as in $x^3-5x^2+7x-3 = (x-1)(x-1)(x-3)$. In this example, we have to look for an answer of the form $A/(x-1)+B/(x-1)^2+C/(x-3)$, the solution being $-.25/(x-1)-.5/(x-1)^2+.25/(x-3)$. (2) The roots may be complex. This is no showstopper if you're using computer software that handles complex numbers gracefully. (You can choose a c that makes the result real.) In fact, as discussed in section 5.3, some beautiful things can happen with complex roots. But as an alternative, any polynomial with real coefficients can be factored into linear and quadratic factors with real coefficients. For each quadratic factor $Q(x)$, we then have a partial fraction of the form $(A+Bx)/Q(x)$, where A and B can be determined by algebra.

5 Complex number techniques

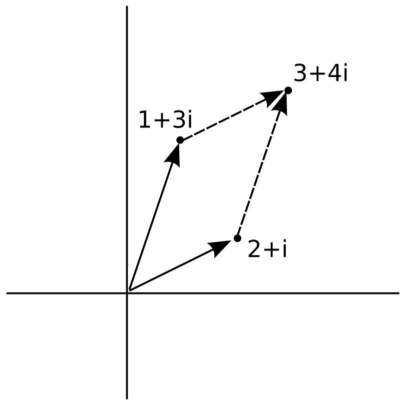
5.1 Review of complex numbers

For a more detailed treatment of complex numbers, see ch. 3 of James Nearing's free book at `http://www.physics.miami.edu/nearing/mathmethods/`.

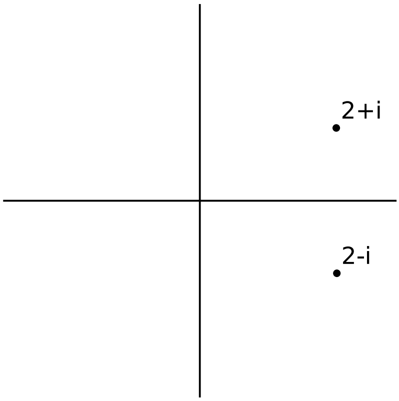


a / Visualizing complex numbers as points in a plane.

We assume there is a number, i, such that $i^2 = -1$. The square roots of -1 are then i and $-i$. (In electrical engineering work, where i stands for current, j is sometimes used instead.) This gives rise to a number system, called the complex numbers, containing the real numbers as a subset. Any complex number z can be written in the form $z = a + bi$, where a and b are real, and a and b are then referred to as the real and imaginary parts of z. A number with a zero real part is called an imaginary number. The complex numbers can be visualized as a plane, figure a, with the real number line placed horizontally like the x axis of the familiar $x-y$ plane, and the imaginary numbers running along the y axis. The complex numbers are complete in a way that the real numbers aren't: every nonzero complex number has two square roots. For example, 1 is a real



b / Addition of complex numbers is just like addition of vectors, although the real and imaginary axes don't actually represent directions in space.



c / A complex number and its conjugate.

number, so it is also a member of the complex numbers, and its square roots are -1 and 1. Likewise, -1 has square roots i and $-i$, and the number i has square roots $1/\sqrt{2}+i/\sqrt{2}$ and $-1/\sqrt{2}-i/\sqrt{2}$.

Complex numbers can be added and subtracted by adding or subtracting their real and imaginary parts, figure b. Geometrically, this is the same as vector addition.

The complex numbers $a+bi$ and $a-bi$, lying at equal distances above and below the real axis, are called complex conjugates. The results of the quadratic formula are either both real, or complex conjugates of each other. The complex conjugate of a number z is notated as $\bar{z}$ or z^*.

The complex numbers obey all the same rules of arithmetic as the reals, except that they can't be ordered along a single line. That is, it's not possible to say whether one complex number is greater than another. We can compare them in terms of their magnitudes (their distances from the origin), but two distinct complex numbers may have the same magnitude, so, for example, we can't say whether 1 is greater than i or i is greater than 1.

Example 43

▷ Prove that $1/\sqrt{2}+i/\sqrt{2}$ is a square root of i.

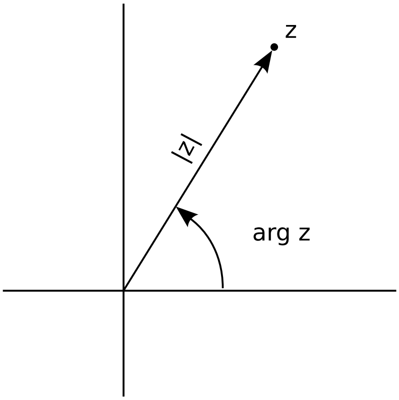
▷ Our proof can use any ordinary rules of arithmetic, except for ordering.

$$\begin{aligned}(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}})^2 &= \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cdot\frac{i}{\sqrt{2}} \\ &\quad+\frac{i}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\cdot\frac{i}{\sqrt{2}} \\ &= \frac{1}{2}(1+i+i-1) \\ &= i\end{aligned}$$

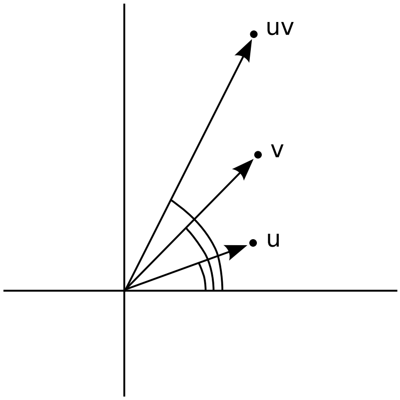
Example 43 showed one method of multiplying complex numbers. However, there is another nice interpretation of complex multiplication. We define the argument of a complex number, figure d, as its angle in the complex plane, measured counterclockwise from the positive real axis. Multiplying two complex numbers then corresponds to multiplying their magnitudes, and adding their arguments, figure e.

Self-Check

Using this interpretation of multiplication, how could you find the square



d / A complex number can be described in terms of its magnitude and argument.



e / The argument of *uv* is the sum of the arguments of *u* and *v*.

roots of a complex number? ▷ Answer, p. 79

Example 44

The magnitude $|z|$ of a complex number z obeys the identity $|z|^2 = z\bar{z}$. To prove this, we first note that $\bar{z}$ has the same magnitude as z, since flipping it to the other side of the real axis doesn't change its distance from the origin. Multiplying z by $\bar{z}$ gives a result whose magnitude is found by multiplying their magnitudes, so the magnitude of $z\bar{z}$ must therefore equal $|z|^2$. Now we just have to prove that $z\bar{z}$ is a positive real number. But if, for example, z lies counterclockwise from the real axis, then $\bar{z}$ lies clockwise from it. If z has a positive argument, then $\bar{z}$ has a negative one, or vice-versa. The sum of their arguments is therefore zero, so the result has an argument of zero, and is on the positive real axis. [1]

This whole system was built up in order to make every number have square roots. What about cube roots, fourth roots, and so on? Does it get even more weird when you want to do those as well? No. The complex number system we've already discussed is sufficient to handle all of them. The nicest way of thinking about it is in terms of roots of polynomials. In the real number system, the polynomial $x^2 - 1$ has two roots, i.e., two values of x (plus and minus one) that we can plug in to the polynomial and get zero. Because it has these two real roots, we can rewrite the polynomial as $(x - 1)(x + 1)$. However, the polynomial $x^2 + 1$ has no real roots. It's ugly that in the real number system, some second-

[1] I cheated a little. If z's argument is 30 degrees, then we could say $\bar{z}$'s was -30, but we could also call it 330. That's OK, because 330+30 gives 360, and an argument of 360 is the same as an argument of zero.

order polynomials have two roots, and can be factored, while others can't. In the complex number system, they all can. For instance, $x^2 + 1$ has roots i and $-i$, and can be factored as $(x - i)(x + i)$. In general, the fundamental theorem of algebra states that in the complex number system, any nth-order polynomial can be factored completely into n linear factors, and we can also say that it has n complex roots, with the understanding that some of the roots may be the same. For instance, the fourth-order polynomial $x^4 + x^2$ can be factored as $(x-i)(x+i)(x-0)(x-0)$, and we say that it has four roots, i, $-i$, 0, and 0, two of which happen to be the same. This is a sensible way to think about it, because in real life, numbers are always approximations anyway, and if we make tiny, random changes to the coefficients of this polynomial, it will have four distinct roots, of which two just happen to be very close to zero. I've given a proof of the fundamental theorem of algebra on page 85.

Example 45

Find $\arg i$, $\arg(-i)$, and $\arg 37$, where $\arg z$ denotes the argument of the complex number z.

Example 46

Visualize the following multiplications in the complex plane using the interpretation of multiplication in terms of multiplying magnitudes and adding arguments: $(i)(i) = -1$, $(i)(-i) = 1$, $(-i)(-i) = -1$.

Example 47

If we visualize z as a point in the complex plane, how should we visualize $-z$?

Example 48

Find four different complex numbers z such that $z^4 = 1$.

Example 49

Compute the following:

$$|1+i| \quad , \quad \arg(1+i) \quad , \quad \left|\frac{1}{1+i}\right| \quad ,$$
$$\arg\left(\frac{1}{1+i}\right) \quad , \quad \frac{1}{1+i}$$

5.2 Euler's formula

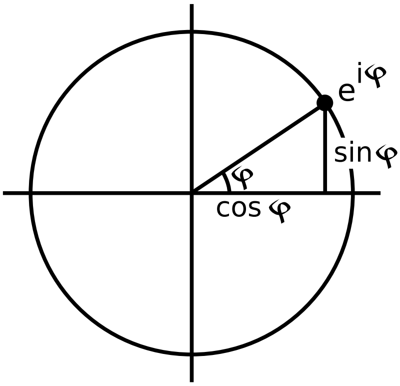
Having expanded our horizons to include the complex numbers, it's natural to want to extend functions we knew and loved from the world of real numbers so that they can also operate on complex numbers. The only really natural way to do this in general is to use Taylor series. A particularly beautiful thing happens with the functions e^x, $\sin x$, and $\cos x$:

$$e^x = 1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$$
$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots$$
$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots$$

If $x = i\phi$ is an imaginary number, we have

$$e^{i\phi} = \cos\phi + i\sin\phi \qquad ,$$

a result known as Euler's formula. The geometrical interpretation in the complex plane is shown in figure f.



f / The complex number $e^{i\phi}$ lies on the unit circle.

Although the result may seem like something out of a freak show at first, applying the definition[2] of the exponential function makes it clear how natural it is:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \qquad .$$

When $x = i\phi$ is imaginary, the quantity $(1 + i\phi/n)$ represents a number lying just above 1 in the complex plane. For large n, $(1 + i\phi/n)$ becomes very close to the unit circle, and its argument is the small angle ϕ/n. Raising this number to the nth power multiplies its argument by n, giving a number with an argument of ϕ.

g / Leonhard Euler (1707-1783)

Euler's formula is used frequently in physics and engineering.

Example 50

▷ Write the sine and cosine functions in terms of exponentials.

▷ Euler's formula for $x = -i\phi$ gives $\cos\phi - i\sin\phi$, since $\cos(-\theta) = \cos\theta$, and $\sin(-\theta) = -\sin\theta$.

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Example 51

▷ Evaluate

$$\int e^x \cos x dx$$

▷ This seemingly impossible integral becomes easy if we rewrite the cosine

[2]See page 83 for an explanation of where this definition comes from and why it makes sense.

in terms of exponentials:

$$\int e^x \cos x \mathrm{d}x = \int e^x \left(\frac{e^{ix}+e^{-ix}}{2}\right) \mathrm{d}x = \frac{1}{2}\int (e^{(1+i)x}+e^{(1-i)x}) \mathrm{d}x = \frac{1}{2}\left(\frac{e^{(1+i)x}}{1+i}+\frac{e^{(1-i)x}}{1-i}\right)+c$$

Since this result is the integral of a real-valued function, we'd like it to be real, and in fact it is, since the first and second terms are complex conjuages of one another. If we wanted to, we could use Euler's theorem to convert it back to a manifestly real result.[3]

5.3 Partial fractions revisited

Suppose we want to evaluate the integral

$$\int \frac{\mathrm{d}x}{x^2+1}$$

by the method of partial fractions. The quadratic formula tells us that the roots are i and $-i$, setting $1/(x^2+1) = A/(x+i)+B/(x-i)$ gives $A = i/2$ and $B = -i/2$, so

$$\int \frac{\mathrm{d}x}{x^2+1} = \frac{i}{2}\int \frac{\mathrm{d}x}{x+i} - \frac{i}{2}\int \frac{\mathrm{d}x}{x-i} = \frac{i}{2}\ln(x+i) - \frac{i}{2}\ln(x-i) = \frac{i}{2}\ln\frac{x+i}{x-i} \quad .$$

The attractive thing about this approach, compared with the method used on page 58, is that it doesn't require any tricks. If you came across this integral ten years from now, you could pull out your old calculus book, flip through it, and say, "Oh, here we go, there's a way to integrate one over a polynomial — partial fractions." On the other hand, it's odd that we started out trying to evaluate an integral that had nothing but real numbers, and came out with an answer that isn't even obviously a real number.

But what about that expression $(x+i)/(x-i)$? Let's give it a name, w. The numerator and denominator are complex conjugates of one another. Since they have the same magnitude, we must have $|w| = 1$, i.e., w is a complex number that lies on the unit circle, the kind of complex number that Euler's formula refers to. The numerator has an argument of $\tan^{-1}(1/x) = \pi/2 - \tan^{-1} x$, and the denominator has the same argument but with the opposite sign. Division

[3] In general, the use of complex number techniques to do an integral could result in a complex number, but that complex number would be a constant, which could be subsumed within the usual constant of integration.

means subtracting arguments, so $\arg w = \pi - 2\tan^{-1} x$. That means that the result can be rewritten using Euler's formula as

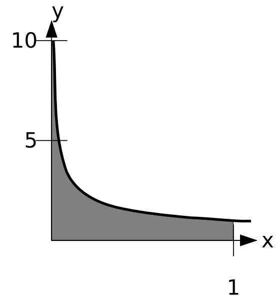
$$\begin{aligned}\int \frac{\mathrm{d}x}{x^2+1} &= \frac{i}{2} \ln e^{i(\pi - 2\tan^{-1} x)} \\ &= \frac{i}{2} \cdot i(\pi - 2\tan^{-1} x) \\ &= \tan^{-1} x + c \qquad .\end{aligned}$$

In other words, it's the same result we found before, but found without the need for trickery.

6 Improper integrals

6.1 Integrating a function that blows up

When we integrate a function that blows up to infinity at some point in the interval we're integrating, the result may be either finite or infinite.



a / The integral $\int_0^1 \mathrm{d}x/\sqrt{x}$ is finite.

Example 52

▷ Integrate the function $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$.

▷ The function blows up to infinity at one end of the region of integration, but let's just try evaluating it, and see what happens.

$$\int_0^1 x^{-1/2}\,\mathrm{d}x = 2x^{1/2}\Big|_0^1$$
$$= 2$$

The result turns out to be finite. Intuitively, the reason for this is that the spike at $x = 0$ is very skinny, and gets skinny fast as we go higher and higher up.

Example 53

▷ Integrate the function $y = 1/x^2$ from $x = 0$ to $x = 1$.

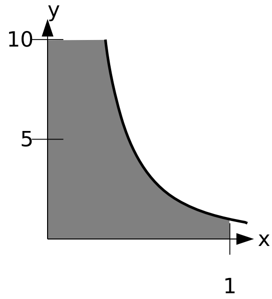
▷

$$\int_0^1 x^{-2}\,\mathrm{d}x = -x^{-1}\Big|_0^1$$
$$= -1 + \frac{1}{0}$$

Division by zero is undefined, so the result is undefined.

Another way of putting it, using the hyperreal number system, is that if we were to integrate from ϵ to 1, where ϵ was an infinitesimal number, then the result would be $-1 + 1/\epsilon$, which is infinite. The smaller we make ϵ, the bigger the infinite result we get out.

Intuitively, the reason that this integral comes out infinite is that the spike at $x = 0$ is fat, and doesn't get skinny fast enough.



b / The integral $\int_0^1 \mathrm{d}x/x^2$ is infinite.

These two examples were examples of improper integrals.

6.2 Limits of integration at infinity

Another type of improper integral is one in which one of the limits of integration is infinite. The notation

$$\int_a^\infty f(x)\,\mathrm{d}x$$

means the limit of $\int_a^H f(x)\,\mathrm{d}x$, where H is made to grow bigger and bigger. Alternatively, we can think of it as an integral in which the top end of the interval of integration is an infinite hyperreal number. A similar interpretation applies when the lower limit is $-\infty$, or when both limits are infinite.

Example 54

▷ Evaluate

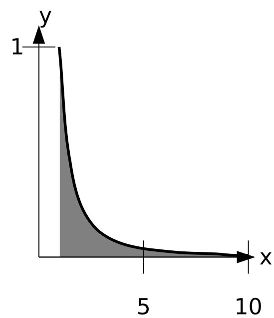
$$\int_1^\infty x^{-2}\,\mathrm{d}x$$

▷

$$\int_1^H x^{-2}\,\mathrm{d}x = -x^{-1}\Big|_1^H$$
$$= -\frac{1}{H} + 1$$

As H gets bigger and bigger, the result gets closer and closer to 1, so the result of the improper integral is 1.

Note that this is the same graph as in example 52, but with the x and y axes interchanged; this shows that the two different types of improper integrals really aren't so different.



c / The integral $\int_1^\infty \mathrm{d}x/x^2$ is finite.

7 Iterated integrals

7.1 Integrals inside integrals

In various applications, you need to do integrals stuck inside other integrals. These are known as iterated integrals, or double integrals, triple integrals, etc. Similar concepts crop up all the time even when you're not doing calculus, so let's start by imagining such an example. Suppose you want to count how many squares there are on a chess board, and you don't know how to multiply eight times eight. You could start from the upper left, count eight squares across, then continue with the second row, and so on, until you how counted every square, giving the result of 64. In slightly more formal mathematical language, we could write the following recipe: for each row, r, from 1 to 8, consider the columns, c, from 1 to 8, and add one to the count for each one of them. Using the sigma notation, this becomes

$$\sum_{r=1}^{8}\sum_{c=1}^{8} 1 \qquad .$$

If you're familiar with computer programming, then you can think of this as a sum that could be calculated using a loop nested inside another loop. To evaluate the result (again, assuming we don't know how to multiply, so we have to use brute force), we can first evaluate the inside sum, which equals 8, giving

$$\sum_{r=1}^{8} 8 \qquad .$$

Notice how the "dummy" variable c has disappeared. Finally we do the outside sum, over r, and find the result of 64.

Now imagine doing the same thing with the pixels on a TV screen. The electron beam sweeps across the screen, painting the pixels in each row, one at a time. This is really no different than the example of the chess board, but because the pixels are so small, you normally think of the image on a TV screen as continuous rather than discrete. This is the idea of an integral in calculus. Suppose we want to find the area of a rectangle of width a and height b, and we don't know that we can just multiply to get the area ab. The brute force way to do this is to break up the rectangle into a grid of infinitesimally small squares, each having width $\mathrm{d}x$ and height $\mathrm{d}y$, and therefore the infinitesimal area $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$. For convenience, we'll imagine that the rectangle's lower left corner is at the origin. Then the area is given by this integral:

$$\begin{aligned} \text{area} &= \int_{y=0}^{b} \int_{x=0}^{a} \mathrm{d}A \\ &= \int_{y=0}^{b} \int_{x=0}^{a} \mathrm{d}x\,\mathrm{d}y \end{aligned}$$

Notice how the leftmost integral sign, over y, and the rightmost differential, $\mathrm{d}y$, act like bookends, or the pieces of bread on a sandwich. Inside them, we have the integral sign that runs over x, and the differential $\mathrm{d}x$ that matches it on the right. Finally, on the innermost layer, we'd normally have the thing we're integrating, but here's it's 1, so I've omitted it. Writing the lower limits of the integrals with $x =$ and $y =$ helps to keep it straight which integral goes with with differential. The result is

$$\begin{aligned} \text{area} &= \int_{y=0}^{b} \int_{x=0}^{a} \mathrm{d}A \\ &= \int_{y=0}^{b} \int_{x=0}^{a} \mathrm{d}x\,\mathrm{d}y \\ &= \int_{y=0}^{b} \left(\int_{x=0}^{a} \mathrm{d}x \right) \mathrm{d}y \\ &= \int_{y=0}^{b} a\,\mathrm{d}y \\ &= a \int_{y=0}^{b} \mathrm{d}y \\ &= ab \quad . \end{aligned}$$

Area of a triangle *Example 55*

▷ Find the area of a 45-45-90 right triangle having legs a.

▷ Let the triangle's hypotenuse run from the origin to the point (a, a), and let its legs run from the origin to $(0, a)$, and then to (a, a). In other words, the triangle sits on top of its hypotenuse. Then the integral can be set up the same way as the one before, but for a particular value of y, values of x only run from 0 (on the y axis) to y (on the hypotenuse). We then have

$$\begin{aligned} \text{area} &= \int_{y=0}^{a} \int_{x=0}^{y} \mathrm{d}A \\ &= \int_{y=0}^{a} \int_{x=0}^{y} \mathrm{d}x\,\mathrm{d}y \\ &= \int_{y=0}^{a} \left(\int_{x=0}^{y} \mathrm{d}x \right) \mathrm{d}y \\ &= \int_{y=0}^{a} y\,\mathrm{d}y \\ &= \frac{1}{2}a^2 \end{aligned}$$

Note that in this example, because the upper end of the x values depends on the value of y, it makes a difference which order we do the integrals in. The x integral has to be on the inside, and we have to do it first.

Volume of a cube *Example 56*

▷ Find the volume of a cube with sides of length a.

▷ This is a three-dimensional example, so we'll have integrals nested three deep, and the thing we're integrating is the volume $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$.

$$
\begin{aligned}
\text{volume} &= \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} \mathrm{d}V \\
&= \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_{z=0}^{a}\int_{y=0}^{a} a\,\mathrm{d}y\,\mathrm{d}z \\
&= a\int_{z=0}^{a}\int_{y=0}^{a} \mathrm{d}y\,\mathrm{d}z \\
&= a\int_{z=0}^{a} a\,\mathrm{d}z \\
&= a^2\int_{z=0}^{a} \mathrm{d}z \\
&= a^3
\end{aligned}
$$

Area of a circle *Example 57*

▷ Find the area of a circle.

▷ To make it easy, let's find the area of a semicircle and then double it. Let the circle's radius be r, and let it be centered on the origin and bounded below by the x axis. Then the curved edge is given by the equation $r^2 = x^2 + y^2$, or $y = \sqrt{r^2 - x^2}$. Since the y integral's limit depends on x, the x integral has to be on the outside. The area is

$$
\begin{aligned}
\text{area} &= \int_{x=-r}^{r}\int_{y=0}^{\sqrt{r^2-x^2}} \mathrm{d}y\,\mathrm{d}x \\
&= \int_{x=-r}^{r} \sqrt{r^2-x^2}\mathrm{d}x \\
&= r\int_{x=-r}^{r} \sqrt{1-(x/r)^2}\,\mathrm{d}x \quad .
\end{aligned}
$$

Substituting $u = x/r$,

$$
\text{area} = r^2\int_{u=-1}^{1} \sqrt{1-u^2}\,\mathrm{d}u
$$

The definite integal equals π, as you can find using a trig substitution or simply by looking it up in a table, and the result is, as expected, $\pi r^2/2$ for the area of the semicircle. Doubling it, we find the expected result of πr^2 for a full circle.

7.2 Applications

Up until now, the integrand of the innermost integral has always been 1, so we really could have done all the double integrals as single integrals. The following example is one in which you really need to do iterated integrals.

a / The famous tightrope walker Charles Blondin uses a long pole for its large moment of inertia.

Moments of inertia *Example 58*

The moment of inertia is a measure of how difficult it is to start an object rotating (or stop it). For example, tightrope walkers carry long poles because they want something with a big moment of inertia. The moment of inertia is defined by $I = \int r^2 \mathrm{d}m$, where $\mathrm{d}m$ is the mass of an infinitesimally small portion of the object, and r is the distance from the axis of rotation.

To start with, let's do an example that doesn't require iterated integrals. Let's calculate the moment of inertia of a thin rod of mass M and length L about a line perpendicular to the rod and passing through its center.

$$\begin{aligned} I &= \int r^2 \mathrm{d}m \\ &= \int_{-L/2}^{L/2} x^2 \frac{M}{L} \mathrm{d}x \qquad [r = |x|, \text{ so } r^2 = x^2] \\ &= \frac{1}{12} ML^2 \end{aligned}$$

Now let's do one that requires iterated integrals: the moment of inertia of a cube of side b, for rotation about an axis that passes through its center and is parallel to four of its faces.

Let the origin be at the center of the cube, and let x be the rotation axis.

$$\begin{aligned} I &= \int r^2 \mathrm{d}m \\ &= \rho \int r^2 \mathrm{d}V \\ &= \rho \int_{b/2}^{b/2} \int_{b/2}^{b/2} \int_{b/2}^{b/2} \left(y^2 + z^2\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \\ &= \rho b \int_{b/2}^{b/2} \int_{b/2}^{b/2} \left(y^2 + z^2\right) \mathrm{d}y\, \mathrm{d}z \end{aligned}$$

The fact that the last step is a trivial integral results from the symmetry of the problem. The integrand of the remaining double integral breaks down into two terms, each of which depends on only one of the variables, so we break it into two integrals,

$$I = \rho b \int_{b/2}^{b/2} \int_{b/2}^{b/2} y^2 \mathrm{d}y \mathrm{d}z + \rho b \int_{b/2}^{b/2} \int_{b/2}^{b/2} z^2 \mathrm{d}y \mathrm{d}z$$

which we know have identical results. We therefore only need to evaluate one of them and double the result:

$$\begin{aligned} I &= 2\rho b \int_{b/2}^{b/2} \int_{b/2}^{b/2} z^2 \mathrm{d}y\, \mathrm{d}\, z \\ &= 2\rho b^2 \int_{b/2}^{b/2} z^2 \mathrm{d}z \\ &= \frac{1}{6} \rho b^5 \\ &= \frac{1}{6} Mb^2 \end{aligned}$$

A Answers to self-checks

Answers to Self-Checks

Answers to self-checks for chapter 5

page 66, self-check 1: Say we're looking for $u=\sqrt{z}$, i.e., we want a number u that, multiplied by itself, equals z. Multiplication multiplies the magnitudes, so the magnitude of u can be found by taking the square root of the magnitude of z. Since multiplication also adds the arguments of the numbers, squaring a number doubles its argument. Therefore we can simply divide the argument of z by two to find the argument of u. This results in one of the square roots of z. There is another one, which is $-u$, since $(-u)^2$ is the same as u^2. This may seem a little odd: if u was chosen so that doubling its argument gave the argument of z, then how can the same be true for $-u$? Well for example, suppose the argument of z is $4\,°$. Then $\arg u = 2\,°$, and $\arg(-u) = 182\,°$. Doubling 182 gives 364, which is actually a synonym for 4 degrees.

B Detours

Formal definition of the tangent line

Let (a, b) be a point on the graph of the function $x(t)$. A line $\ell(t)$ through this point is said not to cut through the graph if there exists some real number d such that either $x(t) \geq \ell(t)$ or $x(t) \leq \ell(t)$ for all t between $a - d$ and $a + d$. The line is said to be the tangent line at this point if it is the only line through this point that doesn't cut through the graph.

Derivatives of polynomials

We want to prove that the derivative of t^k is kt^{k-1}. It suffices to prove that the derivative equals k when evaluated at $t = 1$, since we can then apply the kind of scaling argument used on page 12 to find the derivative of $t^2/2$ was t. The tangent line at $(1, 1)$ has the equation $\ell = k(t-1)+1$, so we only need to prove that the polynomial $t^k - [k(t-1)+1]$ is greater than or equal to zero in some finite region around $t = 1$.

First, let's change variables to $u = t - 1$. Then the polynomial in question becomes $P(u) = (u+1)^k - (ku+1)$, and we want to prove that it's nonnegative in some region around $u = 0$. (We assume $k \geq 2$, since we've already found the derivatives in the cases of $k = 0$ and 1, and in those cases $P(u)$ is identically zero.)

Now the last two terms in the binomial series for $(u + 1)^k$ are just ku and 1, so $P(u)$ is a polynomial whose lowest-order term is a u^2 term. Also, all the nonzero coefficients of the polynomial are positive, so P is positive for $u \geq 0$.

To complete the proof we only need to establish that P is also positive for sufficiently small negative values of u. For negative u, the even-order terms of P are positive, and the odd-order terms negative. To make the idea clear, consider the $k = 5$ case, where $P(u) = u^5 + 5u^4 + 10u^3 + 10u^2$. The idea is to pair off each positive term with the negative one immediately to its left. Although the coefficient of the negative term may, in general, be greater than the coefficient of the positive term with which we've paired it, a property of the binomial coefficients is the ratio of successive coefficients is never greater than k. Thus for $-1/k <$

$u < 0$, each positive term is guaranteed to dominate the negative term immediately to its left.

Details of the proof of the derivative of the sine function Some ideas in this proof are due to Jerome Keisler.

On page **??**, I computed

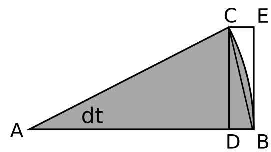
$$\begin{aligned} dx &= \sin(t + dt) - \sin t \qquad , \\ &= \sin t \, \cos dt \\ &\quad + \cos t \, \sin dt - \sin t \\ &\approx \cos t \, dt \qquad . \end{aligned}$$

Here I'll prove that the error introduced by the small-angle approximations really is of order dt^2. We have

$$\sin(t + dt) = \sin t + \cos t dt - E \qquad ,$$

where the error E introduced by the approximations is

$$\begin{aligned} E &= \sin t(1 - \cos dt) \\ &\quad + \cos t(dt - \sin dt) \qquad . \end{aligned}$$



a / Geometrical interpretation of the error term.

Let the radius of the circle in figure a be one, so AD is $\cos dt$ and CD is $\sin dt$. The area of the shaded pie slice is $dt/2$, and the area of triangle ABC is $\sin dt/2$, so the error made in the approximation $\sin dt \approx dt$ equals twice the area of the dish shape formed by line BC and arc BC. Therefore $dt - \sin dt$ is less than the area of rectangle CEBD. But CEBD has both an infinitesimal width and an infinitesimal height, so this error is of no more than order dt^2.

For the approximation $\cos dt \approx 1$, the error (represented by BD) is $1 - \cos dt = 1 - \sqrt{1 - \sin^2 dt}$, which is less than $1 - \sqrt{1 - dt^2}$, since $\sin dt < dt$. Therefore this error is of order dt^2.

Derivative of e^x

All of the reasoning on page 31 have applied equally well to any other exponential function with a different base, such as 2^x or 10^x. Those functions would have different values of c, so if we want to determine the value of c for the base-e case, we need to bring in the definition of e, or of the exponential function e^x, somehow.

We can take the definition of e^x to be

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \qquad .$$

The idea behind this relation is similar to the idea of compound interest. If the interest rate is 10%, compounded annually, then $x = 0.1$, and the balance grows by a factor $(1 + x) = 1.1$ in one year. If, instead, we want to compound the interest monthly, we can set the monthly interest rate to $0.1/12$, and then the growth of the balance over a year is $(1+x/12)^{12} = 1.1047$, which is slightly larger because the interest from the earlier months itself accrues interest in the later months. Continuing this limiting process, we find $e^{1.1} = 1.1052$.

If n is large, then we have a good approximation to the base-e exponential, so let's differentiate this finite-n approximation and try to find an approximation to the derivative of e^x. The chain rule tells is that the derivative of $(1 + x/n)^n$ is the derivative of the raising-to-the-nth-power function, multiplied by the derivative of the inside stuff, $\mathrm{d}(1 + x/n)/\mathrm{d}x = 1/n$. We then have

$$\begin{aligned}\frac{\mathrm{d}\left(1+\frac{x}{n}\right)^n}{\mathrm{d}x} &= \left[n\left(1+\frac{x}{n}\right)^{n-1}\right]\cdot\frac{1}{n} \\ &= \left(1+\frac{x}{n}\right)^{n-1} \qquad .\end{aligned}$$

But evaluating this at $x = 0$ simply gives 1, so at $x = 0$, the approximation to the derivative is exactly 1 for all values of n — it's not even necessary to imagine going to larger and larger values of n. This establishes that $c = 1$, so we have

$$\frac{\mathrm{d}e^x}{\mathrm{d}x} = e^x$$

for all values of x.

Proof of the fundamental theorem of calculus

There are three parts to the proof: (1) Take the equation that states the fundamental theorem, differentiate both sides with respect to b, and show that they're equal. (2) Show that continuous functions with equal derivatives must be essentially the same function, except for an additive constant. (3) Show that the constant in question is zero.

1. By the definition of the indefinite integral, the derivative of $x(b)-x(a)$ with respect to b equals $\dot{x}(b)$. We have to establish that this equals the following:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}b}\int_a^b \dot{x}(t)\mathrm{d}t &= \mathrm{st}\frac{1}{\mathrm{d}b}\left[\int_a^{b+\mathrm{d}b}\dot{x}(t)\mathrm{d}t-\int_a^b \dot{x}(t)\mathrm{d}t\right]\\
&= \mathrm{st}\frac{1}{\mathrm{d}b}\int_b^{b+\mathrm{d}b}\dot{x}(t)\mathrm{d}t\\
&= \mathrm{st}\frac{1}{\mathrm{d}b}\lim_{H\to\infty}\sum_{i=0}^{H}\dot{x}(b+i\,\mathrm{d}b/H)\frac{\mathrm{d}b}{H}\\
&= \mathrm{st}\lim_{H\to\infty}\frac{1}{H}\sum_{i=0}^{H}\dot{x}(b+i\,\mathrm{d}b/H)
\end{aligned}$$

Since $\dot{x}$ is continuous, all the values of $\dot{x}$ occurring inside the sum can differ only infinitesimally from $\dot{x}(b)$. Therefore the quantity inside the limit differs only infinitesimally from $\dot{x}(b)$, and the standard part of its limit must be $\dot{x}(b)$.[1]

2. Suppose f and g are two continuous functions whose derivatives are equal. Then $d=f-g$ is a continuous function whose derivative is zero. But the only continuous function with a derivative of zero is a constant, so f and g differ by at most an additive constant.

3. I've established that the derivatives with respect to b of $x(b)-x(a)$ and $\int_a^b \dot{x}\mathrm{d}t$ are the same, so they differ by at most an additive constant. But at $b=a$, they're both zero, so the constant must be zero.

Proof of the mean value theorem

Suppose that the mean value theorem is violated. Let L by the set of all x in the interval from a to b such that $y(x)<\bar{y}$, and likewise let M be the set with $y(x)>\bar{y}$. If the theorem is violated, then the union of these two sets covers the entire interval from a to b. Neither one can be empty; if, for example, M was empty, then we would have $y<\bar{y}$ everywhere and also $\int_a^b y=\int_a^b \bar{y}$, but it follows directly from the definition of the

[1] If you don't want to use infinitesimals, then you can express the derivative as a limit, and in the final step of the argument use the mean value theorem, introduced later in the chapter.

definite integral that when one function is less than another, its integral is also less than the other's. Since y takes on values less than and greater than $\bar{y}$, it follows from the intermediate value theorem that y takes on the value $\bar{y}$ somewhere (intuitively, at a boundary between L and M).

Proof of the fundamental theorem of algebra

Theorem: In the complex number system, an nth-order polynomial has exactly n roots, i.e., it can be factored into the form $P(z) = (z-a_1)(z-a_2)\ldots(z-a_n)$, where the a_i are complex numbers.

Proof: The proofs in the cases of $n = 0$ and 1 are trivial, so our strategy is to reduce higher-n cases to lower ones. If an nth-degree polynomial P has at least one root, a, then we can always reduce it to a polynomial of degree $n-1$ by dividing it by $(z-a)$. Therefore the theorem is proved by induction provided that we can show that every polynomial of degree greater than zero has at least one root.

Suppose, on the contrary, that there was an nth order polynomial $P(z)$, with $n > 0$, that had no roots at all. Then $|P(z)|$ must have some minimum value, which is achieved at $z = z_o$. (Polynomials don't have asymptotes, so the minimum really does have to occur for some specific, finite z_o.) To make things more simple and concrete, we can construct another polynomial $Q(z) = P(z+z_o)/P(z_o)$, so that $|Q|$ has a minimum value of 1, achieved at $Q(0) = 1$. This means that Q's constant term is 1. What about its other terms? Let $Q(z) = 1+c_1z+\ldots+c_nz^n$. Suppose c_1 was nonzero. Then for infinitesimally small values of z, the terms of order z^2 and higher would be negligible, and we could make $Q(z)$ be a real number less than one by an appropriate choice of z's argument. Therefore c_1 must be zero. But that means that if c_2 is nonzero, then for infinitesimally small z, the z^2 term dominates the z^3 and higher terms, and again this would allow us to make $Q(z)$ be real and less than one for appropriately chosen values of z. Continuing this process, we find that $Q(z)$ has no terms at all beyond the constant term, i.e., $Q(z) = 1$. This contradicts the assumption that n was greater than zero, so we've proved by contradiction that there is no P with the properties claimed.

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