12

MULTIPLE INTEGRALS

The first seven sections of this chapter develop the double and triple integral. They depend on Sections 11.1 and 11.2 on surfaces and continuous functions, but are independent of Chapter 10 on vectors.

Sections 8 through 10 of this chapter discuss the relationship between multiple integrals, line integrals, and surface integrals. Chapters 10 on vectors and 11 on partial derivatives are prerequisites.

12.1 DOUBLE INTEGRALS

The *double integral* is the analogue of the *single integral* (definite integral) suggested by Figure 12.1.1. Figure 12.1.1(a) shows the area A bounded by the interval $[a, b]$ and the curve $y = f(x)$, and corresponds to the single integral

$$A = \int_a^b f(x)\,dx.$$

Figure 12.1.1(b) shows the volume V bounded by the plane region D and the surface $z = f(x, y)$, and corresponds to the double integral

$$V = \iint_D f(x, y)\,dx\,dy.$$

Our development of double integrals will be similar to our development of single integrals in Chapter 4. Before going into detail, we give a brief intuitive preview.

Instead of closed intervals $[u, v]$ in the line, we deal with closed regions D in the plane. A *volume function* for $f(x, y)$ is a function B, which assigns a real number $B(D)$ to each closed region D, and has the following two properties: Addition Property and Cylinder Property.

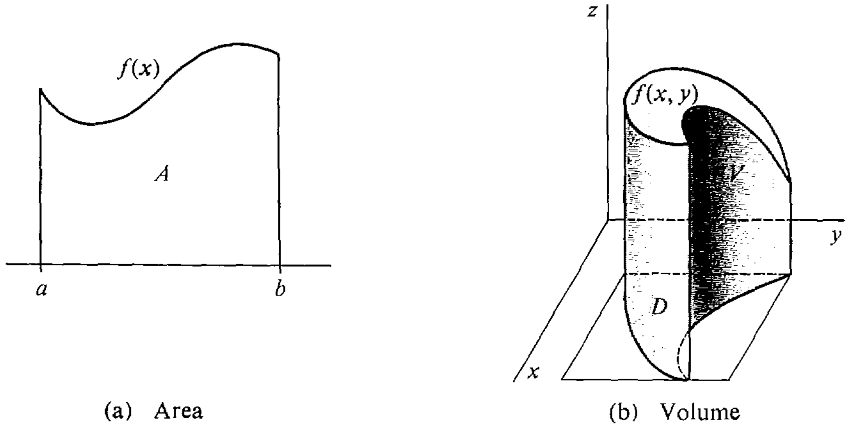


Figure 12.1.1

ADDITION PROPERTY

If D is divided into two regions D_1 and D_2 which meet only on a common boundary curve, then

$$B(D) = B(D_1) + B(D_2).$$

(Intuitively, the volume over D is the sum of the volumes over D_1 and D_2.)

This property is illustrated in Figure 12.1.2(a).

CYLINDER PROPERTY

Let m and M be the minimum and maximum values of $f(x, y)$ on D and let A be the area of D. Then

$$mA \le B(D) \le MA.$$

(Intuitively, the volume over D is between the volumes of the cylinders over D of height m and M. This corresponds to the Rectangle Property for single integrals.)

This property is illustrated in Figure 12.1.2(b).

We shall see at the end of this section that the double integral

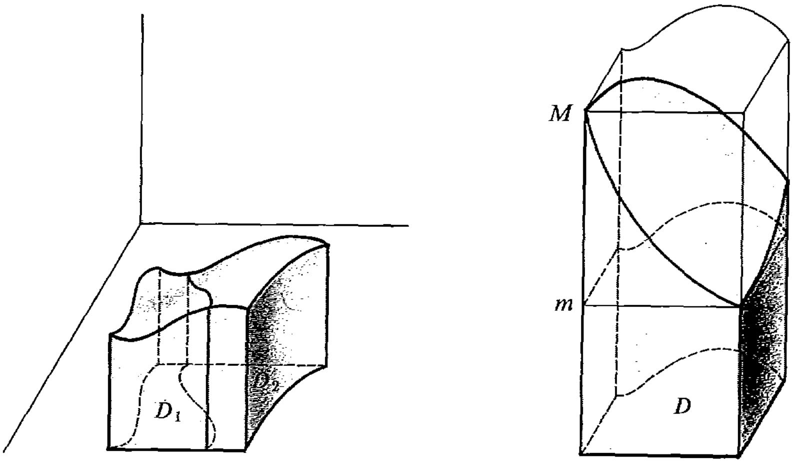
$$\iint_D f(x, y)\,dx\,dy$$

is the unique volume function for a continuous function $f(x, y)$. The double integral will be constructed using double Riemann sums, just as the single integral was constructed from single Riemann sums.

We now begin the construction of the double integral, starting with a careful discussion of closed regions in the plane.

A *closed region* in the (x, y) plane is a set D of real points (x, y) given by inequalities

$$a_1 \le x \le a_2, \qquad b_1(x) \le y \le b_2(x),$$



(a) Addition Property (b) Cylinder Property

Figure 12.1.2

where $b_1(x)$ and $b_2(x)$ are continuous and $b_1(x) \leq b_2(x)$ for x in $[a_1, a_2]$ (Figure 12.1.3(a)). The *boundary* of D is the set of points in D which are on the curves

$$x = a_1, \qquad x = a_2, \qquad y = b_1(x), \qquad y = b_2(x).$$

The simplest type of closed region is a *closed rectangle*

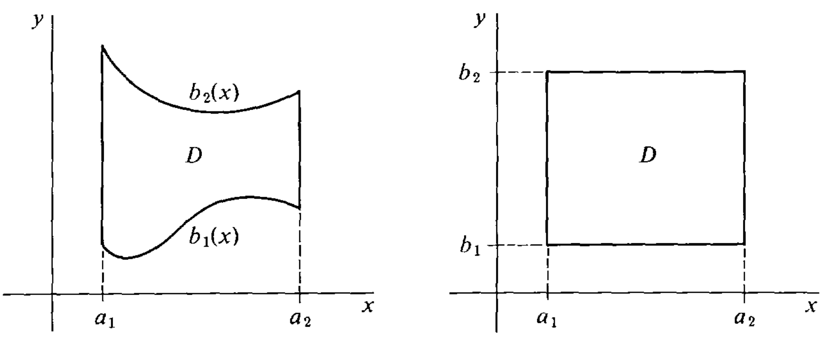
$$a_1 \leq x \leq a_2, \qquad b_1 \leq y \leq b_2,$$

shown in Figure 12.1.3(b).

Remark In this course we are restricting our attention to a very simple type of closed region, sometimes called a basic closed region. In advanced calculus and beyond, a much wider class of closed regions is studied.

An *open region* is a set of real points defined by strict inequalities of the form

$$c_1 < x < c_2, \qquad d_1(x) < y < d_2(x).$$



(a) Closed region (b) Closed rectangle

Figure 12.1.3

We shall usually be working with closed regions. So from now on when we use the word *region* alone we mean *closed region*.

To simplify our treatment we shall consider only continuous functions.

PERMANENT ASSUMPTION FOR CHAPTER 12

Whenever we refer to a function $f(x, y)$ and a region D, we assume that $f(x, y)$ is continuous on some open region containing D.

If $f(x, y) \geq 0$ on D, the double integral is intuitively the volume of the solid over D between the surfaces $z = 0$ and $z = f(x, y)$; i.e., the solid consisting of all points (x, y, z) where (x, y) is in D and

$$0 \leq z \leq f(x, y).$$

If $f(x, y) \leq 0$ on D the double integral is intuitively the negative of the volume of the solid under D between the surfaces $z = f(x, y)$ and $z = 0$. Thus volumes above the plane $z = 0$ are counted positively and volumes below $z = 0$ are counted negatively (Figure 12.1.4).

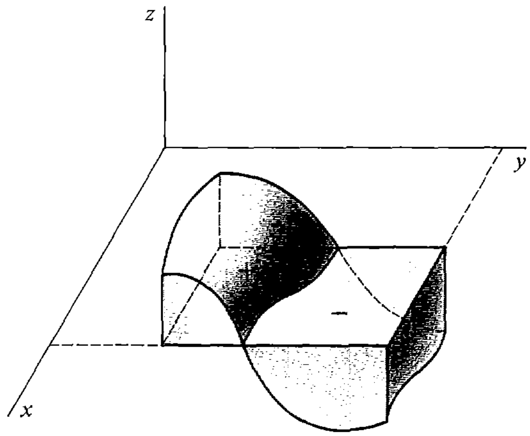


Figure 12.1.4

We now define the double Riemann sum and use it to give a precise definition of the double integral. We first consider the case where D is a rectangle

$$a_1 \leq x \leq a_2, \qquad b_1 \leq y \leq b_2,$$

shown in Figure 12.1.5.

Let Δx and Δy be positive real numbers. We partition the interval $[a_1, a_2]$ into subintervals of length Δx and $[b_1, b_2]$ into subintervals of length Δy. The partition points are

$$x_0 = a_1, \quad x_1 = a_1 + \Delta x, \quad x_2 = a_1 + 2\,\Delta x, \ldots, x_n = a_1 + n\,\Delta x,$$
$$y_0 = b_1, \quad y_1 = b_1 + \Delta y, \quad y_2 = b_1 + 2\,\Delta y, \ldots, y_p = b_1 + p\,\Delta y$$

where $$x_n < a_2 \leq x_n + \Delta x, \qquad y_p < b_2 \leq y_p + \Delta y.$$

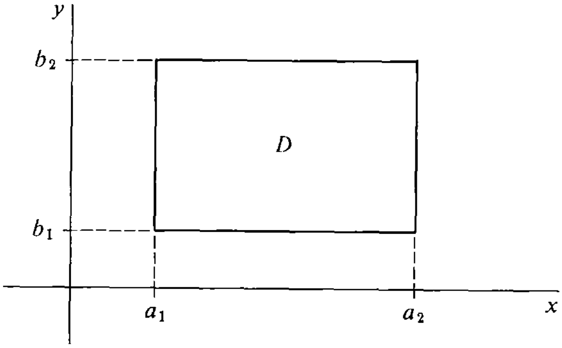


Figure 12.1.5

If Δx and Δy do not evenly divide $a_2 - a_1$ and $b_2 - b_1$, there will be little pieces left over at the end. We have partitioned the rectangle D into Δx by Δy subrectangles with partition points

$$(x_k, y_l), \qquad 0 \le k \le n, \qquad 0 \le l \le p,$$

as in Figure 12.1.6.

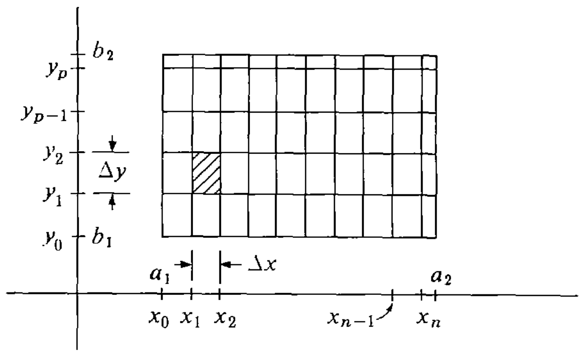


Figure 12.1.6

The *double Riemann sum* for a rectangle D is the sum

$$\sum_D\sum f(x, y)\,\Delta x\,\Delta y = \sum_{k=0}^{n}\sum_{l=0}^{p} f(x_k, y_l)\,\Delta x\,\Delta y.$$

This is the sum of the volume of the rectangular solids with base $\Delta x\,\Delta y$ and height $f(x_k, y_l)$.

As we can see from Figure 12.1.7,

$$\sum_D\sum f(x, y)\,\Delta x\,\Delta y$$

approximates the volume of the solid over D between $z = 0$ and $z = f(x, y)$.

Now let D be a general region

$$a_1 \le x \le a_2, \qquad b_1(x) \le y \le b_2(x).$$

The *circumscribed rectangle* of D is the rectangle

$$a_1 \le x \le a_2, \qquad B_1 \le y \le B_2,$$

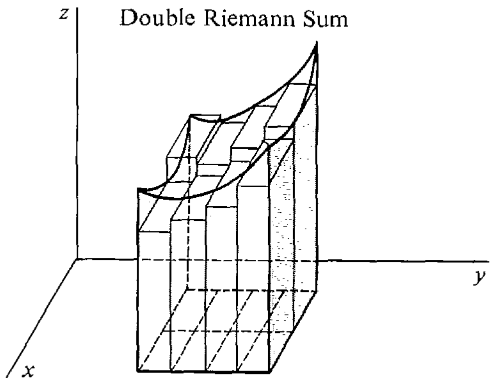


Figure 12.1.7

where B_1 = minimum value of $b_1(x)$,
B_2 = maximum value of $b_2(x)$.

It is shown in Figure 12.1.8.

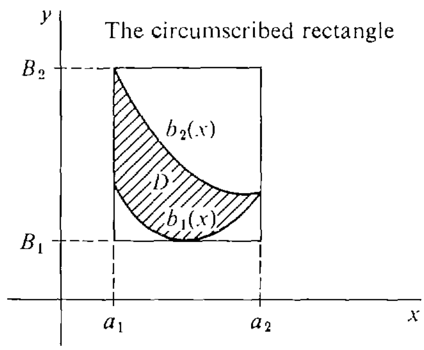


Figure 12.1.8

Given positive real numbers Δx and Δy, we partition the circumscribed rectangle of D into Δx by Δy subrectangles with partition points

$$(x_k, y_l), \qquad 0 \le k \le n, \qquad 0 \le l \le p.$$

DEFINITION

*The **double Riemann sum** over D is defined as the sum of the volumes of the rectangular solids with base $\Delta x\, \Delta y$ and height $f(x_k, y_l)$ corresponding to partition points (x_k, y_l) which belong to D. In symbols,*

$$\sum_D\sum f(x, y)\,\Delta x\,\Delta y = \sum_{(x_k, y_l) \text{ in } D}\sum f(x_k, y_l)\,\Delta x\,\Delta y.$$

Notice that in the double Riemann sum over D, we only use partition points (x_k, y_l) which belong to D (Figure 12.1.9).

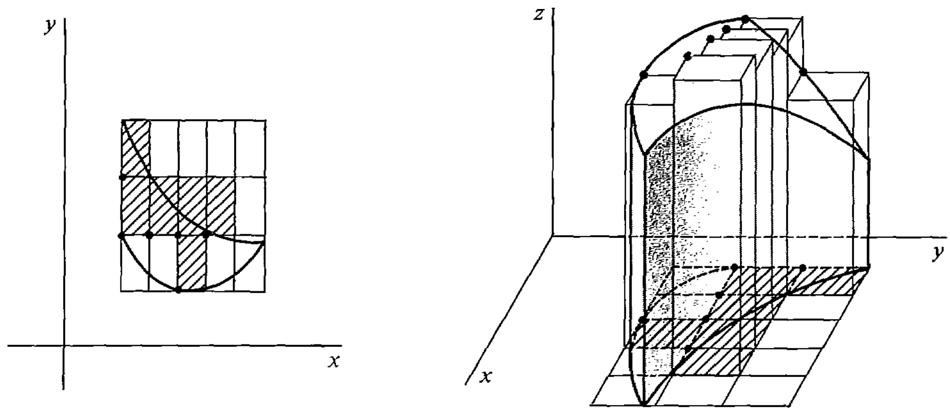


Figure 12.1.9 Double Riemann Sum

EXAMPLE 1 Find the double Riemann sum

$$\sum_{D_1}\sum x^2y\,\Delta x\,\Delta y,$$

where D_1 is the square

$$0 \le x \le 1, \qquad 0 \le y \le 1,$$

and

$$\Delta x = \tfrac{1}{4}, \qquad \Delta y = \tfrac{1}{5}.$$

The partition of D_1 is shown in Figure 12.1.10 and the values of x^2y at the partition points are shown in the table.

x^2y	$y_0 = 0$	$y_1 = \frac{1}{5}$	$y_2 = \frac{2}{5}$	$y_3 = \frac{3}{5}$	$y_4 = \frac{4}{5}$
$x_0 = 0$	0	0	0	0	0
$x_1 = \frac{1}{4}$	0	$\frac{1}{80}$	$\frac{2}{80}$	$\frac{3}{80}$	$\frac{4}{80}$
$x_2 = \frac{1}{2}$	0	$\frac{4}{80}$	$\frac{8}{80}$	$\frac{12}{80}$	$\frac{16}{80}$
$x_3 = \frac{3}{4}$	0	$\frac{9}{80}$	$\frac{18}{80}$	$\frac{27}{80}$	$\frac{36}{80}$

The double Riemann sum is

$$\sum_{D_1}\sum x^2y\,\Delta x\,\Delta y$$

$$= (1 + 2 + 3 + 4 + 4 + 8 + 12 + 16 + 9 + 18 + 27 + 36)\tfrac{1}{80}\cdot\tfrac{1}{4}\cdot\tfrac{1}{5} = 0.0875.$$

A similar computation with $\Delta x = \frac{1}{10}$, $\Delta y = \frac{1}{10}$ gives

$$\sum_{D_1}\sum x^2y\,\Delta x\,\Delta y = 0.12825.$$

EXAMPLE 2 Find the double Riemann sum

$$\sum_{D_2}\sum x^2 y\,\Delta x\,\Delta y,$$

where D_2 is the region

$$0 \le x \le 1, \qquad x^2 \le y \le \sqrt{x}$$

and

$$\Delta x = \tfrac{1}{4}, \qquad \Delta y = \tfrac{1}{5}.$$

The circumscribed rectangle of D_2 is the unit square. The partition and D_2 are shown in Figure 12.1.11 and the partition points which actually belong to D_2 are circled. The table shows the values of x^2y at the partition points which belong to D_2. It is a part of the table from Example 1.

x^2y	$y_0 = 0$	$y_1 = \frac{1}{5}$	$y_2 = \frac{2}{5}$	$y_3 = \frac{3}{5}$	$y_4 = \frac{4}{5}$
$x_0 = 0$	0				
$x_1 = \frac{1}{4}$		$\frac{1}{80}$	$\frac{2}{80}$		
$x_2 = \frac{1}{2}$			$\frac{8}{80}$	$\frac{12}{80}$	
$x_3 = \frac{3}{4}$				$\frac{27}{80}$	$\frac{36}{80}$

The double Riemann sum is

$$\sum_{D_2}\sum x^2 y\,\Delta x\,\Delta y$$
$$= \left(\frac{1}{80} + \frac{2}{80} + \frac{8}{80} + \frac{12}{80} + \frac{27}{80} + \frac{36}{80}\right)\frac{1}{4}\cdot\frac{1}{5} = \frac{86}{80\cdot 4\cdot 5} = 0.05375.$$

A similar computation with $\Delta x = \frac{1}{10}$, $\Delta y = \frac{1}{10}$ gives

$$\sum_{D_2}\sum x^2 y\,\Delta x\,\Delta y = 0.04881.$$

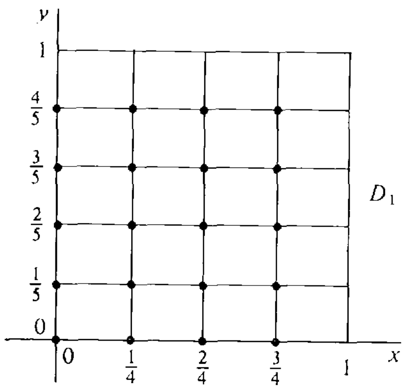


Figure 12.1.10

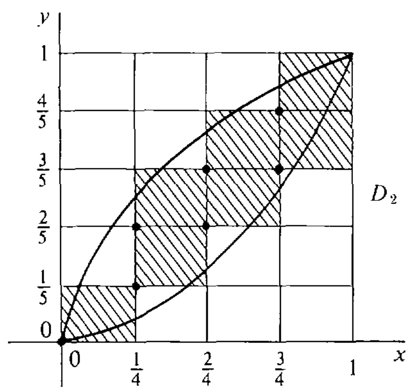


Figure 12.1.11

Given the function $f(x, y)$ and the region D, the double Riemann sum

$$\sum_D\sum f(x, y)\,\Delta x\,\Delta y$$

is a real function of Δx and Δy. When we replace Δx and Δy by positive infinitesimals dx and dy (Figure 12.1.12), we obtain (by the Function Axiom) the *infinite double Riemann sum*

$$\sum_D\sum f(x, y)\,dx\,dy.$$

The infinite double Riemann sum is in general a hyperreal number. Intuitively, it is equal to the sum of the volumes of infinitely many rectangular solids of infinitesimal base $dx\,dy$ and height $f(x_K, y_L)$. The double integral is defined as the standard part of the infinite double Riemann sum. The following lemma, based on our Permanent Assumption for Chapter 12, shows that this sum has a standard part.

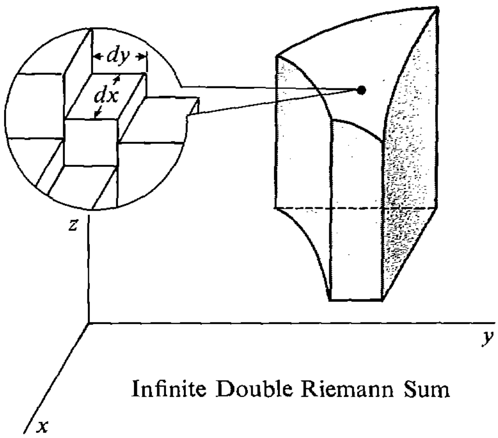


Figure 12.1.12

LEMMA

For any positive infinitesimals dx and dy, the double Riemann sum

$$\sum_D\sum f(x, y)\,dx\,dy$$

is a finite hyperreal number and thus has a standard part.

We omit the proof, which is similar to the proof that single Riemann sums are finite. We are now ready to define the double integral.

DEFINITION

Given positive infinitesimals dx and dy, the ***double integral*** *of a continuous function $f(x, y)$ over D is the standard part of the double Riemann sum·*

$$\iint_D f(x, y)\,dx\,dy = st\left(\sum_D\sum f(x, y)\,dx\,dy\right).$$

Here is a list of properties of the double integral. Each property is analogous to a property of the single integral given in Chapter Four and has a similar proof.

INDEPENDENCE OF *dx* AND *dy*

The value of the double integral $\iint_D f(x, y)\,dx\,dy$ does not depend on dx and dy. That is, if dx, d_1x, dy, and d_1y are positive infinitesimals then

$$\iint_D f(x, y)\,dx\,dy = \iint_D f(x, y)\,d_1x\,d_1y.$$

This theorem shows that the value of the double integral depends only on the function f and the region D. From now on we shall usually use the simpler notation $dA = dx\,dy$ for the area of an infinitesimal dx by dy rectangle, and

$$\iint_D f(x, y)\,dA \qquad \text{for} \qquad \iint_D f(x, y)\,dx\,dy.$$

ADDITION PROPERTY

Let D be divided into two regions D_1 and D_2 which meet only on a common boundary as in Figure 12.1.13. Then

$$\iint_D f(x, y)\,dA = \iint_{D_1} f(x, y)\,dA + \iint_{D_2} f(x, y)\,dA.$$

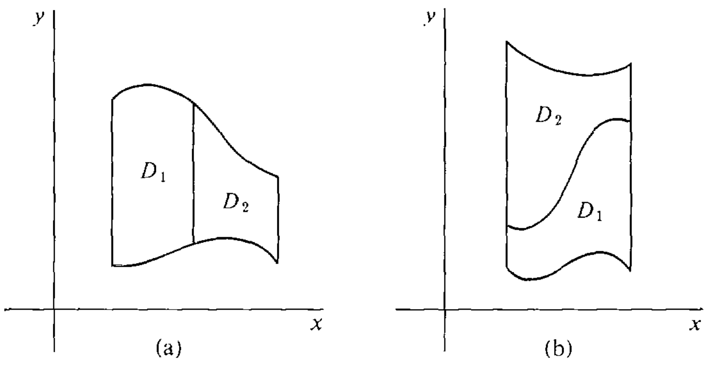


Figure 12.1.13

Interpreting the double integral as a volume, the Addition Property says that the volume of the solid over D is equal to the sum of the volume over D_1 and the volume over D_2, as shown in Figure 12.1.14.

A continuous function $z = f(x, y)$ always has a minimum and maximum value on a closed region D. The proof is similar to the one-variable case.

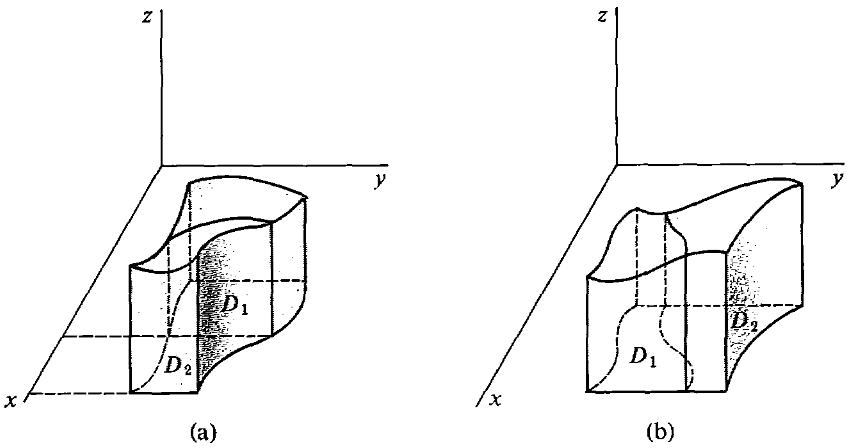


Figure 12.1.14 Addition Property

CYLINDER PROPERTY

Let m and M be the minimum and maximum values of $f(x, y)$ on D and let A be the area of D. Then

$$mA \leq \iint_D f(x, y)\, dA \leq MA.$$

This corresponds to the Rectangle Property for single integrals. The solid with base D and constant height m is called the *inscribed cylinder*, and the solid with base D and height M is called the *circumscribed cylinder*. The inscribed cylinder and the circumscribed cylinder are shown in Figure 12.1.15. Intuitively, the volume of a cylinder is equal to the area of the base A times the height. Thus the Cylinder Property states that the volume of the solid is between the volumes of the inscribed and circumscribed cylinders.

Here are two consequences of the Cylinder Property.

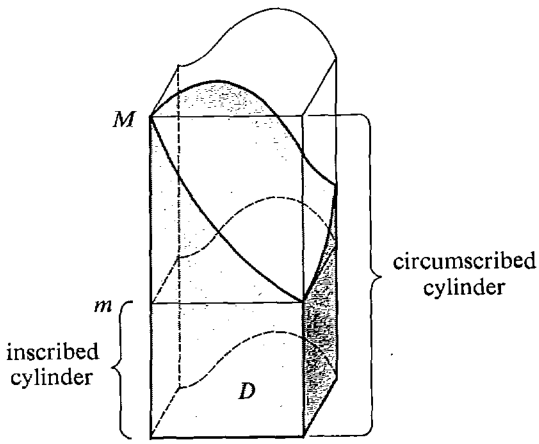


Figure 12.1.15 Cylinder Property

COROLLARY 1

The area of D is equal to the double integral of the constant function 1 *over D*, (Figure 12.1.16):

$$A = \iint_D dA.$$

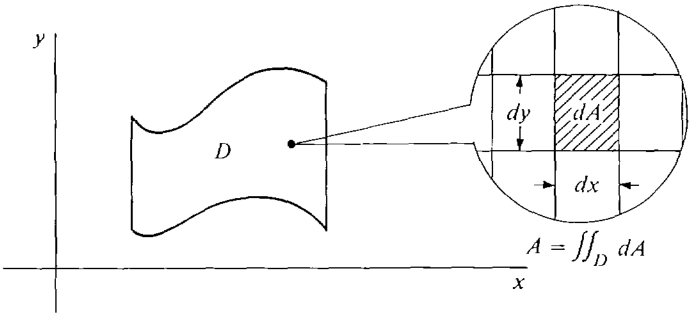


Figure 12.1.16

PROOF Both m and M are equal to 1, so $1 \cdot A \leq \iint_D dA \leq 1 \cdot A$.

COROLLARY 2

If $f(x, y) \geq 0$ *on* D *then* $\iint_D f(x, y)\, dA \geq 0$. *If* $f(x, y) \leq 0$ *on* D *then* $\iint_D f(x, y)\, dA \leq 0$.

To really be sure that the double integral corresponds to the volume, we need to know that it is the only operation that has the Addition and Cylinder Properties. To make this precise, we introduce the notion of a volume function.

We suppose $f(x, y)$ is continuous at every point of an open region D_0, and consider subregions D of D_0. A *volume function* for f is a function B which assigns a real number $B(D)$ to each subregion D of D_0 and has the Addition Property

$$B(D) = B(D_1) + B(D_2)$$

and the Cylinder Property

$$mA \leq B(D) \leq MA,$$

where m is the minimum and M the maximum value of f on D.

UNIQUENESS THEOREM

The double integral $\iint_D f(x, y)\, dA$ *is the only volume function for* f. *That is, if* B *is a function which has the Addition and Cylinder Properties, then*

$$B(D) = \iint_D f(x, y)\, dA \qquad \textit{for every } D.$$

Given a continuous function f such that $f(x, y) \geq 0$ for all (x, y), the function

$$V(D) = \text{volume over } D$$

certainly has the Addition and Cylinder Properties. Thus we are justified in defining the volume as the double integral.

DEFINITION

Let $f(x, y) \geq 0$ for (x, y) in D. Then the volume over D between $z = 0$ and $z = f(x, y)$ is the double integral

$$V = \iint_D f(x, y)\, dA.$$

When $f(x, y)$ is the constant 1, we have

$$A = \iint_D dA = V.$$

That is, the area of D is equal to the volume of the cylinder with base D and height 1, as in Figure 12.1.17.

Given any unit of length (say meters), if the height is one meter then the area is in square meters and the volume has the same value but in cubic meters.

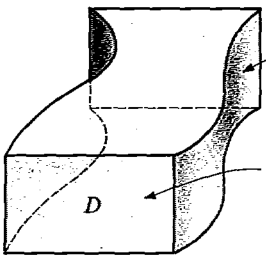


Figure 12.1.17

PROBLEMS FOR SECTION 12.1

Compute the following double Riemann sums.

1 $\sum\sum_D (3x + 4y)\,\Delta x\,\Delta y, \quad \Delta x = \frac{1}{4}, \quad \Delta y = \frac{1}{4}, \quad D: 0 \leq x \leq 1, \quad 0 \leq y \leq 1$

2 $\sum\sum_D (4 + 2x - 5y)\,\Delta x\,\Delta y, \quad \Delta x = \frac{1}{2}, \quad \Delta y = \frac{1}{5}, \quad D: -2 \leq x \leq 2, \quad -1 \leq y \leq 1$

3 $\sum\sum_D (x^2 + y^2)\,\Delta x\,\Delta y, \quad \Delta x = \frac{1}{2}, \quad \Delta y = \frac{1}{2}, \quad D: -2 \leq x \leq 2, \quad -2 \leq y \leq 2$

4 $\sum\sum_D (1 + xy)\,\Delta x\,\Delta y, \quad \Delta x = \frac{1}{3}, \quad \Delta y = \frac{1}{5}, \quad D: 0 \leq x \leq 2, \quad 0 \leq y \leq 1$

5 $\sum\sum_D \frac{x}{y}\,\Delta x\,\Delta y, \quad \Delta x = \frac{1}{4}, \quad \Delta y = \frac{1}{5}, \quad D: 1 \leq x \leq 2, \quad 1 \leq y \leq 2$

6 $\sum\sum_D (\cos x + \sin y)\,\Delta x\,\Delta y$, $\Delta x = \frac{\pi}{6}$, $\Delta y = \frac{\pi}{6}$, $D: -\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $0 \le y \le \pi$

7 $\sum\sum_D (\cos x \sin y)\,\Delta x\,\Delta y$, $\Delta x = \frac{\pi}{6}$, $\Delta y = \frac{\pi}{6}$, $D: -\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $0 \le y \le \pi$

8 $\sum\sum_D xe^y\,\Delta x\,\Delta y$, $\Delta x = \frac{1}{2}$, $\Delta y = 1$, $D: 0 \le x \le 2$, $-2 \le y \le 3$

9 $\sum\sum_D e^{2x-y}\,\Delta x\,\Delta y$, $\Delta x = 1$, $\Delta y = 1$, $D: -2 \le x \le 2$, $-2 \le y \le 2$

10 $\sum\sum_D (x + 2y)\,\Delta x\,\Delta y$, $\Delta x = \frac{1}{4}$, $\Delta y = \frac{1}{4}$, $D: 0 \le x \le 1$, $0 \le y \le x$

11 $\sum\sum_D (2 + x + 3y)\,\Delta x\,\Delta y$, $\Delta x = \frac{1}{4}$, $\Delta y = \frac{1}{4}$, $D: 0 \le x \le 1$, $x \le y \le 1$

12 $\sum\sum_D (x^2 + \sqrt{y})\,\Delta x\,\Delta y$, $\Delta x = \frac{1}{5}$, $\Delta y = \frac{1}{4}$, $D: -1 \le x \le 1$, $0 \le y \le x^2$

13 $\sum\sum_D y \sin x\,\Delta x\,\Delta y$, $\Delta x = \frac{\pi}{4}$, $\Delta y = \frac{1}{4}$, $D: 0 \le x \le \pi$, $\sin^2 x \le y \le 2\sin x$

14 $\sum\sum_D (e^x + e^y)\,\Delta x\,\Delta y$, $\Delta x = 1$, $\Delta y = 1$, $D: -3 \le x \le 3$, $-x \le y \le x$

15 $\sum\sum_D 4\,\Delta x\,\Delta y$, $\Delta x = 1$, $\Delta y = 1$, $D: x^2 + y^2 \le 9$

16 $\sum\sum_D -10\,\Delta x\,\Delta y$, $\Delta x = 1$, $\Delta y = 1$, $D: -3 \le x \le 3$, $x^2 \le y \le 18 - x^2$

□ 17 Show that if D is a region with area A and c is constant, then $\iint_D c\,dA = cA$.

□ 18 Prove the Constant Rule:

$$\sum\sum_D cf(x, y)\,\Delta x\,\Delta y = c\sum\sum_D f(x, y)\,\Delta x\,\Delta y,$$

$$\iint_D cf(x, y)\,dx\,dy = c\iint_D f(x, y)\,dx\,dy.$$

□ 19 Prove the Sum Rule:

$$\sum\sum_D f(x, y) + g(x, y)\,\Delta x\,\Delta y = \sum\sum_D f(x, y)\,\Delta x\,\Delta y + \sum\sum_D g(x, y)\,\Delta x\,\Delta y,$$

$$\iint_D f(x, y) + g(x, y)\,dx\,dy = \iint_D f(x, y)\,dx\,dy + \iint_D g(x, y)\,dx\,dy.$$

12.2 ITERATED INTEGRALS

In this section we shall learn how to evaluate double integrals. A double integral can be evaluated by two single integrations. The Iterated Integral Theorem gives the key formula.

The *iterated integral*

$$\int_{a_1}^{a_2}\left[\int_{b_1(x)}^{b_2(x)} f(x, y)\,dy\right]dx$$

is an integral of an integral of $f(x, y)$. It is evaluated in two stages. First evaluate the inside integral

$$g(x) = \int_{b_1(x)}^{b_2(x)} f(x, y)\,dy$$

by ordinary definite integration, treating x as a constant. This gives us a function of x alone. Second, evaluate the outside integral

$$\int_{a_1}^{a_2} g(x)\,dx = \int_{a_1}^{a_2} \left[\int_{b_1(x)}^{b_2(x)} f(x, y)\,dy \right] dx$$

by a second definite integration.

We shall usually drop the brackets around the inside integral and write the iterated integral as

$$\int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} f(x, y)\,dy\,dx.$$

ITERATED INTEGRAL THEOREM

Let D be a region

$$a_1 \le x \le a_2, \qquad b_1(x) \le y \le b_2(x).$$

The double integral over D is equal to the iterated integral:

$$\iint_D f(x, y)\,dA = \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} f(x, y)\,dy\,dx.$$

Discussion For a fixed x_0, $\int_{b_1(x_0)}^{b_2(x_0)} f(x_0, y)\,dy$ is the area of the cross section shown in Figure 12.2.1. The Iterated Integral Theorem states that the volume is equal to the integral of the areas of the cross sections.

The proof of the Iterated Integral Theorem is given at the end of this section. When using iterated integrals we must be sure that:

(1) $a_1 \le a_2$ and $b_1(x) \le b_2(x)$.
(2) The differentials dx and dy appear in the right order.
(3) The outer integral sign has constant limits.

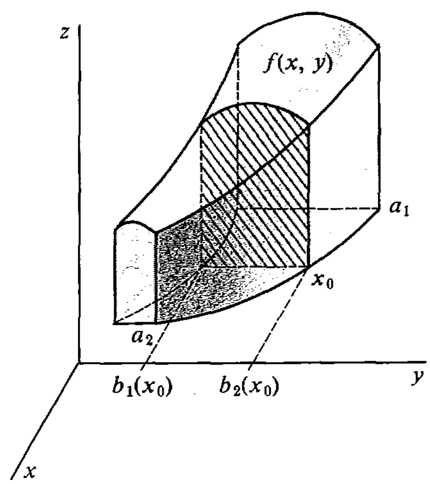


Figure 12.2.1

While the order of the differentials, $dx\,dy$ or $dy\,dx$, does not matter in a double integral, it is important in an iterated integral. The inside integral sign goes with the inside differential, and is performed first.

$$\underbrace{\int_{a_1}^{a_2} \underbrace{\int_{b_1(x)}^{b_2(x)} f(x, y)\,dy}_{\text{do first}}\,dx}_{\text{do second}}$$

When the region D is a rectangle, there are two possible orders of integration, because all the boundaries are constant. Thus there are two different iterated integrals over a rectangle. Integrating first with respect to y we have

$$\int_{a_1}^{a_2} \int_{b_1}^{b_2} f(x, y)\,dy\,dx,$$

and integrating first with respect to x we have

$$\int_{b_1}^{b_2} \int_{a_1}^{a_2} f(x, y)\,dx\,dy.$$

Using the Iterated Integral Theorem twice, we see that both iterated integrals must equal the double integral.

$$\int_{a_1}^{a_2} \int_{b_1}^{b_2} f(x, y)\,dy\,dx = \iint_D f(x, y)\,dA,$$

$$\int_{b_1}^{b_2} \int_{a_1}^{a_2} f(x, y)\,dx\,dy = \iint_D f(x, y)\,dA.$$

Therefore the two iterated integrals are equal to each other. We have proved a corollary.

COROLLARY

The two iterated integrals over a rectangle are equal:

$$\int_{a_1}^{a_2} \int_{b_1}^{b_2} f(x, y)\,dy\,dx = \int_{b_1}^{b_2} \int_{a_1}^{a_2} f(x, y)\,dx\,dy.$$

Discussion This corollary is the simplest form of a result known as Fubini's Theorem. Remember that by our Permanent Assumption, $f(x, y)$ is continuous on D. For an idea of the difficulties that arise when $f(x, y)$ is not assumed to be continuous, see Problem 49 at the end of this section.

There are also other regions besides rectangles over which we can integrate in either of two orders, such as Example 5 in this section.

In the following two examples we evaluate the double integrals which were approximated by double Riemann sums in the preceding section.

EXAMPLE 1 Evaluate
$$\iint_{D_1} x^2 y\,dA$$
where D_1 is the unit square
$$0 \le x \le 1, \qquad 0 \le y \le 1.$$
The limits of the outside integral are given by $0 \le x \le 1$, and those of the inside integral are given by $0 \le y \le 1$. The iterated integral is thus
$$\iint_{D_1} x^2 y\,dA = \int_0^1 \int_0^1 x^2 y\,dy\,dx.$$
The inside integral is
$$\int_0^1 x^2 y\,dy = \tfrac{1}{2}x^2y^2\Big]_{y=0}^{y=1} = \tfrac{1}{2}x^2.$$
Then
$$\iint_{D_1} x^2 y\,dA = \int_0^1 \tfrac{1}{2}x^2\,dx = \tfrac{1}{6}x^3\Big]_0^1 = \tfrac{1}{6} \sim 0.16667.$$
Since D_1 is a rectangle we may also integrate in the other order, and should get the same answer.
$$\iint_{D_1} x^2 y\,dA = \int_0^1 \int_0^1 x^2 y\,dx\,dy.$$
$$\int_0^1 x^2 y\,dx = \tfrac{1}{3}x^3y\Big]_0^1 = \tfrac{1}{3}y.$$
$$\iint_{D_1} x^2 y\,dA = \int_0^1 \tfrac{1}{3}y\,dy = \tfrac{1}{6}y^2\Big]_0^1 = \tfrac{1}{6} \sim 0.16667.$$
The Riemann sums in Section 12.1 were 0.0875, 0.12825.

EXAMPLE 2 Evaluate $\iint_{D_2} x^2 y\,dA$ where D_2 is the region in Figure 12.2.2:
$$0 \le x \le 1, \qquad x^2 \le y \le \sqrt{x}.$$
The limits on the outside integral are given by $0 \le x \le 1$, and those on the inside integral by $x^2 \le y \le \sqrt{x}$, so the iterated integral is

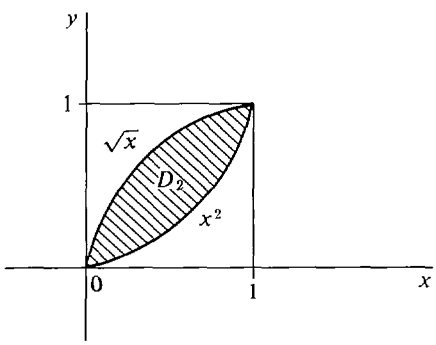


Figure 12.2.2

$$\iint_{D_2} x^2 y \, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} x^2 y \, dy \, dx.$$

$$\int_{x^2}^{\sqrt{x}} x^2 y \, dy = \tfrac{1}{2}x^2y^2 \Big]_{y=x^2}^{y=\sqrt{x}} = \tfrac{1}{2}x^3 - \tfrac{1}{2}x^6.$$

$$\iint_{D_2} x^2 y \, dA = \int_0^1 (\tfrac{1}{2}x^3 - \tfrac{1}{2}x^6) \, dx = \tfrac{1}{8}x^4 - \tfrac{1}{14}x^7 \Big]_0^1$$

$$= \tfrac{3}{56} \sim 0.05357.$$

The Riemann sums in Section 12.1 were 0.05375, 0.04881.

In many applications the region D is given verbally, and part of the problem is to find inequalities which describe D.

EXAMPLE 3 Let D be the region bounded by the curve $xy = 1$ and the line $y = \frac{5}{2} - x$. Find inequalities which describe D, and write down an iterated integral equal to $\iint_D f(x, y) \, dA$.

Step 1 Sketch the region D as in Figure 12.2.3.

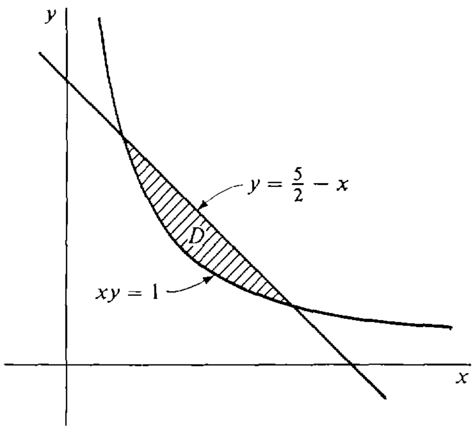


Figure 12.2.3

Step 2 The line and curve intersect where

$$x(\tfrac{5}{2} - x) = 1,$$
$$x^2 - \tfrac{5}{2}x + 1 = 0,$$
$$(x - \tfrac{1}{2})(x - 2) = 0,$$
$$x = \tfrac{1}{2}, \qquad x = 2.$$

For $1/2 \le x \le 2$, the curve $y = 1/x$ is below the line $y = 5/2 - x$. Therefore D is the region

$$\tfrac{1}{2} \le x \le 2, \qquad 1/x \le y \le \tfrac{5}{2} - x.$$

Step 3 The inequalities for x give the limits of the outside integral, and those for y give the limits of the inside integral. Thus

$$\iint_D f(x, y) \, dA = \int_{1/2}^{2} \int_{1/x}^{(5/2)-x} f(x, y) \, dy \, dx.$$

EXAMPLE 4 Find the volume of the solid bounded by the surfaces $z = 0$, $z = y - x^2$, $y = 1$.

Step 1 Sketch the solid and the region D, as in Figure 12.2.4.

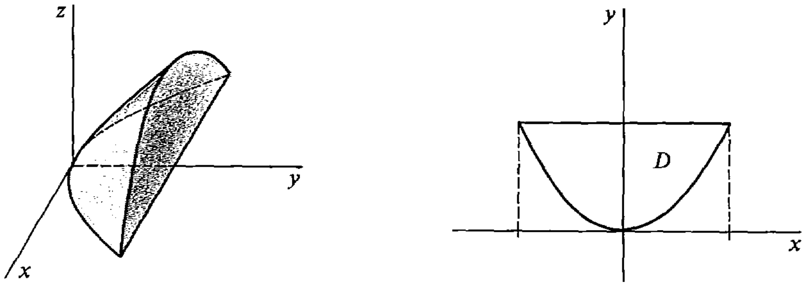


Figure 12.2.4

Step 2 Find the inequalities describing the region D.

This is the hardest step, and gives us the limits of integration. The surfaces $z = 0$ and $z = y - x^2$ intersect at the curve $y = x^2$. We see from the figure that D is the region between the curves $y = x^2$ and $y = 1$, so D is given by

$$-1 \le x \le 1, \qquad x^2 \le y \le 1.$$

Step 3 Set up the iterated integral and evaluate it.

$$V = \iint_D y - x^2\, dA = \int_{-1}^{1}\int_{x^2}^{1} y - x^2\, dy\, dx.$$

$$\int_{x^2}^{1} y - x^2\, dy = \tfrac{1}{2}y^2 - x^2 y\Big]_{x^2}^{1}$$

$$= (\tfrac{1}{2}\cdot 1^2 - x^2\cdot 1) - (\tfrac{1}{2}(x^2)^2 - x^2\cdot x^2)$$

$$= \tfrac{1}{2} - x^2 + \tfrac{1}{2}x^4.$$

$$V = \int_{-1}^{1} \tfrac{1}{2} - x^2 + \tfrac{1}{2}x^4\, dx = \tfrac{16}{30}.$$

Multiple integration problems can be solved by a three-step process as shown in Examples 3 and 4.

Step 1 Sketch the problem.

Step 2 Find the inequalities describing the region D.

Step 3 Set up the iterated integral and evaluate.

We can also integrate over a region in the (y, x) plane instead of the (x, y) plane. A region D in the (y, x) plane has the form

$$b_1 \le y \le b_2, \qquad a_1(y) \le x \le a_2(y),$$

as shown in Figure 12.2.5.

The double integral over D is equal to the iterated integral with dy on the outside and dx inside,

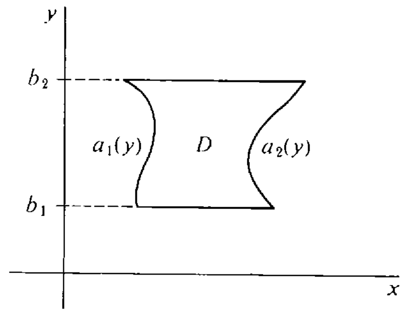
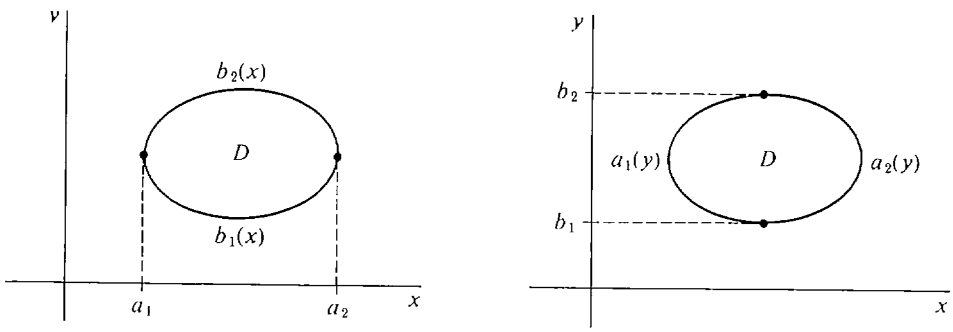


Figure 12.2.5

$$\iint_D f(x, y)\, dA = \int_{b_1}^{b_2} \int_{a_1(y)}^{a_2(y)} f(x, y)\, dx\, dy.$$

Some regions, such as rectangles and ellipses, may be regarded as regions in either the (x, y) plane or the (y, x) plane (Figure 12.2.6).



(a) D as an (x, y) Region

(b) D as a (y, x) Region

Figure 12.2.6

EXAMPLE 5 Let D be the region bounded by the curves

$$x = y^2, \qquad x = y + 2.$$

Evaluate the double integral $\iint_D xy\, dA$.

Step 1 The region D is sketched in Figure 12.2.7.

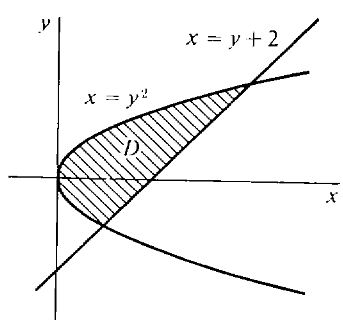


Figure 12.2.7

Step 2 Find inequalities for D. To do this we must find the points where the curves

$$x = y^2, \qquad x = y + 2$$

intersect. Solving for y and then x, we see that they intersect at

$$(1, -1), \qquad (4, 2).$$

We see from the figure that D is a region in either the (x, y) plane or the (y, x) plane. However, the boundary curves are simpler in the (y, x) plane. D is the region

$$-1 \le y \le 2, \qquad y^2 \le x \le y + 2.$$

Step 3 Set up the iterated integral and evaluate.

$$\iint_D xy\,dA = \int_{-1}^{2}\int_{y^2}^{y+2} xy\,dx\,dy.$$

$$\begin{aligned}\int_{y^2}^{y+2} xy\,dx &= \tfrac{1}{2}x^2y\Big]_{y^2}^{y+2}\\ &= \tfrac{1}{2}(y+2)^2y - \tfrac{1}{2}(y^2)^2y\\ &= \tfrac{1}{2}y^3 + 2y^2 + 2y - \tfrac{1}{2}y^5.\end{aligned}$$

$$\iint_D xy\,dA = \int_{-1}^{2} \tfrac{1}{2}y^3 + 2y^2 + 2y - \tfrac{1}{2}y^5\,dy = \tfrac{135}{24}.$$

PROOF OF THE ITERATED INTEGRAL THEOREM For any region D, let $B(D)$ be the iterated integral over D. Our plan is to prove that B has the Addition and Cylinder Properties, so that by the Uniqueness Theorem $B(D)$ will equal the double integral.

PROOF OF ADDITION PROPERTY

Case 1 Let D be divided into D_1 and D_2 as in Figure 12.2.8(a). By the Addition Property for single integrals,

$$\begin{aligned}B(D) &= \int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)} f\,dy\,dx\\ &= \int_{a_1}^{a_3}\int_{b_1(x)}^{b_2(x)} f\,dy\,dx + \int_{a_3}^{a_2}\int_{b_1(x)}^{b_2(x)} f\,dy\,dx\\ &= B(D_1) + B(D_2).\end{aligned}$$

Case 2 Let D be divided into D_1 and D_2 as in Figure 12.2.8(b). Then

$$\begin{aligned}B(D) &= \int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)} f\,dy\,dx\\ &= \int_{a_1}^{a_2}\left[\int_{b_1(x)}^{b_3(x)} f\,dy + \int_{b_3(x)}^{b_2(x)} f\,dy\right]dx\\ &= \int_{a_1}^{a_2}\int_{b_1(x)}^{b_3(x)} f\,dy\,dx + \int_{a_1}^{a_2}\int_{b_3(x)}^{b_2(x)} f\,dy\,dx\\ &= B(D_1) + B(D_2).\end{aligned}$$

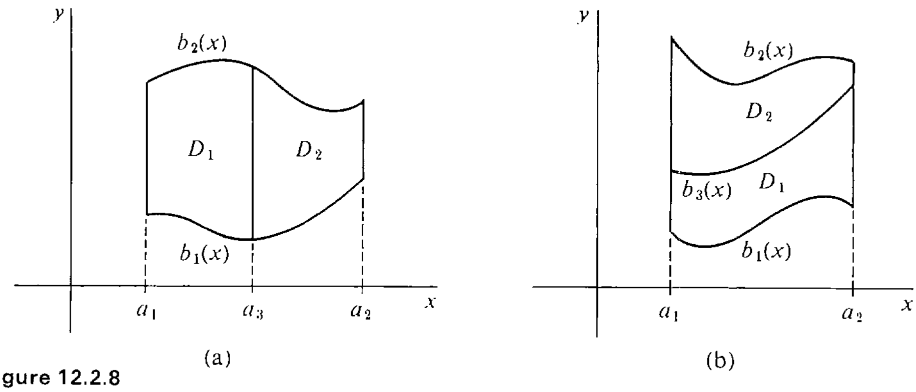


Figure 12.2.8

PROOF OF CYLINDER PROPERTY Let m be the minimum value and M the maximum value of $f(x, y)$ on D For each fixed value of x,

$$\int_{b_1(x)}^{b_2(x)} m\,dy \le \int_{b_1(x)}^{b_2(x)} f(x, y)\,dy.$$

Integrating from a_1 to a_2,

$$\int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)} m\,dy\,dx \le \int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)} f\,dy\,dx = B(D).$$

But
$$\int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)} m\,dy\,dx = \int_{a_1}^{a_2} m(b_2(x) - b_1(x))\,dx$$
$$= m\int_{a_1}^{a_2} (b_2(x) - b_1(x))\,dx = mA.$$

Therefore $$mA \le B(D).$$

By a similar argument, $$B(D) \le MA.$$

Since B has both the Addition and Cylinder Properties,

$$B(D) = \iint_D f(x, y)\,dA.$$

The Constant, Sum, and Inequality Rules for double integrals follow easily from the corresponding rules for single integrals, using the Iterated Integral Theorem.

CONSTANT RULE

$$\iint_D cf(x, y)\,dA = c\iint_D f(x, y)\,dA.$$

SUM RULE

$$\iint_D f(x, y) + g(x, y)\,dA = \iint_D f(x, y)\,dA + \iint_D g(x, y)\,dA.$$

INEQUALITY RULE

If $f(x, y) \le g(x, y)$ *for all* (x, y) *in* D,

$$\iint_D f(x, y)\, dA \le \iint_D g(x, y)\, dA.$$

PROOF As an illustration we prove the Sum Rule.

$$\begin{aligned}\iint_D f + g\, dA &= \int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)} f + g\, dy\, dx\\ &= \int_{a_1}^{a_2}\left[\int_{b_1(x)}^{b_2(x)} f\, dy + \int_{b_1(x)}^{b_2(x)} g\, dy\right] dx\\ &= \int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)} f\, dy\, dx + \int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)} g\, dy\, dx\\ &= \iint_D f\, dA + \iint_D g\, dA.\end{aligned}$$

The Iterated Integral Theorem gives another proof that the area of D is equal to the double integral of 1 over D.

By definition of area between two curves,

$$A = \int_{a_1}^{a_2} (b_2(x) - b_1(x))\, dx.$$

Using iterated integrals,

$$\begin{aligned}\iint_D dA &= \int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)} dy\, dx\\ &= \int_{a_1}^{a_2} (b_2(x) - b_1(x))\, dx = A.\end{aligned}$$

PROBLEMS FOR SECTION 12.2

In Problems 1–16, evaluate the double integrals (compare these with the problems from Section 12.1).

1 $\displaystyle\iint_D (3x + 4y)\, dA, \quad D: 0 \le x \le 1, 0 \le y \le 1$

2 $\displaystyle\iint_D (4 + 2x - 5y)\, dA, \quad D: -2 \le x \le 2, -1 \le y \le 1$

3 $\displaystyle\iint_D (x^2 + y^2)\, dA, \quad D: -2 \le x \le 2, -2 \le y \le 2$

4 $\displaystyle\iint_D (1 + xy)\, dA, \quad D: 0 \le x \le 2, 0 \le y \le 1$

5 $\iint_D \frac{x}{y}\, dA,\quad D: 1 \le x \le 2,\ 1 \le y \le 2$

6 $\iint_D (\cos x + \sin y)\, dA,\quad D: -\pi/2 \le x \le \pi/2,\ 0 \le y \le \pi$

7 $\iint_D (\cos x \sin y)\, dA,\quad D: -\pi/2 \le x \le \pi/2,\ 0 \le y \le \pi$

8 $\iint_D xe^y\, dA,\quad D: 0 \le x \le 2,\ -2 \le y \le 3$

9 $\iint_D e^{2x-y}\, dA,\quad D: -2 \le x \le 2,\ -2 \le y \le 2$

10 $\iint_D (x + 2y)\, dA,\quad D: 0 \le x \le 1,\ 0 \le y \le x$

11 $\iint_D (2 + x + 3y)\, dA,\quad D: 0 \le x \le 1,\ x \le y \le 1$

12 $\iint_D (x^2 + \sqrt{y})\, dA,\quad D: -1 \le x \le 1,\ 0 \le y \le x^2$

13 $\iint_D y \sin x\, dA,\quad D: 0 \le x \le \pi,\ \sin^2 x \le y \le 2 \sin x$

14 $\iint_D (e^x + e^y)\, dA,\quad D: -3 \le x \le 3,\ -x \le y \le x$

15 $\iint_D 4\, dA,\quad D: x^2 + y^2 \le 9$

16 $\iint_D -10\, dA,\quad D: -3 \le x \le 3,\ x^2 \le y \le 18 - x^2$

In Problems 17–24, evaluate the iterated integral. Then check your answer, by evaluating in the other order.

17 $\int_0^1 \int_0^1 (x^2 y - 3xy^2 + 5)\, dy\, dx$

18 $\int_0^1 \int_0^1 xy(2y + 1)\, dy\, dx$

19 $\int_3^6 \int_{-2}^8 dy\, dx$

20 $\int_2^4 \int_1^6 3x\, dy\, dx$

21 $\int_0^{\pi/2} \int_0^{\pi/2} \sin(x + y)\, dy\, dx$

22 $\int_{-1}^1 \int_0^2 \frac{y}{1 + x^2}\, dy\, dx$

23 $\int_0^3 \int_1^6 \sqrt{x + y}\, dy\, dx$

24 $\int_1^2 \int_0^1 \frac{1}{x + y}\, dy\, dx$

In Problems 25–30 evaluate the iterated integral.

25 $\int_0^1 \int_0^{e^x} dy\, dx$

26 $\int_0^\pi \int_{\sin x}^1 dy\, dx$

27 $\int_0^2 \int_0^{\sqrt{4-x^2}} y\,dy\,dx$

28 $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 + y^2\,dy\,dx$

29 $\int_0^3 \int_{y^2}^{3y} x^2 y\,dx\,dy$

30 $\int_0^1 \int_0^y \frac{2}{\sqrt{1-x^2}}\,dx\,dy$

In Problems 31–38, find inequalities which describe the given region D, and write down an iterated integral equal to $\iint_D f(x, y)\,dA$.

31 The triangle with vertices (0, 0), (5, 0), (0, 5).

32 The triangle with vertices (1, −2), (1, 4), (5, 0).

33 The circle of radius 2 with center at the origin.

34 The bottom half of the circle of radius 1 with center at (2, 3).

35 The region bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$.

36 The region above the parabola $y = x^2$ and inside the circle $x^2 + y^2 = 1$.

37 The region bounded by the curves $x = \frac{1}{2}$ and $x = 1/(1 + y^2)$.

38 The region bounded by the curves $x = 12 + y^2$ and $x = y^4$.

39 Find the volume of the solid over the region $x^2 + y^2 \le 1$ and between the surfaces $z = 0, z = x^2$.

40 Find the volume of the solid over the region

$$D: 1 \le x \le 2, x \le y \le x^2$$

and between the surfaces $z = 0$, $z = y/x$.

41 Find the volume of the solid between the surfaces $z = 0$, $z = 2 + 3x - y$, over the region $0 \le x \le 2, 0 \le y \le x$.

42 Find the volume of the solid between the surfaces $z = 0$, $z = \sqrt{y - x}$, over the region $0 \le x \le 1, x \le y \le 1$.

43 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid

$$z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

44 Find the volume of the solid bounded by the three coordinate planes and the plane $ax + by + cz = 1$, where a, b, and c are positive.

□ 45 Show that

$$\int_{a_1}^{a_2} \int_{b_1}^{b_2} f(x)\,dy\,dx = (b_2 - b_1)\int_{a_1}^{a_2} f(x)\,dx.$$

□ 46 Show that

$$\int_{a_1}^{a_2} \int_{b_1}^{b_2} f(x) + g(y)\,dy\,dx = (b_2 - b_1)\int_{a_1}^{a_2} f(x)\,dx + (a_2 - a_1)\int_{b_1}^{b_2} g(y)\,dy.$$

□ 47 Show that

$$\int_{a_1}^{a_2} \int_{b_1}^{b_2} f(x)g(y)\,dy\,dx = \left(\int_{a_1}^{a_2} f(x)\,dx\right)\left(\int_{b_1}^{b_2} g(y)\,dy\right).$$

□ 48 Show that

$$\int_a^b \int_{-g(x)}^{g(x)} y\,dy\,dx = 0.$$

□ 49 Let

$$f(x, y) = \begin{cases} y & \text{if } x \text{ is rational,} \\ 1 - y & \text{if } x \text{ is irrational.} \end{cases}$$

Show that:

(a) $$\int_0^1 \int_0^1 f(x, y)\,dy\,dx = \int_0^1 \tfrac{1}{2}\,dx = \tfrac{1}{2}.$$

(b) For each constant $y_0 \ne \frac{1}{2}$, the function $g(x) = f(x, y_0)$ is everywhere discontinuous, so that the iterated integral $\int_0^1\int_0^1 f(x, y)\,dx\,dy$ is undefined.

12.3 INFINITE SUM THEOREM AND VOLUME

The double integral, like the single integral, has a number of applications to geometry and physics. The basic theorem which justifies these applications is the Infinite Sum Theorem. It shows how to get an integration formula by considering an infinitely small element of area.

An *element of area* is a rectangle ΔD whose sides are infinitesimal and parallel to the x and y axes. Given an element of area ΔD, we let

$$\begin{aligned}(x, y) &= \text{lower left corner of } \Delta D,\\ \Delta x, \Delta y &= \text{dimensions of } \Delta D,\\ \Delta A &= \Delta x\,\Delta y = \text{area of } \Delta D.\end{aligned}$$

ΔD is illustrated in Figure 12.3.1.

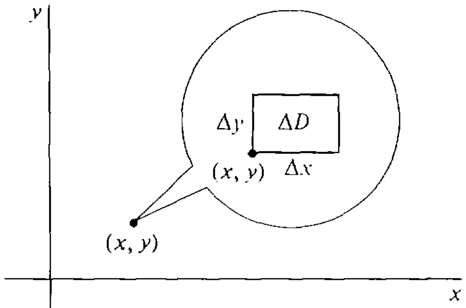


Figure 12.3.1

INFINITE SUM THEOREM

Let $h(x, y)$ be continuous on an open region D_0 and let B be a function which assigns a real number $B(D)$ to each region D contained in D_0. Assume that

(i) *B has the Addition Property $B(D) = B(D_1) + B(D_2)$.*

(ii) *$B(D) \geq 0$ for every D.*

(iii) *For every element of area ΔD, $B(\Delta D) \approx h(x, y)\,\Delta A$ (compared to ΔA).*

Then
$$B(D) = \iint_D h(x, y)\, dA.$$

We shall use the notation

$$\Delta B = B(\Delta D).$$

Given (i) and (ii), the theorem shows that if we always have

$$\Delta B \approx h(x, y)\,\Delta A \qquad \text{(compared to } \Delta A)$$

then
$$B(D) \approx \sum_D\sum h(x, y)\,\Delta A.$$

The proof is simplest in the case that D is a rectangle.

PROOF WHEN D IS A RECTANGLE Choose positive infinitesimal Δx and Δy and partition D into elements of area ΔD (Figure 12.3.2). Since B has the Addition Property, $B(D)$ is the sum of the ΔB's. Let c be any positive real number. For each ΔD we have

$$\Delta B \approx h(x, y)\,\Delta A \qquad \text{(compared to } \Delta A\text{)},$$

$$\frac{\Delta B}{\Delta A} \approx h(x, y),$$

$$\frac{\Delta B}{\Delta A} - c < h(x, y) < \frac{\Delta B}{\Delta A} + c,$$

$$\Delta B - c\,\Delta A < h(x, y)\,\Delta A < \Delta B + c\,\Delta A.$$

Letting A be the area of D and adding up,

$$B(D) - cA < \sum_D\sum h(x, y)\,\Delta A < B(D) + cA.$$

Taking standard parts,

$$B(D) - cA \le \iint_D h(x, y)\,dA \le B(D) + cA,$$

so

$$B(D) = \iint_D h(x, y)\,dA.$$

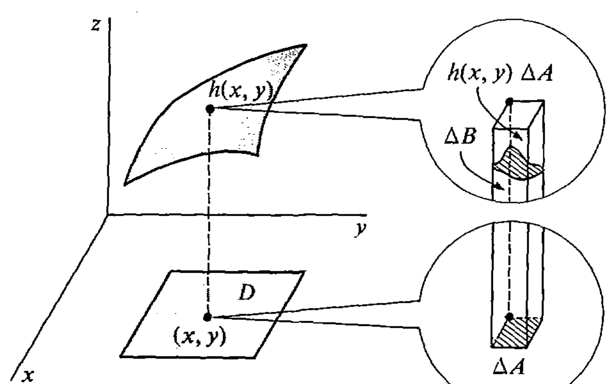


Figure 12.3.2

The proof in the general case is similar except that some of the elements of area ΔD will overlap the boundary of D and thus be only partly within D. (See Figure 12.3.3.) The method of proof is to change D to include all instead of part of each ΔD, use hypothesis (ii) to show that the new $B(D)$ is infinitely close to the old one, and then show as above that the new $B(D)$ is infinitely close to the double integral $\iint_D h(x, y)\,dA$.

In most applications of the Infinite Sum Theorem, hypotheses (i) and (ii) are automatic. To get a formula for $B(D)$ in practice, we take an element of area ΔD and

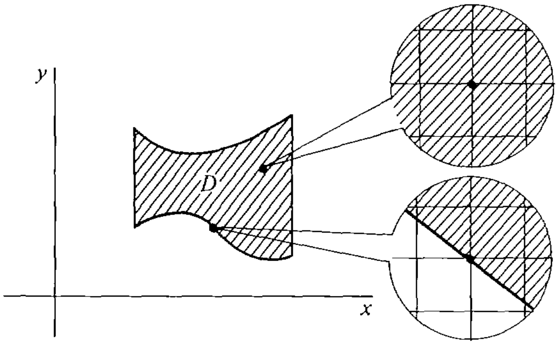


Figure 12.3.3

find an $h(x, y)$ such that

$$\Delta B \approx h(x, y)\,\Delta A \qquad \text{(compared to } \Delta A\text{)}.$$

Our first application is to the volume between two surfaces.

DEFINITION

Let $f(x, y) \le g(x, y)$ for (x, y) in D and let E be the set of all points in space such that

$$(x, y) \text{ is in } D, \qquad f(x, y) \le z \le g(x, y).$$

*The **volume** of E is*

$$V = \iint_D g(x, y) - f(x, y)\, dA.$$

V is called the volume over D between the surfaces $z = f(x, y)$ and $z = g(x, y)$ (Figure 12.3.4).

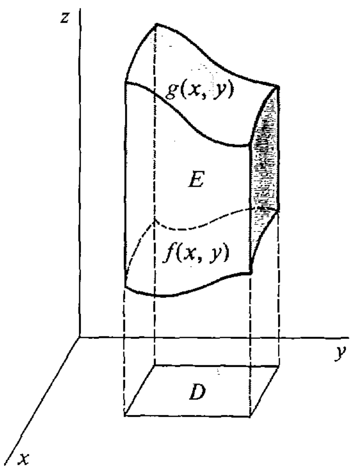


Figure 12.3.4 Volume between two surfaces

JUSTIFICATION The part ΔE of the solid E over an element of area ΔD is a rectangular solid with base ΔA and height $g(x, y) - f(x, y)$, except that the top and bottom surfaces are curved (Figure 12.3.5). Therefore the volume of ΔE is

$$\Delta V \approx (g(x, y) - f(x, y))\,\Delta A \qquad \text{(compared to } \Delta A).$$

By the Infinite Sum Theorem,

$$V = \iint_D g(x, y) - f(x, y)\, dA.$$

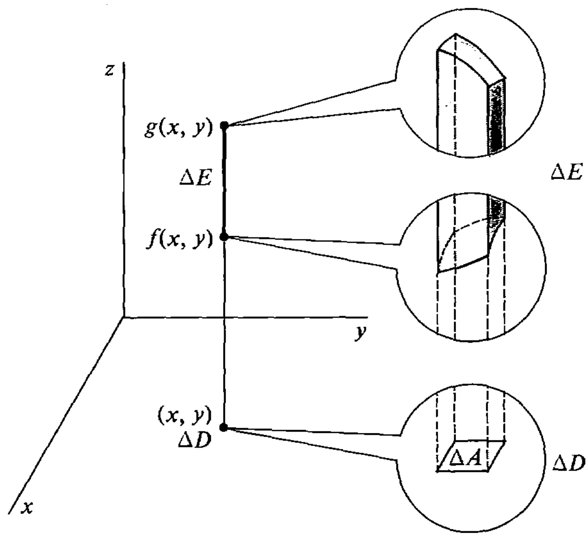


Figure 12.3.5

EXAMPLE 1 Find the volume of the solid

$$0 \le x \le 1, \quad 0 \le y \le x, \qquad x + y \le z \le e^{x+y}.$$

Step 1 D is the triangle shown in Figure 12.3.6.

Step 2 D is the region $0 \le x \le 1, 0 \le y \le x$.

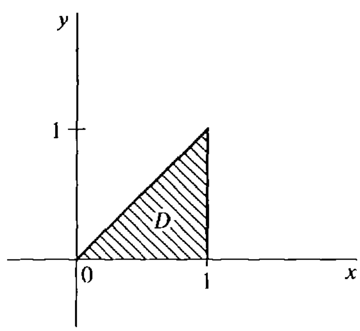


Figure 12.3.6

Step 3

$$V = \iint_D e^{x+y} - (x + y)\, dA$$
$$= \int_0^1 \int_0^x e^{x+y} - (x + y)\, dy\, dx.$$
$$\int_0^x e^{x+y} - (x + y)\, dy = e^{x+y} - xy - \tfrac{1}{2}y^2\Big]_0^x = e^{2x} - e^x - \tfrac{3}{2}x^2.$$
$$V = \int_0^1 e^{2x} - e^x - \tfrac{3}{2}x^2\, dx = \tfrac{1}{2}e^2 - e.$$

EXAMPLE 2 Find the volume of the solid bounded by the four planes

$$x = 0, \qquad y = 0, \qquad z = x + y, \qquad z = 1 - x - y.$$

Step 1 Sketch the planes. We see from Figure 12.3.7 that $z = x + y$ is below $z = 1 - x - y$.

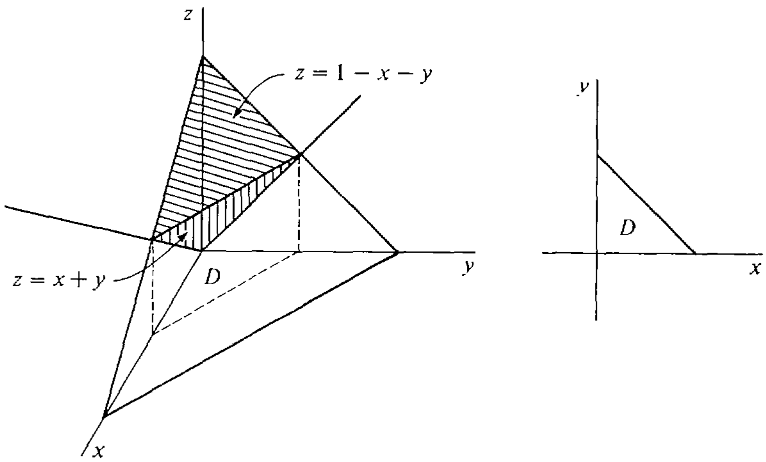


Figure 12.3.7

Step 2 Find inequalities for the region D. Since the two planes

$$z = x + y, \qquad z = 1 - x - y$$

meet at the line $\quad 2x + 2y = 1, \qquad y = \tfrac{1}{2} - x,$

D is the region $\quad 0 \le x \le \tfrac{1}{2}, \qquad 0 \le y \le \tfrac{1}{2} - x.$

Step 3

$$V = \iint_D (1 - x - y) - (x + y)\, dA = \iint_D 1 - 2x - 2y\, dA$$
$$= \int_0^{1/2} \int_0^{1/2 - x} 1 - 2x - 2y\, dy\, dx.$$

$$\int_0^{1/2-x} 1 - 2x - 2y\,dy = y - 2xy - y^2\Big]_0^{1/2-x}$$
$$= \tfrac{1}{2} - x - 2x(\tfrac{1}{2} - x) - (\tfrac{1}{2} - x)^2 = \tfrac{1}{4} - x + x^2.$$
$$V = \int_0^{1/2} \tfrac{1}{4} - x + x^2\,dx = \tfrac{1}{24}.$$

EXAMPLE 3 Find the volume of the solid bounded by the plane $z = 2y$ and the paraboloid $z = 1 - 2x^2 - y^2$.

Step 1 The surfaces and the region D are sketched in Figure 12.3.8.

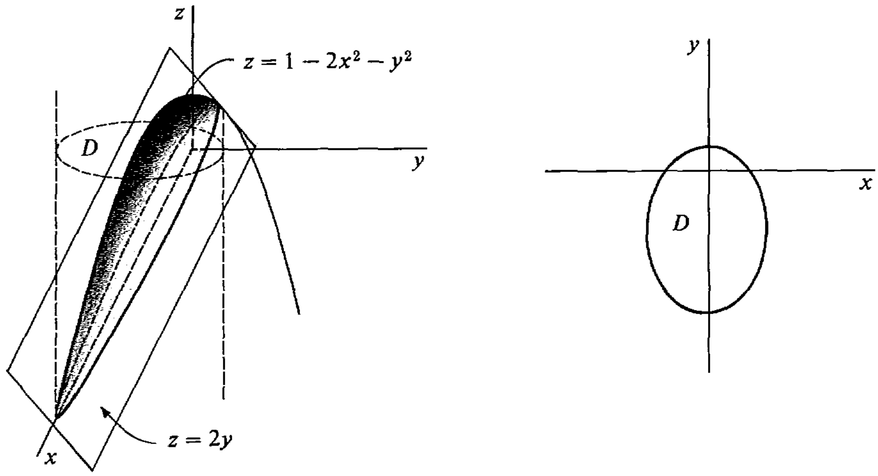


Figure 12.3.8

Step 2 The two surfaces intersect on the curve

$$2y = 1 - 2x^2 - y^2,$$

or solving for y, $\quad y = -1 \pm \sqrt{2 - 2x^2}.$

Therefore D is the region

$$-1 \le x \le 1, \qquad -1 - \sqrt{2 - 2x^2} \le y \le -1 + \sqrt{2 - 2x^2}.$$

Step 3 We see from the figure that the plane is the lower surface and the paraboloid is the upper surface.

$$V = \iint_D (1 - 2x^2 - y^2) - 2y\,dA$$
$$= \int_{-1}^{1} \int_{-1-\sqrt{2-2x^2}}^{-1+\sqrt{2-2x^2}} (1 - 2x^2 - y^2 - 2y)\,dy\,dx.$$

$$\int_{-1-\sqrt{2-2x^2}}^{-1+\sqrt{2-2x^2}} (1 - 2x^2 - y^2 - 2y)\,dy = \int_{-\sqrt{2-2x^2}}^{\sqrt{2-2x^2}} (2 - 2x^2 - u^2)\,du$$
$$= \frac{8\sqrt{2}}{3}(1 - x^2)^{3/2}.$$
$$V = \int_{-1}^{1} \frac{8\sqrt{2}}{3}(1 - x^2)^{3/2}\,dx.$$

Put $x = \sin\theta$, $\sqrt{1 - x^2} = \cos\theta$, $dx = \cos\theta\,d\theta$ (Figure 12.3.9).

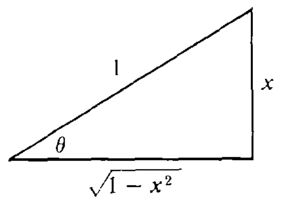


Figure 12.3.9

$$V = \frac{8\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta$$
$$= \frac{8\sqrt{2}}{3}\left(\frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}\left(\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\right)\right)\Big]_{-\pi/2}^{\pi/2}$$
$$= \frac{8\sqrt{2}}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\pi = \sqrt{2}\pi.$$

Answer $V = \sqrt{2}\pi$.

PROBLEMS FOR SECTION 12.3

Find the volumes of the following solids in Problems 1–8.

1 $0 \le x \le 1$, $0 \le y \le 1$, $xy \le z \le 1$

2 $0 \le x \le 2$, $0 \le y \le 2$, $x^2 + y^2 \le z \le 8$

3 $0 \le x \le 1$, $1 \le y \le 2$, $x \le z \le y$

4 $0 \le x \le 4$, $0 \le y \le 1$, $x \le z \le xe^y$

5 $0 \le x \le 2$, $0 \le y \le x$, $y \le z \le x$

6 $1 \le x \le 4$, $x \le y \le 4$, $y \le x \le xy$

7 $-1 \le x \le 1$, $x^2 \le y \le 1$, $x\sqrt{y} \le z \le y$

8 $0 \le x \le \pi$, $-\sin x \le y \le \sin x$, $-\sin x \le z \le \sin x$

In Problems 9–16, find the volume of the solid bounded by the given surfaces.

9 The planes $y = 0$, $x + y = 2$, $z = -x$, $z = x$

10 The planes $x = 0$, $y = 0$, $2x + 3y + z = 4$, $6x + y - z = 8$

11 $z = x^2 + y^2, \quad z = 4$

12 $z = x^2 + y^2 + 1, \quad z = 2x + 2y$

13 $y = 0, \quad z = x^2 + y, \quad z = 1$

14 $x = 0, \quad x = y, \quad z^2 = 1 - y$

15 $x^2 + y^2 = 9, \quad x^2 + z^2 = 9$

16 $z = x^2 + y^2, \quad z = 2 - x^2 - y^2$

17 Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

18 Find the volume of the solid bounded by the paraboloid $z = x^2/a^2 + y^2/b^2$ and the plane $z = c$, where c is positive.

12.4 APPLICATIONS TO PHYSICS

In this section we obtain double integrals for mass, center of mass, and moment of inertia.

DEFINITION

If a plane object fills a region D and has continuous density $\rho(x, y)$, its mass is

$$m = \iint_D \rho(x, y)\, dA.$$

On an element of area ΔD, the density is infinitely close to $\rho(x, y)$ (Figure 12.4.1). Therefore the mass is

$$\Delta m \approx \rho(x, y)\, \Delta A \qquad \text{(compared to } \Delta A).$$

By the Infinite Sum Theorem $m = \iint_D \rho(x, y)\, dx\, dy$.

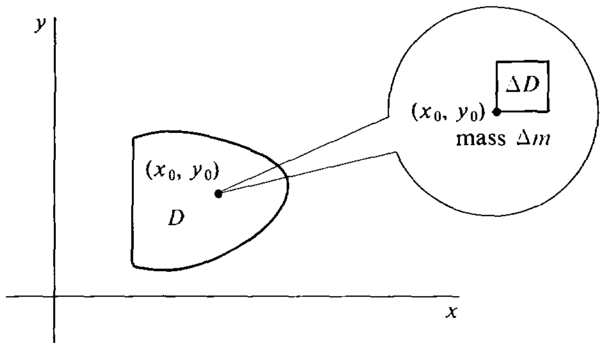


Figure 12.4.1

In Chapter 6 we were able to find the mass of a plane object whose density $\rho(x)$ depends only on x by a single integral,

$$m = \int_{a_1}^{a_2} \rho(x)(b_2(x) - b_1(x))\, dx.$$

Our new formula for mass reduces to the old formula in this case, for by the Iterated Integral Theorem,

$$\begin{aligned} m &= \iint_D \rho(x)\, dA \\ &= \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} \rho(x)\, dy\, dx \\ &= \int_{a_1}^{a_2} \rho(x)(b_2(x) - b_1(x))\, dx. \end{aligned}$$

Now we can find the mass of a plane object whose density $\rho(x, y)$ depends on both x and y instead of on x alone.

EXAMPLE 1 Find the mass of an object in the shape of a unit square whose density is the sum of the distance from one edge and twice the distance from a second perpendicular edge.

Step 1 The region D is shown in Figure 12.4.2.

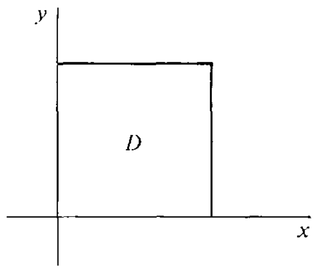


Figure 12.4.2

Step 2 Place the object so the first two edges are on the x and y axes. Then D is the region

$$0 \le x \le 1, \qquad 0 \le y \le 1.$$

Step 3 The density is $\rho(x, y) = y + 2x$.

$$m = \iint_D y + 2x\, dA = \int_0^1 \int_0^1 y + 2x\, dy\, dx.$$

$$\int_0^1 y + 2x\, dy = \tfrac{1}{2}y^2 + 2xy\Big]_0^1 = \tfrac{1}{2} + 2x.$$

$$m = \int_0^1 \tfrac{1}{2} + 2x\, dx = \tfrac{3}{2}.$$

DEFINITION

A plane object which fills a region D and has continuous density $\rho(x, y)$ has ***moments*** *about the x and y axes given by*

$$M_x = \iint_D y\rho(x, y)\, dA.$$

$$M_y = \iint_D x\rho(x, y)\, dA.$$

M_x and M_y are sometimes called first moments to distinguish them from moments of inertia (which are called second moments).

The ***center of mass*** *of the object is the point* $(\bar{x}, \bar{y})$ *with coordinates*

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_D x\rho(x, y)\, dA}{\iint_D \rho(x, y)\, dA},$$

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_D y\rho(x, y)\, dA}{\iint_D \rho(x, y)\, dA}.$$

JUSTIFICATION The piece of the object on an element of area ΔD has mass

$$\Delta m \approx \rho(x, y)\, \Delta A \qquad \text{(compared to } \Delta A\text{)}.$$

A point mass $\bar{m}$ at (x, y) has moments

$$\bar{M}_x = y\bar{m}, \qquad \bar{M}_y = x\bar{m}.$$

Therefore the piece of the object at ΔD has moments

$$\Delta M_x \approx y\, \Delta m \approx y\rho(x, y)\, \Delta A \qquad \text{(compared to } \Delta A\text{)},$$

$$\Delta M_y \approx x\, \Delta m \approx x\rho(x, y)\, \Delta A \qquad \text{(compared to } \Delta A\text{)}.$$

The double integrals for M_x and M_y now follow from the Infinite Sum Theorem.

An object will balance on a pin at its center of mass (Figure 12.4.3). The center of mass is useful in finding the work done against gravity when moving the

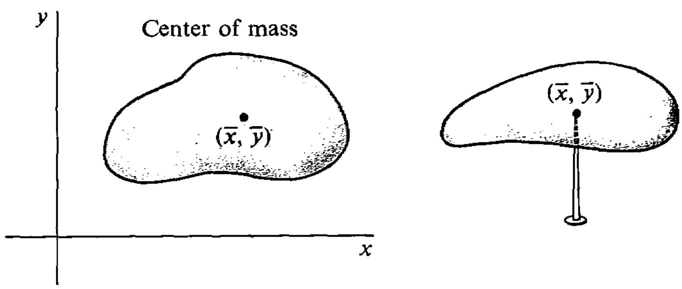


Figure 12.4.3

object. The work is the same as if the mass were all concentrated at the center of mass, and is given by

$$W = mgs$$

where s is the distance the center of mass is raised and g is constant.

EXAMPLE 2 A triangular plate bounded by the lines $x = 0$, $x = y$, $y = 1$ has density $\rho(x, y) = x + y$. Find the moments and center of mass.

Step 1 Sketch the region D, as in Figure 12.4.4.

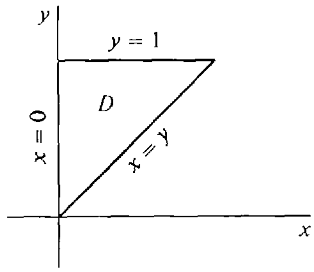


Figure 12.4.4

Step 2 We see from the figure that D is the region

$$0 \le x \le 1, \qquad x \le y \le 1.$$

Step 3 Set up and evaluate the iterated integrals for the mass m and moments M_x and M_y.

$$m = \iint_D x + y\,dA = \int_0^1 \int_x^1 x + y\,dy\,dx.$$

$$\int_x^1 x + y\,dy = x + \tfrac{1}{2} - \tfrac{3}{2}x^2.$$

$$m = \int_0^1 x + \tfrac{1}{2} - \tfrac{3}{2}x^2\,dx = \tfrac{1}{2}.$$

$$M_x = \iint_D y(x + y)\,dA = \int_0^1 \int_x^1 yx + y^2\,dy\,dx.$$

$$\int_x^1 yx + y^2\,dy = \tfrac{1}{2}x + \tfrac{1}{3} - \tfrac{5}{6}x^3.$$

$$M_x = \int_0^1 \tfrac{1}{2}x + \tfrac{1}{3} - \tfrac{5}{6}x^3\,dx = \tfrac{9}{24}.$$

$$M_y = \iint_D x(x + y)\,dA = \int_0^1 \int_x^1 x^2 + xy\,dy\,dx$$

$$\int_x^1 x^2 + xy\,dy = x^2 + \tfrac{1}{2}x - \tfrac{3}{2}x^3.$$

$$M_y = \int_0^1 x^2 + \tfrac{1}{2}x - \tfrac{3}{2}x^3\,dx = \tfrac{5}{24}.$$

The answers are $M_x = \dfrac{9}{24}, \quad M_y = \dfrac{5}{24}.$

$$\bar{x} = \frac{M_y}{m} = \frac{5/24}{1/2} = \frac{5}{12}.$$

$$\bar{y} = \frac{M_x}{m} = \frac{9/24}{1/2} = \frac{9}{12}.$$

The point $(\bar{x}, \bar{y})$ is shown in Figure 12.4.5.

EXAMPLE 3 The object in Example 2 is lying horizontally on the ground. Find the work required to stand the object up with the hypotenuse of the triangle on the ground (Figure 12.4.6).

We use the formula $\quad W = mgs.$

From Example 2, $m = \frac{1}{2}$. We must find s.

$$s = \text{minimum distance from } \left(\frac{5}{12}, \frac{9}{12}\right) \text{ to the line } x = y.$$

$$s = \text{minimum value of } z = \sqrt{\left(x - \frac{5}{12}\right)^2 + \left(x - \frac{9}{12}\right)^2}.$$

$$z = \sqrt{2x^2 - \frac{28}{12}x + \frac{106}{144}}.$$

$$\frac{dz}{dx} = \left(4x - \frac{28}{12}\right)\frac{1}{2}z^{-1/2}.$$

$$\frac{dz}{dx} = 0 \quad \text{at} \quad 4x = \frac{28}{12}, \qquad x = \frac{7}{12}.$$

$$s = \sqrt{2\left(\frac{7}{12}\right)^2 - \frac{28}{12}\cdot\frac{7}{12} + \frac{106}{144}} = \frac{\sqrt{2}}{6}.$$

$$W = mgs = \frac{1}{2}\cdot g \cdot \frac{\sqrt{2}}{6} = \frac{\sqrt{2}}{12}g.$$

The *second moment*, or *moment of inertia*, of a point mass m about the origin is the mass times the square of the distance to the origin,

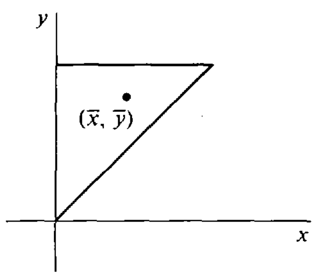


Figure 12.4.5

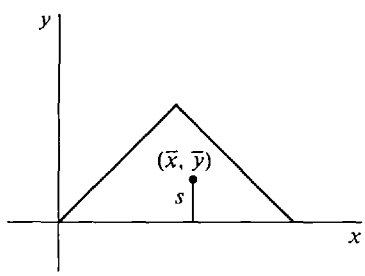


Figure 12.4.6

$$I = m(x^2 + y^2).$$

The moment of inertia is related to the kinetic energy of rotation. A mass m moving at speed v has kinetic energy

$$KE = \tfrac{1}{2}mv^2.$$

Hence if m is rotating about the origin with angular velocity ω radians per second, its speed is $v = \omega\sqrt{x^2 + y^2}$ and

$$KE = \tfrac{1}{2}m(\omega\sqrt{x^2 + y^2})^2 = \tfrac{1}{2}I\omega^2.$$

Thus moment of inertia is the rotational analogue of mass.

DEFINITION

Given a plane object on the region D with continuous density $\rho(x, y)$, the ***moment of inertia*** *about the origin is*

$$I = \iint_D \rho(x, y)(x^2 + y^2)\, dA.$$

JUSTIFICATION On an element of volume ΔD, the moment of inertia is

$$\Delta I \approx (x^2 + y^2)\,\Delta m \approx \rho(x, y)(x^2 + y^2)\,\Delta A \qquad \text{(compared to } \Delta A\text{)}.$$

The integral for I follows by the Infinite Sum Theorem.

EXAMPLE 4 Find the moment of inertia about the origin of an object with constant density $\rho = 1$ which covers the square shown in Figure 12.4.7:

$$-\tfrac{1}{2} \le x \le \tfrac{1}{2}, \qquad -\tfrac{1}{2} \le y \le \tfrac{1}{2}.$$

$$I = \iint_D (x^2 + y^2)\, dA = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} x^2 + y^2\, dy\, dx.$$

$$\int_{-1/2}^{1/2} x^2 + y^2\, dy = x^2y + \tfrac{1}{3}y^3\Big]_{-1/2}^{1/2} = x^2 + \tfrac{1}{12}.$$

$$I = \int_{-1/2}^{1/2} x^2 + \tfrac{1}{12}\, dx = \tfrac{1}{6}.$$

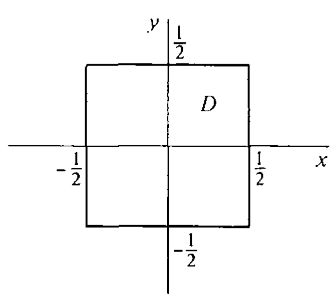


Figure 12.4.7

PROBLEMS FOR SECTION 12.4

In Problems 1–10, find (a) the mass, (b) the center of mass, (c) the moment of inertia about the origin, of the given plane object.

1 $-a \le x \le a, \quad -b \le y \le b, \quad \rho(x, y) = k$

2 $0 \le x \le a, \quad 0 \le y \le b, \quad \rho(x, y) = k$

3 $0 \le x \le 1, \quad x^2 \le y \le 1, \quad \rho(x, y) = k$

4 $0 \le x \le a, \quad 0 \le y \le bx, \quad \rho(x, y) = k$

5 $0 \le x \le 2, \quad x \le y \le 2x, \quad \rho(x, y) = x + y + 1$

6 $0 \le x \le 1, \quad 0 \le y \le x, \quad \rho(x, y) = x - y$

7 $0 \le x \le 1, \quad 0 \le y \le x^2, \quad \rho(x, y) = \sqrt{x} + \sqrt{y}$

8 $1 \le x \le 2, \quad x \le y \le x^2, \quad \rho(x, y) = 1/\sqrt{xy}$

9 $0 \le x \le 2, \quad e^{-x} \le y \le e^x, \quad \rho(x, y) = 1$

10 $-1 \le x \le 1, \quad 0 \le y \le 1/\sqrt{1 + x^2}, \quad \rho(x, y) = y$

11 Find the mass of an object in the shape of a unit square whose density is the sum of the four distances from the sides.

12 Find the mass of an object in the shape of a unit square whose density is the product of the distances from the four sides.

13 An object on the triangle $0 \le x \le 1, 0 \le y \le x$ has density equal to the distance from the hypotenuse $y = x$. Find the amount of work required to stand the object up (a) on one of the short sides, (b) on the hypotenuse.

14 An object in the shape of a unit square has density equal to the distance to the nearest side. Find the mass and the amount of work needed to stand the object up on a side.

15 An object on the plane region $-1 \le x \le 1, x^2 \le y \le 1$ has density $\rho(x, y) = 1 + x + \sqrt{y}$. Find the mass and the work needed to stand the object up on the flat side.

16 An object on the unit square $0 \le x \le 1, 0 \le y \le 1$ has density $\rho(x, y) = ax + by + c$. Find the mass and center of mass.

□ 17 The moment of an object of density $\rho(x, y)$ in the region D about the vertical line $x = a$ is defined as

$$M_{y,x=a} = \iint_D (x - a)\rho(x, y)\, dA.$$

Show that

$$M_{y,x=a} = M_y - a \cdot m$$

where M_y is the moment about the y-axis and m is the mass.

□ 18 The moment of inertia of an object in the region D of density $\rho(x, y)$ about the point $P(a, b)$ is defined as

$$I_P = \iint_D \rho(x, y)((x - a)^2 + (y - b)^2)\, dA.$$

Show that

$$I_P = I - 2aM_x - 2bM_y + m(a^2 + b^2)$$

where I is the moment of inertia about the origin, M_x and M_y are the first moments, and m is the mass.

12.5 DOUBLE INTEGRALS IN POLAR COORDINATES

A point with polar coordinates (θ, r) has rectangular coordinates

$$(x, y) = (r \cos \theta, r \sin \theta).$$

DEFINITION

*A **polar region** is a region D in the (x, y) plane given by polar coordinate inequalities*

$$\alpha \leq \theta \leq \beta, \qquad a(\theta) \leq r \leq b(\theta),$$

where $a(\theta)$ and $b(\theta)$ are continuous. To avoid overlaps, we also require that for all (θ, r) in D,

$$0 \leq \theta \leq 2\pi \quad \text{and} \quad 0 \leq r.$$

The last requirement means that the limits α and β are between 0 and 2π, while the limits $a(\theta)$ and $b(\theta)$ are ≥ 0. Figure 12.5.1 shows a polar region.

The simplest polar regions are the *polar rectangles*

$$\alpha \leq \theta \leq \beta, \qquad a \leq r \leq b.$$

We see in Figure 12.5.2 that the θ boundaries are radii and the r boundaries are circular arcs.

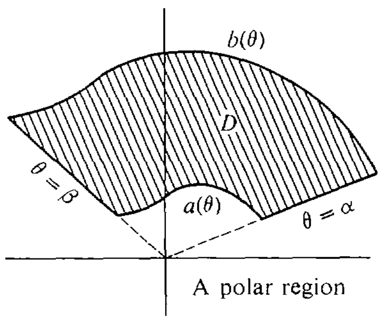


Figure 12.5.1

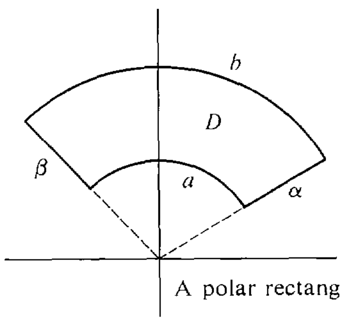


Figure 12.5.2

The polar rectangle

$$\alpha \leq \theta \leq \beta, \qquad 0 \leq r \leq b$$

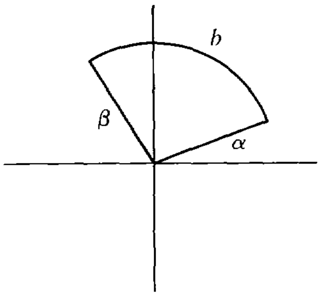
is a sector of a circle of radius b (Figure 12.5.3(a)).

The polar rectangle

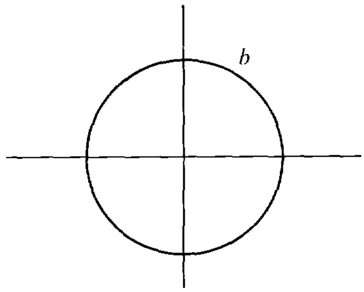
$$0 \leq \theta \leq 2\pi, \qquad 0 \leq r \leq b$$

is a whole circle of radius b (Figure 12.5.3(b)).

Less trivial examples of polar regions are the *circle with diameter from* (0, 0) *to* (0, b),



(a) sector:
$\alpha \le \theta \le \beta,\ 0 \le r \le b$



(b) circle:
$0 \le \theta \le 2\pi,\ 0 \le r \le b$

Figure 12.5.3

$$0 \le \theta \le \pi, \qquad 0 \le r \le b \sin \theta,$$

and the *cardioid* $\qquad 0 \le \theta \le 2\pi, \qquad 0 \le r \le 1 + \cos \theta.$

Both of these regions are shown in Figure 12.5.4.

We shall use the Infinite Sum Theorem to get a formula for the double integral over a polar region. In the proof we take for ΔD an infinitely small polar rectangle.

POLAR INTEGRATION FORMULA

Let D be the polar region

$$\alpha \le \theta \le \beta, \qquad a(\theta) \le r \le b(\theta).$$

The double integral of $f(x, y)$ over D is

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{a(\theta)}^{b(\theta)} f(x, y) r\, dr\, d\theta$$

$$= \int_\alpha^\beta \int_{a(\theta)}^{b(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

Notice that in the iterated integral for a polar region we do not integrate $f(x, y)$ but the product of $f(x, y)$ and r. Intuitively, the extra r comes from the fact that a polar element of area is almost a rectangle of area $r\,\Delta\theta\,\Delta r$ (see Figure 12.5.6(b)).

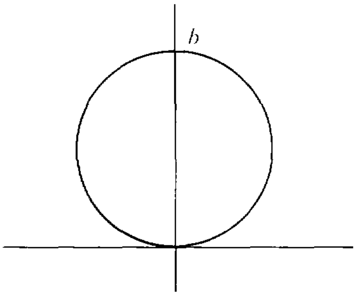
PROOF We shall work with the rectangular (θ, r) plane. Let C be the region in the (θ, r) plane given by the inequalities

$$\alpha \le \theta \le \beta, \qquad a(\theta) \le r \le b(\theta).$$

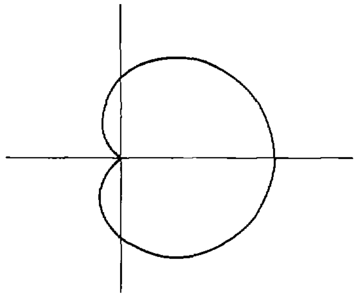
Thus C has the same inequalities as D but they refer to the (θ, r) plane instead of the (x, y) plane. D and C are shown in Figure 12.5.5.

We must prove that

$$\iint_D f(x, y)\, dx\, dy = \iint_C f(x, y) r\, d\theta\, dr.$$

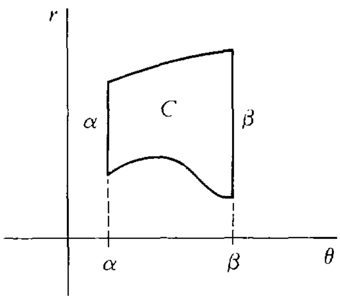


(a) circle:
$0 \le \theta \le \pi$, $0 \le r \le b \sin \theta$



(b) cardioid:
$0 \le \theta \le 2\pi$, $0 \le r \le 1 + \cos \theta$

Figure 12.5.4



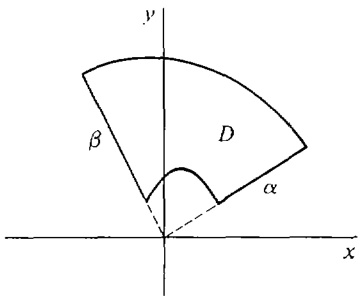


Figure 12.5.5

Our plan is to use the Infinite Sum Theorem in the (θ, r) plane. Assume first that $f(x, y) > 0$ for all (x, y) in D.

For any (θ, r) region C_1 corresponding to a polar region D_1 in the (x, y) plane, let

$$B(C_1) = \iint_{D_1} f(x, y)\, dx\, dy.$$

Then B has the Addition Property and is always ≥ 0. Consider an element of area ΔC in the (θ, r) plane with area $\Delta\theta\, \Delta r$. ΔC corresponds to a polar rectangle ΔD in the (x, y) plane. As we can see from Figure 12.5.6, ΔD is almost a rectangle with sides $r\, \Delta\theta$ and Δr and area $r\, \Delta\theta\, \Delta r$.

The volume over ΔD is almost a rectangular solid with base of area $r\, \Delta\theta\, \Delta r$ and height

$$f(x, y) = f(r \cos \theta, r \sin \theta).$$

Therefore $\quad B(\Delta C) \approx f(x, y) r\, \Delta\theta\, \Delta r \qquad$ (compared to $\Delta\theta\, \Delta r$).

By the Infinite Sum Theorem

$$B(C) = \iint_C f(x, y) r\, d\theta\, dr,$$

and by definition $\qquad B(C) = \iint_D f(x, y)\, dx\, dy.$

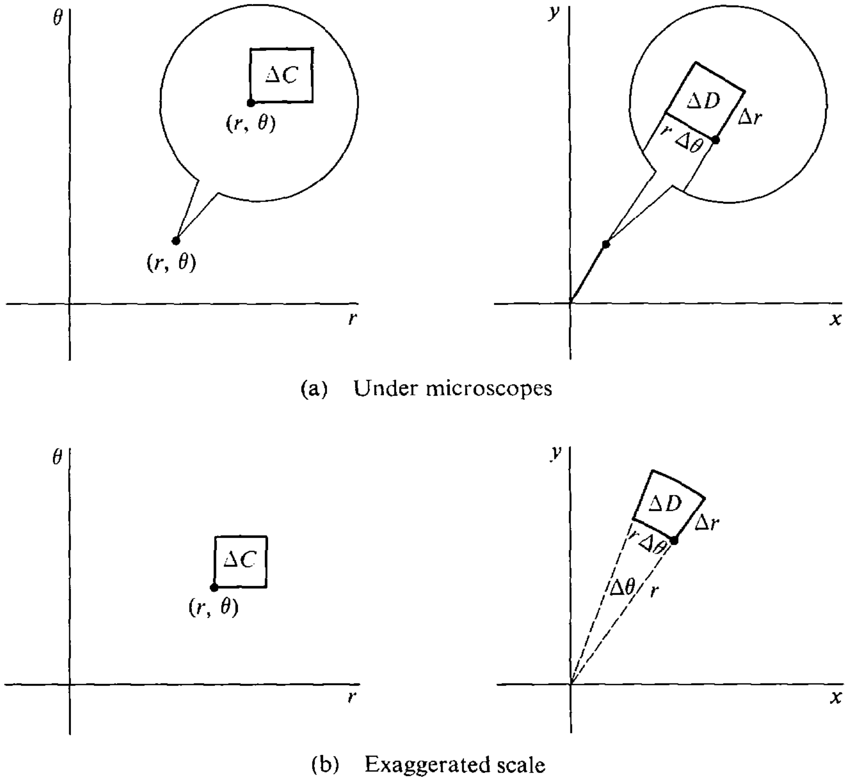


Figure 12.5.6

Finally we consider the case where $f(x, y)$ is not always positive. Pick a real constant $k > 0$ such that $f(x, y) + k$ is always positive for (x, y) in D. By the above proof,

$$\iint_D (f(x, y) + k)\, dx\, dy = \iint_C (f(x, y) + k) r\, d\theta\, dr,$$

$$\iint_D k\, dx\, dy = \iint_C kr\, d\theta\, dr.$$

When we use the Sum Rule and subtract the second equation from the first, we get

$$\iint_D f(x, y)\, dx\, dy = \iint_C f(x, y) r\, d\theta\, dr.$$

In a double integration problem where the region D is a circle or a sector of a circle, it is usually best to take the center as the origin and represent D as a polar rectangle.

EXAMPLE 1 Find the volume over the unit circle $x^2 + y^2 \le 1$ between the surfaces $z = 0$ and $z = x^2$.

Step 1 Sketch D and the solid, as in Figure 12.5.7.

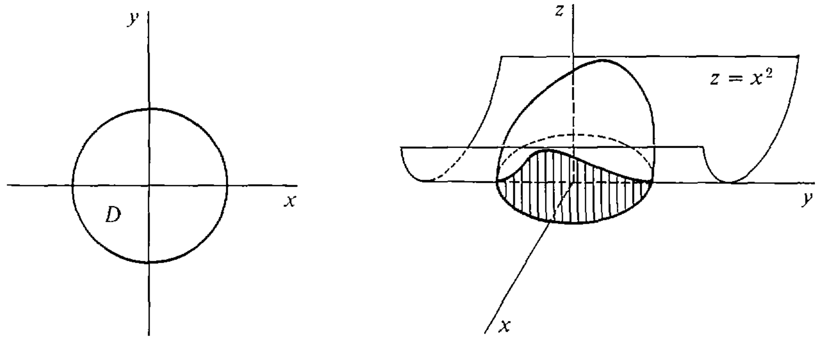


Figure 12.5.7

Step 2 D is the polar region $0 \le \theta \le 2\pi, 0 \le r \le 1$.

Step 3 $$V = \iint_D x^2\,dx\,dy = \int_0^{2\pi}\int_0^1 x^2 r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r\cos\theta)^2 r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3\cos^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{4}\cos^2\theta\,d\theta = \tfrac{1}{4}(\tfrac{1}{2}\sin\theta\cos\theta + \tfrac{1}{2}\theta)\Big]_0^{2\pi} = \pi/4.$$

For comparison let us also work this problem in rectangular coordinates. We can see that it is easier to use polar coordinates.

D is the region $-1 \le x \le 1, \quad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$.

$$V = \iint_D x^2\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2\,dy\,dx = \int_{-1}^{1} 2x^2\sqrt{1-x^2}\,dx.$$

We make the trigonometric substitution shown in Figure 12.5.8:

$$x = \sin\phi, \qquad \sqrt{1-x^2} = \cos\phi, \qquad dx = \cos\phi\,d\phi.$$

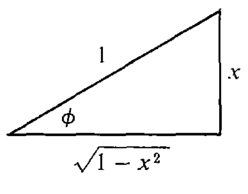


Figure 12.5.8

Then $\phi = -\pi/2$ at $x = -1$ and $\phi = \pi/2$ at $x = 1$, so

$$V = \int_{-\pi/2}^{\pi/2} 2\sin^2\phi\cos^2\phi\,d\phi$$

$$= \int_{-\pi/2}^{\pi/2} 2\left(\frac{1-\cos 2\phi}{2}\right)\left(\frac{1+\cos 2\phi}{2}\right)d\phi$$

$$= \int_{-\pi/2}^{\pi/2} \tfrac{1}{2}(1 - \cos^2 2\phi)\, d\phi$$

$$= \int_{-\pi}^{\pi} \tfrac{1}{4}(1 - \cos^2 u)\, du$$

$$= \tfrac{1}{4}(u - \tfrac{1}{2}\cos u \sin u - \tfrac{1}{2}u)\Big]_{-\pi}^{\pi} = \pi/4.$$

EXAMPLE 2 Find the mass and center of mass of a flat plate in the shape of a semicircle of radius one whose density is equal to the distance from the center of the circle.

Step 1 The region D is sketched in Figure 12.5.9.

Step 2 Take the origin at the center of the circle and the x-axis as the base of the semicircle. D is the polar region $0 \le \theta \le \pi$, $0 \le r \le 1$.

Step 3 The density is

$$\rho(x, y) = \sqrt{x^2 + y^2} = r.$$

$$m = \iint_D \sqrt{x^2 + y^2}\, dA = \int_0^{\pi} \int_0^1 r \cdot r\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^1 r^2\, dr\, d\theta = \int_0^{\pi} \frac{1}{3}\, d\theta = \frac{\pi}{3}.$$

$$M_x = \iint_D y\sqrt{x^2 + y^2}\, dA = \int_0^{\pi} \int_0^1 r \sin\theta \cdot r \cdot r\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^1 r^3 \sin\theta\, dr\, d\theta = \int_0^{\pi} \tfrac{1}{4} \sin\theta\, d\theta = \tfrac{1}{2}.$$

$$M_y = \iint_D x\sqrt{x^2 + y^2}\, dA = \int_0^{\pi} \int_0^1 r \cos\theta \cdot r \cdot r\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^1 r^3 \cos\theta\, dr\, d\theta = \int_0^{\pi} \tfrac{1}{4} \cos\theta\, d\theta = 0.$$

Answer $m = \dfrac{\pi}{3}$, $\quad \bar{x} = \dfrac{M_y}{m} = 0$, $\quad \bar{y} = \dfrac{M_x}{m} = \dfrac{3}{2\pi} \sim 0.477.$

The point $(\bar{x}, \bar{y})$ is shown in Figure 12.5.10.

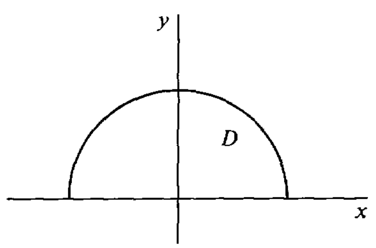


Figure 12.5.9

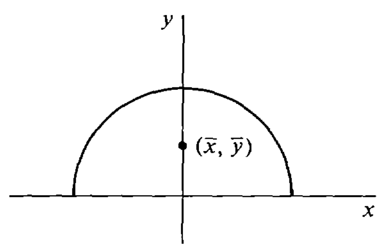


Figure 12.5.10

EXAMPLE 3 Find the moment of inertia of a circle of radius b and constant density ρ about the center of the circle.

Step 1 Draw the region D (Figure 12.5.11).

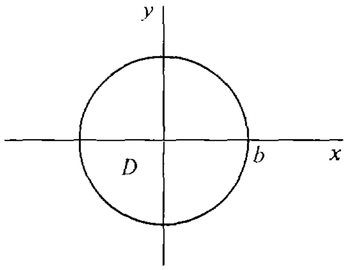


Figure 12.5.11

Step 2 Put the origin at the center, so D is the polar region

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le b.$$

Step 3 $x^2 + y^2 = r^2$, so

$$I = \iint_D \rho \cdot (x^2 + y^2)\, dA = \int_0^{2\pi} \int_0^b \rho r^2 \cdot r\, dr\, d\theta$$

$$= \rho \int_0^{2\pi} \int_0^b r^3\, dr\, d\theta = \rho \int_0^{2\pi} \tfrac{1}{4} b^4\, d\theta$$

$$= \frac{\rho b^4 \pi}{2}.$$

PROBLEMS FOR SECTION 12.5

In Problems 1–16, find the volume using polar coordinates.

1 $x^2 + y^2 \le 1, \quad 0 \le z \le 6$

2 $x^2 + y^2 \le 1, \quad 0 \le z \le x^2 + y^2$

3 $x^2 + y^2 \le 4, \quad 0 \le z \le x + 2$

4 $x^2 + y^2 \le 4, \quad 0 \le z \le \sqrt{x^2 + y^2}$

5 $x^2 + y^2 \le 9, \quad x^2 + y^2 \le z \le 9$

6 $x^2 + y^2 \le 25, \quad 0 \le z \le e^{-x^2 - y^2}$

7 $1 \le x^2 + y^2 \le 4, \quad (x^2 + y^2)^{-1} \le z \le (x^2 + y^2)^{-1/2}$

8 $1 \le x^2 + y^2 \le 9, \quad 1/\sqrt{x^2 + y^2} \le z \le 1$

9 $0 \le x \le 1, \quad 0 \le y \le \sqrt{1 - x^2}, \quad 0 \le z \le x\sqrt{y}$

10 $-2 \le x \le 2, \quad 0 \le y \le \sqrt{4 - x^2}, \quad x \le z \le y + 2$

11 $\pi/4 \le \theta \le \pi/3, \quad 0 \le r \le 1, \quad 0 \le z \le r^2$

12 $0 \le \theta \le \pi/6, \quad 1 \le r \le 2, \quad 0 \le z \le \sqrt{9 - r^2}$

13 $0 \le \theta \le \pi, \quad 0 \le r \le 2\sin\theta, \quad 0 \le z \le r$

14 $0 \le \theta \le \pi/2, \quad 0 \le r \le \cos\theta, \quad r^3 \le z \le r^2$

15 $0 \le \theta \le 2\pi, \quad 0 \le r \le \theta, \quad 0 \le z \le r^2\theta^3 + 2r\theta$

16 $0 \le \theta \le 2\pi, \quad 0 \le r \le e^\theta, \quad 0 \le z \le \sqrt{r}$

17 Find the volume of the solid over the cardioid $r = 1 + \cos\theta$ between the plane $z = 0$ and the cone $z = r$.

18 Find the volume of the solid over the cardioid $r = 1 + \cos\theta$ between the paraboloids $z = r^2$ and $z = 8 - r^2$.

19 Find the volume of the solid over the circle $r = \sin\theta$ between the plane $z = 0$ and the hemisphere $z = \sqrt{1 - r^2}$.

20 Find the volume of the solid over the circle $r = 2\cos\theta$ between the plane $z = 0$ and the cone $z = 2 - r$.

21 Find the volume of the solid over the polar rectangle $\alpha \le \theta \le \beta$, $a \le r \le b$, between the plane $z = 0$ and the cone $z = r$.

22 Find the volume of the portion of the hemisphere $0 \le z \le \sqrt{1 - r^2}$ over the polar rectangle $\alpha \le \theta \le \beta$, $a \le r \le b$ (assuming $b \le 1$).

23 A circular object of radius b has density equal to the distance from the outside of the circle. Find (a) the mass, (b) the moment of inertia about the origin.

24 A circular object of radius b has density equal to the cube of the distance from the center. Find (a) the mass, (b) the moment of inertia about the origin.

25 Find the moment of inertia about the origin of a circular ring $a \le r \le b$, $0 \le \theta \le 2\pi$, of constant density k.

26 Find the moment of inertia of a circular object of radius b and constant density k about a point on its circumference. (The center can be put at $(0, b)$, so the object is on the polar region $0 \le r \le 2b\sin\theta, 0 \le \theta \le \pi$.)

27 An object has constant density k on the circular sector $0 \le x \le 1$, $0 \le y \le \sqrt{1 - x^2}$. Find (a) the center of mass, (b) the moment of inertia about the origin.

28 An object of constant density k covers the cardioid $r \le 1 + \cos\theta, 0 \le \theta \le 2\pi$. Find (a) the center of mass, (b) the moment of inertia about the origin.

29 An object of constant density k covers the region inside the circle $r = 2b\sin\theta$ and outside the circle $r = b$. Find (a) the center of mass, (b) the moment of inertia about the origin.

30 An object of constant density k covers the polar region

$$0 \le \theta \le \pi/2, \qquad 0 \le r \le b\sin 2\theta.$$

Find (a) the center of mass, (b) the moment of inertia about the origin.

□ 31 (a) Use polar coordinates to evaluate $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dy\,dx$.

(b) Show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dy\,dx = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$.

(c) Now evaluate the single integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

12.6 TRIPLE INTEGRALS

A *closed region in space*, or *solid region*, is a set E of points given by inequalities

$$a_1 \le x \le a_2, \qquad b_1(x) \le y \le b_2(x), \qquad c_1(x, y) \le z \le c_2(x, y)$$

where the functions $b_1(x)$, $b_2(x)$ and $c_1(x, y)$, $c_2(x, y)$ are continuous.

The *boundary* of E is the part of E on the following surfaces:

The planes $x = a_1$, $x = a_2$.

The cylinders $y = b_1(x)$, $y = b_2(x)$.
The surfaces $z = c_1(x, y)$, $z = c_2(x, y)$.

The simplest type of closed region is a *rectangular solid*, or *rectangular box*,

$$a_1 \le x \le a_2, \qquad b_1 \le y \le b_2, \qquad c_1 \le z \le c_2.$$

Figure 12.6.1 shows a solid region and a rectangular box.

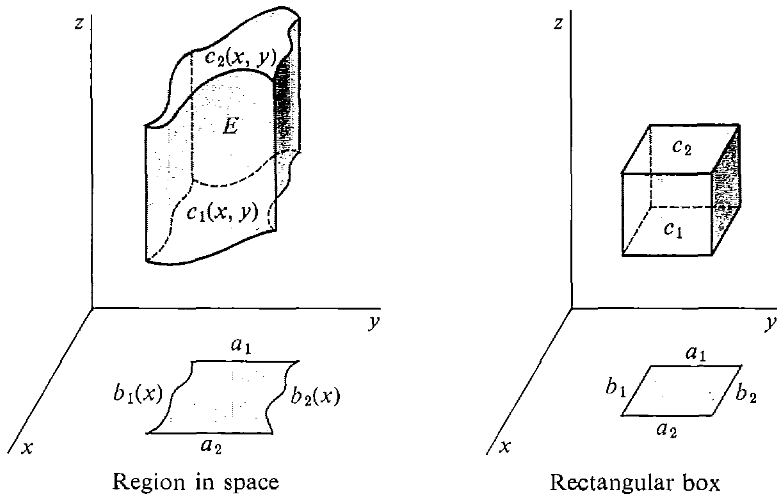


Figure 12.6.1

An *open region* in space is defined in a similar way but with strict inequalities. As in the two-dimensional case, the word *region* alone will mean *closed region*.

PERMANENT ASSUMPTION

Whenever we refer to a function $f(x, y, z)$ and a solid region E, we assume that $f(x, y, z)$ is continuous on some open region containing E.

The triple integral $$\iiint_E f(x, y, z)\,dx\,dy\,dz$$

is analogous to the double integral.

The first step in defining the triple integral is to form the *circumscribed rectangular box* of E (Figure 12.6.2). This is the rectangular box

$$a_1 \le x \le a_2, \qquad B_1 \le y \le B_2, \qquad C_1 \le z \le C_2,$$

where

$$\begin{aligned} B_1 &= \text{minimum value of } b_1(x), \\ B_2 &= \text{maximum value of } b_2(x), \\ C_1 &= \text{minimum value of } c_1(x, y), \\ C_2 &= \text{maximum value of } c_2(x, y). \end{aligned}$$

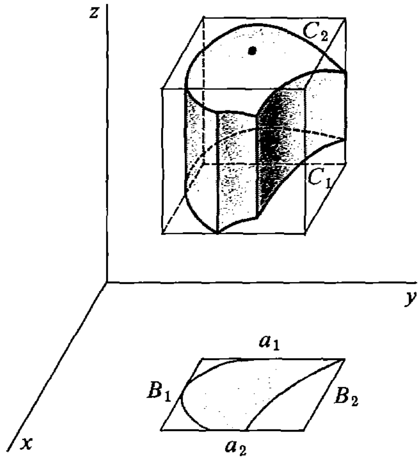


Figure 12.6.2 The circumscribed rectangular box

Our next step is to define the triple Riemann sum. Given positive real numbers Δx, Δy, and Δz, we partition the circumscribed rectangular box of E into rectangular boxes with sides Δx, Δy, and Δz (Figure 12.6.3). The partition points of this three-dimensional partition have the form

$$(x_k, y_l, z_m), \qquad 0 \le k \le n, \qquad 0 \le l \le p, \qquad 0 \le m \le q.$$

The *triple Riemann sum* of $f(x, y, z)\,\Delta x\,\Delta y\,\Delta z$ over E is defined as the sum

$$\sum\sum_E\sum f(x, y, z)\,\Delta x\,\Delta y\,\Delta z = \sum\sum_{(x_k, y_l, z_m) \text{ in } E}\sum f(x_k, y_l, z_m)\,\Delta x\,\Delta y\,\Delta z.$$

When we replace Δx, Δy, Δz by positive infinitesimals dx, dy, dz we obtain an *infinite triple Riemann sum*

$$\sum\sum_E\sum f(x, y, z)\,dx\,dy\,dz.$$

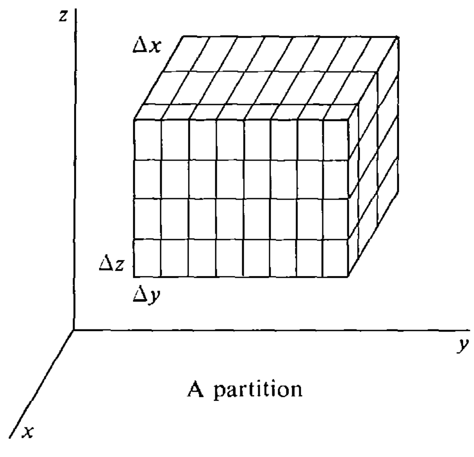


Figure 12.6.3 A partition

LEMMA

For all positive infinitesimals dx, dy, and dz, the triple Riemann sum

$$\sum\sum_E\sum f(x, y, z)\,dx\,dy\,dz$$

is a finite hyperreal number and therefore has a standard part.

We are now ready to define the triple integral (see Figure 12.6.4).

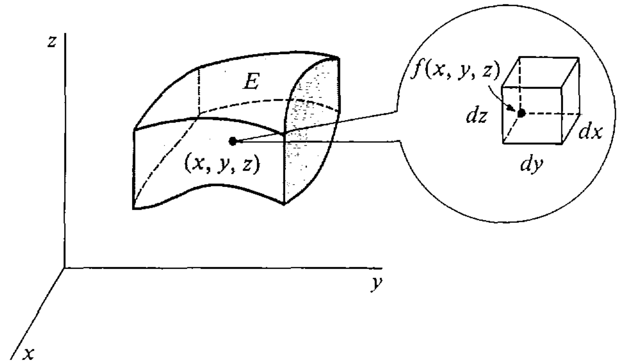


Figure 12.6.4

DEFINITION

*Given positive infinitesimal dx, dy, and dz, the **triple integral** of a continuous function $f(x, y, z)$ over E is*

$$\iiint_E f(x, y, z)\,dx\,dy\,dz = st\left(\sum\sum_E\sum f(x, y, z)\,dx\,dy\,dz\right).$$

We shall now briefly state some basic theorems on triple integrals, which are exactly like the corresponding theorems for double integrals.

INDEPENDENCE OF *dx, dy,* AND *dz*

The value of $\iiint_E f(x, y, z)\,dx\,dy\,dz$ does not depend on dx, dy, or dz.

We shall usually use the notation $dV = dx\,dy\,dz$ for the volume of an infinitesimal dx by dy by dz rectangular box, and write

$$\iiint_E f(x, y, z)\,dV \quad \text{for} \quad \iiint_E f(x, y, z)\,dx\,dy\,dz.$$

ADDITION PROPERTY

If E is divided into two regions E_1 and E_2 which meet only on a common boundary then

$$\iiint_E f(x, y, z)\,dV = \iiint_{E_1} f(x, y, z)\,dV + \iiint_{E_2} f(x, y, z)\,dV.$$

ITERATED INTEGRAL THEOREM

If E is the region

$$a_1 \le x \le a_2, \qquad b_1(x) \le y \le b_2(x), \qquad c_1(x, y) \le z \le c_2(x, y),$$

then $$\iiint_E f(x, y, z)\, dV = \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} \int_{c_1(x,y)}^{c_2(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

If the region E is a rectangular box

$$a_1 \le x \le a_2, \qquad b_1 \le y \le b_2, \qquad c_1 \le z \le c_2,$$

there are six different iterated integrals over E, corresponding to six different orders of integration. Here they are (in "alphabetical" order).

(1) $\int_{a_1}^{a_2} \int_{b_1}^{b_2} \int_{c_1}^{c_2} f(x, y, z)\, dz\, dy\, dx$ (2) $\int_{a_1}^{a_2} \int_{c_1}^{c_2} \int_{b_1}^{b_2} f(x, y, z)\, dy\, dz\, dx$

(3) $\int_{b_1}^{b_2} \int_{a_1}^{a_2} \int_{c_1}^{c_2} f(x, y, z)\, dz\, dx\, dy$ (4) $\int_{b_1}^{b_2} \int_{c_1}^{c_2} \int_{a_1}^{a_2} f(x, y, z)\, dx\, dz\, dy$

(5) $\int_{c_1}^{c_2} \int_{a_1}^{a_2} \int_{b_1}^{b_2} f(x, y, z)\, dy\, dx\, dz$ (6) $\int_{c_1}^{c_2} \int_{b_1}^{b_2} \int_{a_1}^{a_2} f(x, y, z)\, dx\, dy\, dz.$

The Iterated Integral Theorem shows that each of these iterated integrals is equal to the triple integral

$$\iiint_E f(x, y, z)\, dV.$$

EXAMPLE 1 Evaluate $\iiint_E xy^2z^3\, dV$ where E is the rectangular box

$$0 \le x \le 2, \qquad 0 \le y \le 1, \qquad 0 \le z \le 4.$$

There are six iterated integrals which all have the same value. We compute one of them, and then another to check our answer.

FIRST SOLUTION $$\iiint_E xy^2z^3\, dV = \int_0^2 \int_0^1 \int_0^4 xy^2z^3\, dz\, dy\, dx.$$

The inside integral is

$$\int_0^4 xy^2z^3\, dz = \frac{xy^2z^4}{4}\Big]_0^4 = 64xy^2.$$

The second integral is

$$\int_0^1 64xy^2\, dy = \tfrac{64}{3}xy^3\Big]_0^1 = \tfrac{64}{3}x.$$

The final answer is

$$\int_0^2 \tfrac{64}{3}x\, dx = \tfrac{64}{6}x^2\Big]_0^2 = \tfrac{256}{6} = \tfrac{128}{3}.$$

SECOND SOLUTION $\iiint_E xy^2z^3\,dV = \int_0^4\int_0^2\int_0^1 xy^2z^3\,dy\,dx\,dz.$

The inside integral is

$$\int_0^1 xy^2z^3\,dy = \tfrac{1}{3}xy^3z^3\Big]_0^1 = \tfrac{1}{3}xz^3.$$

The second integral is

$$\int_0^2 \tfrac{1}{3}xz^3\,dx = \tfrac{1}{6}x^2z^3\Big]_0^2 = \tfrac{4}{6}z^3.$$

The final answer is

$$\int_0^4 \tfrac{4}{6}z^3\,dz = \tfrac{1}{6}z^4\Big]_0^4 = \tfrac{256}{6} = \tfrac{128}{3}.$$

Triple integrals can be evaluated by iterated integrals.

EXAMPLE 2 Evaluate $\iiint_E y + z\,dV$ where E is the region shown in Figure 12.6.5,

$$0 \le x \le \pi/2, \qquad 0 \le y \le \sin x, \qquad 0 \le z \le y\cos x.$$

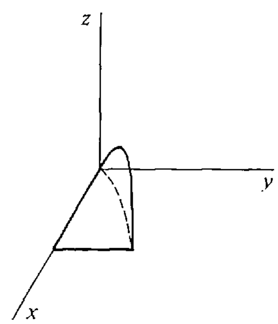


Figure 12.6.5

SOLUTION $\iiint_E y + z\,dV = \int_0^{\pi/2}\int_0^{\sin x}\int_0^{y\cos x} y + z\,dz\,dy\,dx.$

We first evaluate the inside integral.

$$\int_0^{y\cos x} y + z\,dz = yz + \tfrac{1}{2}z^2\Big]_0^{y\cos x} = y^2\cos x + \tfrac{1}{2}y^2\cos^2 x.$$

Now we evaluate the second integral.

$$\int_0^{\sin x} y^2\cos x + \tfrac{1}{2}y^2\cos^2 x\,dy = \tfrac{1}{3}y^3(\cos x + \tfrac{1}{2}\cos^2 x)\Big]_0^{\sin x}$$
$$= \tfrac{1}{3}\sin^3 x(\cos x + \tfrac{1}{2}\cos^2 x).$$

Finally we evaluate the outside integral.

$$\iiint_E y + z\,dV = \int_0^{\pi/2} \tfrac{1}{3}\sin^3 x(\cos x + \tfrac{1}{2}\cos^2 x)\,dx$$

$$= \int_0^{\pi/2} \tfrac{1}{3}(1 - \cos^2 x)(\cos x + \tfrac{1}{2}\cos^2 x)\sin x\,dx$$

$$= \int_1^0 -\tfrac{1}{3}(1 - u^2)(u + \tfrac{1}{2}u^2)\,du$$

$$= \int_0^1 \tfrac{1}{3}(u + \tfrac{1}{2}u^2 - u^3 - \tfrac{1}{2}u^4)\,du = \tfrac{19}{180}.$$

COROLLARY

The volume of a region E in space is equal to the triple integral of the constant 1 *over E* as illustrated in Figure 12.6.6,

$$V = \iiint_E dV.$$

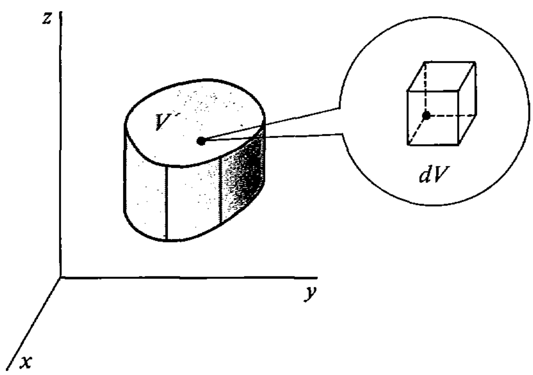


Figure 12.6.6

PROOF E is the solid over the plane region D given by

$$a_1 \le x \le a_2, \qquad b_1(x) \le y \le b_2(x)$$

between the surfaces $z = c_1(x, y)$ and $z = c_2(x, y)$. By definition of the volume between two surfaces,

$$V = \iint_D c_2(x, y) - c_1(x, y)\,dA.$$

Using the Iterated Integral Theorem,

$$\iiint_E dV = \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} \int_{c_1(x,y)}^{c_2(x,y)} dz\,dy\,dx$$

$$= \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} [c_2(x, y) - c_1(x, y)]\,dy\,dx = V.$$

We now come to the Infinite Sum Theorem for triple integrals, which is, again, the key result for applications.

We shall use Δx, Δy, and Δz for positive infinitesimals. By an *element of volume* we mean a rectangular box ΔE with sides Δx, Δy, and Δz (Figure 12.6.7). The volume of ΔE is

$$\Delta V = \Delta x\,\Delta y\,\Delta z.$$

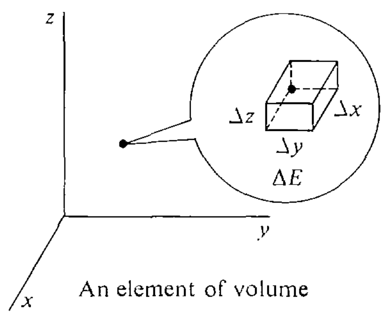


Figure 12.6.7 An element of volume

INFINITE SUM THEOREM

Let $h(x, y, z)$ be continuous on an open region E_0 and let B be a function which assigns a real number $B(E)$ to each region E contained in E_0. Assume that:

(i) *B has the Addition Property.*

(ii) *$B(E) \geq 0$ for every E.*

(iii) *For every element of volume ΔE,*

$$B(\Delta E) \approx h(x, y, z)\,\Delta V \qquad (\text{compared to } \Delta V).$$

Then
$$B(E) = \iiint_E h(x, y, z)\,dV.$$

Here are some applications of the triple Infinite Sum Theorem. Perhaps the simplest physical interpretation of the triple integral is mass as the triple integral of density.

DEFINITION

*The **mass** of an object filling a solid region E with continuous density $\rho(x, y, z)$ is*

$$m = \iiint_E \rho(x, y, z)\,dV.$$

JUSTIFICATION At every point of an element of volume ΔE the density is infinitely close to $\rho(x, y, z)$, so the element of mass is

$$\Delta m \approx \rho(x, y, z)\,\Delta V \qquad (\text{compared to } \Delta V).$$

(See Figure 12.6.8.) By the Infinite Sum Theorem,

$$m = \iiint_E \rho(x, y, z)\,\Delta V.$$

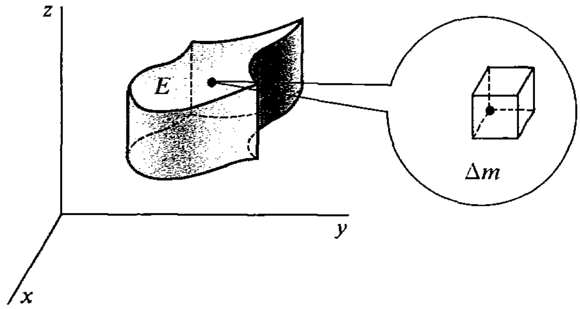


Figure 12.6.8

EXAMPLE 3 Find the mass of an object in the unit cube

$$0 \le x \le 1, \qquad 0 \le y \le 1, \qquad 0 \le z \le 1$$

with density $$\rho(x, y, z) = x + y + z.$$

$$m = \iiint_E x + y + z\, dV$$
$$= \int_0^1 \int_0^1 \int_0^1 x + y + z\, dz\, dy\, dx$$
$$= \int_0^1 \int_0^1 x + y + \tfrac{1}{2}\, dy\, dx = \int_0^1 x + \tfrac{1}{2} + \tfrac{1}{2}\, dx = \tfrac{3}{2}.$$

An object in space has a moment about each coordinate plane.

DEFINITION

*If an object in space fills a region E and has continuous density $\rho(x, y, z)$, its **moments** about the coordinate planes are*

$$M_{xy} = \iiint_E z\rho(x, y, z)\, dV.$$
$$M_{xz} = \iiint_E y\rho(x, y, z)\, dV.$$
$$M_{yz} = \iiint_E x\rho(x, y, z)\, dV.$$

*The **center of mass** of the object is the point $(\bar{x}, \bar{y}, \bar{z})$, where m is mass and*

$$\bar{x} = \frac{M_{yz}}{m}, \qquad \bar{y} = \frac{M_{xz}}{m}, \qquad \bar{z} = \frac{M_{xy}}{m}.$$

JUSTIFICATION A point mass m has moment $M_{xy} = mz$ about the (x, y) plane (Figure 12.6.9). In an element of volume ΔE, the object has moment

$$\Delta M_{xy} \approx z\, \Delta m \approx z\rho(x, y, z)\, \Delta V \qquad \text{(compared to } \Delta V\text{)}.$$

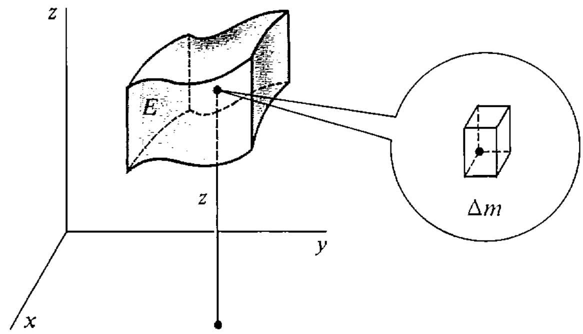


Figure 12.6.9

By the Infinite Sum Theorem,

$$M_{xy} = \iiint_E z\rho(x, y, z)\,dV.$$

EXAMPLE 4 An object has constant density and the shape of a tetrahedron with vertices at the four points

$$(0, 0, 0), \qquad (1, 0, 0), \qquad (0, 1, 0), \qquad (0, 0, 1).$$

Find the center of mass.

Step 1 The region is sketched in Figure 12.6.10.

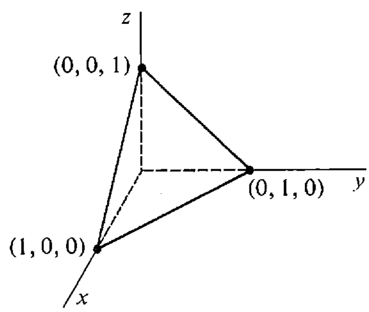


Figure 12.6.10

Step 2 The region E is the solid bounded by the coordinate planes and the plane $x + y + z = 1$ which passes through $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Solving for z, the plane is

$$z = 1 - x - y.$$

This plane meets the plane $z = 0$ at the line $1 - x - y = 0$, or $y = 1 - x$. Therefore E is the region

$$0 \le x \le 1, \qquad 0 \le y \le 1 - x, \qquad 0 \le z \le 1 - x - y.$$

Step 3 Let the density be $\rho = 1$.

$$m = \iiint_E dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} 1 - x - y\, dy\, dx$$

$$= \int_0^1 \tfrac{1}{2}(1-x)^2\, dx = \tfrac{1}{6}.$$

$$M_{yz} = \iiint_E x\, dV$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} x(1 - x - y)\, dy\, dx$$

$$= \int_0^1 \tfrac{1}{2}x(1-x)^2\, dx = \tfrac{1}{24}.$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1/24}{1/6} = \frac{1}{4}.$$

Similarly $$\bar{y} = \frac{1}{4}, \quad \bar{z} = \frac{1}{4}.$$

The center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

An object in space has a moment of inertia about each coordinate axis. Intuitively, the moment of inertia about an axis is the analogue of mass for rotations about the axis.

DEFINITION

*If an object in space fills a region E and has continuous density $\rho(x, y, z)$, its **moments of inertia** about the coordinate axes are*

$$I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)\, dV,$$

$$I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)\, dV,$$

$$I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)\, dV.$$

JUSTIFICATION A point mass m has a moment of inertia about the x-axis of

$$I_x = (y^2 + z^2)m.$$

On an element of volume ΔE, the object has moment of inertia

$$\Delta I_x \approx (y^2 + z^2)\,\Delta m \approx (y^2 + z^2)\rho(x, y, z)\,\Delta V \qquad \text{(compared to } \Delta V\text{)}.$$

The triple integral for I_x follows by the Infinite Sum Theorem.

EXAMPLE 5 Find the moments of inertia about the three axes of an object with constant density 1 filling the cube shown in Figure 12.6.11,

$$0 \le x \le a, \qquad 0 \le y \le a, \qquad 0 \le z \le a.$$

$$I_x = \iiint_E y^2 + z^2\,dV = \int_0^a \int_0^a \int_0^a y^2 + z^2\,dz\,dy\,dx$$

$$= \int_0^a \int_0^a ay^2 + \tfrac{1}{3}a^3\,dy\,dx = \int_0^a \tfrac{2}{3}a^4\,dx = \tfrac{2}{3}a^5.$$

Similarly, $$I_y = \tfrac{2}{3}a^5, \qquad I_z = \tfrac{2}{3}a^5.$$

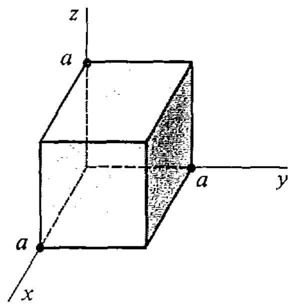


Figure 12.6.11

PROBLEMS FOR SECTION 12.6

In Problems 1–8, evaluate the iterated integral.

1 $$\int_0^1 \int_2^4 \int_1^3 xyz\,dz\,dy\,dx$$

2 $$\int_1^2 \int_0^2 \int_{-1}^4 (x - 2y + 4z)\,dz\,dy\,dx$$

3 $$\int_0^4 \int_0^1 \int_0^1 (3y^2 + 6z^2)\,dz\,dy\,dx$$

4 $$\int_0^2 \int_1^2 \int_1^4 e^{x+z}\,dz\,dy\,dx$$

5 $$\int_0^1 \int_0^x \int_{xy}^1 (x^2 + yz)\,dz\,dy\,dx$$

6 $$\int_0^1 \int_x^1 \int_y^1 2x^2z\,dz\,dy\,dx$$

7 $$\int_0^\pi \int_0^{\sin x} \int_0^{\sin x} \sqrt{yz}\,dz\,dy\,dx$$

8 $$\int_0^{\pi/2} \int_0^{\cos x} \int_0^{y \sin x} (x + 2z)\,dz\,dy\,dx$$

In Problems 9–16, evaluate the triple integral.

9 $\iiint_E (x + 2y)\,dV, \quad E: 0 \le x \le 2, 1 \le y \le 3, 2 \le z \le 4$

10 $\iiint_E x^2yz^3\,dV, \quad E: 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$

11 $\iiint_E (4xy + yz)\,dV, \quad E: 0 \le x \le 10, 0 \le y \le x^2, 0 \le z \le xy$

12 $\iiint_E \left(\frac{1}{x} + \frac{2}{y} + \frac{3}{z}\right) dV, \quad E: 1 \le x \le e, 1 \le y \le x, 1 \le z \le x$

13 $\iiint_E e^{x+2y+3z}\,dV, \quad E: -1 \le x \le 1, x \le y \le 1, x \le z \le y$

14 $\iiint_E xe^{y+z}\,dV, \quad E: 1 \le x \le 2, 0 \le y \le \ln x, 0 \le z \le y$

15 $\iiint_E \sqrt{x + y + z}\,dV, \quad E: 0 \le x \le 1, 0 \le y \le x, y \le z \le 2y$

16 $\iiint_E dV, \quad E: 0 \le x \le 1, x^2 \le y \le x, x^2y \le z \le x\sqrt{y}$

In Problems 17–26, find (a) the mass, (b) the center of mass, (c) the moments of inertia about the three coordinate axes, of an object with density $\rho(x, y, z)$ filling the region E.

17 $E: 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1 \quad \rho(x, y, z) = x + 2y + 3z$

18 $E: 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1 \quad \rho(x, y, z) = x^2 + y^2 + z^2$

19 $E: 0 \le x \le 1, 0 \le y \le x, 0 \le z \le x + y \quad \rho(x, y, z) = 2$

20 $E: -1 \le x \le 1, x^2 \le y \le 1, x^2 \le z \le 1 \quad \rho(x, y, z) = z$

21 $E: 0 \le x \le 1, 0 \le y \le 1, \sqrt{xy} \le z \le 1 \quad \rho(x, y, z) = xyz$

22 $E: 0 \le x \le 1, x \le y \le 1, x \le z \le y \quad \rho(x, y, z) = 10$

23 E is the tetrahedron with vertices at $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, $\rho(x, y, z) = k$.

24 E is the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$, $\rho(x, y, z) = x + y + z$.

25 E is the rectangular box $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, $\rho(x, y, z) = k$.

26 E is the rectangular box $-a \le x \le a$, $-b \le y \le b$, $-c \le z \le c$, $\rho(x, y, z) = k$.

12.7 CYLINDRICAL AND SPHERICAL COORDINATES

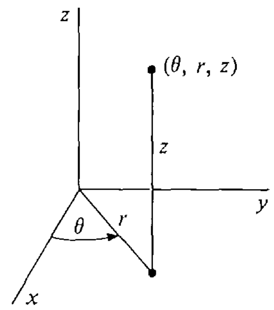
In evaluating triple integrals it is sometimes easier to use cylindrical or spherical coordinates instead of rectangular coordinates.

A point (x, y, z) has *cylindrical coordinates* (θ, r, z) if

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

That is, as we see in Figure 12.7.1, (θ, r) is a polar coordinate representation of (x, y), and z is the height above the (x, y) plane.

The name cylindrical coordinates is used because the graph of the cylindrical coordinate equation $r =$ constant is a circular cylinder as shown in Figure 12.7.2.



Cylindrical coordinates

Figure 12.7.1

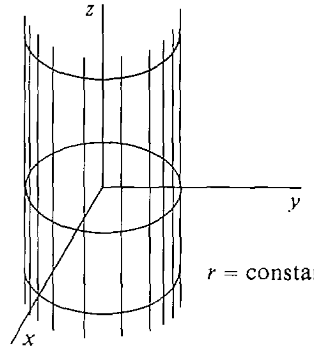


Figure 12.7.2

DEFINITION

*A **cylindrical region** is a region E in (x, y, z) space given by cylindrical coordinate inequalities*

$$\alpha \leq \theta \leq \beta, \qquad a(\theta) \leq r \leq b(\theta), \qquad c_1(\theta, r) \leq z \leq c_2(\theta, r),$$

where all the functions are continuous. To avoid overlaps we also require that for (θ, r, z) in E,

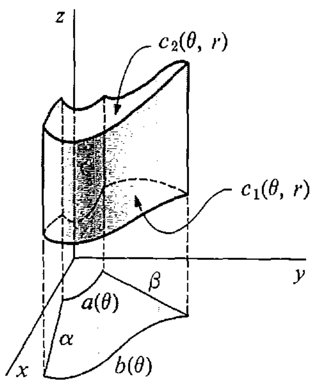
$$0 \leq \theta \leq 2\pi \quad \text{and} \quad 0 \leq r.$$

A cylindrical region is shown in Figure 12.7.3.

The simplest kind of cylindrical region is the *cylindrical box*

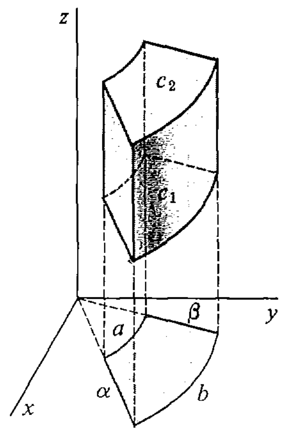
$$\alpha \leq \theta \leq \beta, \qquad a \leq r \leq b, \qquad c_1 \leq z \leq c_2.$$

This is a cylinder whose base is a polar rectangle and whose upper and lower faces are horizontal, as in Figure 12.7.4.



A cylindrical region

Figure 12.7.3



A cylindrical box

Figure 12.7.4

The cylinder box

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le b, \qquad c_1 \le z \le c_2$$

is a cylinder whose base is a circle of radius b and whose top and bottom faces are horizontal (Figure 12.7.5).

The cylindrical box

$$0 \le \theta \le 2\pi, \qquad a \le r \le b, \qquad c_1 \le z \le c_2$$

is a circular pipe with inner radius a and outer radius b (Figure 12.7.6).

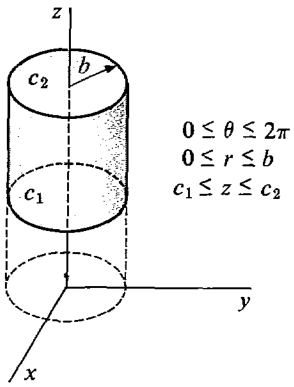


Figure 12.7.5

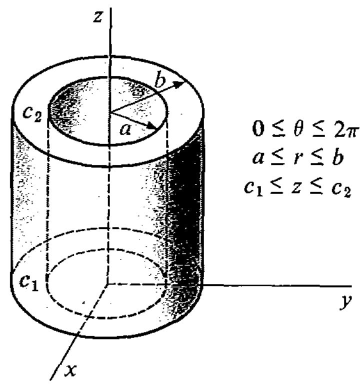


Figure 12.7.6

To get a formula for the triple integral over a cylindrical region E, we use the Infinite Sum Theorem but take for ΔE an infinitely small cylindrical box instead of rectangular box.

CYLINDRICAL INTEGRATION FORMULA

Let E be the cylindrical region

$$\alpha \le \theta \le \beta, \qquad a(\theta) \le r \le b(\theta), \qquad c_1(\theta, r) \le z \le c_2(\theta, r).$$

The triple integral of $f(x, y, z)$ over E is

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{a(\theta)}^{b(\theta)} \int_{c_1(\theta,r)}^{c_2(\theta,r)} f(x, y, z) r\, dz\, dr\, d\theta.$$

To evaluate the triple integral we substitute

$$f(x, y, z) = f(r\cos\theta, r\sin\theta, z).$$

This is like the Polar Integration Formula but has an extra variable z. In the iterated integral we do not integrate $f(x, y, z)$ but the product of $f(x, y, z)$ and r.

PROOF Let C be the region in the rectangular (θ, r, z) space given by

$$\alpha \le \theta \le \beta, \qquad a(\theta) \le r \le b(\theta), \qquad c_1(\theta, r) \le z \le c_2(\theta, r).$$

The region C is shown in Figure 12.7.7.

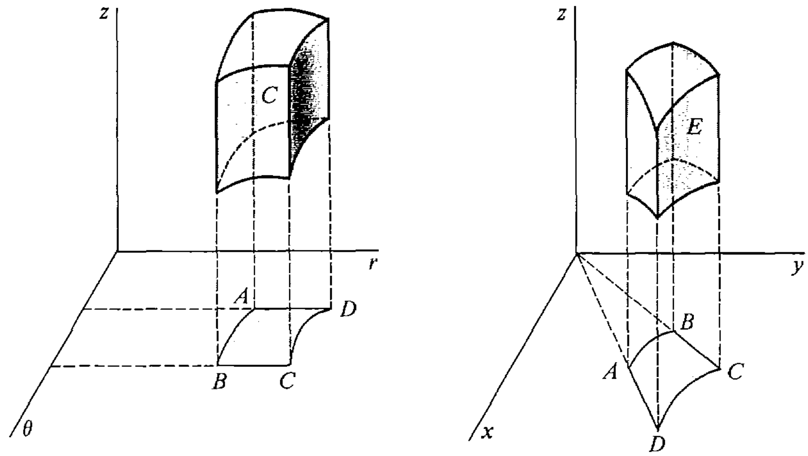


Figure 12.7.7

We must prove that

$$\iiint_E f(x, y, z)\,dx\,dy\,dz = \iiint_C f(x, y, z) r\,d\theta\,dr\,dz.$$

Assume first that $f(x, y, z) > 0$ on E. For any (θ, r, z) region C_1 corresponding to a cylindrical region E_1, define

$$B(C_1) = \iiint_{E_1} f(x, y, z)\,dx\,dy\,dz.$$

B has the Addition Property and is ≥ 0. An element of volume ΔC in the (θ, r, z) space has volume $\Delta\theta\,\Delta r\,\Delta z$. As we can see from Figure 12.7.8, ΔC corresponds to a cylindrical box ΔE. ΔE is almost a rectangular box with sides $r\,\Delta\theta$, Δr, and Δz, and volume $r\,\Delta\theta\,\Delta r\,\Delta z$.

At any point of ΔE, f has value infinitely close to

$$f(x, y, z) = f(r\cos\theta, r\sin\theta, z).$$

Therefore $\qquad B(\Delta C) \approx f(x, y, z) r\,\Delta\theta\,\Delta r\,\Delta z \qquad$ (compared to $\Delta\theta\,\Delta r\,\Delta z$).

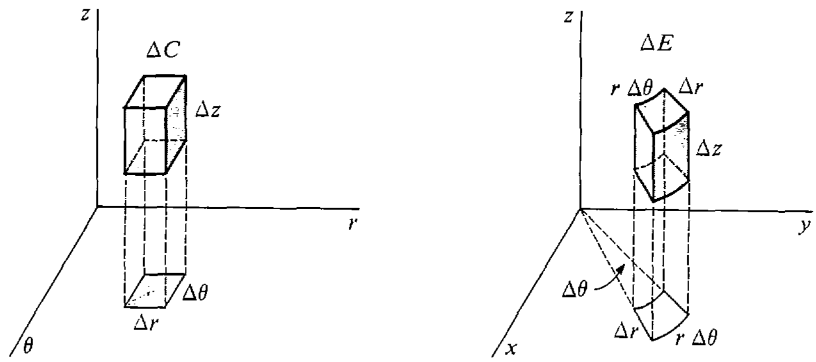


Figure 12.7.8

By the Infinite Sum Theorem

$$B(C) = \iiint_C f(x, y, z) r \, d\theta \, dr \, dr,$$

and by definition

$$B(C) = \iiint_E f(x, y, z) \, dx \, dy \, dz.$$

The general case where $f(x, y, z)$ is not always positive is dealt with as in the Polar Integration Formula proof.

When integrating over a solid region E whose base is a circle or polar rectangle, it is often easier to use cylindrical instead of rectangular coordinates.

EXAMPLE 1 Find the moment of inertia of a cylinder of height h, base a circle of radius b, and constant density 1, about its axis.

Step 1 Draw the region as in Figure 12.7.9.

Step 2 The problem is greatly simplified by a wise choice of coordinate axes. Let the z-axis be the axis of the cylinder and put the origin at the center of the base. Then the region E in rectangular coordinates is

$$-b \le x \le b, \qquad -\sqrt{b^2 - x^2} \le y \le \sqrt{b^2 - x^2}, \qquad 0 \le z \le h,$$

and in cylindrical coordinates is

$$0 \le \theta \le 2\pi, \quad 0 \le r \le b, \quad 0 \le z \le h.$$

Step 3 The problem looks easier in cylindrical coordinates.

$$x^2 + y^2 = r^2.$$

$$\begin{aligned} I_z &= \iiint_E (x^2 + y^2) \, dV = \int_0^{2\pi} \int_0^b \int_0^h r^2 r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^b \int_0^h r^3 \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^b r^3 h \, dr \, d\theta = \int_0^{2\pi} \frac{1}{4} b^4 h \, d\theta = \frac{\pi b^4 h}{2}. \end{aligned}$$

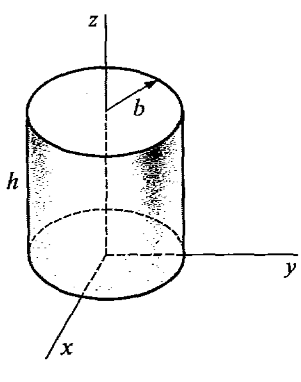


Figure 12.7.9

EXAMPLE 2 Find the center of mass of a cone of constant density with height h and base a circle of radius b.

Step 1 The region is sketched in Figure 12.7.10.

Step 2 Put the origin at the center of the base and let the z-axis be the axis of the cone. E is the cylindrical region

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le b, \qquad 0 \le z \le h - \frac{h}{b}r.$$

Step 3 Let the density be 1.

$$m = \iiint_E dV = \int_0^{2\pi}\int_0^b\int_0^{h-hr/b} r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^b r\left(h - \frac{h}{b}r\right)dr\,d\theta$$
$$= \int_0^{2\pi}\left(\frac{1}{2}b^2h - \frac{1}{3}b^3\frac{h}{b}\right)d\theta = \int_0^{2\pi}\frac{1}{6}b^2h\,d\theta = \frac{\pi b^2h}{3}.$$

$$M_{xy} = \iiint_E z\,dV = \int_0^{2\pi}\int_0^b\int_0^{h-hr/b} zr\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^b \frac{1}{2}r\left(h - \frac{hr}{b}\right)^2 dr\,d\theta$$
$$= \frac{1}{2}h^2\int_0^{2\pi}\int_0^b r - \frac{2r^2}{b} + \frac{r^3}{b^2}\,dr\,d\theta$$
$$= \frac{1}{2}h^2\int_0^{2\pi}\frac{1}{2}b^2 - \frac{2}{3}\cdot\frac{b^3}{b} + \frac{1}{4}\frac{b^4}{b^2}\,d\theta$$
$$= \frac{1}{2}h^2\int_0^{2\pi}\frac{1}{12}b^2\,d\theta = \frac{\pi b^2h^2}{12}.$$

Since the cone is symmetric about the z-axis, $\bar{x} = 0$ and $\bar{y} = 0$.

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{4}h, \qquad (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{1}{4}h\right).$$

The point $(\bar{x}, \bar{y}, \bar{z})$ is shown in Figure 12.7.11.

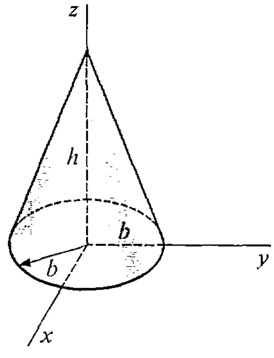


Figure 12.7.10

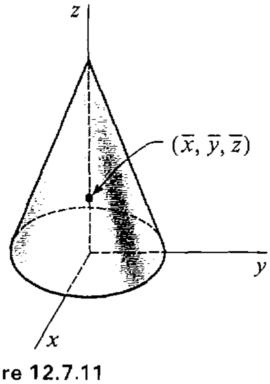


Figure 12.7.11

To express a point $P(x, y, z)$ in *spherical coordinates* we let ρ (rho) be the distance from the origin to P, let θ be the same angle as in cylindrical coordinates, and let ϕ be the angle between the positive z-axis and the line OP. Note that ϕ can always be chosen between 0 and π.

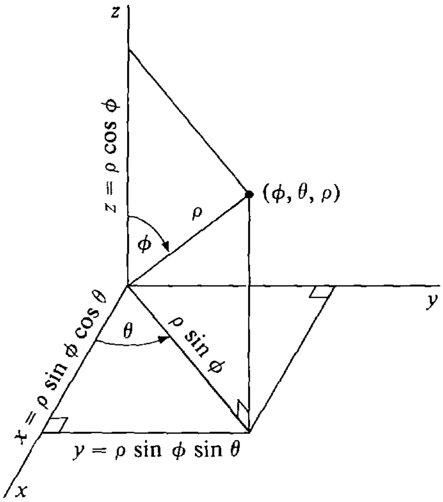


Figure 12.7.12

We see from Figure 12.7.12 that a point (x, y, z) has spherical coordinates (θ, ϕ, ρ) if

$$x = \rho \sin \phi \cos \theta, \qquad y = \rho \sin \phi \sin \theta, \qquad z = \rho \cos \phi.$$

The graph of the equation ρ = constant is a sphere with center at the origin (hence the name spherical coordinates). The graph of ϕ = constant is a vertical cone with vertex at the origin. The graph of θ = constant is a half-plane through the z-axis. These surfaces are shown in Figure 12.7.13.

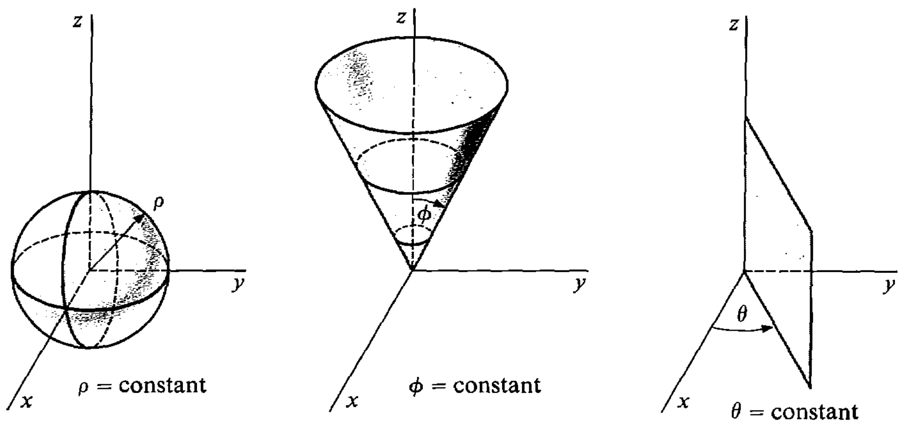


Figure 12.7.13

DEFINITION

*A **spherical region** E is a region in (x, y, z) space given by spherical coordinate inequalities*

$$\alpha_1 \le \theta \le \alpha_2, \qquad \beta_1(\theta) \le \phi \le \beta_2(\theta), \qquad c_1(\theta, \phi) \le \rho \le c_2(\theta, \phi),$$

where all the functions are continuous. To avoid overlaps we also require that for (θ, ϕ, ρ) *in* E,

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi, \qquad 0 \le \rho.$$

A *spherical box* is a spherical region of the simple form

$$\alpha_1 \le \theta \le \alpha_2, \qquad \beta_1 \le \phi \le \beta_2, \qquad c_1 \le \rho \le c_2.$$

The θ-boundaries are planes, the ϕ-boundaries are portions of cone surfaces, and the ρ-boundaries are portions of spherical surfaces. Figure 12.7.14 shows a spherical box.

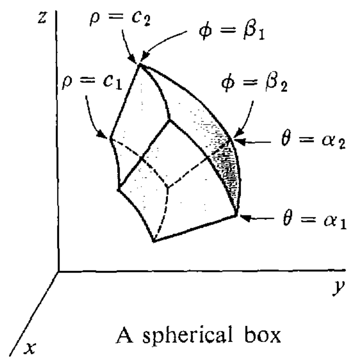


Figure 12.7.14

The spherical box

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi, \qquad 0 \le \rho \le c$$

is a sphere of radius c with center at the origin.

The spherical box

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \beta, \qquad 0 \le \rho \le c$$

is a cone whose vertex is at the origin and whose top is spherical instead of flat. (See Figure 12.7.15.)

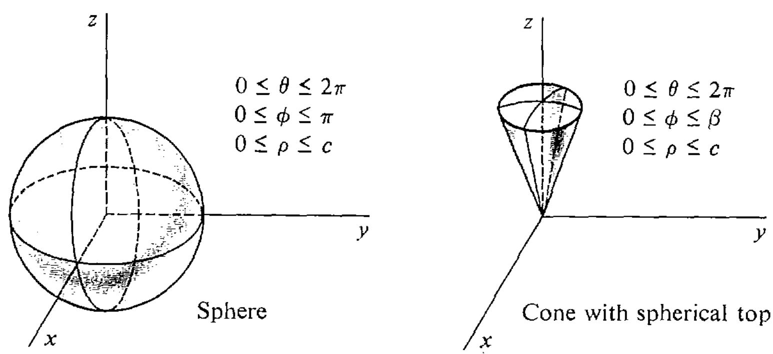


Figure 12.7.15

Another important example is the spherical region

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi/2, \qquad 0 \le \rho \le c\cos\phi,$$

which is a sphere of radius $\frac{1}{2}c$ whose center is on the z-axis at $z = \frac{1}{2}c$ (Figure 12.7.16).

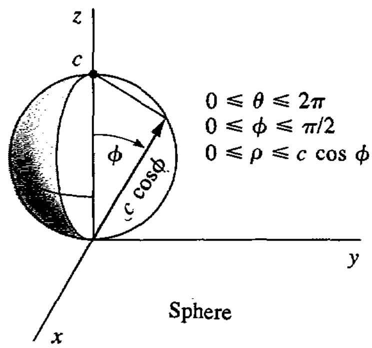


Figure 12.7.16

When integrating over a solid region E made up of spheres or cones, it is often easiest to use spherical coordinates.

SPHERICAL INTEGRATION FORMULA

Let E be a spherical region

$$\alpha_1 \le \theta \le \alpha_2, \qquad \beta_1(\theta) \le \phi \le \beta_2(\theta), \qquad c_1(\theta, \phi) \le \rho \le c_2(\theta, \phi).$$

The triple integral of $f(x, y, z)$ over E is

$$\iiint_E f(x, y, z)\, dV = \int_{\alpha_1}^{\alpha_2} \int_{\beta_1(\theta)}^{\beta_2(\theta)} \int_{c_1(\phi,\theta)}^{c_2(\phi,\theta)} f(x, y, z)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta.$$

In practice we make the substitution

$$f(x, y, z) = f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$$

before integrating.

PROOF Let C be the region in the rectangular (θ, ϕ, ρ) space which has the same inequalities as E. We prove

$$\iiint_E f(x, y, z)\, dx\, dy\, dz = \iiint_C f(x, y, z)\rho^2 \sin\phi \, d\theta\, d\phi\, d\rho.$$

As usual we let $f(x, y, z) > 0$ on E and put

$$B(C_1) = \iiint_{E_1} f(x, y, z)\, dx\, dy\, dz.$$

Consider an element of volume ΔC. As we see from Figure 12.7.17, ΔC corresponds to a spherical box ΔE. ΔE is almost a rectangular box with sides

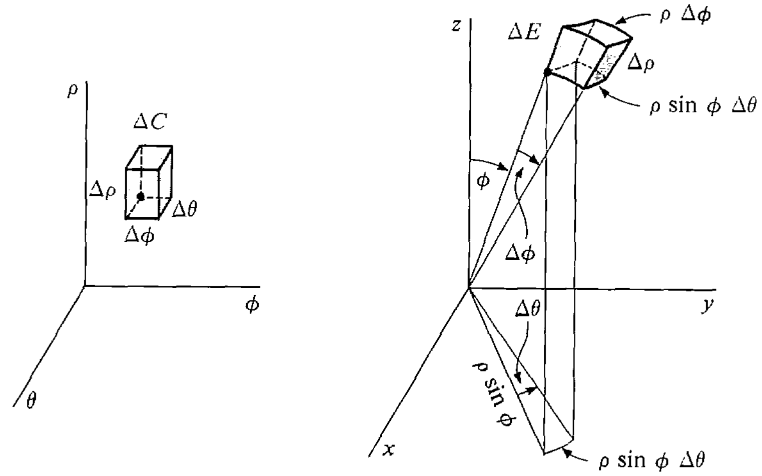


Figure 12.7.17 Spherical Element of Volume

$$\Delta\rho, \quad \rho\,\Delta\phi, \quad \rho \sin\phi\,\Delta\theta$$

and volume $\rho^2 \sin\phi\,\Delta\theta\,\Delta\phi\,\Delta\rho.$

Thus $B(\Delta C) \approx f(x, y, z)\rho^2 \sin\phi\,\Delta\theta\,\Delta\phi\,\Delta\rho$ (compared to $\Delta\theta\,\Delta\phi\,\Delta\rho$).

By the Infinite Sum Theorem

$$B(C) = \iiint_C f(x, y, z)\rho^2 \sin\phi\, d\theta\, d\phi\, d\rho,$$

and by definition

$$B(C) = \iiint_E f(x, y, z)\, dx\, dy\, dz.$$

The triple integral for volume,

$$V = \iiint_E dV,$$

gives us iterated integral formulas for volume in rectangular, cylindrical, and spherical coordinates.

Rectangular $$V = \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} \int_{c_1(x,y)}^{c_2(x,y)} dz\, dy\, dx.$$

Cylindrical $$V = \int_{\alpha}^{\beta} \int_{a(\theta)}^{b(\theta)} \int_{c_1(\theta, r)}^{c_2(\theta, r)} r\, dz\, dr\, d\theta.$$

Spherical $$V = \int_{\alpha_1}^{\alpha_2} \int_{\beta_1(\theta)}^{\beta_2(\theta)} \int_{c_1(\theta,\phi)}^{c_2(\theta,\phi)} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

The rectangular formula is really equivalent to the double integral for the volume between two surfaces. Similarly, the cylindrical formula is equivalent to the double integral in polar coordinates for the volume between two surfaces.

On the other hand, the volume formula in spherical coordinates is something new which is useful for finding volumes of spherical regions.

EXAMPLE 3 Find the volume of the region above the cone $\phi = \beta$ and inside the sphere $\rho = c$.

The region, shown in Figure 12.7.18, is given by

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \beta, \qquad 0 \le \rho \le c.$$

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\beta} \int_0^{c} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\beta} \frac{c^3}{3} \sin\phi \, d\phi \, d\theta \\ &= \int_0^{2\pi} (1 - \cos\beta)\frac{c^3}{3} \, d\theta = \frac{2\pi}{3}(1 - \cos\beta)c^3. \end{aligned}$$

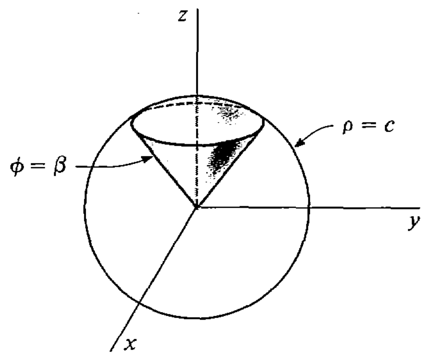


Figure 12.7.18

EXAMPLE 4 A sphere of diameter a passes through the center of a sphere of radius b, and $a > b$. Find the volume of the region inside the sphere of diameter a and outside the sphere of radius b.

Step 1 The region is sketched in Figure 12.7.19.

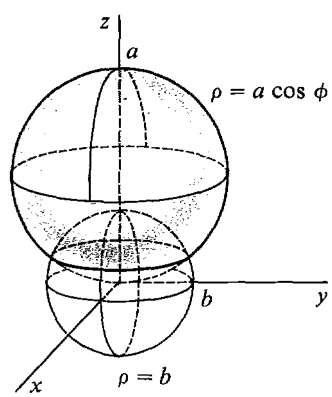


Figure 12.7.19

Step 2 We let the z-axis be the line through the two centers and put the origin at the center of the sphere of radius b. The two spheres have the spherical equations

$$\rho = a\cos\phi, \qquad \rho = b.$$

They intersect at

$$\cos\phi = \frac{b}{a}.$$

Thus E is the region

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \arccos\frac{b}{a}, \qquad b \le \rho \le a\cos\phi.$$

Step 3

$$V = \int_0^{2\pi}\int_0^{\arccos(b/a)}\int_b^{a\cos\phi} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\arccos(b/a)} \frac{1}{3}(a^3\cos^3\phi - b^3)\sin\phi\, d\phi\, d\theta.$$

Put $u = \cos\phi$, $du = -\sin\phi\, d\phi$. Then

$$V = \int_0^{2\pi}\int_1^{b/a} -\frac{1}{3}(a^3u^3 - b^3)\, du\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_{b/a}^{1} a^3u^3 - b^3\, du\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \frac{a^3}{4} - b^3 + \frac{3}{4}\frac{b^4}{a}\, d\theta = \frac{\pi}{6}\left(a^3 - 4b^3 + 3\frac{b^4}{a}\right).$$

EXAMPLE 5 Find the mass of a sphere of radius c whose density is equal to the distance from the surface. The sphere is shown in Figure 12.7.20.

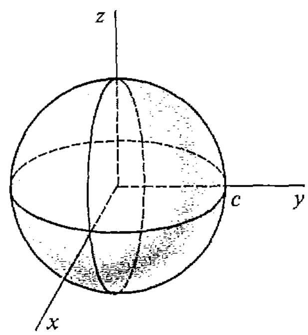


Figure 12.7.20

Put the center at the origin. The sphere is then given by

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi, \qquad 0 \le \rho \le c.$$

The density at (θ, ϕ, ρ) is

$$\text{density} = c - \rho.$$

The mass is

$$\begin{aligned} m &= \int_0^{2\pi}\int_0^{\pi}\int_0^{c}(c-\rho)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi}\left(\frac{c}{3}c^3-\frac{1}{4}c^4\right)\sin\phi\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi}\frac{1}{12}c^4\sin\phi\,d\phi\,d\theta \\ &= \int_0^{2\pi}\frac{1}{6}c^4\,d\theta = \frac{\pi}{3}c^4. \end{aligned}$$

PROBLEMS FOR SECTION 12.7

In Problems 1–6, evaluate the integral using cylindrical coordinates.

1 $\iiint_E \sqrt{x^2+y^2}\,z\,dV$, E is the cylinder $x^2+y^2\le 1, 0\le z\le 2$

2 $\iiint_E x^2+z\,dV$, E is the cylinder $x^2+y^2\le 9, 0\le z\le 6$

3 $\iiint_E x^2+y^2\,dV$, E is the cone $x^2+y^2\le 1, 0\le z\le 1-\sqrt{x^2+y^2}$

4 $\iiint_E 4+\sqrt{z}\,dV$, E is the cone $x^2+y^2\le 1, \sqrt{x^2+y^2}\le z\le 1$

5 $\iiint_E (x+y)z\,dV$, E is the region $0\le x\le 2, 0\le y\le\sqrt{4-x^2}, 0\le z\le x^2+y^2$

6 $\iiint_E \frac{z}{\sqrt{x^2+y^2}}\,dV$, E is the region $1\le x^2+y^2\le 4, 0\le z\le |x|$

7 Find the mass of an object in the shape of a cylinder of radius b and height h whose density is equal to the distance from the axis.

8 Find the mass of an object in the shape of a cylinder of radius b and height h whose density is equal to the distance from the base.

9 Find the mass of an object in the shape of a cone of radius b and height h whose density is equal to the square of the distance from the axis.

10 Find the mass of an object in the shape of a cone of radius b and height h whose density is equal to the sum of the distance from the base and the distance from the axis.

11 Find the center of mass of an object of constant density filling the region above the paraboloid $z=x^2+y^2$ and below the plane $z=1$.

12 Find the center of mass of an object of constant density filling the region

$$x^2+y^2\le b, \qquad 0\le z\le\sqrt{x^2+y^2}.$$

13 Find the moment of inertia of an object of constant density k in the cylinder $0\le r\le b$, $-c\le z\le c$, about the x-axis.

14 Find the moment of inertia of an object of constant density k in the cylindrical shell $a\le r\le b, -c\le z\le c$, about the z-axis.

15 Find the moment of inertia of an object of constant density k in a cone of radius b and height h about its axis.

16 Find the moment of inertia of an object of constant density k in a cone of radius b and height h about a line through its apex and perpendicular to its axis.

In Problems 17–24, evaluate the integral using spherical coordinates.

17 $\iiint_E x^2 + y^2 + z^2\,dV$, E is the sphere $x^2 + y^2 + z^2 \le b^2$

18 $\iiint_E \sqrt{x^2 + y^2 + z^2}\,dV$, E is the sphere $x^2 + y^2 + z^2 \le b^2$

19 $\iiint_E x^2\,dV$, E is the sphere $x^2 + y^2 + z^2 \le 1$

20 $\iiint_E z^2\,dV$, E is the sphere $x^2 + y^2 + z^2 \le 1$

21 $\iiint_E z\,dV$, E is the sphere $\rho \le 2b\cos\phi$

22 $\iiint_E (x^2 + y^2 + z^2)^{3/2}\,dV$, E is the intersection of the spheres $\rho \le 2b\cos\phi$, $\rho \le b$

23 $\iiint_E z\sqrt{x^2 + y^2 + z^2}\,dV$, E is the region above the cone $\phi = \alpha$ and inside the sphere $\rho = b$

24 $\iiint_E \dfrac{1}{x^2 + y^2 + z^2}\,dV$, E is the spherical shell $a \le \rho \le b$

25 Find the volume of the spherical shell $a \le \rho \le b$.

26 Find the volume of the spherical box $\alpha_1 \le \theta \le \alpha_2$, $\beta_1 \le \phi \le \beta_2$, $c_1 \le \rho \le c_2$.

27 Find the volume of the region above the cone $\phi = \beta$ and inside the sphere $\rho = b\cos\phi$.

28 Find the volume of the spherical region $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, $0 \le \rho \le \sin\phi$.

29 Find the mass of an object in the shape of a sphere of radius c whose density is equal to the distance from the center.

30 Find the mass of a spherical shell $a \le \rho \le b$ whose density is equal to the reciprocal of the distance from the center.

31 Find the moment of inertia of a spherical object of radius b and constant density k about a diameter of the sphere.

32 Find the moment of inertia of a spherical shell $a \le \rho \le b$ of constant density k about any diameter.

33 A hole of radius a is bored through a sphere of radius b, and the surface of the hole passes through the center of the sphere, $a = \frac{1}{2}b$. Find the volume removed.

34 A hole of radius a is bored through a cone of height h and base of radius b, and the axis of the cone is on the surface of the hole ($a \le \frac{1}{2}b$). Find the volume removed.

35 Find the center of mass of a hemisphere of constant density and radius b.

36 Find the moment of inertia of an object of constant density k in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

about the z-axis. *Hint*: Change variables $x_1 = x/a$, $y_1 = y/b$, $z_1 = z/c$ and use spherical coordinates.

EXTRA PROBLEMS FOR CHAPTER 12

1 Compute the Riemann sum

$$\sum_D\sum x^2 - y^2\,\Delta x\,\Delta y, \Delta x = \tfrac{1}{2}, \Delta y = \tfrac{1}{2}, \quad D: -2 \le x \le 2, -2 \le y \le 2.$$

2 Compute the Riemann sum

$$\sum_D\sum x^2 - \sqrt{y}\,\Delta x\,\Delta y, \Delta x = \tfrac{1}{3}, \Delta y = \tfrac{1}{4}, \quad D: -1 \le x \le 1, 0 \le y \le 1 - x^2.$$

3 Evaluate $\iint_D x^2 - y^2\,dA, \quad D: -2 \le x \le 2, -2 \le y \le 2.$

4 Evaluate $\iint_D x^2 - \sqrt{y}\,dA, \quad D: -1 \le x \le 1, 0 \le y \le 1 - x^2.$

5 Evaluate $\int_0^{\sqrt{2}/2}\int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} x\,dy\,dx.$

6 Evaluate $\int_0^1\int_{\sqrt{x}}^1 ye^x\,dy\,dx.$

7 Find the volume of the solid over the region $-1 \le x \le 1, 0 \le y \le 1 - x^2$ and between the surfaces $z = 0, z = 1 - y$.

8 Find the volume of the solid over the region $x^2 + y^2 = 4$ and between the surfaces $z = 0$ and $z = y^2 + x + 2$.

9 Find the volume of the solid $1 \le x \le 2, 0 \le y \le \ln x, y/x \le z \le 1/x$.

10 Find the volume of the solid $x^2 + y^2 \le 1, x^2y^3 \le z \le 1$.

11 Find the volume of the solid bounded by the planes

$$x = 1, \qquad x = y, \qquad z = x + y, \qquad z = x + 2.$$

12 Find the volume of the solid bounded by the cylinders

$$x^2 + y^2 = 1, \quad x^2 + z^2 = 1.$$

13 Find the mass, center of mass, and moment of inertia about the origin of the plane object

$$0 \le x \le \pi, \qquad 0 \le y \le \sin x, \qquad \rho(x, y) = k.$$

14 Find the mass, center of mass, and moment of inertia about the origin of the plane object

$$0 \le x \le 1, \qquad x \le y \le 1, \qquad \rho(x, y) = x^2y.$$

15 A circular disc filling the region $x^2 + y^2 \le r^2$ has density $\rho(x, y) = y^2$. Find the mass, center of mass, and moment of inertia about the origin.

16 A semicircular object on the region

$$-r \le x \le r, \qquad 0 \le y \le \sqrt{r^2 - x^2}$$

has density $\rho(x, y) = y$. Find the work required to stand the object up on its flat side.

17 Using polar coordinates, find the volume of the solid

$$x^2 + y^2 \le 9, \qquad y \le z \le x + 5.$$

18 Find the volume of the solid over the region $0 \le r \le 3 + \cos\theta$ between the plane $z = 0$ and the cone $z = r$.

19 Find the volume of the solid over the circle $0 \le r \le a$ between the plane $z = 0$ and the surface $z = 1/r$.

20 Find the mass and the moment of inertia about the origin of a semicircular object $0 \le r \le 1, 0 \le \theta \le \pi$ whose density is $\rho(r, \theta) = r\theta$.

21 A plane object covers the circle $0 \le r \le a$ and its density depends only on the distance r from the center, $\rho(r, \theta) = f(r)$. Show that the center of mass is at the origin.

22 Evaluate $\int_0^{\pi}\int_0^{\pi/2}\int_0^1 z\sin x + z\cos y\,dz\,dy\,dx.$

23 Evaluate $\int_0^1 \int_0^{x^2} \int_0^{y^2} x + y + z\,dz\,dy\,dx$.

24 Evaluate the triple integral

$$\iiint_E \frac{y+z}{x}\,dV, \quad E: 1 \le x \le 4, 1 \le y \le x, 1 \le z \le y.$$

25 An object has constant density k in the region

$$E: 0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le xy.$$

Find its center of mass and its moments of inertia about the coordinate axes.

26 Use cylindrical coordinates to evaluate $\iiint_E z\,dV$, where E is the region inside the cylinder $x^2 + y^2 = 1$ which is above the plane $z = 0$ and within the sphere $x^2 + y^2 + z^2 = 9$.

27 An object of constant density k has the shape of a parabolic bowl

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le b, \qquad r^2 \le z \le r^2 + c.$$

Find its center of mass and its moment of inertia about the z-axis.

28 Use spherical coordinates to evaluate the integral

$$\iiint_E x + y + z\,dV,$$

E is the spherical octant

$$x^2 + y^2 + z^2 \le 1, \qquad 0 \le x, \qquad 0 \le y, \qquad 0 \le z.$$

29 A spherical shell $a \le \rho \le b$ has density equal to the distance from the center. Find its mass and its moment of inertia about a diameter.

☐ 30 Prove that the double Riemann sum $\sum\sum_D f(x, y)\,dx\,dy$ is finite whenever $f(x, y)$ is continuous, D is a closed region, and dx, dy are positive infinitesimals.

☐ 31 Suppose a plane object is symmetric about the x-axis, that is, it covers a region D of the form

$$D: a \le x \le b, -g(x) \le y \le g(x)$$

and has density $\rho(x, y) = \rho(x, -y)$. Prove that the center of mass is on the x-axis.

☐ 32 The moment of inertia about the x-axis of a point in the plane of mass m is $I_x = my^2$. Use the Infinite Sum Theorem to show that the moment of inertia about the x-axis of a plane object with density $\rho(x, y)$ in the region D is $I_x = \iint_D \rho(x, y)y^2\,dA$.

☐ 33 The kinetic energy of a point of mass m moving at speed v is $KE = \frac{1}{2}mv^2$. A rigid object of density $\rho(x, y)$ in the plane region D is rotating about the origin with angular velocity ω (so a point at distance d from the origin has speed ωd). Use the Infinite Sum Theorem to show that the kinetic energy of the object is

$$KE = \iint_D \tfrac{1}{2}\omega^2(x^2 + y^2)\rho(x, y)\,dA = \tfrac{1}{2}\omega^2 I.$$

☐ 34 Suppose a plane object is symmetric about the origin; that is, it fills a polar region $0 \le r \le g(\theta)$, $-\pi \le \theta \le \pi$, such that $g(\theta \pm \pi) = g(\theta)$, and its density has the property $\rho(r, \theta) = \rho(r, \theta \pm \pi)$. Show that the center of mass is at the origin.

☐ 35 Use the Infinite Sum Theorem to show that if D is a polar region of the form $a \le r \le b$, $\alpha(r) \le \theta \le \beta(r)$, then

$$\iint_D f(x, y)\,dA = \int_a^b \int_{\alpha(r)}^{\beta(r)} f(r, \theta)r\,d\theta\,dr.$$