# *INTRODUCTION TO VECTORS AND TENSORS*

## *Vector and Tensor Analysis*

## *Volume 2*

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# *PREFACE*

## *To Volume 2*

This is the second volume of a two-volume work on vectors and tensors. Volume 1 is concerned with the algebra of vectors and tensors, while this volume is concerned with the geometrical aspects of vectors and tensors. This volume begins with a discussion of Euclidean manifolds. The principal mathematical entity considered in this volume is a *field*, which is defined on a domain in a Euclidean manifold. The values of the field may be vectors or tensors. We investigate results due to the distribution of the vector or tensor values of the field on its domain. While we do not discuss general differentiable manifolds, we do include a chapter on vector and tensor fields defined on hypersurfaces in a Euclidean manifold.

This volume contains frequent references to Volume 1. However, references are limited to basic algebraic concepts, and a student with a modest background in linear algebra should be able to utilize this volume as an independent textbook. As indicated in the preface to Volume 1, this volume is suitable for a one-semester course on vector and tensor analysis. On occasions when we have taught a one –semester course, we covered material from Chapters 9, 10, and 11 of this volume. This course also covered the material in Chapters 0,3,4,5, and 8 from Volume 1.

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*Houston, Texas* *R.M.B.*
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*Selected Reading for Part III*


BISHOP, R. L., and R. J. CRITTENDEN, *Geometry of Manifolds*, Academic Press, New York, 1964

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# Chapter 9

# EUCLIDEAN MANIFOLDS

This chapter is the first where the algebraic concepts developed thus far are combined with ideas from analysis. The main concept to be introduced is that of a *manifold*. We will discuss here only a special case cal1ed a Euclidean manifold. The reader is assumed to be familiar with certain elementary concepts in analysis, but, for the sake of completeness, many of these shall be inserted when needed.

## Section 43 Euclidean Point Spaces

Consider an inner produce space $\mathscr{V}$ and a set $\mathscr{E}$. The set $\mathscr{E}$ is a Euclidean point space if there exists a function $f:\mathscr{E}\times\mathscr{E}\to\mathscr{V}$ such that:

(a) $f(\mathbf{x},\mathbf{y}) = f(\mathbf{x},\mathbf{z}) + f(\mathbf{z},\mathbf{y}), \qquad \mathbf{x},\mathbf{y},\mathbf{z}\in\mathscr{E}$

and

(b) For every $\mathbf{x}\in\mathscr{E}$ and $\mathbf{v}\in\mathscr{V}$ there exists a unique element $\mathbf{y}\in\mathscr{E}$ such that $f(\mathbf{x},\mathbf{y})=\mathbf{v}$.

The elements of $\mathscr{E}$ are called *points*, and the inner product space $\mathscr{V}$ is called the *translation space*. We say that $f(\mathbf{x},\mathbf{y})$ is the vector determined by the end point $\mathbf{x}$ and the initial point $\mathbf{y}$. Condition b) above is equivalent to requiring the function $f_{\mathbf{x}}:\mathscr{E}\to\mathscr{V}$ defined by $f_{\mathbf{x}}(\mathbf{y})=f(\mathbf{x},\mathbf{y})$ to be one to one for each $\mathbf{x}$. The *dimension* of $\mathscr{E}$, written $\dim\mathscr{E}$, is defined to be the dimension of $\mathscr{V}$. If $\mathscr{V}$ does not have an inner product, the set $\mathscr{E}$ defined above is called an *affine space*.

A Euclidean point space is not a vector space but a vector space with inner product is made a Euclidean point space by defining $f(\mathbf{v}_1,\mathbf{v}_2)\equiv\mathbf{v}_1-\mathbf{v}_2$ for all $\mathbf{v}\in\mathscr{V}$. For an arbitrary point space the function $f$ is called the *point difference*, and it is customary to use the suggestive notation

$$f(\mathbf{x},\mathbf{y})=\mathbf{x}-\mathbf{y} \tag{43.1}$$

In this notation (a) and (b) above take the forms

$$\mathbf{x}-\mathbf{y}=\mathbf{x}-\mathbf{z}+\mathbf{z}-\mathbf{y} \tag{43.2}$$

and

$$\mathbf{x}-\mathbf{y}=\mathbf{v} \tag{43.3}$$

**Theorem 43.1**. In a Euclidean point space $\mathscr{E}$

$$\begin{aligned} &(i) && \mathbf{x}-\mathbf{x}=\mathbf{0} \\ &(ii) && \mathbf{x}-\mathbf{y}=-(\mathbf{y}-\mathbf{x}) \\ &(iii) && \text{if } \mathbf{x}-\mathbf{y}=\mathbf{x}'-\mathbf{y}', \text{ then } \mathbf{x}-\mathbf{x}'=\mathbf{y}-\mathbf{y}' \end{aligned} \tag{43.4}$$

*Proof*. For (*i*) take $\mathbf{x}=\mathbf{y}=\mathbf{z}$ in (43.2); then

$$\mathbf{x}-\mathbf{x}=\mathbf{x}-\mathbf{x}+\mathbf{x}-\mathbf{x}$$

which implies $\mathbf{x}-\mathbf{x}=\mathbf{0}$. To obtain (*ii*) take $\mathbf{y}=\mathbf{x}$ in (43.2) and use (*i*). For (*iii*) observe that

$$\mathbf{x}-\mathbf{y}'=\mathbf{x}-\mathbf{y}+\mathbf{y}-\mathbf{y}'=\mathbf{x}-\mathbf{x}'+\mathbf{x}'-\mathbf{y}'$$

from (43.2). However, we are given $\mathbf{x}-\mathbf{y}=\mathbf{x}'-\mathbf{y}'$ which implies (*iii*).

The equation

$$\mathbf{x}-\mathbf{y}=\mathbf{v}$$

has the property that given any $\mathbf{v}$ and $\mathbf{y}$, $\mathbf{x}$ is uniquely determined. For this reason it is customary to write

$$\mathbf{x}=\mathbf{y}+\mathbf{v} \tag{43.5}$$

for the point $\mathbf{x}$ uniquely determined by $\mathbf{y} \in \mathscr{E}$ and $\mathbf{v} \in \mathscr{V}$.

The *distance* from $\mathbf{x}$ to $\mathbf{y}$, written $d(\mathbf{x}, \mathbf{y})$, is defined by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \left\{ (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \right\}^{1/2} \tag{43.6}$$

It easily fol1ows from the definition (43.6) and the properties of the inner product that

$$d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x}) \tag{43.7}$$

and

$$d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y}) \tag{43.8}$$

for all $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ in $\mathscr{E}$. Equation (43.8) is simply rewritten in terms of the points $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ rather than the vectors $\mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{z}$, and $\mathbf{z} - \mathbf{y}$. It is also apparent from (43.6) that

$$d(\mathbf{x}, \mathbf{y}) \geq 0 \qquad \text{and} \qquad d(\mathbf{x}, \mathbf{y}) = 0 \Leftrightarrow \mathbf{x} = \mathbf{y} \tag{43.9}$$

The properties (43.7)-(43.9) establish that $\mathscr{E}$ is a *metric space.*

There are several concepts from the theory of metric spaces which we need to summarize. For simplicity the definitions are sated here in terms of Euclidean point spaces only even though they can be defined for metric spaces in general.

In a Euclidean point space $\mathscr{E}$ an *open ball* of *radius* $\varepsilon > 0$ centered at $\mathbf{x}_0 \in \mathscr{E}$ is the set

$$B(\mathbf{x}_0, \varepsilon) = \left\{ \mathbf{x} \,\middle|\, d(\mathbf{x}, \mathbf{x}_0) < \varepsilon \right\} \tag{43.10}$$

and a *closed ball* is the set

$$\overline{B}(\mathbf{x}_0,\varepsilon)=\left\{\mathbf{x}\middle|d(\mathbf{x},\mathbf{x}_0)\le\varepsilon\right\} \tag{43.11}$$

A *neighborhood* of $\mathbf{x}\in\mathscr{E}$ is a set which contains an open ball centered at $\mathbf{x}$. A subset $\mathscr{U}$ of $\mathscr{E}$ is open if it is a neighborhood of each of its points. The empty set $\varnothing$ is trivially open because it contains no points. It also follows from the definitions that $\mathscr{E}$ is open.

**Theorem 43.2.** An open ball is an open set.

*Proof.* Consider the open ball $B(\mathbf{x}_0,\varepsilon)$. Let $\mathbf{x}$ be an arbitrary point in $B(\mathbf{x}_0,\varepsilon)$. Then $\varepsilon-d(\mathbf{x},\mathbf{x}_0)>0$ and the open ball $B(\mathbf{x},\varepsilon-d(\mathbf{x},\mathbf{x}_0))$ is in $B(\mathbf{x}_0,\varepsilon)$, because if $\mathbf{y}\in B(\mathbf{x},\varepsilon-d(\mathbf{x},\mathbf{x}_0))$, then $d(\mathbf{y},\mathbf{x})<\varepsilon-d(\mathbf{x}_0,\mathbf{x})$ and, by (43.8), $d(\mathbf{x}_0,\mathbf{y})\le d(\mathbf{x}_0,\mathbf{x})+d(\mathbf{x},\mathbf{y})$, which yields $d(\mathbf{x}_0,\mathbf{y})<\varepsilon$ and, thus, $\mathbf{y}\in B(\mathbf{x}_0,\varepsilon)$.

A subset $\mathscr{U}$ of $\mathscr{E}$ is closed if its complement, $\mathscr{E}/\mathscr{U}$, is open. It can be shown that closed balls are indeed closed sets. The empty set, $\varnothing$, is closed because $\mathscr{E}=\mathscr{E}/\varnothing$ is open. By the same logic $\mathscr{E}$ is closed since $\mathscr{E}/\mathscr{E}=\varnothing$ is open. In fact $\varnothing$ and $\mathscr{E}$ are the only subsets of $\mathscr{E}$ which are both open and closed. A subset $\mathscr{U}\subset\mathscr{E}$ is *bounded* if it is contained in some open ball. A subset $\mathscr{U}\subset\mathscr{E}$ is *compact* if it is closed and bounded.

**Theorem 43.3.** The union of any collection of open sets is open.

*Proof.* Let $\left\{\mathscr{U}_\alpha\middle|\alpha\in I\right\}$ be a collection of open sets, where $I$ is an index set. Assume that $\mathbf{x}\in\bigcup_{\alpha\in I}\mathscr{U}_\alpha$. Then $\mathbf{x}$ must belong to at least one of the sets in the collection, say $\mathscr{U}_{\alpha_0}$. Since $\mathscr{U}_{\alpha_0}$ is open, there exists an open ball $B(\mathbf{x},\varepsilon)\subset\mathscr{U}_{\alpha_0}\subset\bigcup_{\alpha\in I}\mathscr{U}_\alpha$. Thus, $B(\mathbf{x},\varepsilon)\subset\bigcup_{\alpha\in I}\mathscr{U}_\alpha$. Since $\mathbf{x}$ is arbitrary, $\bigcup_{\alpha\in I}\mathscr{U}_\alpha$ is open.

**Theorem43.4.** The intersection of a finite collection of open sets is open.

*Proof.* Let $\left\{\mathscr{U}_1,\dots,\mathscr{U}_\alpha\right\}$ be a finite family of open sets. If $\bigcap_{i=1}^n\mathscr{U}_i$ is empty, the assertion is trivial. Thus, assume $\bigcap_{i=1}^n\mathscr{U}_i$ is not empty and let $\mathbf{x}$ be an arbitrary element of $\bigcap_{i=1}^n\mathscr{U}_i$. Then $\mathbf{x}\in\mathscr{U}_i$ for $i=1,\dots,n$ and there is an open ball $B(\mathbf{x},\varepsilon_i)\subset\mathscr{U}_i$ for $i=1,\dots,n$. Let $\varepsilon$ be the smallest of the positive numbers $\varepsilon_1,\dots,\varepsilon_n$. Then $\mathbf{x}\in B(\mathbf{x},\varepsilon)\subset\bigcap_{i=1}^n\mathscr{U}_i$. Thus $\bigcap_{i=1}^n\mathscr{U}_i$ is open.

It should be noted that arbitrary intersections of open sets will not always lead to open sets. The standard counter example is given by the family of open sets of $\mathscr{R}$ of the form $(-1/n, 1/n)$, $n = 1, 2, 3....$ The *intersection* $\bigcap_{n=1}^{\infty}(-1/n, 1/n)$ is the set $\{0\}$ which is not open.

By a *sequence* in $\mathscr{E}$ we mean a function on the positive integers $\{1, 2, 3, ..., n, ...\}$ with values in $\mathscr{E}$. The notation $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 ..., \mathbf{x}_n, ...\}$, or simply $\{\mathbf{x}_n\}$, is usually used to denote the values of the sequence. A sequence $\{\mathbf{x}_n\}$ is said to converge to a *limit* $\mathbf{x} \in \mathscr{E}$ if for every open ball $B(\mathbf{x}, \varepsilon)$ centered at $\mathbf{x}$, there exists a positive integer $n_0(\varepsilon)$ such that $\mathbf{x}_n \in B(x, \varepsilon)$ whenever $n \geq n_0(\varepsilon)$. Equivalently, a sequence $\{\mathbf{x}_n\}$ converges to $\mathbf{x}$ if for every real number $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that $d(\mathbf{x}_n, \mathbf{x}) < \varepsilon$ for all $n > n_0(\varepsilon)$. If $\{\mathbf{x}_n\}$ converges to $\mathbf{x}$, it is conventional to write

$$\mathbf{x} = \lim_{n \to \infty} \mathbf{x}_n \qquad \text{or} \qquad \mathbf{x} \to \mathbf{x}_n \quad \text{as} \quad n \to \infty$$

**Theorem 43.5.** If $\{\mathbf{x}_n\}$ converges to a limit, then the limit is unique.

*Proof.* Assume that

$$\mathbf{x} = \lim_{n \to \infty} \mathbf{x}_n, \qquad \mathbf{y} = \lim_{n \to \infty} \mathbf{x}_n$$

Then, from (43.8)

$$d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{x}_n) + d(\mathbf{x}_n, \mathbf{y})$$

for every $n$. Let $\varepsilon$ be an arbitrary positive real number. Then from the definition of convergence of $\{\mathbf{x}_n\}$, there exists an integer $n_0(\varepsilon)$ such that $n \geq n_0(\varepsilon)$ implies $d(\mathbf{x}, \mathbf{x}_n) < \varepsilon$ and $d(\mathbf{x}_n, \mathbf{y}) < \varepsilon$. Therefore,

$$d(\mathbf{x}, \mathbf{y}) \leq 2\varepsilon$$

for arbitrary $\varepsilon$. This result implies $d(\mathbf{x}, \mathbf{y}) = 0$ and, thus, $\mathbf{x} = \mathbf{y}$.

A point $\mathbf{x} \in \mathscr{E}$ is a limit point of a subset $\mathscr{U} \subset \mathscr{E}$ if every neighborhood of $\mathbf{x}$ contains a point of $\mathscr{U}$ distinct from $\mathbf{x}$. Note that $\mathbf{x}$ need not be in $\mathscr{U}$. For example, the sphere $\{\mathbf{x} \mid d(\mathbf{x},\mathbf{x}_0)=\varepsilon\}$ are limit points of the open ball $B(\mathbf{x}_0,\varepsilon)$. The *closure* of $\mathscr{U} \subset \mathscr{E}$, written $\overline{\mathscr{U}}$, is the union of $\mathscr{U}$ and its limit points. For example, the closure of the open ball $B(\mathbf{x}_0,\varepsilon)$ is the closed ball $\bar{B}(\mathbf{x}_0,\varepsilon)$. It is a fact that the closure of $\mathscr{U}$ is the smallest closed set containing $\mathscr{U}$. Thus $\mathscr{U}$ is closed if and only if $\mathscr{U}=\overline{\mathscr{U}}$.

The reader is cautioned not to confuse the concepts limit of a sequence and limit point of a subset. A sequence is *not* a subset of $\mathscr{E}$; it is a function with values in $\mathscr{E}$. A sequence may have a limit when it has no limit point. Likewise the set of pints which represent the values of a sequence may have a limit point when the sequence does not converge to a limit. However, these two concepts are related by the following result from the theory of metric spaces: a point $\mathbf{x}$ is a limit point of a set $\mathscr{U}$ if and only if there exists a convergent sequence of *distinct* points of $\mathscr{U}$ with $\mathbf{x}$ as a limit.

A mapping $f:\mathscr{U} \to \mathscr{E}'$, where $\mathscr{U}$ is an open set in $\mathscr{E}$ and $\mathscr{E}'$ is a Euclidean point space or an inner product space, is *continuous* at $\mathbf{x}_0 \in \mathscr{U}$ if for every real number $\varepsilon > 0$ there exists a real number $\delta(\mathbf{x}_0,\varepsilon) > 0$ such that $d(\mathbf{x},\mathbf{x}_0) < \delta(\varepsilon,\mathbf{x}_0)$ implies $d'(f(\mathbf{x}_0), f(\mathbf{x})) < \varepsilon$. Here $d'$ is the distance function for $\mathscr{E}'$. When $f$ is continuous at $\mathbf{x}_0$, it is conventional to write

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) \qquad \text{or} \qquad f(\mathbf{x}) \to f(\mathbf{x}_0) \quad \text{as} \quad \mathbf{x}\to\mathbf{x}_0$$

The mapping $f$ is continuous on $\mathscr{U}$ if it is continuous at every point of $\mathscr{U}$. A continuous mapping is called a *homomorphism* if it is one-to-one and if its inverse is also continuous. What we have just defined is sometimes called a *homeomorphism into*. If a homomorphism is also onto, then it is called specifically a *homeomorphism onto*. It is easily verified that a composition of two continuous maps is a continuous map and the composition of two homomorphisms is a homomorphism.

A mapping $f:\mathscr{U} \to \mathscr{E}'$, where $\mathscr{U}$ and $\mathscr{E}'$ are defined as before, is *differentiable* at $\mathbf{x} \in \mathscr{U}$ if there exists a linear transformation $\mathbf{A}_\mathbf{x} \in \mathscr{L}(\mathscr{V};\mathscr{V}')$ such that

$$f(\mathbf{x}+\mathbf{v}) = f(\mathbf{x}) + \mathbf{A}_\mathbf{x}\mathbf{v} + o(\mathbf{x},\|\mathbf{v}\|) \tag{43.12}$$

where

$$\lim_{\mathbf{v}\to\mathbf{0}} \frac{o(\mathbf{x},\|\mathbf{v}\|)}{\|\mathbf{v}\|} = \mathbf{0} \tag{43.13}$$

In the above definition $\mathscr{V}'$ denotes the translation space of $\mathscr{E}'$.

**Theorem 43.6.** The linear transformation $\mathbf{A}_\mathbf{x}$ in (43.12) is unique.

*Proof.* If (43.12) holds for $\mathbf{A}_\mathbf{x}$ and $\bar{\mathbf{A}}_\mathbf{x}$, then by subtraction we find

$$\left(\mathbf{A}_\mathbf{x} - \bar{\mathbf{A}}_\mathbf{x}\right)\mathbf{v} = \bar{o}(\mathbf{x},\|\mathbf{v}\|) - o(\mathbf{x},\|\mathbf{v}\|)$$

By (43.13), $\left(\mathbf{A}_\mathbf{x} - \bar{\mathbf{A}}_\mathbf{x}\right)\mathbf{e}$ must be zero for each unit vector $\mathbf{e}$, so that $\mathbf{A}_\mathbf{x} = \bar{\mathbf{A}}_\mathbf{x}$.

If $f$ is differentiable at *every* point of $\mathscr{U}$, then we can define a mapping $\operatorname{grad} f : \mathscr{U} \to \mathscr{L}(\mathscr{V};\mathscr{V}')$, called the *gradient* of $f$, by

$$\operatorname{grad} f(\mathbf{x}) = \mathbf{A}_\mathbf{x}, \qquad \mathbf{x} \in \mathscr{U} \tag{43.14}$$

If $\operatorname{grad} f$ is continuous on $\mathscr{U}$, then $f$ is said to be of class $C^1$. If $\operatorname{grad} f$ exists and is itself of class $C^1$, then $f$ is of class $C^2$. More generally, $f$ is of class $C^r$, $r > 0$, if it is of class $C^{r-1}$ and its $(r-1)$ st gradient, written $\overset{r-1}{\operatorname{grad}} f$, is of class $C^1$. Of course, $f$ is of class $C^0$ if it is continuous on $\mathscr{U}$. If $f$ is a $C^r$ one-to-one map with a $C^r$ inverse $f^{-1}$ defined on $f(\mathscr{U})$, then $f$ is called a $C^r$ diffeomorphism.

If $f$ is differentiable at $\mathbf{x}$, then it follows from (43.12) that

$$\mathbf{A}_\mathbf{x}\mathbf{v} = \lim_{\tau\to 0} \frac{f(\mathbf{x}+\tau\mathbf{v}) - f(\mathbf{x})}{\tau} = \left.\frac{d}{d\tau} f(\mathbf{x}+\tau\mathbf{v})\right|_{\tau=0} \tag{43.15}$$

for all $\mathbf{v} \in \mathscr{V}$. To obtain (43.15) replace $\mathbf{v}$ by $\tau\mathbf{v}$, $\tau > 0$ in (43.12) and write the result as

$$\mathbf{A}_{\mathbf{x}}\mathbf{v} = \frac{f(\mathbf{x}+\tau\mathbf{v}) - f(\mathbf{x})}{\tau} - \frac{o(\mathbf{x}, \|\tau\mathbf{v}\|)}{\tau} \tag{43.16}$$

By (43.13) the limit of the last term is zero as $\tau \to 0$, and (43.15) is obtained. Equation (43.15) holds for all $\mathbf{v} \in \mathscr{V}$ because we can always choose $\tau$ in (43.16) small enough to ensure that $\mathbf{x}+\tau\mathbf{v}$ is in $\mathscr{U}$, the domain of $f$. If $f$ is differentiable at every $\mathbf{x} \in \mathscr{U}$, then (43.15) can be written

$$\left(\operatorname{grad} f(\mathbf{x})\right)\mathbf{v} = \frac{d}{d\tau} f(\mathbf{x}+\tau\mathbf{v})\bigg|_{\tau=0} \tag{43.17}$$

A function $f : \mathscr{U} \to \mathscr{R}$, where $\mathscr{U}$ is an open subset of $\mathscr{E}$, is called a *scalar field*. Similarly, $f : \mathscr{U} \to \mathscr{V}$ is a *vector field*, and $f : \mathscr{U} \to \mathscr{T}_q(\mathscr{V})$ is a *tensor field* of order $q$. It should be noted that the term *field* is defined here is not the same as that in Section 7.

Before closing this section there is an important theorem which needs to be recorded for later use. We shall not prove this theorem here, but we assume that the reader is familiar with the result known as the inverse mapping theorem in multivariable calculus.

**Theorem 43.7.** Let $f : \mathscr{U} \to \mathscr{E}'$ be a $C^r$ mapping and assume that $\operatorname{grad} f(\mathbf{x}_0)$ is a linear isomorphism. Then there exists a neighborhood $\mathscr{U}_1$ of $\mathbf{x}_0$ such that the restriction of $f$ to $\mathscr{U}_1$ is a $C^r$ diffeomorphism. In addition

$$\operatorname{grad} f^{-1}(f(\mathbf{x}_0)) = \left(\operatorname{grad} f(\mathbf{x}_0)\right)^{-1} \tag{43.18}$$

This theorem provides a condition under which one can asert the existence of a local inverse of a smooth mapping.

## *Exercises*

43.1 Let a sequence $\{\mathbf{x}_n\}$ converge to $\mathbf{x}$. Show that every subsequence of $\{\mathbf{x}_n\}$ also converges to $\mathbf{x}$.

43.2 Show that arbitrary intersections and finite unions of closed sets yields closed sets.

43.3 Let $f:\mathscr{U}\rightarrow\mathscr{E}'$, where $\mathscr{U}$ is open in $\mathscr{E}$, and $\mathscr{E}'$ is either a Euclidean point space or an inner produce space. Show that $f$ is continuous on $\mathscr{U}$ if and only if $f^{-1}(\mathscr{D})$ is open in $\mathscr{E}$ for all $\mathscr{D}$ open in $f(\mathscr{U})$.

43.4 Let $f:\mathscr{U}\rightarrow\mathscr{E}'$ be a homeomorphism. Show that $f$ maps any open set in $\mathscr{U}$ onto an open set in $\mathscr{E}'$.

43.5 If $f$ is a differentiable scalar valued function on $\mathscr{L}(\mathscr{V};\mathscr{V})$, show that the gradient of $f$ at $\mathbf{A}\in\mathscr{L}(\mathscr{V};\mathscr{V})$, written

$$\frac{\partial f}{\partial \mathbf{A}}(\mathbf{A})$$

is a linear transformation in $\mathscr{L}(\mathscr{V};\mathscr{V})$ defined by

$$\operatorname{tr}\left(\frac{\partial f}{\partial \mathbf{A}}(\mathbf{A})\mathbf{B}^T\right)=\frac{df}{d\tau}\left(\mathbf{A}+\tau\mathbf{B}\right)\bigg|_{\tau=0}$$

for all $\mathbf{B}\in\mathscr{L}(\mathscr{V};\mathscr{V})$

43.6 Show that

$$\frac{\partial \mu_1}{\partial \mathbf{A}}(\mathbf{A})=\mathbf{I} \qquad \text{and} \qquad \frac{\partial \mu_N}{\partial \mathbf{A}}(\mathbf{A})=\left(\operatorname{adj}\mathbf{A}\right)^T$$

## Section 44 Coordinate Systems

Given a Euclidean point space $\mathscr{E}$ of dimension $N$, we define a $C^r$*-chart* at $\mathbf{x} \in \mathscr{E}$ to be a pair $(\mathscr{U}, \hat{x})$, where $\mathscr{U}$ is an open set in $\mathscr{E}$ containing $\mathbf{x}$ and $\hat{x} : \mathscr{U} \to \mathscr{R}^N$ is a $C^r$ diffeomorphism. Given any chart $(\mathscr{U}, \hat{x})$, there are $N$ scalar fields $\hat{x}^i : \mathscr{U} \to \mathscr{R}$ such that

$$\hat{x}(\mathbf{x}) = \left(\hat{x}^1(\mathbf{x}), \ldots, \hat{x}^N(\mathbf{x})\right) \tag{44.1}$$

for all $\mathbf{x} \in \mathscr{U}$. We call these fields the *coordinate functions* of the chart, and the mapping $\hat{x}$ is also called a *coordinate map* or a *coordinate system* on $\mathscr{U}$. The set $\mathscr{U}$ is called the *coordinate neighborhood*.

Two charts $\hat{x} : \mathscr{U}_1 \to \mathscr{R}^N$ and $\hat{y} : \mathscr{U}_2 \to \mathscr{R}^N$, where $\mathscr{U}_1 \cap \mathscr{U}_2 \neq \varnothing$, yield the *coordinate transformation* $\hat{y} \circ \hat{x}^{-1} : \hat{x}(\mathscr{U}_1 \cap \mathscr{U}_2) \to \hat{y}(\mathscr{U}_1 \cap \mathscr{U}_2)$ and its inverse $\hat{x} \circ \hat{y}^{-1} : \hat{y}(\mathscr{U}_1 \cap \mathscr{U}_2) \to \hat{x}(\mathscr{U}_1 \cap \mathscr{U}_2)$. Since

$$\hat{y}(\mathbf{x}) = \left(\hat{y}^1(\mathbf{x}), \ldots, \hat{y}^N(\mathbf{x})\right) \tag{44.2}$$

The coordinate transformation can be written as the equations

$$\left(\hat{y}^1(\mathbf{x}), \ldots, \hat{y}^N(\mathbf{x})\right) = \hat{y} \circ \hat{x}^{-1}\left(\hat{x}^1(\mathbf{x}), \ldots, \hat{x}^N(\mathbf{x})\right) \tag{44.3}$$

and the inverse can be written

$$\left(\hat{x}^1(\mathbf{x}), \ldots, \hat{x}^N(\mathbf{x})\right) = \hat{x} \circ \hat{y}^{-1}\left(\hat{y}^1(\mathbf{x}), \ldots, \hat{y}^N(\mathbf{x})\right) \tag{44.4}$$

The component forms of (44.3) and (44.4) can be written in the simplified notation

$$y^j = y^j(x^1, \ldots, x^N) \equiv y^j(x^k) \tag{44.5}$$

and

$$x^j = x^j(y^1,...,y^N) \equiv x^j(y^k) \tag{44.6}$$

The two $N$-tuples $(y^1,...,y^N)$ and $(x^1,...,x^N)$, where $y^j = \hat{y}^j(\mathbf{x})$ and $x^j = \hat{x}^j(\mathbf{x})$, are the *coordinates* of the point $\mathbf{x} \in \mathscr{U}_1 \cap \mathscr{U}_2$. Figure 6 is useful in understanding the coordinate transformations.

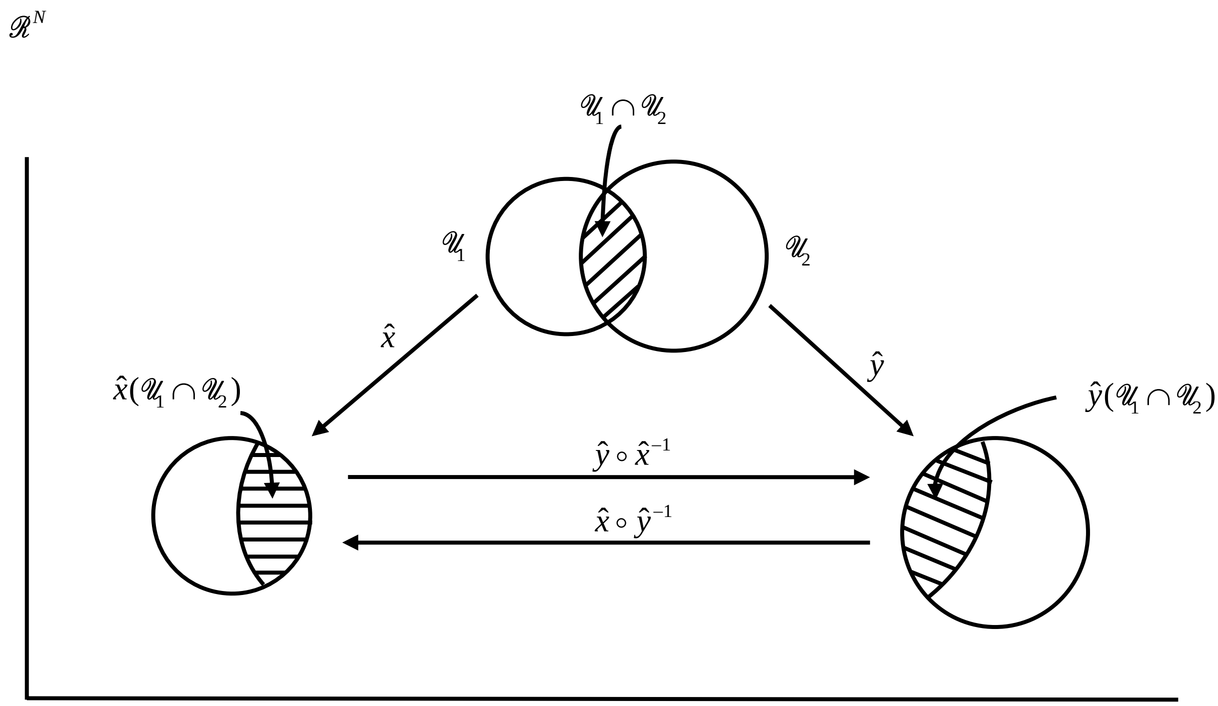


Figure 6

It is important to note that the quantities $x^1,...,x^N$, $y^1,...,y^N$ are *scalar fields*, i.e., real-valued functions defined on certain subsets of $\mathscr{E}$. Since $\mathscr{U}_1$ and $\mathscr{U}_2$ are open sets, $\mathscr{U}_1 \cap \mathscr{U}_2$ is open, and because $\hat{x}$ and $\hat{y}$ are $C^r$ diffeomorphisms, $\hat{x}\left(\mathscr{U}_1 \cap \mathscr{U}_2\right)$ and $\hat{y}\left(\mathscr{U}_1 \cap \mathscr{U}_2\right)$ are open subsets of $\mathscr{R}^N$. In addition, the mapping $\hat{y} \circ \hat{x}^{-1}$ and $\hat{x} \circ \hat{y}^{-1}$ are $C^r$ diffeomorphisms. Since $\hat{x}$ is a diffeomorphism, equation (44.1) written in the form

$$\hat{x}(\mathbf{x}) = (x^1,...,x^N) \tag{44.7}$$

can be inverted to yie1d

$$\mathbf{x} = \tilde{\mathbf{x}}(x^1, \ldots, x^N) = \tilde{\mathbf{x}}(x^j) \tag{44.8}$$

where $\tilde{\mathbf{x}}$ is a diffeomorphism $\tilde{\mathbf{x}} : \hat{x}(\mathscr{U}) \to \mathscr{U}$.

A $C^r$*-atlas* on $\mathscr{E}$ is a family (not necessarily countable) of $C^r$*-charts* $\left\{ \left(\mathscr{U}_\alpha, \hat{x}_\alpha\right) \middle| \alpha \in I \right\}$, where $I$ is an index set, such that

$$\mathscr{E} = \bigcup_{\alpha \in I} \mathscr{U}_\alpha \tag{44.9}$$

Equation (44.9) states that $\mathscr{E}$ is covered by the family of open sets $\left\{ \mathscr{U}_\alpha \middle| \alpha \in I \right\}$. A $C^r$*-Euclidean manifold* is a Euclidean point space equipped with a $C^r$*-atlas*. A $C^\infty$*-atlas* and a $C^\infty$*-Euclidean manifold* are defined similarly. For simplicity, we shall assume that $\mathscr{E}$ is $C^\infty$.

A $C^\infty$ curve in $\mathscr{E}$ is a $C^\infty$ mapping $\boldsymbol{\lambda} : (a,b) \to \mathscr{E}$, where $(a,b)$ is an open interval of $\mathscr{R}$. A $C^\infty$ curve $\boldsymbol{\lambda}$ passes through $\mathbf{x}_0 \in \mathscr{E}$ if there exists a $c \in (a,b)$ such that $\boldsymbol{\lambda}(c) = \mathbf{x}_0$. Given a chart $\left(\mathscr{U}, \hat{x}\right)$ and a point $\mathbf{x}_0 \in \mathscr{U}$, the $j^{\text{th}}$ *coordinate curve* passing through $\mathbf{x}_0$ is the curve $\boldsymbol{\lambda}_j$ defined by

$$\boldsymbol{\lambda}_j(t) = \tilde{\mathbf{x}}(x_0^1, \ldots, x_0^{j-1}, x_0^j + t, x_0^{j+1}, \ldots, x_0^N) \tag{44.10}$$

for all $t$ such that $(x_0^1, \ldots, x_0^{j-1}, x_0^j + t, x_0^{j+1}, \ldots, x_0^N) \in \hat{x}(\mathscr{U})$, where $(x_0^k) = \hat{x}(\mathbf{x}_0)$. The subset of $\mathscr{U}$ obtained by requiring

$$x^j = \hat{x}^j(\mathbf{x}) = \text{ const} \tag{44.11}$$

is called the $j^{\text{th}}$ *coordinate surface* of the chart

Euclidean manifolds possess certain special coordinate systems of major interest. Let $\left\{\mathbf{i}^1, \ldots, \mathbf{i}^N\right\}$ be an arbitrary basis, not necessarily orthonormal, for $\mathscr{V}$. We define $N$ *constant vector fields* $\mathbf{i}^j : \mathscr{E} \to \mathscr{V}$, $j = 1, \ldots, N$, by the formulas

$$\mathbf{i}^j(\mathbf{x}) = \mathbf{i}^j, \qquad \mathbf{x} \in \mathscr{U} \tag{44.12}$$

The use of the same symbol for the vector field and its value will cause no confusion and simplifies the notation considerably. If $\mathbf{0}_{\mathscr{E}}$ denotes a fixed element of $\mathscr{E}$, then a *Cartesian coordinate system* on $\mathscr{E}$ is defined by the $N$ scalar fields $\hat{z}^1, \hat{z}^2, ..., \hat{z}^N$ such that

$$z^j = \hat{z}^j(\mathbf{x}) = (\mathbf{x} - \mathbf{0}_{\mathscr{E}}) \cdot \mathbf{i}^j, \qquad \mathbf{x} \in \mathscr{E} \tag{44.13}$$

If the basis $\{\mathbf{i}^1, ..., \mathbf{i}^N\}$ is orthonormal, the Cartesian system is called a *rectangular Cartesian system*. The point $\mathbf{0}_{\mathscr{E}}$ is called the *origin* of the Cartesian coordinate system. The vector field defined by

$$\mathbf{r}(\mathbf{x}) = \mathbf{x} - \mathbf{0}_{\mathscr{E}} \tag{44.14}$$

for all $\mathbf{x} \in \mathscr{E}$ is the *position vector field* relative to $\mathbf{0}_{\mathscr{E}}$. The value $\mathbf{r}(\mathbf{x})$ is the position vector of $\mathbf{x}$. If $\{\mathbf{i}_1, ..., \mathbf{i}_N\}$ is the basis reciprocal to $\{\mathbf{i}^1, ..., \mathbf{i}^N\}$, then (44.13) implies

$$\mathbf{x} - \mathbf{0}_{\mathscr{E}} = \tilde{\mathbf{x}}(z^1, ..., z^N) - \mathbf{0}_{\mathscr{E}} = \hat{z}^j(\mathbf{x})\mathbf{i}_j = z^j \mathbf{i}_j \tag{44.15}$$

Defining constant vector fields $\mathbf{i}_1, ..., \mathbf{i}_N$ as before, we can write (44.15) as

$$\mathbf{r} = \hat{z}^j \mathbf{i}_j \tag{44.16}$$

The product of the scalar field $\hat{z}^j$ with the vector field $\mathbf{i}_j$ in (44.16) is defined pointwise; i.e., if $f$ is a scalar field and $\mathbf{v}$ is a vector field, then $f\mathbf{v}$ is a vector field defined by

$$f\mathbf{v}(\mathbf{x}) = f(\mathbf{x})\mathbf{v}(\mathbf{x}) \tag{44.17}$$

for all $\mathbf{x}$ in the intersection of the domains of $f$ and $\mathbf{v}$. An equivalent version of (44.13) is

$$\hat{z}^j = \mathbf{r} \cdot \mathbf{i}^j \tag{44.18}$$

where the operator $\mathbf{r}\cdot\mathbf{i}^j$ between vector fields is defined pointwise in a similar fashion as in (44.17). As an illustration, let $\{\overline{\mathbf{i}}^1,\overline{\mathbf{i}}^2,\ldots,\overline{\mathbf{i}}^N\}$ be another basis for $\mathscr{V}$ related to the original basis by

$$\overline{\mathbf{i}}^j = Q_k^j \mathbf{i}^k \tag{44.19}$$

and let $\overline{\mathbf{0}}_{\mathscr{E}}$ be another fixed point of $\mathscr{E}$. Then by (44.13)

$$\begin{aligned}\overline{z}^j &= (\mathbf{x}-\overline{\mathbf{0}}_{\mathscr{E}})\cdot\overline{\mathbf{i}}^j = (\mathbf{x}-\mathbf{0}_{\mathscr{E}})\cdot\overline{\mathbf{i}}^j + (\mathbf{0}_{\mathscr{E}}-\overline{\mathbf{0}}_{\mathscr{E}})\cdot\overline{\mathbf{i}}^j\\ &= Q_k^j(\mathbf{x}-\mathbf{0}_{\mathscr{E}})\cdot\mathbf{i}^k + (\mathbf{0}_{\mathscr{E}}-\overline{\mathbf{0}}_{\mathscr{E}})\cdot\overline{\mathbf{i}}^j = Q_k^j z^k + c^j\end{aligned} \tag{44.20}$$

where (44.19) has been used. Also in (44.20) the constant scalars $c^k$, $k=1,\ldots,N$, are defined by

$$c^k = (\mathbf{0}_{\mathscr{E}}-\overline{\mathbf{0}}_{\mathscr{E}})\cdot\overline{\mathbf{i}}^k \tag{44.21}$$

If the bases $\{\mathbf{i}^1,\mathbf{i}^2,\ldots,\mathbf{i}^N\}$ and $\{\overline{\mathbf{i}}^1,\overline{\mathbf{i}}^2,\ldots,\overline{\mathbf{i}}^N\}$ are both orthonormal, then the matrix $\left[Q_k^j\right]$ is orthogonal. Note that the coordinate neighborhood is the entire space $\mathscr{E}$.

The $j$th coordinate curve which passes through $\mathbf{0}_{\mathscr{E}}$ of the Cartesian coordinate system is the curve

$$\boldsymbol{\lambda}(t) = t\mathbf{i}_j + \mathbf{0}_{\mathscr{E}} \tag{44.22}$$

Equation (44.22) follows from (44.10), (44.15), and the fact that for $\mathbf{x}=\mathbf{0}_{\mathscr{E}}$, $z^1=z^2=\cdots=z^N=0$. As (44.22) indicates, the coordinate curves are straight lines passing through $\mathbf{0}_{\mathscr{E}}$. Similarly, the $j$th coordinate surface is the plane defined by

$$(\mathbf{x}-\mathbf{0}_{\mathscr{E}})\cdot\mathbf{i}^j = \text{const}$$

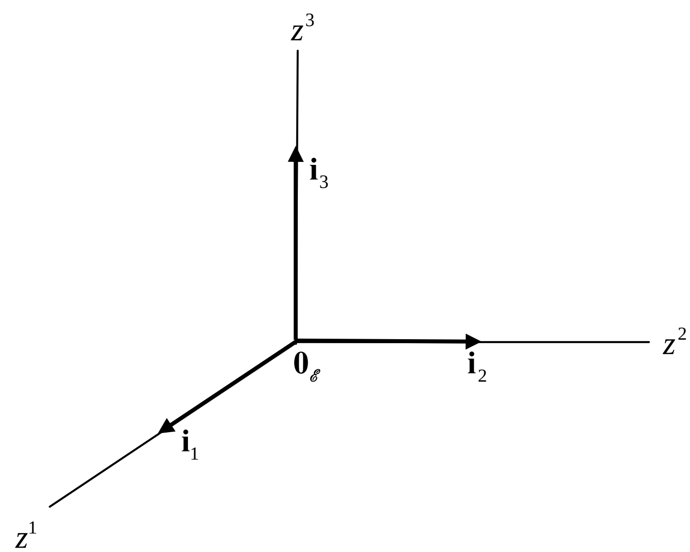


Figure 7

Geometrically, the Cartesian coordinate system can be represented by Figure 7 for $N$=3. The coordinate transformation represented by (44.20) yields the result in Figure 8 (again for $N$=3). Since every inner product space has an orthonormal basis (see Theorem 13.3), there is no loss of generality in assuming that associated with every point of $\mathscr{E}$ as origin we can introduce a rectangular Cartesian coordinate system.

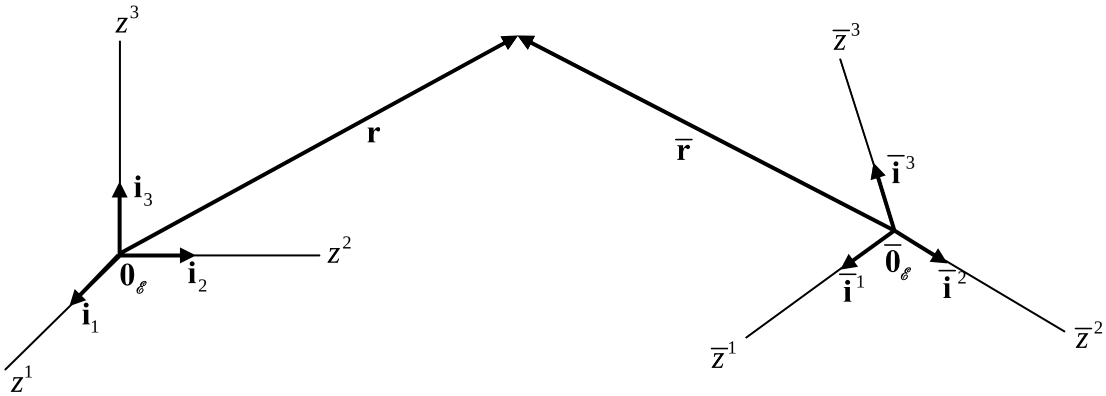


Figure 8

Given any rectangular coordinate system $(\hat{z}^1,...,\hat{z}^N)$, we can characterize a general or a *curvilinear* coordinate system as follows: Let $(\mathscr{U},\hat{x})$ be a chart. Then it can be specified by the coordinate transformation from $\hat{z}$ to $\hat{x}$ as described earlier, since in this case the overlap of the coordinate neighborhood is $\mathscr{U} = \mathscr{E} \cap \mathscr{U}$. Thus we have

$$\begin{aligned}(z^1,...,z^N) &= \hat{z}\circ\hat{x}^{-1}(x^1,...,x^N)\\ (x^1,...,x^N) &= \hat{x}\circ\hat{z}^{-1}(z^1,...,z^N)\end{aligned} \tag{44.23}$$

where $\hat{z}\circ\hat{x}^{-1}:\hat{x}(\mathscr{U})\to\hat{z}(\mathscr{U})$ and $\hat{x}\circ\hat{z}^{-1}:\hat{z}(\mathscr{U})\to\hat{x}(\mathscr{U})$ are diffeomorphisms. Equivalent versions of (44.23) are

$$\begin{aligned}z^j &= z^j(x^1,...,x^N) = z^j(x^k)\\ x^j &= x^j(z^1,...,z^N) = x^j(z^k)\end{aligned} \tag{44.24}$$

As an example of the above ideas, consider the *cylindrical coordinate system*[1]. In this case $N$=3 and equations (44.23) take the special form

$$(z^1, z^2, z^3) = (x^1 \cos x^2, x^1 \sin x^2, x^3) \tag{44.25}$$

In order for (44.25) to qualify as a coordinate transformation, it is necessary for the transformation functions to be $C^\infty$. It is apparent from (44.25) that $\hat{z} \circ \hat{x}^{-1}$ is $C^\infty$ on every open subset of $\mathscr{R}^3$. Also, by examination of (44.25), $\hat{z} \circ \hat{x}^{-1}$ is one-to-one if we restrict it to an appropriate domain, say $(0,\infty)\times(0,2\pi)\times(-\infty,\infty)$. The image of this subset of $\mathscr{R}^3$ under $\hat{z} \circ \hat{x}^{-1}$ is easily seen to be the set $\mathscr{R}^3 \big/ \left\{(z^1, z^2, z^3) \middle| z^1 \geq 0, z^2 = 0\right\}$ and the inverse transformation is

$$(x^1, x^2, x^3) = \left( \left[ (z^1)^2 + (z^2)^2 \right]^{1/2}, \tan^{-1} \frac{z^2}{z^1}, z^3 \right) \tag{44.26}$$

which is also $C^\infty$. Consequently we can choose the coordinate neighborhood to be any open subset $\mathscr{U}$ in $\mathscr{E}$ such that

$$\hat{z}(\mathscr{U}) \subset \mathscr{R}^3 \big/ \left\{(z^1, z^2, z^3) \middle| z^1 \geq 0, z^2 = 0\right\} \tag{44.27}$$

or, equivalently,

$$\hat{x}(\mathscr{U}) \subset (0,\infty)\times(0,2\pi)\times(-\infty,\infty)$$

Figure 9 describes the cylindrical system. The coordinate curves are a straight line (for $x^1$), a circle lying in a plane parallel to the $(z^1, z^2)$ plane (for $x^2$), and a straight line coincident with $z^3$ (for $x^3$). The coordinate surface $x^1 =$ const is a circular cylinder whose generators are the $z^3$ lines. The remaining coordinate surfaces are planes.

---

[1] The computer program Maple has a plot command, *coordplot3d*, that is useful when trying to visualize coordinate curves and coordinate surfaces. The program MATLAB will also produce useful and instructive plots.

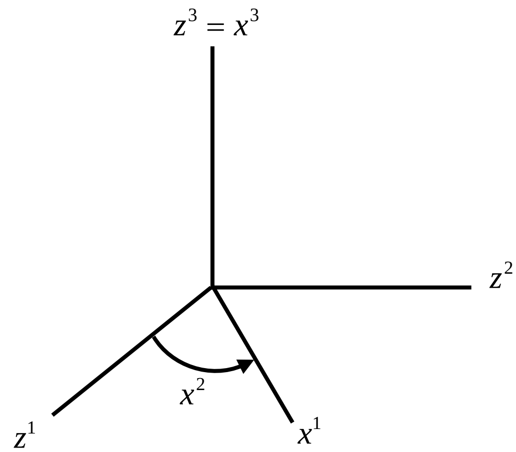


Figure 9

Returning to the general transformations (44.5) and (44.6), we can substitute the second into the first and differentiate the result to find

$$\frac{\partial y^i}{\partial x^k}(x^1,...,x^N)\frac{\partial x^k}{\partial y^j}(y^1,...,y^N) = \delta^i_j \tag{44.28}$$

By a similar argument with $x$ and $y$ interchanged,

$$\frac{\partial y^k}{\partial x^j}(x^1,...,x^N)\frac{\partial x^i}{\partial y^k}(y^1,...,y^N) = \delta^i_j \tag{44.29}$$

Each of these equations ensures that

$$\det\left[\frac{\partial y^i}{\partial x^k}(x^1,...,x^N)\right] = \frac{1}{\det\left[\dfrac{\partial x^j}{\partial y^l}(y^1,...,y^N)\right]} \neq 0 \tag{44.30}$$

For example, $\det\left[\left(\partial z^j/\partial x^k\right)\left(x^1,x^2,x^3\right)\right] = x^1 \neq 0$ for the cylindrical coordinate system. The determinant $\det\left[\left(\partial x^j/\partial y^k\right)\left(y^1,...,y^N\right)\right]$ is the *Jacobian* of the coordinate transformation (44.6).

Just as a vector space can be assigned an orientation, a Euclidean manifold can be oriented by assigning the orientation to its translation space $\mathscr{V}$. In this case $\mathscr{E}$ is called an *oriented Euclidean manifold*. In such a manifold we use Cartesian coordinate systems associated with positive basis only and these coordinate systems are called *positive*. A curvilinear coordinate system is positive if its coordinate transformation relative to a positive Cartesian coordinate system has a positive Jacobian.

Given a chart $(\mathscr{U}, \hat{x})$ for $\mathscr{E}$, we can compute the gradient of each coordinate function $\hat{x}^i$ and obtain a $C^\infty$ vector field on $\mathscr{U}$. We shall denote each of these fields by $\mathbf{g}^i$, namely

$$\mathbf{g}^i = \operatorname{grad} \hat{x}^i \tag{44.31}$$

for $i = 1, \dots, N$. From this definition, it is clear that $\mathbf{g}^i(\mathbf{x})$ is a vector in $\mathscr{V}$ *normal* to the $i^{\text{th}}$ coordinate surface. From (44.8) we can define $N$ vector fields $\mathbf{g}_1, \dots, \mathbf{g}_N$ on $\mathscr{U}$ by

$$\mathbf{g}_i = \left[\operatorname{grad} \tilde{\mathbf{x}}\right](0, \dots, \underset{i}{1}, \dots, 0) \tag{44.32}$$

Or, equivalently,

$$\mathbf{g}_i = \lim_{t \to 0} \frac{\tilde{\mathbf{x}}(x^1, \dots, x^i + t, \dots, x^N) - \tilde{\mathbf{x}}(x^1, \dots, x^N)}{t} \equiv \frac{\partial \tilde{\mathbf{x}}}{\partial x^i}(x^1, \dots, x^N) \tag{44.33}$$

for all $\mathbf{x} \in \mathscr{U}$. Equations (44.10) and (44.33) show that $\mathbf{g}_i(\mathbf{x})$ is *tangent* to the $i^{\text{th}}$ coordinate curve. Since

$$\hat{x}^i(\tilde{\mathbf{x}}(x^1, \dots, x^N)) = x^i \tag{44.34}$$

The chain rule along with the definitions (44.31) and (44.32) yield

$$\mathbf{g}^i(\mathbf{x}) \cdot \mathbf{g}_j(\mathbf{x}) = \delta^i_j \tag{44.35}$$

as they should.

The values of the vector fields $\{\mathbf{g}_1,...,\mathbf{g}_N\}$ form a linearly independent set of vectors $\{\mathbf{g}_1(\mathbf{x}),...,\mathbf{g}_N(\mathbf{x})\}$ at each $\mathbf{x}\in\mathscr{U}$. To see this assertion, assume $\mathbf{x}\in\mathscr{U}$ and

$$\lambda^1\mathbf{g}_1(\mathbf{x})+\lambda^2\mathbf{g}_2(\mathbf{x})+\cdots+\lambda^N\mathbf{g}_N(\mathbf{x})=\mathbf{0}$$

for $\lambda^1,...,\lambda^N\in\mathscr{R}$. Taking the inner product of this equation with $\mathbf{g}^j(\mathbf{x})$ and using equation (44.35) , we see that $\lambda^j=0$, $j=1,...,N$ which proves the assertion.

Because $\mathscr{V}$ has dimension *N*, $\{\mathbf{g}_1(\mathbf{x}),...,\mathbf{g}_N(\mathbf{x})\}$ forms a basis for $\mathscr{V}$ at each $\mathbf{x}\in\mathscr{U}$. Equation (44.35) shows that $\{\mathbf{g}^1(\mathbf{x}),...,\mathbf{g}^N(\mathbf{x})\}$ is the basis reciprocal to $\{\mathbf{g}_1(\mathbf{x}),...,\mathbf{g}_N(\mathbf{x})\}$. Because of the special geometric interpretation of the vectors $\{\mathbf{g}_1(\mathbf{x}),...,\mathbf{g}_N(\mathbf{x})\}$ and $\{\mathbf{g}^1(\mathbf{x}),...,\mathbf{g}^N(\mathbf{x})\}$ mentioned above, these bases are called the *natural bases* of $\hat{x}$ at $\mathbf{x}$. Any other basis field which cannot be determined by either (44.31) or (44.32) relative to any coordinate system is called an *anholonomic* or *nonintegrable* basis. The constant vector fields $\{\mathbf{i}_1,...,\mathbf{i}_N\}$ and $\{\mathbf{i}^1,...,\mathbf{i}^N\}$ yield the natural bases for the Cartesian coordinate systems.

If $(\mathscr{U}_1,\hat{x})$ and $(\mathscr{U}_2,\hat{y})$ are two charts such that $\mathscr{U}_1\cap\mathscr{U}_2\neq\varnothing$, we can determine the transformation rules for the changes of natural bases at $\mathbf{x}\in\mathscr{U}_1\cap\mathscr{U}_2$ in the following way: We shall let the vector fields $\mathbf{h}^j$, $j=1,...,N$, be defined by

$$\mathbf{h}^j=\operatorname{grad}\hat{y}^j \tag{44.36}$$

Then, from (44.5) and (44.31),

$$\begin{aligned}\mathbf{h}^j(\mathbf{x})=\operatorname{grad}\hat{y}^j(\mathbf{x})&=\frac{\partial y^j}{\partial x^i}(x^1,...,x^N)\operatorname{grad}\hat{x}^i(\mathbf{x})\\&=\frac{\partial y^j}{\partial x^i}(x^1,...,x^N)\mathbf{g}^i(\mathbf{x})\end{aligned}\tag{44.37}$$

for all $\mathbf{x}\in\mathscr{U}_1\cap\mathscr{U}_2$. A similar calculation shows that

$$\mathbf{h}_j(\mathbf{x}) = \frac{\partial x^i}{\partial y^j}(y^1,\ldots,y^N)\mathbf{g}_i(\mathbf{x}) \tag{44.38}$$

for all $\mathbf{x} \in \mathscr{U}_1 \cap \mathscr{U}_2$. Equations (44.37) and (44.38) are the desired transformations.

Given a chart $(\mathscr{U}_1, \hat{x})$, we can define $2N^2$ scalar fields $g_{ij} : \mathscr{U} \to \mathscr{R}$ and $g^{ij} : \mathscr{U} \to \mathscr{R}$ by

$$g_{ij}(\mathbf{x}) = \mathbf{g}_i(\mathbf{x}) \cdot \mathbf{g}_j(\mathbf{x}) = g_{ji}(\mathbf{x}) \tag{44.39}$$

and

$$g^{ij}(\mathbf{x}) = \mathbf{g}^i(\mathbf{x}) \cdot \mathbf{g}^j(\mathbf{x}) = g^{ji}(\mathbf{x}) \tag{44.40}$$

for all $\mathbf{x} \in \mathscr{U}$. It immediately follows from (44.35) that

$$\left[g^{ij}(\mathbf{x})\right] = \left[g_{ij}(\mathbf{x})\right]^{-1} \tag{44.41}$$

since we have

$$\mathbf{g}^i = g^{ij}\mathbf{g}_j \tag{44.42}$$

and

$$\mathbf{g}_j = g_{ji}\mathbf{g}^i \tag{44.43}$$

where the produce of the scalar fields with vector fields is defined by (44.17). If $\theta_{ij}$ is the angle between the $i^{\text{th}}$ and the $j^{\text{th}}$ coordinate curves at $\mathbf{x} \in \mathscr{U}$, then from (44.39)

$$\cos\theta_{ij} = \frac{g_{ij}(\mathbf{x})}{\left[g_{ii}(\mathbf{x})g_{jj}(\mathbf{x})\right]^{1/2}} \qquad \text{(no sum)} \tag{44.44}$$

Based upon (44.44), the curvilinear coordinate system is *orthogonal* if $g_{ij}=0$ when $i\neq j$. The symbol $g$ denotes a scalar field on $\mathscr{U}$ defined by

$$g(\mathbf{x})=\det\left[g_{ij}(\mathbf{x})\right] \tag{44.45}$$

for all $\mathbf{x}\in\mathscr{U}$.

At the point $\mathbf{x}\in\mathscr{U}$, the differential element of arc $ds$ is defined by

$$ds^2=d\mathbf{x}\cdot d\mathbf{x} \tag{44.46}$$

and, by (44.8), (44.33), and (44.39),

$$\begin{aligned} ds^2 &= \frac{\partial\tilde{\mathbf{x}}}{\partial x^i}(\hat{x}(\mathbf{x}))\cdot\frac{\partial\tilde{\mathbf{x}}}{\partial x^j}(\hat{x}(\mathbf{x}))dx^i dx^j \\ &= g_{ij}(\mathbf{x})dx^i dx^j \end{aligned} \tag{44.47}$$

If $(\mathscr{U}_1,\hat{x})$ and $(\mathscr{U}_2,\hat{y})$ are charts where $\mathscr{U}_1\cap\mathscr{U}_2\neq\varnothing$, then at $\mathbf{x}\in\mathscr{U}_1\cap\mathscr{U}_2$

$$\begin{aligned} h_{ij}(\mathbf{x}) &= \mathbf{h}_i(\mathbf{x})\cdot\mathbf{h}_j(\mathbf{x}) \\ &= \frac{\partial x^k}{\partial y^i}(y^1,\ldots,y^N)\frac{\partial x^l}{\partial y^j}(y^1,\ldots,y^N)g_{kl}(\mathbf{x}) \end{aligned} \tag{44.48}$$

Equation (44.48) is helpful for actual calculations of the quantities $g_{ij}(\mathbf{x})$. For example, for the transformation (44.23), (44.48) can be arranged to yield

$$g_{ij}(\mathbf{x})=\frac{\partial z^k}{\partial x^i}(x^1,\ldots,x^N)\frac{\partial z^k}{\partial x^j}(x^1,\ldots,x^N) \tag{44.49}$$

since $\mathbf{i}_k\cdot\mathbf{i}_l=\delta_{kl}$. For the cylindrical coordinate system defined by (44.25) a simple calculation based upon (44.49) yields

$$\left[g_{ij}(\mathbf{x})\right]=\begin{bmatrix}1 & 0 & 0\\ 0 & (x^1)^2 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{44.50}$$

Among other things, (44.50) shows that this coordinate system is orthogonal. By (44.50) and (44.41)

$$\left[g^{ij}(\mathbf{x})\right]=\begin{bmatrix}1 & 0 & 0\\ 0 & 1/(x^1)^2 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{44.51}$$

And, from (44.50) and (44.45),

$$g(\mathbf{x})=(x^1)^2 \tag{44.52}$$

It follows from (44.50) and (44.47) that

$$ds^2=(dx^1)^2+(x^1)^2(dx^2)^2+(dx^3)^2 \tag{44.53}$$

## *Exercises*

44.1 Show that

$$\mathbf{i}_k=\frac{\partial\tilde{\mathbf{x}}}{\partial \mathrm{z}^k} \qquad \text{and} \qquad \mathbf{i}^k=\operatorname{grad}\hat{z}^k$$

for any Cartesian coordinate system $\hat{z}$ associated with $\left\{\mathbf{i}^j\right\}$.

44.2 Show that

$$\mathbf{I}=\operatorname{grad}\mathbf{r}(\mathbf{x})$$

44.3 Spherical coordinates $(x^1, x^2, x^3)$ are defined by the coordinate transformation

$$
\begin{aligned}
z^1 &= x^1 \sin x^2 \cos x^3 \\
z^2 &= x^1 \sin x^2 \sin x^3 \\
z^3 &= x^1 \cos x^2
\end{aligned}
$$

relative to a rectangular Cartesian coordinate system $\hat{z}$. How must the quantity $(x^1, x^2, x^3)$ be restricted so as to make $\hat{z} \circ \hat{x}^{-1}$ one-to-one? Discuss the coordinate curves and the coordinate surfaces. Show that

$$
\left[ g_{ij}(\mathbf{x}) \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & (x^1 \sin x^2)^2 \end{bmatrix}
$$

44.4 Paraboloidal coordinates $(x^1, x^2, x^3)$ are defined by

$$
\begin{aligned}
z^1 &= x^1 x^2 \cos x^3 \\
z^2 &= x^1 x^2 \sin x^3 \\
z^3 &= \frac{1}{2}\left((x^1)^2 - (x^2)^2\right)
\end{aligned}
$$

Relative to a rectangular Cartesian coordinate system $\hat{z}$. How must the quantity $(x^1, x^2, x^3)$ be restricted so as to make $\hat{z} \circ \hat{x}^{-1}$ one-to-one. Discuss the coordinate curves and the coordinate surfaces. Show that

$$
\left[ g_{ij}(\mathbf{x}) \right] = \begin{bmatrix} (x^1)^2 + (x^2)^2 & 0 & 0 \\ 0 & (x^1)^2 + (x^2)^2 & 0 \\ 0 & 0 & (x^1 x^2)^2 \end{bmatrix}
$$

44.5 A bispherical coordinate system $(x^1, x^2, x^3)$ is defined relative to a rectangular Cartesian coordinate system by

$$z^1 = \frac{a \sin x^2 \cos x^3}{\cosh x^1 - \cos x^2}$$
$$z^2 = \frac{a \sin x^2 \sin x^3}{\cosh x^1 - \cos x^2}$$

and

$$z^3 = \frac{a \sinh x^1}{\cosh x^1 - \cos x^2}$$

where $a > 0$. How must $(x^1, x^2, x^3)$ be restricted so as to make $\hat{z} \circ \hat{x}^{-1}$ one-to-one? Discuss the coordinate curves and the coordinate surfaces. Also show that

$$\left[ g_{ij}(\mathbf{x}) \right] = \begin{bmatrix} \dfrac{a^2}{\left(\cosh x^1 - \cos x^2\right)^2} & 0 & 0 \\ 0 & \dfrac{a^2}{\left(\cosh x^1 - \cos x^2\right)^2} & 0 \\ 0 & 0 & \dfrac{a^2 (\sin x^2)^2}{\left(\cosh x^1 - \cos x^2\right)^2} \end{bmatrix}$$

44.6 Prolate spheroidal coordinates $(x^1, x^2, x^3)$ are defined by

$$z^1 = a \sinh x^1 \sin x^2 \cos x^3$$
$$z^2 = a \sinh x^1 \sin x^2 \sin x^3$$
$$z^3 = a \cosh x^1 \cos x^2$$

relative to a rectangular Cartesian coordinate system $\hat{z}$, where $a > 0$. How must $(x^1, x^2, x^3)$ be restricted so as to make $\hat{z} \circ \hat{x}^{-1}$ one-to-one? Also discuss the coordinate curves and the coordinate surfaces and show that

$$\left[g_{ij}(\mathbf{x})\right]=\begin{bmatrix} a^2\left(\cosh^2 x^1-\cos^2 x^2\right) & 0 & 0 \\ 0 & a^2\left(\cosh^2 x^1-\cos^2 x^2\right) & 0 \\ 0 & 0 & a^2\sinh^2 x^1 \sin^2 x^2 \end{bmatrix}$$

44.7 Elliptical cylindrical coordinates $(x^1,x^2,x^3)$ are defined relative to a rectangular Cartesian coordinate system by

$$\begin{aligned} z^1 &= a\cosh x^1 \cos x^2 \\ z^2 &= a\sinh x^1 \sin x^2 \\ z^3 &= x^3 \end{aligned}$$

where $a>0$. How must $(x^1,x^2,x^3)$ be restricted so as to make $\hat{z}\circ\hat{x}^{-1}$ one-to-one? Discuss the coordinate curves and coordinate surfaces. Also, show that

$$\left[g_{ij}(\mathbf{x})\right]=\begin{bmatrix} a^2\left(\sinh^2 x^1+\sin^2 x^2\right) & 0 & 0 \\ 0 & a^2\left(\sinh^2 x^1+\sin^2 x^2\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

44.8 For the cylindrical coordinate system show that

$$\begin{aligned} \mathbf{g}_1 &= (\cos x^2)\mathbf{i}_1+(\sin x^2)\mathbf{i}_2 \\ \mathbf{g}_2 &= -x^1(\sin x^2)\mathbf{i}_1+x^1(\cos x^2)\mathbf{i}_2 \\ \mathbf{g}_3 &= \mathbf{i}_3 \end{aligned}$$

44.9 At a point $\mathbf{x}$ in $\mathscr{E}$, the components of the position vector $\mathbf{r}(\mathbf{x})=\mathbf{x}-\mathbf{0}_{\mathscr{E}}$ with respect to the basis $\{\mathbf{i}_1,...,\mathbf{i}_N\}$ associated with a rectangular Cartesian coordinate system are $z^1,...,z^N$. This observation follows, of course, from (44.16). Compute the components of $\mathbf{r}(\mathbf{x})$ with respect to the basis $\{\mathbf{g}_1(\mathbf{x}),\mathbf{g}_2(\mathbf{x}),\mathbf{g}_3(\mathbf{x})\}$ for (a) cylindrical coordinates, (b) spherical coordinates, and (c) parabolic coordinates. You should find that

$$\mathbf{r}(\mathbf{x}) = x^1\mathbf{g}_1(\mathbf{x}) + x^3\mathbf{g}_3(\mathbf{x}) \qquad \text{for (a)}$$
$$\mathbf{r}(\mathbf{x}) = x^1\mathbf{g}_1(\mathbf{x}) \qquad \text{for (b)}$$
$$\mathbf{r}(\mathbf{x}) = \frac{1}{2}x^1\mathbf{g}_1(\mathbf{x}) + \frac{1}{2}x^2\mathbf{g}_2(\mathbf{x}) \qquad \text{for (c)}$$

44.10 Toroidal coordinates $(x^1, x^2, x^3)$ are defined relative to a rectangular Cartesian coordinate system by

$$z^1 = \frac{a\sinh x^1 \cos x^3}{\cosh x^1 - \cos x^2}$$
$$z^2 = \frac{a\sinh x^1 \sin x^3}{\cosh x^1 - \cos x^2}$$

and

$$z^3 = \frac{a\sin x^2}{\cosh x^1 - \cos x^2}$$

where $a > 0$. How must $(x^1, x^2, x^3)$ be restricted so as to make $\hat{z} \circ \hat{x}^{-1}$ one to one? Discuss the coordinate surfaces. Show that

$$\left[g_{ij}(\mathbf{x})\right] = \begin{bmatrix} \dfrac{a^2}{\left(\cosh x^1 - \cos x^2\right)^2} & 0 & 0 \\ 0 & \dfrac{a^2}{\left(\cosh x^1 - \cos x^2\right)^2} & 0 \\ 0 & 0 & \dfrac{a^2 \sinh^2 x^2}{\left(\cos x^1 - \cos x^2\right)^2} \end{bmatrix}$$

## *Section 45. Transformation Rules for Vectors and Tensor Fields*

In this section, we shall formalize certain ideas regarding fields on $\mathscr{E}$ and then investigate the transformation rules for vectors and tensor fields. Let $\mathscr{U}$ be an open subset of $\mathscr{E}$; we shall denote by $F^\infty(\mathscr{U})$ the set of $C^\infty$ functions $f:\mathscr{U}\to\mathscr{R}$. First we shall study the algebraic structure of $F^\infty(\mathscr{U})$. If $f_1$ and $f_2$ are in $F^\infty(\mathscr{U})$, then their sum $f_1+f_2$ is an element of $F^\infty(\mathscr{U})$ defined by

$$(f_1+f_2)(\mathbf{x}) = f_1(\mathbf{x})+f_2(\mathbf{x}) \tag{45.1}$$

and their produce $f_1 f_2$ is also an element of $F^\infty(\mathscr{U})$ defined by

$$(f_1 f_2)(\mathbf{x}) = f_1(\mathbf{x}) f_2(\mathbf{x}) \tag{45.2}$$

for all $\mathbf{x}\in\mathscr{U}$. For any real number $\lambda\in\mathscr{R}$ the *constant* function is defined by

$$\lambda(\mathbf{x}) = \lambda \tag{45.3}$$

for all $\mathbf{x}\in\mathscr{U}$. For simplicity, the function and the value in (45.3) are indicated by the same symbol. Thus, the zero function in $F^\infty(\mathscr{U})$ is denoted simply by $0$ and for every $f\in F^\infty(\mathscr{U})$

$$f+0=f \tag{45.4}$$

It is also apparent that

$$1f = f \tag{45.5}$$

In addition, we define

$$-f = (-1)f \tag{45.6}$$

It is easily shown that the operations of addition and multiplication obey commutative, associative, and distributive laws. These facts show that $F^{\infty}(\mathscr{U})$ is a commutative ring (see Section 7).

An important collection of scalar fields can be constructed as follows: Given two charts $(\mathscr{U}_1,\hat{x})$ and $(\mathscr{U}_2,\hat{y})$, where $\mathscr{U}_1\cap\mathscr{U}_2\neq\varnothing$, we define the $N^2$ partial derivatives $\left(\partial y^i/\partial x^j\right)\left(x^1,...,x^N\right)$ at every $\left(x^1,...,x^N\right)\in\hat{x}\left(\mathscr{U}_1\cap\mathscr{U}_2\right)$. Using a suggestive notation, we can define $N^2$ $C^{\infty}$ functions $\partial y^i/\partial x^j:\mathscr{U}_1\cap\mathscr{U}_2\to\mathscr{R}$ by

$$\frac{\partial y^i}{\partial x^j}(\mathbf{x})=\frac{\partial y^i}{\partial x^j}\circ\hat{x}(\mathbf{x}) \tag{45.7}$$

for all $\mathbf{x}\in\mathscr{U}_1\cap\mathscr{U}_2$.

As mentioned earlier, a $C^{\infty}$ vector field on an open set $\mathscr{U}$ of $\mathscr{E}$ is a $C^{\infty}$ map $\mathbf{v}:\mathscr{U}\to\mathscr{V}$, where $\mathscr{V}$ is the translation space of $\mathscr{E}$. The fields defined by (44.31) and (44.32) are special cases of vector fields. We can express $\mathbf{v}$ in component forms on $\mathscr{U}_1\cap\mathscr{U}_2$,

$$\mathbf{v}=\upsilon^i\mathbf{g}_i=\upsilon_j\mathbf{g}^j \tag{45.8}$$

$$\upsilon_j=g_{ji}\upsilon^i \tag{45.9}$$

As usual, we can computer $\upsilon^i:\mathscr{U}\cap\mathscr{U}_1\to\mathscr{R}$ by

$$\upsilon^i(\mathbf{x})=\mathbf{v}(\mathbf{x})\cdot\mathbf{g}^i(\mathbf{x}),\qquad\qquad \mathbf{x}\in\mathscr{U}_1\cap\mathscr{U}_2 \tag{45.10}$$

and $\upsilon_i$ by (45.9). In particular, if $(\mathscr{U}_2,\hat{y})$ is another chart such that $\mathscr{U}_1\cap\mathscr{U}_2\neq\varnothing$, then the component form of $\mathbf{h}_j$ relative to $\hat{x}$ is

$$\mathbf{h}_j=\frac{\partial x^i}{\partial y^j}\mathbf{g}_i \tag{45.11}$$

With respect to the chart $(\mathscr{U}_2, \hat{y})$, we have also

$$\mathbf{v} = \bar{\upsilon}^k \mathbf{h}_k \tag{45.12}$$

where $\bar{\upsilon}^k : \mathscr{U} \cap \mathscr{U}_2 \to \mathscr{R}$. From (45.12), (45.11), and (45.8), the transformation rule for the components of $\mathbf{v}$ relative to the two charts $(\mathscr{U}_1, \hat{x})$ and $(\mathscr{U}_2, \hat{y})$ is

$$\upsilon^i = \frac{\partial x^i}{\partial y^j} \bar{\upsilon}^j \tag{45.13}$$

for all $\mathbf{x} \in \mathscr{U}_1 \cap \mathscr{U}_2 \cap \mathscr{U}$.

As in (44.18), we can define an inner product operation between vector fields. If $\mathbf{v}_1 : \mathscr{U}_1 \to \mathscr{V}$ and $\mathbf{v}_2 : \mathscr{U}_2 \to \mathscr{V}$ are vector fields, then $\mathbf{v}_1 \cdot \mathbf{v}_2$ is a scalar field defined on $\mathscr{U}_1 \cap \mathscr{U}_2$ by

$$\mathbf{v}_1 \cdot \mathbf{v}_2(\mathbf{x}) = \mathbf{v}_1(\mathbf{x}) \cdot \mathbf{v}_2(\mathbf{x}), \qquad \mathbf{x} \in \mathscr{U}_1 \cap \mathscr{U}_2 \tag{45.14}$$

Then (45.10) can be written

$$\upsilon^i = \mathbf{v} \cdot \mathbf{g}^i \tag{45.15}$$

Now let us consider tensor fields in general. Let $T_q^\infty(\mathscr{U})$ denote the set of all tensor fields of order $q$ defined on an open set $\mathscr{U}$ in $\mathscr{E}$. As with the set $F^\infty(\mathscr{U})$, the set $T_q^\infty(\mathscr{U})$ can be assigned an algebraic structure. The *sum* of $\mathbf{A} : \mathscr{U} \to \mathscr{T}_q(\mathscr{V})$ and $\mathbf{B} : \mathscr{U} \to \mathscr{T}_q(\mathscr{V})$ is a $C^\infty$ tensor field $\mathbf{A} + \mathbf{B} : \mathscr{U} \to \mathscr{T}_q(\mathscr{V})$ defined by

$$(\mathbf{A} + \mathbf{B})(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \mathbf{B}(\mathbf{x}) \tag{45.16}$$

for all $\mathbf{x} \in \mathscr{U}$. If $f \in F^\infty(\mathscr{U})$ and $\mathbf{A} \in T_q^\infty(\mathscr{U})$, then we can define $f\mathbf{A} \in T_q^\infty(\mathscr{U})$ by

$$f\mathbf{A}(\mathbf{x}) = f(\mathbf{x})\mathbf{A}(\mathbf{x}) \tag{45.17}$$

Clearly this multiplication operation satisfies the usual associative and distributive laws with respect to the sum for all $\mathbf{x} \in \mathscr{U}$. As with $F^\infty(\mathscr{U})$, constant tensor fields in $T_q^\infty(\mathscr{U})$ are given the same symbol as their value. For example, the *zero* tensor field is $\mathbf{0}: \mathscr{U} \to \mathscr{T}_q(\mathscr{V})$ and is defined by

$$\mathbf{0}(\mathbf{x}) = \mathbf{0} \tag{45.18}$$

for all $\mathbf{x} \in \mathscr{U}$. If 1 is the constant function in $F^\infty(\mathscr{U})$, then

$$-\mathbf{A} = (-1)\mathbf{A} \tag{45.19}$$

The algebraic structure on the set $T_q^\infty(\mathscr{V})$ just defined is called a module over the ring $F^\infty(\mathscr{U})$.

The *components* of a tensor field $\mathbf{A}: \mathscr{U} \to \mathscr{T}_q(\mathscr{V})$ with respect to a chart $(\mathscr{U}_1, \hat{x})$ are the $N^q$ scalar fields $A_{i_1 \ldots i_q}: \mathscr{U} \cap \mathscr{U}_1 \to \mathscr{R}$ defined by

$$A_{i_1 \ldots i_q}(\mathbf{x}) = \mathbf{A}(\mathbf{x})(\mathbf{g}_{i_1}(\mathbf{x}), \ldots, \mathbf{g}_{i_q}(\mathbf{x})) \tag{45.20}$$

for all $\mathbf{x} \in \mathscr{U} \cap \mathscr{U}_1$. Clearly we can regard tensor fields as multilinear mappings on vector fields with values as scalar fields. For example, $\mathbf{A}(\mathbf{g}_{i_1}, \ldots, \mathbf{g}_{i_q})$ is a scalar field defined by

$$\mathbf{A}(\mathbf{g}_{i_1}, \ldots, \mathbf{g}_{i_q})(\mathbf{x}) = \mathbf{A}(\mathbf{x})(\mathbf{g}_{i_1}(\mathbf{x}), \ldots, \mathbf{g}_{i_q}(\mathbf{x}))$$

for all $\mathbf{x} \in \mathscr{U} \cap \mathscr{U}_1$. In fact we can, and shall, carry over to tensor fields the many algebraic operations previously applied to tensors. In particular a tensor field $\mathbf{A}: \mathscr{U} \to \mathscr{T}_q(\mathscr{V})$ has the representation

$$\mathbf{A} = A_{i_1 \ldots i_q} \mathbf{g}^{i_1} \otimes \cdots \otimes \mathbf{g}^{i_q} \tag{45.21}$$

for all $\mathbf{x} \in \mathscr{U} \cap \mathscr{U}_1$, where $\mathscr{U}_1$ is the coordinate neighborhood for a chart $(\mathscr{U}_1, \hat{x})$. The scalar fields $A_{i_1 \ldots i_q}$ are the *covariant components* of $\mathsf{A}$ and under a change of coordinates obey the transformation rule

$$\bar{A}_{k_1 \ldots k_q} = \frac{\partial x^{i_1}}{\partial y^{k_1}} \cdots \frac{\partial x^{i_q}}{\partial y^{k_q}} A_{i_1 \ldots i_q} \tag{45.22}$$

Equation (45.22) is a relationship among the component fields and holds at all points $\mathbf{x} \in \mathscr{U}_1 \cap \mathscr{U}_2 \cap \mathscr{U}$ where the charts involved are $(\mathscr{U}_1, \hat{x})$ and $(\mathscr{U}_2, \hat{y})$. We encountered an example of (45.22) earlier with (44.48). Equation (44.48) shows that the $g_{ij}$ are the covariant components of a tensor field $\mathbf{I}$ whose value is the identity or metric tensor, namely

$$\mathbf{I} = g_{ij} \mathbf{g}^i \otimes \mathbf{g}^j = \mathbf{g}_j \otimes \mathbf{g}^j = \mathbf{g}^j \otimes \mathbf{g}_j = g^{ij} \mathbf{g}_i \otimes \mathbf{g}_j \tag{45.23}$$

for points $\mathbf{x} \in \mathscr{U}_1$, where the chart in question is $(\mathscr{U}_1, \hat{x})$. Equations (45.23) show that the components of a constant tensor field are not necessarily constant scalar fields. It is only in Cartesian coordinates that constant tensor fields have constant components.

Another important tensor field is the one constructed from the positive unit volume tensor $\mathsf{E}$. With respect to an orthonormal basis $\{\mathbf{i}_j\}$, which has positive orientation, $\mathsf{E}$ is given by (41.6), i.e.,

$$\mathsf{E} = \mathbf{i}_1 \wedge \cdots \wedge \mathbf{i}_N = \varepsilon_{i_1 \cdots i_N} \mathbf{i}_{i_1} \otimes \cdots \otimes \mathbf{i}_{i_N} \tag{45.24}$$

Given this tensor, we define as usual a constant tensor field $\mathsf{E} : \mathscr{E} \to \hat{\mathscr{T}}_N(\mathscr{V})$ by

$$\mathsf{E}(\mathbf{x}) = \mathsf{E} \tag{45.25}$$

for all $\mathbf{x} \in \mathscr{E}$. With respect to a chart $(\mathscr{U}_1, \hat{x})$, it follows from the general formula (42.27) that

$$\mathsf{E} = E_{i_1 \cdots i_N} \mathbf{g}^{i_1} \otimes \cdots \otimes \mathbf{g}^{i_N} = E^{i_1 \cdots i_N} \mathbf{g}_{i_1} \otimes \cdots \otimes \mathbf{g}_{i_N} \tag{45.26}$$

where $E_{i_1 \cdots i_N}$ and $E^{i_1 \cdots i_N}$ are scalar fields on $\mathscr{U}_1$ defined by

$$E_{i_1 \cdots i_N} = e\sqrt{g}\,\varepsilon_{i_1 \cdots i_N} \tag{45.27}$$

and

$$E^{i_1 \cdots i_N} = \frac{e}{\sqrt{g}}\varepsilon^{i_1 \cdots i_N} \tag{45.28}$$

where, as in Section 42, $e$ is $+1$ if $\{\mathbf{g}_i(\mathbf{x})\}$ is positively oriented and $-1$ if $\{\mathbf{g}_i(\mathbf{x})\}$ is negatively oriented, and where $g$ is the determinant of $\left[g_{ij}\right]$ as defined by (44.45). By application of (42.28), it follows that

$$E^{i_1 \cdots i_N} = g^{i_1 j_1} \cdots g^{i_N j_N} E_{j_1 \cdots j_N} \tag{45.29}$$

An interesting application of the formulas derived thus far is the derivation of an expression for the differential element of volume in curvilinear coordinates. Given the position vector $\mathbf{r}$ defined by (44.14) and a chart $(\mathscr{U}_1, \hat{x})$, the differential of $\mathbf{r}$ can be written

$$d\mathbf{r} = d\mathbf{x} = \mathbf{g}_i(\mathbf{x})dx^i \tag{45.30}$$

where (44.33) has been used. Given $N$ differentials of $\mathbf{r}$, $d\mathbf{r}_1, d\mathbf{r}_2, \cdots, d\mathbf{r}_N$, the differential volume element $d\upsilon$ generated by them is defined by

$$d\upsilon = \left|\mathbf{E}\left(d\mathbf{r}_1, \cdots, d\mathbf{r}_N\right)\right| \tag{45.31}$$

or, equivalently,

$$d\upsilon = \left|d\mathbf{r}_1 \wedge \cdots \wedge d\mathbf{r}_N\right| \tag{45.32}$$

If we select $d\mathbf{r}_1 = \mathbf{g}_1(\mathbf{x})dx^1, d\mathbf{r}_2 = \mathbf{g}_2(\mathbf{x})dx^2, \cdots, d\mathbf{r}_N = \mathbf{g}_N(\mathbf{x})dx^N$, we can write (45.31)) as

$$d\upsilon = \left| \mathbf{E}\left(\mathbf{g}_1(\mathbf{x}), \cdots, \mathbf{g}_N(\mathbf{x})\right) dx^1 dx^2 \cdots dx^N \right| \tag{45.33}$$

By use of (45.26) and (45.27), we then get

$$\mathbf{E}\left(\mathbf{g}_1(\mathbf{x}), \cdots, \mathbf{g}_N(\mathbf{x})\right) = E_{12\cdots N} = e\sqrt{g} \tag{45.34}$$

Therefore,

$$d\upsilon = \left| \sqrt{g}\, dx^1 dx^2 \cdots dx^N \right| \tag{45.35}$$

For example, in the parabolic coordinates mentioned in Exercise 44.4,

$$d\upsilon = \left| \left( \left(x^1\right)^2 + \left(x^2\right)^2 \right) x^1 x^2 dx^1 dx^2 dx^3 \right| \tag{45.36}$$

## *Exercises*

45.1 Let $\mathbf{v}$ be a $C^\infty$ vector field and $f$ be a $C^\infty$ function both defined on $\mathscr{U}$ an open set in $\mathscr{E}$. We define $\mathbf{v} \circ f : \mathscr{U} \to \mathscr{R}$ by

$$\mathbf{v} \circ f(\mathbf{x}) = \mathbf{v}(\mathbf{x}) \cdot \operatorname{grad} f(\mathbf{x}), \qquad \mathbf{x} \in \mathscr{U} \tag{45.37}$$

Show that

$$\mathbf{v} \circ \left(\lambda f + \mu g\right) = \lambda\left(\mathbf{v} \circ f\right) + \mu\left(\mathbf{v} \circ g\right)$$

and

$$\mathbf{v}\circ(fg) = (\mathbf{v}\circ f)g + f(\mathbf{v}\circ g)$$

For all constant functions $\lambda$, $\mu$ and all $C^\infty$ functions $f$ and $g$. In differential geometry, an operator on $F^\infty(\mathscr{U})$ with the above properties is called a *derivation*. Show that, conversely, every derivation on $F^\infty(\mathscr{U})$ corresponds to a unique vector field on $\mathscr{U}$ by (45.37).

45.2 By use of the definition (45.37), the *Lie bracket* of two vector fields $\mathbf{v}:\mathscr{U}\to\mathscr{V}$ and $\mathbf{u}:\mathscr{U}\to\mathscr{V}$, written $[\mathbf{u},\mathbf{v}]$, is a vector field defined by

$$[\mathbf{u},\mathbf{v}]\circ f = \mathbf{u}\circ(\mathbf{v}\circ f) - \mathbf{v}\circ(\mathbf{u}\circ f) \tag{45.38}$$

For all scalar fields $f\in F^\infty(\mathscr{U})$. Show that $[\mathbf{u},\mathbf{v}]$ is well defined by verifying that (45.38) defines a derivation on $[\mathbf{u},\mathbf{v}]$. Also, show that

$$[\mathbf{u},\mathbf{v}] = (\operatorname{grad}\mathbf{v})\mathbf{u} - (\operatorname{grad}\mathbf{u})\mathbf{v}$$

and then establish the following results:

(a) $[\mathbf{u},\mathbf{v}] = -[\mathbf{v},\mathbf{u}]$

(b) $[\mathbf{v},[\mathbf{u},\mathbf{w}]] + [\mathbf{u},[\mathbf{w},\mathbf{v}]] + [\mathbf{w},[\mathbf{v},\mathbf{u}]] = \mathbf{0}$

for all vector fields $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$.

(c) Let $(\mathscr{U}_1,\hat{x})$ be a chart with natural basis field $\{\mathbf{g}_i\}$. Show that $[\mathbf{g}_i,\mathbf{g}_j] = \mathbf{0}$.

The results (a) and (b) are known as *Jacobi's identities*.

45.3 In a three-dimensional Euclidean space the differential element of area normal to the plan formed from $d\mathbf{r}_1$ and $d\mathbf{r}_2$ is defined by

$$d\boldsymbol{\sigma} = d\mathbf{r}_1 \times d\mathbf{r}_2$$

Show that

$$d\boldsymbol{\sigma} = E_{ijk}\,dx_1^j\,dx_2^k\,\mathbf{g}^i(\mathbf{x})$$

## *Section 46. Anholonomic and Physical Components of Tensors*

In many applications, the components of interest are not always the components with respect to the natural basis fields $\{\mathbf{g}_i\}$ and $\{\mathbf{g}^j\}$. For definiteness let us call the components of a tensor field $\mathsf{A} \in T_q^\infty(\mathscr{U})$ is defined by (45.20) the *holonomic components* of $\mathsf{A}$. In this section, we shall consider briefly the concept of the *anholonomic components* of $\mathsf{A}$; i.e., the components of $\mathsf{A}$ taken with respect to an anholonomic basis of vector fields. The concept of the *physical components* of a tensor field is a special case and will also be discussed.

Let $\mathscr{U}_1$ be an open set in $\mathscr{E}$ and let $\{\mathbf{e}_a\}$ denote a set of $N$ vectors fields on $\mathscr{U}_1$, which are linearly independent, i.e., at each $\mathbf{x} \in \mathscr{U}_1$, $\{\mathbf{e}_a\}$ is a basis of $\mathscr{V}$. If $\mathsf{A}$ is a tensor field in $T_q^\infty(\mathscr{U})$ where $\mathscr{U}_1 \cap \mathscr{U} \neq \varnothing$, then by the same type of argument as used in Section 45, we can write

$$\mathsf{A} = A_{a_1 a_2 \ldots a_q} \mathbf{e}^{a_1} \otimes \cdots \otimes \mathbf{e}^{a_q} \tag{46.1}$$

or, for example,

$$\mathsf{A} = A^{b_1 \ldots b_q} \mathbf{e}_{b_1} \otimes \cdots \otimes \mathbf{e}_{b_q} \tag{46.2}$$

where $\{\mathbf{e}^a\}$ is the reciprocal basis field to $\{\mathbf{e}_a\}$ defined by

$$\mathbf{e}^a(\mathbf{x}) \cdot \mathbf{e}_b(\mathbf{x}) = \delta_b^a \tag{46.3}$$

for all $\mathbf{x} \in \mathscr{U}_1$. Equations (46.1) and (46.2) hold on $\mathscr{U} \cap \mathscr{U}_1$, and the component fields as defined by

$$A_{a_1 a_2 \ldots a_q} = \mathsf{A}(\mathbf{e}_{a_1}, \ldots, \mathbf{e}_{a_q}) \tag{46.4}$$

and

$$A^{b_1 \ldots b_q} = \mathsf{A}(\mathbf{e}^{b_1}, \ldots, \mathbf{e}^{b_q}) \tag{46.5}$$

are scalar fields on $\mathscr{U} \cap \mathscr{U}_1$. These fields are the *anholonomic components* of $\mathbf{A}$ when the bases $\{\mathbf{e}^a\}$ and $\{\mathbf{e}_a\}$ are not the natural bases of any coordinate system.

Given a set of $N$ vector fields $\{\mathbf{e}_a\}$ as above, one can show that a necessary and sufficient condition for $\{\mathbf{e}_a\}$ to be the natural basis field of some chart is

$$[\mathbf{e}_a, \mathbf{e}_b] = \mathbf{0}$$

for all $a, b = 1, ..., N$, where the bracket product is defined in Exercise 45.2. We shall prove this important result in Section 49. Formulas which generalize (45.22) to anholonomic components can easily be derived. If $\{\hat{\mathbf{e}}_a\}$ is an anholonomic basis field defined on an open set $\mathscr{U}_2$ such that $\mathscr{U}_1 \cap \mathscr{U}_2 \neq \varnothing$, then we can express each vector field $\hat{\mathbf{e}}_b$ in anholonomic component form relative to the basis $\{\mathbf{e}_a\}$, namely

$$\hat{\mathbf{e}}_b = T_b^a \mathbf{e}_a \tag{46.6}$$

where the $T_b^a$ are scalar fields on $\mathscr{U}_1 \cap \mathscr{U}_2$ defined by

$$T_b^a = \hat{\mathbf{e}}_b \cdot \mathbf{e}^a$$

The inverse of (46.6) can be written

$$\mathbf{e}_a = \hat{T}_a^b \hat{\mathbf{e}}_b \tag{46.7}$$

where

$$\hat{T}_a^b(\mathbf{x}) T_c^a(\mathbf{x}) = \delta_c^b \tag{46.8}$$

and

$$T^a_c(\mathbf{x})\hat{T}^c_b(\mathbf{x}) = \delta^a_b \tag{46.9}$$

for all $\mathbf{x} \in \mathscr{U}_1 \cap \mathscr{U}_2$. It follows from (46.4) and (46.7) that

$$A_{a_1 \ldots a_q} = \hat{T}^{b_1}_{a_1} \cdots \hat{T}^{b_q}_{a_q} \hat{A}_{b_1 \ldots b_q} \tag{46.10}$$

where

$$\hat{A}_{b_1 \ldots b_q} = \mathsf{A}(\hat{\mathbf{e}}_{b_1}, \ldots, \hat{\mathbf{e}}_{b_q}) \tag{46.11}$$

Equation (46.10) is the transformation rule for the anholonomic components of $\mathsf{A}$. Of course, (46.10) is a field equation which holds at every point of $\mathscr{U}_1 \cap \mathscr{U}_2 \cap \mathscr{U}$. Similar transformation rules for the other components of $\mathsf{A}$ can easily be derived by the same type of argument used above.

We define the *physical components* of $\mathsf{A}$, denoted by $A_{\langle a_1, \ldots, a_q \rangle}$, to be the anholonomic components of $\mathsf{A}$ relative to the field of orthonomal basis $\{\mathbf{g}_{\langle i \rangle}\}$ whose basis vectors $\mathbf{g}_{\langle i \rangle}$ are unit vectors in the direction of the natural basis vectors $\mathbf{g}_i$ of an orthogonal coordinate system. Let $(\mathscr{U}_1, \hat{x})$ by such a coordinate system with $g_{ij} = 0,\ i \neq j$. Then we define

$$\mathbf{g}_{\langle i \rangle}(\mathbf{x}) = \mathbf{g}_i(\mathbf{x}) / \|\mathbf{g}_i(\mathbf{x})\| \qquad \text{(no sum)} \tag{46.12}$$

at every $\mathbf{x} \in \mathscr{U}_1$. By (44.39), an equivalent version of (46.12) is

$$\mathbf{g}_{\langle i \rangle}(\mathbf{x}) = \mathbf{g}_i(\mathbf{x}) / (g_{ii}(\mathbf{x}))^{1/2} \qquad \text{(no sum)} \tag{46.13}$$

Since $\{\mathbf{g}_i\}$ is orthogonal, it follows from (46.13) and (44.39) that $\{\mathbf{g}_{\langle i \rangle}\}$ is orthonormal:

$$\mathbf{g}_{\langle a \rangle} \cdot \mathbf{g}_{\langle b \rangle} = \delta_{ab} \tag{46.14}$$

as it should, and it follows from (44.41) that

$$\left[g^{ij}(\mathbf{x})\right]=\begin{bmatrix} 1/g_{11}(\mathbf{x}) & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1/g_{22}(\mathbf{x}) & & & & 0 \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1/g_{NN}(\mathbf{x}) \end{bmatrix} \tag{46.15}$$

This result shows that $\mathbf{g}_{\langle i \rangle}$ can also be written

$$\mathbf{g}_{\langle i \rangle}(\mathbf{x})=\frac{\mathbf{g}^{i}(\mathbf{x})}{(g^{ii}(\mathbf{x}))^{1/2}}=(g_{ii}(\mathbf{x}))^{1/2}\mathbf{g}^{i}(\mathbf{x}) \qquad \text{(no sum)} \tag{46.16}$$

Equation (46.13) can be viewed as a special case of (46.7), where

$$\left[\hat{T}_{a}^{b}\right]=\begin{bmatrix} 1/\sqrt{g_{11}} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1/\sqrt{g_{22}} & & & & 0 \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1/\sqrt{g_{NN}} \end{bmatrix} \tag{46.17}$$

By the transformation rule (46.10), the physical components of $\mathbf{A}$ are related to the covariant components of $\mathbf{A}$ by

$$A_{\langle a_1 a_2 \ldots a_q \rangle} \equiv \mathbf{A}(\mathbf{g}_{\langle a_1 \rangle},\mathbf{g}_{\langle a_2 \rangle},\ldots,\mathbf{g}_{\langle a_q \rangle})=\left(g_{a_1 a_1}\cdots g_{a_q a_q}\right)^{-1/2} A_{a_1 \ldots a_q} \qquad \text{(no sum)} \tag{46.18}$$

Equation (46.18) is a field equation which holds for all $\mathbf{x}\in\mathscr{U}$. Since the coordinate system is orthogonal, we can replace $(46.18)_1$ with several equivalent formulas as follows:

$$
\begin{aligned}
A_{\langle a_1 a_2 \ldots a_q \rangle} &= \left( g_{a_1 a_1} \cdots g_{a_q a_q} \right)^{1/2} A^{a_1 \ldots a_q} \\
&= \left( g_{a_1 a_1} \right)^{-1/2} \left( g_{a_2 a_2} \cdots g_{a_q a_q} \right)^{1/2} A_{a_1}{}^{a_2 \ldots a_q} \\
&\ \cdot \\
&\ \cdot \\
&\ \cdot \\
&= \left( g_{a_1 a_1} \cdots g_{a_{q-1} a_{q-1}} \right)^{-1/2} \left( g_{a_q a_q} \right)^{1/2} A_{a_1 \ldots a_{q-1}}{}^{a_q}
\end{aligned}
\tag{46.19}
$$

In mathematical physics, tensor fields often arise naturally in component forms relative to product bases associated with several bases. For example, if $\{\mathbf{e}_a\}$ and $\{\hat{\mathbf{e}}_b\}$ are fields of bases, possibly anholonomic, then it might be convenient to express a second-order tensor field $\mathbf{A}$ as a field of linear transformations such that

$$
\mathbf{A}\mathbf{e}_a = A^{\hat{b}}{}_a \hat{\mathbf{e}}_b, \qquad a = 1, \ldots, N \tag{46.20}
$$

In this case $\mathbf{A}$ has naturally the component form

$$
\mathbf{A} = A^{\hat{b}}{}_a \hat{\mathbf{e}}_b \otimes \mathbf{e}^a \tag{46.21}
$$

Relative to the product basis $\{\hat{\mathbf{e}}_b \otimes \mathbf{e}^a\}$ formed by $\{\hat{\mathbf{e}}_b\}$ and $\{\mathbf{e}^a\}$, the latter being the reciprocal basis of $\{\mathbf{e}_a\}$ as usual. For definiteness, we call $\{\hat{\mathbf{e}}_b \otimes \mathbf{e}^a\}$ a *composite product* basis associated with the bases $\{\hat{\mathbf{e}}_b\}$ and $\{\mathbf{e}^a\}$. Then the scalar fields $A^{\hat{b}}{}_a$ defined by (46.20) or (46.21), may be called the *composite components* of $\mathbf{A}$, and they are given by

$$
A^{\hat{b}}{}_a = \mathbf{A}(\hat{\mathbf{e}}^b \otimes \mathbf{e}_a) \tag{46.22}
$$

Similarly we may define other types of composite components, e.g.,

$$
A_{\hat{b}a} = \mathbf{A}(\hat{\mathbf{e}}_b, \mathbf{e}_a), \qquad A^{\hat{b}a} = \mathbf{A}(\hat{\mathbf{e}}^b, \mathbf{e}^a) \tag{46.23}
$$

etc., and these components are related to one another by

$$A_{\hat{b}a} = A^{\hat{c}}{}_{a}\hat{g}_{\hat{c}\hat{b}} = A_{\hat{b}}{}^{c}g_{ca} = A^{\hat{c}d}\hat{g}_{\hat{b}\hat{c}}g_{da} \tag{46.24}$$

etc. Further, the composite components are related to the regular tensor components associated with a single basis field by

$$A_{\hat{b}a} = A_{ca}T^{c}_{\hat{b}} = A_{\hat{b}\hat{c}}\hat{T}^{\hat{c}}_{a}, \qquad \text{etc.} \tag{46.25}$$

where $T^{a}_{\hat{b}}$ and $\hat{T}^{\hat{a}}_{b}$ are given by (46.6) and (46.7) as before, In the special case where $\{\mathbf{e}_a\}$ and $\{\hat{\mathbf{e}}_b\}$ are orthogonal but not orthonormal, we define the normalized basis vectors $\mathbf{e}_{\langle a\rangle}$ and $\hat{\mathbf{e}}_{\langle a\rangle}$ as before. Then the *composite physical components* $A_{\langle \hat{b},a\rangle}$ of $\mathbf{A}$ are given by

$$A_{\langle \hat{b},a\rangle} = \mathbf{A}(\hat{\mathbf{e}}_{\langle \hat{b}\rangle}, \mathbf{e}_{\langle a\rangle}) \tag{46.26}$$

and these are related to the composite components by

$$A_{\langle \hat{b},a\rangle} = \left(\hat{g}_{\hat{b}\hat{b}}\right)^{1/2} A^{\hat{b}}{}_{a}\left(g^{aa}\right)^{1/2} \qquad \text{(no sum)} \tag{46.27}$$

Clearly the concepts of composite components and composite physical components can be defined for higher order tensors also.

## *Exercises*

46.1 In a three-dimensional Euclidean space the covariant components of a tensor field $\mathbf{A}$ relative to the cylindrical coordinate system are $A_{ij}$. Determine the physical components of $\mathbf{A}$.

46.2 Relative to the cylindrical coordinate system the *helical basis* $\{\mathbf{e}_a\}$ has the component form

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{g}_{\langle 1\rangle} \\ \mathbf{e}_2 &= (\cos\alpha)\mathbf{g}_{\langle 2\rangle} + (\sin\alpha)\mathbf{g}_{\langle 3\rangle} \\ \mathbf{e}_3 &= -(\sin\alpha)\mathbf{g}_{\langle 2\rangle} + (\cos\alpha)\mathbf{g}_{\langle 3\rangle} \end{aligned} \tag{46.28}$$

where $\alpha$ is a constant called the *pitch*, and where $\{\mathbf{g}_{\langle\alpha\rangle}\}$ is the orthonormal basis associated with the natural basis of the cylindrical system. Show that $\{\mathbf{e}_a\}$ is anholonomic. Determine the anholonomic components of the tensor field $\mathbf{A}$ in the preceding exercise relative to the helical basis.

46.3 Determine the composite physical components of $\mathbf{A}$ relative to the composite product basis $\{\mathbf{e}_{\langle a\rangle} \otimes \mathbf{g}_{\langle b\rangle}\}$

## *Section 47. Christoffel Symbols and Covariant Differentiation*

In this section we shall investigate the problem of representing the gradient of various tensor fields in components relative to the natural basis of arbitrary coordinate systems. We consider first the simple case of representing the tangent of a smooth curve in $\mathscr{E}$. Let $\boldsymbol{\lambda}:(a,b)\rightarrow\mathscr{E}$ be a smooth curve passing through a point $\mathbf{x}$, say $\mathbf{x}=\boldsymbol{\lambda}(c)$. Then the tangent vector of $\boldsymbol{\lambda}$ at $\mathbf{x}$ is defined by

$$\dot{\boldsymbol{\lambda}}\Big|_{\mathbf{x}}=\frac{d\boldsymbol{\lambda}(t)}{dt}\bigg|_{t=c} \tag{47.1}$$

Given the chart $(\mathscr{U},\hat{x})$ covering $\mathbf{x}$, we can project the vector equation (47.1) into the natural basis $\{\mathbf{g}_i\}$ of $\hat{x}$. First, the coordinates of the curve $\boldsymbol{\lambda}$ are given by

$$\hat{x}(\boldsymbol{\lambda}(t))=(\lambda^1(t),...,\lambda^N(t)),\qquad \boldsymbol{\lambda}(t)=\tilde{\mathbf{x}}(\lambda^1(t),...,\lambda^N(t)) \tag{47.2}$$

for all $t$ such that $\boldsymbol{\lambda}(t)\in\mathscr{U}$. Differentiating $(47.2)_2$ with respect to $t$, we get

$$\dot{\boldsymbol{\lambda}}\Big|_{\mathbf{x}}=\frac{\partial\tilde{\mathbf{x}}}{\partial x^j}\frac{d\lambda^j}{dt}\bigg|_{t=c} \tag{47.3}$$

By (47.3) this equation can be rewritten as

$$\dot{\boldsymbol{\lambda}}\Big|_{\mathbf{x}}=\frac{d\lambda^j}{dt}\bigg|_{t=c}\mathbf{g}_j(\mathbf{x}) \tag{47.4}$$

Thus, the components of $\dot{\boldsymbol{\lambda}}$ relative to $\{\mathbf{g}_j\}$ are simply the derivatives of the coordinate representations of $\boldsymbol{\lambda}$ in $\hat{x}$. In fact (44.33) can be regarded as a special case of (47.3) when $\boldsymbol{\lambda}$ coincides with the $i^{\text{th}}$ coordinate curve of $\hat{x}$.

An important consequence of (47.3) is that

$$\hat{x}\left(\boldsymbol{\lambda}(t+\Delta t)\right) = \left(\lambda^1(t) + \upsilon^1\Delta t + o(\Delta t), \ldots, \lambda^N(t) + \upsilon^N\Delta t + o(\Delta t)\right) \tag{47.5}$$

where $\upsilon^i$ denotes a component of $\dot{\boldsymbol{\lambda}}$, i.e.,

$$\upsilon^i = d\lambda^i / dt \tag{47.6}$$

In particular, if $\boldsymbol{\lambda}$ is a straight line segment, say

$$\boldsymbol{\lambda}(t) = \mathbf{x} + t\mathbf{v} \tag{47.7}$$

so that

$$\dot{\boldsymbol{\lambda}}(t) = \mathbf{v} \tag{47.8}$$

for all $t$, then (47.5) becomes

$$\hat{x}\left(\boldsymbol{\lambda}(t)\right) = \left(x^1 + \upsilon^1 t, \ldots, x^N + \upsilon^N t\right) + o(t) \tag{47.9}$$

for sufficiently small $t$.

Next we consider the gradient of a smooth function $f$ defined on an open set $\mathscr{U} \subset \mathscr{E}$. From (43.17) the gradient of $f$ is defined by

$$\operatorname{grad} f(\mathbf{x}) \cdot \mathbf{v} = \frac{d}{d\tau} f(\mathbf{x} + \tau\mathbf{v})\Big|_{\tau=0} \tag{47.10}$$

for all $\mathbf{v} \in \mathscr{V}$. As before, we choose a chart near $\mathbf{x}$; then $f$ can be represented by the function

$$f(x^1, \ldots, x^N) \equiv f \circ \tilde{\mathbf{x}}(x^1, \ldots, x^N) \tag{47.11}$$

For definiteness, we call the function $f \circ \tilde{\mathbf{x}}$ the *coordinate representation* of $f$. From (47.9), we see that

$$f(\mathbf{x}+\tau\mathbf{v}) = f(x^1 + \upsilon^1\tau + o(\tau), \ldots, x^N + \upsilon^N\tau + o(\tau)) \tag{47.12}$$

As a result, the right-hand side of (47.10) is given by

$$\frac{d}{d\tau} f(\mathbf{x}+\tau\mathbf{v})\Big|_{\tau=0} = \frac{\partial f(\mathbf{x})}{\partial x^j}\upsilon^j \tag{47.13}$$

Now since $\mathbf{v}$ is arbitrary, (47.13) can be combined with (47.10) to obtain

$$\operatorname{grad} f = \frac{\partial f}{\partial x^j}\mathbf{g}^j \tag{47.14}$$

where $\mathbf{g}^j$ is a natural basis vector associated with the coordinate chart as defined by (44.31). In fact, that equation can now be regarded as a special case of (47.14) where $f$ reduces to the coordinate function $\hat{x}^i$.

Having considered the tangent of a curve and the gradient of a function, we now turn to the problem of representing the gradient of a tensor field in general. Let $\mathbf{A} \in T_q^\infty(\mathscr{U})$ be such a field and suppose that $\mathbf{x}$ is an arbitrary point in its domain $\mathscr{U}$. We choose an arbitrary chart $\hat{x}$ covering $\mathbf{x}$. Then the formula generalizing (47.4) and (47.14) is

$$\operatorname{grad}\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{A}(\mathbf{x})}{\partial x^j} \otimes \mathbf{g}^j(\mathbf{x}) \tag{47.15}$$

where the quantity $\partial\mathbf{A}(\mathbf{x})/\partial x^j$ on the right-hand side is the partial derivative of the *coordinate representation* of $\mathbf{A}$, i.e.,

$$\frac{\partial \mathbf{A}(\mathbf{x})}{\partial x^j} = \frac{\partial}{\partial x^j}\mathbf{A} \circ \tilde{\mathbf{x}}(x^1, \ldots, x^N) \tag{47.16}$$

From (47.15) we see that $\operatorname{grad}\mathbf{A}$ is a tensor field of order $q+1$ on $\mathscr{U}$, $\operatorname{grad}\mathbf{A} \in T_{q+1}^\infty(\mathscr{U})$.

To prove (47.15) we shall first regard $\operatorname{grad}\mathbf{A}(\mathbf{x})$ as in $\mathscr{L}(\mathscr{V};\mathscr{T}_q(\mathscr{V}))$. Then by (43.15), when $\mathbf{A}$ is smooth, we get

$$\left(\operatorname{grad}\mathbf{A}(\mathbf{x})\right)\mathbf{v}=\frac{d}{d\tau}\mathbf{A}(\mathbf{x}+\tau\mathbf{v})\Big|_{\tau=0} \tag{47.17}$$

for all $\mathbf{v}\in\mathscr{V}$. By using exactly the same argument from (47.10) to (47.13), we now have

$$\left(\operatorname{grad}\mathbf{A}(\mathbf{x})\right)\mathbf{v}=\frac{\partial\mathbf{A}(\mathbf{x})}{\partial x^j}\upsilon^j \tag{47.18}$$

Since this equation must hold for all $\mathbf{v}\in\mathscr{V}$, we may take $\mathbf{v}=\mathbf{g}_k(\mathbf{x})$ and find

$$\left(\operatorname{grad}\mathbf{A}(\mathbf{x})\right)\mathbf{g}_k=\frac{\partial\mathbf{A}(\mathbf{x})}{\partial x^k} \tag{47.19}$$

which is equivalent to (47.15) by virtue of the canonical isomorphism between $\mathscr{L}(\mathscr{V};\mathscr{T}_q(\mathscr{V}))$ and $\mathscr{T}_{q+1}(\mathscr{V})$.

Since $\operatorname{grad}\mathbf{A}(\mathbf{x})\in\mathscr{T}_{q+1}(\mathscr{V})$, it can be represented by its component form relative to the natural basis, say

$$\operatorname{grad}\mathbf{A}(\mathbf{x})=\left(\operatorname{grad}\mathbf{A}(\mathbf{x})\right)^{i_1\ldots i_q}{}_j\,\mathbf{g}_{i_1}(\mathbf{x})\cdots\mathbf{g}_{i_q}(\mathbf{x})\otimes\mathbf{g}^j(\mathbf{x}) \tag{47.20}$$

Comparing this equation with (47.15), we se that

$$\left(\operatorname{grad}\mathbf{A}(\mathbf{x})\right)^{i_1\ldots i_q}{}_j\,\mathbf{g}_{i_1}(\mathbf{x})\cdots\mathbf{g}_{i_q}(\mathbf{x})=\frac{\partial\mathbf{A}(\mathbf{x})}{\partial x^j} \tag{47.21}$$

for all $j = 1, \dots, N$. In applications it is convenient to express the components of $\operatorname{grad} \mathbf{A}(\mathbf{x})$ in terms of the components of $\mathbf{A}(\mathbf{x})$ relative to the same coordinate chart. If we write the component form of $\mathbf{A}(\mathbf{x})$ as usual by

$$\mathbf{A}(\mathbf{x}) = A^{i_1 \dots i_q}(\mathbf{x}) \mathbf{g}_{i_1}(\mathbf{x}) \otimes \cdots \otimes \mathbf{g}_{i_q}(\mathbf{x}) \tag{47.22}$$

then the right-hand side of (47.21) is given by

$$\begin{aligned} \frac{\partial \mathbf{A}(\mathbf{x})}{\partial x^j} &= \frac{\partial A^{i_1 \dots i_q}(\mathbf{x})}{\partial x^j} \mathbf{g}_{i_1}(\mathbf{x}) \otimes \cdots \otimes \mathbf{g}_{i_q}(\mathbf{x}) \\ &\quad + A^{i_1 \dots i_q}(\mathbf{x}) \left[ \frac{\partial \mathbf{g}_{i_1}(\mathbf{x})}{\partial x^j} \otimes \cdots \otimes \mathbf{g}_{i_q}(\mathbf{x}) + \mathbf{g}_{i_1}(\mathbf{x}) \otimes \cdots \otimes \frac{\partial \mathbf{g}_{i_q}(\mathbf{x})}{\partial x^j} \right] \end{aligned} \tag{47.23}$$

From this representation we see that it is important to express the gradient of the basis vector $\mathbf{g}_i$ in component form first, since from (47.21) for the case $\mathbf{A} = \mathbf{g}_i$, we have

$$\frac{\partial \mathbf{g}_i(\mathbf{x})}{\partial x^j} = \left( \operatorname{grad} \mathbf{g}_i(\mathbf{x}) \right)^k_j \mathbf{g}_k(\mathbf{x}) \tag{47.24}$$

or, equivalently,

$$\frac{\partial \mathbf{g}_i(\mathbf{x})}{\partial x^j} = \begin{Bmatrix} k \\ ij \end{Bmatrix} \mathbf{g}_k(\mathbf{x}) \tag{47.25}$$

We call $\begin{Bmatrix} k \\ ij \end{Bmatrix}$ the *Christoffel symbol* associated with the chart $\hat{x}$. Notice that, in general, $\begin{Bmatrix} k \\ ij \end{Bmatrix}$ is a function of $\mathbf{x}$, but we have suppressed the argument $\mathbf{x}$ in the notation. More accurately, (47.25) should be replaced by the field equation

$$\frac{\partial \mathbf{g}_i}{\partial x^j} = \begin{Bmatrix} k \\ ij \end{Bmatrix} \mathbf{g}_k \tag{47.26}$$

which is valid at each point $\mathbf{x}$ in the domain of the chart $\hat{x}$, for all $i, j = 1,..., N$. We shall consider some preliminary results about the Christoffel symbols first.

From (47.26) and (44.35) we have

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = \frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}^k \tag{47.27}$$

By virtue of (44.33), this equation can be rewritten as

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = \frac{\partial^2 \tilde{\mathbf{x}}}{\partial x^i \partial x^j} \cdot \mathbf{g}^k \tag{47.28}$$

It follows from (47.28) that the Christoffel symbols are symmetric in the pair $(ij)$, namely

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = \begin{Bmatrix} k \\ ji \end{Bmatrix} \tag{47.29}$$

for all $i, j, k = 1,..., N$. Now by definition

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j \tag{47.30}$$

Taking the partial derivative of (47.30) with respect to $x^k$ and using the component form (47.26), we get

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial \mathbf{g}_i}{\partial x^k} \cdot \mathbf{g}_j + \mathbf{g}_i \cdot \frac{\partial \mathbf{g}_j}{\partial x^k} \tag{47.31}$$

When the symmetry property (47.29) is used, equation (47.31) can be solved for the Christoffel symbols:

$$\left\{ \begin{matrix} k \\ ij \end{matrix} \right\} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right) \tag{47.32}$$

where $g^{kl}$ denotes the contravariant components of the metric tensor, namely

$$g^{kl} = \mathbf{g}^k \cdot \mathbf{g}^l \tag{47.33}$$

The formula (47.32) is most convenient for the calculation of the Christoffel symbols in any given chart.

As an example, we now compute the Christoffel symbols for the cylindrical coordinate system in a three-dimensional space. In Section 44 we have shown that the components of the metric tensor are given by (44.50) and (44.51) relative to this coordinate system. Substituting those components into (47.32), we obtain

$$\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} = -x^1, \qquad \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} = \frac{1}{x^1} \tag{47.34}$$

and all other Christoffel symbols are equal to zero.

Given two charts $\hat{x}$ and $\hat{y}$ with natural bases $\{\mathbf{g}_i\}$ and $\{\mathbf{h}_i\}$, respectively, the transformation rule for the Christoffel symbols can be derived in the following way:

$$\begin{aligned} \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} &= \mathbf{g}^k \cdot \frac{\partial \mathbf{g}_i}{\partial x^j} = \frac{\partial x^k}{\partial y^l} \mathbf{h}^l \cdot \frac{\partial}{\partial x^j} \left( \frac{\partial y^s}{\partial x^i} \mathbf{h}_s \right) \\ &= \frac{\partial x^k}{\partial y^l} \mathbf{h}^l \cdot \left( \frac{\partial^2 y^s}{\partial x^j \partial x^i} \mathbf{h}_s + \frac{\partial y^s}{\partial x^i} \frac{\partial \mathbf{h}_s}{\partial y^l} \frac{\partial y^l}{\partial x^j} \right) \\ &= \frac{\partial x^k}{\partial y^l} \frac{\partial^2 y^l}{\partial x^j \partial x^i} + \frac{\partial x^k}{\partial y^l} \frac{\partial y^s}{\partial x^i} \frac{\partial y^l}{\partial x^j} \mathbf{h}^l \cdot \frac{\partial \mathbf{h}_s}{\partial y^l} \\ &= \frac{\partial x^k}{\partial y^l} \frac{\partial^2 y^l}{\partial x^j \partial x^i} + \frac{\partial x^k}{\partial y^l} \frac{\partial y^s}{\partial x^i} \frac{\partial y^l}{\partial x^j} \overline{\left\{ \begin{matrix} l \\ st \end{matrix} \right\}} \end{aligned} \tag{47.35}$$

where $\left\{ \begin{matrix} k \\ ij \end{matrix} \right\}$ and $\overline{\left\{ \begin{matrix} l \\ st \end{matrix} \right\}}$ denote the Christoffel symbols associated with $\hat{x}$ and $\hat{y}$, respectively. Since (47.35) is different from the tensor transformation rule (45.22), it follows that the Christoffel symbols are not the components of a particular tensor field. In fact, if $\hat{y}$ is a Cartesian chart, then $\overline{\left\{ \begin{matrix} l \\ st \end{matrix} \right\}}$ vanishes since the natural basis vectors $\mathbf{h}_s$, are constant. In that case (47.35) reduces to

$$\left\{ \begin{matrix} k \\ ij \end{matrix} \right\} = \frac{\partial x^k}{\partial y^l} \frac{\partial^2 y^l}{\partial x^j \partial x^i} \tag{47.36}$$

and $\left\{ \begin{matrix} k \\ ij \end{matrix} \right\}$ need not vanish unless $\hat{x}$ is also a Cartesian chart. The formula (47.36) can also be used to calculate the Christoffel symbols when the coordination transformation from $\hat{x}$ to a Cartesian system $\hat{y}$ is given.

Having presented some basis properties of the Christoffel symbols, we now return to the general formula (47.23) for the components of the gradient of a tensor field. Substituting (47.26) into (47.23) yields

$$\frac{\partial \mathsf{A}(\mathbf{x})}{\partial x^j} = \left[ \frac{\partial A^{i_1 \cdots i_q}(\mathbf{x})}{\partial x^j} + A^{k i_2 \cdots i_q} \left\{ \begin{matrix} i_1 \\ kj \end{matrix} \right\} + \cdots + A^{i_1 \cdots i_{q-1} k} \left\{ \begin{matrix} i_q \\ kj \end{matrix} \right\} \right] \mathbf{g}_{i_1}(\mathbf{x}) \otimes \cdots \otimes \mathbf{g}_{i_q}(\mathbf{x}) \tag{47.37}$$

Comparing this result with (47.21), we finally obtain

$$A^{i_1 \cdots i_q}{}_{,j} \equiv \left( \operatorname{grad} \mathsf{A} \right)^{i_1 \cdots i_q}{}_{j} = \frac{\partial A^{i_1 \cdots i_q}}{\partial x^j} + A^{k i_2 \cdots i_q} \left\{ \begin{matrix} i_1 \\ kj \end{matrix} \right\} + \cdots + A^{i_1 \cdots i_{q-1} k} \left\{ \begin{matrix} i_q \\ kj \end{matrix} \right\} \tag{47.38}$$

This particular formula gives the components $A^{i_1 \cdots i_q}{}_{,j}$ of the gradient of $\mathsf{A}$ in terms of the contravariant components $A^{i_1 \cdots i_q}$ of $\mathsf{A}$. If the mixed components of $\mathsf{A}$ are used, the formula becomes

$$
\begin{aligned}
A^{i_1\ldots i_r}{}_{j_1\ldots j_s},_k \equiv \left(\operatorname{grad}\mathbf{A}\right)^{i_1\ldots i_r}{}_{j_1\ldots j_s,k} &= \frac{\partial A^{i_1\ldots i_r}{}_{j_1\ldots j_s}}{\partial x^k} + A^{li_2\ldots i_r}{}_{j_1\ldots j_s}\begin{Bmatrix} i_1 \\ lk \end{Bmatrix} + \cdots + A^{i_1\ldots i_{r-1}l}{}_{j_1\ldots j_s}\begin{Bmatrix} i_r \\ lk \end{Bmatrix} \\
&\quad - A^{i_1\ldots i_r}{}_{lj_2\ldots j_s}\begin{Bmatrix} l \\ j_1k \end{Bmatrix} - \cdots - A^{i_1\ldots i_r}{}_{j_1\ldots j_{s-1}l}\begin{Bmatrix} l \\ j_sk \end{Bmatrix}
\end{aligned}
\tag{47.39}
$$

We leave the proof of this general formula as an exercise. From (47.39), if $\mathbf{A} \in T_q^\infty(\mathscr{U})$, where $q = r + s$, then $\operatorname{grad}\mathbf{A} \in T_{q+1}^\infty(\mathscr{U})$. Further, if the coordinate system is Cartesian, then $A^{i_1\ldots i_r}{}_{j_1\ldots j_s},_k$ reduces to the ordinary partial derivative of $A^{i_1\ldots i_r}{}_{j_1\ldots j_s}$ wih respect to $x^k$.

Some special cases of (47.39) should be noted here. First, since the metric tensor is a constant second-order tensor field, we have

$$
g_{ij},_k = g^{ij},_k = \delta^i_j,_k = 0 \tag{47.40}
$$

for all $i, l, k = 1, \ldots, N$. In fact, (47.40) is equivalent to (47.31), which we have used to obtain the formula (47.32) for the Christoffel symbols. An important consequence of (47.40) is that the operations of raising and lowering of indices commute with the operation of gradient or covariant differentiation.

Another constant tensor field on $\mathscr{E}$ is the tensor field $\mathbf{E}$ defined by (45.26). While the sign of $\mathbf{E}$ depends on the orientation, we always have

$$
E_{i_1\ldots i_N},_k = E^{i_1\ldots i_N},_k = 0 \tag{47.41}
$$

If we substitute (45.27) and (45.28) into (47.41), we can rewrite the result in the form

$$
\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^i} = \begin{Bmatrix} k \\ lk \end{Bmatrix} \tag{47.42}
$$

where $g$ is the determinant of $\left[g_{ij}\right]$ as defined by (44.45).

Finally, the operation of skew-symmetrization $\mathbf{K}_r$ introduced in Section 37 is also a constant tensor. So we have

$$\delta^{i_1 \ldots i_r}_{j_1 \ldots j_r},_k = 0 \tag{47.43}$$

in any coordinate system. Thus, the operations of skew-symmetrization and covariant differentiation commute, provided that the indices of the covariant differentiations are not affected by the skew-symmetrization.

Some classical differential operators can be derived from the gradient. First, if $\mathbf{A}$ is a tensor field of order $q \geq 1$, then the *divergence* of $\mathbf{A}$ is defined by

$$\operatorname{div} \mathbf{A} = \mathbf{C}_{q,q+1}(\operatorname{grad} \mathbf{A})$$

where $\mathbf{C}$ denotes the contraction operation. In component form we have

$$\operatorname{div} \mathbf{A} = A^{i_1 \ldots i_{q-1} k},_k \mathbf{g}_{i_1} \otimes \cdots \otimes \mathbf{g}_{i_{q-1}} \tag{47.44}$$

so that $\operatorname{div} \mathbf{A}$ is a tensor field of order $q-1$. In particular, for a vector field $\mathbf{v}$, (47.44) reduces to

$$\operatorname{div} \mathbf{v} = \upsilon^i,_i = g^{ij} \upsilon_i,_j \tag{47.45}$$

By use of (47.42) and (47.40), we an rewrite this formula as

$$\operatorname{div} \mathbf{v} = \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g}\,\upsilon^i\right)}{\partial x^i} \tag{47.46}$$

This result is useful since it does not depend on the Christoffel symbols explicitly.

The Laplacian of a tensor field of order $q+1$ is a tensor field of the same order $q$ defined by

$$\Delta \mathbf{A} = \operatorname{div}\left(\operatorname{grad} \mathbf{A}\right) \tag{47.47}$$

In component form we have

$$\Delta \mathbf{A} = g^{kl} A^{i_1 \dots i_q}{}_{,kl}\, \mathbf{g}_{i_1} \otimes \cdots \otimes \mathbf{g}_{i_q} \tag{47.48}$$

For a scalar field $f$ the Laplacian is given by

$$\Delta f = g^{kl} f_{,kl} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^l}\left( \sqrt{g}\, g^{kl} \frac{\partial f}{\partial x^k} \right) \tag{47.49}$$

where (47.42), (47.14) and (47.39) have been used. Like (47.46), the formula (47.49) does not depend explicitly on the Christoffel symbols. In (47.48), $A^{i_1 \dots i_q}{}_{,kl}$ denotes the components of the second gradient $\operatorname{grad}\left(\operatorname{grad} \mathbf{A}\right)$ of $\mathbf{A}$. The reader will verify easily that $A^{i_1 \dots i_q}{}_{,kl}$, like the ordinary second partial derivative, is symmetric in the pair $(k,l)$. Indeed, if the coordinate system is Cartesian, then $A^{i_1 \dots i_q}{}_{,kl}$ reduces to $\partial^2 A^{i_1 \dots i_q} / \partial x^k \partial x^l$.

Finally, the classical curl operator can be defined in the following way. If $\mathbf{v}$ is a vector field, then $\operatorname{curl} \mathbf{v}$ is a skew-symmetric second-order tensor field defined by

$$\operatorname{curl} \mathbf{v} \equiv \mathbf{K}_2 \left(\operatorname{grad} \mathbf{v}\right) \tag{47.50}$$

where $\mathbf{K}_2$ is the skew-symmetrization operator. In component form (47.50) becomes

$$\operatorname{curl} \mathbf{v} = \frac{1}{2}\left( \upsilon_{i,j} - \upsilon_{j,i} \right) \mathbf{g}^i \otimes \mathbf{g}^j \tag{47.51}$$

By virtue of (47.29) and (47.39) this formula can be rewritten as

$$\operatorname{curl} \mathbf{v} = \frac{1}{2}\left( \frac{\partial \upsilon_i}{\partial x^j} - \frac{\partial \upsilon_j}{\partial x^i} \right) \mathbf{g}^i \otimes \mathbf{g}^j \tag{47.52}$$

which no longer depends on the Christoffel symbols. We shall generalize the curl operator to arbitrary skew-symmetric tensor fields in the next chapter.

## Exercises

47.1 Show that in spherical coordinates on a three-dimensional Euclidean manifold the nonzero Christoffel symbols are

$$\begin{aligned}
&\left\{\begin{matrix} 2 \\ 21 \end{matrix}\right\} = \left\{\begin{matrix} 2 \\ 12 \end{matrix}\right\} = \left\{\begin{matrix} 3 \\ 31 \end{matrix}\right\} = \left\{\begin{matrix} 3 \\ 13 \end{matrix}\right\} = \frac{1}{x^1} \\
&\left\{\begin{matrix} 1 \\ 22 \end{matrix}\right\} = -x^1 \\
&\left\{\begin{matrix} 1 \\ 33 \end{matrix}\right\} = -x^1(\sin x^2)^2 \\
&\left\{\begin{matrix} 3 \\ 32 \end{matrix}\right\} = \left\{\begin{matrix} 3 \\ 23 \end{matrix}\right\} = \cot x^2 \\
&\left\{\begin{matrix} 2 \\ 33 \end{matrix}\right\} = -\sin x^2 \cos x^2
\end{aligned}$$

47.2 On an oriented three-dimensional Euclidean manifold the curl of a vector field can be regarded as a vector field by

$$\operatorname{curl} \mathbf{v} \equiv -E^{ijk} \upsilon_{j,k} \mathbf{g}_i = -E^{ijk} \frac{\partial \upsilon_j}{\partial x^k} \mathbf{g}_i \tag{47.53}$$

where $E^{ijk}$ denotes the components of the positive volume tensor $\mathbf{E}$. Show that $\operatorname{curl}(\operatorname{curl} \mathbf{v}) = \operatorname{grad}(\operatorname{div} \mathbf{v}) - \operatorname{div}(\operatorname{grad} \mathbf{v})$. Also show that

$$\operatorname{curl}(\operatorname{grad} f) = 0 \tag{47.54}$$

For any scalar field $f$ and that

$$\operatorname{div}\left(\operatorname{curl} \mathbf{v}\right) = 0 \tag{47.55}$$

for any vector field $\mathbf{v}$.

47.3 Verify that

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = -\frac{\partial \mathbf{g}^k}{\partial x^j} \cdot \mathbf{g}_i \tag{47.56}$$

Therefore,

$$\frac{\partial \mathbf{g}^k}{\partial x^j} = -\begin{Bmatrix} k \\ ij \end{Bmatrix} \mathbf{g}^i \tag{47.57}$$

47.4 Prove the formula (47.37).

47.5 Prove the formula (47.42) and show that

$$\operatorname{div} \mathbf{g}_j = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^j} \tag{47.58}$$

47.6 Show that for an orthogonal coordinate system

$$\begin{aligned}
&\begin{Bmatrix} k \\ ij \end{Bmatrix} = 0, && i \neq j, i \neq k, j \neq k \\
&\begin{Bmatrix} j \\ ii \end{Bmatrix} = -\frac{1}{2g_{jj}} \frac{\partial g_{ii}}{\partial x^j} && \text{if} \quad i \neq j \\
&\begin{Bmatrix} i \\ ij \end{Bmatrix} = \begin{Bmatrix} i \\ ji \end{Bmatrix} = -\frac{1}{2g_{ii}} \frac{\partial g_{ii}}{\partial x^j} && \text{if} \quad i \neq j \\
&\begin{Bmatrix} i \\ ii \end{Bmatrix} = \frac{1}{2g_{ii}} \frac{\partial g_{ii}}{\partial x^i}
\end{aligned}$$

Where the indices $i$ and $j$ are not summed.

47.7 Show that

$$\frac{\partial}{\partial x^j}\begin{Bmatrix} k \\ il \end{Bmatrix} - \frac{\partial}{\partial x^l}\begin{Bmatrix} k \\ ij \end{Bmatrix} + \begin{Bmatrix} t \\ il \end{Bmatrix}\begin{Bmatrix} k \\ tj \end{Bmatrix} - \begin{Bmatrix} t \\ ij \end{Bmatrix}\begin{Bmatrix} k \\ tl \end{Bmatrix} = 0$$

The quantity on the left-hand side of this equation is the component of a fourth-order tensor $\mathbf{R}$, called the curvature tensor, which is zero for any Euclidean manifold[2].

[2] The Maple computer program contains a package *tensor* that will produce Christoffel symbols and other important tensor quantities associated with various coordinate systems.

## *Section 48. Covariant Derivatives along Curves*

In the preceding section we have considered covariant differentiation of tensor fields which are defined on open submanifolds in $\mathscr{E}$. In applications, however, we often encounter vector or tensor fields defined only on some smooth curve in $\mathscr{E}$. For example, if $\boldsymbol{\lambda}:(a,b)\rightarrow\mathscr{E}$ is a smooth curve, then the tangent vector $\dot{\boldsymbol{\lambda}}$ is a vector field on $\mathscr{E}$. In this section we shall consider the problem of representing the gradients of arbitrary tensor fields defined on smooth curves in a Euclidean space.

Given any smooth curve $\boldsymbol{\lambda}:(a,b)\rightarrow\mathscr{E}$ and a field $\mathbf{A}:(a,b)\rightarrow\mathscr{T}_q(\mathscr{V})$ we can regard the value $\mathbf{A}(t)$ as a tensor of order $q$ at $\boldsymbol{\lambda}(t)$. Then the *gradient* or *covariant derivative* of $\mathbf{A}$ along $\boldsymbol{\lambda}$ is defined by

$$\frac{d\mathbf{A}(t)}{dt}\equiv\lim_{\Delta t\rightarrow 0}\frac{\mathbf{A}(t+\Delta t)-\mathbf{A}(t)}{\Delta t} \tag{48.1}$$

for all $t\in(a,b)$. If the limit on the right-hand side of (48.1) exists, then $d\mathbf{A}(t)/dt$ is itself also a tensor field of order $q$ on $\boldsymbol{\lambda}$. Hence, we can define the second gradient $d^2\mathbf{A}(t)/dt^2$ by replacing $\mathbf{A}$ by $d\mathbf{A}(t)/dt$ in (48.1). Higher gradients of $\mathbf{A}$ are defined similarly. If all gradients of $\mathbf{A}$ exist, then $\mathbf{A}$ is $C^\infty$-smooth on $\boldsymbol{\lambda}$. We are interested in representing the gradients of $\mathbf{A}$ in component form.

Let $\hat{x}$ be a coordinate system covering some point of $\boldsymbol{\lambda}$. Then as before we can characterize $\boldsymbol{\lambda}$ by its coordinate representation $\left(\lambda^i(t), i=1,\ldots,N\right)$. Similarly, we can express $\mathbf{A}$ in component form

$$\mathbf{A}(t)=A^{i_1\ldots i_q}(t)\mathbf{g}_{i_1}(\boldsymbol{\lambda}(t))\otimes\cdots\otimes\mathbf{g}_{i_q}(\boldsymbol{\lambda}(t)) \tag{48.2}$$

where the product basis is that of $\hat{x}$ at $\boldsymbol{\lambda}(t)$, the point where $\mathbf{A}(t)$ is defined. Differentiating (48.2) with respect to $t$, we obtain

$$\begin{aligned}\frac{d\mathbf{A}(t)}{dt} &= \frac{dA^{i_1\ldots i_q}(t)}{dt}\mathbf{g}_{i_1}(\boldsymbol{\lambda}(t))\otimes\cdots\otimes\mathbf{g}_{i_q}(\boldsymbol{\lambda}(t)) \\ &+A^{i_1\ldots i_q}(t)\left[\frac{d\mathbf{g}_{i_1}(\boldsymbol{\lambda}(t))}{dt}\otimes\cdots\otimes\mathbf{g}_{i_q}(\boldsymbol{\lambda}(t))+\cdots+\mathbf{g}_{i_1}(\boldsymbol{\lambda}(t))\otimes\cdots\otimes\frac{d\mathbf{g}_{i_q}(\boldsymbol{\lambda}(t))}{dt}\right]\end{aligned} \tag{48.3}$$

By application of the chain rule, we obtain

$$\frac{d\mathbf{g}_i(\boldsymbol{\lambda}(t))}{dt} = \frac{\partial\mathbf{g}_i(\boldsymbol{\lambda}(t))}{\partial x^j}\frac{d\lambda^j(t)}{dt} \tag{48.4}$$

where (47.5) and (47.6) have been used. In the preceding section we have represented the partial derivative of $\mathbf{g}_i$ by the component form (47.26). Hence we can rewrite (48.4) as

$$\frac{d\mathbf{g}_i(\boldsymbol{\lambda}(t))}{dt} = \begin{Bmatrix} k \\ ij \end{Bmatrix}\frac{d\lambda^j(t)}{dt}\mathbf{g}_k(\boldsymbol{\lambda}(t)) \tag{48.5}$$

Substitution of (48.5) into (48.3) yields the desired component representation

$$\begin{aligned}\frac{d\mathbf{A}(t)}{dt} = &\left\{\frac{dA^{i_1\ldots i_q}(t)}{dt}+\left[A^{ki_2\ldots i_q}(t)\begin{Bmatrix} i_1 \\ kj \end{Bmatrix}+\cdots+A^{i_1\ldots i_{q-1}k}(t)\begin{Bmatrix} i_q \\ kj \end{Bmatrix}\right]\frac{d\lambda^j(t)}{dt}\right\} \\ &\times\mathbf{g}_{i_1}(\boldsymbol{\lambda}(t))\otimes\cdots\otimes\mathbf{g}_{i_q}(\boldsymbol{\lambda}(t))\end{aligned} \tag{48.6}$$

where the Christoffel symbols are evaluated at the position $\boldsymbol{\lambda}(t)$. The representation (48.6) gives the contravariant components $\left(d\mathbf{A}(t)/dt\right)^{i_1\ldots i_q}$ of $d\mathbf{A}(t)/dt$ in terms of the contravariant components $A^{i_1\ldots i_q}(t)$ of $\mathbf{A}(t)$ relative to the same coordinate chart $\hat{x}$. If the mixed components are used, the representation becomes

$$\begin{aligned}\frac{d\mathbf{A}(t)}{dt} = &\left\{\frac{dA^{i_1\ldots i_r}{}_{j_1\ldots j_s}(t)}{dt}+\left[A^{ki_2\ldots i_r}{}_{j_1\ldots j_s}(t)\begin{Bmatrix} i_1 \\ kl \end{Bmatrix}+\cdots+A^{i_1\ldots i_{r-1}k}{}_{j_1\ldots j_s}(t)\begin{Bmatrix} i_r \\ kl \end{Bmatrix}\right.\right. \\ &\left.\left.-A^{i_1\ldots i_r}{}_{kj_2\ldots j_s}(t)\begin{Bmatrix} k \\ j_1 l \end{Bmatrix}-\cdots-A^{i_1\ldots i_r}{}_{j_1\ldots j_{s-1}k}(t)\begin{Bmatrix} k \\ j_s l \end{Bmatrix}\right]\frac{d\lambda^l(t)}{dt}\right\} \\ &\times\mathbf{g}_{i_1}(\boldsymbol{\lambda}(t))\otimes\cdots\otimes\mathbf{g}_{i_r}(\boldsymbol{\lambda}(t))\otimes\mathbf{g}^{j_1}(\boldsymbol{\lambda}(t))\otimes\cdots\otimes\mathbf{g}^{j_s}(\boldsymbol{\lambda}(t))\end{aligned} \tag{48.7}$$

We leave the proof of this general formula as an exercise. In view of the representations (48.6) and (48.7), we see that it is important to distinguish the notation

$$\left(\frac{d\mathbf{A}}{dt}\right)^{i_1\cdots i_r}{}_{j_1\cdots j_s}$$

which denotes a component of the covariant derivative $d\mathbf{A}/dt$, from the notation

$$\frac{dA^{i_1\cdots i_r}{}_{j_1\cdots j_s}}{dt}$$

which denotes the derivative of a component of $\mathbf{A}$. For this reason we shall denote the former by the new notation

$$\frac{DA^{i_1\cdots i_r}{}_{j_1\cdots j_s}}{Dt}$$

As an example we compute the components of the gradient of a vector field $\mathbf{v}$ along $\boldsymbol{\lambda}$. By (48.6)

$$\frac{d\mathbf{v}(t)}{dt}=\left\{\frac{d\upsilon^i(t)}{dt}+\upsilon^k(t)\begin{Bmatrix} i \\ kj\end{Bmatrix}\frac{d\lambda^j(t)}{dt}\right\}\mathbf{g}_i(\boldsymbol{\lambda}(t)) \tag{48.8}$$

or, equivalently,

$$\frac{D\upsilon^i(t)}{Dt}=\frac{d\upsilon^i(t)}{dt}+\upsilon^k(t)\begin{Bmatrix} i \\ kj\end{Bmatrix}\frac{d\lambda^j(t)}{dt} \tag{48.9}$$

In particular, when $\mathbf{v}$ is the $i^{\text{th}}$ basis vector $\mathbf{g}_i$ and then $\boldsymbol{\lambda}$ is the $j^{\text{th}}$ coordinate curve, then (48.8) reduces to

$$\frac{\partial \mathbf{g}_i}{\partial x^j} = \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} \mathbf{g}_k$$

which is the previous equation (47.26).

Next if $\mathbf{v}$ is the tangent vector $\dot{\boldsymbol{\lambda}}$ of $\boldsymbol{\lambda}$, then (48.8) reduces to

$$\frac{d^2\boldsymbol{\lambda}(t)}{dt^2} = \left\{ \frac{d^2\lambda^i(t)}{dt^2} + \frac{d\lambda^k(t)}{dt} \left\{ \begin{matrix} i \\ kj \end{matrix} \right\} \frac{d\lambda^j(t)}{dt} \right\} \mathbf{g}_i(\boldsymbol{\lambda}(t)) \tag{48.10}$$

where (47.4) has been used. In particular, if $\boldsymbol{\lambda}$ is a straight line with homogeneous parameter, i.e., if $\dot{\boldsymbol{\lambda}} = \mathbf{v} = \text{const}$, then

$$\frac{d^2\lambda^i(t)}{dt^2} + \frac{d\lambda^k(t)}{dt} \frac{d\lambda^j(t)}{dt} \left\{ \begin{matrix} i \\ kj \end{matrix} \right\} = 0, \qquad i = 1, \ldots, N \tag{48.11}$$

This equation shows that, for the straight line $\boldsymbol{\lambda}(t) = \mathbf{x} + t\mathbf{v}$ given by (47.7), we can sharpen the result (47.9) to

$$\hat{x}\left(\boldsymbol{\lambda}(t)\right) = \left( x^1 + \upsilon^1 t - \upsilon^k \upsilon^j \frac{1}{2} \left\{ \begin{matrix} 1 \\ kj \end{matrix} \right\} t^2, \ldots, x^N + \upsilon^N t - \frac{1}{2} \upsilon^k \upsilon^j \left\{ \begin{matrix} N \\ kj \end{matrix} \right\} t^2 \right) + o(t^2) \tag{48.12}$$

for sufficiently small $t$, where the Christoffel symbols are evaluated at the point $\mathbf{x} = \boldsymbol{\lambda}(0)$.

Now suppose that $\mathsf{A}$ is a tensor field on $\mathscr{U}$, say $\mathsf{A} \in T_q^\infty$, and let $\boldsymbol{\lambda} : (a,b) \to \mathscr{U}$ be a curve in $\mathscr{U}$. Then the restriction of $\mathsf{A}$ on $\boldsymbol{\lambda}$ is a tensor field

$$\hat{\mathsf{A}}(t) \equiv \mathsf{A}(\boldsymbol{\lambda}(t)), \qquad t \in (a,b) \tag{48.13}$$

In this case we can compute the gradient of $\mathsf{A}$ along $\boldsymbol{\lambda}$ either by (48.6) or directly by the chain rule of (48.13). In both cases the result is

$$\frac{d\mathbf{A}(\boldsymbol{\lambda}(t))}{dt}=\left[\frac{\partial A^{i_1\ldots i_q}(\boldsymbol{\lambda}(t))}{\partial x^j}+A^{ki_2\ldots i_q}(t)\begin{Bmatrix} i_1 \\ kj \end{Bmatrix}+\cdots+A^{i_1\ldots i_{q-1}k}(t)\begin{Bmatrix} i_q \\ kj \end{Bmatrix}\right]\frac{d\lambda^j(t)}{dt}$$
$$\times\mathbf{g}_{i_1}(\boldsymbol{\lambda}(t))\otimes\cdots\otimes\mathbf{g}_{i_q}(\boldsymbol{\lambda}(t)) \tag{48.14}$$

or, equivalently,

$$\frac{d\mathbf{A}(\boldsymbol{\lambda}(t))}{dt}=\left(\operatorname{grad}\mathbf{A}(\boldsymbol{\lambda}(t))\right)\dot{\boldsymbol{\lambda}}(t) \tag{48.15}$$

where $\operatorname{grad}\mathbf{A}(\boldsymbol{\lambda}(t))$ is regarded as an element of $\mathscr{L}(\mathscr{V};\mathscr{T}_q(\mathscr{V}))$ as before. Since $\operatorname{grad}\mathbf{A}$ is given by (47.15), the result (48.15) also can be written as

$$\frac{d\mathbf{A}(\boldsymbol{\lambda}(t))}{dt}=\frac{\partial\mathbf{A}(\boldsymbol{\lambda}(t))}{\partial x^j}\frac{d\lambda^i(t)}{dt} \tag{48.16}$$

A special case of (48.16), when $\mathbf{A}$ reduces to $\mathbf{g}_i$, is (48.4).

By (48.16) it follows that the gradient of the metric tensor, the volume tensor, and the skew-symmetrization operator all vanish along any curve.

## *Exercises*

48.1 Prove the formula (48.7).

48.2 If the parameter $t$ of a curve $\boldsymbol{\lambda}$ is regarded as time, then the tangent vector

$$\mathbf{v}=\frac{d\boldsymbol{\lambda}}{dt}$$

is the *velocity* and the gradient of $\mathbf{v}$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

is the *acceleration*. For a curve $\boldsymbol{\lambda}$ in a three-dimensional Euclidean manifold, express the acceleration in component form relative to the spherical coordinates.

# Chapter 10

## VECTOR FIELDS AND DIFFERENTIAL FORMS

### Section 49. Lie Derivatives

Let $\mathscr{E}$ be a Euclidean manifold and let $\mathbf{u}$ and $\mathbf{v}$ be vector fields defined on some open set $\mathscr{U}$ in $\mathscr{E}$. In Exercise 45.2 we have defined the *Lie bracket* $[\mathbf{u},\mathbf{v}]$ by

$$[\mathbf{u},\mathbf{v}]=(\operatorname{grad}\mathbf{v})\mathbf{u}-(\operatorname{grad}\mathbf{u})\mathbf{v} \tag{49.1}$$

In this section first we shall explain the geometric meaning of the formula (49.1), and then we generalize the operation to the *Lie derivative* of arbitrary tensor fields.

To interpret the operation on the right-hand side of (49.1), we start from the concept of the flow generated by a vector field. We say that a curve $\boldsymbol{\lambda}:(a,b)\to\mathscr{U}$ is an *integral curve* of a vector field $\mathbf{v}$ if

$$\frac{d\boldsymbol{\lambda}(t)}{dt}=\mathbf{v}(\boldsymbol{\lambda}(t)) \tag{49.2}$$

for all $t\in(a,b)$. By (47.1), the condition (49.2) means that $\boldsymbol{\lambda}$ is an integral curve of $\mathbf{v}$ if and only if its tangent vector coincides with the value of $\mathbf{v}$ at every point $\boldsymbol{\lambda}(t)$. An integral curve may be visualized as the orbit of a point flowing with velocity $\mathbf{v}$. Then the *flow* generated by $\mathbf{v}$ is defined to be the mapping that sends a point $\boldsymbol{\lambda}(t_0)$ to a point $\boldsymbol{\lambda}(t)$ along any integral curve of $\mathbf{v}$.

To make this concept more precise, let us introduce a local coordinate system $\hat{x}$. Then the condition (49.2) can be represented by

$$d\lambda^i(t)/dt=\upsilon^i(\boldsymbol{\lambda}(t)) \tag{49.3}$$

where (47.4) has been used. This formula shows that the coordinates $(\lambda^i(t), i=1,\dots,N)$ of an integral curve are governed by a system of first-order differential equations. Now it is proved in the theory of differential equations that if the fields $\upsilon^i$ on the right-hand side of (49.3) are smooth, then corresponding to any initial condition, say

$$\boldsymbol{\lambda}(0)=\mathbf{x}_0 \tag{49.4}$$

or equivalently

$$\lambda^i(0)=x_0^i, \qquad i=1,\ldots,N \tag{49.5}$$

a unique solution of (49.3) exists on a certain interval $(-\delta,\delta)$, where $\delta$ may depend on the initial point $\mathbf{x}_0$ but it may be chosen to be a fixed, positive number for all initial points in a sufficiently small neighborhood of $\mathbf{x}_0$. For definiteness, we denote the solution of (49.2) corresponding to the initial point $\mathbf{x}$ by $\boldsymbol{\lambda}(t,\mathbf{x})$; then it is known that the mapping from $\mathbf{x}$ to $\boldsymbol{\lambda}(t,\mathbf{x})$; then it is known that the mapping from $\mathbf{x}$ to $\boldsymbol{\lambda}(t,\mathbf{x})$ is smooth for each *t* belonging to the interval of existence of the solution. We denote this mapping by $\boldsymbol{\rho}_t$, namely

$$\boldsymbol{\rho}_t(\mathbf{x})=\boldsymbol{\lambda}(t,\mathbf{x}) \tag{49.6}$$

and we call it the *flow* generated by $\mathbf{v}$. In particular,

$$\boldsymbol{\rho}_0(\mathbf{x})=\mathbf{x}, \qquad\qquad \mathbf{x}\in\mathscr{U} \tag{49.7}$$

reflecting the fact that $\mathbf{x}$ is the initial point of the integral curve $\boldsymbol{\lambda}(t,\mathbf{x})$.

Since the fields $\upsilon^i$ are independent of t, they system (49.3) is said to be autonomous. A characteristic property of such a system is that the flow generated by $\mathbf{v}$ forms a local one-parameter group. That is, locally,

$$\boldsymbol{\rho}_{t_1+t_2}=\boldsymbol{\rho}_{t_1}\circ\boldsymbol{\rho}_{t_2} \tag{49.8}$$

or equivalently

$$\boldsymbol{\lambda}(t_1+t_2,\mathbf{x})=\boldsymbol{\lambda}(t_2,\boldsymbol{\lambda}(t_1,\mathbf{x})) \tag{49.9}$$

for all $t_1,t_2,\mathbf{x}$ such that the mappings in (49.9) are defined. Combining (49.7) with (49.9), we see that the flow $\boldsymbol{\rho}_t$ is a local diffeomorphism and, locally,

$$\boldsymbol{\rho}_t^{-1}=\boldsymbol{\rho}_{-t} \tag{49.10}$$

Consequently the gradient, grad $\boldsymbol{\rho}_t$, is a linear isomorphism which carries a vector at any point $\mathbf{x}$ to a vector at the point $\boldsymbol{\rho}_t(\mathbf{x})$. We call this linear isomorphism the *parallelism* generated by the flow and, for brevity, denote it by $\mathbf{P}_t$. The parallelism $\mathbf{P}_t$ is a typical *two-point tensor* whose component representation has the form

$$\mathbf{P}_t(\mathbf{x}) = P_t(\mathbf{x})^i_{\ j}\, \mathbf{g}_i(\boldsymbol{\rho}_t(\mathbf{x})) \otimes \mathbf{g}^j(\mathbf{x}) \tag{49.11}$$

or, equivalently,

$$[\mathbf{P}_t(\mathbf{x})](\mathbf{g}_j(\mathbf{x})) = P_t(\mathbf{x})^i_{\ j}\, \mathbf{g}_i(\boldsymbol{\rho}_t(\mathbf{x})) \tag{49.12}$$

where $\{\mathbf{g}_i(\mathbf{x})\}$ and $\{\mathbf{g}_j(\boldsymbol{\rho}_t(\mathbf{x}))\}$ are the natural bases of $\hat{x}$ at the points $\mathbf{x}$ and $\boldsymbol{\rho}_t(\mathbf{x})$, respectively.

Now the parallelism generated by the flow of $\mathbf{v}$ gives rise to a difference quotient of the vector field $\mathbf{u}$ in the following way: At any point $\mathbf{x} \in \mathscr{U}$, $\mathbf{P}_t(\mathbf{x})$ carries the vector $\mathbf{u}(\mathbf{x})$ at $\mathbf{x}$ to the vector $(\mathbf{P}_t(\mathbf{x}))(\mathbf{u}(\mathbf{x}))$ at $\boldsymbol{\rho}_t(\mathbf{x})$, which can then be compared with the vector $\mathbf{u}(\boldsymbol{\rho}_t(\mathbf{x}))$ also at $\boldsymbol{\rho}_t(\mathbf{x})$. Thus we define the difference quotient

$$\frac{1}{t}\left[\mathbf{u}(\boldsymbol{\rho}_t(\mathbf{x})) - (\mathbf{P}_t(\mathbf{x}))(\mathbf{u}(\mathbf{x}))\right] \tag{49.13}$$

The limit of this difference quotient as *t* approaches zero is called the *Lie derivative* of $\mathbf{u}$ with respect to $\mathbf{v}$ and is denoted by

$$\underset{\mathbf{v}}{\mathscr{L}}\mathbf{u}(\mathbf{x}) \equiv \lim_{t\to 0} \frac{1}{t}\left[\mathbf{u}(\boldsymbol{\rho}_t(\mathbf{x})) - (\mathbf{P}_t(\mathbf{x}))(\mathbf{u}(\mathbf{x}))\right] \tag{49.14}$$

We now derive a representation for the Lie derivative in terms of a local coordinate system $\hat{x}$. In view of (49.14) we see that we need an approximate representation for $\mathbf{P}_t$ to within first-order terms in *t*. Let $\mathbf{x}_0$ be a particular reference point. From (49.3) we have

$$\lambda^i(t, \mathbf{x}_0) = x_0^i + \upsilon^i(\mathbf{x}_0)t + o(t) \tag{49.15}$$

Suppose that $\mathbf{x}$ is an arbitrary neighboring point of $\mathbf{x}_0$, say $x^i = x_0^i + \Delta x^i$. Then by the same argument as (49.15) we have also

$$\begin{aligned}\lambda^i(t,\mathbf{x}) &= x^i + \upsilon^i(\mathbf{x})t + o(t) \\ &= x_0^i + \Delta x^i + \upsilon^i(\mathbf{x}_0)t + \frac{\partial \upsilon^i(\mathbf{x}_0)}{\partial x^j}(\Delta x^j)t + o(\Delta x^k) + o(t)\end{aligned} \tag{49.16}$$

Subtracting (49.15) from (49.16), we obtain

$$\lambda^i(t,\mathbf{x}) - \lambda^i(t,\mathbf{x}_0) = \Delta x^i + \frac{\partial \upsilon^i(\mathbf{x}_0)}{\partial x^j}(\Delta x^j)t + o(\Delta x^k) + o(t) \tag{49.17}$$

It follows from (49.17) that

$$P_t(\mathbf{x}_0)^i_j \equiv (\operatorname{grad} \boldsymbol{\rho}_t(\mathbf{x}_o))^i_j = \delta^i_j + \frac{\partial \upsilon^i(\mathbf{x}_0)}{\partial x^j} t + 0(t) \tag{49.18}$$

Now assuming that $\mathbf{u}$ is also smooth, we can represent $\mathbf{u}(\boldsymbol{\rho}_t(\mathbf{x}_0))$ approximately by

$$u^i(\boldsymbol{\rho}_t(\mathbf{x}_0)) = u^i(\mathbf{x}_0) + \frac{\partial u^i(\mathbf{x}_0)}{\partial x^j}\upsilon^j(\mathbf{x}_0)t + o(t) \tag{49.19}$$

where (49.15) has been used. Substituting (49.18) and (49.19) into (49.14) and taking the limit, we finally obtain

$$\underset{\mathbf{v}}{\mathscr{L}}\,\mathbf{u}(\mathbf{x}_0) = \left[\frac{\partial u^i(\mathbf{x}_0)}{\partial x^j}\upsilon^j - \frac{\partial \upsilon^i(\mathbf{x}_0)}{\partial x^j}u^j(\mathbf{x}_0)\right]\mathbf{g}_i(\mathbf{x}_0) \tag{49.20}$$

where $\mathbf{x}_0$ is arbitrary. Thus the field equation for (49.20) is

$$\underset{\mathbf{v}}{\mathscr{L}}\,\mathbf{u} = \left[\frac{\partial u^i}{\partial x^j}\upsilon^j - \frac{\partial \upsilon^i}{\partial x^j}u^j\right]\mathbf{g}_i \tag{49.21}$$

By (47.39) or by using a Cartesian system, we can rewrite (49.21) as

$$\underset{\mathbf{v}}{\mathscr{L}}\,\mathbf{u} = \left(u^i{}_{,j}\,\upsilon^j - \upsilon^i{}_{,j}\,u^j\right)\mathbf{g}_i \tag{49.22}$$

Consequently the Lie derivative has the following coordinate-free representation:

$$\underset{\mathbf{v}}{\mathscr{L}}\,\mathbf{u} = (\operatorname{grad}\mathbf{u})\mathbf{v} - (\operatorname{grad}\mathbf{v})\mathbf{u} \tag{49.23}$$

Comparing (49.23) with (49.1), we obtain

$$\underset{\mathbf{v}}{\mathscr{L}}\mathbf{u} = [\mathbf{v}, \mathbf{u}] \tag{49.24}$$

Since the Lie derivative is defined by the limit of the difference quotient (49.13), $\underset{\mathbf{v}}{\mathscr{L}}\mathbf{u}$ vanishes if and only if $\mathbf{u}$ commutes with the flow of $\mathbf{v}$ in the following sense:

$$\mathbf{u} \circ \boldsymbol{\rho}_t = \mathbf{P}_i \mathbf{u} \tag{49.25}$$

When $\mathbf{u}$ satisfies this condition, it may be called *invariant* with respect to $\mathbf{v}$. Clearly, this condition is symmetric for the pair $(\mathbf{u}, \mathbf{v})$, since the Lie bracket is skew-symmetric, namely

$$[\mathbf{u}, \mathbf{v}] = -[\mathbf{v}, \mathbf{u}] \tag{49.26}$$

In particular, (49.25) is equivalent to

$$\mathbf{v} \circ \boldsymbol{\varphi}_t = \mathbf{Q}_t \mathbf{v} \tag{49.27}$$

where $\boldsymbol{\varphi}_t$ and $\mathbf{Q}_t$ denote the flow and the parallelism generated by $\mathbf{u}$.

Another condition equivalent to (49.25) and (49.27) is

$$\boldsymbol{\varphi}_s \circ \boldsymbol{\rho}_t = \boldsymbol{\rho}_t \circ \boldsymbol{\varphi}_s \tag{49.28}$$

which means that

$$\begin{aligned} &\boldsymbol{\rho}_t\,(\text{an integral curve of } \mathbf{u}) = \text{an integral curve of } \mathbf{u} \\ &\boldsymbol{\varphi}_s\,(\text{an integral curve of } \mathbf{v}) = \text{an integral curve of } \mathbf{v} \end{aligned} \tag{49.29}$$

To show that (49.29) is necessary and sufficient for

$$[\mathbf{u}, \mathbf{v}] = \mathbf{0} \tag{49.30}$$

we choose any particular integral curve $\boldsymbol{\lambda} : (-\delta, \delta) \to \mathscr{U}$ for $\mathbf{v}$, At each point $\boldsymbol{\lambda}(t)$ we define an integral curve $\boldsymbol{\mu}(s,t)$ for $\mathbf{u}$ such that

$$\boldsymbol{\mu}(0,t) = \boldsymbol{\lambda}(t) \tag{49.31}$$

The integral curves $\boldsymbol{\mu}(\cdot,t)$ with $t\in(-\delta,\delta)$ then sweep out a surface parameterized by $(s,t)$. We shall now show that (49.30) requires that the curves $\boldsymbol{\mu}(s,\cdot)$ be integral curves of $\mathbf{v}$ for all $s$.

As before, let $\hat{x}$ be a local coordinate system. Then by definition

$$\partial\mu^i(s,t)/\partial s = u^i(\boldsymbol{\mu}(s,t)) \tag{49.32}$$

and

$$\partial\mu^i(0,t)/\partial s = \upsilon^i(\boldsymbol{\mu}(0,t)) \tag{49.33}$$

and we need to prove that

$$\partial\mu^i(s,t)/\partial s = \upsilon^i(\boldsymbol{\mu}(s,t)) \tag{49.34}$$

for $s\neq 0$. We put

$$\zeta^i(s,t) \equiv \frac{\partial\mu^i(s,t)}{\partial t} - \upsilon^i(\boldsymbol{\mu}(s,t)) \tag{49.35}$$

By (49.33)

$$\zeta^i(0,t) = 0 \tag{49.36}$$

We now show that $\zeta^i(s,t)$ vanishes identically. Indeed, if we differentiate (49.35) with respect to $s$ and use (49.32), we obtain

$$\begin{aligned}
\frac{\partial\zeta^i}{\partial s} &= \frac{\partial}{\partial t}\left(\frac{\partial\mu^i}{\partial s}\right) - \frac{\partial\upsilon^i}{\partial x^j}\frac{\partial\mu^j}{\partial s} \\
&= \frac{\partial u^i}{\partial x^j}\frac{\partial\mu^j}{\partial t} - \frac{\partial\upsilon^i}{\partial x^j}u^j \\
&= \frac{\partial u^i}{\partial x^j}\zeta^j + \left(\frac{\partial u^i}{\partial x^j}\upsilon^i - \frac{\partial\upsilon^i}{\partial x^j}u^j\right)
\end{aligned} \tag{49.37}$$

As a result, when (49.30) holds, $\zeta^i$ are governed by the system of differential equations

$$\frac{\partial\zeta^i}{\partial s} = \frac{\partial u^i}{\partial x^j}\zeta^j \tag{49.38}$$

and subject to the initial condition (49.36) for each fixed *t*. Consequently, $\zeta^i = 0$ is the only solution. Conversely, when $\zeta^i$ vanishes identically on the surface, (49.37) implies immediately that the Lie bracket of $\mathbf{u}$ and $\mathbf{v}$ vanishes. Thus the condition (49.28) is shown to be equivalent to the condition (49.30).

The result just established can be used to prove the theorem mentioned in Section 46 that a field of basis is holonomic if and only if

$$\left[\mathbf{h}_i, \mathbf{h}_j\right] = 0 \tag{49.39}$$

for all $i, j - 1, \ldots, N$. Necessity is obvious, since when $\{\mathbf{h}_i\}$ is holonomic, the components of each $\mathbf{h}_i$ relative to the coordinate system corresponding to $\{\mathbf{h}_i\}$ are the constants $\delta^i_j$. Hence from (49.21) we must have (49.39). Conversely, suppose (49.39) holds. Then by (49.28) there exists a surface swept out by integral curves of the vector fields $\mathbf{h}_1$ and $\mathbf{h}_2$. We denote the surface parameters by $x_1$ and $x_2$. Now if we define an integral curve for $\mathbf{h}_3$ at each surface point $(x_1, x_2)$, then by the conditions

$$\left[\mathbf{h}_1, \mathbf{h}_3\right] = \left[\mathbf{h}_2, \mathbf{h}_3\right] = 0$$

we see that the integral curves of $\mathbf{h}_1$, $\mathbf{h}_2, \mathbf{h}_3$ form a three-dimensional net which can be regarded as a "surface coordinate system" $(x_1, x_2, x_3)$ on a three-dimensional hypersurface in the N-dimensional Euclidean manifold $\mathscr{E}$. By repeating the same process based on the condition (49.39), we finally arrive at an N-dimensional net formed by integral curves of $\mathbf{h}_1, \ldots, \mathbf{h}_N$. The corresponding N-dimensional coordinate system $x_1, \ldots, x_N$ now forms a chart in $\mathscr{E}$ and its natural basis is the given basis $\{\mathbf{h}_i\}$. Thus the theorem is proved. In the next section we shall make use of this theorem to prove the classical Frobenius theorem.

So far we have considered the Lie derivative $\underset{\mathbf{v}}{\mathscr{L}}\mathbf{u}$ of a vector field $\mathbf{u}$ relative to $\mathbf{v}$ only. Since the parallelism $\mathbf{P}_t$ generated by $\mathbf{v}$ is a linear isomorphism, it can be extended to tensor fields. As usual for simple tensors we define

$$\mathbf{P}_t\left(\mathbf{a} \otimes \mathbf{b} \otimes \cdots\right) = \left(\mathbf{P}_t\mathbf{a}\right) \otimes \left(\mathbf{P}_t\mathbf{b}\right) \otimes \cdots \tag{49.40}$$

Then we extend $\mathbf{P}_t$ to arbitrary tensors by linearity. Using this extended parallelism, we define the Lie derivative of a tensor field $\mathbf{A}$ with respect to $\mathbf{v}$ by

$$\underset{\mathbf{v}}{\mathscr{L}}\mathbf{A}(\mathbf{x}) \equiv \lim_{t \to 0} \frac{1}{t}\left[\mathbf{A}\left(\boldsymbol{\rho}_t(\mathbf{x})\right) - \left(\mathbf{P}_t(\mathbf{x})\right)\left(\mathbf{A}(\mathbf{x})\right)\right] \tag{49.41}$$

which is clearly a generalization of (49.14). In terms of a coordinate system it can be shown that

$$
\begin{aligned}
\left(\mathop{\mathscr{L}}_{\mathbf{v}}\mathbf{A}\right)^{i_1\ldots i_r}{}_{j_1\ldots j_s} &= \frac{\partial \mathrm{A}^{i_1\ldots i_r}{}_{j_1\ldots j_s}}{\partial x^k}\upsilon^k \\
&\quad - \mathrm{A}^{k i_2\ldots i_r}{}_{j_1\ldots j_s}\frac{\partial \upsilon^{i_1}}{\partial x^k} - \cdots - \mathrm{A}^{i_1\ldots i_{r-1}k}{}_{j_1\ldots j_s}\frac{\partial \upsilon^{i_k}}{\partial x^k} \\
&\quad + \mathrm{A}^{i_1\ldots i_r}{}_{k\ldots j_s}\frac{\partial \upsilon^k}{\partial x^{j1}} + \cdots \mathrm{A}^{i_1\ldots i_r}{}_{j_1\ldots j_{s-1}k}\frac{\partial \upsilon^k}{\partial x^{j_s}}
\end{aligned}
\tag{49.42}
$$

which generalizes the formula (49.21). We leave the proof of (49.42) as an exercise. By the same argument as (49.22), the partial derivatives in (49.42) can be replaced by covariant derivatives.

It should be noted that the operations of raising and lowering of indices by the Euclidean metric do not commute with the Lie derivative, since the parallelism $\mathbf{P}_t$ generated by the flow generally does not preserve the metric. Consequently, to compute the Lie derivative of a tensor field $\mathbf{A}$, we must assign a particular contravariant order and covariant order to $\mathbf{A}$. The formula (49.42) is valid when $\mathbf{A}$ is regarded as a tensor field of contravariant order $r$ and covariant order $s$.

By the same token, the Lie derivative of a constant tensor such as the volume tensor or the skew-symmetric operator generally need not vanish.

*Exercises*

49.1 Prove the general representation formula (49.42) for the Lie derivative.

49.2 Show that the right-hand side of (49.42) obeys the transformation rule of the components of a tensor field.

49.3 In the two-dimensional Euclidean plane $\mathscr{E}$, consider the vector field $\mathbf{v}$ whose components relative to a rectangular Cartesian coordinate system $\left(x^1, x^2\right)$ are

$$
\upsilon^1 = \alpha x^1, \qquad \upsilon^2 = \alpha x^2
$$

where $\alpha$ is a positive constant. Determine the flow generated by $\mathbf{v}$. In particular, find the integral curve passing through the initial point

$$
\left(x_0^1,\ x_0^2\right) = (1,1)
$$

49.4 In the same two-dimensional plane $\mathscr{E}$ consider the vector field such that

$$\upsilon^1 = x^2, \qquad \upsilon^2 = x^1$$

Show that the flow generated by this vector field is the group of rotations of $\mathscr{E}$. In particular, show that the Euclidean metric is invariant with respect to this vector field.

49.5 Show that the flow of the autonomous system (49.3) possesses the local one-parameter group property (49.8).

## *Section 50. The Frobenius Theorem*

The concept of the integral curve of a vector field has been considered in some detail in the preceding section. In this section we introduce a somewhat more general concept. If $\mathbf{v}$ is a non-vanishing vector field in $\mathscr{E}$, then at each point $\mathbf{x}$ in the domain of $\mathbf{v}$ we can define a one-dimensional Euclidean space $\mathscr{D}(\mathbf{x})$ spanned by the vector $\mathbf{v}(\mathbf{x})$. In this sense an integral curve of $\mathbf{v}$ corresponds to a one-dimensional hypersurface in $\mathscr{E}$ which is tangent to $\mathscr{D}(\mathbf{x})$ at each point of the hypersurface. Clearly this situation can be generalized if we allow the field of Euclidean spaces $\mathscr{D}$ to be multidimensional. For definiteness, we call such a field $\mathscr{D}$ a distribution in $\mathscr{E}$, say of dimension $D$. Then a $D$-dimensional hypersurface $\mathscr{L}$ in $\mathscr{E}$ is called an *integral surface* of $\mathscr{D}$ if $\mathscr{L}$ is tangent to $\mathscr{D}$ at every point of the hypersurface.

Unlike a one-dimensional distribution, which corresponds to some non-vanishing vector field and hence always possesses many integral curves, a $D$-dimensional distribution $\mathscr{D}$ in general need not have any integral hypersurface at all. The problem of characterizing those distributions that do possess integral hypersurfaces is answered precisely by the Frobenius theorem. Before entering into the details of this important theorem, we introduce some preliminary concepts first.

Let $\mathscr{D}$ be a $D$-dimensional distribution and let $\mathbf{v}$ be a vector field. Then $\mathbf{v}$ is said to be *contained* in $\mathscr{D}$ if $\mathbf{v}(\mathbf{x}) \in \mathscr{D}(\mathbf{x})$ at each $\mathbf{x}$ in the domain of $\mathbf{v}$. Since $\mathscr{D}$ is $D$-dimensional, there exist $D$ vector fields $\{\mathbf{v}_\alpha, \alpha = 1,\dots,D\}$ contained in $\mathscr{D}$ such that $\mathscr{D}(\mathbf{x})$ is spanned by $\{\mathbf{v}_\alpha(\mathbf{x}), \alpha = 1,\dots,D\}$ at each point $\mathbf{x}$ in the domain of the vector fields. We call $\{\mathbf{v}_\alpha, \alpha = 1,\dots,D\}$ a *local basis* for $\mathscr{D}$. We say that $\mathscr{D}$ is smooth at some point $\mathbf{x}_0$ if there exists a local basis defined on a neighborhood of $\mathbf{x}_0$ formed by smooth vector fields $\mathbf{v}_\alpha$ contained in $\mathscr{D}$. Naturally, $\mathscr{D}$ is said to be smooth if it is smooth at each point of its domain. We shall be interested in *smooth* distributions only.

We say that $\mathscr{D}$ is *integrable* at a point $\mathbf{x}_0$ if there exists a local coordinate system $\hat{x}$ defined on a neighborhood of $\mathbf{x}_0$ such that the vector fields $\{\mathbf{g}_\alpha, \alpha = 1,\dots,D\}$ form a local basis for $\mathscr{D}$. Equivalently, this condition means that the hypersurfaces characterized by the conditions

$$x^{D+1} = \text{const},\dots,\quad x^N = \text{const} \tag{50.1}$$

are integral hypersurfaces of $\mathscr{D}$. Since the natural basis vectors $\mathbf{g}_i$ of any coordinate system are smooth, by the very definition $\mathscr{D}$ must be smooth at $\mathbf{x}_0$ if it is integrable there. Naturally, $\mathscr{D}$ is said to be *integrable* if it is integrable at each point of its domain. Consequently, every integrable distribution must be smooth.

Now we are ready to state and prove the Frobenius theorem, which characterizes integrable distributions.

**Theorem 50.1.** A smooth distribution $\mathscr{D}$ is integrable if and only if it is closed with respect to the Lie bracket, i.e.,

$$\mathbf{u},\mathbf{v} \in \mathscr{D} \Rightarrow [\mathbf{u},\mathbf{v}] \in \mathscr{D} \tag{50.2}$$

*Proof.* Necessity can be verified by direct calculation. If $\mathbf{u},\mathbf{v} \in \mathscr{D}$ and supposing that $\hat{x}$ is a local coordinate system such that $\{\mathbf{g}_\alpha, \alpha = 1,\dots,D\}$ forms a local basis for $\mathscr{D}$, then $\mathbf{u}$ and $\mathbf{v}$ have the component forms

$$\mathbf{u} = u^\alpha \mathbf{g}_\alpha, \qquad \mathbf{v} = \upsilon^\alpha \mathbf{g}_\alpha \tag{50.3}$$

where the Greek index α is summed from 1 to *D*. Substituting (50.3) into (49.21), we see that

$$[\mathbf{u},\mathbf{v}] = \left( \frac{\partial \upsilon^\alpha}{\partial x^\beta} u^\beta - \frac{\partial u^\alpha}{\partial x^\beta} \upsilon^\beta \right) \mathbf{g}_\alpha \tag{50.4}$$

Thus (50.2) holds.

Conversely, suppose that (50.2) holds. Then for any local basis $\{\mathbf{v}_\alpha, \alpha = 1,\dots,D\}$ for $\mathscr{D}$ we have

$$\left[\mathbf{v}_\alpha, \mathbf{v}_\beta\right] = C^\gamma_{\alpha\beta} \mathbf{v}_\gamma \tag{50.5}$$

where $C^\gamma_{\alpha\beta}$ are some smooth functions. We show first that there exists a local basis $\{\mathbf{u}_\alpha, \alpha = 1,\dots,D\}$ which satisfies the somewhat stronger condition

$$\left[\mathbf{u}_\alpha, \mathbf{u}_\beta\right] = 0, \qquad \alpha,\beta = 1,\dots,D \tag{50.6}$$

To construct such a basis, we choose a local coordinate system $\hat{y}$ and represent the basis $\{\mathbf{v}_\alpha\}$ by the component form

$$\begin{aligned} \mathbf{v}_\alpha &= \upsilon^1_\alpha \mathbf{k}_1 + \cdots + \upsilon^D_\alpha \mathbf{k}_D + \upsilon^{D+1}_\alpha \mathbf{k}_{D+1} + \cdots + \upsilon^N_\alpha \mathbf{k}_N \\ &\equiv \upsilon^\beta_\alpha \mathbf{k}_\beta + \upsilon^\Delta_\alpha \mathbf{k}_\Delta \end{aligned} \tag{50.7}$$

where $\{\mathbf{k}_i\}$ denotes the natural basis of $\hat{y}$, and whre the repeated Greek indices $\beta$ and $\Delta$ are summed from $1$ to $D$ and $D+1$ to $N$, respectively. Since the local basis $\{\mathbf{v}_\alpha\}$ is linearly independent, without loss of generality we can assume that the $D\times D$ matrix $\left[\upsilon_\alpha^\beta\right]$ is nonsingular, namely

$$\det\left[\upsilon_\alpha^\beta\right]\neq 0 \tag{50.8}$$

Now we define the basis $\{\mathbf{u}_\alpha\}$ by

$$\mathbf{u}_\alpha \equiv \upsilon^{-1\beta}_{\ \ \alpha}\mathbf{v}_\beta, \qquad \alpha=1,\ldots,D \tag{50.9}$$

where $\left[\upsilon^{-1\beta}_{\ \ \alpha}\right]$ denotes the inverse matrix of $\left[\upsilon_\alpha^\beta\right]$, as usual. Substituting (50.7) into (50.9), we see that the component representation of $\mathbf{u}_\alpha$ is

$$\mathbf{u}_\alpha = \delta_\alpha^\beta\mathbf{k}_\beta + u_\alpha^\Delta\mathbf{k}_\Delta = \mathbf{k}_\alpha + u_\alpha^\Delta\mathbf{k}_\Delta \tag{50.10}$$

We now show that the basis $\{\mathbf{u}_\alpha\}$ has the property (50.6). From (50.10), by direct calculation based on (49.21), we can verify easily that the first $D$ components of $\left[\mathbf{u}_\alpha,\mathbf{u}_\beta\right]$ are zero, i.e., $\left[\mathbf{u}_\alpha,\mathbf{u}_\beta\right]$ has the representation

$$\left[\mathbf{u}_\alpha,\mathbf{u}_\beta\right] = K_{\alpha\beta}^\Delta\mathbf{k}_\Delta \tag{50.11}$$

but, by assumption, $\mathscr{D}$ is closed with respect to the Lie bracket; it follows that

$$\left[\mathbf{u}_\alpha,\mathbf{u}_\beta\right] = K_{\alpha\beta}^\gamma\mathbf{u}_\gamma \tag{50.12}$$

Substituting (50.10) into (50.12), we then obtain

$$\left[\mathbf{u}_\alpha,\mathbf{u}_\beta\right] = K_{\alpha\beta}^\gamma\mathbf{k}_\gamma + K_{\alpha\beta}^\Delta\mathbf{k}_\Delta \tag{50.13}$$

Comparing this representation with (50.11), we see that $K_{\alpha\beta}^\gamma$ must vanish and hence, by (50.12), (50.6) holds.

Now we claim that the local basis $\{\mathbf{u}_\alpha\}$ for $\mathscr{D}$ can be extended to a field of basis $\{\mathbf{u}_i\}$ for $\mathscr{V}$ and that

$$\left[\mathbf{u}_i, \mathbf{u}_j\right] = \mathbf{0}, \qquad i, j = 1, \ldots, N \tag{50.14}$$

This fact is more or less obvious. From (50.6), by the argument presented in the preceding section, the integral curves of $\{\mathbf{u}_\alpha\}$ form a "coordinate net" on a $D$-dimensional hypersurface defined in the neighborhood of any reference point $\mathbf{x}_0$. To define $\mathbf{u}_{D+1}$, we simply choose an arbitrary smooth curve $\boldsymbol{\lambda}(t, \mathbf{x}_0)$ passing through $\mathbf{x}_0$, having a nonzero tangent, and not belonging to the hyper surface generated by $\{\mathbf{u}_1, \ldots, \mathbf{u}_D\}$. We regard the points of the curve $\boldsymbol{\lambda}(t, \mathbf{x}_0)$ to have constant coordinates in the coordinate net on the $D$-dimensional hypersurfaces, say $\left(x_o^1, \ldots, x_o^D\right)$. Then we define the curves $\boldsymbol{\lambda}(t, \mathbf{x})$ for all neighboring points $\mathbf{x}$ on the hypersurface of $\mathbf{x}_o$, by exactly the same condition with constant coordinates $\left(x^1, \ldots, x^D\right)$. Thus, by definition, the flow generated by the curves $\boldsymbol{\lambda}(t, \mathbf{x})$ preserves the integral curves of any $\mathbf{u}_\alpha$. Hence if we define $\mathbf{u}_{D+1}$ to be the tangent vector field of the curves $\boldsymbol{\lambda}(t, \mathbf{x})$, then

$$\left[\mathbf{u}_{D+1}, \mathbf{u}_\alpha\right] = \mathbf{0}, \qquad \alpha = 1, \ldots, D \tag{50.15}$$

where we have used the necessary and sufficient condition (49.29) for the condition (49.30).

Having defined the vector fields $\{\mathbf{u}_1, \ldots, \mathbf{u}_{D+1}\}$ which satisfy the conditions (50.6) and (50.15), we repeat that same procedure and construct the fields $\mathbf{u}_{D+2}, \mathbf{u}_{D+3}, \ldots,$ until we arrive at a field of basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_N\}$. Now from a theorem proved in the preceding section [cf. (49.39)] the condition (50.14) is necessary and sufficient that $\{\mathbf{u}_i\}$ be the natural basis of a coordinate system $\hat{x}$. Consequently, $\mathscr{D}$ is integrable and the proof is complete.

From the proof of the preceding theorem it is clear that an integrable distribution $\mathscr{D}$ can be characterized by the opening remark: In the neighborhood of any point $\mathbf{x}_0$ in the domain of $\mathscr{D}$ there exists a $D$-dimensional hypersurface $\mathscr{S}$ such that $\mathscr{D}(\mathbf{x})$ is the $D$-dimensional tangent hyperplane of $\mathscr{S}$ at each point $\mathbf{x}$ in $\mathscr{S}$.

We shall state and prove a dual version of the Frobenius theorem in Section 52.

## Exercises

50.1 Let $\mathscr{D}$ be an integrable distribution defined on $\mathscr{U}$. Then we define the following equivalence relation on $\mathscr{U}$: $\mathbf{x}_0 \sim \mathbf{x}_1 \Leftrightarrow$ there exists a smooth curve $\boldsymbol{\lambda}$ in $\mathscr{U}$ joining $\mathbf{x}_0$ to $\mathbf{x}_1$ and tangent to $\mathscr{D}$ at each point, i.e., $\dot{\boldsymbol{\lambda}}(t) \in \mathscr{D}(\boldsymbol{\lambda}(t))$, for all $t$. Suppose that $\mathscr{S}$ is an equivalence set relative to the preceding equivalence relation. Show that $\mathscr{S}$ is an integral surface of $\mathscr{D}$. (Such an integral surface is called a *maximal integral surface* or a *leaf* of $\mathscr{D}$.)

## *Section 51. Differential Forms and Exterior Derivative*

In this section we define a differential operator on skew-symmetric covariant tensor fields. This operator generalizes the classical curl operator defined in Section 47.

Let $\mathscr{U}$ be an open set in $\mathscr{E}$ and let $\mathbf{A}$ be a skew-symmetric covariant tensor field of order $r$ on $\mathscr{U}$, i.e.,

$$\mathbf{A}: \mathscr{U} \rightarrow \hat{\mathscr{T}}_r(\mathscr{V}) \tag{51.1}$$

Then for any point $\mathbf{x} \in \mathscr{U}$, $\mathbf{A}(\mathbf{x})$ is a skew-symmetric tensor of order $r$ (cf. Chapter 8). Choosing a coordinate chart $\hat{x}$ on $\mathscr{U}$ as before, we can express $\mathbf{A}$ in the component form

$$\mathbf{A} = A_{i_1 \dots i_r} \mathbf{g}^{i_1} \otimes \cdots \otimes \mathbf{g}^{i_r} \tag{51.2}$$

where $\{\mathbf{g}^i\}$ denotes the natural basis of $\hat{x}$. Since $\mathbf{A}$ is skew-symmetric, its components obey the identify

$$A_{i_1 \dots j \dots k \dots i_r} = -A_{i_1 \dots k \dots j \dots i_r} \tag{51.3}$$

for any pair of indices $(j,k)$ in $(i_1, \dots, i_r)$. As explained in Section 39, we can then represent $\mathbf{A}$ by

$$\mathbf{A} = \sum_{i_1 < \cdots < i_r} A_{i_1 \dots i_r} \mathbf{g}^{i_1} \wedge \cdots \wedge \mathbf{g}^{i_r} = \frac{1}{r!} A_{i_1 \dots i_r} \mathbf{g}^{i_1} \wedge \cdots \wedge \mathbf{g}^{i_r} \tag{51.4}$$

where $\wedge$ denotes the exterior product.

Now if $\mathbf{A}$ is smooth, then it is called a *differential form of order r*, or simply an *r-form*. In the theory of differentiable manifolds, differential forms are important geometric entities, since they correspond to linear combinations of volume tensors on various hypersurfaces (cf. the book by Flanders[1]). For our purpose, however, we shall consider only some elementary results about differential forms.

---

[1] H. Flanders, *Differential Forms with Applications to the Physical Sciences*, Academic Press, New York-London, 1963.

We define first the notion of the exterior derivative of a differential form. Let $\mathbf{A}$ be the *r-form* represented by (51.2) or (51.4). Then the *exterior derivative* $d\mathbf{A}$ is an $(r+1)$–form given by any one of the following three equivalent formulas:

$$\begin{aligned} d\mathbf{A} &= \sum_{i_1<\cdots<i_r}\sum_{k=1}^{N}\frac{\partial A_{i_1\ldots i_r}}{\partial x^k}\mathbf{g}^k\wedge\mathbf{g}^{i_1}\wedge\cdots\wedge\mathbf{g}^{i_r} \\ &= \frac{1}{r!}\frac{\partial A_{i_1\ldots i_r}}{\partial x^k}\mathbf{g}^k\wedge\mathbf{g}^{i_1}\wedge\cdots\wedge\mathbf{g}^{i_r} \\ &= \frac{1}{r!(r+1)!}\delta^{ki_1\ldots i_r}_{j_1\cdots j_{r+1}}\frac{\partial A_{i_1\ldots i_r}}{\partial x^k}\mathbf{g}^{j_1}\wedge\cdots\wedge\mathbf{g}^{j_{r+1}} \end{aligned} \tag{51.5}$$

Of course, we have to show that $d\mathbf{A}$ as defined by (51.5), is independent of the choice of the chart $\hat{x}$. If this result is granted, then (51.5) can be written in the coordinate-free form

$$d\mathbf{A} = (-1)^r(r+1)!\mathbf{K}_{r+1}(\operatorname{grad}\mathbf{A}) \tag{51.6}$$

where $\mathbf{K}_{r+1}$ is the skew-symmetric operator introduced in Section 37.

We shall now show that $d\mathbf{A}$ is well-defined by (51.5), i.e., if $(\overline{x}^i)$ is another coordinate system, then

$$\frac{\partial A_{i_1\ldots i_r}}{\partial x^k}\mathbf{g}^k\wedge\mathbf{g}^{i_1}\wedge\cdots\wedge\mathbf{g}^{i_r} = \frac{\partial \overline{A}_{i_1\ldots i_r}}{\partial \overline{x}^k}\overline{\mathbf{g}}^k\wedge\overline{\mathbf{g}}^{i_1}\wedge\cdots\overline{\mathbf{g}}^{i_r} \tag{51.7}$$

or, equivalently,

$$\delta^{ki_1\ldots i_r}_{j_1\cdots j_{r+1}}\frac{\partial A_{i_1\ldots i_r}}{\partial x^k} = \delta^{ki_1\ldots i_r}_{l_1\ldots l_{r+1}}\frac{\partial \overline{A}_{i_1\ldots i_r}}{\partial \overline{x}^k}\frac{\partial \overline{x}^{l_1}}{\partial x^{j_1}}\cdots\frac{\partial \overline{x}^{l_{r+1}}}{\partial x^{j_{r+1}}} \tag{51.8}$$

for all $j_1,\ldots,j_{r+1}$, where $\overline{A}_{i_1\ldots i_r}$ and $\{\overline{\mathbf{g}}^i\}$ denote the components of $\mathbf{A}$ and the natural basis corresponding to $(\overline{x}^i)$. To prove (51.8), we recall first the transformation rule

$$\overline{A}_{i_1\cdots i_r} = A_{j_1\cdots j_r}\frac{\partial x^{j_1}}{\partial \overline{x}^{i_1}}\cdots\frac{\partial x^{j_r}}{\partial \overline{x}^{i_r}} \tag{51.9}$$

for any covariant tensor components. Now differentiating (51.9) with respect to $\overline{x}^k$, we obtain

$$\begin{aligned}\frac{\partial \overline{A}_{i_1\ldots i_r}}{\partial \overline{x}^k} &= \frac{\partial \overline{A}_{j_1\cdots j_r}}{\partial x^l}\frac{\partial x^l}{\partial \overline{x}^k}\cdots\frac{\partial x^{j_1}}{\partial \overline{x}^{i_1}}\cdots\frac{\partial x^{j_r}}{\partial \overline{x}^{i_r}}\\ &+A_{j_1\cdots j_r}\frac{\partial^2 x^{j_1}}{\partial \overline{x}^k\,\partial \overline{x}^{i_1}}\frac{\partial x^{j_2}}{\partial \overline{x}^{i_2}}\cdots\frac{\partial x^{j_r}}{\partial \overline{x}^{i_r}}\\ &+\cdots+A_{j_1\cdots j_r}\frac{\partial x^{j_1}}{\partial \overline{x}^{i_1}}\cdots\frac{\partial x^{j_{r-1}}}{\partial \overline{x}^{i_{r-1}}}\frac{\partial^2 x^{j_r}}{\partial \overline{x}^k\,\partial \overline{x}^{i_r}}\end{aligned} \tag{51.10}$$

Since the second derivative $\partial^2 x^j/\partial\overline{x}^k\partial\overline{x}^i$ is symmetric with respect to the pair $(k,i)$, when we form the contraction of (51.10) with the skew-symmetric operator $\mathbf{K}_{r+1}$ the result is (51.8). Here we have used the fact that $\mathbf{K}_{r+1}$ is a tensor of order $2(r+1)$, so that we have the identities

$$\begin{aligned}\delta^{ki_1\ldots i_r}_{j_1\cdots j_{r+1}} &= \delta^{lm_1\ldots m_r}_{p_1\cdots p_{r+1}}\frac{\partial x^k}{\partial \overline{x}^1}\frac{\partial x^{i_1}}{\partial \overline{x}^{m_1}}\cdots\frac{\partial x^{i_r}}{\partial \overline{x}^{m_r}}\frac{\partial \overline{x}^{p_1}}{\partial x^{j_1}}\cdots\frac{\partial \overline{x}^{p_{r+1}}}{\partial x^{j_{r+1}}}\\ &= \delta^{lm_1\ldots m_r}_{p_1\cdots p_{r+1}}\frac{\partial \overline{x}^k}{\partial x^l}\frac{\partial \overline{x}^{i_1}}{\partial x^{m_1}}\cdots\frac{\partial \overline{x}^{i_r}}{\partial x^{m_r}}\frac{\partial x^{p_1}}{\partial \overline{x}^{j_1}}\cdots\frac{\partial x^{p_{r+1}}}{\partial \overline{x}^{j_{r+1}}}\end{aligned} \tag{51.11}$$

From (38.2) and (51.5), if $\mathbf{A}$ is a 1-form, say $\mathbf{A}=\mathbf{u}$, then

$$d\mathbf{u}=\left(\frac{\partial u_i}{\partial x^j}-\frac{\partial u_j}{\partial x^i}\right)\mathbf{g}^j\otimes\mathbf{g}^i \tag{51.12}$$

Comparing this representation with (47.52), we see that the exterior derivative and the curl operator are related by

$$2\ \text{curl}\ \mathbf{u}=-d\mathbf{u} \tag{51.13}$$

for any 1-form $\mathbf{u}$. Equation (51.13) also follows from (51.6) and (47.50). In the sense of (51.13), the exterior derivative is a generalization of the curl operator from 1-forms to *r*-forms in general. We shall now consider some basic properties of the exterior derivative.

I. If *f* is a smooth function (i.e., a 0-form), then the exterior derivative of *f* coincides with the gradient of *f*,

$$df=\text{grad}\ f \tag{51.14}$$

This result follows readily from (51.5) and (47.14).

II. If $\mathbf{w}$ is a smooth covariant vector field (i.e., a 1-form), and if $\mathbf{u}$ and $\mathbf{v}$ are smooth vector fields, then

$$\mathbf{u}\cdot d(\mathbf{v}\cdot\mathbf{w})-\mathbf{v}\cdot d(\mathbf{u}\cdot\mathbf{w})=[\mathbf{u},\mathbf{v}]\cdot\mathbf{w}+d\mathbf{w}(\mathbf{u},\mathbf{v}) \tag{51.15}$$

We can prove this formula by direct calculation. Let the component representations of $\mathbf{u},\mathbf{v},\mathbf{w}$ in a chart $\hat{x}$ be

$$\mathbf{u}=u^i\,\mathbf{g}_i, \qquad \mathbf{v}=\upsilon^i\,\mathbf{g}_i, \qquad \mathbf{w}=w_i\,\mathbf{g}^i \tag{51.16}$$

Then

$$\mathbf{u}\cdot\mathbf{w}=u^i w_i, \qquad \mathbf{v}\cdot\mathbf{w}=\upsilon^i w_i \tag{51.17}$$

From I it follows that

$$d(\mathbf{u}\cdot\mathbf{w})=\frac{\partial(u^i w_i)}{\partial x^k}\mathbf{g}^k=\left(u^i\frac{\partial w_i}{\partial x^k}+w_i\frac{\partial u^i}{\partial x^k}\right)\mathbf{g}^k \tag{51.18}$$

and similarly

$$d(\mathbf{v}\cdot\mathbf{w})=\frac{\partial(\upsilon^i w_i)}{\partial x^k}\mathbf{g}^k=\left(\upsilon^i\frac{\partial w_i}{\partial x^k}+w_i\frac{\partial \upsilon^i}{\partial x^k}\right)\mathbf{g}^k \tag{51.19}$$

Consequently,

$$\mathbf{u}\cdot d(\mathbf{v}\cdot\mathbf{w})-\mathbf{v}\cdot d(\mathbf{u}\cdot\mathbf{w})=\left(u^k\frac{\partial \upsilon^i}{\partial x^k}-\upsilon^k\frac{\partial u^i}{\partial x^k}\right)w_i+\left(u^k\upsilon^i-u^i\upsilon^k\right)\frac{\partial w_i}{\partial x^k} \tag{51.20}$$

Now from (49.21) the first term on the right-hand side of (51.20) is simply $[\mathbf{u},\mathbf{v}]\cdot\mathbf{w}$. Since the coefficient of the second tensor on the right-hand side of (51.20) is skew-symmetric, that term can be rewritten as

$$u^k\upsilon^i\left(\frac{\partial w_i}{\partial x^k}-\frac{\partial w_k}{\partial x^i}\right)$$

or, equivalently,

$$d\mathbf{w}(\mathbf{u},\mathbf{v})$$

when the representation (51.12) is used. Thus the identity (51.15) is proved.

III If $\mathbf{A}$ is an $r$-form and $\mathbf{B}$ is an $s$-form, then

$$d\left(\mathbf{A}\wedge\mathbf{B}\right)=d\mathbf{A}\wedge\mathbf{B}+\left(-1\right)^{r}\mathbf{A}\wedge d\mathbf{B} \tag{51.21}$$

This formula can be proved by direct calculation also, so we leave it as an exercise.

IV. If $\mathbf{A}$ and $\mathbf{B}$ are any $r$-forms, then

$$d\left(\mathbf{A}+\mathbf{B}\right)=d\mathbf{A}+d\mathbf{B} \tag{51.22}$$

The proof of this result is obvious from the representation (51.5). Combining (51.21) and (51.22) , we have

$$d\left(\alpha\mathbf{A}+\beta\mathbf{B}\right)=\alpha d\mathbf{A}+\beta d\mathbf{B} \tag{51.23}$$

for any $r$-forms $\mathbf{A}$ and $\mathbf{B}$ and scalars α and β.

V. For any $r$-form $\mathbf{A}$

$$d^{2}\mathbf{A}\equiv d\left(d\mathbf{A}\right)=\mathbf{0} \tag{51.24}$$

This result is a consequence of the symmetry of the second derivative.

Indeed, from (51.5) for the $\left(r+1\right)-$ form $d\mathbf{A}$ we have

$$\begin{aligned}
d^{2}\mathbf{A}&=\frac{1}{r!}\,\frac{1}{\left(r+2\right)!}\,\frac{1}{\left(\left(r+1\right)!\right)^{2}}\,\delta^{lj_1\ldots j_{r+1}}_{p_1\cdots p_{r+2}}\,\delta^{ki_1\ldots i_r}_{j_1\cdots j_{r+1}}\,\frac{\partial^{2}A_{i_1\ldots i_r}}{\partial x^{l}\,\partial x^{k}}\,\mathbf{g}^{p_1}\wedge\cdots\wedge\mathbf{g}^{p_{r+2}}\\
&=\frac{1}{r!}\frac{1}{\left(r+1\right)!}\,\frac{1}{\left(r+2\right)!}\delta^{lki_1\ldots i_r}_{p_1\cdots p_{r+2}}\,\frac{\partial^{2}A_{i_1\ldots i_r}}{\partial x^{l}\,\partial x^{k}}\,\mathbf{g}^{p_1}\wedge\cdots\wedge\mathbf{g}^{p_{r+2}}\\
&=\mathbf{0}
\end{aligned} \tag{51.25}$$

where we have used the identity (20.14).

VI. Let $\mathbf{f}$ be a smooth mapping from an open set $\bar{\mathscr{U}}$ in $\bar{\mathscr{E}}$ into an open set $\mathscr{U}$ in $\mathscr{E}$, and suppose that $\mathbf{A}$ is an $r$-form on $\mathscr{U}$. Then

$$\bar{d}\left(\mathbf{f}^{*}\left(\mathbf{A}\right)\right)=\mathbf{f}^{*}\left(d\mathbf{A}\right) \tag{51.26}$$

where $\mathbf{f}^{*}$ denotes the induced linear map (defined below) corresponding to $\mathbf{f}$, so that $\mathbf{f}^{*}\left(\mathbf{A}\right)$ is an $r$-form on $\bar{\mathscr{U}}$ , and where $\bar{d}$ denotes the exterior derivative on $\bar{\mathscr{U}}$. To establish (51.26), let

$\dim \mathscr{E} = N$ and $\dim \overline{\mathscr{E}} = M$. We choose coordinate systems $\left(x^i, i = 1,\ldots,N\right)$ and $\left(\overline{x}^\alpha, \alpha = 1,\ldots,M\right)$ on $\mathscr{U}$ and $\overline{\mathscr{U}}$, respectively. Then the mapping $\mathbf{f}$ can be characterized by

$$x^i = f^i\left(\overline{x}^1,\ldots,\overline{x}^M\right), \qquad i = 1,\ldots,N \tag{51.27}$$

Suppose that $\mathbf{A}$ has the representation $(51.4)_2$ in $\left(x^i\right)$. Then by definition $\mathbf{f}^*\left(\mathbf{A}\right)$ has the representation

$$\mathbf{f}^*\left(\mathbf{A}\right) = \frac{1}{r!} A_{i_1\ldots i_r} \frac{\partial x^{i_1}}{\partial \overline{x}^{\alpha_1}} \cdots \frac{\partial x^{i_r}}{\partial \overline{x}^{\alpha_r}} \overline{\mathbf{g}}^{\alpha_1} \wedge \cdots \wedge \overline{\mathbf{g}}^{\alpha_r} \tag{51.28}$$

where $\left\{\overline{\mathbf{g}}^\alpha\right\}$ denotes the natural basis of $\left(\overline{x}^\alpha\right)$. Now by direct calculation of the exterior derivatives of $\mathbf{A}$ and $\mathbf{f}^*\left(\mathbf{A}\right)$, and by using the symmetry of the second derivative, we obtain the representation

$$\begin{aligned} \overline{d}\left(\mathbf{f}^*\left(\mathbf{A}\right)\right) &= \mathbf{f}^*\left(d\mathbf{A}\right) \\ &= \frac{1}{r!} \frac{\partial A_{i_1\ldots i_r}}{\partial x^k} \frac{\partial x^k}{\partial \overline{x}^\beta} \frac{\partial x^{i_1}}{\partial \overline{x}^{\alpha_1}} \cdots \frac{\partial x^{i_r}}{\partial \overline{x}^{\alpha_r}} \overline{\mathbf{g}}^\beta \wedge \overline{\mathbf{g}}^{\alpha_1} \wedge \cdots \wedge \overline{\mathbf{g}}^{\alpha_r} \end{aligned} \tag{51.29}$$

Thus (51.26) is proved.

Applying the result (51.26) to the special case when $\mathbf{f}$ is the flow $\boldsymbol{\rho}_t$ generated by a vector field $\mathbf{v}$ as defined in Section 49, we obtain the following property.

VII. The exterior derivative and the Lie derivative commute, i.e., for any differential form $\mathbf{A}$ and any smooth vector field $\mathbf{v}$

$$d\left(\mathscr{L}_{\mathbf{v}} \mathbf{A}\right) = \mathscr{L}_{\mathbf{v}}\left(d\mathbf{A}\right) \tag{51.30}$$

We leave the proof of (51.30), by direct calculation based on the representations (51.5) and (49.42), as an exercise.

The results I-VII summarized above are the basic properties of the exterior derivative. In fact, results I and III-V characterize the exterior derivative completely. This converse assertion can be stated formally in the following way.

*Coordinate-free definition of the exterior derivative*. The exterior derivative $d$ is an operator from an $r$-form to an $\left(r+1\right)-$ form and satisfies the conditions, I, III-V above.

To see that this definition is equivalent to the representations (51.5), we consider any $r$-form $\mathbf{A}$ given by the representation $(51.4)_2$. Applying the operator $d$ to both sides of that representation and making use of conditions I and III-V, we get

$$
\begin{aligned}
r!\, d\mathbf{A} &= d\left(A_{i_1\dots i_r}\ \mathbf{g}^{i_1}\wedge\cdots\wedge\mathbf{g}^{i_r}\right)\\
&=\left(d\,A_{i_1\dots i_r}\right)\wedge\mathbf{g}^{i_1}\wedge\cdots\wedge\mathbf{g}^{i_r}+A_{i_1\dots i_r}\,d\,\mathbf{g}^{i_1}\wedge\mathbf{g}^{i_2}\wedge\cdots\wedge\mathbf{g}^{i_r}\\
&\quad -A_{i_1\dots i_r}\,\mathbf{g}^{i_1}\wedge d\mathbf{g}^{i_2}\wedge\mathbf{g}^{i_3}\wedge\cdots\wedge\mathbf{g}^{i_r}+\cdots\\
&=\left(\frac{\partial A_{i_1\dots i_r}}{\partial x^k}\mathbf{g}^k\right)\wedge\mathbf{g}^{i_1}\wedge\cdots\wedge\mathbf{g}^{i_r}+\mathbf{0}-\mathbf{0}+\cdots\\
&=\frac{\partial A_{i_1\dots i_r}}{\partial x^k}\mathbf{g}^k\wedge\mathbf{g}^{i_1}\wedge\cdots\wedge\mathbf{g}^{i_r}
\end{aligned}
\tag{51.31}
$$

where we have used the fact that the natural basis vector $\mathbf{g}^i$ is the gradient of the coordinate function $x^i$, so that, by I,

$$\mathbf{g}^i=\operatorname{grad} x^i = dx^i \tag{51.32}$$

and thus, by V,

$$d\,\mathbf{g}^i = d^2x^i=\mathbf{0} \tag{51.33}$$

Consequently (51.5) is equivalent to the coordinate-free definition just stated.

## *Exercises*

51.1 Prove the product rule (51.21) for the exterior derivative.

51.2 Given (47.53), the classical definition of the curl of a vector field, prove that

$$\operatorname{curl}\mathbf{v}=\mathbf{D}_2\left(d\mathbf{v}\right) \tag{51.34}$$

where $\mathbf{D}$ is the duality operator introduced in Section 41.

51.3 Let $f$ be a 0-form. Prove that

$$\Delta f=\mathbf{D}_3 d\left(\mathbf{D}_1\, df\right) \tag{51.35}$$

in the case where $\dim\mathscr{E}=3$.

51.4 Let $\mathbf{f}$ be the smooth mapping from an open set $\bar{\mathscr{U}}$ in $\bar{\mathscr{E}}$ onto an open set $\mathscr{U}$ in $\mathscr{E}$ discussed in VI. First, show that

$$\operatorname{grad} \mathbf{f} = \frac{\partial x^i}{\partial \overline{x}^\alpha} \mathbf{g}_i \otimes \overline{\mathbf{g}}^\alpha$$

Next show that $\mathbf{f}^*$, which maps $r$-forms on $\mathscr{U}$ into $r$-forms on $\overline{\mathscr{U}}$, can be defined by

$$\mathbf{f}^*(\mathbf{A})(\overline{\mathbf{u}}^1, \ldots, \overline{\mathbf{u}}^r) = \mathbf{A}((\operatorname{grad} \mathbf{f})\overline{\mathbf{u}}^1, \ldots, (\operatorname{grad} \mathbf{f})\overline{\mathbf{u}}^r)$$

for all $\overline{\mathbf{u}}^1, \ldots, \overline{\mathbf{u}}^r$ in the translation space of $\overline{\mathscr{E}}$. Note that for $r = 1$, $\mathbf{f}^*$ is the transpose of grad $\mathbf{f}$.

51.5 Check the commutation relation (51.30) by component representations.

## Section 52. The Dual Form of the Frobenius Theorem; the Poincaré Lemma

The Frobenius theorem as stated and proved in Section 50 characterizes the integrability of a distribution by a condition on the Lie bracket of the generators of the distribution. In this section we shall characterize the same by a condition on the exterior derivative of the generators of the orthogonal complement of the distribution. This condition constitutes the dual form of the Frobenius theorem.

As in Section 50, let $\mathscr{D}$ be a distribution of dimension $D$ defined on a domain $\mathscr{U}$ in $\mathscr{E}$. Then at each point $\mathbf{x} \in \mathscr{U}$, $\mathscr{D}(\mathbf{x})$ is a $D$-dimensional subspace of $\mathscr{V}$. We put $\mathscr{D}^{\perp}(\mathbf{x})$ to be the orthogonal complement of $\mathscr{D}(\mathbf{x})$. Then $\dim \mathscr{D}^{\perp}(\mathbf{x}) = N - D$ and the field $\mathscr{D}^{\perp}$ on $\mathscr{U}$ defined in this way is itself a distribution of dimension $N - D$. In particular, locally, there exist 1-forms $\{\mathbf{z}^{\Gamma}, \Gamma = 1, \ldots, N - D\}$ which generate $\mathscr{D}^{\perp}$. The dual form of the Frobenius theorem is the following theorem.

**Theorem 52.1**. The distribution $\mathscr{D}$ is integrable if and only if

$$d\,\mathbf{z}^{\Gamma} \wedge \mathbf{z}^{1} \wedge \cdots \wedge \mathbf{z}^{N-D} = \mathbf{0}, \qquad \Gamma = 1, \ldots, N - D \tag{52.1}$$

*Proof*. As in Section 50, let $\{\mathbf{v}_{\alpha}, \alpha = 1, \ldots, D\}$ be a local basis for $\mathscr{D}$. Then

$$\mathbf{v}_{\alpha} \cdot \mathbf{z}^{\Gamma} = 0, \qquad \alpha = 1, \ldots, D, \quad \Gamma = 1, \ldots, N - D \tag{52.2}$$

By the Frobenius theorem $\mathscr{D}$ is integrable if and only if (50.5) holds. Substituting (50.5) and (52.2) into (51.15) with $\mathbf{u} = \mathbf{v}_{\alpha}, \mathbf{v} = \mathbf{v}_{\beta},$ *and* $\mathbf{w} = \mathbf{z}^{\Gamma}$, we see that (50.5) is equivalent to

$$d\,\mathbf{z}^{\Gamma}\left(\mathbf{v}_{\alpha}, \mathbf{v}_{\beta}\right) = 0, \qquad \alpha, \beta = 1, \ldots, D, \quad \Gamma = 1, \ldots, N - D \tag{52.3}$$

We now show that this condition is equivalent to (52.1).

To prove the said equivalence, we extend $\{\mathbf{v}_{\alpha}\}$ into a basis $\{\mathbf{v}_{i}\}$ in such a way that its dual basis $\{\mathbf{v}^{i}\}$ satisfies $\mathbf{v}^{D+\Gamma} = \mathbf{z}^{\Gamma}$ for all $\Gamma = 1, \ldots, N - D$. This extension is possibly by virtue of (52.2). Relative to the basis $\{\mathbf{v}_{i}\}$, the condition (52.3) means that in the representation of $d\,\mathbf{z}^{\Gamma}$ by

$$2d\,\mathbf{z}^{\Gamma} = \zeta_{ij}^{\Gamma} \mathbf{v}^{i} \wedge \mathbf{v}^{j} = 2\zeta_{ij}^{\Gamma} \mathbf{v}^{i} \otimes \mathbf{v}^{j} \tag{52.4}$$

the components $\zeta_{ij}^{\Gamma}$ with $1 \le i, j \le D$ must vanish. Thus

$$2d\,\mathbf{z}^{\Gamma} = \zeta^{\Gamma}_{i\Delta+D}\mathbf{v}^{i} \wedge \mathbf{z}^{\Delta} \tag{52.5}$$

which is clearly equivalent to (52.1).

As an example of the integrability condition (52.1), we can derive a necessary and sufficient condition for a vector field $\mathbf{z}$ to be orthogonal to a family of surfaces. That is, there exist smooth functions *h* and *f* such that

$$\mathbf{z} = h \text{ grad } f \tag{52.6}$$

Such a vector field is called *complex-lamellar* in the classical theory. Using the terminology here, we see that a complex-lamellar vector field $\mathbf{z}$ is a vector field such that $\mathbf{z}^{\perp}$ is integrable. Hence by (52.1), $\mathbf{z}$ is complex-lamellar if and only if $d\mathbf{z} \wedge \mathbf{z} = \mathbf{0}$. In a three-dimensional space this condition can be written as

$$\left(\operatorname{curl}\mathbf{z}\right)\cdot\mathbf{z} = 0 \tag{52.7}$$

which was first noted by Kelvin (1851). In component form, if $\mathbf{z}$ is represented by

$$\mathbf{z} = z_i\,\mathbf{g}^i = z_i dx^i \tag{52.8}$$

then $d\mathbf{z}$ is represented by

$$d\mathbf{z} = \frac{1}{2}\frac{\partial z_i}{\partial x^k} dx^k \wedge dx^i \tag{52.9}$$

and thus $d\mathbf{z} \wedge \mathbf{z} = \mathbf{0}$ means

$$z_j \frac{\partial z_i}{\partial x^k} dx^k \wedge dx^i \wedge dx^j = 0 \tag{52.10}$$

or, equivalently,

$$\delta^{kij}_{pqr} z_j \frac{\partial z_i}{\partial x^k} = 0, \qquad p,q,r = 1,\ldots,N \tag{52.11}$$

Since it suffices to consider the special cases with $p<q<r$ in (52.11), when $N$=3 we have

$$0 = \delta^{kij}_{123}\, z_j \frac{\partial z_i}{\partial x^k} = z_j \varepsilon^{kij} \frac{\partial z_i}{\partial x^k} \tag{52.12}$$

which is equivalent to the component version of (52.7). In view of this special case we see that the dual form of the Frobenius theorem is a generalization of Kelvin's condition from 1-forms to *r*-forms in general.

In the classical theory a vector field (1-form) $\mathbf{z}$ is called *lamellar* if locally $\mathbf{z}$ is the gradient of a smooth function (0-form) *f*, namely

$$\mathbf{z} = \operatorname{grad} f = df \tag{52.13}$$

Then it can be shown that $\mathbf{z}$ is lamellar if and only if $d\mathbf{z}$ vanishes, i.e.,

$$\operatorname{curl} \mathbf{z} = \mathbf{0} \tag{52.14}$$

The generalization of lamellar fields from 1-forms to *r*-forms in general is obvious: We say that an *r*-form $\mathsf{A}$ is closed if its exterior derivative vanishes

$$d\mathsf{A} = \mathbf{0} \tag{52.15}$$

and we say that $\mathsf{A}$ is *exact* if it is the exterior derivatives of an (*r*-1)-form, say

$$\mathsf{A} = d\mathsf{B} \tag{52.16}$$

From (51.24) any exact form is necessarily closed. The theorem that generalizes the classical result is the Poincaré lemma, which implies that (52.16) and (52.15) are locally equivalent.

Before stating the Poincaré lemma, we defined first the notion of a *star-shaped* (or *retractable*) open set $\mathscr{U}$ in $\mathscr{E}$ : $\mathscr{U}$ is star-shaped if there exists a smooth mapping

$$\boldsymbol{\rho}: \ \mathscr{U} \times \mathscr{R} \to \mathscr{U} \tag{52.17}$$

such that

$$\boldsymbol{\rho}(\mathbf{x},t) = \begin{cases} \mathbf{x} & \text{when } \ t \le 0 \\ \mathbf{x}_0 & \text{when } \ t \ge 1 \end{cases} \tag{52.18}$$

where $\mathbf{x}_0$ is a particular point in $\mathscr{U}$. In a star-shaped open set $\mathscr{U}$ all closed hypersurfaces of dimensions 1 to $N-1$ are retractable in the usual sense. For example, an open ball is star-shaped but the open set between two concentric spheres is not, since a closed sphere in the latter is not retractable.

The equivalence of (52.13) and (52.14) requires only that the domain $\mathscr{U}$ of $\mathbf{z}$ be simply connected, i.e., every closed curve in $\mathscr{U}$ be retractable. When we generalize the result to the equivalence of (52.15) and (52.16), the domain $\mathscr{U}$ of $\mathbf{A}$ must be retractable relative to all $r$-dimensional closed hypersurfaces. For simplicity, we shall assume that $\mathscr{U}$ is star-shaped.

**Theorem 52.2.** If $\mathbf{A}$ is a closed $r$-form defined on a star-shaped domain, then $\mathbf{A}$ is exact, i.e., there exists an (r-1)-form $\mathbf{B}$ on $\mathscr{U}$ such that (52.16) holds.

*Proof.* We choose a coordinate system $\left(x^i\right)$ on $\mathscr{U}$. Then a coordinate system $\left(y^\alpha, \alpha=1,\ldots,N+1\right)$ on $\mathscr{U}\times\mathscr{R}$ is given by

$$\left(y^1,\ldots,y^{N+1}\right)=\left(x^1,\ldots,x^N,t\right) \tag{52.19}$$

From (52.18) the mapping $\boldsymbol{\rho}$ has the property

$$\boldsymbol{\rho}(\cdot,t)=id_{\mathscr{U}} \qquad \text{when} \quad t\le 0 \tag{52.20}$$

and

$$\boldsymbol{\rho}(\cdot,t)=x_o \qquad \text{when} \quad t\ge 0 \tag{52.21}$$

Hence if the coordinate representation of $\boldsymbol{\rho}$ is

$$x^i=\rho^i\left(y^\alpha\right), \qquad i=1,\ldots,N \tag{52.22}$$

Then

$$\frac{\partial\rho^i}{\partial y_j}=\delta^i_j, \qquad \frac{\partial\rho^i}{\partial y^{N+1}}=0 \qquad \text{when} \quad t\le 0 \tag{52.23}$$

and

$$\frac{\partial\rho^i}{\partial y^j}=0 \qquad \frac{\partial\rho^i}{\partial y^{N+1}}=0 \qquad \text{when} \quad t\ge 1 \tag{52.24}$$

As usual we can express $\mathbf{A}$ in component form relative to $\left(x^i\right)$

$$\mathbf{A}=\frac{1}{r!}A_{i_1\ldots i_r}\mathbf{g}^{i_1}\wedge\cdots\wedge\mathbf{g}^{i_r} \tag{52.25}$$

Then $\boldsymbol{\rho}^*(\mathbf{A})$ is an $r$-form on $\mathscr{U}\times\mathscr{R}$ defined by

$$\boldsymbol{\rho}^*(\mathbf{A})=\frac{1}{r!}A_{i_1\dots i_r}\frac{\partial\rho^{i_1}}{\partial y^{\alpha_1}}\cdots\frac{\partial\rho^{i_r}}{\partial y^{\alpha_r}}\mathbf{h}^{\alpha_1}\wedge\cdots\wedge\mathbf{h}^{\alpha_r} \tag{52.26}$$

where $\{\mathbf{h}^\alpha,\alpha=1,\dots,N+1\}$ denotes the natural basis of $(y^\alpha)$, i.e.,

$$\mathbf{h}^\alpha=\bar{d}y^\alpha \tag{52.27}$$

where $\bar{d}$ denotes the exterior derivative on $\mathscr{U}\times\mathscr{R}$.

Now from (52.19) we can rewrite (52.26) as

$$r!\boldsymbol{\rho}^*(\mathbf{A})=X_{i_1\dots i_r}\mathbf{h}^{i_1}\wedge\cdots\wedge\mathbf{h}^{i_r}+r\mathrm{Y}_{i_1\dots i_{r-1}}\mathbf{h}^{i_1}\wedge\cdots\wedge\mathbf{h}^{i_{r-1}}\wedge\mathbf{h}^{N+1} \tag{52.28}$$

where

$$X_{i_1\dots i_r}=A_{j_1\dots j_r}\frac{\partial\rho^{j_1}}{\partial y^{i_1}}\cdots\frac{\partial\rho^{j_r}}{\partial y^{i_r}} \tag{52.29}$$

and

$$Y_{i_1\dots i_{r-1}}=A_{j_1\dots j_r}\frac{\partial\rho^{j_1}}{\partial y^{i_1}}\cdots\frac{\partial\rho^{j_{r-1}}}{\partial y^{i_{r-1}}}\frac{\partial\rho^{j_r}}{\partial y^{N+1}} \tag{52.30}$$

We put

$$\mathbf{X}=\frac{1}{r!}X_{i_1\dots i_r}\mathbf{h}^{i_1}\wedge\cdots\wedge\mathbf{h}^{i_r} \tag{52.31}$$

and

$$\mathbf{Y}=\frac{1}{(r-1)!}Y_{j_1\dots j_{r-1}}\mathbf{h}^{j_1}\wedge\cdots\wedge\mathbf{h}^{j_{r-1}} \tag{52.32}$$

Then $\mathbf{X}$ is an $r$-form and $\mathbf{Y}$ is an $(r-1)-$form on $\mathscr{U}\times\mathscr{R}$. Further, from (52.28) we have

$$\boldsymbol{\rho}^*(\mathbf{A})=\mathbf{X}+\mathbf{Y}\wedge\mathbf{h}^{N+1}=\mathbf{X}+\mathbf{Y}\wedge\bar{d}t \tag{52.33}$$

From (52.23), (52.24), (52.29), and (52.30), $\mathbf{X}$ and $\mathbf{Y}$ satisfy the end conditions

$$X_{i_1\dots i_r}(\mathbf{x},\mathrm{t}) = A_{i_1\dots i_r}(\mathbf{x}), \quad Y_{i_1\dots i_{r-1}}(\mathbf{x},\mathrm{t}) = 0 \quad \text{when} \quad t \le 0 \tag{52.34}$$

and

$$X_{i_1\dots i_r}(\mathbf{x},\mathrm{t}) = 0, \qquad Y_{i_1\dots i_{r-1}}(\mathbf{x},\mathrm{t}) = 0 \qquad \text{when} \quad t \ge 1 \tag{52.35}$$

for all $\mathbf{x} \in \mathscr{U}$. We define

$$\mathbf{B} \equiv \frac{(-1)}{(r-1)!}\left(\int_0^1 Y_{i_1\dots i_{r-1}}(\cdot,t)\,dt\right)\mathbf{g}^{i_1} \wedge \cdots \wedge \mathbf{g}^{i_{r-1}} \tag{52.36}$$

and we claim that this $(r-1)$-form $\mathbf{B}$ satisfies the condition (52.16).

To prove this, we take the exterior derivative of (52.33). By (52.26) and the fact that $\mathbf{A}$ is closed, the result is

$$\bar{d}\mathbf{X} + \bar{d}\mathbf{Y} \wedge \bar{d}t = \mathbf{0} \tag{52.37}$$

where we have used also (51.21) and (51.24) on the term $\mathbf{Y} \wedge \bar{d}t$. From (52.31) and (52.32) the exterior derivatives of $\mathbf{X}$ and $\mathbf{Y}$ have the representations

$$r!\,\bar{d}\mathbf{X} = \frac{\partial X_{i_1\dots i_r}}{\partial x^j}\mathbf{h}^j \wedge \mathbf{h}^{i_1} \wedge \cdots \wedge \mathbf{h}^{i_r} + \frac{\partial X_{i_1\dots i_r}}{\partial t}\bar{d}t \wedge \mathbf{h}^{i_1} \wedge \cdots \wedge \mathbf{h}^{i_r} \tag{52.38}$$

and

$$(r-1)!\,\bar{d}\mathbf{Y} = \frac{\partial Y_{i_1\dots i_{r-1}}}{\partial x^j}\mathbf{h}^j \wedge \mathbf{h}^{i_1} \wedge \cdots \wedge \mathbf{h}^{i_{r-1}} + \frac{\partial Y_{i_1\dots i_{r-1}}}{\partial t}\bar{d}t \wedge \mathbf{h}^{i_1} \wedge \cdots \wedge \mathbf{h}^{i_r} \tag{52.39}$$

Substituting these into (52.37), we then get

$$\frac{\partial X_{i_1\dots i_r}}{\partial x^j}\mathbf{h}^j \wedge \mathbf{h}^{i_1} \wedge \cdots \wedge \mathbf{h}^{i_r} = \mathbf{0} \tag{52.40}$$

and

$$\left((-1)^r \frac{\partial X_{i_1\ldots i_r}}{\partial t} + r\,\frac{\partial Y_{i_2\ldots i_r}}{\partial x^{i_1}}\right)\mathbf{h}^{i_1}\wedge\cdots\wedge\mathbf{h}^{i_r}\wedge dt = \mathbf{0} \tag{52.41}$$

Consequently

$$-\frac{1}{(r-1)!}\,\delta^{i_1\ldots i_r}_{j_1\cdots j_r}\,\frac{\partial Y_{i_2\ldots i_r}}{\partial x^{i_1}} = (-1)^r\,\frac{\partial X_{j_1\cdots j_r}}{\partial t} \tag{52.42}$$

Now from (52.36) if we take the exterior derivative of the $(r-1)$-form $\mathsf{B}$ on $\mathscr{U}$, the result is

$$\begin{aligned} d\mathsf{B} &= \frac{(-1)^r}{(r-1)!}\left(\int_0^1 \frac{\partial Y_{i_2\ldots i_r}}{\partial x^{i_1}}\,dt\right)\mathbf{g}^{i_1}\wedge\cdots\wedge\mathbf{g}^{i_r} \\ &= \frac{(-1)^r}{(r-1)!}\,\frac{1}{r!}\,\delta^{i_1\ldots i_r}_{j_1\cdots j_r}\left(\int_0^1 \frac{\partial Y_{i_2\ldots i_r}}{\partial x^{i_1}}\,dt\right)\mathbf{g}^{j_1}\wedge\cdots\wedge\mathbf{g}^{j_r} \end{aligned} \tag{52.43}$$

Substituting (52.42) into this equation yields

$$\begin{aligned} d\mathsf{B} &= \frac{1}{r!}\left(-\int_0^1 \frac{\partial X_{j_1\ldots j_r}}{\partial t}\,dt\right)\mathbf{g}^{j_1}\wedge\cdots\wedge\mathbf{g}^{j_r} \\ &= \frac{1}{r!}\left(X_{j_1\ldots j_r}(\cdot,0) - X_{j_1\ldots j_r}(\cdot,1)\right)\mathbf{g}^{j_1}\wedge\cdots\wedge\mathbf{g}^{j_r} \\ &= \frac{1}{r!}\,A_{j_1\ldots j_r}\,\mathbf{g}^{j_1}\wedge\cdots\wedge\mathbf{g}^{j_r} = \mathsf{A} \end{aligned} \tag{52.44}$$

where we have used the end conditions (52.34) and (52.35) on $\mathbf{X}$.

The $(r-1)$-form $\mathsf{B}$, whose existence has just been proved by (52.44), is not unique of course. Since

$$d\mathsf{B} = d\hat{\mathsf{B}} \Leftrightarrow d\left(\mathsf{B}-\hat{\mathsf{B}}\right) = \mathbf{0} \tag{52.45}$$

$\mathsf{B}$ is unique to within an arbitrary closed $(r-1)$-form on $\mathscr{U}$.

## *Exercises*

52.1 In calculus a “differential”

$$P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz \tag{52.46}$$

is called *exact* if there exists a function $U(x,y,z)$ such that

$$\begin{aligned} dU &= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \\ &= P\,dx + Q\,dy + R\,dz \end{aligned} \tag{52.47}$$

Use the Poincaré lemma and show that (52.46) is exact if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \tag{52.48}$$

52.2 The "differential" (52.46) is called *integrable* if there exists a nonvanishing function $\mu(x,y,z)$, called an *integration factor*, such that the differential

$$\mu(P\,dx + Q\,dy + R\,dz) \tag{52.49}$$

is exact. Show that (52.46) is integrable if and only if the two-dimensional distribution orthogonal to the 1-form (52.46) is integrable in the sense defined in this section. Then use the dual form of the Frobenius theorem and show that (52.46) is integrable if and only if

$$P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0 \tag{52.50}$$

## Section 53. Vector Fields in a Three-Dimensional Euclidean Manifold, I. Invariants and Intrinsic Equations

The preceding four sections of this chapter concern vector fields, distributions, and differential forms, defined on domains in an *N*-dimensional Euclidean manifold $\mathscr{E}$ in general. In applications, of course, the most important case is when $\mathscr{E}$ is three-dimensional. Indeed, classical vector analysis was developed just for this special case. In this section we shall review some of the most important results in the classical theory from the more modern point of view, as we have developed so far in this text.

We recall first that when $\mathscr{E}$ is three-dimensional the exterior product may be replaced by the cross product [cf.(41.19)]. Specially, relative to a positive orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathscr{V}$, the component representation of $\mathbf{u}\times\mathbf{v}$ for any $\mathbf{u}, \mathbf{v}, \in \mathscr{V}$ is

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= \varepsilon_{ijk} u^i v^j \mathbf{e}^k \\ &= \left(u^2\upsilon^3 - u^3\upsilon^2\right)\mathbf{e}_1 + \left(u^3\upsilon^1 - u^1\upsilon^3\right)\mathbf{e}_2 + \left(u^1\upsilon^2 - u^2\upsilon^1\right)\mathbf{e}_3\end{aligned} \tag{53.1}$$

Where

$$\mathbf{u} = u^i\mathbf{e}_i, \qquad \mathbf{v} = \upsilon^j\mathbf{e}_j \tag{53.2}$$

are the usual component representations of $\mathbf{u}$ and $\mathbf{v}$ and where the reciprocal basis $\{\mathbf{e}^i\}$ coincides with $\{\mathbf{e}_i\}$. It is important to note that the representation (53.1) is valid relative to a positive orthonormal basis only; if the orthonormal basis $\{\mathbf{e}_i\}$ is negative, the signs on the right-hand side of (53.1) must be reversed. For this reason, $\mathbf{u}\times\mathbf{v}$ is called an axial vector in the classical theory.

We recall also that when $\mathscr{E}$ is three-dimensional, then the curl of a vector field can be represented by a vector field [cf. (47.53) or (51.34)]. Again, if a positive rectangular Cartesian coordinate system $\left(x^1, x^2, x^3\right)$ induced by $\{\mathbf{e}_i\}$ is used, then $\operatorname{curl}\mathbf{v}$ has the components

$$\begin{aligned}\operatorname{curl}\mathbf{v} &= \varepsilon_{ijk}\frac{\partial \upsilon_j}{\partial x^i}\mathbf{e}_k \\ &= \left(\frac{\partial\upsilon_3}{\partial x^2} - \frac{\partial\upsilon_2}{\partial x^3}\right)\mathbf{e}_1 + \left(\frac{\partial\upsilon_1}{\partial x^3} - \frac{\partial\upsilon_3}{\partial x^1}\right)\mathbf{e}_2 + \left(\frac{\partial\upsilon_2}{\partial x^1} - \frac{\partial\upsilon_1}{\partial x^2}\right)\mathbf{e}_3\end{aligned} \tag{53.3}$$

where $\mathbf{v}$ is now a smooth vector field having the representation

$$\mathbf{v} = \upsilon_i\mathbf{e}^i = \upsilon^i\mathbf{e}_i \tag{53.4}$$

Since $\left(x^i\right)$ is rectangular Cartesian, the natural basis vectors and the component fields satisfy the usual conditions

$$\upsilon^i = \upsilon_i, \qquad \mathbf{e}^i = \mathbf{e}_i, \qquad i = 1,2,3 \tag{53.5}$$

By the same remark as before, $\operatorname{curl} \mathbf{v}$ is an axial vector field, so the signs on the right-hand side must be reversed when $\left(x^i\right)$ is negative.

Now consider a nonvanishing smooth vector field $\mathbf{v}$. We put

$$\mathbf{s} = \mathbf{v} / \left\| \mathbf{v} \right\| \tag{53.6}$$

Then $\mathbf{s}$ is a unit vector field in the same direction as $\mathbf{v}$. In Section 49 we have introduced the notions of internal curves and flows corresponding to any smooth vector field. We now apply these to the vector field $\mathbf{s}$. Since $\mathbf{s}$ is a unit vector, its integral curves are parameterized by the *arc length*[1] *s*. A typical integral curve of $\mathbf{s}$ is

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}(s) \tag{53.7}$$

where

$$d\boldsymbol{\lambda} / ds = \mathbf{s}\left(\boldsymbol{\lambda}(s)\right) \tag{53.8}$$

at each point $\boldsymbol{\lambda}(s)$ of the curve. From (53.6) and (53.8) we have

$$\frac{d\boldsymbol{\lambda}}{ds} \text{ parallel to } \mathbf{v}\left(\boldsymbol{\lambda}(s)\right) \tag{53.9}$$

but generally $d\boldsymbol{\lambda} / ds$ is not equal to $\mathbf{v}\left(\boldsymbol{\lambda}(s)\right)$, so $\boldsymbol{\lambda}$ is not an integral curve of $\mathbf{v}$ as defined in Section 49. In the classical theory the locus of $\boldsymbol{\lambda}$ without any particular parameterization is called a *vector line* of $\mathbf{v}$.

Now assuming that $\boldsymbol{\lambda}$ is not a straight line, i.e., $\mathbf{s}$ is not invariant on $\boldsymbol{\lambda}$, we can take the covariant derivative of $\mathbf{s}$ on $\boldsymbol{\lambda}$ (cf. Section 48 and write the result in the form

[1] Arc length shall be defined in a general in Section 68. Here *s* can be regarded as a parameter such that $\left\| d\boldsymbol{\lambda} / ds \right\| = 1$.

$$d\mathbf{s}/ds = \kappa\mathbf{n} \tag{53.10}$$

where $\kappa$ and $\mathbf{n}$ are called the *curvature* and the *principal normal* of $\boldsymbol{\lambda}$, and they are characterized by the condition

$$\kappa = \left\| \frac{d\mathbf{s}}{ds} \right\| > 0 \tag{53.11}$$

It should be noted that (53.10) defines both $\kappa$ and $\mathbf{n}$: $\kappa$ is the norm of $d\mathbf{s}/ds$ and $\mathbf{n}$ is the unit vector in the direction of the nonvanishing vector field $d\mathbf{s}/ds$. The reciprocal of $\kappa$,

$$r = 1/\kappa \tag{53.12}$$

is called the *radius of curvature* of $\boldsymbol{\lambda}$.

Since $\mathbf{s}$ is a vector, we have

$$0 = \frac{d(\mathbf{s}\cdot\mathbf{s})}{ds} = 2\mathbf{s}\cdot\frac{d\mathbf{s}}{ds} = 2\kappa\mathbf{s}\cdot\mathbf{n} \tag{53.13}$$

or, equivalently,

$$\mathbf{s}\cdot\mathbf{n} = 0 \tag{53.14}$$

Thus $\mathbf{n}$ is normal to $\mathbf{s}$, as it should be. In view of (53.14) the cross product of $\mathbf{s}$ with $\mathbf{n}$ is a unit vector

$$\mathbf{b} \equiv \mathbf{s}\times\mathbf{n} \tag{53.15}$$

which is called the *binormal* of $\boldsymbol{\lambda}$. The triad $\{\mathbf{s},\mathbf{n},\mathbf{b}\}$ now forms a field of positive orthonormal basis in the domain of $\mathbf{v}$. In general $\{\mathbf{s},\mathbf{n},\mathbf{b}\}$ is anholonomic, of course.

Now we compute the covariant derivative of $\mathbf{n}$ and $\mathbf{b}$ along the curve $\boldsymbol{\lambda}$. Since $\mathbf{b}$ is a unit vector, by the same argument as (53.13) we have

$$\frac{d\mathbf{b}}{ds}\cdot\mathbf{b} = 0 \tag{53.16}$$

Similarly, since $\mathbf{b}\cdot\mathbf{s} = 0,$ on differentiating with respect to $s$ along $\boldsymbol{\lambda},$ we obtain

$$\frac{d\mathbf{b}}{ds}\cdot\mathbf{s} = -\mathbf{b}\cdot\frac{d\mathbf{s}}{ds} = -\kappa\mathbf{b}\cdot\mathbf{n} = 0 \tag{53.17}$$

where we have used (53.10). Combining (53.16) and (53.17), we see that $d\mathbf{b}/ds$ is parallel to $\mathbf{n}$, say

$$\frac{d\mathbf{b}}{ds} = -\tau\mathbf{n} \tag{53.18}$$

where $\tau$ is called the *torsion* of the curve $\boldsymbol{\lambda}$. From (53.10) and (53.18) the gradient of $\mathbf{n}$ along $\boldsymbol{\lambda}$ can be computed easily by the representation

$$\mathbf{n} = \mathbf{b}\times\mathbf{s} \tag{53.19}$$

so that

$$\frac{d\mathbf{n}}{ds} = \mathbf{b}\times\frac{d\mathbf{s}}{ds} + \frac{d\mathbf{b}}{ds}\times\mathbf{s} = \mathbf{b}\times\kappa\mathbf{n} - \tau\mathbf{n}\times\mathbf{s} = -\kappa\mathbf{s} + \tau\mathbf{b} \tag{53.20}$$

The results (53.10), (53.18), and (53.20) are the *Serret-Frenet formulas* for the curve $\boldsymbol{\lambda}$.

So far we have introduced a field of basis $\{\mathbf{s},\mathbf{n},\mathbf{b}\}$ associated with the nonzero and nonrectilinear vector field $\mathbf{v}$. Moreover, the Serret-Frenet formulas give the covariant derivative of the basis along the vector lines of $\mathbf{v}$. In order to make full use of the basis, however, we need a complete representation of the covariant derivative of that basis along all curves. Then we can express the gradient of the vector fields $\mathbf{s},\mathbf{n},\mathbf{b}$ in component forms relative to the anholonomic basis $\{\mathbf{s},\mathbf{n},\mathbf{b}\}$. These components play the same role as the Christoffel symbols for a holonomic basis. The component forms of $\operatorname{grad}\mathbf{s}$, $\operatorname{grad}\mathbf{n}$ and $\operatorname{grad}\mathbf{b}$ have been derived by Bjørgum.[2] We shall summarize his results without proofs here.

Bjørgum shows first that the components of the vector fields $\operatorname{curl}\mathbf{s}$, $\operatorname{curl}\mathbf{n}$, and $\operatorname{curl}\mathbf{b}$ relative to the basis $\{\mathbf{s},\mathbf{n},\mathbf{b}\}$ are given by

$$\begin{aligned}\operatorname{curl}\mathbf{s} &= \Omega_{\mathbf{s}}\mathbf{s} + \kappa\mathbf{b}\\ \operatorname{curl}\mathbf{n} &= -(\operatorname{div}\mathbf{b})\mathbf{s} + \Omega_{\mathbf{n}}\mathbf{n} + \theta\mathbf{b}\\ \operatorname{curl}\mathbf{b} &= (\kappa + \operatorname{div}\mathbf{n})\mathbf{s} + \eta\mathbf{n} + \Omega_{\mathbf{b}}\mathbf{b}\end{aligned} \tag{53.21}$$

where $\Omega_{\mathbf{s}}, \Omega_{\mathbf{n}}, \Omega_{\mathbf{b}}$ are given by

[2] O. Bjørgum, "On Beltrami Vector Fields and Flows, Part I. A Comparative Study of Some Basic Types of Vector Fields," *Universitetet I Bergen, Arbok* 1951, *Naturvitenskapelig rekke* Nr. 1.

$$\mathbf{s}\cdot\operatorname{curl}\mathbf{s}=\Omega_{\mathbf{s}},\qquad \mathbf{n}\cdot\operatorname{curl}\mathbf{n}=\Omega_{\mathbf{n}},\qquad \mathbf{b}\cdot\operatorname{curl}\mathbf{b}=\Omega_{\mathbf{b}} \tag{53.22}$$

and are called the *abnormality* of $\mathbf{s},\mathbf{n},\mathbf{b}$, respectively. From Kelvin's theorem (cf. Section 52) we know that the abnormality measures, in some sense, the departure of a vector field from a complex-lamellar field. Since $\mathbf{b}$ is given by (53.15), the abnormalities $\Omega_{\mathbf{s}}$, $\Omega_{\mathbf{n}}$, $\Omega_{\mathbf{b}}$ are not independent. Bjørgum shows that

$$\Omega_{\mathbf{n}}+\Omega_{\mathbf{b}}=\Omega_{\mathbf{s}}-2\tau \tag{53.23}$$

where $\tau$ is the torsion of $\boldsymbol{\lambda}$ as defined by (53.18). The quantities $\theta$ and $\eta$ in (53.21) are defined by

$$\mathbf{b}\cdot\operatorname{curl}\mathbf{n}=\theta,\qquad \mathbf{n}\cdot\operatorname{curl}\mathbf{b}=\eta \tag{53.24}$$

and Bjørgum shows that

$$\theta-\eta=\operatorname{div}\mathbf{s} \tag{53.25}$$

Notice that (53.21) implies that

$$\mathbf{n}\cdot\operatorname{curl}\mathbf{s}=0 \tag{53.26}$$

but the remaining eight components in (53.21) are generally nonzero.

Next Bjørgum shows that the components of the second-order tensor fields $\operatorname{grad}\mathbf{s}$, $\operatorname{grad}\mathbf{n}$, and $\operatorname{grad}\mathbf{b}$ relative to the basis $\{\mathbf{s},\mathbf{n},\mathbf{b}\}$ are given by

$$\begin{aligned}
\operatorname{grad}\mathbf{s}&=\kappa\mathbf{n}\otimes\mathbf{s}+\theta\mathbf{n}-(\Omega_{\mathbf{n}}+\tau)\mathbf{n}\otimes\mathbf{b}\\
&\quad+(\Omega_{\mathbf{b}}+\tau)\mathbf{b}\otimes\mathbf{n}-\eta\mathbf{b}\otimes\mathbf{b}\\
\operatorname{grad}\mathbf{n}&=-\kappa\mathbf{s}\otimes\mathbf{s}-\theta\mathbf{s}\otimes\mathbf{n}+(\Omega_{\mathbf{n}}+\tau)\mathbf{s}\otimes\mathbf{b}\\
&\quad+\tau\mathbf{b}\otimes\mathbf{s}-(\operatorname{div}\mathbf{b})\mathbf{b}\otimes\mathbf{n}+(\kappa+\operatorname{div}\mathbf{n})\mathbf{b}\otimes\mathbf{b}\\
\operatorname{grad}\mathbf{b}&=-(\Omega_{\mathbf{b}}+\tau)\mathbf{s}\otimes\mathbf{n}+\eta\mathbf{s}\otimes\mathbf{b}-\tau\mathbf{n}\otimes\mathbf{s}\\
&\quad+(\operatorname{div}\mathbf{b})\mathbf{n}\otimes\mathbf{n}-(\kappa+\operatorname{div}\mathbf{n})\mathbf{n}\otimes\mathbf{b}
\end{aligned} \tag{53.27}$$

These representations are clearly consistent with the representations (53.21) through the general formula (47.53). Further, from (48.15) the covariant derivatives of $\mathbf{s},\mathbf{n}$, and $\mathbf{b}$ along the integral curve $\boldsymbol{\lambda}$ of $\mathbf{s}$ are

$$\begin{aligned}\frac{d\mathbf{s}}{ds} &= (\operatorname{grad}\mathbf{s})\mathbf{s} = \kappa\mathbf{n}\\ \frac{d\mathbf{n}}{ds} &= (\operatorname{grad}\mathbf{n})\mathbf{s} = -\kappa\mathbf{s} + \tau\mathbf{b}\\ \frac{d\mathbf{b}}{ds} &= (\operatorname{grad}\mathbf{b})\mathbf{s} = -\tau\mathbf{n}\end{aligned} \tag{53.28}$$

which are consistent with the Serret-Frenet formulas (53.10), (53.20), and (53.18).

The representations (53.27) tell us also the gradients of $\{\mathbf{s},\mathbf{n},\mathbf{b}\}$ along any integral curves $\boldsymbol{\mu} = \boldsymbol{\mu}(n)$ of $\mathbf{n}$ and $\mathbf{v} = \mathbf{v}(b)$ of $\mathbf{b}$. Indeed, we have

$$\begin{aligned} d\mathbf{s}/dn &= (\operatorname{grad}\mathbf{s})\mathbf{n} = \theta\mathbf{n} + (\Omega_{\mathbf{b}} + \tau)\mathbf{b}\\ d\mathbf{n}/dn &= (\operatorname{grad}\mathbf{n})\mathbf{n} = -\theta\mathbf{s} - (\operatorname{div}\mathbf{b})\mathbf{b}\\ d\mathbf{b}/dn &= (\operatorname{grad}\mathbf{b})\mathbf{n} = -(\Omega_{\mathbf{b}} + \tau)\mathbf{s} + (\operatorname{div}\mathbf{b})\mathbf{n}\end{aligned} \tag{53.29}$$

and

$$\begin{aligned} d\mathbf{s}/db &= (\operatorname{grad}\mathbf{s})\mathbf{b} = -(\Omega_{\mathbf{n}} + \tau)\mathbf{n} - \eta\mathbf{b}\\ d\mathbf{n}/db &= (\operatorname{grad}\mathbf{n})\mathbf{b} = (\Omega_{\mathbf{n}} + \tau)\mathbf{s} + (\kappa + \operatorname{div}\mathbf{n})\mathbf{b}\\ d\mathbf{b}/db &= (\operatorname{grad}\mathbf{b})\mathbf{b} = \eta\mathbf{s} - (\kappa + \operatorname{div}\mathbf{n})\mathbf{n}\end{aligned} \tag{53.30}$$

Since the basis $\{\mathbf{s},\mathbf{n},\mathbf{b}\}$ is anholonomic in general, the parameters $(s,n,b)$ are not local coordinates. In particular, the differential operators $d/ds$, $d/dn$, $d/db$ do not commute. We derive first the commutation formulas[3] for a scalar function *f*.

From (47.14) and (46.10) we verify easily that the anholonomic representation of grad *f* is

$$\operatorname{grad} f = \frac{df}{ds}\mathbf{s} + \frac{df}{dn}\mathbf{n} + \frac{df}{db}\mathbf{b} \tag{53.31}$$

for any smooth function *f* defined on the domain of the basis $\{\mathbf{s},\mathbf{n},\mathbf{b}\}$. Taking the gradient of (53.31) and using (53.27), we obtain

[3] See A.W. Marris and C.-C. Wang, "Solenoidal Screw Fields of Constant Magnitude," *Arch. Rational Mech.* Anal. **39**, 227-244 (1970).

$$
\begin{aligned}
\operatorname{grad}\left(\operatorname{grad} f\right) = {} & \mathbf{s}\otimes\left(\operatorname{grad}\frac{df}{ds}\right)+\frac{df}{ds}\left(\operatorname{grad} s\right)+\mathbf{n}\otimes\left(\operatorname{grad}\frac{df}{dn}\right) \\
& +\frac{df}{dn}\left(\operatorname{grad}\mathbf{n}\right)+\mathbf{b}\otimes\left(\operatorname{grad}\frac{df}{db}\right)+\frac{df}{db}\left(\operatorname{grad}\mathbf{b}\right) \\
= {} & \left[\frac{d}{ds}\frac{df}{ds}-\kappa\frac{df}{dn}\right]\mathbf{s}\otimes\mathbf{s} \\
& +\left[\frac{d}{dn}\frac{df}{ds}-\theta\frac{df}{dn}-\left(\Omega_b+\tau\right)\frac{df}{db}\right]\mathbf{s}\otimes\mathbf{n} \\
& +\left[\frac{d}{db}\frac{df}{ds}+\left(\Omega_{\mathbf{n}}+\tau\right)\frac{df}{dn}+\eta\frac{df}{db}\right]\mathbf{s}\otimes\mathbf{b} \\
& +\left[\kappa\frac{df}{ds}+\frac{d}{ds}\frac{df}{dn}-\tau\frac{df}{db}\right]\mathbf{n}\otimes\mathbf{s} \\
& +\left[\theta\frac{df}{dn}+\frac{d}{dn}\frac{df}{dn}+\left(\operatorname{div}\mathbf{b}\right)\frac{df}{db}\right]\mathbf{n}\otimes\mathbf{n} \\
& +\left[-\left(\Omega_{\mathbf{n}}+\tau\right)\frac{df}{ds}+\frac{d}{db}\frac{df}{dn}-\left(\kappa+\operatorname{div}\mathbf{n}\right)\frac{df}{db}\right]\mathbf{n}\otimes\mathbf{b} \\
& +\left[\tau\frac{df}{dn}+\frac{d}{ds}\frac{df}{db}\right]\mathbf{b}\otimes\mathbf{s} \\
& +\left[\left(\Omega_{\mathbf{b}}+\tau\right)\frac{df}{ds}-\left(\operatorname{div}\mathbf{b}\right)\frac{df}{dn}+\frac{d}{dn}\frac{df}{db}\right]\mathbf{b}\otimes\mathbf{n} \\
& +\left[-\eta\frac{df}{ds}+\left(\kappa+\operatorname{div}\mathbf{n}\right)\frac{df}{dn}+\frac{d}{db}\frac{df}{db}\right]\mathbf{b}\otimes\mathbf{b}
\end{aligned}
\tag{53.32}
$$

Now since $\operatorname{grad}\left(\operatorname{grad} f\right)$ is symmetric, (53.32) yields

$$
\begin{aligned}
\frac{d}{ds}\frac{df}{ds}-\frac{d}{ds}\frac{df}{dn} &= \kappa\frac{df}{ds}+\theta\frac{df}{dn}+\Omega_{\mathbf{b}}\frac{df}{db} \\
\frac{d}{ds}\frac{df}{db}-\frac{d}{db}\frac{df}{ds} &= \Omega_n\frac{df}{dn}+\eta\frac{df}{db} \\
\frac{d}{db}\frac{df}{dn}-\frac{d}{dn}\frac{df}{db} &= \Omega_s\frac{df}{ds}-\left(\operatorname{div}\mathbf{b}\right)\frac{df}{dn}+\left(\kappa+\operatorname{div}\mathbf{n}\right)\frac{df}{db}
\end{aligned}
\tag{53.33}
$$

where we have used (53.23). The Formulas $(53.33)_{1\text{-}3}$ are the desired *commutation rules*.

Exactly the same technique can be applied to the identities

$$
\operatorname{curl}\left(\operatorname{grad}\mathbf{s}\right)=\operatorname{curl}\left(\operatorname{grad}\mathbf{n}\right)=\operatorname{curl}\left(\operatorname{grad}\mathbf{b}\right)=\mathbf{0}
\tag{53.34}
$$

and the results are the following nine *intrinsic equations*[4] for the basis $\{\mathbf{s},\mathbf{n},\mathbf{b}\}$:

$$
\begin{aligned}
&-\frac{d}{dn}(\Omega_{\mathbf{n}}+\tau)-\frac{d\theta}{db}+(\kappa+\operatorname{div}\mathbf{n})(\Omega_{\mathbf{b}}-\Omega_{\mathbf{n}})-(\theta+\eta)\operatorname{div}\mathbf{b}+\Omega_{\mathbf{s}}\kappa=0\\
&-\frac{d\eta}{dn}-\frac{d}{db}(\Omega_{\mathbf{b}}+\tau)-(\Omega_{\mathbf{b}}-\Omega_{\mathbf{n}})\operatorname{div}\mathbf{b}-(\theta+\eta)(\kappa+\operatorname{div}\mathbf{n})=0\\
&\frac{d\kappa}{db}+\frac{d}{ds}(\Omega_{n}+\tau)-\eta(\Omega_{\mathbf{s}}-\Omega_{\mathbf{b}})+\Omega_{\mathbf{n}}\theta=0\\
&\frac{d\tau}{db}-\frac{d}{ds}(\kappa+\operatorname{div}\mathbf{n})-\Omega_{\mathbf{n}}\operatorname{div}\mathbf{b}+\eta(2\kappa+\operatorname{div}\mathbf{n})=0\\
&-\frac{d\eta}{ds}+\eta^{2}-\kappa(\kappa+\operatorname{div}\mathbf{n})-\tau^{2}-\Omega_{\mathbf{n}}(\Omega_{\mathbf{s}}-\Omega_{\mathbf{n}})=0\\
&\frac{d\kappa}{dn}-\frac{d\theta}{ds}-\kappa^{2}-\theta^{2}+(2\Omega_{s}-3\tau)+\Omega_{\mathbf{n}}(\Omega_{\mathbf{s}}-\Omega_{\mathbf{n}}-4\tau)=0\\
&\frac{d\tau}{dn}+\frac{d}{ds}\operatorname{div}\mathbf{b}-\kappa(\Omega_{\mathbf{s}}-\Omega_{\mathbf{n}})+\theta\operatorname{div}\mathbf{b}-\Omega_{\mathbf{b}}(\kappa+\operatorname{div}\mathbf{n})=0\\
&\frac{d}{dn}(\kappa+\operatorname{div}\mathbf{n})+\frac{d}{db}\operatorname{div}\mathbf{b}-\theta_{\mathbf{n}}+(\operatorname{div}\mathbf{b})^{2}+(\kappa+\operatorname{div}\mathbf{n})^{2}\\
&\qquad+\Omega_{\mathbf{s}}\tau+(\Omega_{\mathbf{n}}+\tau)(\Omega_{\mathbf{b}}+\tau)=0\\
&\frac{d\Omega_{\mathbf{s}}}{ds}+\frac{d\kappa}{db}+\Omega_{\mathbf{s}}(\theta-\eta)+\kappa\operatorname{div}\mathbf{b}=0
\end{aligned}
\tag{53.35}
$$

Having obtained the representations (53.21) and (53.27), the commutation rules (53.33), and the intrinsic equations (53.35) for the basis $\{\mathbf{s},\mathbf{n},\mathbf{b}\}$, we can now return to the original relations (53.6) and derive various representations for the invariants of $\mathbf{v}$. For brevity, we shall now denote the norm of $\mathbf{v}$ by $\upsilon$. Then (53.6) can be rewritten as

$$\mathbf{v}=\upsilon\mathbf{s} \tag{53.36}$$

Taking the gradient of this equation yields

$$
\begin{aligned}
\operatorname{grad}\mathbf{v}&=\mathbf{s}\otimes\operatorname{grad}\upsilon+\upsilon\operatorname{grad}\mathbf{s}\\
&=\frac{d\upsilon}{ds}\mathbf{s}\otimes\mathbf{s}+\frac{d\upsilon}{dn}\mathbf{s}\otimes\mathbf{n}+\frac{d\upsilon}{db}\mathbf{s}\otimes\mathbf{b}\\
&\quad+\upsilon\kappa\mathbf{n}\otimes\mathbf{s}+\upsilon\theta\mathbf{n}\otimes\mathbf{n}-\upsilon(\Omega_{\mathbf{n}}+\tau)\mathbf{n}\otimes\mathbf{b}\\
&\quad+\upsilon(\Omega_{\mathbf{b}}+\tau)\mathbf{b}\otimes\mathbf{n}-\upsilon\eta\mathbf{b}\otimes\mathbf{b}
\end{aligned}
\tag{53.37}
$$

[4] A. W. Marris and C.-C. Wang, see footnote 3.

where we have used $(53.27)_1$. Notice that the component of $\operatorname{grad}\mathbf{v}$ in $\mathbf{b}\otimes\mathbf{s}$ vanishes identically, i.e.,

$$\mathbf{b}\cdot\left((\operatorname{grad}\mathbf{v})\mathbf{s}\right)=[\operatorname{grad}\mathbf{v}](\mathbf{b},\mathbf{s})=0 \tag{53.38}$$

or, equivalently,

$$\mathbf{b}\cdot\frac{d\mathbf{v}}{ds}=0 \tag{53.39}$$

so that the covariant derivative of $\mathbf{v}$ along its vector lines stays on the plane spanned by $\mathbf{s}$ and $\mathbf{n}$. In differential geometry this plane is called the *osculating plane* of the said vector line.

Next we can read off from (53.37) the representations[5]

$$\begin{aligned}\operatorname{div}\mathbf{v}&=\frac{d\upsilon}{ds}+\upsilon\theta-\upsilon\eta\\&=\frac{d\upsilon}{ds}+\upsilon\operatorname{div}\mathbf{s}\\\operatorname{curl}\mathbf{v}&=\upsilon\left(\Omega_{\mathbf{b}}+\Omega_{\mathbf{n}}+2\tau\right)\mathbf{s}+\frac{d\upsilon}{db}\mathbf{n}+\left(\upsilon\kappa-\frac{d\upsilon}{dn}\right)\mathbf{b}\\&=\upsilon\Omega_{\mathbf{s}}\mathbf{s}+\frac{d\upsilon}{db}\mathbf{n}+\left(\upsilon\kappa-\frac{d\upsilon}{dn}\right)\mathbf{b}\end{aligned} \tag{53.40}$$

where we have used (53.23) and (53.25). From $(53.40)_4$ the scalar field $\Omega=\mathbf{v}\cdot\operatorname{curl}\mathbf{v}$ is given by

$$\Omega=\upsilon\mathbf{s}\cdot\operatorname{curl}\mathbf{v}=\upsilon^2\Omega_{\mathbf{s}} \tag{53.41}$$

The representation (53.37) and its consequences (53.40) and (53.41) have many important applications in hydrodynamics and continuum mechanics. It should be noted that $(53.21)_1$ and $(53.27)_1$ now can be regarded as special cases of $(53.40)_4$ and $(53.37)_2$, respectively, with $\upsilon=1$.

## *Exercises*

53.1 Prove the intrinsic equations (53.35).

53.2 Show that the Serret-Frenet formulas can be written

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[5] O. Bjørgum, see footnote 2. See also J.L. Ericksen, "Tensor Fields," Handbuch der Physik, Vol. III/1, Appendix, Edited by Flügge, Springer-Verlag (1960).

$$d\mathbf{s}/ds = \boldsymbol{\omega}\times\mathbf{s}, \qquad d\mathbf{n}/ds = \boldsymbol{\omega}\times\mathbf{n}, \qquad d\mathbf{b}/ds = \boldsymbol{\omega}\times\mathbf{b} \tag{53.42}$$

where $\boldsymbol{\omega} = \tau\mathbf{s} + \kappa\mathbf{b}$ .

## *Section 54. Vector Fields in a Three-Dimensional Euclidean Manifold, II. Representations for Special Classes of Vector Fields*

In Section 52 we have proved the Poincaré lemma, which asserts that, locally, a differential form is exact if and only if it is closed. This result means that we have the local representation

$$\mathbf{f} = d\mathbf{g} \tag{54.1}$$

for any $\mathbf{f}$ such that

$$d\mathbf{f} = 0 \tag{54.2}$$

In a three-dimensional Euclidean manifold the local representation (54.1) has the following two special cases.

*(i)* *Lamellar Fields*

$$\mathbf{v} = \operatorname{grad} f \tag{54.3}$$

is a local representation for any vector field $\mathbf{v}$ such that

$$\operatorname{curl} \mathbf{v} = 0 \tag{54.4}$$

Such a vector field $\mathbf{v}$ is called a *lamellar field* in the classical theory, and the scalar function $f$ is called the *potential* of $\mathbf{v}$. Clearly, the potential is locally unique to within an arbitrary additive constant.

*(ii)* *Solenoidal Fields*

$$\mathbf{v} = \operatorname{curl} \mathbf{u} \tag{54.5}$$

is a local representation for any vector field $\mathbf{v}$ such that

$$\operatorname{div} \mathbf{v} = 0 \tag{54.6}$$

Such a vector field $\mathbf{v}$ is called a *solenoidal field* in the classical theory, and the vector field $\mathbf{u}$ is called the *vector potential* of $\mathbf{v}$. The vector potential is locally unique to within an arbitrary additive lamellar field.

In the representation (54.3), $f$ is regarded as a 0-form and $\mathbf{v}$ is regarded as a 1-form, while in the representation (54.5), $\mathbf{u}$ is regarded as a 1-form and $\mathbf{v}$ is regarded as the dual of a 2-form; duality being defined by the canonical positive unit volume tensor of the 3-dimensional Euclidean manifold $\mathscr{E}$.

In Section 52 we remarked also that the dual form of the Frobenius theorem implies the following representation.

(*iii*) *Complex-Lamellar Fields*

$$\mathbf{v} = h \text{ grad } f \tag{54.7}$$

is a local representation for any vector field $\mathbf{v}$ such that

$$\mathbf{v} \cdot \operatorname{curl} \mathbf{v} = 0 \tag{54.8}$$

Such a vector field $\mathbf{v}$ is called *complex-lamellar* in the classical theory. In the representation (54.7) the surfaces defined by

$$f(\mathbf{x}) = \text{const} \tag{54.9}$$

are orthogonal to the vector field $\mathbf{v}$.

We shall now derive some other well-known representations in the classical theory.

*A. Euler's Representation for Solenoidal Fields*

Every solenoidal vector field $\mathbf{v}$ may be represented locally by

$$\mathbf{v} = (\operatorname{grad} h) \times (\operatorname{grad} f) \tag{54.10}$$

*Proof.* We claim that $\mathbf{v}$ has a particular vector potential $\hat{\mathbf{u}}$ which is complex-lamellar. From the remark on (54.5), we may choose $\hat{\mathbf{u}}$ by

$$\hat{\mathbf{u}} = \mathbf{u} + \operatorname{grad} k \tag{54.11}$$

where $\mathbf{u}$ is any vector potential of $\mathbf{v}$. In order for $\hat{\mathbf{u}}$ to be complex-lamellar, it must satisfy the condition (54.8), i.e.,

$$\begin{aligned} 0 &= (\mathbf{u} + \operatorname{grad} k) \cdot \operatorname{curl}(\mathbf{u} + \operatorname{grad} k) \\ &= (\mathbf{u} + \operatorname{grad} k) \cdot \operatorname{curl} \mathbf{u} = \mathbf{v} \cdot (\mathbf{u} + \operatorname{grad} k) \end{aligned} \tag{54.12}$$

Clearly, this equation possesses infinitely many solutions for the scalar function $k$, since it is a first-order partial differential equation with smooth coefficients. Hence by the representation (54.7) we may write

$$\hat{\mathbf{u}} = h \text{ grad } f \tag{54.13}$$

Taking the curl of this equation, we obtain the Euler representation (54.10):

$$\mathbf{v} = \text{curl}\,\hat{\mathbf{u}} = \text{curl}\left(h \,\text{grad } f\right) = \left(\text{grad}\, h\right) \times \left(\text{grad } f\right) \tag{54.14}$$

It should be noted that in the Euler representation (54.10) the vector $\mathbf{v}$ is orthogonal to $\text{grad}\, h$ as well as to $\text{grad } f$, namely

$$\mathbf{v} \cdot \text{grad}\, h = \mathbf{v} \cdot \text{grad } f = 0 \tag{54.15}$$

Consequently, $\mathbf{v}$ is tangent to the surfaces

$$h\left(\mathbf{x}\right) = \text{const} \tag{54.16}$$

or

$$f\left(\mathbf{x}\right) = \text{const} \tag{54.17}$$

For this reason, these surfaces are then called *vector sheets* of $\mathbf{v}$. If $\mathbf{v} \neq \mathbf{0}$, then from (54.10), $\text{grad}\, h$ and $\text{grad } f$ are not parallel, so that *h* and *f* are functionally independent. In this case the intersections of the surfaces (54.16) and (54.17) are the vector lines of $\mathbf{v}$.

Euler's representation for solenoidal fields implies the following results.

*B. Monge's Representation for Arbitrary Smooth Vector Fields*

Every smooth vector field $\mathbf{v}$ may be represented locally by

$$\mathbf{v} = \text{grad } h + k \,\text{grad } f \tag{54.18}$$

where the scalar functions, *h*, *k*, *f* are called the *Monge potentials* (not unique) of $\mathbf{v}$.

*Proof.* Since (54.10) is a representation for any solenoidal vector field, from (54.5) we can write $\text{curl}\,\mathbf{v}$ as

$$\text{curl}\,\mathbf{v} = \left(\text{grad}\, k\right) \times \left(\text{grad } f\right) \tag{54.19}$$

It follows that (54.19) that

$$\text{curl}\left(\mathbf{v} - k \,\text{grad } f\right) = 0 \tag{54.20}$$

Thus $\mathbf{v} - k \text{ grad } f$ is a lamellar vector field. From (54.3) we then we have the local representation

$$\mathbf{v} - k \text{ grad } f = \text{grad}\, h \tag{54.21}$$

which is equivalent to (54.18).

Next we prove another well-known representation for arbitrary smooth vector fields in the classical theory.

*C. Stokes' Representation for Arbitrary Smooth Vector Fields*

Every smooth vector field $\mathbf{v}$ may be represented locally by

$$\mathbf{v} = \text{grad } h + \text{curl}\, \mathbf{u} \tag{54.22}$$

where $h$ and $\mathbf{u}$ are called the *Stokes potential* (not unique) of $\mathbf{v}$.

*Proof.* We show that there exists a scalar function $h$ such that $\mathbf{v} - \text{grad } h$ is solenoidal. Equivalently, this condition means

$$\text{div}\left(\mathbf{v} - \text{grad } h\right) = 0 \tag{54.23}$$

Expanding (54.23), we get

$$\Delta h = \text{div}\, \mathbf{v} \tag{54.24}$$

where $\Delta$ denotes the Laplacian [cf. (47.49)]. Thus $h$ satisfies the Poisson equation (54.24). It is well known that, locally, there exist infinitely many solutions (54.24). Hence the representation (54.22) is valid.

Notice that, from (54.19), Stokes' representation (54.22) also can be put in the form

$$\mathbf{v} = \text{grad } h + \left(\text{grad}\, k\right) \times \left(\text{grad } f\right) \tag{54.25}$$

Next we consider the intrinsic conditions for the various special classes of vector fields. First, from $(53.40)_2$ a vector field $\mathbf{v}$ is solenoidal if and only if

$$d\upsilon / ds = -\upsilon \text{ div } \mathbf{s} \tag{54.26}$$

where $\mathbf{s}, \upsilon$, and $s$ are defined in the preceding section. Integrating (54.26) along any vector line $\boldsymbol{\lambda} = \boldsymbol{\lambda}(s)$ defined before, we obtain

$$\upsilon(s) = \upsilon_0 \exp\left(-\int_{s_0}^{s} \operatorname{div} \mathbf{s}\, ds\right) \tag{54.27}$$

where $\upsilon_0$ is the value of $\upsilon$ at any reference point $\boldsymbol{\lambda}(s_0)$. Thus[1] in a solenoidal vector field $\mathbf{v}$ the vector magnitude is determined to within a constant factor along any vector line $\boldsymbol{\lambda} = \boldsymbol{\lambda}(s)$ by the vector line pattern of $\mathbf{v}$.

From $(53.40)_4$, a vector field $\mathbf{v}$ is complex-lamellar if and only if

$$\upsilon \Omega_s = 0 \tag{54.28}$$

or, equivalently,

$$\Omega_s = 0 \tag{54.29}$$

since in the intrinsic representation $\mathbf{v}$ is assumed to be nonvanishing. The result (54.29) is entirely obvious, because it defines the unit vector field $\mathbf{s}$, parallel to $\mathbf{v}$, to be complex-lamellar.

From $(53.40)_4$ again, a vector field $\mathbf{v}$ is lamellar if and only if, in addition to (54.28) or (54.29), we have also

$$d\upsilon / db = 0, \qquad d\upsilon / dn = \upsilon \kappa \tag{54.30}$$

It should be noted that, when $\mathbf{v}$ is lamellar, it can be represented by (54.3), and thus the potential surfaces defined by

$$f(\mathbf{x}) = \text{const} \tag{54.31}$$

are formed by the integral curves of $\mathbf{n}$ and $\mathbf{b}$. From $(54.30)_1$ along any $\mathbf{b}-$line

$$\upsilon(b) = \upsilon(b_0) = \text{const} \tag{54.32}$$

while from $(54.30)_2$ along any $\mathbf{n}-$line

$$\upsilon(n) = \upsilon(n_0) \exp\left(\int_{n_0}^{n} \kappa\, dn\right) \tag{54.33}$$

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[1] O. Bj∅rgum, see footnote 2 in Section 53.

Finally, in the classical theory a vector field $\mathbf{v}$ is called a *screw field* or a *Beltrami field* if $\mathbf{v}$ is parallel to its curl, namely

$$\mathbf{v} \times \operatorname{curl} \mathbf{v} = \mathbf{0} \tag{54.34}$$

or, equivalently,

$$\operatorname{curl} \mathbf{v} = \Omega_s \mathbf{v} \tag{54.35}$$

where $\Omega_s$ is the abnormality of $\mathbf{s}$, defined by $(53.22)_1$. In some sense a screw field is just the opposite of a complex-lamellar field, which is defined by the condition that the vector field is orthogonal to its curl [cf. (54.8)]. Unfortunately, there is no known simple direct representation for screw fields. We must refer the reader to the three long articles by Bjørgum and Godal (see footnote 1 above and footnotes 4 and 5 below), which are devoted entirely to the study of these fields.

We can, of course, use some general representations for arbitrary smooth vector fields, such as Monge's representation or Stokes' representation, to express a screw field first. Then we impose some additional restrictions on the scalar fields involved in the said representations. For example, if we use Monge's representation (54.18) for $\mathbf{v}$, then curl $\mathbf{v}$ is given by (54.19). In this case $\mathbf{v}$ is a screw field if and only if the constant potential surfaces of $k$ and $f$ are vector sheets of $\mathbf{v}$, i.e.,

$$\mathbf{v} \cdot \operatorname{grad} k = \mathbf{v} \cdot \operatorname{grad} f = 0 \tag{54.36}$$

From $(53.40)_4$ the intrinsic conditions for a screw field are easily fround to be simply the conditions (54.30). So the integrals (54.32) and (54.33) remain valid in this case, along the $\mathbf{b}-$lines and the $\mathbf{n}-$lines. When the abnormality $\Omega_s$ is nonvanishing, the integral of $\upsilon$ along any $\mathbf{s}-$line can be found in the following way: From (53.40) we have

$$d\upsilon / ds = \operatorname{div} \mathbf{v} - \upsilon \operatorname{div} \mathbf{s} \tag{54.37}$$

Now from the basic conditions (54.35) for a screw field we obtain

$$0 = \operatorname{div}\left(\Omega_s \mathbf{v}\right) = \Omega_s \operatorname{div} \mathbf{v} + \upsilon \frac{d\Omega_s}{ds} \tag{54.38}$$

So $\operatorname{div} \mathbf{v}$ can be represented by

$$\operatorname{div} \mathbf{v} = -\frac{\upsilon}{\Omega_s} \frac{d\Omega_s}{ds} \tag{54.39}$$

Substituting (54.39) into (54.37) yields

$$\frac{d\upsilon}{ds} = -\upsilon\left(\operatorname{div}\mathbf{s} + \frac{1}{\Omega_s}\frac{d\Omega_s}{ds}\right) \tag{54.40}$$

or, equivalently,

$$\frac{d\left(\upsilon\Omega_s\right)}{ds} = -\upsilon\Omega_s \operatorname{div}\mathbf{s} \tag{54.41}$$

The last equation can be integrated at once, and the result is

$$\upsilon(s) = \frac{\upsilon(s_0)\Omega_s(s_0)}{\Omega_s(s_0)}\exp\left(-\int_{s_0}^{s}(\operatorname{div}\mathbf{s})\,ds\right) \tag{54.42}$$

which may be rewritten as

$$\frac{\upsilon(s)}{\upsilon(s_0)} = \frac{\Omega_s(s_0)}{\Omega_s(s)}\exp\left(-\int_{s_0}^{s}(\operatorname{div}\mathbf{s})\,ds\right) \tag{54.43}$$

since $\upsilon$ is nonvanishing. From (54.43), (54.33), and (54.32) we see that the magnitude of a screw field, except for a constant factor, is determined along any $\mathbf{s}-$line, $\mathbf{n}-$line, or $\mathbf{b}-$line by the vector line pattern of the field.[2]

A screw field whose curl is also a screw field is called a *Trkalian field*. According to a theorem of Mémenyi and Prim (1949), a screw field is a Trkalian field if and only if its abnormality is a constant. Further, all Trkalian fields are solenoidal and successive curls of the field are screw fields, all having the same abnormality.[3]

The proof of this theorem may be found also in Bjørgum's article. Trkalian fields are considered in detail in the subsequent articles of Bjørgum and Godal[4] and Godal[5]

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[2] O. Bjørgum, see footnote 2, Section 53.

[3] P. Nemenyi and R. Prim, "Some Properties of Rotational Flow of a Perfect Gas," *Proc. Nat. Acad. Sci.* **34**, 119-124; *Erratum* **35**, 116 (1949).

[4] O. Bjørgum and T. Godal, "On Beltrami Vector Fields and Flows, Part II. The Case when Ω is Constant in Space," *Universitetet i Bergen, Arbok 1952, Naturvitenskapelig rekke* Nr. 13.

[5] T. Godal, "On Beltrami Vector Fields and Flows, Part III. Some Considerations on the General Case," *Universitete i Bergen, Arbok 1957, Naturvitenskapelig rekke* Nr.12.

# Chapter 11

# HYPERSURFACES ON A EUCLIDEAN MANIFOLD

In this chapter we consider the theory of ($N$-1)-dimensional hypersurfaces embedded in an $N$-dimensional Euclidean manifold $\mathscr{E}$. We shall not treat hypersurfaces of dimension less than $N$-1, although many results of this chapter can be generalized to results valid for those hypersurfaces also.

## Section 55. Normal Vector, Tangent Plane, and Surface metric

A hypersurface of dimension $N$-1 in $\mathscr{E}$ a set $\mathscr{S}$ of points in $\mathscr{E}$ which can be characterized locally by an equation

$$\mathbf{x} \in \mathscr{N} \subset \mathscr{S} \Leftrightarrow f(\mathbf{x}) = 0 \tag{55.1}$$

where $f$ is a smooth function having nonvanishing gradient. The unit vector field on $\mathscr{N}$

$$\mathbf{n} = \frac{\operatorname{grad} f}{\|\operatorname{grad} f\|} \tag{55.2}$$

is called a *unit normal* of $\mathscr{S}$, since from (55.1) and (48.15) for any smooth curve $\boldsymbol{\lambda} = \boldsymbol{\lambda}(t)$ in $\mathscr{S}$ we have

$$0 = \frac{df \circ \boldsymbol{\lambda}}{dt} = (\operatorname{grad} f) \cdot \dot{\boldsymbol{\lambda}} = \frac{1}{\|\operatorname{grad} f\|} \mathbf{n} \cdot \dot{\boldsymbol{\lambda}} \tag{55.3}$$

The local representation (55.1) of $\mathscr{S}$ is not unique, or course. Indeed, if $f$ satisfies (55.1), so does $-f$, the induced unit normal of $-f$ being $-\mathbf{n}$. If the hypersurface $\mathscr{S}$ can be represented *globally* by (55.1), i.e., there exists a smooth function whose domain contains the entire hypersurface such that

$$\mathbf{x} \in \mathscr{S} \Leftrightarrow f(\mathbf{x}) = 0 \tag{55.4}$$

then $\mathscr{S}$ is called *orientable*. In this case $\mathscr{S}$ can be equipped with a smooth global unit normal field $\mathbf{n}$. (Of course, $-\mathbf{n}$ is also a smooth global unit normal field.) We say that $\mathscr{S}$ is *oriented* if a particular smooth global unit normal field $\mathbf{n}$ has been selected and designated as the *positive* unit normal of $\mathscr{S}$. We shall consider oriented hypersurfaces only in this chapter.

Since grad $f$ is nonvanishing, $\mathscr{S}$ can be characterized locally also by

$$\mathbf{x} = \mathbf{x}(y^1, \ldots, y^{N-1}) \tag{55.5}$$

in such a way that $(y^1, \ldots, y^{N-1}, f)$ forms a local coordinate system in $\mathscr{S}$. If $\mathbf{n}$ is the positive unit normal and $f$ satisfies (55.2), then the parameters $(y^1, \ldots, y^{N-1})$ are said to form a *positive local coordinate system* in $\mathscr{S}$ when $(y^1, \ldots, y^{N-1}, f)$ is a positive local coordinate system in $\mathscr{E}$. Since the coordinate curves of $y^\Gamma, \Gamma = 1, \ldots, N-1,$ are contained in $\mathscr{S}$ the natural basis vectors

$$\mathbf{h}_\Gamma = \partial \mathbf{x} / \partial y^\Gamma, \quad \Gamma = 1, \ldots, N-1 \tag{55.6}$$

are tangent to $\mathscr{S}$. Moreover, the basis $\{\mathbf{h}_\Gamma, \mathbf{n}\}$ is positive in $\mathscr{E}$. We call the $(N-1)$-dimensional hyperplane $\mathscr{S}_\mathbf{x}$ spanned by $\{\mathbf{h}_\Gamma(\mathbf{x})\}$ the *tangent plane* of $\mathscr{S}$ at the point $\mathbf{x} \in \mathscr{S}$.

Since

$$\mathbf{h}_\Gamma \cdot \mathbf{n} = 0, \qquad \Gamma = 1, \ldots, N-1 \tag{55.7}$$

The reciprocal basis of $\{\mathbf{h}_\Gamma, \mathbf{n}\}$ has the form $\{\mathbf{h}^\Gamma, \mathbf{n}\}$ where $\mathbf{h}^\Gamma$ are also tangent to $\mathscr{S}$, namely

$$\mathbf{h}^\Gamma \cdot \mathbf{n} = 0, \qquad \Gamma = 1, \ldots, N-1 \tag{55.8}$$

and

$$\mathbf{h}^\Gamma \cdot \mathbf{h}_\Delta = \delta^\Gamma_\Delta, \qquad \Gamma, \Delta = 1, \ldots, N-1 \tag{55.9}$$

In view of (55.9) we call $\{\mathbf{h}_\Gamma\}$ and $\{\mathbf{h}^\Gamma\}$ reciprocal natural bases of $(y^\Gamma)$ on $\mathscr{S}$.

Let $\mathbf{v}$ be a vector field on $\mathscr{S}$, i. e., a function $\mathbf{v}; \mathscr{S} \rightarrow \mathscr{V}$. Then, for each $\mathbf{x} \in \mathscr{S}$, $\mathbf{v}$ can be represented in terms of the bases $\{\mathbf{h}_\Gamma, \mathbf{n}\}$ and $\{\mathbf{h}^\Gamma, \mathbf{n}\}$ by

$$\mathbf{v} = \upsilon^\Gamma \mathbf{h}_\Gamma + \upsilon^N \mathbf{n} = \upsilon_\Gamma \mathbf{h}^\Gamma + \upsilon_N \mathbf{n} \tag{55.10}$$

where, from (55.7)-(55.9),

$$\upsilon^\Gamma = \mathbf{v} \cdot \mathbf{h}^\Gamma, \qquad \upsilon_\Gamma = \mathbf{v} \cdot \mathbf{h}_\Gamma, \qquad \upsilon_N = \upsilon^N = \mathbf{v} \cdot \mathbf{n} \tag{55.11}$$

We call the vector field

$$\mathbf{v}_{\mathscr{S}} \equiv \upsilon^{\Gamma}\mathbf{h}_{\Gamma} = \upsilon_{\Gamma}\mathbf{h}^{\Gamma} = \mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n} \tag{55.12}$$

the *tangential projection* of $\mathbf{v}$, and we call the vector field

$$\mathbf{v}_{n} \equiv \upsilon^{N}\mathbf{n} = \upsilon_{N}\mathbf{n} = \mathbf{v} - \mathbf{v}_{\mathscr{S}} \tag{55.13}$$

the *normal projection* of $\mathbf{v}$. Notice that in (55.10)-(55.12) the repeated Greek index is summed from 1 to $N$-1. We say that $\mathbf{v}$ is a *tangential vector field* on $\mathscr{S}$ if $\mathbf{v}_n = \mathbf{0}$ and a *normal vector field* if $\mathbf{v}_{\mathscr{S}} = \mathbf{0}$.

If we introduce a local coordinate system $\left(x^i\right)$ in $\mathscr{E}$, then the representation (55.5) may be written

$$x^i = x^i(y^1,\ldots,y^{N-1}), \qquad i = 1,\ldots,N \tag{55.14}$$

From (55.14) the surface basis $\{\mathbf{h}_\Gamma\}$ is related to the natural basis $\{\mathbf{g}_i\}$ of $\left(x^i\right)$ by

$$\mathbf{h}_{\Gamma} = h_{\Gamma}^{i}\mathbf{g}_i = \frac{\partial x^i}{\partial y^{\Gamma}}\mathbf{g}_i, \qquad \Gamma = 1,\ldots,N-1 \tag{55.15}$$

while from (55.2) the unit normal $\mathbf{n}$ has the component form

$$\mathbf{n} = \frac{\partial f/\partial x^i}{(g^{ab}(\partial f/\partial x^a)(\partial f/\partial x^b))^{1/2}}\mathbf{g}^i \tag{55.16}$$

where, as usual $\{\mathbf{g}^i\}$, is the reciprocal basis of $\{\mathbf{g}_i\}$ and $\{g^{ab}\}$ is the component of the Euclidean metric, namely

$$g^{ab} = \mathbf{g}^a \cdot \mathbf{g}^b \tag{55.17}$$

The component representation for the surface reciprocal basis $\{\mathbf{h}^\Gamma\}$ is somewhat harder to find. We find first the components of the surface metric.

$$a_{\Gamma\Delta} \equiv \mathbf{h}_{\Gamma}\cdot\mathbf{h}_{\Delta} = g_{ij}\frac{\partial x^i}{\partial y^{\Gamma}}\frac{\partial x^i}{\partial y^{\Delta}} \tag{55.18}$$

where $g_{ij}$ is a component of the Euclidean metric

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j \tag{55.19}$$

Now let $\left[a^{\Gamma\Delta}\right]$ be the inverse of $\left[a_{\Gamma\Delta}\right]$, i.e.,

$$a^{\Gamma\Delta} a_{\Delta\Sigma} = \delta^{\Gamma}_{\Sigma}, \qquad \Gamma, \Sigma = 1, \ldots, N-1 \tag{55.20}$$

The inverse matrix exists because from (55.18), $\left[a_{\Gamma\Delta}\right]$ is positive-definite and symmetric. In fact, from (55.18) and (55.9)

$$a^{\Gamma\Delta} = \mathbf{h}^{\Gamma} \cdot \mathbf{h}^{\Delta} \tag{55.21}$$

so that $a^{\Gamma\Delta}$ is also a component of the surface metric. From (55.21) and (55.15) we then have

$$\mathbf{h}^{\Gamma} = a^{\Gamma\Delta} \frac{\partial x^i}{\partial y^{\Delta}} \mathbf{g}_i, \qquad \Gamma = 1, \ldots, N-1 \tag{55.22}$$

which is the desired representation for $\mathbf{h}^{\Gamma}$.

At each point $\mathbf{x} \in \mathscr{S}$ the components $a_{\Gamma\Delta}(\mathbf{x})$ and $a^{\Gamma\Delta}(\mathbf{x})$ defined by (55.18) and (55.21) are those of an inner product on $\mathscr{S}_{\mathbf{x}}$ relative to the surface coordinate system $(y^{\Gamma})$, the inner product being the one with induced by that of $\mathscr{V}$ since $\mathscr{S}_{\mathbf{x}}$ is a subspace of $\mathscr{V}$. In other words if $\mathbf{u}$ and $\mathbf{v}$ are tangent to $\mathscr{S}$ at $\mathbf{x}$, say

$$\mathbf{u} = u^{\Gamma} \mathbf{h}_{\Gamma}(\mathbf{x}) \qquad \mathbf{v} = \upsilon^{\Delta} \mathbf{h}_{\Delta}(\mathbf{x}) \tag{55.23}$$

then

$$\mathbf{u} \cdot \mathbf{v} = a_{\Gamma\Delta}(\mathbf{x}) u^{\Gamma} \upsilon^{\Delta} \tag{55.24}$$

This inner product gives rise to the usual operations of rising and lowering of indices for tangent vectors of $\mathscr{S}$. Thus $(55.23)_1$ is equivalent to

$$\mathbf{u} = a_{\Gamma\Delta}(\mathbf{x}) u^{\Gamma} \mathbf{h}^{\Delta}(\mathbf{x}) = u_{\Delta} \mathbf{h}^{\Delta}(\mathbf{x}) \tag{55.25}$$

Obviously we can also extend the operations to tensor fields on $\mathscr{S}$ having nonzero components in the product basis of the surface bases $\{\mathbf{h}_\Gamma\}$ and $\{\mathbf{h}^\Gamma\}$ only. Such a tensor field $\mathsf{A}$ may be called a *tangential tensor field* of $\mathscr{S}$ and has the representation

$$\begin{aligned}\mathsf{A} &= A^{\Gamma_1\ldots\Gamma_r}\mathbf{h}_{\Gamma_1}\otimes\cdots\otimes\mathbf{h}_{\Gamma_2} \\ &= A_{\Gamma_1}{}^{\Gamma_2\ldots\Gamma_r}\mathbf{h}^{\Gamma_1}\otimes\mathbf{h}_{\Gamma_2}\otimes\cdots\otimes\mathbf{h}_{\Gamma_r}, \quad \text{etc.}\end{aligned} \tag{55.26}$$

Then

$$A_{\Gamma_1}{}^{\Gamma_2\ldots\Gamma_r} = a_{\Gamma_1\Delta}A^{\Delta\Gamma_2\ldots\Gamma_r}, \quad \text{etc.} \tag{55.27}$$

There is a fundamental difference between the surface metric $\mathbf{a}$ on $\mathscr{S}$ and the Euclidean metric $\mathbf{g}$ on $\mathscr{E}$ however. In the Euclidean space $\mathscr{E}$ there exist coordinate systems in which the components of $\mathbf{g}$ are constant. Indeed, if the coordinate system is a rectangular Cartesian one, then $g_{ij}$ is $\delta_{ij}$ at all points of the domain of the coordinate system. On the hypersurface $\mathscr{S}$, generally, there need not be any coordinate system in which the components $a_{\Gamma\Delta}$ or $a^{\Gamma\Delta}$ are constant unless $\mathscr{S}$ happens to be a hyperplane. As we shall see in a later section, the departure of $\mathbf{a}$ from $\mathbf{g}$ in this regard can be characterized by the *curvature* of $\mathscr{S}$.

Another important difference between $\mathscr{S}$ and $\mathscr{E}$ is the fact that in $\mathscr{S}$ the tangent planes at different points generally are different ($N$-1)-dimensional subspaces of $\mathscr{V}$. Hence a vector in $\mathscr{V}$ may be tangent to $\mathscr{S}$ at one point but not at another point. For this reason there is no canonical parallelism which connects the tangent planes of $\mathscr{S}$ at distinct points. As a result, the notions of *gradient* or *covariant derivative* of a tangential vector or tensor field on $\mathscr{S}$ must be carefully defined, as we shall do in the next section.

The notions of Lie derivative and exterior derivative introduced in Sections 49 and 51, however, can be readily defined for tangential fields of $\mathscr{S}$. We consider first the Lie derivative.

Let $\mathbf{v}$ be a smooth tangent field defined on a domain in $\mathscr{S}$. Then as before we say that a smooth curve $\boldsymbol{\lambda} = \boldsymbol{\lambda}(t)$ in the domain of $\mathbf{v}$ is an integral curve if

$$d\boldsymbol{\lambda}(t)/dt = \mathbf{v}(\boldsymbol{\lambda}(t)) \tag{55.28}$$

at all points of the curve. If we represent $\boldsymbol{\lambda}$ and $\mathbf{v}$ in component forms

$$\hat{y}(\boldsymbol{\lambda}(t)) = (\lambda^1(t),\ldots,\lambda^{N-1}(t)) \text{ and } \mathbf{v} = \upsilon^\Gamma\mathbf{h}_\Gamma \tag{55.29}$$

relative to a surface coordinate system $(y^\Gamma)$, then (55.28) can be expressed by

$$d\lambda^\Gamma(t)/dt = \upsilon^\Gamma(\boldsymbol{\lambda}(t)) \tag{55.30}$$

Hence integral curves exist for any smooth tangential vector field $\mathbf{v}$ and they generate flow, and hence a parallelism, along any integral curve. By the same argument as in Section 49, we define the Lie derivative of a smooth tangential vector field $\mathbf{u}$ relative to $\mathbf{v}$ by the limit (49.14), except that now $\boldsymbol{\rho}_t$ and $\mathbf{P}_t$ are the flow and the parallelism in $\mathscr{S}$. Following exactly the same derivation as before, we then obtain

$$\underset{\mathbf{v}}{\mathscr{L}}\mathbf{u} = \left(\frac{\partial u^\Gamma}{\partial y^\Delta}\upsilon^\Delta - \frac{\partial \upsilon^\Gamma}{\partial y^\Delta}u^\Delta\right)\mathbf{h}_\Gamma \tag{55.31}$$

which generalizes (49.21). Similarly if $\mathsf{A}$ is a smooth tangential tensor field on $\mathscr{S}$, then $\underset{\mathbf{v}}{\mathscr{L}}\mathsf{A}$ is defined by (49.41) with $\boldsymbol{\rho}_t$ and $\mathbf{P}_t$ as just explained, and the component representation for $\underset{\mathbf{v}}{\mathscr{L}}\mathsf{A}$ is

$$\begin{aligned}(\underset{\mathbf{v}}{\mathscr{L}}\mathsf{A})^{\Gamma_1\cdots\Gamma_r}{}_{\Delta_1\cdots\Delta_s} &= \frac{\partial A^{\Gamma_1\cdots\Gamma_r}{}_{\Delta_1\cdots\Delta_s}}{\partial y^\Sigma}\upsilon^\Sigma - A^{\Sigma\Gamma_2\cdots\Gamma_r}{}_{\Delta_1\cdots\Delta_s}\frac{\partial \upsilon^{\Gamma_1}}{\partial y^\Sigma} \\ &\quad - \cdots - A^{\Gamma_1\cdots\Gamma_{r-1}\Sigma}{}_{\Delta_1\cdots\Delta_s}\frac{\partial \upsilon^{\Gamma_r}}{\partial y^\Sigma} + A^{\Gamma_1\cdots\Gamma_r}{}_{\Sigma\Delta_2\cdots\Delta_s}\frac{\partial \upsilon^\Sigma}{\partial y^{\Delta_1}} \\ &\quad + \cdots + A^{\Gamma_1\cdots\Gamma_r}{}_{\Delta_1\cdots\Delta_{s-1}\Sigma}\frac{\partial \upsilon^\Sigma}{\partial y^{\Delta_s}}\end{aligned} \tag{55.32}$$

which generalizes (49.42).

Next we consider the exterior derivative. Let $\mathsf{A}$ be a tangential differential form on $\mathscr{S}$, i.e., $\mathsf{A}$ is skew-symmetric and has the representations

$$\begin{aligned}\mathsf{A} &= A_{\Gamma_1\cdots\Gamma_r}\mathbf{h}^{\Gamma_1}\otimes\cdots\otimes\mathbf{h}^{\Gamma_r} \\ &= \sum_{\Gamma_1<\cdots<\Gamma_r} A_{\Gamma_1\cdots\Gamma_r}\mathbf{h}^{\Gamma_1}\wedge\cdots\wedge\mathbf{h}^{\Gamma_r} \\ &= \frac{1}{r!}A_{\Gamma_1\cdots\Gamma_r}\mathbf{h}^{\Gamma_1}\wedge\cdots\wedge\mathbf{h}^{\Gamma_r}\end{aligned} \tag{55.33}$$

Then we define the surface exterior derivative as $d\mathsf{A}$

$$
\begin{aligned}
d\mathbf{A} &= \sum_{\Gamma_1<\cdots<\Gamma_r} \sum_{\Delta=1}^{N-1} \frac{A_{\Gamma_1\ldots\Gamma_r}}{\partial y^\Delta} \mathbf{h}^\Delta \wedge \mathbf{h}^{\Gamma_1} \wedge \cdots \wedge \mathbf{h}^{\Gamma_r} \\
&= \frac{1}{r!} \frac{\partial A_{\Gamma_1\ldots\Gamma_r}}{\partial y^\Delta} \mathbf{h}^\Delta \wedge \cdots \wedge \mathbf{h}^{\Gamma_r} \qquad (55.34) \\
&= \frac{1}{r!(r+1)!} \delta^{\Delta\Gamma_1\ldots\Gamma_r}_{\Sigma_1\ldots\Sigma_{r+1}} \frac{\partial A_{\Gamma_1\ldots\Gamma_r}}{\partial y^\Delta} \mathbf{h}^{\Sigma_1} \wedge \cdots \wedge \mathbf{h}^{\Sigma_{r+1}}
\end{aligned}
$$

which generalizes (51.5). However, since the surface covariant derivative $\nabla\mathbf{A}$ has not yet been defined, we cannot write down the equation that generalizes (51.6) to the surface exterior derivative. But we shall achieve this generalization in the next section. Other than this exception, all results of Sections 49-52 can now be generalized and restated in an obvious way for tangential fields on the hypersurface. In fact, those results are valid for differentiable manifolds in general, so that they can be applied to Euclidean manifolds as well as to hypersurfaces therein.

## *Excercises*

55.1 For each $\mathbf{x}\in\mathscr{S}$ define a linear transformation $\mathbf{L}_\mathbf{x}:\mathscr{V}\to\mathscr{V}$ by

$$\mathbf{L}_\mathbf{x}\mathbf{v} = \mathbf{v}_\mathscr{S}$$

for all $\mathbf{v}\in\mathscr{V}$. Show that $\mathbf{L}_\mathbf{x}$ is an orthogonal projection whose image space is $\mathscr{S}_\mathbf{x}$ and whose kernel is the one-dimensional subspace generated by $\mathbf{n}$ at $\mathbf{x}$.

55.2 Show that

$$
\begin{aligned}
\mathbf{L}_\mathbf{x} &= \mathbf{h}^\Gamma(\mathbf{x})\otimes\mathbf{h}_\Gamma(\mathbf{x}) = a_{\Gamma\Delta}(\mathbf{x})\mathbf{h}^\Gamma(\mathbf{x})\otimes\mathbf{h}^\Delta(\mathbf{x}) \\
&= a^{\Gamma\Delta}(\mathbf{x})\frac{\partial x^i}{\partial y^\Delta}\frac{\partial x^j}{\partial y^\Gamma}\mathbf{g}_i(\mathbf{x})\otimes\mathbf{g}_j(\mathbf{x})
\end{aligned}
$$

Thus $\mathbf{L}_\mathbf{x}$ is the linear transformation naturally isomorphic to the surface matric tensor at $\mathbf{x}$.

55.3 Show that

$$\mathbf{I} = \mathbf{L}_\mathbf{x} + \mathbf{n}\otimes\mathbf{n}$$

and

$$g^{ij}(\mathbf{x}) = a^{\Gamma\Delta}(\mathbf{x})\frac{\partial x^i}{\partial y^\Delta}\frac{\partial x^j}{\partial y^\Gamma} + n^i(\mathbf{x})n^j(\mathbf{x})$$

55.4 Show that

$$\mathbf{L}_{\mathbf{x}}\mathbf{g}_j(\mathbf{x}) = g_{jl}(\mathbf{x})\frac{\partial x^l}{\partial y^\Sigma}\mathbf{h}^\Sigma(\mathbf{x})$$

and

$$\mathbf{L}_{\mathbf{x}}\mathbf{g}^j(\mathbf{x}) = \frac{\partial x^j}{\partial y^\Sigma}\mathbf{h}^\Sigma(\mathbf{x})$$

55.5 Compute the components $a_{\Gamma\Delta}$ for (a) the spherical surface defined by

$$\begin{aligned} x^1 &= c\sin y^1 \cos y^2 \\ x^2 &= c\sin y^1 \sin y^2 \\ x^3 &= c\cos y^1 \end{aligned}$$

and (b) the cylindrical surface

$$x^1 = c\cos y^1, \quad x^2 = c\sin y^1, \quad x^3 = y^2$$

where c is a constant.

55.6 For N=3 show that

$$a^{11} = a_{22}/a, \quad a^{12} = -a_{12}/a, \quad a^{22} = a_{11}/a$$

where $a = a_{11}a_{22} - a_{12}^2 = \det\left[a_{\Delta\Gamma}\right]$.

55.7 Given an ellipsoid of revolution whose surface is determined by

$$\begin{aligned} x^1 &= b\cos y^1 \sin y^2 \\ x^2 &= b\sin y^1 \sin y^2 \\ x^3 &= c\cos y^1 \end{aligned}$$

where $b$ and $c$ are constants and $b^2 > c^2$ show that

$$a_{11} = b^2 \sin^2 y^2, \qquad a_{12} = 0$$

and

$$a_{22} = b^2 \cos^2 y^2 + c^2 \sin^2 y^2$$

## Section 56. Surface Covariant Derivatives

As mentioned in the preceding section, the tangent planes at different points of a hypersurface generally do not coincide as subspaces of the translation space $\mathscr{V}$ of $\mathscr{E}$. Consequently, it is no longer possible to define the gradient or covariant derivative of a tangential tensor field by a condition that formally generalizes (47.17). However, we shall see that it is possible to define a type of differentiation which makes (47.17) formally unchanged. For a tangential tensor field $\mathbf{A}$ represented by (55.26) we define the *surface gradient* $\nabla\mathbf{A}$ by

$$\nabla\mathbf{A} = A^{\Gamma_1\dots\Gamma_r}{}_{,\Delta}\, \mathbf{h}_{\Gamma_1} \otimes \cdots \otimes \mathbf{h}_{\Gamma_r} \otimes \mathbf{h}^{\Delta} \tag{56.1}$$

where

$$A^{\Gamma_1\dots\Gamma_r}{}_{,\Delta} = \frac{\partial A^{\Gamma_1\dots\Gamma_r}}{\partial y^{\Delta}} + A^{\Sigma\Gamma_2\dots\Gamma_r}\begin{Bmatrix}\Gamma_1\\ \Sigma\Delta\end{Bmatrix} + \cdots + A^{\Gamma_1\dots\Gamma_{r-1}\Sigma}\begin{Bmatrix}\Gamma_r\\ \Sigma\Delta\end{Bmatrix} \tag{56.2}$$

and

$$\begin{Bmatrix}\Omega\\ \Gamma\Delta\end{Bmatrix} = \frac{1}{2}a^{\Omega\Sigma}\left(\frac{\partial a_{\Gamma\Sigma}}{\partial y^{\Delta}} + \frac{\partial a_{\Delta\Sigma}}{\partial y^{\Gamma}} - \frac{\partial a_{\Gamma\Delta}}{\partial y^{\Sigma}}\right) \tag{56.3}$$

The quantities $\begin{Bmatrix}\Omega\\ \Gamma\Delta\end{Bmatrix}$ are the *surface Christoffel symbols.* They obey the symmetric condition

$$\begin{Bmatrix}\Omega\\ \Gamma\Delta\end{Bmatrix} = \begin{Bmatrix}\Omega\\ \Delta\Gamma\end{Bmatrix} \tag{56.4}$$

We would like to have available a formula like (47.26) which would characterize the surface Christoffel symbols as components of $\partial\mathbf{h}_{\Delta}/\partial y^{\Gamma}$. However, we have no assurance that $\partial\mathbf{h}_{\Delta}/\partial y^{\Gamma}$ is a tangential vector field. In fact, as we shall see in Section 58, $\partial\mathbf{h}_{\Delta}/\partial y^{\Gamma}$ is *not* generally a tangential vector field. However, given the surface Christoffel symbols, we can *formally* write

$$\frac{D\mathbf{h}_{\Delta}}{Dy^{\Gamma}} = \begin{Bmatrix}\Omega\\ \Gamma\Delta\end{Bmatrix}\mathbf{h}_{\Omega} \tag{56.5}$$

With this definition, we can combine (56.2) with (56.1) and obtain

$$
\begin{aligned}
\nabla \mathbf{A} &= \frac{\partial A^{\Gamma_1 \cdots \Gamma_r}}{\partial y^\Delta} \mathbf{h}_{\Gamma_1} \otimes \cdots \otimes \mathbf{h}_{\Gamma_r} \otimes \mathbf{h}^\Delta \\
&+ A^{\Gamma_1 \cdots \Gamma_r} \left\{ \frac{D\mathbf{h}_{\Gamma_1}}{Dy^\Delta} \otimes \cdots \otimes \mathbf{h}_{\Gamma_r} + \cdots + \mathbf{h}_{\Gamma_1} \otimes \cdots \otimes \frac{D\mathbf{h}_{\Gamma_r}}{Dy^\Delta} \right\} \otimes \mathbf{h}^\Delta
\end{aligned}
\tag{56.6}
$$

If we adopt the notional convention

$$
\frac{DA^{\Gamma_1 \cdots \Gamma_r}}{Dy^\Delta} = \frac{\partial A^{\Gamma_1 \cdots \Gamma_r}}{\partial y^\Delta} \tag{56.7}
$$

then (56.6) takes the suggestive form

$$
\nabla \mathbf{A} = \frac{D\mathbf{A}}{Dy^\Delta} \otimes \mathbf{h}^\Delta \tag{56.8}
$$

The components $A^{\Gamma_1 \cdots \Gamma_r}{}_{,\Delta}$ of $\nabla \mathbf{A}$ represents the *surface covariant derivative.* If the mixed components of $\mathbf{A}$ are used, say with

$$
\mathbf{A} = A^{\Gamma_1 \cdots \Gamma_r}{}_{\Delta_1 \cdots \Delta_s} \mathbf{h}_{\Gamma_1} \otimes \cdots \otimes \mathbf{h}_{\Gamma_r} \otimes \mathbf{h}^{\Delta_1} \otimes \cdots \otimes \mathbf{h}^{\Delta_s} \tag{56.9}
$$

then the components of $\nabla \mathbf{A}$ are given by

$$
\begin{aligned}
A^{\Gamma_1 \cdots \Gamma_r}{}_{\Delta_1 \cdots \Delta_s , \Sigma} &= \frac{\partial A^{\Gamma_1 \cdots \Gamma_r}{}_{\Delta_1 \cdots \Delta_s}}{\partial y^\Sigma} \\
&+ A^{\Omega \Gamma_2 \cdots \Gamma_r}{}_{\Delta_1 \cdots \Delta_s} \begin{Bmatrix} \Gamma_1 \\ \Omega\Sigma \end{Bmatrix} + \cdots + A^{\Gamma_1 \cdots \Gamma_{r-1} \Omega}{}_{\Delta_1 \cdots \Delta_s} \begin{Bmatrix} \Gamma_r \\ \Omega\Delta \end{Bmatrix} \\
&- A^{\Gamma_1 \cdots \Gamma_r}{}_{\Omega \Delta_2 \cdots \Delta_s} \begin{Bmatrix} \Omega \\ \Delta_1 \Sigma \end{Bmatrix} - \cdots - A^{\Gamma_1 \cdots \Gamma_r}{}_{\Delta_1 \cdots \Delta_{s-1} \Omega} \begin{Bmatrix} \Omega \\ \Delta_s \Sigma \end{Bmatrix}
\end{aligned}
\tag{56.10}
$$

which generalizes (47.39). Equation (56.10) can be formally derived from (56.8) if we adopt the definition

$$
\frac{D\mathbf{h}^\Gamma}{Dy^\Sigma} = -\begin{Bmatrix} \Gamma \\ \Sigma\Delta \end{Bmatrix} \mathbf{h}^\Delta \tag{56.11}
$$

When we apply (56.10) to the surface metric, we obtain

$$a_{\Gamma\Delta,\Sigma} = a^{\Gamma\Delta},_{\Sigma} = \delta^{\Gamma}_{\Delta,\Sigma} = 0 \tag{56.12}$$

which generalizes (47.40).

The formulas (56.2) and (56.10) give the covariant derivative of a tangential tensor field in component form. To show that $A^{\Gamma_1\ldots\Gamma_r},_{\Delta}$ and $A^{\Gamma_1\ldots\Gamma_r}{}_{\Delta_1\ldots\Delta_s},_{\Sigma}$ are the components of some tangential tensor fields, we must show that they obey the tensor transformation rule, e.g., if $\left(\overline{y}^{\Gamma}\right)$ is another surface coordinate system, then

$$\overline{A}^{\Gamma_1\ldots\Gamma_r},_{\Delta} = \frac{\partial\overline{y}^{\Gamma_1}}{\partial y^{\Sigma_1}}\cdots\frac{\partial\overline{y}^{\Gamma_r}}{\partial y^{\Sigma_r}}\frac{\partial y^{\Omega}}{\partial\overline{y}^{\Delta}}A^{\Sigma_1\ldots\Sigma_r},_{\Omega} \tag{56.13}$$

where $\overline{A}^{\Gamma_1\ldots\Gamma_r},_{\Delta}$ are obtained from (56.2) when all the fields on the right-hand side are referred to $\left(\overline{y}^{\Gamma}\right)$. To prove (56.13), we observe first that from (56.3) and the fact that $a_{\Gamma\Delta}$ and $a^{\Gamma\Delta}$ are components of the surface metric, so that

$$\overline{a}^{\Gamma\Delta} = \frac{\partial\overline{y}^{\Gamma}}{\partial y^{\Sigma}}\frac{\partial\overline{y}^{\Delta}}{\partial y^{\Theta}}a^{\Sigma\Theta}, \qquad \overline{a}_{\Gamma\Delta} = \frac{\partial y^{\Sigma}}{\partial\overline{y}^{\Gamma}}\frac{\partial y^{\Omega}}{\partial\overline{y}^{\Delta}}a_{\Sigma\Omega} \tag{56.14}$$

we have the transformation rule

$$\begin{Bmatrix}\Sigma\\ \Gamma\Delta\end{Bmatrix} = \frac{\partial y^{\Sigma}}{\partial\overline{y}^{\Omega}}\frac{\partial^2\overline{y}^{\Omega}}{\partial y^{\Gamma}\partial y^{\Delta}} + \frac{\partial y^{\Sigma}}{\partial\overline{y}^{\Omega}}\frac{\partial\overline{y}^{\Theta}}{\partial y^{\Gamma}}\frac{\partial\overline{y}^{\Phi}}{\partial y^{\Delta}}\overline{\begin{Bmatrix}\Omega\\ \Theta\Phi\end{Bmatrix}} \tag{56.15}$$

which generalizes (47.35). Now using (56.15), (56.3), and the transformation rule

$$\overline{A}^{\Gamma_1\ldots\Gamma_r} = \frac{\partial\overline{y}^{\Gamma_1}}{\partial y^{\Delta_1}}\cdots\frac{\partial\overline{y}^{\Gamma_r}}{\partial y^{\Delta_r}}A^{\Delta_1\ldots\Delta_r} \tag{56.16}$$

we can verify that (56.13) is valid. Thus $\nabla\mathbf{A}$, as defined by (56.1), is indeed a tangential tensor field.

The surface covariant derivative $\nabla\mathbf{A}$ just defined is not the same as the covariant derivative defined in Section 47. First, the domain of $\mathbf{A}$ here is contained in the hypersurface $\mathscr{S}$, which is not an open set in $\mathscr{E}$. Second, the surface Christoffel symbols $\begin{Bmatrix}\Sigma\\ \Gamma\Delta\end{Bmatrix}$ are generally nonvanishing relative to any surface coordinate system $\left(y^{\Gamma}\right)$ unless $\mathscr{S}$ happens to be a hyperplane on which the metric components $a^{\Gamma\Delta}$ and $a_{\Gamma\Delta}$ are constant relative to certain

"Cartesian" coordinate systems. Other than these two points the formulas for the surface covariant derivative are formally the same as those for the covariant derivative on $\mathscr{E}$.

In view of (56.3) and (56.10), we see that the surface covariant derivative and the surface exterior derivative are still related by a formula formally the same as (51.6), namely

$$d\mathbf{A} = (-1)^r (r+1)!\mathbf{K}_{r+1}(\nabla\mathbf{A}) \tag{56.17}$$

for any surface $r$-form. Here $\mathbf{K}_{r+1}$ denotes the surface skew-symmetric operator. That is, in terms of any surface coordinate system $\left(y^\Gamma\right)$

$$\mathbf{K}_p = \frac{1}{p!}\delta^{\Gamma_1\cdots\Gamma_p}_{\Delta_1\cdots\Delta_p}\mathbf{h}^{\Delta_1}\otimes\cdots\otimes\mathbf{h}^{\Delta_p}\otimes\mathbf{h}_{\Gamma_1}\otimes\cdots\otimes\mathbf{h}_{\Gamma_p} \tag{56.18}$$

for any integer $p$ from 1 to N-1. Notice that from (56.18) and (56.10) the operator $\mathbf{K}_p$, like the surface metric $\mathbf{a}$, is a constant tangential tensor field relative to the surface covariant derivative, i.e.,

$$\delta^{\Gamma_1\ldots\Gamma_p}_{\Delta_1\ldots\Delta_p},_{\Sigma} = 0 \tag{56.19}$$

As a result, the skew-symmetric operator as well as the operators of raising and lowering of indices for tangential fields both commute with the surface covariant derivative.

In Section 48 we have defined the covariant derivative along a smooth curve $\boldsymbol{\lambda}$ in $\mathscr{E}$. That concept can be generalized to the surface covariant derivative along a smooth curve $\boldsymbol{\lambda}$ in $\mathscr{S}$. Specifically, let $\boldsymbol{\lambda}$ be represented by $\left(\lambda^\Gamma(t)\right)$ relative to a surface coordinate system $\left(y^\Gamma\right)$ in $\mathscr{S}$, and suppose that $\mathbf{A}$ is a tangential tensor field on $\boldsymbol{\lambda}$ represented by

$$\mathbf{A}(t) = A^{\Gamma_1\ldots\Gamma_r}(t)\mathbf{h}_{\Gamma_1}\left(\boldsymbol{\lambda}(t)\right)\otimes\cdots\otimes\mathbf{h}_{\Gamma_r}\left(\boldsymbol{\lambda}(t)\right) \tag{56.20}$$

Then we define the surface covariant derivative by the formula

$$\frac{D\mathbf{A}}{Dt} = \left[\frac{dA^{\Gamma_1\ldots\Gamma_r}}{dt} + \left(A^{\Delta\Gamma_2\ldots\Gamma_r}\begin{Bmatrix}\Gamma_1\\ \Delta\Sigma\end{Bmatrix} + \cdots + A^{\Gamma_1\ldots\Gamma_r}\begin{Bmatrix}\Gamma_1\\ \Delta\Sigma\end{Bmatrix}\right)\frac{d\lambda^\Sigma}{dt}\right]\mathbf{h}_{\Gamma_1}\otimes\cdots\otimes\mathbf{h}_{\Gamma_r} \tag{56.21}$$

which formally generalizes (48.6). By use of (56.15) and (56.16) we can show that the surface covariant derivative $D\mathbf{A}/Dt$ along $\boldsymbol{\lambda}$ is a tangential tensor field on $\boldsymbol{\lambda}$ independent of the choice of the surface coordinate system $\left(y^\Gamma\right)$ employed in the representation (56.21). Like the surface covariant derivative of a field in $\mathscr{S}$, the surface covariant derivative along a curve commutes with the operations of raising and lowering of indices. Consequently, when the mixed component representation is used, say with $\mathbf{A}$ given by

$$\mathbf{A}(t) = A^{\Gamma_1\ldots\Gamma_r}{}_{\Delta_1\ldots\Delta_s}(t)\,\mathbf{h}_{\Gamma_1}(\boldsymbol{\lambda}(t))\otimes\cdots\otimes\mathbf{h}_{\Gamma_1}(\boldsymbol{\lambda}(t))\otimes\mathbf{h}^{\Delta_1}(\boldsymbol{\lambda}(t))\otimes\cdots\otimes\mathbf{h}^{\Delta_s}(\boldsymbol{\lambda}(t)) \tag{56.22}$$

the formula for $D\mathbf{A}/Dt$ becomes

$$\begin{aligned}\frac{D\mathbf{A}}{Dt} = \Bigg(&\frac{dA^{\Gamma_1\ldots\Gamma_r}{}_{\Delta_1\ldots\Delta_s}}{dt} + (A^{\Sigma\Gamma_2\ldots\Gamma_r}{}_{\Delta_1\ldots\Delta_s}\begin{Bmatrix}\Gamma_1\\ \Sigma\Omega\end{Bmatrix} + \cdots + A^{\Gamma_1\ldots\Gamma_{r-1}\Sigma}{}_{\Delta_1\ldots\Delta_s}\begin{Bmatrix}\Gamma_r\\ \Sigma\Omega\end{Bmatrix}\\ &-A^{\Gamma_1\ldots\Gamma_r}{}_{\Sigma\Delta_2\ldots\Delta_s}\begin{Bmatrix}\Sigma\\ \Delta_1\Omega\end{Bmatrix} - \cdots - A^{\Gamma_1\ldots\Gamma_r}{}_{\Delta_1\ldots\Delta_{s-1}\Sigma}\begin{Bmatrix}\Sigma\\ \Delta_s\Omega\end{Bmatrix})\frac{d\lambda^\Omega}{dt}\Bigg)\mathbf{h}_{\Gamma_1}\otimes\cdots\otimes\mathbf{h}_{\Gamma_1}\otimes\mathbf{h}^{\Delta_1}\otimes\cdots\otimes\mathbf{h}^{\Delta_s}\end{aligned} \tag{56.23}$$

which formally generalizes (48.7).

As before, if $\mathbf{A}$ is a tangential tensor field and $\boldsymbol{\lambda}$ is a curve in the domain of $\mathbf{A}$, then the restriction of $\mathbf{A}$ on $\boldsymbol{\lambda}$ is a tensor of the form (56.20) or (56.22). In this case the covariant derivative of $\mathbf{A}$ along $\boldsymbol{\lambda}$ is given by

$$D\mathbf{A}(\boldsymbol{\lambda}(t))/Dt = \left[\nabla\mathbf{A}(\boldsymbol{\lambda}(t))\right]\dot{\boldsymbol{\lambda}}(t) \tag{56.24}$$

which generalizes (48.15). The component form of (56.24) is formally the same as (48.14). A special case of (56.24) is equation (56.5), which is equivalent to

$$D\mathbf{h}_\Gamma/Dy^\Delta = \left[\nabla\mathbf{h}_\Gamma\right]\mathbf{h}_\Delta \tag{56.25}$$

since from (56.10)

$$\nabla\mathbf{h}_\Gamma = \begin{Bmatrix}\Sigma\\ \Gamma\Omega\end{Bmatrix}\mathbf{h}_\Sigma\otimes\mathbf{h}^\Omega \tag{56.26}$$

If $\mathbf{v}$ is a tangential vector field defined on $\boldsymbol{\lambda}$ such that

$$D\mathbf{v}/Dt = \mathbf{0} \tag{56.27}$$

then $\mathbf{v}$ may be called a *constant field* or a *parallel field* on $\boldsymbol{\lambda}$. From (56.21), $\mathbf{v}$ is a parallel field if and only if its components satisfy the *equations of parallel transport:*

$$\frac{d\upsilon^\Gamma}{dt} + \upsilon^\Delta\begin{Bmatrix}\Gamma\\ \Delta\Sigma\end{Bmatrix}\dot{\boldsymbol{\lambda}}^\Sigma = 0, \qquad \Gamma = 1,\ldots,N-1 \tag{56.28}$$

Since $\dot{\lambda}^{\Sigma}(t)$ and $\left\{ {\Gamma \atop \Delta\Sigma} \right\}(\boldsymbol{\lambda}(t))$ are smooth functions of *t*, it follows from a theorem in ordinary differential equations that (56.28) possesses a unique solution

$$\upsilon^{\Delta} = \upsilon^{\Delta}(t), \qquad \Delta = 1, \ldots, N-1 \tag{56.29}$$

provided that a suitable initial condition

$$\upsilon^{\Delta}(0) = \upsilon_0^{\Delta}, \qquad \Delta = 1, \ldots, N-1 \tag{56.30}$$

is specified. Since (56.28) is linear in $\mathbf{v}$, the solution $\mathbf{v}(t)$ of (56.27) depends linearly on $\mathbf{v}(0)$. Thus there exists a linear isomorphism

$$\boldsymbol{\rho}_{0,t} : \mathscr{S}_{\boldsymbol{\lambda}(0)} \to \mathscr{S}_{\boldsymbol{\lambda}(t)} \tag{56.31}$$

defined by

$$\boldsymbol{\rho}_{0,t}(\mathbf{v}(0)) = \mathbf{v}(t) \tag{56.32}$$

for all parallel fields $\mathbf{v}$ on $\boldsymbol{\lambda}$. Naturally, we call $\boldsymbol{\rho}_{0,t}$ the *parallel transport* along $\boldsymbol{\lambda}$ induced by the surface covariant derivative.

The parallel transport $\boldsymbol{\rho}_{0,t}$ preserves the surface metric in the sense that

$$\boldsymbol{\rho}_{0,t}(\mathbf{v}(0)) \cdot \boldsymbol{\rho}_{0,t}(\mathbf{u}(0)) = \mathbf{v}(0) \cdot \mathbf{u}(0) \tag{56.33}$$

for all $\mathbf{u}(0)$, $\mathbf{v}(0)$ in $\mathscr{S}_{\boldsymbol{\lambda}(0)}$ but generally $\boldsymbol{\rho}_{0,t}$ does not coincide with the Euclidean parallel transport on $\mathscr{E}$ through the translation space $\mathscr{V}$. In fact since $\mathscr{S}_{\boldsymbol{\lambda}(0)}$ and $\mathscr{S}_{\boldsymbol{\lambda}(t)}$ need not be the same subspace in $\mathscr{V}$, it is not always possible to compare $\boldsymbol{\rho}_{0,t}$ with the Euclidean parallelism. As we shall prove later, the parallel transport $\boldsymbol{\rho}_{0,t}$ depends not only on the end points $\boldsymbol{\lambda}(0)$ and $\boldsymbol{\lambda}(t)$ but also on the particular curve joining the two points. When the same two end points $\boldsymbol{\lambda}(0)$ and $\boldsymbol{\lambda}(t)$ are joined by another curve $\boldsymbol{\mu}$ in $\mathscr{S}$, generally the parallel transport along $\boldsymbol{\mu}$ from $\boldsymbol{\lambda}(0)$ to $\boldsymbol{\lambda}(t)$ need not coincide with that along $\boldsymbol{\lambda}$.

If the parallel transport $\boldsymbol{\rho}_{t,\bar{t}}$ along $\boldsymbol{\lambda}$ from $\boldsymbol{\lambda}(t)$ to $\boldsymbol{\lambda}(\bar{t})$ is used, the covariant derivative $D\mathbf{v}/Dt$ of a vector field on $\boldsymbol{\lambda}$ can be defined also by the limit of a difference quotient, namely

$$\frac{D\mathbf{v}}{Dt} = \lim_{\Delta t\to 0} \frac{\mathbf{v}(t+\Delta t) - \boldsymbol{\rho}_{t,t+\Delta t}\left(\mathbf{v}(t)\right)}{\Delta t} \tag{56.34}$$

To prove this we observe first that

$$\mathbf{v}(t+\Delta t) = \left(\upsilon^{\Gamma}(t) + \frac{d\upsilon^{\Gamma}(t)}{dt}\Delta t\right)\mathbf{h}_{\Gamma}\left(\boldsymbol{\lambda}(t+\Delta t)\right) + o(\Delta t) \tag{56.35}$$

From (56.28) we have also

$$\begin{aligned}\boldsymbol{\rho}_{t,t+\Delta t}&\left(\mathbf{v}(t)\right)\\ &= \left(\upsilon^{\Gamma}(t) - \upsilon^{\Delta}(t)\begin{Bmatrix}\Gamma\\ \Delta\Sigma\end{Bmatrix}\left(\boldsymbol{\lambda}(t)\right)\dot{\lambda}^{\Sigma}(t)\Delta t\right)\mathbf{h}_{\Gamma}\left(\boldsymbol{\lambda}(t+\Delta t)\right) + o(\Delta t)\end{aligned} \tag{56.36}$$

Substituting these approximations into (56.34), we see that

$$\lim_{\Delta t\to 0}\frac{\mathbf{v}(t+\Delta t) - \boldsymbol{\rho}_{t,t+\Delta t}\left(\mathbf{v}(t)\right)}{\Delta t} = \left(\frac{d\upsilon^{\Gamma}}{dt} + \upsilon^{\Delta}\begin{Bmatrix}\Gamma\\ \Delta\Sigma\end{Bmatrix}\dot{\lambda}^{\Sigma}\right)\mathbf{h}_{\Gamma} \tag{56.37}$$

which is consistant with the previous formula (56.21) for the surface covariant derivative $D\mathbf{v}/Dt$ of $\mathbf{v}$ along $\boldsymbol{\lambda}$.

Since the parallel transport $\boldsymbol{\rho}_{t,\bar{t}}$ is a linear isomorphism from $\mathscr{S}_{\boldsymbol{\lambda}(t)}$ to $\mathscr{S}_{\boldsymbol{\lambda}(\bar{t})}$, it gives rise to various induced parallel transport for tangential tensors. We define a tangential tensor field $\mathsf{A}$ on $\boldsymbol{\lambda}$ a *constant field* or a *parallel field* if

$$D\mathsf{A}/Dt = 0 \tag{56.38}$$

Then the equations of parallel transport along $\boldsymbol{\lambda}$ for tensor fields of the forms (56.20) are

$$\frac{dA^{\Gamma_1\ldots\Gamma_r}}{dt} + \left(A^{\Delta\Gamma_2\ldots\Gamma_r}\begin{Bmatrix}\Gamma_1\\ \Delta\Sigma\end{Bmatrix} + \cdots + A^{\Gamma_1\ldots\Gamma_{r-1}\Delta}\begin{Bmatrix}\Gamma_r\\ \Delta\Sigma\end{Bmatrix}\right)\frac{d\lambda^{\Sigma}}{dt} = 0 \tag{56.39}$$

If we donate the induced parallel transport by $\mathbf{P}_{0,t}$, or more generally by $\mathbf{P}_{t,\bar{t}}$, then as before we have

$$\frac{D\mathsf{A}}{Dt} = \lim_{\Delta t\to 0}\frac{\mathsf{A}(t+\Delta t) - \mathbf{P}_{t,t+\Delta t}\left(\mathsf{A}(t)\right)}{\Delta t} \tag{56.40}$$

The coordinate-free definitions (56.34) and (56.40) demonstrate clearly the main difference between the surface covariant derivative on $\mathscr{S}$ and the ordinary covariant derivative on the Euclidean manifold $\mathscr{E}$. Specifically, in the former case the parallelism used to compute the difference quotient is the path-dependent surface parallelism, while in the latter case the parallelism is simply the Euclidean parallelism, which is path-independent between any pair of points in $\mathscr{E}$.

In classical differential geometry the surface parallelism $\boldsymbol{\rho}$, or more generally **P,** defined by the equation (56.28) or (56.39) is known as the *Levi-Civita parallelism* or the *Riemannian parallelism.* This parallelism is generally path-dependent and is determined completely by the surface Christoffel symbols.

## *Exercises*

56.1 Use (55.18), (56.5), and (56.4) and formally derive the formula (56.3).

56.2 Use (55.9) and (56.5) and the assumption that $D\mathbf{h}^{\Gamma} / Dy^{\Sigma}$ is a surface vector field and formally derive the formula (56.11).

56.3 Show that

$$\mathbf{h}^{\Omega} \cdot \frac{\partial \mathbf{h}_{\Delta}}{\partial y^{\Gamma}} = \mathbf{h}^{\Omega} \cdot \frac{D\mathbf{h}_{\Delta}}{Dy^{\Gamma}} = \begin{Bmatrix} \Omega \\ \Gamma\Delta \end{Bmatrix}$$

and

$$\mathbf{h}_{\Delta} \cdot \frac{\partial \mathbf{h}^{\Gamma}}{\partial y^{\Sigma}} = \mathbf{h}_{\Delta} \cdot \frac{D\mathbf{h}^{\Gamma}}{Dy^{\Sigma}} = -\begin{Bmatrix} \Gamma \\ \Sigma\Delta \end{Bmatrix}$$

These formulas show that the tangential parts of $\partial \mathbf{h}_{\Delta} \big/ \partial y^{\Gamma}$ and $\partial \mathbf{h}^{\Gamma} \big/ \partial y^{\Sigma}$ coincide with $D\mathbf{h}_{\Delta} \big/ Dy^{\Gamma}$ and $D\mathbf{h}^{\Gamma} \big/ Dy^{\Sigma}$, respectively.

56.4 Show that the results of Exercise 56.3 can be written

$$\mathsf{L}\left(\frac{\partial \mathbf{h}_{\Delta}}{\partial y^{\Gamma}}\right) = \frac{D\mathbf{h}_{\Delta}}{Dy^{\Gamma}} \qquad \text{and} \qquad \mathsf{L}\left(\frac{\partial \mathbf{h}^{\Gamma}}{\partial y^{\Sigma}}\right) = \frac{D\mathbf{h}^{\Gamma}}{Dy^{\Sigma}}$$

where $\mathsf{L}$ is the field whose value at each $\mathbf{x} \in \mathscr{S}$ is the projection $\mathbf{L}_{\mathbf{x}}$ defined in Exercise 55.1.

56.5 Use the results of Exercise 56.3 and show that

$$\nabla \mathbf{A} = \mathbf{L}^* \left( \frac{\partial \mathbf{A}}{\partial y^\Delta} \right) \otimes \mathbf{h}^\Delta$$

where $\mathbf{L}^*$ is the linear mapping induced by $\mathbf{L}$.

56.6 Adopt the result of Exercise 56.5 and derive (56.10). This result shows that the above formula for $\nabla \mathbf{A}$ can be adopted as the definition of the surface gradient. One advantage of this approach is that one does not need to introduce the formal operation $D\mathbf{A}/Dy^\Delta$. If needed, it can simply be defined to be $\mathbf{L}^*\left(\partial \mathbf{A}/\partial y^\Delta\right)$.

56.7 Another advantage of adopting the result of Exercise 56.5 as a definition is that it can be used to compute the surface gradient of field on $\mathscr{S}$ which are not tangential. For example, each $\mathbf{g}_k$ can be restricted to a field on $\mathscr{S}$ but it does not have a component representation of the form (56.9). As an illustration of this concept, show that

$$\nabla \mathbf{g}_k = \begin{Bmatrix} j \\ kl \end{Bmatrix} g_{jq} \frac{\partial x^q}{\partial y^\Sigma} \frac{\partial x^l}{\partial y^\Delta} \mathbf{h}^\Sigma \otimes \mathbf{h}^\Delta$$

and

$$\nabla \mathbf{L} = \left( \frac{\partial^2 x^k}{\partial y^\Gamma \partial y^\Delta} - \begin{Bmatrix} \Sigma \\ \Delta\Gamma \end{Bmatrix} \frac{\partial x^k}{\partial y^\Sigma} + \begin{Bmatrix} k \\ jl \end{Bmatrix} \frac{\partial x^l}{\partial y^\Delta} \frac{\partial x^j}{\partial y^\Gamma} \right) g_{ks} \frac{\partial x^s}{\partial y^\Phi} \mathbf{h}^\Phi \otimes \mathbf{h}^\Delta \otimes \mathbf{h}^\Gamma = \mathbf{0}$$

Note that $\nabla \mathbf{L} = \mathbf{0}$ is no more than the result (56.12).

56.8 Compute the surface Christoffel symbols for the surfaces defined in Excercises 55.5 and 55.7.

56.9 If **A** is a tensor field on $\mathscr{E}$, then we can, of course, calculate its spatial gradient, grad **A**. Also, we can restrict its domain $\mathscr{S}$ and compute the surface gradient $\nabla \mathbf{A}$. Show that

$$\nabla \mathbf{A} = \mathbf{L}^* \left( grad \mathbf{A} \right)$$

56.10 Show that

$$\frac{\partial \left( \det \left[ a_{\Delta\Gamma} \right] \right)^{1/2}}{\partial y^\Sigma} = \left( \det \left[ a_{\Delta\Gamma} \right] \right)^{1/2} \begin{Bmatrix} \Phi \\ \Phi\Sigma \end{Bmatrix}$$

## Section 57. Surface Geodesics and the Exponential Map

In Section 48 we pointed out that a straight line $\boldsymbol{\lambda}$ in $\mathscr{E}$ with homogeneous parameter can be characterized by equation (48.11), which means that the tangent vector $\dot{\boldsymbol{\lambda}}$ of $\boldsymbol{\lambda}$ is constant along $\boldsymbol{\lambda}$, namely

$$d\dot{\boldsymbol{\lambda}}/dt = \mathbf{0} \tag{57.1}$$

The same condition for a curve in $\mathscr{S}$ defines a *surface geodesic*. In terms of any surface coordinate system $\left(y^{\Gamma}\right)$ the *equations of geodesics* are

$$\frac{d^2\lambda^{\Gamma}}{dt^2} + \frac{d\lambda^{\Sigma}}{dt}\frac{d\lambda^{\Delta}}{dt}\begin{Bmatrix} \Gamma \\ \Sigma\Delta \end{Bmatrix} = 0, \qquad \Gamma = 1,\ldots,N-1 \tag{57.2}$$

Since (57.2) is a system of second-order differential equations with smooth coefficients, at each point

$$\mathbf{x}_0 = \boldsymbol{\lambda}(0) \in \mathscr{S} \tag{57.3}$$

and in each tangential direction

$$\mathbf{v}_0 = \dot{\boldsymbol{\lambda}}(0) \in \mathscr{S}_{\mathbf{x}0} \tag{57.4}$$

there exists a unique geodesic $\boldsymbol{\lambda} = \boldsymbol{\lambda}(t)$ satisfying the initial conditions (57.3) and (57.4).

In classical calculus of variations it is known that the geodesic equations (57.2) represent the *Euler-Lagrange* equations for the *arc length integral*

$$\begin{aligned} s(\boldsymbol{\lambda}) &= \int_{t_0}^{t_1} \left(\dot{\boldsymbol{\lambda}}(t)\cdot\dot{\boldsymbol{\lambda}}(t)\right)^{1/2} dt \\ &= \int_{t_0}^{t_1} \left(a_{\Gamma\Delta}\left(\boldsymbol{\lambda}(t)\right)\frac{d\lambda^{\Gamma}}{dt}\frac{d\lambda^{\Delta}}{dt}\right)^{1/2} dt \end{aligned} \tag{57.5}$$

between any two fixed points

$$\mathbf{x} = \boldsymbol{\lambda}(t_0), \qquad \mathbf{y} = \boldsymbol{\lambda}(t_1) \tag{57.6}$$

on $\mathscr{S}$. If we consider all smooth curves $\boldsymbol{\lambda}$ in $\mathscr{S}$ joining the fixed end points $\mathbf{x}$ and $\mathbf{y}$, then the ones satisfying the geodesic equations are curves whose arc length integral is an extremum in the class of variations of curves.

To prove this, we observe first that the arc length integral is invariant under any change of parameter along the curve. This condition is only natural, since the arc length is a geometric property of the point set that constitutes the curve, independent of how that point set is parameterized. From (57.1) it is obvious that the tangent vector of a geodesic must have constant norm, since

$$\frac{D\left(\dot{\boldsymbol{\lambda}}\cdot\dot{\boldsymbol{\lambda}}\right)}{Dt} = 2\frac{D\dot{\boldsymbol{\lambda}}}{Dt}\cdot\dot{\boldsymbol{\lambda}} = 0\cdot\dot{\boldsymbol{\lambda}} = 0 \tag{57.7}$$

Consequently, we seek only those extremal curves for (57.5) on which the integrand on the right-hand of (57.5) is constant.

Now if we denote that integrand by

$$L\left(\lambda^\Gamma,\dot{\lambda}^\Gamma\right) \equiv \left(a_{\Gamma\Delta}\left(\lambda^\Omega\right)\dot{\lambda}^\Gamma\dot{\lambda}^\Delta\right)^{1/2} \tag{57.8}$$

then it is known that the Euler-Lagrange equations for (57.5) are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\lambda}^\Delta}\right) - \frac{\partial L}{\partial\lambda^\Delta} = 0, \qquad \Delta = 1,\ldots,N-1 \tag{57.9}$$

From (57.8) we have

$$\frac{\partial L}{\partial\dot{\lambda}^\Delta} = \frac{1}{2L}\left(a_{\Gamma\Delta}\dot{\lambda}^\Gamma + a_{\Delta\Gamma}\dot{\lambda}^\Gamma\right) = \frac{1}{L}a_{\Gamma\Delta}\dot{\lambda}^\Gamma \tag{57.10}$$

Hence on the extermal curves we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\lambda}^\Delta}\right) = \frac{1}{L}\left\{\frac{\partial a_{\Gamma\Delta}}{\partial y^\Omega}\dot{\lambda}^\Omega\dot{\lambda}^\Gamma + a_{\Gamma\Delta}\ddot{\lambda}^\Gamma\right\} \tag{57.11}$$

where we have used the condition that $L$ is constant on those curves. From (57.8) we have also

$$\frac{\partial L}{\partial\lambda^\Delta} = \frac{1}{2L}\frac{\partial a_{\Gamma\Omega}}{\partial y_\Delta}\dot{\lambda}^\Gamma\dot{\lambda}^\Omega \tag{57.12}$$

Combining (57.11) and (57.12), we see that the Euler-Lagrange equations have the explicit form

$$\frac{1}{L}\left[a_{\Gamma\Delta}\ddot{\lambda}+\frac{1}{2}\left(\frac{\partial a_{\Gamma\Delta}}{\partial y^{\Omega}}+\frac{\partial a_{\Omega\Delta}}{\partial y^{\Gamma}}-\frac{\partial a_{\Gamma\Omega}}{\partial y^{\Delta}}\right)\dot{\lambda}^{\Gamma}\dot{\lambda}^{\Omega}\right]=0 \tag{57.13}$$

where we have used the symmetry of the product $\dot{\lambda}^{\Gamma}\dot{\lambda}^{\Omega}$ with respect to $\Gamma$ and $\Omega$. Since $L \neq 0$ (otherwise the extremal curve is just one point), (57.13) is equivalent to

$$\ddot{\lambda}^{\Theta}+\frac{1}{2}a^{\Theta\Delta}\left(\frac{\partial a_{\Gamma\Delta}}{\partial y^{\Omega}}+\frac{\partial a_{\Omega\Delta}}{\partial y^{\Gamma}}-\frac{\partial a_{\Gamma\Omega}}{\partial y^{\Delta}}\right)\dot{\lambda}^{\Gamma}\dot{\lambda}^{\Omega}=0 \tag{57.14}$$

which is identical to (57.2) upon using the formula (56.3) for the surface Christoffel symbols.

The preceding result in the calculus of variations shows only that a geodesic is a curve whose arc length is an *extremum* in a class of variations of curves. In terms of a fixed surface coordinate system $\left(y^{\Gamma}\right)$ we can characterize a typical variation of the curve $\boldsymbol{\lambda}$ by $N-1$ smooth functions $\eta^{\Gamma}(t)$ such that

$$\eta^{\Gamma}(t_0)=\eta^{\Gamma}(t_1)=0, \qquad \Gamma=1,\ldots,N-1 \tag{57.15}$$

A one-parameter family of variations of $\boldsymbol{\lambda}$ is then given by the curses $\boldsymbol{\lambda}_{\alpha}$ with representations.

$$\lambda_{\alpha}^{\Gamma}(t)=\lambda^{\Gamma}(t)+\alpha\eta^{\Gamma}(t), \qquad \Gamma=1,\ldots,N-1 \tag{57.16}$$

From (57.15) the curves $\boldsymbol{\lambda}_{\alpha}$ satisfy the same end conditions (57.6) as the curve $\boldsymbol{\lambda}$, and $\boldsymbol{\lambda}_{\alpha}$ reduces to $\boldsymbol{\lambda}$ when $\alpha=0$. The Euler-Lagrange equations express simply the condition that

$$\left.\frac{ds\left(\boldsymbol{\lambda}_{a}\right)}{d\alpha}\right|_{\alpha=0}=0 \tag{57.17}$$

for all choice of $\eta^{\Gamma}$ satisfying (57.15). We note that (57.17) allows $\alpha=0$ to be a local minimum, a local maximum, or a local minimax point in the class of variations. In order for the arc length integral to be a local minimum, additional conditions must be imposed on the geodesic.

It can be shown, however, that in a sufficient small surface neighborhood $\mathscr{N}_{\mathbf{x}_0}$ of any point $\mathbf{x}_0$ in $\mathscr{S}$ every geodesic is, in fact, a curve of minimum arc length. We shall omit the proof of this result since it is not simple. A consequence of this result is that between any pair of points $\mathbf{x}$ and $\mathbf{y}$ in the surface neighborhood $\mathscr{N}_{\mathbf{x}_0}$ there exists one and only one geodesic $\boldsymbol{\lambda}$ (aside from a change of parameter) which lies entirely in $\mathscr{N}_{\mathbf{x}_0}$. That geodesic has the minimum arc kength among *all* curves in $\mathscr{S}$ not just curves in $\mathscr{N}_{\mathbf{x}_0}$, joining $\mathbf{x}$ to $\mathbf{y}$.

Now, as we have remarked earlier in this section, at any point $\mathbf{x}_0 \in \mathscr{S}$, and in each tangential direction $\mathbf{v}_0 \in \mathscr{S}_{\mathbf{x}_0}$ there exists a unique geodesic $\boldsymbol{\lambda}$ satisfying the conditions (57.3) and (57.4). For definiteness, we donate this particular geodesic by $\boldsymbol{\lambda}_{\mathbf{v}_0}$. Since $\boldsymbol{\lambda}_{\mathbf{v}_0}$ is smooth and, when the norm of $\mathbf{v}_0$ is sufficiently small, the geodesic $\boldsymbol{\lambda}_{\mathbf{v}_0}(t)$, $t \in [0,1]$, is contained entirely in the surface neighborhood $\mathscr{N}_{\mathbf{x}_0}$ of $\mathbf{x}_0$. Hence there exists a one-to-one mapping

$$\exp_{\mathbf{x}_o} : \mathscr{B}_{\mathbf{x}_o} \to \mathscr{N}_{\mathbf{x}_o} \tag{57.18}$$

defined by

$$\exp_{\mathbf{x}_0}(\mathbf{v}) \equiv \boldsymbol{\lambda}_{\mathbf{v}}(1) \tag{57.19}$$

for all $\mathbf{v}$ belonging to a certain small neighborhood $\mathscr{B}_{\mathbf{x}_0}$ of $\mathbf{0}$ in $\mathscr{S}_{\mathbf{x}_0}$. We call this injection the *exponential map* at $\mathbf{x}_0$.

Since $\boldsymbol{\lambda}_{\mathbf{v}}$ is the solution of (57.2), the surface coordinates $\left(y^\Gamma\right)$ of the point

$$\mathbf{y} \equiv \exp_{\mathbf{x}_0}(\mathbf{v}) = \boldsymbol{\lambda}_{\mathbf{v}}(1) \tag{57.20}$$

depends smoothly on the components $\upsilon^\Gamma$ of $\mathbf{v}$ relative to $\left(y^\Gamma\right)$,i.e., there exists smooth functions

$$y^\Gamma = \exp^\Gamma_{\mathbf{x}_0}\left(\upsilon^1,\ldots,\upsilon^{N-1}\right), \qquad \Gamma = 1,\ldots,N-1 \tag{57.21}$$

where $\left(\upsilon^\Gamma\right)$ can be regarded as a Cartesian coordinate system on $\mathscr{B}_{\mathbf{x}_0}$ induced by the basis $\left\{\mathbf{h}_\Gamma\left(\mathbf{x}_0\right)\right\}$, namely,

$$\mathbf{v} = \upsilon^\Gamma \mathbf{h}_\Gamma\left(\mathbf{x}_0\right) \tag{57.22}$$

In the sense of (57.21) we say that the exponential map $\exp_{\mathbf{x}0}$ is *smooth.*

Smoothness of the exponential map can be visualized also from a slightly different point of view. Since $\mathscr{N}_{\mathbf{x}_0}$ is contained in $\mathscr{S}$, which is contained in $\mathscr{E}$, $\exp_{\mathbf{x}_0}$ can be regarded also as a mapping from a domain $\mathscr{B}_{\mathbf{x}_0}$ in an Euclidean space $\mathscr{S}_{\mathbf{x}_0}$ to the Euclidean space $\mathscr{E}$, namely

$$\exp_{\mathbf{x}_0} : \mathscr{B}_{\mathbf{x}_0} \to \mathscr{E} \tag{57.23}$$

Now the smoothness of $\exp_{\mathbf{x}_0}$ has the usual meaning as defined in Section 43. Since the surface coordinates $\left(y^\Gamma\right)$ can be extended to a local coordinate system $\left(y^\Gamma, f\right)$ as explained in Section 55, smoothness in the sense of (57.21) is consistant with that of (57.23).

As explained in Section 43, the smooth mapping $\exp_{\mathbf{x}_0}$ has a gradient at any point $\mathbf{v}$ in the domain $\mathscr{B}_{\mathbf{x}_0}$. In particular, at $\mathbf{v} = \mathbf{0}$, $\operatorname{grad}\left(\exp_{\mathbf{x}_0}(\mathbf{0})\right)$ exists and corresponds to a linear map

$$\operatorname{grad}\exp_{\mathbf{x}_0}(\mathbf{0}) : \mathscr{S}_{\mathbf{x}_0} \to \mathscr{V} \tag{57.24}$$

We claim that the image of this linear map is precisely the tangent plane $\mathscr{S}_{\mathbf{x}_0}$, considered as a subspace of $\mathscr{V}$; moreover, $\operatorname{grad}\left(\exp_{\mathbf{x}_0}(\mathbf{0})\right)$ is simply the identity map on $\mathscr{S}_{\mathbf{x}_0}$. This fact is more or less obvious, since by definition the linear map $\operatorname{grad}\left(\exp_{\mathbf{x}_0}(\mathbf{0})\right)$ is characterized by the condition that

$$\left[\operatorname{grad}\left(\exp_{\mathbf{x}_0}(\mathbf{0})\right)\right]\left(\dot{\mathbf{v}}(0)\right) = \frac{d}{dt}\exp_{\mathbf{x}_0}\left(\mathbf{v}(t)\right)\Big|_{t=0} \tag{57.25}$$

for any curve $\mathbf{v} = \mathbf{v}(t)$ such that $\mathbf{v}(0) = \mathbf{0}$. In particular, for the straight lines

$$\mathbf{v}(t) = \mathbf{v}t \tag{57.26}$$

with $\mathbf{v} \in \mathscr{B}_{\mathbf{x}_0}$, we have $\dot{\mathbf{v}}(0) = \mathbf{v}$ and

$$\exp_{\mathbf{x}_0}(\mathbf{v}t) = \boldsymbol{\lambda}_{\mathbf{v}t}(1) = \boldsymbol{\lambda}_{\mathbf{v}}(t) \tag{57.27}$$

so that

$$\frac{d}{dt}\exp_{\mathbf{x}_0}(\mathbf{v}t) = \mathbf{v} \tag{57.28}$$

Hence

$$\left[\operatorname{grad}\left(\exp_{\mathbf{x}_0}(\mathbf{0})\right)\right](\mathbf{v}) = \mathbf{v}, \qquad \mathbf{v} \in \mathscr{B}_{\mathbf{x}_0} \tag{57.29}$$

But since $\operatorname{grad}\left(\exp_{\mathbf{x}_0}(\mathbf{0})\right)$ is a linear map, (57.29) implies that the same holds for all $\mathbf{v} \in \mathscr{S}_{\mathbf{x}_0}$, and thus

$$\operatorname{grad}\left(\exp_{\mathbf{x}_0}(\mathbf{0})\right) = id_{\mathscr{S}_{\mathbf{x}_0}} \tag{57.30}$$

It should be noted, however, that the condition (57.30) is valid only at the origin $\mathbf{0}$ of $\mathscr{S}_{\mathbf{x}_0}$; the same condition generally is *not* true at any other point $\mathbf{v} \in \mathscr{B}_{\mathbf{x}_0}$.

Now suppose that $\{\mathbf{e}_\Gamma, \Gamma = 1,\ldots,N-1\}$ is any basis of $\mathscr{S}_{\mathbf{x}_0}$. Then as usual it gives rise to a Cartesian coordinate system $(w^\Gamma)$ on $\mathscr{S}_{\mathbf{x}_0}$ by the component representation

$$\mathbf{w} = w^\Gamma \mathbf{e}_\Gamma \tag{57.31}$$

for any $\mathbf{w} \in \mathscr{S}_{\mathbf{x}_0}$. Since $\exp_{\mathbf{x}_0}$ is a smooth one-to-one map, it carries the Cartesian system $(w^\Gamma)$ on $\mathscr{B}_{\mathbf{x}_0}$ to a surface coordinate system $(z^\Gamma)$ on the surface neighborhood $\mathscr{N}_{\mathbf{x}_0}$ of $\mathbf{x}_0$ in $\mathscr{S}$. For definiteness, this coordinate system $(z^\Gamma)$ is called a *canonical* surface coordinate system at $\mathbf{x}_0$. Thus a point $\mathbf{z} \in \mathscr{N}_{\mathbf{x}_0}$ has the coordinates $(z^\Gamma)$ if and only if

$$\mathbf{z} = \exp_{\mathbf{x}_0}\left(z^\Gamma \mathbf{e}_\Gamma\right) \tag{57.32}$$

Relative to a canonical surface coordinate system a curve $\boldsymbol{\lambda}$ passing through $\mathbf{x}_0$ at $t = 0$ is a surface geodesic if and only if its representation $\lambda^\Gamma(t)$ in terms of $(z^\Gamma)$ has the form

$$\lambda^\Gamma(t) = \upsilon^\Gamma t, \qquad \Gamma = 1,\ldots,N-1 \tag{57.33}$$

for some constant $\upsilon^\Gamma$. From (57.32) the geodesic $\boldsymbol{\lambda}$ is simply the one denoted earlier by $\boldsymbol{\lambda}_\mathbf{v}$, where

$$\mathbf{v} = \upsilon^\Gamma \mathbf{e}_\Gamma \tag{57.34}$$

An important property of a canonical surface coordinate system at $\mathbf{x}_0$ is that the corresponding surface Christoffel symbols are all equal to zero at $\mathbf{x}_0$. Indeed, since any curve $\boldsymbol{\lambda}$ given by (57.33) is a surface geodesic, the geodesic equations imply

$$\begin{Bmatrix} \Sigma \\ \Gamma\Delta \end{Bmatrix} \upsilon^\Gamma \upsilon^\Delta = 0 \tag{57.35}$$

where the Christoffel symbols are evaluated at any point of the curve $\boldsymbol{\lambda}$. In particular, at $t = 0$ we get

$$\begin{Bmatrix} \Sigma \\ \Gamma\Delta \end{Bmatrix}(\mathbf{x}_0)\upsilon^\Gamma\upsilon^\Delta = 0 \tag{57.36}$$

Now since $\upsilon^\Gamma$ is arbitrary, by use of the symmetry condition (56.4) we conclude that

$$\begin{Bmatrix} \Sigma \\ \Gamma\Delta \end{Bmatrix}(\mathbf{x}_0) = 0 \tag{57.37}$$

In general, a surface coordinate system $\left(y^\Gamma\right)$ is called a *geodesic coordinate system* at a point $\mathbf{x}_0$ if the Christoffel symbols corresponding to $\left(y^\Gamma\right)$ vanish at $\mathbf{x}_0$. From (57.37), we see that a canonical surface coordinate system at $\mathbf{x}_0$ is a geodesic coordinate system at the same point. The converse, of course, is not true. In fact, from the transformation rule (56.15) a surface coordinate system $\left(y^\Delta\right)$ is geodesic at $\mathbf{x}_0$ if and only if its coordinate transformation relative to a canonical surface coordinate system $\left(z^\Gamma\right)$ satisfies the condition

$$\left.\frac{\partial^2 y^\Delta}{\partial z^\Gamma \partial z^\Omega}\right|_{\mathbf{x}_0} = 0 \tag{57.38}$$

which is somewhat weaker that the transformation rule

$$y^\Delta = e^\Delta_\Gamma z^\Gamma \tag{57.39}$$

for some nonsingular matrix $\left[e^\Delta_\Gamma\right]$ when $\left(y^\Delta\right)$ is also a canonical surface coordinate system at $\mathbf{x}_0$, the $\left[e^\Delta_\Gamma\right]$ being simply the transformation matrix connecting the basis $\{\mathbf{e}_\Gamma\}$ for $\left(z^\Gamma\right)$ and thebasis $\{\overline{\mathbf{e}}_\Gamma\}$ for $\left(y^\Gamma\right)$.

A geodesic coordinate system at $\mathbf{x}_0$ plays a role similar to that of a Cartesian coordinate system in $\mathscr{E}$. In view of the condition (57.37), we see that the representation of the surface covariant derivative at $\mathbf{x}_0$ reduces simply to the partial derivative at $\mathbf{x}_0$, namely

$$\nabla\mathbf{A}(\mathbf{x}_0) = \frac{\partial A^{\Gamma_1\ldots\Gamma_r}(\mathbf{x}_0)}{\partial z^\Delta}\mathbf{e}_{\Gamma_1}\otimes\cdots\otimes\mathbf{e}_{\Gamma_1}\otimes\mathbf{e}^\Delta \tag{57.40}$$

It should be noted, however, that (57.40) is valid at the point $\mathbf{x}_0$ only, since a geodesic coordinate system at $\mathbf{x}_0$ generally does not remain a geodesic coordinate system at any neighboring point of $\mathbf{x}_0$. Notice also that the basis $\{\mathbf{e}_\Gamma\}$ in $\mathscr{S}_{\mathbf{x}_0}$ is the natural basis of $(z^\Delta)$ at $\mathbf{x}_0$, this fact being a direct consequence of the condition (57.30).

In closing, we remark that we can choose the basis $\{\mathbf{e}_\Gamma\}$ in $\mathscr{S}_{\mathbf{x}_0}$ to be an orthonormal basis. In this case the corresponding canonical surface coordinate system $(z^\Gamma)$ satisfies the additional condition

$$a_{\Gamma\Delta}(\mathbf{x}_0) = \delta_{\Gamma\Delta} \tag{57.41}$$

Then we do not even have to distinguish the contravariant and the covariant component of a tangential tensor at $\mathbf{x}_0$. In classical differential geometry such a canonical surface coordinate system is called a *normal coordinate system* or a *Riemannian coordinate system* at the surface point under consideration.

## *Section 58. Surface Curvature, I. The Formulas of Weingarten and Gauss*

In the preceding sections we have considered the surface covariant derivative at tangential vector and tensor fields on $\mathscr{S}$. We have pointed out that this covariant derivative is defined relative to a particular path-dependent parallelism on $\mathscr{S}$, namely the parallelism of Levi-Civita as defined by (56.28) and (56.39). We have remarked repeatedly that this parallelism is not the same as the Euclidean parallelism on the underlying space $\mathscr{E}$ in which $\mathscr{S}$ is embedded. Now a tangential tensor on $\mathscr{S}$, or course, is also a tensor over $\mathscr{E}$ being merely a tensor having nonzero components only in the product basis of $\{\mathbf{h}_\Gamma\}$, which can be regarded as a subset of the basis $\{\mathbf{h}_\Gamma, \mathbf{n}\}$ for $\mathscr{V}$. Consequently, the spatial covariant derivative of a tangential tensor field along a curve in $\mathscr{S}$ is defined. Similarly, the special covariant derivative of the unit normal $\mathbf{n}$ of $\mathscr{S}$ along a curve in $\mathscr{S}$ is also defined. We shall study these spatial covariant derivatives in this section.

In classical differential geometry the spatial covariant derivative of a tangential field is a special case of the *total covariant derivative,* which we shall consider in detail later. Since the spatial covariant derivative and the surface covariant derivative along a surface curve often appear in the same equation, we use the notion $d/dt$ for the former and $D/Dt$ for the latter. However, when the curve is the coordinate curve of $y^\Gamma$, we shall write the covariant derivative as $\partial/\partial y^\Gamma$ and $D/Dy^\Gamma$, respectively. It should be noted also that $d/dt$ is defined for all tensor fields on $\boldsymbol{\lambda}$, whether or not the field is tangential, while $D/Dt$ is defined only for tangential fields.

We consider first the covariant derivative $d\mathbf{n}/dt$ of the unit normal field of $\mathscr{S}$ on any curve $\boldsymbol{\lambda} \in \mathscr{S}$. Since $\mathbf{n}$ is a unit vector field and since $d/dt$ preserves the spatial metric on $\mathscr{S}$, we have

$$d\mathbf{n}/dt \cdot \mathbf{n} = 0 \tag{58.1}$$

Thus $d\mathbf{n}/dt$ is a tangential vector field. We claim that there exists a symmetric second-order tangential tensor field $\mathbf{B}$ on $\mathscr{S}$ such that

$$d\mathbf{n}/dt = -\mathbf{B}\dot{\boldsymbol{\lambda}} \tag{58.2}$$

for all surface curves $\boldsymbol{\lambda}$. The proof of (58.2) is more or less obvious. From (55.2), the unit normal $\mathbf{n}$ on $\mathscr{S}$ is parallel to the gradient of a certain smooth function $f$, and locally $\mathscr{S}$ can be characterized by (55.1). By normalizing the function $f$ to a function

$$w(\mathbf{x}) = \frac{f(\mathbf{x})}{\left\|\operatorname{grad} f(\mathbf{x})\right\|} \tag{58.3}$$

we find

$$\mathbf{n}(\mathbf{x}) = \operatorname{grad} w(x), \qquad \mathbf{x} \in \mathscr{S} \tag{58.4}$$

Now for the smooth vector field $\operatorname{grad} w$ we can apply the usual formula (48.15) to compute the spatial covariant derivative $(d/dt)(\operatorname{grad} w)$ along any smooth curve in $\mathscr{E}$. In particular, along the surface curve $\boldsymbol{\lambda}$ under consideration we have the formula (58.2), where

$$\mathbf{B} = -\operatorname{grad}(\operatorname{grad}\ w)\big|_{\mathscr{S}} \tag{58.5}$$

As a result, $\mathbf{B}$ is symmetric. The fact that $\mathbf{B}$ is a tangential tensor has been remarked after (58.1).

From (58.2) the surface tensor $\mathbf{B}$ characterizes the spatial change of the unit normal $\mathbf{n}$ of $\mathscr{S}$. Hence in some sense $\mathbf{B}$ is a measurement of the *curvature* of $\mathscr{S}$ in $\mathscr{E}$. In classical differential geometry, $\mathbf{B}$ is called the *second fundamental form* of $\mathscr{S}$, the surface metric $\mathbf{a}$ defined by (55.18) being the *first fundamental form*. In component form relative to a surface coordinate system $(y^\Gamma)$, $\mathbf{B}$ can be represented as usual

$$\mathbf{B} = b_{\Gamma\Delta}\mathbf{h}^\Gamma \otimes \mathbf{h}^\Delta = b^{\Gamma\Delta}\mathbf{h}_\Gamma \otimes \mathbf{h}_\Delta = b^\Gamma_\Delta \mathbf{h}_\Gamma \otimes \mathbf{h}^\Delta \tag{58.6}$$

From (58.2) the components of $\mathbf{B}$ are those of $\partial\mathbf{n}/\partial y^\Gamma$ taken along the $y^\Gamma$-curve in $\mathscr{S}$, namely

$$\partial\mathbf{n}/\partial y^\Gamma = -b^\Delta_\Gamma\, \mathbf{h}_\Delta = -b_{\Gamma\Delta}\mathbf{h}^\Delta \tag{58.7}$$

This equation is called *Weingarten's formula* in classical differential geometry.

Next we consider the covariant derivatives $d\mathbf{h}_\Gamma/dt$ and $d\mathbf{h}^\Gamma/dt$ of the surface natural basis vectors $\mathbf{h}_\Gamma$ and $\mathbf{h}^\Gamma$ along any curve $\boldsymbol{\lambda}$ in $\mathscr{S}$. From (56.21) and (56.23) we have

$$\frac{D\mathbf{h}_\Gamma}{Dt} = \begin{Bmatrix} \Delta \\ \Gamma\Sigma \end{Bmatrix} \dot{\lambda}^\Sigma \mathbf{h}_\Delta \tag{58.8}$$

and

$$\frac{D\mathbf{h}^\Gamma}{Dt} = -\begin{Bmatrix} \Gamma \\ \Delta\Sigma \end{Bmatrix} \dot{\lambda}^\Sigma \mathbf{h}^\Delta \tag{58.9}$$

By a similar argument as (58.2), we have first

$$\frac{d\mathbf{h}^\Gamma}{dt} = -\mathbf{C}^\Gamma \dot{\boldsymbol{\lambda}} \tag{58.10}$$

where $\mathbf{C}^\Gamma$ is a symmetric spatial tensor field on $\mathscr{S}$. In particular, when $\boldsymbol{\lambda}$ is the coordinate curve of $y^\Delta$, (58.10) reduces to

$$\frac{\partial \mathbf{h}^\Gamma}{\partial y^\Delta} = -\mathbf{C}^\Gamma \mathbf{h}_\Delta \tag{58.11}$$

Since $\mathbf{C}^\Gamma$ is symmetric, this equation implies

$$-\mathbf{h}_\Sigma \cdot \frac{\partial \mathbf{h}^\Gamma}{\partial y^\Delta} = -\mathbf{h}_\Delta \cdot \frac{\partial \mathbf{h}^\Gamma}{\partial y^\Sigma} = \mathbf{C}^\Gamma \left( \mathbf{h}_\Delta, \mathbf{h}_\Sigma \right) \tag{58.12}$$

We claim that the quantity given by this equation is simply the surface Christoffel symbol, $\begin{Bmatrix} \Gamma \\ \Delta\Sigma \end{Bmatrix}$

$$-\mathbf{h}_\Sigma \cdot \frac{\partial \mathbf{h}^\Gamma}{\partial y^\Delta} = \begin{Bmatrix} \Gamma \\ \Delta\Sigma \end{Bmatrix} \tag{58.13}$$

Indeed, since both $d/dt$ and $D/Dt$ preserve the surface metric, we have

$$0 = \frac{\partial \delta_\Sigma^\Gamma}{\partial y^\Delta} = \frac{\partial \left( \mathbf{h}_\Sigma \cdot \mathbf{h}^\Gamma \right)}{\partial y^\Delta} = \mathbf{h}_\Sigma \cdot \frac{\partial \mathbf{h}^\Gamma}{\partial y^\Delta} + \frac{\partial \mathbf{h}_\Sigma}{\partial y^\Delta} \cdot \mathbf{h}^\Gamma \tag{58.14}$$

Thus (58.13) is equivalent to

$$\mathbf{h}^\Gamma \cdot \frac{\partial \mathbf{h}_\Sigma}{\partial y^\Delta} = \begin{Bmatrix} \Gamma \\ \Sigma\Delta \end{Bmatrix} \tag{58.15}$$

But as in (58.14) we have also

$$\begin{aligned} \frac{\partial a_{\Gamma\Delta}}{\partial y^\Sigma} &= \frac{\partial \left( \mathbf{h}_\Gamma \cdot \mathbf{h}_\Delta \right)}{\partial y^\Sigma} = \mathbf{h}_\Gamma \cdot \frac{\partial \mathbf{h}_\Delta}{\partial y^\Sigma} + \mathbf{h}_\Delta \cdot \frac{\partial \mathbf{h}_\Gamma}{\partial y^\Sigma} \\ &= a_{\Gamma\Omega} \left( \mathbf{h}^\Omega \cdot \frac{\partial \mathbf{h}_\Delta}{\partial y^\Sigma} \right) + a_{\Delta\Omega} \left( \mathbf{h}^\Omega \cdot \frac{\partial \mathbf{h}_\Gamma}{\partial y^\Sigma} \right) \end{aligned} \tag{58.16}$$

Further, from (58.14) and (58.12) we have

$$\mathbf{h}^{\Omega} \cdot \frac{\partial \mathbf{h}_{\Delta}}{\partial y^{\Sigma}} = \mathbf{h}^{\Omega} \cdot \frac{\partial \mathbf{h}_{\Sigma}}{\partial y^{\Delta}} \tag{58.17}$$

Comparing (58.17) and (58.16) with (56.4) and (56.12), respectively, we see that (58.15) holds.

On differentiating (55.8), we get

$$0 = \frac{\partial \left( \mathbf{n} \cdot \mathbf{h}^{\Gamma} \right)}{\partial y^{\Delta}} = \mathbf{n} \cdot \frac{\partial \mathbf{h}^{\Gamma}}{\partial y^{\Delta}} + \mathbf{h}^{\Gamma} \cdot \frac{\partial \mathbf{n}}{\partial y^{\Delta}} \tag{58.18}$$

Substituting Weingarten's formula (58.7) into (58.18), we then obtain

$$\mathbf{n} \cdot \frac{\partial \mathbf{h}^{\Gamma}}{\partial y^{\Delta}} = b_{\Delta}^{\Gamma} \tag{58.19}$$

The formulas (58.19) and (58.13) determine completely the spatial covariant derivative of $\mathbf{h}^{\Gamma}$ along any $y^{\Delta}$-curve:

$$\frac{\partial \mathbf{h}^{\Gamma}}{\partial y^{\Delta}} = b_{\Delta}^{\Gamma} \mathbf{n} - \begin{Bmatrix} \Gamma \\ \Delta\Sigma \end{Bmatrix} \mathbf{h}^{\Sigma} = b_{\Delta}^{\Gamma} \mathbf{n} + \frac{D\mathbf{h}^{\Gamma}}{Dy^{\Delta}} \tag{58.20}$$

where we have used (58.9) for $(58.20)_2$. By exactly the same argument we have also

$$\frac{\partial \mathbf{h}_{\Gamma}}{\partial y^{\Delta}} = b_{\Gamma\Delta} \mathbf{n} + \begin{Bmatrix} \Sigma \\ \Gamma\Delta \end{Bmatrix} \mathbf{h}_{\Sigma} = b_{\Gamma\Delta} \mathbf{n} + \frac{D\mathbf{h}_{\Gamma}}{Dy^{\Delta}} \tag{58.21}$$

As we shall see, (58.20) and (58.21) are equivalent to the *formula of Gauss* in classical differential geometry.

The formulas (58.20), (58.21), and (58.7) determine completely the spatial covariant derivatives of the bases $\{\mathbf{h}_{\Gamma}, \mathbf{n}\}$ and $\{\mathbf{h}^{\Gamma}, \mathbf{n}\}$ along any curve $\boldsymbol{\lambda}$ in $\mathscr{S}$. Indeed, if the coordinates of $\boldsymbol{\lambda}$ in $\left(y^{\Gamma}\right)$ are $\left(\lambda^{\Gamma}\right)$, then we have

$$\begin{aligned}\frac{d\mathbf{n}}{dt} &= -b_{\Gamma\Delta}\dot{\lambda}^{\Delta}\mathbf{h}^{\Gamma} = -b_{\Delta}^{\Gamma}\dot{\lambda}^{\Delta}\mathbf{h}_{\Gamma}\\ \frac{d\mathbf{h}^{\Gamma}}{dt} &= b_{\Delta}^{\Gamma}\dot{\lambda}^{\Delta}\mathbf{n} - \begin{Bmatrix}\Gamma\\ \Delta\Sigma\end{Bmatrix}\dot{\lambda}^{\Delta}\mathbf{h}^{\Sigma} = b_{\Delta}^{\Gamma}\dot{\lambda}^{\Delta}\mathbf{n} + \frac{D\mathbf{h}^{\Gamma}}{Dt}\\ \frac{d\mathbf{h}_{\Gamma}}{dt} &= b_{\Gamma\Delta}\dot{\lambda}^{\Delta}\mathbf{n} + \begin{Bmatrix}\Sigma\\ \Gamma\Delta\end{Bmatrix}\dot{\lambda}^{\Delta}\mathbf{h}_{\Sigma} = b_{\Gamma\Delta}\dot{\lambda}^{\Delta}\mathbf{n} + \frac{D\mathbf{h}_{\Gamma}}{Dt}\end{aligned} \tag{58.22}$$

From these representations we can compute the spatial covariant derivative of any vector or tensor fields along $\boldsymbol{\lambda}$ when their components relative to $\{\mathbf{h}_{\Gamma}, \mathbf{n}\}$ or $\{\mathbf{h}^{\Gamma}, \mathbf{n}\}$ are given. For example, if $\mathbf{v}$ is a vector field having the component form

$$\mathbf{v}(t) = \upsilon_n(t)\,\mathbf{n}(\boldsymbol{\lambda}(t)) + \upsilon^{\Gamma}(t)\,\mathbf{h}_{\Gamma}(\boldsymbol{\lambda}(t)) \tag{58.23}$$

along $\boldsymbol{\lambda}$, then

$$\frac{d\mathbf{v}}{dt} = \left(\dot{\upsilon}_n + b_{\Gamma\Delta}\upsilon^{\Gamma}\dot{\lambda}^{\Delta}\right)\mathbf{n} + \left(\dot{\upsilon}^{\Gamma} - \upsilon_n b_{\Delta}^{\Gamma}\dot{\lambda}^{\Delta} + \begin{Bmatrix}\Gamma\\ \Sigma\Delta\end{Bmatrix}\upsilon^{\Sigma}\dot{\lambda}^{\Delta}\right)\mathbf{h}_{\Gamma} \tag{58.24}$$

In particular, if $\mathbf{v}$ is a tangential field, then (58.24) reduces to

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= b_{\Gamma\Delta}\upsilon^{\Gamma}\dot{\lambda}^{\Delta}\mathbf{n} + \left(\dot{\upsilon}^{\Gamma} + \begin{Bmatrix}\Gamma\\ \Sigma\Delta\end{Bmatrix}\upsilon^{\Sigma}\dot{\lambda}^{\Delta}\right)\mathbf{h}_{\Gamma}\\ &= b_{\Gamma\Delta}\upsilon^{\Gamma}\dot{\lambda}^{\Delta}\mathbf{n} + \frac{D\mathbf{v}}{Dt}\end{aligned} \tag{58.25}$$

where we have used (56.21). Equation (58.25) shows clearly the difference between $d/dt$ and $D/Dt$ for any tangential field.

Applying (58.25) to the tangent vector $\dot{\boldsymbol{\lambda}}$ of $\boldsymbol{\lambda}$, we get

$$\frac{d\dot{\boldsymbol{\lambda}}}{dt} = \mathbf{B}(\dot{\boldsymbol{\lambda}}, \dot{\boldsymbol{\lambda}})\mathbf{n} + \frac{D\dot{\boldsymbol{\lambda}}}{Dt} \tag{58.26}$$

In particular, when $\boldsymbol{\lambda}$ is a surface geodesic, then (58.26) reduces to

$$d\dot{\boldsymbol{\lambda}}/dt = \mathbf{B}(\dot{\boldsymbol{\lambda}}, \dot{\boldsymbol{\lambda}})\mathbf{n} \tag{58.27}$$

Comparing this result with the classical notions of curvature and principal normal of a curve (cf. Section 53), we see that the surface normal is the principal normal of a surface curve $\boldsymbol{\lambda}$ if and only if $\boldsymbol{\lambda}$ is a surface geodesic; further, in this case the curvature of $\boldsymbol{\lambda}$ is the quadratic form $\mathbf{B}(\mathbf{s},\mathbf{s})$ of $\mathbf{B}$ in the direction of the unit tangent $\mathbf{s}$ of $\boldsymbol{\lambda}$.

In general, if $\boldsymbol{\lambda}$ is not a surface geodesic but it is parameterized by the arc length, then (58.26) reads

$$\frac{d\mathbf{s}}{ds} = \mathbf{B}(\mathbf{s},\mathbf{s})\,\mathbf{n} + \frac{D\mathbf{s}}{Ds} \tag{58.28}$$

where $\mathbf{s}$ denotes the unit tangent of $\boldsymbol{\lambda}$, as usual. We call the norms of the three vectors in (58.28) the *spatial curvature, the normal curvature,* and the *geodesic curvature* of $\boldsymbol{\lambda}$ and denote them by $\kappa, \kappa_n$, and $\kappa_g$, respectively, namely

$$\kappa = \left\| \frac{d\mathbf{s}}{ds} \right\|, \qquad \kappa_n = \mathbf{B}(\mathbf{s},\mathbf{s}), \qquad \kappa_g = \left\| \frac{D\mathbf{s}}{Ds} \right\| \tag{58.29}$$

Then (58.28) implies

$$\kappa^2 = \kappa_n^2 + \kappa_g^2 \tag{58.30}$$

Further, if $\mathbf{n}_0$ and $\mathbf{n}_g$ are unit vectors in the directions of $d\mathbf{s}/ds$ and $D\mathbf{s}/Ds$, then

$$\kappa\mathbf{n}_0 = \kappa_n\mathbf{n} + \kappa_g\mathbf{n}_g \tag{58.31}$$

Or equivalently,

$$\mathbf{n}_0 = (\kappa_n/\kappa)\,\mathbf{n} + (\kappa_g/\kappa)\,\mathbf{n}_g \tag{58.32}$$

which is called *Meunier's equation.*

At this point we can define another kind of covariant derivative for tensor fields on $\mathscr{S}$. As we have remarked before, the tangent plane $\mathscr{S}_{\mathbf{x}}$ of $\mathscr{S}$ is a subspace of $\mathscr{V}$. Hence a tangential vector $\mathbf{v}$ can be regarded either as a surface vector in $\mathscr{S}_{\mathbf{x}}$ or as a special vector in $\mathscr{V}$. In the former sense it is natural to use the surface covariant derivative $D\mathbf{v}/Dt$, while in the latter sense we may consider the spatial covariant derivative $d\mathbf{v}/dt$ along any smooth curve $\boldsymbol{\lambda}$ in $\mathscr{S}$. For a tangential tensor field $\mathbf{A}$, such as the one represented (56.20) or (56.22), we may choose to recognize certain indices as *spatial indices* and the remaining ones as *surface indices.* In this case it becomes possible to define a covariant derivative tha is a mixture of $d/dt$ and $D/Dt$. In classical differential geometry this new kind of covariant derivative is called the *total covariant derivative.*

More specifically, we first consider a concrete example to explain this concept. By definition, the surface metric tensor $\mathbf{a}$ is a constant tangential tensor relative to the surface covariant derivative. In terms of any surface coordinate system $\left(y^\Gamma\right)$, $\mathbf{a}$ can be represented by

$$\mathbf{a} = a_{\Gamma\Delta}\mathbf{h}^\Gamma \otimes \mathbf{h}^\Delta = \mathbf{h}_\Delta \otimes \mathbf{h}^\Delta = a^{\Gamma\Delta}\mathbf{h}_\Gamma \otimes \mathbf{h}_\Delta \tag{58.33}$$

and we have

$$\mathbf{0} = \frac{D\mathbf{a}}{Dt} = \frac{D\left(\mathbf{h}_\Delta \otimes \mathbf{h}^\Delta\right)}{Dt} = \mathbf{h}_\Delta \otimes \frac{D\mathbf{h}^\Delta}{Dt} + \frac{D\mathbf{h}_\Delta}{Dt} \otimes \mathbf{h}^\Delta \tag{58.34}$$

The tensor $\mathbf{a}$, however, can be regarded also as the inclusion map $\mathbf{A}$ of $\mathscr{S}_\mathbf{x}$ in $\mathscr{V}$, namely

$$\mathbf{A} : \mathscr{S}_\mathbf{x} \to \mathscr{V} \tag{58.35}$$

in the sense that for any surface vector $\mathbf{v} \in \mathscr{S}_\mathbf{x}$, $\mathbf{A}\mathbf{v} \in \mathscr{V}$ is given by

$$\mathbf{A}\mathbf{v} = \left(\mathbf{h}_\Delta \otimes \mathbf{h}^\Delta\right)(\mathbf{v}) = \mathbf{h}_\Delta\left(\mathbf{h}^\Delta \cdot \mathbf{v}\right) = \upsilon^\Delta \mathbf{h}_\Delta \tag{58.36}$$

In (58.36) it is more natural to regard the first basis vector $\mathbf{h}_\Delta$ in the product basis $\mathbf{h}_\Delta \otimes \mathbf{h}^\Delta$ as a spatial vector and the second basis vector $\mathbf{h}^\Delta$ as a surface vector. In fact, in classical differential geometry the tensor $\mathbf{A}$ given by (58.35) is often denoted by

$$\mathbf{A} = h^i_\Delta \mathbf{g}_i \otimes \mathbf{h}^\Delta = \frac{\partial x^i}{\partial y^\Delta}\mathbf{g}_i \otimes \mathbf{h}^\Delta \tag{58.37}$$

where $\left(x^i\right)$ is a spatial coordinate system whose natural basis is $\{\mathbf{g}_i\}$. When the indices are recognized in this way, it is natural to define a *total covariant derivative* of $\mathbf{A}$, denoted by $\delta\mathbf{A}/\delta t$, by

$$\begin{aligned}\frac{\delta \mathbf{A}}{\delta t} &= \frac{d\mathbf{h}_\Delta}{dt} \otimes \mathbf{h}^\Delta + \mathbf{h}_\Delta \otimes \frac{D\mathbf{h}^\Delta}{Dt} \\ &= \frac{d}{dt}\left(\frac{\partial x^i}{\partial y^\Delta}\mathbf{g}_i\right) \otimes \mathbf{h}^\Delta + \mathbf{h}_\Delta \otimes \frac{D\mathbf{h}^\Delta}{Dt}\end{aligned} \tag{58.38}$$

Since a mixture of $d/dt$ and $D/Dt$ is used in defining $\delta/\delta t$, it is important to recognize the "spatial" or "surface" designation of the indices of the components of a tensor before the total covariant derivative is computed.

In classical differential geometry the indices are distinguished directly in the notation of the components. Thus a tensor $\mathbf{A}$ represented by

$$\mathbf{A}(t) = A^{ik\ldots\Gamma\ldots}_{j\ldots\Delta\ldots}(t)\mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}_k \otimes \cdots \otimes \mathbf{h}_\Gamma \otimes \mathbf{h}^\Delta \cdots \tag{58.39}$$

has an obvious interpretation: the Latin indices $i, j, k\ldots$ are designated as "spatial" and the Greek indices $\Gamma, \Delta, \ldots$ are "surface." So $\delta\mathbf{A}/\delta t$ is defined by

$$\begin{aligned}
\frac{\delta \mathbf{A}}{\delta t} &= \frac{d}{dt}\left(A^{ik\ldots\Gamma\ldots}_{j\ldots\Delta\ldots}\mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}_k \otimes \cdots\right) \otimes \mathbf{h}_\Gamma \otimes \mathbf{h}^\Delta \cdots \\
&\quad + A^{ik\ldots\Gamma\ldots}_{j\ldots\Delta\ldots}\mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}_k \otimes \cdots \otimes \frac{D}{Dt}\left(\mathbf{h}_\Gamma \otimes \mathbf{h}^\Delta \cdots\right) \\
&= \left(\frac{d}{dt} A^{ik\ldots\Gamma\ldots}_{j\ldots\Delta\ldots}\right)\mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}_k \otimes \cdots \otimes \mathbf{h}_\Gamma \otimes \mathbf{h}^\Delta \cdots \\
&\quad + A^{ik\ldots\Gamma\ldots}_{j\ldots\Delta\ldots}\left[\frac{d}{dt}\left(\mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}_k \otimes \cdots\right) \otimes \mathbf{h}_\Gamma \otimes \mathbf{h}^\Delta \cdots \right. \\
&\quad \left. + \mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}_k \otimes \cdots \otimes \frac{D}{Dt}\left(\mathbf{h}_\Gamma \otimes \mathbf{h}^\Delta \cdots\right)\right]
\end{aligned} \tag{58.40}$$

The derivative $dA^{ik\ldots\Gamma\ldots}_{j\ldots\Delta\ldots}/dt$ is the same as $DA^{ik\ldots\Gamma\ldots}_{j\ldots\Delta\ldots}/Dt$, of course, since for scalars there is but one kind of parallelism along any curve. In particular, we can compute explicitly the representation for $\delta\mathbf{A}/\delta t$ as defined by (58.38):

$$\begin{aligned}
\frac{\delta \mathbf{A}}{\delta t} &= \frac{d}{dt}\left(\frac{\partial x^i}{\partial y^\Delta}\right)\mathbf{g}_i \otimes \mathbf{h}^\Delta + \frac{\partial x^i}{\partial y^\Delta}\left(\frac{d\mathbf{g}_i}{dt} \otimes \mathbf{h}^\Delta + \mathbf{g}_i \otimes \frac{D\mathbf{h}^\Delta}{Dt}\right) \\
&= \left(\frac{\partial^2 x^i}{\partial y^\Delta \partial y^\Gamma} + \frac{\partial x^j}{\partial y^\Delta}\frac{\partial x^k}{\partial y^\Gamma}\begin{Bmatrix} i \\ jk \end{Bmatrix} - \frac{\partial x^i}{\partial y^\Sigma}\begin{Bmatrix} \Sigma \\ \Gamma\Delta \end{Bmatrix}\dot{\lambda}^\Gamma \mathbf{g}_i \otimes \mathbf{h}^\Delta\right)
\end{aligned} \tag{58.41}$$

As a result, when $\boldsymbol{\lambda}$ is the coordinate curve of $(y^\Gamma)$, (58.41) reduces to

$$\frac{\delta \mathbf{A}}{\delta y^\Gamma} = \left(\frac{\partial^2 x^i}{\partial y^\Delta \partial y^\Gamma} + \frac{\partial x^j}{\partial y^\Delta}\frac{\partial x^k}{\partial y^\Gamma}\begin{Bmatrix} i \\ jk \end{Bmatrix} - \frac{\partial x^i}{\partial y^\Sigma}\begin{Bmatrix} \Sigma \\ \Gamma\Delta \end{Bmatrix}\right)\mathbf{g}_i \otimes \mathbf{h}^\Delta \tag{58.42}$$

Actually, the quantity on the right-hand side of (58.42) has a very simple representation. Since $\delta \mathbf{A} / \delta y^{\Gamma}$ can be expressed also by $(58.38)_1$, namely

$$\frac{\delta \mathbf{A}}{\delta y^{\Gamma}} = \frac{\partial \mathbf{h}_{\Delta}}{\partial y^{\Gamma}} \otimes \mathbf{h}^{\Delta} + \mathbf{h}_{\Delta} \otimes \frac{D\mathbf{h}^{\Delta}}{Dy^{\Gamma}} \tag{58.43}$$

from (58.21) we have

$$\begin{aligned} \frac{\delta \mathbf{A}}{\delta y^{\Gamma}} &= b_{\Delta\Gamma}\mathbf{n} \otimes \mathbf{h}^{\Delta} + \frac{D\mathbf{h}_{\Delta}}{Dy^{\Gamma}} \otimes \mathbf{h}^{\Delta} + \mathbf{h}_{\Delta} \otimes \frac{D\mathbf{h}^{\Delta}}{Dy^{\Gamma}} \\ &= b_{\Delta\Gamma}\mathbf{n} \otimes \mathbf{h}^{\Delta} + \frac{D}{Dy^{\Gamma}}\left(\mathbf{h}_{\Delta} \otimes \mathbf{h}^{\Delta}\right) \\ &= b_{\Delta\Gamma}\mathbf{n} \otimes \mathbf{h}^{\Delta} \end{aligned} \tag{58.44}$$

where we have used (58.34). The formula $(58.44)_3$ is known as *Gauss' formula* in classical differential geometry. It is often written in component form

$$x^{i}{}_{;\Gamma\Delta} = b_{\Gamma\Delta} n^{i} \tag{58.45}$$

where the semicolon in the subscript on the left-hand side denotes the total covariant derivative.

## *Exercises*

58.1 Show that

(a) $b_{\Gamma\Delta} = -\dfrac{1}{2}\left( \dfrac{\partial \mathbf{n}}{\partial y^{\Delta}} \cdot \dfrac{\partial \mathbf{x}}{\partial y^{\Gamma}} + \dfrac{\partial \mathbf{x}}{\partial y^{\Gamma}} \cdot \dfrac{\partial \mathbf{n}}{\partial y^{\Gamma}} \right)$

(b) $b_{\Delta\Gamma} = g_{ij} x^{i}{}_{;\Delta\Gamma}\, n^{j}$

(c) $b_{\Delta\Gamma} = \mathbf{n} \cdot \dfrac{\partial^{2} \mathbf{x}}{\partial y^{\Delta} \partial y^{\Gamma}}$

58.2 Compute the quantities $b_{\Delta\Gamma}$ for the surfaces defined in Exercises 55.5 and 55.7.

58.3 Let $\mathbf{A}$ be a tensor field of the form

$$\mathbf{A} = A^{j_1 \ldots j_r}{}_{\Gamma_1 \ldots \Gamma_s} \mathbf{g}_{j_1} \otimes \cdots \otimes \mathbf{g}_{j_r} \otimes \mathbf{h}^{\Gamma_1} \otimes \cdots \otimes \mathbf{h}^{\Gamma_s}$$

show that

$$\frac{\delta \mathbf{A}}{\delta y^{\Delta}} = A^{j_1 \dots j_r}{}_{\Gamma_1 \dots \Gamma_s;\Delta} \mathbf{g}_{j_1} \otimes \cdots \otimes \mathbf{g}_{j_r} \otimes \mathbf{h}^{\Gamma_1} \otimes \cdots \otimes \mathbf{h}^{\Gamma_s}$$

where

$$\begin{aligned} A^{j_1 \dots j_r}{}_{\Gamma_1 \dots \Gamma_s;\Delta} = & \frac{\partial A^{j_1 \dots j_r}{}_{\Gamma_1 \dots \Gamma_s}}{\partial y^{\Delta}} \\ & + \sum_{\beta=1}^{r} \left\{ \begin{matrix} j_\beta \\ lk \end{matrix} \right\} A^{j_1 \dots j_{\beta-1} l j_{\beta+1} \dots j_r}{}_{\Gamma_1 \dots \Gamma_s} \frac{\partial x^k}{\partial y^{\Delta}} \\ & - \sum_{\beta=1}^{s} \left\{ \begin{matrix} \Lambda \\ \Gamma_\beta \Delta \end{matrix} \right\} A^{j_1 \dots j_r}{}_{\Gamma_1 \dots \Gamma_{\beta-1} \Lambda \Gamma_{\beta+1} \dots \Gamma_s} \end{aligned}$$

Similar formulas can be derived for other types of mixed tensor fields defined on $\mathscr{S}$ .

58.4 Show that (58.7) can be written

$$n^i{}_{;\Gamma} = -b^{\Delta}_{\Gamma} \frac{\partial x^i}{\partial y^{\Delta}}$$

## *Section 59. Surface Curvature, II. The Riemann-Christoffel Tensor and the Ricci Identities*

In the preceding section we have considered the curvature of $\mathscr{S}$ by examining the change of the unit normal **n** of $\mathscr{S}$ in $\mathscr{E}$. This approach is natural for a hypersurface, since the metric on $\mathscr{S}$ is induced by that of $\mathscr{E}$. The results of this approach, however, are not entirely intrinsic to $\mathscr{S}$, since they depend not only on the surface metric but also on the particular imbedding of $\mathscr{S}$ into $\mathscr{E}$. In this section, we shall consider curvature from a more intrinsic point of view. We seek results which depend only on the surface metric. Our basic idea is that curvature on $\mathscr{S}$ corresponds to the departure of the Levi-Civita parallelism on $\mathscr{S}$ from a Euclidean parallelism.

We recall that relative to a Cartesian coordinate system $\hat{x}$ on $\mathscr{E}$ the covariant derivative of a vector field **v** has the simplest representation

$$\operatorname{grad} v = \upsilon^i{,}_j \, \mathbf{g}_i \otimes \mathbf{g}^j = \frac{\partial \upsilon^i}{\partial x^j} \mathbf{g}_i \otimes \mathbf{g}^j \tag{59.1}$$

Hence if we take the second covariant derivatives, then in the same coordinate system we have

$$\operatorname{grad}\left(\operatorname{grad} \mathbf{v}\right) = \upsilon^i{,}_{jk} \, \mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}^k = \frac{\partial^2 \upsilon^i}{\partial x^j \partial x^k} \mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \tag{59.2}$$

In particular, the second covariant derivatives satisfy the same symmetry condition as that of the ordinary partial derivatives:

$$\upsilon^i{,}_{jk} = \upsilon^i{,}_{kj} \tag{59.3}$$

Note that the proof of (59.3) depends crucially on the existence of a Cartesian coordinate system relative to which the Christoffel symbols of the Euclidean parallelism vanish identically. For the hypersurface $\mathscr{S}$ in general, the geodesic coordinate system at a reference point is the closest counterpart of a Cartesian system. However, in a geodesic coordinate system the surface Christoffel symbols vanish at the reference point only. As a result the surface covariant derivative at the reference point $\mathbf{x}_0$ still has a simple representation like (59.1)

$$\begin{aligned} \operatorname{grad} \mathbf{v}\left(\mathbf{x}_0\right) &= \upsilon^\Gamma{,}_\Delta \left(\mathbf{x}_0\right) \mathbf{e}_\Gamma \left(\mathbf{x}_0\right) \otimes \mathbf{e}^\Delta \left(\mathbf{x}_0\right) \\ &= \left.\frac{\partial \upsilon^\Gamma}{\partial z^\Delta}\right|_{\mathbf{x}_0} \mathbf{e}_\Gamma \left(\mathbf{x}_0\right) \otimes \mathbf{e}^\Delta \left(\mathbf{x}_0\right) \end{aligned} \tag{59.4}$$

[cf. (57.40)]. But generally, in the same coordinate system, the representation (59.4)(59.4) does not hold any neighboring point of $\mathbf{x}_0$. This situation has been explained in detail in Section 57.

In particular, there is no counterpart for the representation $(59.2)_2$ on $\mathscr{S}$. Indeed the surface second covariant derivatives generally fail to satisfy the symmetry condition (59.3) valid for the spatial covariant derivatives.

To see this fact we choose an arbitrary surface coordinate system $\left(y^\Gamma\right)$ with natural basis $\left\{\mathbf{h}_\Gamma\right\}$ and $\left\{\mathbf{h}^\Gamma\right\}$ on $\mathscr{S}$ as before. From (56.10) the surface covariant derivative of a tangential vector field

$$\mathbf{v} = \upsilon^\Gamma \mathbf{h}_\Gamma \tag{59.5}$$

is given by

$$\upsilon^\Gamma{}_{,\Delta} = \frac{\partial \upsilon^\Gamma}{\partial y^\Delta} + \upsilon^\Sigma \begin{Bmatrix} \Gamma \\ \Sigma\Delta \end{Bmatrix} \tag{59.6}$$

Applying the same formula again, we obtain

$$\begin{aligned}
\upsilon^\Gamma{}_{,\Delta\Omega} &= \frac{\partial}{\partial y^\Omega}\left(\frac{\partial \upsilon^\Gamma}{\partial y^\Delta} + v^\Sigma \begin{Bmatrix} \Gamma \\ \Sigma\Delta \end{Bmatrix}\right) + \left(\frac{\partial \upsilon^\Phi}{\partial y^\Delta} + \upsilon^\Sigma \begin{Bmatrix} \Phi \\ \Sigma\Delta \end{Bmatrix}\right)\begin{Bmatrix} \Gamma \\ \Phi\Omega \end{Bmatrix} \\
&\quad -\left(\frac{\partial \upsilon^\Gamma}{\partial y^\Psi} + \upsilon^\Sigma \begin{Bmatrix} \Gamma \\ \Sigma\Psi \end{Bmatrix}\right)\begin{Bmatrix} \Psi \\ \Delta\Omega \end{Bmatrix} \\
&= \frac{\partial^2 \upsilon^\Gamma}{\partial y^\Delta \partial y^\Omega} + \frac{\partial \upsilon^\Sigma}{\partial y^\Omega}\begin{Bmatrix} \Gamma \\ \Sigma\Delta \end{Bmatrix} + \frac{\partial \upsilon^\Phi}{\partial y^\Delta}\begin{Bmatrix} \Gamma \\ \Phi\Omega \end{Bmatrix} - \frac{\partial \upsilon^\Gamma}{\partial y^\Psi}\begin{Bmatrix} \Psi \\ \Delta\Omega \end{Bmatrix} \\
&\quad +\upsilon^\Sigma\left(\begin{Bmatrix} \Phi \\ \Sigma\Delta \end{Bmatrix}\begin{Bmatrix} \Gamma \\ \Phi\Omega \end{Bmatrix} - \begin{Bmatrix} \Gamma \\ \Sigma\Psi \end{Bmatrix}\begin{Bmatrix} \Psi \\ \Delta\Omega \end{Bmatrix} + \frac{\partial}{\partial y^\Omega}\begin{Bmatrix} \Gamma \\ \Sigma\Delta \end{Bmatrix}\right)
\end{aligned} \tag{59.7}$$

In particular, even if the surface coordinate system reduces to a geodesic coordinate system $\left(y^\Gamma\right)$ at a reference point $\mathbf{x}_0$, (59.7) can only be simplified to

$$\upsilon^\Gamma{}_{,\Delta\Omega}\left(\mathbf{x}_0\right) = \left.\frac{\partial^2 \upsilon^\Gamma}{\partial z^\Delta \partial z^\Omega}\right|_{\mathbf{x}_0} + \upsilon^\Sigma\left(\mathbf{x}_0\right) \left.\frac{\partial}{\partial z^\Omega}\begin{Bmatrix} \Gamma \\ \Sigma\Delta \end{Bmatrix}\right|_{\mathbf{x}_0} \tag{59.8}$$

which contains a term in addition to the spatial case $(59.2))_2$. Since that additional term generally need not be symmetric in the pair $(\Delta, \Omega)$, we have shown that the surface second covariant derivatives do not necessarily obey the symmetry condition (59.3).

From (59.7) if we subtract $\upsilon^{\Gamma}_{,\Omega\Delta}$ from $\upsilon^{\Gamma}_{,\Delta\Omega}$ then the result is the following commutation rule:

$$\upsilon^{\Gamma}{}_{,\Delta\Omega} - \upsilon^{\Gamma}{}_{,\Omega\Delta} = -\upsilon^{\Sigma} R^{\Gamma}{}_{\Sigma\Delta\Omega} \tag{59.9}$$

where

$$R^{\Gamma}{}_{\Sigma\Delta\Omega} \equiv \frac{\partial}{\partial y^{\Delta}} \begin{Bmatrix} \Gamma \\ \Sigma\Omega \end{Bmatrix} - \frac{\partial}{\partial y^{\Omega}} \begin{Bmatrix} \Gamma \\ \Sigma\Delta \end{Bmatrix} + \begin{Bmatrix} \Phi \\ \Sigma\Omega \end{Bmatrix} \begin{Bmatrix} \Gamma \\ \Phi\Delta \end{Bmatrix} - \begin{Bmatrix} \Phi \\ \Sigma\Delta \end{Bmatrix} \begin{Bmatrix} \Gamma \\ \Phi\Omega \end{Bmatrix} \tag{59.10}$$

Since the commutation rule is valid for all tangential vector fields $\mathbf{v}$, (59.9) implies that under a change of surface coordinate system the fields $R^{\Gamma}{}_{\Sigma\Delta\Omega}$ satisfy the transformation rule for the components of a fourth-order tangential tensor. Thus we define

$$\mathsf{R} \equiv R^{\Gamma}{}_{\Sigma\Delta\Omega} \mathbf{h}_{\Gamma} \otimes \mathbf{h}^{\Sigma} \otimes \mathbf{h}^{\Delta} \otimes \mathbf{h}^{\Omega} \tag{59.11}$$

and we call the tensor $\mathsf{R}$ the *Riemann-Christoffel* tensor of $\mathscr{S}$. Notice that $\mathsf{R}$ depends only on the surface metrica $\mathbf{a}$, since its components are determined completely by the surface Christoffel symbols. In particular, in a geodesic coordinates system $\left(z^{\Gamma}\right)$ at $\mathbf{x}_0$ (59.10) simplifies to

$$R^{\Gamma}{}_{\Sigma\Delta\Omega}\left(\mathbf{x}_0\right) = \frac{\partial}{\partial z^{\Delta}} \begin{Bmatrix} \Gamma \\ \Sigma\Omega \end{Bmatrix}\Bigg|_{\mathbf{x}_0} - \frac{\partial}{\partial z^{\Omega}} \begin{Bmatrix} \Gamma \\ \Sigma\Delta \end{Bmatrix}\Bigg|_{\mathbf{x}_0} \tag{59.12}$$

In a certain sense the Riemann-Christoffel tensor $\mathsf{R}$ characterizes locally the departure of the Levi-Civita parallelism from a Euclidean one. We have the following result.

**Theorem 59.1**. The Riemann-Christoffel tensor $\mathsf{R}$ vanishes identically on a neighborhood of a point $\mathbf{x}_0 \in \mathscr{S}$ if and only if there exists a surface coordinate system $\left(z^{\Gamma}\right)$ covering $\mathbf{x}_0$ relativwe to which the surface Christoffel symbols $\begin{Bmatrix} \Gamma \\ \Delta\Sigma \end{Bmatrix}$ vanishing identically on a neighborhood of $\mathbf{x}_0$.

In view of the representation (59.10), the sufficiency part of the preceding theorem is obvious. To prove the necessity part, we observe first the following lemma.

**Lemma**. The Riemann-Christoffel tensor $\mathsf{R}$ vanishes identically near $\mathbf{x}_0$ if and only if the system of first-order partial differential equations

$$0 = \upsilon^{\Gamma}{}_{,\Delta} = \frac{\partial \upsilon^{\Gamma}}{\partial y^{\Delta}} + \upsilon^{\Omega} \begin{Bmatrix} \Gamma \\ \Omega\Delta \end{Bmatrix}, \qquad \Gamma = 1, \ldots N-1 \tag{59.13}$$

can be integrated near $\mathbf{x}_0$ for each prescribed initial value

$$\upsilon^\Gamma(\mathbf{x}_0) = \upsilon_0^\Gamma, \qquad \Gamma = 1, \ldots N-1 \tag{59.14}$$

*Proof. Necessity.* Let $\{\mathbf{v}_\Sigma, \Sigma = 1, \ldots, N-1\}$ be linearly independent and satisfy (59.13). Then we can obtain the surface Christoffel symbols from

$$0 = \frac{\partial \upsilon_\Sigma^\Gamma}{\partial y^\Delta} + \upsilon_\Sigma^\Omega \begin{Bmatrix} \Gamma \\ \Omega\Delta \end{Bmatrix} \tag{59.15}$$

and

$$\begin{Bmatrix} \Gamma \\ \Omega\Delta \end{Bmatrix} = -u_\Omega^\Sigma \frac{\partial \upsilon_\Sigma^\Gamma}{\partial y^\Delta} \tag{59.16}$$

where $\left[u_\Omega^\Sigma\right]$ denotes the inverse matrix of $\left[\upsilon_\Sigma^\Omega\right]$. Substituting (59.16) into (59.10), we obtain directly

$$R^\Gamma{}_{\Sigma\Delta\Omega} = 0 \tag{59.17}$$

*Sufficiency*. It follows from the Frobenius theorem that the conditions of integrability for the system of first-order partial differential equations

$$\frac{\partial \upsilon^\Gamma}{\partial y^\Delta} = -\upsilon^\Omega \begin{Bmatrix} \Gamma \\ \Omega\Delta \end{Bmatrix} \tag{59.18}$$

are

$$\frac{\partial}{\partial y^\Sigma}\left(\upsilon^\Omega \begin{Bmatrix} \Gamma \\ \Omega\Delta \end{Bmatrix}\right) - \frac{\partial}{\partial y^\Delta}\left(\upsilon^\Omega \begin{Bmatrix} \Gamma \\ \Omega\Sigma \end{Bmatrix}\right) = 0, \qquad \Gamma = 1, \ldots N-1 \tag{59.19}$$

If we expand the partial derivatives in (59.19) and use (59.18), then (59.17) follows as a result of (59.10). Thus the lemma is proved.

Now we return to the proof of the necessity part of the theorem. From (59.16) and the condition (56.4) we see that the basis $\{\mathbf{v}_\Sigma\}$ obeys the rule

$$\frac{\partial \upsilon_\Sigma^\Gamma}{\partial y^\Delta}\upsilon_\Omega^\Delta - \frac{\partial \upsilon_\Omega^\Gamma}{\partial y^\Delta}\upsilon_\Sigma^\Delta = 0 \tag{59.20}$$

These equations are the coordinate forms of

$$\left[\mathbf{v}_\Sigma, \mathbf{v}_\Omega\right] = \mathbf{0}, \qquad \Sigma, \Omega = 1, \ldots, N-1 \tag{59.21}$$

As a result there exists a coordinate system $\left(z^\Gamma\right)$ whose natural basis coincides with $\left\{\mathbf{v}_\Sigma\right\}$. In particular, relative to $\left(z^\Gamma\right)$ (59.16) reduces to

$$\left\{ \begin{matrix} \Gamma \\ \Omega\Delta \end{matrix} \right\} = -\delta_\Sigma^\Omega \frac{\partial \delta_\Sigma^\Gamma}{\partial z^\Delta} = 0 \tag{59.22}$$

Thus the theorem is proved.

It should be noted that tangential vector fields satisfying (59.13) are *parallel fields* relative to the Levi-Civita parallelism, since from (56.24) we have the representation

$$D\mathbf{v}\left(\boldsymbol{\lambda}\left(t\right)\right)/Dt = \upsilon^\Gamma{}_{,\Delta}\, \dot{\lambda}^\Delta \mathbf{h}_\Gamma \tag{59.23}$$

along any curve $\boldsymbol{\lambda}$. Consequently, the Levi-Civita parallelism becomes locally *path-independent*. For definiteness, we call this a *locally Euclidean parallelism* or a *flat parallelism*. Then the preceding theorem asserts that the Levi-Civita parallelism on $\mathscr{S}$ is locally Euclidean if and only if the Riemann-Christoffel tensor $\mathbf{R}$ based on the surface metric vanishes.

The commutation rule (59.9) is valid for tangential vector fields only. However, we can easily generalize that rule to the following Ricci identities:

$$\begin{aligned} A^{\Gamma_1 \ldots \Gamma_r}{}_{\Delta_1 \ldots \Delta_s, \Sigma\Omega} - A^{\Gamma_1 \ldots \Gamma_r}{}_{\Delta_1 \ldots \Delta_s, \Omega\Sigma} & \\ = -A^{\Phi\Gamma_2 \ldots \Gamma_r}{}_{\Delta_1 \ldots \Delta_s} R^{\Gamma_1}{}_{\Phi\Sigma\Omega} & \\ - \cdots - A^{\Gamma_1 \ldots \Gamma_{r-1}\Phi}{}_{\Delta_1 \ldots \Delta_s} R^{\Gamma_r}{}_{\Phi\Sigma\Omega} + A^{\Gamma_1 \ldots \Gamma_r}{}_{\Psi\Delta_2 \ldots \Delta_s} R^{\Psi}{}_{\Delta_1\Sigma\Omega} & \\ + \cdots + A^{\Gamma_1 \ldots \Gamma_r}{}_{\Delta_1 \ldots \Delta_{s-1}\Psi} R^{\Psi}{}_{\Delta_s\Sigma\Omega} & \end{aligned} \tag{59.24}$$

As a result, (59.17) is also the integrability condition of the system

$$A^{\Gamma_1 \ldots \Gamma_r}{}_{\Delta_1 \ldots \Delta_s, \Sigma} = 0 \tag{59.25}$$

for each prescribed initial value at any reference point $\mathbf{x}_0$; further, a solution of (59.25) corresponds to a parallel tangential tensor field on the developable hyper surface $\mathscr{S}$.

In classical differential geometry a surface $\mathscr{S}$ is called *developable* if its Riemann-Christoffel curvature tensore vanishes identically.

## Section 60. Surface Curvature, III. The Equations of Gauss and Codazzi

In the preceding two sections we have considered the curvature of $\mathscr{S}$ from both the extrinsic point of view and the intrinsic point of view. In this section we shall unite the results of these two approaches.

Our starting point is the general representation (58.25) applied to the $y^\Delta$-coordinate curve:

$$\frac{\partial \mathbf{v}}{\partial y^\Delta} = b_{\Gamma\Delta}\upsilon^\Gamma \mathbf{n} + \frac{D\mathbf{v}}{Dy^\Delta} \tag{60.1}$$

for an arbitrary tangential vector field $\mathbf{v}$. This representation gives the natural decomposition of the spatial vector field $\partial\mathbf{v}/\partial y^\Delta$ on $\mathscr{S}$ into a normal projection $b_{\Gamma\Delta}\upsilon^\Gamma\mathbf{n}$ and a tangential projection $D\mathbf{v}/Dy^\Delta$. Applying the spatial covariant derivative $\partial/\partial y^\Sigma$ along the $y^\Sigma$- coordinate curve to (60.1)(60.1), we obtain

$$\begin{aligned}\frac{\partial}{\partial y^\Sigma}\left(\frac{\partial \mathbf{v}}{\partial y^\Delta}\right) &= \frac{\partial}{\partial y^\Sigma}\left(b_{\Gamma\Delta}\upsilon^\Gamma\mathbf{n}\right) + \frac{\partial}{\partial y^\Sigma}\left(\frac{D\mathbf{v}}{Dy^\Delta}\right)\\ &= \frac{\partial}{\partial y^\Sigma}\left(b_{\Gamma\Delta}\upsilon^\Gamma\mathbf{n}\right) + b_{\Gamma\Sigma}\upsilon^\Gamma{}_{,\Delta}\mathbf{n} + \frac{D}{Dy^\Sigma}\left(\frac{D\mathbf{v}}{Dy^\Delta}\right)\end{aligned} \tag{60.2}$$

where we have applied (60.1) in $(60.2)_2$ to the tangential vector field $D\mathbf{v}/Dy^\Delta$. Now, since the spatial parallelism is Euclidean, the lefthand side of (60.2) is symmetric in the pair $(\Sigma,\Delta)$, namely

$$\frac{\partial}{\partial y^\Sigma}\left(\frac{\partial \mathbf{v}}{\partial y^\Delta}\right) = \frac{\partial}{\partial y^\Delta}\left(\frac{\partial \mathbf{v}}{\partial y^\Sigma}\right) \tag{60.3}$$

Hence from (60.2) we have

$$\begin{aligned}\frac{D}{Dy^\Delta}\left(\frac{D\mathbf{v}}{Dy^\Sigma}\right) - \frac{D}{Dy^\Sigma}\left(\frac{D\mathbf{v}}{Dy^\Delta}\right) &= \frac{\partial}{\partial y^\Sigma}\left(b_{\Gamma\Delta}\upsilon^\Gamma\mathbf{n}\right) - \left(b_{\Gamma\Sigma}\upsilon^\Gamma\mathbf{n}\right)\\ &\quad + \left(b_{\Gamma\Sigma}\upsilon^\Gamma{}_{,\Delta} - b_{\Gamma\Delta}\upsilon^\Gamma{}_{,\Sigma}\right)\mathbf{n}\end{aligned} \tag{60.4}$$

This is the basic formula from which we can extract the relation between the Riemann-Christoffel tensor $\mathsf{R}$ and the second fundamental form $\mathbf{B}$.

Specifically, the left-hand side of (60.4) is a tangaential vector having the component form

$$\frac{D}{Dy^\Delta}\left(\frac{D\mathbf{v}}{Dy^\Sigma}\right)-\frac{D}{Dy^\Sigma}\left(\frac{D\mathbf{v}}{Dy^\Delta}\right)=\left(\upsilon^\Gamma{}_{,\Delta\Sigma}-\upsilon^\Gamma{}_{,\Sigma\Delta}\right)\mathbf{h}_\Gamma \qquad (60.5)$$
$$=-\upsilon^\Omega R^\Gamma{}_{\Omega\Sigma\Delta}\mathbf{h}_\Gamma$$

as required by (59.9), while the right-hand side is a vector having the normal projection

$$\left(\frac{\partial\left(b_{\Gamma\Delta}\upsilon^\Gamma\right)}{\partial y^\Sigma}-\frac{\partial\left(b_{\Gamma\Sigma}\upsilon^\Gamma\right)}{\partial y^\Delta}+b_{\Gamma\Sigma}\upsilon^\Gamma{}_{,\Delta}-b_{\Gamma\Delta}\upsilon^\Gamma{}_{,\Sigma}\right)\mathbf{n} \qquad (60.6)$$

and the tangential projection

$$\left(-b_{\Gamma\Delta}\upsilon^\Gamma b^\Phi_\Sigma+b_{\Gamma\Sigma}\upsilon^\Gamma b^\Phi_\Delta\right)\mathbf{h}_\Phi \qquad (60.7)$$

where we have used Weingarten's formula (58.7) to compute the covariant derivatives $\partial\mathbf{n}/\partial y^\Sigma$ and $\partial\mathbf{n}/\partial y^\Delta$. Consequently, (60.5) implies that

$$\frac{\partial\left(b_{\Gamma\Delta}\upsilon^\Gamma\right)}{\partial y^\Sigma}-\frac{\partial\left(b_{\Gamma\Sigma}\upsilon^\Gamma\right)}{\partial y^\Delta}+b_{\Gamma\Sigma}\upsilon^\Gamma{}_{,\Delta}-b_{\Gamma\Delta}\upsilon^\Gamma{}_{,\Sigma}=0 \qquad (60.8)$$

and that

$$\upsilon^\Gamma R^\Phi{}_{\Gamma\Sigma\Delta}=b_{\Gamma\Delta}\upsilon^\Gamma b^\Phi_\Sigma-b_{\Gamma\Sigma}\upsilon^\Gamma b^\Phi_\Delta \qquad (60.9)$$

for all tangential vector fields $\mathbf{v}$.

If we choose $\mathbf{v}=\mathbf{h}_\Gamma$, then (60.9) becomes

$$R^\Phi{}_{\Gamma\Sigma\Delta}=b_{\Gamma\Delta}b^\Phi_\Sigma-b_{\Gamma\Sigma}b^\Phi_\Delta \qquad (60.10)$$

or, equivalently,

$$R_{\Phi\Gamma\Sigma\Delta}=b_{\Gamma\Delta}b_{\Phi\Sigma}-b_{\Gamma\Sigma}b_{\Phi\Delta} \qquad (60.11)$$

Thus the Riemann-Christoffel tensor $\mathbf{R}$ is completely determined by the second fundamental form. The important result (60.11) is called the *equations of Gauss* in classical differential geometry. A similar choice for $\mathbf{v}.$ in (60.8) yields

$$\frac{\partial b_{\Gamma\Delta}}{\partial y^{\Sigma}} - b_{\Phi\Delta}\begin{Bmatrix}\Phi\\ \Gamma\Sigma\end{Bmatrix} - \frac{\partial b_{\Gamma\Sigma}}{\partial y^{\Delta}} + b_{\Phi\Sigma}\begin{Bmatrix}\Phi\\ \Gamma\Delta\end{Bmatrix} = 0 \tag{60.12}$$

By use of the symmetry condition (56.4) we can rewrite (60.12) in the more elegant form

$$b_{\Gamma\Delta,\Sigma} - b_{\Gamma\Sigma,\Delta} = 0 \tag{60.13}$$

which are the *equations of Codazzi*.

The importance of the equations of Gauss and Codazzi lies not only in their uniting the second fundamental form with the Riemann Christoffel tensor for any hypersurfaces $\mathscr{S}$ in $\mathscr{E}$, but also in their being the conditions of integrability as asserted by the following theorem.

**Theorem 60.1**. Suppose that $a_{\Gamma\Delta}$ and $b_{\Gamma\Delta}$ are any presceibed smooth functions of $\left(y^{\Omega}\right)$ such that $\left[a_{\Gamma\Delta}\right]$ is positive-definite and symmetric, $\left[b_{\Gamma\Delta}\right]$ is symmetric, and together $\left[a_{\Gamma\Delta}\right]$ and $\left[b_{\Gamma\Delta}\right]$ satisfy the equations of Gauss and Codazzi. Then locally there exists a hyper surface $\mathscr{S}$ with representation

$$x^i = x^i\left(y^{\Omega}\right) \tag{60.14}$$

on which the prescribed $a_{\Gamma\Delta}$ and $b_{\Gamma\Delta}$ are the first and second fundamental forms.

*Proof*. For simplicity we choose a rectangular Cartesian coordinate system $x^i$ in $\mathscr{E}$. Then in component form (58.21) and (58.7) are represented by

$$\partial h^i_{\Gamma} / \partial y^{\Delta} = b_{\Gamma\Delta} n^i + \begin{Bmatrix}\Sigma\\ \Gamma\Delta\end{Bmatrix} h^i_{\Sigma}, \quad \partial n^i / \partial y^{\Gamma} = -b^{\Delta}_{\Gamma} h^i_{\Delta} \tag{60.15}$$

We claim that this system can be integrated and the solution preserves the algebraic conditions

$$\delta_{ij} h^i_{\Gamma} h^j_{\Delta} = a_{\Gamma\Delta}, \qquad \delta_{ij} h^i_{\Gamma} n^j = 0, \qquad \delta_{ij} n^i n^j = 1 \tag{60.16}$$

In particular, if (60.16) are imposed at any one reference point, then they hold identically on a neighborhood of the reference point.

The fact that the solution of (60.15) preserves the conditions (60.16) is more or less obvious. We put

$$f_{\Gamma\Delta} \equiv \delta_{ij} h^i_\Gamma h^j_\Delta - a_{\Gamma\Delta}, \qquad f_\Gamma \equiv \delta_{ij} h^i_\Gamma n^j, \qquad f \equiv \delta_{ij} n^i n^j - 1 \tag{60.17}$$

Then initially at some point $\left(y^1_0,\, y^2_0\right)$ we have

$$f_{\Gamma\Delta}\left(y^1_0,\, y^2_0\right) = f_\Gamma\left(y^1_0,\, y^2_0\right) = f\left(y^1_0,\, y^2_0\right) = 0 \tag{60.18}$$

Now from (60.15) and (60.17) we can verify easily that

$$\frac{\partial f_{\Gamma\Delta}}{\partial y^\Sigma} = b_{\Gamma\Sigma} f_\Delta + b_{\Delta\Sigma} f_\Gamma, \quad \frac{\partial f_\Gamma}{\partial y^\Sigma} = -b^\Delta_\Sigma f_{\Gamma\Delta} + \left\{ \begin{matrix} \Delta \\ \Gamma\Sigma \end{matrix} \right\} f_\Delta + b_{\Gamma\Sigma} f, \quad \frac{\partial f}{\partial y^\Sigma} = -2 b^\Delta_\Sigma f_\Delta \tag{60.19}$$

along the coordinate curve of any $y^\Sigma$. From (60.18) and (60.19) we see that $f_{\Gamma\Delta}$, $f_\Gamma$, and $f$ must vanish identically.

Now the system (60.15) is integrable, since by use of the equations of Gauss and Codazzi we have

$$\begin{aligned} \frac{\partial}{\partial y^\Sigma}\left(\frac{\partial h^i_\Gamma}{\partial y^\Delta}\right) - \frac{\partial}{\partial y^\Delta}\left(\frac{\partial h^i_\Gamma}{\partial y^\Sigma}\right) &= \left(R^\Omega{}_{\Gamma\Sigma\Delta} - b_{\Gamma\Sigma} b^\Omega_\Delta + b_{\Gamma\Delta} b^\Omega_\Sigma\right) h^i_\Omega \\ &+ \left(b_{\Gamma\Sigma,\Delta} - b_{\Gamma\Delta,\Sigma}\right) n^i = 0 \end{aligned} \tag{60.20}$$

and

$$\frac{\partial}{\partial y^\Sigma}\left(\frac{\partial n^i}{\partial y^\Gamma}\right) - \frac{\partial}{\partial y^\Gamma}\left(\frac{\partial n^i}{\partial y^\Sigma}\right) = \left(b^\Omega_{\Sigma,\Gamma} - b^\Omega_{\Gamma,\Sigma}\right) h^i_\Omega = 0 \tag{60.21}$$

Hence locally there exist functions $h^i_\Gamma\left(y^\Omega\right)$ and $n^i\left(y^\Omega\right)$ which verify (60.15) and (60.16).

Next we set up the system of first-order partial differential equations

$$\partial x^i / \partial y^\Delta = h^i_\Delta\left(y^\Omega\right) \tag{60.22}$$

where the right-hand side is the solution of (60.15) and (60.16). Then (60.22) is also integrable, since we have

$$\frac{\partial}{\partial y^{\Sigma}}\left(\frac{\partial x^i}{\partial y^{\Delta}}\right)=\frac{\partial h^i_{\Delta}}{\partial y^{\Sigma}}=b_{\Delta\Sigma}n^i+\left\{\begin{matrix}\Omega\\ \Delta\Sigma\end{matrix}\right\}h^i_{\Omega}=\frac{\partial}{\partial y^{\Delta}}\left(\frac{\partial x^i}{\partial y^{\Sigma}}\right) \tag{60.23}$$

Consequently the solution (60.14) exists; further, from (60.22), (60.16), and (60.15) the first and the second fundamental forms on the hypersurface defined by (60.14) are precisely the prescribed fields $a_{\Gamma\Delta}$ and $b_{\Gamma\Delta}$, respectively. Thus the theorem is proved.

It should be noted that, since the algebraic conditions (60.16) are preserved by the solution of (60.15), any two solutions $\left(h^i_{\Gamma}, n^j\right)$ and $\left(\overline{h}^i_{\Gamma}, \overline{n}^j\right)$ of (60.15) can differ by at most a transformation of the form

$$\overline{h}^i_{\Gamma}=Q^i_j\,h^j_{\Gamma}, \qquad \overline{n}^i=Q^i_j\,n^j \tag{60.24}$$

where $\left[Q^i_j\right]$ is a constant orthogonal matrix. This transformation corresponds to a change of rectangular Cartesian coordinate system on $\mathscr{E}$ used in the representation (60.14). In this sense the fundamental forms $a_{\Gamma\Delta}$ and $b_{\Gamma\Delta}$, together with the equations of Gauss andCodazzi, determine the hyper surface $\mathscr{S}$ locally to within a rigid displacement in $\mathscr{E}$.

While the equations of Gauss show that the Riemann Christoffel tensor $\mathbf{R}$ of $\mathscr{S}$ is completely determined by the second fundamental form $\mathbf{B}$, the converse is generally not true. Thus mathematically the extrinsic characterization of curvature is much stronger than the intrinsic one, as it should be. In particular, if $\mathbf{B}$ vanishes, then $\mathscr{S}$ reduces to a hyperplane which is trivially developable so that $\mathbf{R}$ vanishes. On the other hand, if $\mathbf{R}$ vanishes, then $\mathscr{S}$ can be developed into a hyperplane, but generally $\mathbf{B}$ does not vanish, so $\mathscr{S}$ need not itself be a hyperplane. For example, in a three-dimensional Euclidean space a cylinder is developable, but it is not a plane.

## *Exercise*

60.1 In the case where $N=3$ show that the only independent equations of Codazzi are

$$b_{\Delta\Delta,\Gamma}-b_{\Delta\Gamma,\Delta}=0$$

for $\Delta\neq\Gamma$ and where the summation convention has been abandoned. Also show that the only independent equation of Gauss is the single equation

$$R_{1212}=b_{11}b_{22}-b_{12}^2=\det\mathbf{B}$$

## *Section 61. Surface Area, Minimal Surface*

Since we assume that $\mathscr{S}$ is oriented, the tangent plane $\mathscr{S}_{\mathbf{x}}$ of each point $\mathbf{x} \in \mathscr{S}$ is equipped with a volume tensor $\mathbf{S}$. Relative to any positive surface coordinate system $\left(y^\Gamma\right)$ with natural basis $\{\mathbf{h}_\Gamma\}$, $\mathbf{S}$ can be represented by

$$\mathbf{S} = \sqrt{a}\,\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_{N-1} \tag{61.1}$$

where

$$a = \det\left[a_{\Gamma\Delta}\right] \tag{61.2}$$

If $\mathscr{U}$ is a domain in $\mathscr{S}$ covered by $\left(y^\Gamma\right)$, then the *surface area* $\sigma(\mathscr{U})$ is defined by

$$\sigma(\mathscr{U}) = \int_{\mathscr{U}} \cdots \int \sqrt{a}\, dy^1 \cdots dy^{N-1} \tag{61.3}$$

where the $(N-1)$-fold integral is taken over the coordinates $\left(y^\Gamma\right)$ on the domain $\mathscr{U}$. Since $\sqrt{a}$ obeys the transformation rule

$$\sqrt{\bar{a}} = \sqrt{a}\,\det\left[\frac{\partial y^\Gamma}{\partial \bar{y}^\Delta}\right] \tag{61.4}$$

relative to any positive coordinate systems $\left(y^\Gamma\right)$ and $\left(\bar{y}^\Gamma\right)$, the value of $\sigma(\mathscr{U})$ is independent of the choice of coordinates. Thus σ can be extended into a positive measure on $\mathscr{S}$ in the sense of measure theory.

We shall consider the theory of integration relative to $\sigma$ in detail in Chapter 13. Here we shall note only a particular result wich connects a property of the surface area to the mean curvature of $\mathscr{S}$. Namely, we seek a geometric condition for $\mathscr{S}$ to be a *minimal surface*, which is defined by the condition that the surface area $\sigma(\mathscr{U})$ be an extremum in the class of variations of hypersurfaces having the same boundary as $\mathscr{U}$. The concept of minimal surface is similar to that of a geodesic whichwe have explored in Section 57, except that here we are interested in the variation of the integral $\sigma$ given by (61.3) instead of the integral $s$ given by (57.5).

As before, the geometric condition for a minimal surface follows from the Euler-Lagrange equation for (61.3), namely,

$$\frac{\partial}{\partial y^\Gamma}\left(\frac{\partial\sqrt{a}}{\partial h^i_\Gamma}\right)-\frac{\partial\sqrt{a}}{\partial x^i}=0 \tag{61.5}$$

where $a$ is regarded as a function of $x^i$ and $h^i_\Gamma$ through the representation

$$a_{\Gamma\Delta}=\delta_{ij}\,h^i_\Gamma\,h^i_\Delta=\delta_{ij}\,\frac{\partial x^i}{\partial y^\Gamma}\frac{\partial x^j}{\partial y^\Delta} \tag{61.6}$$

For simplicity we have chosen the spatial coordinates to be rectangular Cartesian, so that $a_{\Gamma\Delta}$ does not depend explicitly on $x^i$. From (61.6) the partial derivative of $\sqrt{a}$ with respect to $h^i_\Gamma$ is given by

$$\frac{\partial\sqrt{a}}{\partial h^i_\Gamma}=\sqrt{a}\,a^{\Gamma\Delta}\,\delta_{ij}\;h^i_\Delta=\sqrt{a}\,\delta_{ij}\,h^{\Gamma j} \tag{61.7}$$

Substituting this formula into (61.5), we see that the condition for $\mathscr{S}$ to be a minimal surface is

$$\sqrt{a}\,\delta_{ij}\left(\begin{Bmatrix}\Omega\\ \Omega\Gamma\end{Bmatrix}h^{\Gamma j}+\frac{\partial h^{\Gamma j}}{\partial y^\Gamma}\right)=0,\qquad i=1,\ldots,N \tag{61.8}$$

or, equivalently, in vector notation

$$\sqrt{a}\,\left(\begin{Bmatrix}\Omega\\ \Omega\Gamma\end{Bmatrix}\mathbf{h}^\Gamma+\frac{\partial\mathbf{h}^\Gamma}{\partial y^\Gamma}\right)=\mathbf{0} \tag{61.9}$$

In deriving (61.8) and (61.9), we have used the identity

$$\frac{\partial\sqrt{a}}{\partial y^\Gamma}=\begin{Bmatrix}\Omega\\ \Omega\Gamma\end{Bmatrix}\sqrt{a} \tag{61.10}$$

which we have noted in Exercise 56.10. Now from (58.20) we can rewrite (61.9) in the simple form

$$\sqrt{a}\;b^\Gamma_\Gamma\mathbf{n}=0 \tag{61.11}$$

As a result, the geometric condition for a minimal surface is

$$\mathrm{I}_{\mathbf{B}} = \operatorname{tr} \mathbf{B} = b_{\Gamma}^{\Gamma} = 0 \tag{61.12}$$

In general, the first invariant $\mathrm{I}_{\mathbf{B}}$ of the second fundamental form $\mathbf{B}$ is called the *mean curvature* of the hyper surface. Equation (61.12) then asserts that $\mathscr{S}$ is a minimal surface if and only if its mean curvature vanishes.

Since in general $\mathbf{B}$ is symmetric, it has the usual spectral decomposition

$$\mathbf{B} = \sum_{\Gamma=1}^{N-1} \beta_{\Gamma} \mathbf{c}_{\Gamma} \otimes \mathbf{c}_{\Gamma} \tag{61.13}$$

where the principal basis $\{\mathbf{c}_{\Gamma}\}$ is orthonormal on the surface. In differential geometry the direction of $\mathbf{c}_{\Gamma}(\mathbf{x})$ at each point $\mathbf{x} \in \mathscr{S}$ is called a *principal direction* and the corresponding proper number $\beta_{\Gamma}(\mathbf{x})$ is called a *principal (normal) curvature* at $\mathbf{x}$. As usual, the principal (normal) curvatures are the extrema of the normal curvature

$$\kappa_n = \mathbf{B}(\mathbf{s}, \mathbf{s}) \tag{61.14}$$

in all unit tangents $\mathbf{s}$ at any point $\mathbf{x}$ [cf. $(58.29)_2$]. The mean curvature $\mathrm{I}_{\mathbf{B}}$, of course, is equal to the sum of the principal curvatures, i.e.,

$$\mathrm{I}_{\mathbf{B}} = \sum_{\Gamma=1}^{N-1} \beta_{\Gamma} \tag{61.15}$$

## *Section 62. Surfaces in a Three-Dimensional Euclidean Manifold*

In this section we shall apply the general theory of hypersurfaces to the special case of a two-dimensional surface imbedded in a three-dimensional Euclidean manifold. This special case is the commonest case in application, and it also provides a good example to illustrate the general results.

As usual we can represent $\mathscr{S}$ by

$$x^i = x^i\left(y^\Gamma\right) \tag{62.1}$$

where $i$ ranges from 1 to 3 and Γ ranges from 1 to 2. We choose $\left(x^i\right)$ and $\left(y^\Gamma\right)$ to be positive spatial and surface coordinate systems, respectively. Then the tangent plane of $\mathscr{S}$ is spanned by the basis

$$\mathbf{h}_\Gamma \equiv \frac{\partial x^i}{\partial y^\Gamma}\mathbf{g}_i, \qquad \Gamma = 1,2 \tag{62.2}$$

and the unit normal $\mathbf{n}$ is given by

$$\mathbf{n} = \frac{\mathbf{h}_1 \times \mathbf{h}_2}{\left\|\mathbf{h}_1 \times \mathbf{h}_2\right\|} \tag{62.3}$$

The first and the second fundamental forms $a_{\Gamma\Delta}$ and $b_{\Gamma\Delta}$ relative to $\left(y^\Gamma\right)$ are defined by

$$a_{\Gamma\Delta} = \mathbf{h}_\Gamma \cdot \mathbf{h}_\Delta. \qquad b_{\Gamma\Delta} = \mathbf{n}\cdot\frac{\partial \mathbf{h}_\Gamma}{\partial y^\Delta} = -\mathbf{h}_\Gamma \cdot \frac{\partial \mathbf{n}}{\partial y^\Delta} \tag{62.4}$$

These forms satisfy the equations of Gauss and Codazzi:

$$R_{\Phi\Gamma\Sigma\Delta} = b_{\Gamma\Delta}\, b_{\Phi\Sigma} - b_{\Gamma\Sigma} b_{\Phi\Delta}, \qquad b_{\Gamma\Delta,\Sigma} - b_{\Gamma\Sigma,\Delta} = 0 \tag{62.5}$$

Since $b_{\Gamma\Delta}$ is symmetric, at each point $\mathbf{x}\in\mathscr{S}$ there exists a positive orthonormal basis $\left\{\mathbf{c}_\Gamma\left(\mathbf{x}\right)\right\}$ relative to which $\mathbf{B}\left(\mathbf{x}\right)$ can be represented by the spectral form

$$\mathbf{B}(\mathbf{x}) = \beta_1(\mathbf{x})\mathbf{c}_1(\mathbf{x})\otimes\mathbf{c}_1(\mathbf{x}) + \beta_2(\mathbf{x})\mathbf{c}_2(\mathbf{x})\otimes\mathbf{c}_2(\mathbf{x}) \tag{62.6}$$

The proper numbers $\beta_1(\mathbf{x})$ and $\beta_2(\mathbf{x})$ are the *principal (normal) curvatures* of $\mathscr{S}$ at $\mathbf{x}$. In general $\{\mathbf{c}_\Gamma(\mathbf{x})\}$ is not the natural basis of a surface coordinate system unless

$$[\mathbf{c}_1, \mathbf{c}_2] = 0 \tag{62.7}$$

However, locally we can always choose a coordinate system $(z^\Gamma)$ in such a way that the natural basis $\{\mathbf{h}_\Gamma\}$ is parallel to $\{\mathbf{c}_\Gamma(\mathbf{x})\}$. Naturally, such a coordinate system is called a *principal coordinate system* and its coordinate curves are called the *lines of curvature*. Relative to a principal coordinate system the components $a_{\Gamma\Delta}$ and $b_{\Gamma\Delta}$ satisfy

$$b_{12} = a_{12} = 0 \qquad \text{and} \qquad \mathbf{h}_1 = \sqrt{a_{11}}\,\mathbf{c}_1, \quad \mathbf{h}_2 = \sqrt{a_{22}}\,\mathbf{c}_2 \tag{62.8}$$

The principal invariants of $\mathbf{B}$ are

$$\begin{aligned} \mathrm{I}_{\mathbf{B}} &= \operatorname{tr}\mathbf{B} = \beta_1 + \beta_2 = a^{\Gamma\Delta} b_{\Gamma\Delta} \\ \mathrm{II}_{\mathbf{B}} &= \det\mathbf{B} = \beta_1\beta_2 = \det\left[b^\Gamma_\Delta\right] \end{aligned} \tag{62.9}$$

In the preceding section we have defined $\mathrm{I}_{\mathbf{B}}$ to be the *mean curvature*. Now $\mathrm{II}_{\mathbf{B}}$ is called the *Gaussian curvature*. Since

$$b^\Gamma_\Delta = a^{\Gamma\Omega} b_{\Delta\Omega} \tag{62.10}$$

we have also

$$\mathbf{II}_{\mathbf{B}} = \frac{\det\left[b_{\Delta\Omega}\right]}{\det\left[a_{\Delta\Omega}\right]} = \frac{\det\left[b_{\Delta\Omega}\right]}{a} \tag{62.11}$$

where the numerator on the right-hand side is given by the equation of Gauss:

$$\det\left[b_{\Delta\Omega}\right] = b_{11}b_{22} - b_{12}^2 = R_{1212} \tag{62.12}$$

Notice that for the two-dimensional case the tensor $\mathbf{R}$ is completely determined by $R_{1212}$, since from $(62.5)_1$, or indirectly from (59.10), $R_{\Phi\Gamma\Sigma\Delta}$ vanishes when $\Phi = \Gamma$ or when $\Sigma = \Delta$.

We call a point $\mathbf{x} \in \mathscr{S}$ *elliptic, hyperbolic, or parabolic* if the Gaussian curvature threre is positive, negative, or zero, respectively. These terms are suggested by the following geometric considerations: We choose a fixed reference point $\mathbf{x}_0 \in \mathscr{S}$ and define a surface coordinate system $\left(y^\Gamma\right)$ such that the coordinates of $\mathbf{x}_0$ are $(0,0)$ ) and the basis $\left\{\mathbf{h}_\Gamma(\mathbf{x})\right\}$ is orthonormal, i.e.,

$$a_{\Gamma\Delta}(0,0)=\delta_{\Gamma\Delta} \tag{62.13}$$

In this coordinate system a point $\mathbf{x}\in\mathscr{S}$ having coordinate $\left(y^1,y^2\right)$ near $(0,0)$ is located at a normal distance $d\left(y^1,y^2\right)$ from the tangenet plane $\mathscr{S}_{\mathbf{x}_o}$ with $d\left(y^1,y^2\right)$ given approximately by

$$\begin{aligned} d\left(y^1,y^2\right)&=\mathbf{n}(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)\\ &\cong\mathbf{n}(\mathbf{x}_0)\cdot\left(\mathbf{h}_\Gamma(\mathbf{x}_0)y^\Gamma+\frac{1}{2}\left.\frac{D\mathbf{h}_\Gamma}{Dy^\Delta}\right|_{\mathbf{x}_0}y^\Gamma y^\Delta\right)\\ &\cong\frac{1}{2}b_{\Gamma\Delta}(\mathbf{x}_0)y^\Gamma y^\Delta \end{aligned} \tag{62.14}$$

where (62.14)$_{2,3}$ are valid to within an error of third or higher order in $y^\Gamma$. From this estimate we see that the intersection of $\mathscr{S}$ with a plane parallel to $\mathscr{S}_{\mathbf{x}_o}$ is represented approximately by the curve

$$b_{\Gamma\Delta}(\mathbf{x}_0)y^\Gamma y^\Delta=\text{const} \tag{62.15}$$

which is an ellipse, a hyperbola, or parallel lines when $\mathbf{x}_0$ is an elliptic, hyperbolic, or parabolic point, respectively. Further, in each case the principal basis $\left\{\mathbf{c}_\Gamma\right\}$ of $\mathbf{B}(\mathbf{x}_0)$ coincides with the principal axes of the curves in the sense of conic sections in analytical geometry. The estimate (62.14) means also that in the rectangular Cartesian coordinate system $\left(x^i\right)$ induced by the orthonormal basis $\left\{\mathbf{h}_1(\mathbf{x}_0),\,\mathbf{h}_2(\mathbf{x}_0),\mathbf{n}\right\}$, the surface $\mathscr{S}$ can be represented locally by the equations

$$x^1=y^1,\qquad x^2=y^2,\qquad x^3\cong\frac{1}{2}b_{\Gamma\Delta}(\mathbf{x}_0)y^\Gamma y^\Delta \tag{62.16}$$

As usual we can define the notion of *conjugate directions* relative to the symmetric bilinear form of $\mathbf{B}(\mathbf{x}_0)$. We say that the tangential vectors $\mathbf{u},\mathbf{v}\in\mathscr{S}_{\mathbf{x}_0}$ are *conjugate* at $\mathbf{x}_0$ if

$$\mathbf{B}(\mathbf{u},\mathbf{v})=\mathbf{u}\cdot(\mathbf{B}\mathbf{v})=\mathbf{v}\cdot(\mathbf{B}\mathbf{u})=0 \tag{62.17}$$

For example, the principal basis vectors $\mathbf{c}_1(\mathbf{x}_0)$ and $\mathbf{c}_2(\mathbf{x}_0)$ are conjugate since the components of $\mathbf{B}$ form a diagonal matrix relative to $\{\mathbf{c}_\Gamma\}$. Geometrically, conjugate directions may be explained in the following way: We choose any curve $\boldsymbol{\lambda}(t) \in \mathscr{S}$ such that

$$\dot{\boldsymbol{\lambda}}(0) = \mathbf{u} \in \mathscr{S}_{\mathbf{x}_o} \tag{62.18}$$

Then the conjugate direction $\mathbf{v}$ of $\mathbf{u}$ corresponds to the limit of the intersection of $\mathscr{S}_{\boldsymbol{\lambda}(t)}$ with $\mathscr{S}_{\mathbf{x}_0}$ as *t* tends to zero. We leave the proof of this geometric interpretation for conjugate directions as an exercise.

A direction represented by a self-conjugate vector $\mathbf{v}$ is called an *asymptotic direction*. In this case $\mathbf{v}$ satisfies the equation

$$\mathbf{B}(\mathbf{v},\mathbf{v}) = \mathbf{v}\cdot(\mathbf{B}\mathbf{v}) = 0 \tag{62.19}$$

Clearly, asymptotic directions exit at a point $\mathbf{x}_o$ if and only if $\mathbf{x}_o$ is hyperbolic or parabolic. In the former case the asympototic directions are the same as those for the hyperbola given by (62.15), while in the latter case the asymptotic direction is unique and coincides with the direction of the parallel lines given by (62.15).

If every point of $\mathscr{S}$ is hyperbolic, then the asymptotic lines form a coordinate net, and we can define an *asymptotic coordinate system*. Relative to such a coordinate system the components $b_{\Gamma\Delta}$ satisfy the condition

$$b_{11} = b_{22} = 0 \tag{62.20}$$

and the Gaussian curvature is given by

$$\mathrm{II}_{\mathbf{B}} = -b_{12}^2 / a \tag{62.21}$$

A minimal surface which does not reduce to a plane is necessarily hyperbolic at every point, since when $\beta_1 + \beta_2 = 0$ we must have $\beta_1\beta_2 < 0$ unless $\beta_1 = \beta_2 = 0$.

From (62.12), $\mathscr{S}$ is parabolic at every point if and only if it is developable. In this case the asymptotic lines are straight lines. In fact, it can be shown that there are only three kinds of developable surfaces, namely cylinders, cones, and tangent developables. Their asymptotic lines are simply their generators. Of course, these generators are also lines of curvature, since they are in the principal direction corresponding to the zero principal curvature.

## *Exercises*

62.1 Show that

$$\sqrt{a} = \mathbf{n} \cdot \left( \mathbf{h}_1 \times \mathbf{h}_2 \right)$$

62.2 Show that

$$b_{\Gamma\Delta} = \frac{1}{\sqrt{a}} \frac{\partial^2 \mathbf{x}}{\partial y^\Delta \partial y^\Gamma} \cdot \left( \frac{\partial \mathbf{x}}{\partial y^1} \times \frac{\partial \mathbf{x}}{\partial y^2} \right)$$

62.3 Compute the principal curvatures for the surfaces defined in Exercises 55.5 and 55.7.

## *Chapter 12*

# *ELEMENTS OF CLASSICAL CONTINUOUS GROUPS*

In this chapter we consider the structure of various classical continuous groups which are formed by linear transformations of an inner product space $\mathscr{V}$. Since the space of all linear transformation of $\mathscr{V}$ is itself an inner product space, the structure of the continuous groups contained in it can be exploited by using ideas similar to those developed in the preceding chapter. In addition, the group structure also gives rise to a special parallelism on the groups.

## *Section 63. The General Linear Group and Its Subgroups*

In section 17 we pointed out that the vector space $\mathscr{L}(\mathscr{V};\mathscr{V})$ has the structure of an *algebra*, the product operation being the composition of linear transformations. Since $\mathscr{V}$ is an inner product space, the *transpose* operation

$$T:\mathscr{L}(\mathscr{V};\mathscr{V})\rightarrow\mathscr{L}(\mathscr{V};\mathscr{V}) \tag{63.1}$$

is defined by (18.1); i.e.,

$$\mathbf{u}\cdot\mathbf{A}^{T}\,\mathbf{v}=\mathbf{Au}\cdot\mathbf{v},\quad \mathbf{u},\mathbf{v}\in\mathscr{V} \tag{63.2}$$

for any $\mathbf{A}\in\mathscr{L}(\mathscr{V};\mathscr{V})$. The inner product of any $\mathbf{A},\mathbf{B}\in\mathscr{L}(\mathscr{V};\mathscr{V})$ is then defined by (cf. Exercise 19.4))

$$\mathbf{A}\cdot\mathbf{B}=tr\left(\mathbf{AB}^{T}\right) \tag{63.3}$$

From Theorem 22.2, a linear transformation $\mathbf{A}\in\mathscr{L}(\mathscr{V};\mathscr{V})$ is an isomorphism of $\mathscr{V}$ if and only if

$$\det\mathbf{A}\neq 0 \tag{63.4}$$

Clearly, if **A** and **B** are isomorphisms, then **AB** and **BA** are also isomorphisms, and from Exercise 22.1 we have

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = (\det \mathbf{A})(\det \mathbf{B}) \tag{63.5}$$

As a result, the set of all linear isomorphisms of $\mathscr{V}$ forms a group $\mathscr{GL}(\mathscr{V})$, called the *general linear group* of $\mathscr{V}$. This group was mentioned in Section 17. We claim that $\mathscr{GL}(\mathscr{V})$ is the disjoint union of two connected open sets in $\mathscr{L}(\mathscr{V};\mathscr{V})$. This fact is more of less obvious since $\mathscr{GL}(\mathscr{V})$ is the reimage of the disjoint union $(-\infty,0)\cup(0,\infty)$ under the continuous map

$$\det : \mathscr{L}(\mathscr{V};\mathscr{V}) \to \mathscr{R}$$

If we denote the primeages of $(-\infty,0)$ and $(0,\infty)$ under the mapping $\det$ by $\mathscr{GL}(\mathscr{V})^-$ and $\mathscr{GL}(\mathscr{V})^+$, respectively, then they are disjoint connected open sets in $\mathscr{L}(\mathscr{V};\mathscr{V})$, and

$$\mathscr{GL}(\mathscr{V}) = \mathscr{GL}(\mathscr{V})^- \cup \mathscr{GL}(\mathscr{V})^+ \tag{63.6}$$

Notice that from (63.5) $\mathscr{GL}(\mathscr{V})^+$ is, but $\mathscr{GL}(\mathscr{V})^-$ is not, closed with respect to the group operation. So $\mathscr{GL}(\mathscr{V})^+$ is, but $\mathscr{GL}(\mathscr{V})^-$ is not, a subgroup of $\mathscr{GL}(\mathscr{V})$. We call the elements of $\mathscr{GL}(\mathscr{V})^+$ and $\mathscr{GL}(\mathscr{V})^-$ *proper* and *improper* transformations of $\mathscr{V}$, respectively. They are separated by the hypersurface $\mathscr{S}$ defined by

$$\mathbf{A} \in \mathscr{S} \Leftrightarrow \mathbf{A} \in \mathscr{L}(\mathscr{V};\mathscr{V}) \qquad \text{and} \qquad \det \mathbf{A} = 0$$

Now since $\mathscr{GL}(\mathscr{V})$ is an open set in $\mathscr{L}(\mathscr{V};\mathscr{V})$, any coordinate system in $\mathscr{L}(\mathscr{V};\mathscr{V})$ corresponds to a coordinate system in $\mathscr{GL}(\mathscr{V})$ when its coordinate neighborhood is restricted to a subset of $\mathscr{GL}(\mathscr{V})$. For example, if $\{\mathbf{e}_i\}$ and $\{\mathbf{e}^i\}$ are reciprocal bases for $\mathscr{V}$, then $\{\mathbf{e}_i \otimes \mathbf{e}^i\}$ is a basis for $\mathscr{L}(\mathscr{V};\mathscr{V})$ which gives rise to the Cartesian coordinate system $\{X_i^j\}$. The restriction of $\{X_i^j\}$ to $\mathscr{GL}(\mathscr{V})$ is then a coordinate system on $\mathscr{GL}(\mathscr{V})$. A Euclidean geometry can then be defined on $\mathscr{GL}(\mathscr{V})$ as we have done in general on an arbitrary open set in a Euclidean space. The Euclidean geometry on $\mathscr{GL}(\mathscr{V})$ is not of much interest, however, since it is independent of the group structure on $\mathscr{GL}(\mathscr{V})$.

The group structure on $\mathscr{GL}(\mathscr{V})$ is characterized by the following operations.

1. *Left-multiplication* by any $\mathbf{A} \in \mathscr{GL}(\mathscr{V})$,

$$\boldsymbol{L}_{\mathbf{A}} : \mathscr{GL}(\mathscr{V}) \to \mathscr{GL}(\mathscr{V})$$

is defined by

$$\boldsymbol{L}_{\mathbf{A}}(\mathbf{X}) \equiv \mathbf{AX}, \quad \mathbf{X} \in \mathscr{GL}(\mathscr{V}) \tag{63.7}$$

2. *Right-multiplication* by any $\mathbf{A} \in \mathscr{GL}(\mathscr{V})$

$$\boldsymbol{R}_{\mathbf{A}} : \mathscr{GL}(\mathscr{V}) \to \mathscr{GL}(\mathscr{V})$$

is defined by

$$\boldsymbol{R}_{\mathbf{A}}(\mathbf{X}) \equiv \mathbf{XA}, \quad \mathbf{X} \in \mathscr{GL}(\mathscr{V}) \tag{63.8}$$

3. *Inversion*

$$\boldsymbol{J} : \mathscr{GL}(\mathscr{V}) \to \mathscr{GL}(\mathscr{V})$$

is defined by

$$\boldsymbol{J}(\mathbf{X}) = \mathbf{X}^{-1}, \quad \mathbf{X} \in \mathscr{GL}(\mathscr{V}) \tag{63.9}$$

Clearly these operations are smooth mappings, so they give rise to various gradients which are fields of linear transformations of the underlying inner product space $\mathscr{L}(\mathscr{V};\mathscr{V})$. For example, the gradient of $\boldsymbol{L}_{\mathbf{A}}$ is a constant field given by

$$\nabla \boldsymbol{L}_{\mathbf{A}}(\mathbf{Y}) = \mathbf{AY}, \quad \mathbf{Y} \in \mathscr{L}(\mathscr{V};\mathscr{V}) \tag{63.10}$$

since for any $\boldsymbol{Y} \in \mathscr{L}(\mathscr{V};\mathscr{V})$ we have

$$\frac{d}{dt}\boldsymbol{L}_{\mathbf{A}}(\mathbf{X}+\mathbf{Y}t)\Big|_{t=0} = \frac{d}{dt}(\boldsymbol{A}\mathbf{X}+\boldsymbol{A}\mathbf{Y}t)\Big|_{t=0} = \mathbf{AY}$$

By the same argument $\nabla \boldsymbol{R}_{\mathbf{A}}$ is also a constant field and is given by

$$\nabla \boldsymbol{R}_{\mathbf{A}}(\mathbf{Y}) = \mathbf{YA}, \qquad \mathbf{Y} \in \mathscr{L}(\mathscr{V};\mathscr{V}) \tag{63.11}$$

On the other hand, $\nabla \boldsymbol{J}$ is not a constant field; its value at any point $\mathbf{X} \in \mathscr{GL}(\mathscr{V})$ is given by

$$\left[\nabla \boldsymbol{J}(\mathbf{X})\right](\mathbf{Y}) = -\mathbf{X}^{-1}\mathbf{YX}, \qquad \mathbf{Y} \in \mathscr{L}(\mathscr{V};\mathscr{V}) \tag{63.12}$$

In particular, at $\mathbf{X} = \mathbf{I}$ the value of $\nabla \boldsymbol{J}(\mathbf{I})$ is simply the negation operation,

$$\left[\nabla \boldsymbol{J}(\mathbf{I})\right](\mathbf{Y}) = -\mathbf{Y}, \qquad \mathbf{Y} \in \mathscr{L}(\mathscr{V};\mathscr{V}) \tag{63.13}$$

We leave the proof of (63.13) as an exercise.

The gradients $\nabla \boldsymbol{L}_{\mathbf{A}}$, $\mathbf{A} \in \mathscr{GL}(\mathscr{V})$, can be regarded as a parallelism on $\mathscr{GL}(\mathscr{V})$ in the following way: For any two points $\mathbf{X}$ and $\mathbf{Y}$ in $\mathscr{GL}(\mathscr{V})$ there exists a unique $\mathbf{A} \in \mathscr{GL}(\mathscr{V})$ such that $\mathbf{Y} = \boldsymbol{L}_{\mathbf{A}}\mathbf{X}$. The corresponding gradient $\nabla \boldsymbol{L}_{\mathbf{A}}$ at the point $\mathbf{X}$ is a linear isomorphism

$$\left[\nabla \boldsymbol{L}_{\mathbf{A}}(\mathbf{X})\right] : \mathscr{GL}(\mathscr{V})_{\mathbf{X}} \to \mathscr{GL}(\mathscr{V})_{\mathbf{Y}} \tag{63.14}$$

where $\mathscr{GL}(\mathscr{V})_{\mathbf{X}}$ denotes the tangent space of $\mathscr{GL}(\mathscr{V})$ at $\mathbf{X}$, a notation consistent with that introduced in the preceding chapter. Here, of course, $\mathscr{GL}(\mathscr{V})_{\mathbf{X}}$ coincides with $\mathscr{L}(\mathscr{V};\mathscr{V})_{\mathbf{X}}$, which is a copy of $\mathscr{L}(\mathscr{V};\mathscr{V})$. We inserted the argument $\mathbf{X}$ in $\nabla \boldsymbol{L}_{\mathbf{A}}(\mathbf{X})$ to emphasize the fact that $\nabla \boldsymbol{L}_{\mathbf{A}}(\mathbf{X})$ is a linear map from the tangent space of $\mathscr{GL}(\mathscr{V})$ at $\mathbf{X}$ to the tangent space of $\mathscr{GL}(\mathscr{V})$ at the image point $\mathbf{Y}$. This mapping is given by (63.10) via the isomorphism of $\mathscr{GL}(\mathscr{V})_{\mathbf{X}}$ and $\mathscr{GL}(\mathscr{V})_{\mathbf{Y}}$ with $\mathscr{L}(\mathscr{V};\mathscr{V})$.

From (63.10) we see that the parallelism defined by (63.14) is not the same as the Euclidean parallelism. Also, $\nabla \boldsymbol{L}_{\mathbf{A}}$ is not the same kind of parallelism as the Levi-Civita parallelism on a hypersurface because it is independent of any path joining $\mathbf{X}$ and $\mathbf{Y}$. In fact, if $\mathbf{X}$ and $\mathbf{Y}$ do not belong to the same connected set of $\mathscr{GL}(\mathscr{V})$, then there exists no smooth curve joining them at all, but the parallelism $\nabla \boldsymbol{L}_{\mathbf{A}}(\mathbf{X})$ is still defined. We call the parallelism $\nabla \boldsymbol{L}_{\mathbf{A}}(\mathbf{X})$ with $\mathbf{A} \in \mathscr{GL}(\mathscr{V})$ the *Cartan parallelism* on $\mathscr{GL}(\mathscr{V})$, and we shall study it in detail in the next section.

The choice of left multiplication rather than the right multiplication is merely a convention. The gradient $\boldsymbol{R}_{\mathbf{A}}$ also defined a parallelism on the group. Further, the two parallelisms $\nabla \boldsymbol{L}_{\mathbf{A}}$ and $\nabla \boldsymbol{R}_{\mathbf{A}}$ are related by

$$\nabla \boldsymbol{L}_{\mathbf{A}} = \nabla \boldsymbol{J} \circ \nabla \boldsymbol{R}_{\mathbf{A}^{-1}} \circ \nabla \boldsymbol{J} \tag{63.15}$$

Since $\boldsymbol{L}_{\mathbf{A}}$ and $\boldsymbol{R}_{\mathbf{A}}$ are related by

$$\boldsymbol{L}_{\mathbf{A}} = \boldsymbol{J} \circ \boldsymbol{R}_{\mathbf{A}^{-1}} \circ \boldsymbol{J} \tag{63.16}$$

for all $\mathbf{A} \in \mathscr{GL}(\mathscr{V})$.

Before closing the section, we mention here that besides the proper general linear group $\mathscr{GL}(\mathscr{V})^{+}$, several other subgroups of $\mathscr{GL}(\mathscr{V})$ are important in the applications. First, the *special linear group* $\mathscr{SL}(\mathscr{V})$ is defined by the condition

$$\mathbf{A} \in \mathscr{SL}(\mathscr{V}) \Leftrightarrow \det \mathbf{A} = 1 \tag{63.17}$$

Clearly $\mathscr{SL}(\mathscr{V})$ is a hypersurface of dimension $N^2 - 1$ in the inner product space $\mathscr{L}(\mathscr{V};\mathscr{V})$. Unlike $\mathscr{GL}(\mathscr{V})$, $\mathscr{SL}(\mathscr{V})$ is a connected continuous group. The *unimodular group* $\mathscr{UM}(\mathscr{V})$ is defined similiarly by

$$\mathbf{A} \in \mathscr{UM}(\mathscr{V}) \Leftrightarrow |\det \mathbf{A}| = 1 \tag{63.18}$$

Thus $\mathscr{SL}(\mathscr{V})$ is the proper subgroup of $\mathscr{UM}(\mathscr{V})$, namely

$$\mathscr{UM}(\mathscr{V})^{+} = \mathscr{SL}(\mathscr{V}) \tag{63.19}$$

Next the *orthogonal group* $\mathscr{O}(\mathscr{V})$ is defined by the condition

$$\mathbf{A} \in \mathscr{O}(\mathscr{V}) \Leftrightarrow \mathbf{A}^{-1} = \mathbf{A}^{T} \tag{63.20}$$

From (18.18) or (63.2) we see that $\mathbf{A}$ belongs to $\mathscr{O}(\mathscr{V})$ if and only if it preserves the inner product of $\mathscr{V}$, i.e.,

$$\mathbf{Au} \cdot \mathbf{Av} = \mathbf{u} \cdot \mathbf{v}, \qquad \mathbf{u}, \mathbf{v} \in \mathscr{V} \tag{63.21}$$

Further, from (63.20) if $\mathbf{A} \in \mathscr{O}(\mathscr{V})$, then det $\mathbf{A}$ has absolute value 1. Consequently, $\mathscr{O}(\mathscr{V})$ is a subgroup of $\mathscr{UM}(\mathscr{V})$. As usual, $\mathscr{O}(\mathscr{V})$ has a proper component and an improper component, the former being a subgroup of $\mathscr{O}(\mathscr{V})$, denoted by $\mathscr{SO}(\mathscr{V})$, called the *special orthogonal group* or the *rotational group* of $\mathscr{V}$. As we shall see in Section 65, $\mathscr{O}(\mathscr{V})$ is a hypersurface of dimension $\frac{1}{2}N(N-1)$. From (63.20), $\mathscr{O}(\mathscr{V})$ is contained in the sphere of radius $\sqrt{N}$ in $\mathscr{L}(\mathscr{V};\mathscr{V})$ since if $\mathbf{A}\in\mathscr{O}(\mathscr{V})$, then

$$\mathbf{A}\cdot\mathbf{A}=tr\,\mathbf{A}\mathbf{A}^{T}=tr\,\mathbf{A}\mathbf{A}^{-1}=tr\,\mathbf{I}=N \tag{63.22}$$

As a result, $\mathscr{O}(\mathscr{V})$ is bounded in $\mathscr{L}(\mathscr{V};\mathscr{V})$.

Since $\mathscr{UM}(\mathscr{V})$, $\mathscr{SL}(\mathscr{V})$, $\mathscr{O}(\mathscr{V})$, and $\mathscr{SO}(\mathscr{V})$ are subgroups of $\mathscr{GL}(\mathscr{V})$, the restrictions of the group operations of $\mathscr{GL}(\mathscr{V})$ can be identified as the group operations of the subgroups. In particular, if $\mathbf{X},\mathbf{Y}\in\mathscr{SL}(\mathscr{V})$ and $\mathbf{Y}=\boldsymbol{L}_{\mathbf{A}}(\mathbf{X})$, then the restriction of the mapping $\nabla\boldsymbol{L}_{\mathbf{A}}(\mathbf{X})$ defined by (63.14) is a mapping

$$\left[\nabla\boldsymbol{L}_{A}(\mathbf{X})\right]:\mathscr{SL}(\mathscr{V})_{\mathbf{X}}\to\mathscr{SL}(\mathscr{V})_{\mathbf{Y}} \tag{63.23}$$

Thus there exists a Cartan parallelism on the subgroups as well as on $\mathscr{GL}(\mathscr{V})$. The tangent spaces of the subgroups are subspaces of $\mathscr{L}(\mathscr{V};\mathscr{V})$ of course; further , as explained in the preceding chapter, they vary from point to point. We shall characterize the tangent spaces of $\mathscr{SL}(\mathscr{V})$ and $\mathscr{SO}(\mathscr{V})$ at the identity element $\mathbf{I}$ in Section 66. The tangent space at any other point can then be obtained by the Cartan parallelism, e.g.,

$$\mathscr{SL}(\mathscr{V})_{\mathbf{X}}=\left[\nabla\boldsymbol{L}_{\mathbf{X}}(\mathbf{I})\right]\left(\mathscr{SL}(\mathscr{V})_{\mathbf{I}}\right)$$

for any $\mathbf{X}\in\mathscr{SL}(\mathscr{V})$.

## *Exercise*

63.1 Verify (63.12) and (63.13).

## Section 64. The Parallelism of Cartan

The concept of Cartan parallelism on $\mathscr{GL}(\mathscr{V})$ and on its various subgroups was introduced in the preceding section. In this section, we develop the concept in more detail. As explained before, the Cartan parallelism is path-independent. To emphasize this fact, we now replace the notation $\nabla \boldsymbol{L}_{\mathbf{A}}(\mathbf{X})$ by the notation $\boldsymbol{C}(\mathbf{X},\mathbf{Y})$, where $\mathbf{Y}=\mathbf{AX}$.
Then we put

$$\boldsymbol{C}(\mathbf{X}) \equiv \boldsymbol{C}(\mathbf{I},\mathbf{X}) \equiv \nabla \boldsymbol{L}_{\mathbf{X}}(\mathbf{I}) \tag{64.1}$$

for any $\mathbf{X} \in \mathscr{GL}(\mathscr{V})$. It can be verified easily that

$$\boldsymbol{C}(\mathbf{X},\mathbf{Y}) = \boldsymbol{C}(\mathbf{Y}) \circ \boldsymbol{C}(\mathbf{X})^{-1} \tag{64.2}$$

for all **X** and **Y**. We use the same notation $\boldsymbol{C}(\mathbf{X},\mathbf{Y})$ for the Cartan parallelism from **X** to **Y** for pairs **X**, **Y** in $\mathscr{GL}(\mathscr{V})$ or in any continuous subgroup of $\mathscr{GL}(\mathscr{V})$, such as $\mathscr{SL}(\mathscr{V})$. Thus $\boldsymbol{C}(\mathbf{X},\mathbf{Y})$ denotes either the linear isomorphism

$$\boldsymbol{C}(\mathbf{X},\mathbf{Y}) : \mathscr{GL}(\mathscr{V})_{\mathbf{X}} \to \mathscr{GL}(\mathscr{V})_{\mathbf{Y}} \tag{64.3}$$

or its various restrictions such as

$$\boldsymbol{C}(\mathbf{X},\mathbf{Y}) : \mathscr{SL}(\mathscr{V})_{\mathbf{X}} \to \mathscr{SL}(\mathscr{V})_{\mathbf{Y}} \tag{64.4}$$

when **X**, **Y** belong to $\mathscr{SL}(\mathscr{V})$.

Let **V** be a vector field on $\mathscr{GL}(\mathscr{V})$, that is

$$\mathbf{V} : \mathscr{GL}(\mathscr{V}) \to \mathscr{L}(\mathscr{V};\mathscr{V}) \tag{64.5}$$

Then we say that **V** is a *left-invarient field* if its values are parallel vectors relative to the Cartan parallelism, i.e.,

$$\left[\boldsymbol{C}(\mathbf{X},\mathbf{Y})\right]\left(\mathbf{V}(\mathbf{X})\right) = \mathbf{V}(\mathbf{Y}) \tag{64.6}$$

for any **X**, **Y** in $\mathscr{GL}(\mathscr{V})$. Since the Cartan parallelism is not the same as the Euclidean parallelism induced by the inner product space $\mathscr{L}(\mathscr{V};\mathscr{V})$, a left-invarient field is not a constant field. From (64.2) we have the following representations for a left-invarient field.

**Theorem 64.1.** A vector field $\mathbf{V}$ is left-invariant if and only if it has the representation

$$\mathbf{V}(\mathbf{X})=\left[\boldsymbol{C}(\mathbf{X})\right]\left(\mathbf{V}(\mathbf{I})\right) \tag{64.7}$$

for all $\mathbf{X}\in\mathscr{GL}(\mathscr{V})$.

As a result, each tangent vector $\mathbf{V}(\mathbf{I})$ at the identity element $\mathbf{I}$ has a unique extension into a left-variant field. Consequently, the set of all left-invariant fields, denoted by $\mathfrak{gl}(\mathscr{V})$, is a copy of the tangent space $\mathscr{GL}(\mathscr{V})_{\mathbf{I}}$ which is canonically isomorphic to $\mathscr{L}(\mathscr{V};\mathscr{V})$. We call the restriction map

$$\big|_{\mathbf{I}}:\mathfrak{gl}(\mathscr{V})\rightarrow\mathscr{GL}(\mathscr{V})_{\mathbf{I}}\cong\mathscr{L}(\mathscr{V};\mathscr{V}) \tag{64.8}$$

the *standard representation* of $\mathfrak{gl}(\mathscr{V})$. Since the elements of $\mathfrak{gl}(\mathscr{V})$ satisfy the representation (64.7), they characterize the Caratan parallelism completely by the condition (64.6) for all $\mathbf{V}\in\mathfrak{gl}(\mathscr{V})$. In the next section we shall show that $\mathfrak{gl}(\mathscr{V})$ has the structure of a Lie algebra, so we call it the *Lie Algebra* of $\mathscr{GL}(\mathscr{V})$.

Now using the Cartan parallelism $\boldsymbol{C}(\mathbf{X},\mathbf{Y})$, we can define an opereation of covariant derivative by the limit of difference similar to (56.34), which defines the covariant derivative relative to the Levi-Civita parallelism. Specifically, let $\mathbf{X}(t)$ be a smooth curve in $\mathscr{GL}(\mathscr{V})$ and let $\mathbf{U}(t)$ be a vector field on $\mathbf{X}(t)$. Then we defince the covariant derivative of $\mathbf{U}(t)$ along $\mathbf{X}(t)$ relative to the Cartan parallelism by

$$\frac{D\mathbf{U}(t)}{Dt}=\frac{\mathbf{U}(t+\Delta t)-\left[\boldsymbol{C}\left(\mathbf{X}(t),\mathbf{X}(t+\Delta t)\right)\right]\left(\mathbf{U}(t)\right)}{\Delta t} \tag{64.9}$$

Since the Cartan parallelism is defined not just on $\mathscr{GL}(\mathscr{V})$ but also on the various continuous subgroups of $\mathscr{GL}(\mathscr{V})$, we may use the same formula (64.9) to define the covariant derivative of tangent vector fields along a smooth curve in the subgroups also. For this reason we shall now derive a general representation for the covariant derivative relative to the Cartan parallelism without restricting the underlying continuous group to be the general linear group $\mathscr{GL}(\mathscr{V})$. Then we can apply the representation to vector fields on $\mathscr{GL}(\mathscr{V})$ as well as to vector fields on the various continuous subgroups of $\mathscr{GL}(\mathscr{V})$, such as the special linear group $\mathscr{SL}(\mathscr{V})$.

The simplest way to represent the covariant derivative defined by (64.9) is to first express the vector field $\mathbf{U}(t)$ in component form relative to a basis of the Lie algebra of the underlying continuous group. From (64.7) we know that the values $\{\mathbf{E}_\Gamma(\mathbf{X}), \Gamma=1,\ldots,M\}$ of a left-invariant field of basis $\{\mathbf{E}_\Gamma(\mathbf{X}), \Gamma=1,\ldots,M\}$ form a basis of the tangent space at $\mathbf{X}$ for all $\mathbf{X}$ belonging to the underlying group. Here $M$ denotes the dimension of the group; it is purposely left arbitrary so as to achieve generality in the representation. Now since $\mathbf{U}(t)$ is a tangent vector at $\mathbf{X}(t)$, it can be represented as usual by the component form relative to the basis $\{\mathbf{E}_\Gamma(\mathbf{X}(t))\}$, say

$$\mathbf{U}(t) = \hat{U}^\Gamma(t)\mathbf{E}_\Gamma(\mathbf{X}(t)) \tag{64.10}$$

where $\Gamma$ is summed from 1 to $M$. Substituting (64.10) into (64.9) and using the fact that the basis $\{\mathbf{E}_\Gamma\}$ is formed by parallel fields relative to the Cartan parallelism, we obtain directly

$$\frac{D\mathbf{U}(t)}{Dt} = \frac{d\hat{U}^\Gamma(t)}{dt}\mathbf{E}_\Gamma(\mathbf{X}(t)) \tag{64.11}$$

This formula is comparable to the representation of the covariant derivative relative to the Euclidean parallelism when the vector field is expressed in component form in the terms of a Cartesian basis.

As we shall see in the next section, the left-variant basis $\{\mathbf{E}_\Gamma\}$ used in the representation (64.11) is not the natural basis of any coordinate system. Indeed, this point is the major difference between the Cartan parallelism and the Euclidean parallelism, since relative to the latter a parallel basis is a constant basis which is the natural basis of a Cartesian coordinate system. If we introduce a local coordinate system with natural basis $\{\mathbf{H}_\Gamma, \Gamma = 1,\ldots,M\}$, then as usual we can represent $\mathbf{X}(t)$ by its coordinate functions $(X^\Gamma(t))$, and $\mathbf{E}_\Gamma$ and $\mathbf{U}(t)$ by their components

$$\mathbf{E}_\Gamma = E_\Gamma^\Delta \mathbf{H}_\Delta \qquad \text{and} \qquad \mathbf{U}(t) = U^\Gamma(t)\mathbf{H}_\Gamma(\mathbf{X}(t)) \tag{64.12}$$

By using the usual transformation rule, we then have

$$\hat{U}^\Gamma(t) = U^\Delta(t) F_\Delta^\Gamma(\mathbf{X}(t)) \tag{64.13}$$

where $\left[F_\Delta^\Gamma\right]$ is the inverse of $\left[E_\Delta^\Gamma\right]$. Substituting (64.13) into (64.11), we get

$$\frac{D\mathbf{U}}{Dt} = \left( \frac{dU^\Delta}{dt} + U^\Omega \frac{\partial F_\Omega^\Gamma}{\partial X^\Sigma} \frac{dX^\Sigma}{dt} E_\Gamma^\Delta \right) \mathbf{H}_\Delta \tag{64.14}$$

We can rewrite this formula as

$$\frac{D\mathbf{U}}{Dt} = \left( \frac{dU^\Delta}{dt} + U^\Omega L_{\Omega\Sigma}^\Delta \frac{dX^\Sigma}{dt} \right) \mathbf{H}_\Delta \tag{64.15}$$

where

$$L_{\Omega\Sigma}^\Delta = E_\Gamma^\Delta \frac{\partial F_\Omega^\Gamma}{\partial X^\Sigma} = -F_\Omega^\Gamma \frac{\partial E_\Gamma^\Delta}{\partial X^\Sigma} \tag{64.16}$$

Now the formula (64.15) is comparable to (56.37) with $L_{\Omega\Sigma}^\Delta$ playing the role of the Christoffel symbols, except that $L_{\Omega\Sigma}^\Delta$ is not symmetric with respect to the indices $\Omega$ and $\Sigma$. For definiteness, we call $L_{\Omega\Sigma}^\Delta$ the *Cartan symbols*. From (64.16) we can verify easily that they do not depend on the choice of the basis $\{\mathbf{E}_\Gamma\}$.

It follows from $(64.12)_1$ and (64.16) that the Cartan symbols obey the same transformation rule as the Christoffel symbols. Specifically, if $\left(\bar{X}^\Gamma\right)$ is another coordinate system in which the Cartan symbols are $\bar{L}_{\Omega\Sigma}^\Delta$, then

$$\bar{L}_{\Omega\Sigma}^\Delta = L_{\Psi\Theta}^\Phi \frac{\partial \bar{X}^\Delta}{\partial X^\Phi} \frac{\partial X^\Psi}{\partial \bar{X}^\Omega} \frac{\partial X^\Theta}{\partial \bar{X}^\Sigma} + \frac{\partial \bar{X}^\Delta}{\partial X^\Phi} \frac{\partial^2 X^\Phi}{\partial \bar{X}^\Omega \partial \bar{X}^\Sigma} \tag{64.17}$$

This formula is comparable to (56.15). In view of (64.15) and (64.17) we can define the covariant derivative of a vector field relative to the Cartan parallelism by

$$\nabla\mathbf{U} = \left( \frac{\partial U^\Delta}{\partial X^\Sigma} + U^\Omega L_{\Omega\Sigma}^\Delta \right) \mathbf{H}_\Delta \otimes \mathbf{H}^\Sigma \tag{64.18}$$

where $\left\{\mathbf{H}^\Sigma\right\}$ denotes the dual basis of $\left\{\mathbf{H}_\Delta\right\}$. The covariant derivative defined in this way clearly possesses the following property.

**Theorem 64.2.** A vector field **U** is left-invariant if and only if its covariant derivative relative to the Cartan parallelism vanishes.

The proof of this proposition if more or less obvious, since the condition

$$\frac{\partial U^{\Delta}}{\partial X^{\Sigma}} + U^{\Omega} L^{\Delta}_{\Omega\Sigma} = 0 \tag{64.19}$$

is equivalent to the condition

$$\frac{\partial \hat{U}^{\Delta}}{\partial X^{\Sigma}} = 0 \tag{64.20}$$

where $\hat{U}^{\Delta}$ denotes the components of $\mathbf{U}$ relative to the parallel basis $\{\mathbf{E}_{\Gamma}\}$. The condition (64.20) means simply that $\hat{U}^{\Delta}$ are constant, or, equivalent, $\mathbf{U}$ is left-invariant.

Comparing (64.16) with (59.16), we see that the Cartan parallelism also possesses the following property.

**Theorem 64.3.** The curvature tensor whose components are defined by

$$R^{\Gamma}{}_{\Sigma\Delta\Omega} \equiv \frac{\partial}{\partial X^{\Delta}} L^{\Gamma}_{\Sigma\Omega} - \frac{\partial}{\partial X^{\Omega}} L^{\Gamma}_{\Sigma\Delta} + L^{\Phi}_{\Sigma\Omega} L^{\Gamma}_{\Phi\Delta} - L^{\Phi}_{\Sigma\Delta} L^{\Gamma}_{\Phi\Omega} \tag{64.21}$$

vanishes identically.

Notice that (64.21) is comparable with (59.10) where the Christoffel symbols are replaced by the Cartan symbols. The vanishing of the curvature tensor

$$R^{\Gamma}{}_{\Sigma\Delta\Omega} = 0 \tag{64.22}$$

is simply the condition of integrability of equation (64.19) whose solutions are left-invariant fields.

From the transformation rule (64.17) and the fact that the second derivative therein is symmetric with respect to the indices $\Omega$ and $\Sigma$, we obtain

$$\bar{L}^{\Delta}_{\Omega\Sigma} - \bar{L}^{\Delta}_{\Sigma\Omega} = \left( L^{\Phi}_{\Psi\Theta} - L^{\Phi}_{\Theta\Psi} \right) \frac{\partial \bar{X}^{\Delta}}{\partial X^{\Phi}} \frac{\partial X^{\Psi}}{\partial \bar{X}^{\Omega}} \frac{\partial X^{\Theta}}{\partial \bar{X}^{\Sigma}} \tag{64.23}$$

which shows that the quanties defined by

$$T^{\Delta}_{\Omega\Sigma} \equiv L^{\Delta}_{\Omega\Sigma} - L^{\Delta}_{\Sigma\Omega} \tag{64.24}$$

are the components of a third-order tensor field. We call the $\mathbf{T}$ the *torsion tensor* of the Cartan parallelism. To see the geometric meaning of this tensor, we substitute the formula (64.16) into (64.24) and rewrite the result in the following equivalent form:

$$T^{\Delta}_{\Omega\Sigma} E^{\Omega}_{\Phi} E^{\Sigma}_{\Psi} = E^{\Omega}_{\Phi} \frac{\partial E^{\Delta}_{\Psi}}{\partial X^{\Omega}} - E^{\Omega}_{\Psi} \frac{\partial E^{\Delta}_{\Phi}}{\partial X^{\Omega}} \tag{64.25}$$

which is the component representation of

$$\mathbf{T}\left(\mathbf{E}_{\Phi}, \mathbf{E}_{\Psi}\right) = \left[\mathbf{E}_{\Phi}, \mathbf{E}_{\Psi}\right] \tag{64.26}$$

where the right-hand side is the Lie bracket of $\mathbf{E}_{\Phi}$ and $\mathbf{E}_{\Psi}$, i.e.,

$$\left[\mathbf{E}_{\Phi}, \mathbf{E}_{\Psi}\right] \equiv \underset{\mathbf{E}_{\Phi}}{\mathscr{L}} \mathbf{E}_{\Psi} = \left( E^{\Omega}_{\Phi} \frac{\partial E^{\Delta}_{\Psi}}{\partial X^{\Omega}} - E^{\Omega}_{\Psi} \frac{\partial E^{\Delta}_{\Phi}}{\partial X^{\Omega}} \right) \mathbf{H}_{\Delta} \tag{64.27}$$

Equation (64.27) is comparable to (49.21) and (55.31).

As we shall see in the next section, the Lie bracket of any pair of left-invariant fields is itself also a left-invariant field. Indeed, this is the very reason that the set of all left-invariant fields is so endowed with the structure of a Lie algebra. As a result, the components of $\left[\mathbf{E}_{\Phi}, \mathbf{E}_{\Psi}\right]$ relative to the left-invariant basis $\{\mathbf{E}_{\Gamma}\}$ are constant scalar fields, namely

$$\left[\mathbf{E}_{\Phi}, \mathbf{E}_{\Psi}\right] = C^{\Gamma}_{\Phi\Psi} \mathbf{E}_{\Gamma} \tag{64.28}$$

We call $C^{\Gamma}_{\Phi\Psi}$ the *structure constants* of the left-invariant basis $\{\mathbf{E}_{\Gamma}\}$. From (64.26), $C^{\Gamma}_{\Phi\Psi}$ are nothing but the components of the torsion tensor $\mathbf{T}$ relative to the basis $\{\mathbf{E}_{\Gamma}\}$, namely

$$\mathbf{T} = C^{\Gamma}_{\Phi\Psi} \mathbf{E}_{\Gamma} \otimes \mathbf{E}^{\Phi} \otimes \mathbf{E}^{\Psi} \tag{64.29}$$

Now the covariant derivative defined by (64.9) for vector fields can be generalized as in the preceding chapter to covariant derivatives of arbitrary tangential tensor fields. Specifically, the general formula is

$$\frac{D\mathbf{A}}{Dt} = \left\{ \frac{d A^{\Gamma_1 \ldots \Gamma_r}_{\Delta_1 \ldots \Delta_s}}{dt} + \begin{pmatrix} A^{\Sigma\Gamma_2 \ldots \Gamma_r}_{\Delta_1 \ldots \Delta_s} L^{\Gamma_1}_{\Sigma\Omega} + \cdots + A^{\Gamma_1 \ldots \Gamma_{r-1}\Sigma}_{\Delta_1 \ldots \Delta_s} L^{\Gamma_r}_{\Sigma\Omega} \\ - A^{\Gamma_1 \cdots \Gamma_r}_{\Sigma\Delta_2 \ldots \Delta_s} L^{\Sigma}_{\Delta_1\Omega} - \cdots - A^{\Gamma_1 \cdots \Gamma_r}_{\Delta_1 \ldots \Delta_{s-1}\Sigma} L^{\Sigma}_{\Delta_s\Omega} \end{pmatrix} \frac{d X^{\Omega}}{dt} \right\} \mathbf{H}_{\Gamma_1} \otimes \cdots \otimes \mathbf{H}^{\Delta_s} \tag{64.30}$$

which is comparable to (48.7) and (56.23). The formula (64.30) represents the covariant derivative in terms of a coordinate system $\left(X^{\Gamma}\right)$. If we express $\mathbf{A}$ in component form relative to a left-variant basis $\{\mathbf{E}_{\Gamma}\}$, say

$$\mathbf{A}(t)=\hat{A}^{\Gamma_1\ldots\Gamma_r}_{\Delta_1\ldots\Delta_s}(t)\mathbf{E}_{\Gamma_1}\left(\mathbf{X}(\mathbf{t})\right)\otimes\cdots\otimes\mathbf{E}^{\Delta_s}\left(\mathbf{X}(t)\right) \tag{64.31}$$

then the representation of the covariant derivative is simply

$$\frac{D\mathbf{A}(t)}{Dt}=\frac{d\hat{A}^{\Gamma_1\ldots\Gamma_r}_{\Delta_1\ldots\Delta_s}(t)}{dt}\mathbf{E}_{\Gamma_1}\left(\mathbf{X}(t)\right)\otimes\cdots\otimes\mathbf{E}^{\Delta_s}\left(\mathbf{X}(t)\right) \tag{64.32}$$

The formulas (64.30) and (64.32) represent the covariant derivative of a tangential tensor field $\mathbf{A}$ along a smooth curve $\mathbf{X}(t)$. Now if $\mathbf{A}$ is a tangential tensor field defined on the continuous group, then we define the covariant derivative of $\mathbf{A}$ relative to the Cartan parallelism by a formula similar to (56.10) except that we replace the Christoffel symbols there by the Cartan symbols.

Naturally we say that a tensor field $\mathbf{A}$ is *left-invariant* if $\nabla\mathbf{A}$ vanishes. The torsion tensor field $\mathbf{T}$ given by (64.29) is an example of a left-invariant third-order tensor field. Since a tensor field is left invariant if and only if its components relative to the product basis of a left-invariant basis are constants, a representation formally generalizing (64.7) can be stated for left-invariant tensor fields in general. In particular, if $\{\mathbf{E}_{\Gamma}\}$ is a left-invariant field of bases, then

$$\mathbf{E}=\mathbf{E}_1\wedge\cdots\wedge\mathbf{E}_M \tag{64.33}$$

is a left-invariant field of density tensors on the group.

## *Section 65. One-Parameter Groups and the Exponential Map*

In section 57 of the preceding chapter we have introduced the concepts of geodesics and exponential map relative to the Levi-Civita parallelism on a hypersurface. In this section we consider similar concepts relative to the Cartan parallelism. As before, we define a *geodesic* to be a smooth curve $\mathbf{X}(t)$ such that

$$\frac{D\dot{\mathbf{X}}}{Dt}=\mathbf{0} \tag{65.1}$$

where $\dot{\mathbf{X}}$ denotes the tangent vector $\mathbf{X}$ and where the covariant derivative is taken relative to the Cartan parallelism.

Since (65.1) is formally the same as (57.1), relative to a coordinate system $\left(X^{\Gamma}\right)$ we have the following *equations of geodesics*

$$\frac{d^2X^{\Gamma}}{dt^2}+\frac{dX^{\Sigma}}{dt}\frac{dX^{\Delta}}{dt}L^{\Gamma}_{\Sigma\Delta}=0,\ \ \Gamma=1,\ldots,M \tag{65.2}$$

which are comparable to (57.2). However, the equations of geodesics here are no longer the Euler-Lagrange equations of the arc length integral, since the Cartan parallelism is not induced by a metric and the arc length integral is not defined. To interpret the geometric meaning of a geodesic relative to the Cartan parallelism, we must refer to the definition (64.9) of the covariant derivative.

We notice first that if we express the tangent vector $\dot{\mathbf{X}}(t)$ of any smooth curve $\mathbf{X}(t)$ in component form relative to a left-invariant basis $\{\mathbf{E}_{\Gamma}\}$,

$$\dot{\mathbf{X}}(t)=G^{\Gamma}(t)\mathbf{E}_{\Gamma}\left(\mathbf{X}(t)\right) \tag{65.3}$$

then from (64.11) a necessary and sufficient condition for $\mathbf{X}(t)$ to be a geodesic is that the components $G^{\Gamma}(t)$ be constant independent of $t$. Equivalently, this condition means that

$$\dot{\mathbf{X}}(t)=\mathbf{G}\left(\mathbf{X}(t)\right) \tag{65.4}$$

where $\mathbf{G}$ is a left-invariant field having the component form

$$\mathbf{G}=G^{\Gamma}\mathbf{E}_{\Gamma} \tag{65.5}$$

where $G^{\Gamma}$ are constant. In other words, a curve $\mathbf{X}(t)$ is a geodesic relative to the Cartan parallelism if and only if it is an integral curve of a left-invariant vector field.

This characteristic property of a geodesic implies immediately the following result.

**Theorem 65.1.** If $\mathbf{X}(t)$ is a geodesic tangent to the left-invariant field $\mathbf{G}$, then $\boldsymbol{L}_{\mathbf{A}}(\mathbf{X}(t))$ is also a geodesic tangent to the same left-invariant field $\mathbf{G}$ for all $\mathbf{A}$ in the underlying continuous group.

A corollary of this theorem is that every geodesic can be extended indefinitely from $t=-\infty$ to $t=+\infty$. Indeed, if $\mathbf{X}(t)$ is a geodesic defined for an interval, say $t\in[0,1]$, then we can extend $\mathbf{X}(t)$ to the interval $t\in[1,2]$ by

$$\mathbf{X}(t+1)\equiv \boldsymbol{L}_{\mathbf{X}(\mathbf{1})\mathbf{X}^{-1}(\mathbf{0})}(\mathbf{X}(t)), \quad t\in[0,1] \tag{65.6}$$

and so forth. An important consequence of this extension is the following result.

**Theorem 65.2.** A smooth curve $\mathbf{X}(t)$ passing through the identity element $\mathbf{I}$ at $t=0$ is a geodesic if and only if it forms a one-parameter group, i.e.,

$$\mathbf{X}(t_1+t_2)=\mathbf{X}(t_1)\mathbf{X}(t_2), \qquad t_1,t_2\in\mathscr{R} \tag{65.7}$$

The necessity of (65.7) is a direct consequence of (65.6), which may be generalized to

$$\mathbf{X}(t_1+t_2)=\boldsymbol{L}_{\mathbf{X}(t_1)}(\mathbf{X}(t_2)) \tag{65.8}$$

for all $t_1$ and $t_2$. Here we have used the condition that

$$\mathbf{X}(0)=\mathbf{I} \tag{65.9}$$

Conversely, if (65.7) holds, then by differentiating with respect to $t_2$ and evaluating the result at $t_2=0$, we obtain

$$\dot{\mathbf{X}}(t_1)=\left[\boldsymbol{C}(\mathbf{X}(t_1))\right](\dot{\mathbf{X}}(0)) \tag{65.10}$$

which shows that $\dot{\mathbf{X}}(t)$ is the value of a particular left-invariant field at all $\mathbf{X}(t)$, and thus $\mathbf{X}(t)$ is a geodesic relative to the Cartan parallelism.

Now combining the preceding propositions, we see that the class of all geodesics can be characterized in the following way. First, for each left-invariant field $\mathbf{G}$ there exists a unique one-parameter group $\mathbf{X}(t)$ such that

$$\dot{\mathbf{X}}(0)=\mathbf{G}(\mathbf{I}) \tag{65.11}$$

where $\mathbf{G}(\mathbf{I})$ is the standard representation of $\mathbf{G}$. Next, the set of all geodesics tangent to $\mathbf{G}$ can be represented by $\boldsymbol{L}_{\mathbf{A}}(\mathbf{X}(t))$ for all $\mathbf{A}$ belonging to the underlying group.

As in Section 57, the one-to-one correspondence between $\mathbf{G}(\mathbf{I})$ and $\mathbf{X}(t)$ gives rise to the notion of the exponential map at the indentity element $\mathbf{I}$. For brevity, let $\mathbf{A}$ be the standard representation of $\mathbf{G}$, i.e.,

$$\mathbf{A}\equiv\mathbf{G}(\mathbf{I}) \tag{65.12}$$

Then we define

$$\mathbf{X}(1)=\exp_{\mathbf{I}}(\mathbf{A})\equiv\exp\mathbf{A} \tag{65.13}$$

which is comparable to (57.19). As explained in Section 57, (65.13) implies that

$$\mathbf{X}(t)=\exp(\mathbf{A}t) \tag{65.14}$$

for all $t\in\mathscr{R}$. Here we have used the extension property of the geodesic. Equation (65.14) formally represents the one-parameter group whose initial tangent vector at the identity element $\mathbf{I}$ is $\mathbf{A}$.

We claim that the exponential map defined by (65.13) can be represented explicitly by the exponential series

$$\exp(\mathbf{A})=\mathbf{I}+\mathbf{A}+\frac{1}{2!}\mathbf{A}^2+\frac{1}{3!}\mathbf{A}^3+\cdots+\frac{1}{n!}\mathbf{A}^n+\cdots \tag{65.15}$$

Clearly, this series converges for each $\mathbf{A}\in\mathscr{L}(\mathscr{V};\mathscr{V})$. Indeed, since we have

$$\left\|\mathbf{A}^n\right\|\le\left\|\mathbf{A}\right\|^n \tag{65.16}$$

for all positive integers $n$, the partial sums of (65.15) form a Cauchy sequence in the inner product space $\mathscr{L}(\mathscr{V};\mathscr{V})$. That is,

$$\left\| \frac{1}{n!}\mathbf{A}^n + \cdots + \frac{1}{m!}\mathbf{A}^m \right\| \le \frac{1}{n!}\|\mathbf{A}\|^n + \cdots + \frac{1}{m!}\|\mathbf{A}\|^m \tag{65.17}$$

and the right-hand side of (65.17) converges to zero as $n$ and $m$ approach infinity.

Now to prove that (65.15) is the correct representation for the exponential map, we have to show that the series

$$\exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \cdots + \frac{1}{n!}\mathbf{A}^n t^n + \cdots \tag{65.18}$$

defines a one-parameter group. This fact is more or less obvious since the exponential series satisfies the usual power law

$$\exp(\mathbf{A}t)\exp(\mathbf{A}t_2) = \exp(\mathbf{A}(t_1 + t_2)) \tag{65.19}$$

which can be verified by direct multiplication of the power series for $\exp(\mathbf{A}t_1)$ and $\exp(\mathbf{A}t_2)$. Finally, it is easily seen that the initial tangent of the curve $\exp(\mathbf{A}t)$ is $\mathbf{A}$ since

$$\frac{d}{dt}\left( \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2 t^2 + \cdots \right)\Bigg|_{t=0} = \mathbf{A} \tag{65.20}$$

This completes the proof of the representation (65.18).

We summarize our results as follows.

**Theorem 65.3.** Let $\mathbf{G}$ be a left-invariant field with standard representation $\mathbf{A}$. Then the geodesics $\mathbf{X}(t)$ tangent to $\mathbf{G}$ can be expressed by

$$\mathbf{X}(t) = \mathbf{X}(0)\exp(\mathbf{A}t) = \mathbf{X}(0)\left( \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 \cdots \right) \tag{65.21}$$

where the initial point $\mathbf{X}(0)$ is arbitrary.

In the view of this representation, we see that the flow generated by the left-invariant field $\mathbf{G}$ is simply the right multiplication by $\exp(\mathbf{A}t)$, namely

$$\boldsymbol{\rho}_t = R_{\exp(\mathbf{A}t)} \tag{65.22}$$

for all $t$. As a result, if $\mathbf{K}$ is another left-invariant field, then the Lie bracket of $\mathbf{G}$ with $\mathbf{K}$ is given by

$$[\mathbf{G},\mathbf{K}](\mathbf{X})=\lim_{t\to 0}\frac{\mathbf{K}\left(\mathbf{X}\exp(\mathbf{A}t)\right)-\mathbf{K}(\mathbf{X})\exp(\mathbf{A}t)}{t} \tag{65.23}$$

This formula implies immediately that $[\mathbf{G},\mathbf{K}]$ is also a left-invariant field. Indeed, if the standard representation of $\mathbf{K}$ is $\mathbf{B}$, then from the representation (64.7) we have

$$\mathbf{K}(\mathbf{X})=\mathbf{X}\mathbf{B} \tag{65.24}$$

and therefore

$$\mathbf{K}\left(\mathbf{X}\exp(\mathbf{A}t)\right)=\mathbf{X}\exp(\mathbf{A}t)\mathbf{B} \tag{65.25}$$

Substituting (65.24) and (65.25) into (65.23) and using the power series representation (65.18), we obtain

$$[\mathbf{G},\mathbf{K}](\mathbf{X})=\mathbf{X}(\mathbf{AB}-\mathbf{BA}) \tag{65.26}$$

which shows that $[\mathbf{G},\mathbf{K}]$ is left-invariant with the standard representation $\mathbf{AB}-\mathbf{BA}$. Hence in terms of the standard representation the Lie bracket on the Lie algebra is given by

$$[\mathbf{A},\mathbf{B}]=\mathbf{AB}-\mathbf{BA} \tag{65.27}$$

So far we have shown that the set of left-invariant fields is closed with respect to the operation of the Lie bracket. In general a vector space equipped with a bilinear bracket product which obeys the Jacobi identities

$$[\mathbf{A},\mathbf{B}]=-[\mathbf{B},\mathbf{A}] \tag{65.28}$$

and

$$\left[\mathbf{A},[\mathbf{B},\mathbf{C}]\right]+\left[\mathbf{B},[\mathbf{C},\mathbf{A}]\right]+\left[\mathbf{C},[\mathbf{A},\mathbf{B}]\right]=\mathbf{0} \tag{65.29}$$

is called a *Lie Algebra*. From (65.27) the Lie bracket of left-invariant fields clearly satisfies the identities (65.28) and (65.29). As a result, the set of all left-invariant fields has the structure of a Lie algebra with respect to the Lie bracket. This is why that set is called the Lie algebra of the underlying group, as we have remarked in the preceding section.

Before closing the section, we remark that the Lie algebra of a continuous group depends only on the identity component of the group. If two groups share the same identity component, then their Lie algebras are essentially the same. For example, the Lie algebra of $\mathscr{GL}(\mathscr{V})$ and $\mathscr{GL}(\mathscr{V})^{+}$ are both representable by $\mathscr{L}(\mathscr{V};\mathscr{V})$ with the Lie bracket given by (65.27). The fact that $\mathscr{GL}(\mathscr{V})$ has two components, namely $\mathscr{GL}(\mathscr{V})^{+}$ and $\mathscr{GL}(\mathscr{V})^{-}$ cannot be reflected in any way by the Lie algebra $\mathscr{gl}(\mathscr{V})$. We shall consider the relation between the Lie algebra and the identity component of the underlying group in more detail in the next section.

## Exercises

65.1. Establish the following properties of the function exp on $\mathscr{L}(\mathscr{V};\mathscr{V})$:

(a) $\exp(\mathbf{A}+\mathbf{B})=(\exp\mathbf{A})(\exp\mathbf{B})$ if $\mathbf{AB}=\mathbf{BA}$.

(b) $\exp(-\mathbf{A})=(\exp\mathbf{A})^{-1}$.

(c) $\exp\mathbf{0}=\mathbf{I}$

(d) $\mathbf{B}(\exp\mathbf{A})\mathbf{B}^{-1}=\exp\mathbf{BAB}^{-1}$ for regular $\mathbf{B}$.

(e) $\mathbf{A}=\mathbf{A}^{\mathrm{T}}$ if and only if $\exp\mathbf{A}=(\exp\mathbf{A})^{\mathrm{T}}$.

(f) $\mathbf{A}=-\mathbf{A}^{\mathrm{T}}$ if and only if $\exp\mathbf{A}$ is orthogonal.

(g) $\det(\exp\mathbf{A})=e^{tr\,\mathbf{A}}$.

65.2 If $\mathbf{P}$ is a projection, show that

$$\exp\lambda\mathbf{P}=\mathbf{I}+(e^{\lambda}-1)\mathbf{P}$$

for $\lambda\in\mathscr{R}$.

## Section 66. Subgroups and Subalgebras

In the preceding section we have shown that the set of left invariant fields is closed with respect to the Lie bracket. This is an important property of a continuous group. In this section we shall elaborate further on this property by proving that there exists a one-to-one correspondence between a connected continuous subgroup of $\mathscr{GL}(\mathscr{V})$ and a subalgebra of $\mathscr{gl}(\mathscr{V})$. Naturally we call a subspace $\mathscr{h}$ of $\mathscr{gl}(\mathscr{V})$ a subalgebra if $\mathscr{h}$ is closed with respect to the Lie bracket, i.e., $[\mathbf{G},\mathbf{H}]\in\mathscr{h}$ whenever both $\mathbf{G}$ and $\mathbf{H}$ belong to $\mathscr{h}$. We claim that for each subalgebra $\mathscr{h}$ of $\mathscr{gl}(\mathscr{V})$ there corresponds uniquely a connected continuous subgroup $\mathscr{H}$ of $\mathscr{GL}(\mathscr{V})$ whose Lie algebra coincides with the restriction of $\mathscr{h}$ on $\mathscr{H}$.

First, suppose that $\mathscr{H}$ is a continuous subgroup of $\mathscr{GL}(\mathscr{V})$, i.e., $\mathscr{H}$ is algebraically a subgroup of $\mathscr{GL}(\mathscr{V})$ and geometrically a smooth hyper surface in $\mathscr{GL}(\mathscr{V})$. For example, $\mathscr{H}$ may be the orthogonal group or the special linear group. Then the set of all left-invariant tangent vector fields on $\mathscr{H}$ forms a Lie algebra $\mathscr{h}$. We claim that every element $\mathbf{V}$ in $\mathscr{h}$ can be extended uniquely into a left-invariant vector field $\bar{\mathbf{V}}$ on $\mathscr{GL}(\mathscr{V})$. Indeed, this extension is provided by the representation (64.7). That is, we simply define

$$\bar{\mathbf{V}}(\mathbf{X})=\left[\boldsymbol{C}(\mathbf{X})\right]\left(\mathbf{V}(\mathbf{I})\right) \tag{66.1}$$

for all $\mathbf{X}\in\mathscr{GL}(\mathscr{V})$. Here we have used the fact that the Cartan parallelism on $\mathscr{H}$ is the restriction of that on $\mathscr{GL}(\mathscr{V})$ to $\mathscr{H}$. Therefore, when $\mathbf{X}\in\mathscr{H}$, the representation (66.1) reduces to

$$\mathbf{V}(\mathbf{X})=\left[\boldsymbol{C}(\mathbf{X})\right]\left(\mathbf{V}(\mathbf{I})\right)$$

since $\mathbf{V}$ is left-invariant on $\mathscr{H}$.

From (66.1), $\mathbf{V}$ and $\bar{\mathbf{V}}$ share the same standard representation:

$$\mathbf{A}\equiv\mathbf{V}(\mathbf{I})$$

As a result, from (65.27) the Lie bracket on $\mathscr{H}$ is related to that on $\mathscr{GL}(\mathscr{V})$ by

$$\overline{[\mathbf{V},\mathbf{U}]}=\left[\bar{\mathbf{V}},\bar{\mathbf{U}}\right] \tag{66.2}$$

for all $\mathbf{U}$ and $\mathbf{V}$ in $\mathscr{h}$. This condition shows that the extension $\bar{\mathscr{h}}$ of $\mathscr{h}$ consisting of all left-invariant fields $\bar{\mathbf{V}}$ with $\mathbf{V}$ in $\mathscr{h}$ is a Lie subalgebra of $\mathscr{gl}(\mathscr{V})$. In view of (66.1) and (66.2), we simplify the notation by suppressing the overbar. If this convention is adopted, then $\mathscr{h}$ becomes a Lie subalgebra of $\mathscr{gl}(\mathscr{V})$.

Since the inclusion $\mathscr{h} \subset \mathscr{gl}(\mathscr{V})$ is based on the extension (66.1), if $\mathscr{H}_1$ and $\mathscr{H}_2$ are continuous subgroups of $\mathscr{GL}(\mathscr{V})$ having the same identity components, then their Lie algebras $\mathscr{h}_1$ and $\mathscr{h}_2$ coincide as Lie subalgebras of $\mathscr{gl}(\mathscr{V})$. In this sense we say that the Lie algebra characterizes only the identity component of the underlying group, as we have remarked at the end of the preceding section. In particular, the Lie algebra of $\mathscr{UM}(\mathscr{V})$ coincides with that of $\mathscr{SL}(\mathscr{V})$.

It turns out that every Lie subalgebra of $\mathscr{gl}(\mathscr{V})$ can be identified as the Lie algebra of a unique connected continuous subgroup of $\mathscr{gl}(\mathscr{V})$. To prove this, let $\mathscr{h}$ be an arbitrary Lie subalgebra of $\mathscr{gl}(\mathscr{V})$. Then the values of the left-invariant field belonging to $\mathscr{h}$ form a linear subspace of $\mathscr{L}(\mathscr{V};\mathscr{V})$ at each point of $\mathscr{GL}(\mathscr{V})$. This field of subspaces is a distribution on $\mathscr{GL}(\mathscr{V})$ as defined in Section 50. According to the Frobenius theorem, the distribution is integrable if and only if it is closed with respect to the Lie bracket. This condition is clearly satisfied since $\mathscr{h}$ is a Lie subalgebra. As a result, there exists an integral hypersurface of the distribution at each point in $\mathscr{GL}(\mathscr{V})$.

We denote the maximal connected integral hypersurface of the distribution at the identity by $\mathscr{H}$. Here maximality means that $\mathscr{H}$ is not a proper subset of any other connected integral hypersurface of the distribution. This condition implies immediately that $\mathscr{H}$ is also the maximal connected integral hypersurface at any point $\mathbf{X}$ which belongs to $\mathscr{H}$. By virtue of this fact we claim that

$$\boldsymbol{L}_{\mathbf{X}}(\mathscr{H}) = \mathscr{H} \tag{66.3}$$

for all $\mathbf{X} \in \mathscr{H}$. Indeed, since the distribution is generated by left-invariant fields, its collection of maximal connected integral hypersurfaces is invariant under any left multiplication. In particular, $\boldsymbol{L}_{\mathbf{X}}(\mathscr{H})$ is the maximal connected integral hypersurface at the point $\mathbf{X}$, since $\mathscr{H}$ contains the identity $\mathbf{I}$. As a result, (66.3) holds.

Now from (66.3) we see that $\mathbf{X} \in \mathscr{H}$ implies $\mathbf{X}^{-1} \in \mathscr{H}$ since $\mathbf{X}^{-1}$ is the only possible element such that $\boldsymbol{L}_{\mathbf{X}}(\mathbf{X}^{-1}) = \mathbf{I}$. Similiarly, if $\mathbf{X}$ and $\mathbf{Y}$ are contained in $\mathscr{H}$, then $\mathbf{XY}$ must also be contained in $\mathscr{H}$ since $\mathbf{XY}$ is the only possible element such that $\boldsymbol{L}_{\mathbf{Y}^{-1}}\boldsymbol{L}_{\mathbf{X}^{-1}}(\mathbf{XY}) = \mathbf{I}$. For the last condition we have used the fact that

$$\boldsymbol{L}_{\mathbf{Y}^{-1}}\boldsymbol{L}_{\mathbf{X}^{-1}}\left(\mathscr{H}\right)=\boldsymbol{L}_{\mathbf{Y}^{-1}}\left(\mathscr{H}\right)=\mathscr{H}$$

which follows from (66.3) and the fact that $\mathbf{X}^{-1}$ and $\mathbf{Y}^{-1}$ are both in $\mathscr{H}$. Thus we have shown that $\mathscr{H}$ is a connected continuous subgroup of $\mathscr{GL}(\mathscr{V})$ having $\mathscr{h}$ as its Lie algebra.

Summarizing the results obtained so far, we can state the following theorem.

**Theorem 66.1.** There exists a one-to-one correspondence between the set of Lie subalgebras of $\mathscr{gl}(\mathscr{V})$ and the set of connected continuous subgroups of $\mathscr{GL}(\mathscr{V})$ in such a way that each Lie subalgebra $\mathscr{h}$ of $\mathscr{gl}(\mathscr{V})$ is the Lie algebra of a unique connected continuous subgroup $\mathscr{H}$ of $\mathscr{GL}(\mathscr{V})$.

To illustrate this theorem, we now determine explicitly the Lie algebras $\mathscr{sl}(\mathscr{V})$ and $\mathscr{so}(\mathscr{V})$ of the subgroups $\mathscr{SL}(\mathscr{V})$ and $\mathscr{SO}(\mathscr{V})$. We claim first

$$\mathbf{A}\in\mathscr{sl}(\mathscr{V})\Leftrightarrow tr\,\mathbf{A}=0 \tag{66.4}$$

where $tr\,\mathbf{A}$ denotes the trace of $\mathbf{A}$. To prove this, we consider the one-parameter group $\exp(\mathbf{A}t)$ for any $\mathbf{A}\in\mathscr{L}(\mathscr{V};\mathscr{V})$. In order that $\mathbf{A}\in\mathscr{sl}(\mathscr{V})$, we must have

$$\det\left(\exp\left(\mathbf{A}t\right)\right)=1,\qquad t\in\mathscr{R} \tag{66.5}$$

Differentiating this condition with respect to $t$ and evaluating the result at $t=0$, we obtain [cf. Exercise 65.1(g)]

$$0=\frac{d}{dt}\left[\det\left(\exp\left(\mathbf{A}t\right)\right)\right]\Big|_{t=0}=tr\,\mathbf{A} \tag{66.6}$$

Conversely, if $tr\,\mathbf{A}$ vanishes, then (66.5) holds because the one-parameter group property of $\exp(\mathbf{A}t)$ implies

$$\frac{d}{dt}\left[\det\left(\exp\left(\mathbf{A}t\right)\right)\right]=\det\left(\exp\left(\mathbf{A}t\right)\right)tr\,\mathbf{A}=0,\quad t\in\mathscr{R}$$

while the initial condition at $t=0$,

$$\det(\exp \mathbf{0}) = \det \mathbf{I} = 1$$

is obvious. Thus we have completed the proof of (66.4).

From the representation (65.27) the reader will verify easily that the subspace of $\mathscr{L}(\mathscr{V};\mathscr{V})$ characterized by the right-hand side of (66.4) is indeed a Lie subalgebra, as it should be.

Next we claim that

$$\mathbf{A} \in so(\mathscr{V}) \Leftrightarrow \mathbf{A}^T = -\mathbf{A} \tag{66.7}$$

where $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$. Again we consider the one-parameter group $\exp(\mathbf{A}t)$ for any $\mathbf{A} \in \mathscr{L}(\mathscr{V};\mathscr{V})$. In order that $\mathbf{A} \in so(\mathscr{V})$, we must have

$$\exp(-\mathbf{A}t) = \left[\exp(\mathbf{A}t)\right]^T = \exp\left(\mathbf{A}^T t\right) \tag{66.8}$$

Here we have used the identities

$$\left[\exp(\mathbf{A})\right]^{-1} = \exp(-\mathbf{A}), \qquad \left[\exp(\mathbf{A})\right]^T = \exp\left(\mathbf{A}^T\right) \tag{66.9}$$

which can be verified directly from (65.15). The condition (66.7) clearly follows from the condition (66.8). From (65.27) the reader also will verify the fact that the subpace of $\mathscr{L}(\mathscr{V};\mathscr{V})$ characterized by the right-hand side is a Lie subalgebra.

The conditions (66.4) and (66.7) characterize completely the tangent spaces of $\mathscr{SL}(\mathscr{V})$ and $\mathscr{SO}(\mathscr{V})$ at the identity element $\mathbf{I}$. These conditions verify the claims on the dimensions of $\mathscr{SL}(\mathscr{V})$ and $\mathscr{SO}(\mathscr{V})$ made in Section 63.

## Section 67. Maximal Abelian Subgroups and Subalgebras

In this section we consider the problem of determining the Abelian subgroups of $\mathscr{GL}(\mathscr{V})$. Since we shall use the Lie algebras to characterize the subgroups, our results are necessarily restricted to connected continuous Abelian subgroups only. We define first the tentative notion of a *maximal Abelian subset* $\mathscr{H}$ of $\mathscr{GL}(\mathscr{V})$. The subset $\mathscr{H}$ is required to satisfy the following two conditions:

*(i)* Any pair of elements $\mathbf{X}$ and $\mathbf{Y}$ belonging to $\mathscr{H}$ commute.

*(ii)* $\mathscr{H}$ is not a proper subset of any subset of $\mathscr{GL}(\mathscr{V})$ satisfying condition *(i)*.

**Theorem 67.1.** A maximal Abelian subset is necessarily a subgroup.

The proof is more or less obvious. Clearly, the identity element $\mathbf{I}$ is a member of every maximal Abelian subset. Next, if $\mathbf{X}$ belongs to a certain maximal Abelian subset $\mathscr{H}$, then $\mathbf{X}^{-1}$ also belongs to $\mathscr{H}$. Indeed, $\mathbf{X}\in\mathscr{H}$ means that

$$\mathbf{XY}=\mathbf{YX}, \qquad \mathbf{Y}\in\mathscr{H} \tag{67.1}$$

Multiplying this equation on the left and on the right by $\mathbf{X}^{-1}$, we get

$$\mathbf{YX}^{-1}=\mathbf{X}^{-1}\mathbf{Y}, \quad \mathbf{Y}\in\mathscr{H} \tag{67.2}$$

As a result, $\mathbf{X}^{-1}\in\mathscr{H}$ since $\mathscr{H}$ is maximal. By the same argument we can prove also that $\mathbf{XY}\in\mathscr{H}$ whenever $\mathbf{X}\in\mathscr{H}$ and $\mathbf{Y}\in\mathscr{H}$. Thus, $\mathscr{H}$ is a subgroup of $\mathscr{GL}(\mathscr{V})$.

In view of this theorem and the opening remarks we shall now consider the maximal, connected, continuous, Abelian subgroups of $\mathscr{GL}(\mathscr{V})$. Our first result is the following.

**Theorem 67.2.** The one-parameter groups $\exp(\mathbf{A}t)$ and $\exp(\mathbf{B}t)$ commute if and only if their initial tangents $\mathbf{A}$ and $\mathbf{B}$ commute.

Sufficiency is obvious, since when $\mathbf{AB}=\mathbf{BA}$ the series representations for $\exp(\mathbf{A}t)$ and $\exp(\mathbf{B}t)$ imply directly that $\exp(\mathbf{A}t)\exp(\mathbf{B}t) = \exp(\mathbf{B}t)\exp(\mathbf{A}t)$. In fact, we have

$$\exp(\mathbf{A}t)\exp(\mathbf{B}t)=\exp((\mathbf{A}+\mathbf{B})t)=\exp(\mathbf{B}t)\exp(\mathbf{A}t)$$

in this case. Conversely, if $\exp(\mathbf{A}t)$ and $\exp(\mathbf{B}t)$ commute, then their initial tangents $\mathbf{A}$ and $\mathbf{B}$ must also commute, since we can compare the power series expansions for $\exp(\mathbf{A}t)\exp(\mathbf{B}t)$ and $\exp(\mathbf{B}t)\exp(\mathbf{A}t)$ for sufficiently small $t$. Thus the proposition is proved.

We note here a word of caution: While the assertion

$$\mathbf{AB}=\mathbf{BA}\Rightarrow\exp(\mathbf{A})\exp(\mathbf{B})=\exp(\mathbf{B})\exp(\mathbf{A}) \tag{67.3}$$

is true, its converse is not true in general. This is due to the fact that the exponential map is local diffeomorphism, but globally it may or may not be one-to-one. Thus there exists a nonzero solution $\mathbf{A}$ for the equation

$$\exp(\mathbf{A})=\mathbf{I} \tag{67.4}$$

For example, in the simplest case when $\mathscr{V}$ is a two-dimensional spce, we can check directly from (65.15) that

$$\exp\begin{bmatrix}0 & -\theta\\ 0 & 0\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \tag{67.5}$$

In particular, a possible solution for (67.4) is the matrix

$$\mathbf{A}=\begin{bmatrix}0 & -2\pi\\ 2\pi & 0\end{bmatrix} \tag{67.6}$$

For this solution $\exp(\mathbf{A})$ clearly commutes with $\exp(\mathbf{B})$ for all $\mathbf{B}$, even though $\mathbf{A}$ may or may not commute with $\mathbf{B}$. Thus the converse of (67.3) does not hold in general.

The main result of this section is the following theorem.

**Theorem 67.3.** $\mathscr{H}$ is a maximal connected, continuous Abelian subgroup of $\mathscr{GL}(\mathscr{V})$ if and only if it is the subgroup corresponding to a maximal Abelian subalgebra $\mathscr{h}$ of $\mathscr{gl}(\mathscr{V})$.

Naturally, a *maximal* Abelian subalgebra $\mathscr{h}$ of $\mathscr{gl}(\mathscr{V})$ is defined by the following two conditions:

*(i)* Any pair of elements $\mathbf{A}$ and $\mathbf{B}$ belonging to $\mathscr{h}$ commute, i.e., $[\mathbf{A},\mathbf{B}]=\mathbf{0}$.

*(ii)* $\mathscr{h}$ is not a proper subset of any subalgebra of $\mathscr{gl}(\mathscr{V})$ satisfying condition *(i)*.

To prove the preceding theorem, we need the following lemma.

**Lemma. Let** $\mathscr{H}$ be an arbitrary connected continuous subgroup of $\mathscr{GL}(\mathscr{V})$ and let $\mathscr{N}$ be a neighborhood of $\mathbf{I}$ in $\mathscr{H}$. Then $\mathscr{H}$ is generated by $\mathscr{N}$, i.e., every element $\mathbf{X}$ of $\mathscr{H}$ can be expressed as a product (not unique)

$$\mathbf{X} = \mathbf{Y}_1 \mathbf{Y}_2 \cdots \mathbf{Y}_k \tag{67.7}$$

where $\mathbf{Y}_i$ or $\mathbf{Y}_i^{-1}$ belongs to $\mathscr{N}$. [The number of factors $k$ in the representation (67.7) is arbitrary.]

*Note*. Since $\mathscr{H}$ is a hypersurface in the inner product space $\mathscr{L}(\mathscr{V};\mathscr{V})$, we can define a neighborhood system on $\mathscr{H}$ simply by the intersection of the Euclidean neighborhood system on $\mathscr{L}(\mathscr{V};\mathscr{V})$ with $\mathscr{H}$. The topology defined in this way on $\mathscr{H}$ is called the *induced topology*.

To prove the lemma, let $\mathscr{H}_0$ be the subgroup generated by $\mathscr{N}$. Then $\mathscr{H}_0$ is an open set in $\mathscr{H}$ since from (67.7) every point $\mathbf{X} \in \mathscr{H}_0$ has a neighborhood $\boldsymbol{L}_{\mathbf{X}}(\mathscr{N})$ in $\mathscr{H}$. On the other hand, $\mathscr{H}_0$ is also a closed set in $\mathscr{H}$ because the complement of $\mathscr{H}_0$ in $\mathscr{H}$ is the union of $\boldsymbol{L}_{\mathbf{Y}}(\mathscr{H}_0)$, $\mathbf{Y} \in \mathscr{H}/\mathscr{H}_0$ which are all open sets in $\mathscr{H}$. As a result $\mathscr{H}_0$ must coincide with $\mathscr{H}$ since by hypothesis $\mathscr{H}$ has only one component.

By virtue of the lemma $\mathscr{H}$ is Abelian if and only if $\mathscr{N}$ is an Abelian set. Combining this remark with Theorem 67.2, and using the fact that the exponential map is a local diffeomorphism at the identity element, we can conclude immediately that $\mathscr{H}$ is a maximal connected continuous Abelian subgroup of $\mathscr{GL}(\mathscr{V})$ if and only if $\mathfrak{h}$ is a maximal Abelian Lie subalgebra of $\mathfrak{gl}(\mathscr{V})$. This completes the proof.

It should be noticed that on a connected Abelian continuous subgroup of $\mathscr{GL}(\mathscr{V})$ the Cartan parallelism reduced to a Euclidean parallelism. Indeed, since the Lie bracket vanishes identically on $\mathfrak{h}$, any invariant basis $\{\mathbf{E}_\Gamma\}$ is also the natural basis of a coordinate system $\{X^\Gamma\}$ on $\mathscr{H}$. Further, the coordinate map defined by

$$\mathbf{X} \equiv \exp\left(X^\Gamma \mathbf{E}_\Gamma\right) \tag{67.8}$$

is a homomorphism of the additive group $\mathscr{R}^M$ with the Abelian group $\mathscr{H}$. This coordinate system plays the role of a local Cartesian coordinate system on a neighborhood of the identity element of $\mathscr{H}$. The mapping defined by (67.8) may or may not be one-to-one. In the former case $\mathscr{H}$ is isomorphic to $\mathscr{R}^M$, in the latter case $\mathscr{H}$ is isomorphic to a cylinder or a torus of dimension $M$.

We say that the Cartan parallelism on the Abelian group $\mathscr{H}$ is a Euclidean parallelism because there exists a local Cartesian coordinate system relative to which the Cartan symbols vanish identically. This Euclidean parallelsin on $\mathscr{H}$ should not be confused with the Euclidean parallelism on the underlying inner product space $\mathscr{L}(\mathscr{V};\mathscr{V})$ in which $\mathscr{H}$ is a hypersurface. In genereal, even if $\mathscr{H}$ is Abelian, the tangent spaces at different points of $\mathscr{H}$ are still different subspaces of $\mathscr{L}(\mathscr{V};\mathscr{V})$. Thus the Euclidean parallelism on $\mathscr{H}$ is not the restriction of the Euclidean parallelism of $\mathscr{L}(\mathscr{V};\mathscr{V})$ to $\mathscr{H}$.

An example of a maximal connected Abelian continuous subgroup of $\mathscr{GL}(\mathscr{V})$ is a dilatation group defined as follows: Let $\{\mathbf{e}_i, i=1,\ldots,N\}$ be the basis of $\mathscr{V}$. Then a linear transformation $\mathbf{X}$ of $\mathscr{V}$ is a *dilatation with axes* $\{\mathbf{e}_i\}$ if each $\mathbf{e}_i$ is an eigenvector of $\mathbf{X}$ and the corresponding eigenvalue is positive. In other words, the component matrix of $\mathbf{X}$ relative to $\{\mathbf{e}_i\}$ is a diagonal matrix with positive diagonal components, say

$$\left[X^i_j\right]=\begin{bmatrix}\lambda_1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \lambda_N\end{bmatrix}, \lambda_i>0 \quad i=1,\ldots,N \tag{67.9}$$

where $\lambda_i$ may or may not be distinct. The *dilatation group* with axes $\{\mathbf{e}_i\}$ is the group of all dilatations $\mathbf{X}$. We leave the proof of the fact that a dilatation group is a maximal connected Abelian continuous subgroup of $\mathscr{GL}(\mathscr{V})$ as an exercise.

Dilatation groups are not the only class of maximal Abelian subgroup of $\mathscr{GL}(\mathscr{V})$, of course. For example, when $\mathscr{V}$ is three-dimensional we choose a basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ for $\mathscr{V}$; then the subgroup consisting of all linear transformations having component matrix relative to $\{\mathbf{e}_i\}$ of the form

$$\begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ b & c & a \end{bmatrix}$$

with positive $a$ and arbitrary $b$ and $c$ is also a maximal connected Abelian continuous subgroup of $\mathscr{GL}(\mathscr{V})$. Again, we leave the proof of this fact as an exercise.

For inner product spaces of lower dimensions a complete classification of the Lie subalgebars of the Lie algebra $\mathscr{gl}(\mathscr{V})$ is known. In such cases a corresponding classification of connected continuous subgroups of $\mathscr{GL}(\mathscr{V})$ can be obtained by using the main result of the preceding section. Then the set of maximal connected Abelian continuous subgroups can be determined completely. These results are beyond the scope of this chapter, however.

# Chapter 13

# INTEGRATION OF FIELDS ON EUCLIDEAN MANIFORDS, HYPERSURFACES, AND CONTINUOUS GROUPS

In this chapter we consider the theory of integration of vector and tensor fields defined on various geometric entities introduced in the preceding chapters. We assume that the reader is familiar with the basic notion of the Riemann integral for functions of several real variables. Since we shall restrict our attention to the integration of continuous fields only, we do not need the more general notion of the Lebesgue integral.

## *Section 68. Arc Length, Surface Area, and Volume*

Let $\mathscr{E}$ be a Euclidean maniforld and let $\boldsymbol{\lambda}$ be a smooth curve in $\mathscr{E}$. Then the tangent of $\boldsymbol{\lambda}$ is a vector in the translation space $\mathscr{V}$ of $\mathscr{E}$ defined by

$$\dot{\boldsymbol{\lambda}}(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{\lambda}(t + \Delta t) - \boldsymbol{\lambda}(t)}{\Delta t} \tag{68.1}$$

As usual we denote the norm of $\dot{\boldsymbol{\lambda}}$ by $\|\dot{\boldsymbol{\lambda}}\|$,

$$\|\dot{\boldsymbol{\lambda}}(t)\| = \left[\dot{\boldsymbol{\lambda}}(t) \cdot \dot{\boldsymbol{\lambda}}(t)\right]^{1/2} \tag{68.2}$$

which is a continuous function of $t$, the parameter of $\boldsymbol{\lambda}$. Now suppose that $\boldsymbol{\lambda}$ is defined for $t$ from $a$ to $b$. Then we define the *arc length* of $\boldsymbol{\lambda}$ between $\boldsymbol{\lambda}(a)$ and $\boldsymbol{\lambda}(b)$ by

$$l = \left| \int_a^b \|\dot{\boldsymbol{\lambda}}(t)\| dt \right| \tag{68.3}$$

We claim that the arc length possesses the following properties which justify the definition (68.3).

*(i)* The arc length depends only on the path of $\boldsymbol{\lambda}$ joining $\boldsymbol{\lambda}(a)$ and $\boldsymbol{\lambda}(b)$, independent of the choice of parameterization on the path.

Indeed, if the path is parameterized by $\bar{t}$ so that

$$\boldsymbol{\lambda}(t) = \bar{\boldsymbol{\lambda}}(\bar{t}) \tag{68.4}$$

with

$$\bar{t} = \bar{t}(t) \tag{68.5}$$

then the tangent vectors $\dot{\boldsymbol{\lambda}}$ *and* $\dot{\bar{\boldsymbol{\lambda}}}$ are related by

$$\dot{\boldsymbol{\lambda}}(t) = \dot{\bar{\boldsymbol{\lambda}}}(\bar{t})\frac{d\bar{t}}{dt} \tag{68.6}$$

As a result, we have

$$\left|\int_a^b \left\|\dot{\boldsymbol{\lambda}}(t)\right\| dt\right| = \left|\int_{\bar{a}}^{\bar{b}} \left\|\dot{\boldsymbol{\lambda}}(t)\right\| d\bar{t}\right| \tag{68.7}$$

which proves property (i).

*(ii)* When the path joining $\boldsymbol{\lambda}(a)$ and $\boldsymbol{\lambda}(b)$ is a straight line segment, the arc length is given by

$$l = \left\|\boldsymbol{\lambda}(b) - \boldsymbol{\lambda}(a)\right\| \tag{68.8}$$

This property can be verified by using the parameterization

$$\bar{\boldsymbol{\lambda}}(t) = \left(\boldsymbol{\lambda}(b) - \boldsymbol{\lambda}(a)\right)\bar{t} + \boldsymbol{\lambda}(a) \tag{68.9}$$

where $\bar{t}$ ranges from 0 to 1 since $\bar{\boldsymbol{\lambda}}(0) = \boldsymbol{\lambda}(a)$ and $\bar{\boldsymbol{\lambda}}(1) = \boldsymbol{\lambda}(b)$. In view of (68.7), $l$ is given by

$$l = \left|\int_0^1 \left\|\boldsymbol{\lambda}(b) - \boldsymbol{\lambda}(a)\right\| d\bar{t}\right| = \left\|\boldsymbol{\lambda}(b) - \boldsymbol{\lambda}(a)\right\| \tag{68.10}$$

*(iii)* The arc length integral is additive, i.e., the sum of the arc lengths from $\boldsymbol{\lambda}(a)$ to $\boldsymbol{\lambda}(b)$ and from $\boldsymbol{\lambda}(b)$ to $\boldsymbol{\lambda}(c)$ is equal to the arc length from $\boldsymbol{\lambda}(a)$ to $\boldsymbol{\lambda}(c)$.

Now using property *(i)*, we can parameterize the path of $\boldsymbol{\lambda}$ by the arc length relative to a certain reference point on the path. As usual, we assume that the path is oriented. Then we assign a positive parameter $s$ to a point on the positive side and a negative parameter $s$ to a point on the negative side of the reference point, the absolute value $|s|$ being the arc length between the point and the reference point. Hence when the parameter $t$ is positively oriented, the arc length parameter $s$ is related to $t$ by

$$s = s(t) = \int_0^1 \|\dot{\boldsymbol{\lambda}}(t)\| dt \tag{68.11}$$

where $\boldsymbol{\lambda}(0)$ is chosen as the reference point. From (68.11) we get

$$ds/dt = \|\dot{\boldsymbol{\lambda}}(t)\| \tag{68.12}$$

Substituting this formula into the general transformation rule (68.6), we see that the tangent vector relative to $s$ is a unit vector pointing in the positive direction of the path, as it should be.

Having defined the concept of arc length, we consider next the concept of surface area. For simplicity we begin with the area of a two-dimensional smooth surface $\mathscr{S}$ in $\mathscr{E}$. As usual, we can characterize $\mathscr{S}$ in terms of a pair of parameters $(u^\Gamma, \Gamma = 1,2)$ which form a local coordinate system on $\mathscr{S}$

$$\mathbf{x} \in \mathscr{S} \Leftrightarrow \mathbf{x} = \boldsymbol{\zeta}(u^1, u^2) \tag{68.13}$$

where $\boldsymbol{\zeta}$ is a smooth mapping. We denote the tangent vector of the coordinate curves by

$$\mathbf{h}_\Gamma \equiv \partial\boldsymbol{\zeta}/\partial u^\Gamma, \qquad \Gamma = 1,2 \tag{68.14}$$

Then $\{\mathbf{h}_\Gamma\}$ is a basis of the tangent plane $\mathscr{S}_{\mathbf{x}}$ of $\mathscr{S}$ at any $\mathbf{x}$ given by (68.13). We assume that $\mathscr{S}$ is oriented and that $(u^\Gamma)$ is a positive coordinate system. Thus $\{\mathbf{h}_\Gamma\}$ is also positive for $\mathscr{S}_{\mathbf{x}}$.

Now let $\mathscr{U}$ be a domain in $\mathscr{S}$ with piecewise smooth boundary. We consider first the simple case when $\mathscr{U}$ can be covered entirely by the coordinate system $(u^\Gamma)$. Then we define the *surface area* of $\mathscr{U}$ by

$$\sigma = \left| \iint_{\boldsymbol{\zeta}^{-1}(\mathscr{U})} e(u^1, u^2)\, du^1\, du^2 \right| \tag{68.15}$$

where $e(u^1, u^2)$ is defined by

$$e \equiv \sqrt{a =} \left(\det[a_{\Gamma\Delta}]\right)^{1/2} = \left(\det[\mathbf{h}_\Gamma \cdot \mathbf{h}_\Delta]\right)^{1/2} \tag{68.16}$$

The double integral in (68.15) is taken over $\boldsymbol{\zeta}^{-1}(\mathscr{U})$, which denotes the set of coordinates $(u^\Gamma)$ for points belonging to $\mathscr{U}$.

By essentially the same argument as before, we can prove that the surface area has the following properties which justify the definition (68.15).

*(iv)* The surface area depends only on the domain $\mathscr{U}$, independent of the choice of parameterization on $\mathscr{U}$.

To prove this, we note that under a change of surface coordinates the integrand $e$ of (68.15) obeys the transformation rule [cf. (61.4)]

$$e = \bar{e}\left|\det\left[\frac{\partial \bar{u}^\Gamma}{\partial u^\Delta}\right]\right| \tag{68.17}$$

As a result, we have

$$\left|\iint_{\zeta^{-1}(\mathscr{U})} e\left(u^1,u^2\right) du^1\, du^2\right| = \left|\iint_{\zeta^{-1}(\mathscr{U})} \bar{e}\left(\bar{u}^1,\bar{u}^2\right) d\bar{u}^1\, d\bar{u}^2\right| \tag{68.18}$$

which proves property *(iv).*

*(v)* When $\mathscr{S}$ is a plane and $\mathscr{U}$ is a square spanned by the vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ at the point $\mathbf{x}_0 \in \mathscr{S}$, the surface area of $\mathscr{U}$ is

$$\sigma = \|\mathbf{h}_1\|\|\mathbf{h}_2\| \tag{68.19}$$

The proof is essentially the same as before. We use the parameterization

$$\boldsymbol{\zeta}\left(u^1,u^2\right) = \mathbf{x}_0 + u^\Gamma \mathbf{h}_\Gamma \tag{68.20}$$

From (68.16), $e$ is a constant

$$e = \|\mathbf{h}_1\|\|\mathbf{h}_2\| \tag{68.21}$$

and from (68.20), $\boldsymbol{\zeta}^{-1}(\mathscr{U})$ is the square $[0,1]\times[0,1]$. Hence by (68.15) we have

$$\sigma = \int_0^1 \int_0^1 \|\mathbf{h}_1\|\|\mathbf{h}_2\|\, du^1\, du^2 = \|\mathbf{h}_1\|\|\mathbf{h}_2\| \tag{68.22}$$

*(vi)* The surface area integral is additive in the same sence as *(iii).*

Like the arc length parameter $s$ on a path, a local coordinate system $\left(\bar{u}^\Gamma\right)$ on $\mathscr{S}$ is called an *isochoric coordinate* system if the surface area density $\bar{e}\left(\bar{u}^1,\bar{u}^2\right)$ is identical to 1 for all $\left(\bar{u}^\Gamma\right)$. We can define an isochoric coordinate system $\left(\bar{u}^\Gamma\right)$ in terms of an arbitrary surface coordinate system $\left(u^\Gamma\right)$ in the following way. We put

$$\bar{u}^1 = \bar{u}^1\left(u^1,u^2\right) \equiv u^1 \tag{68.23}$$

and

$$\bar{u}^2 = \bar{u}^2\left(u^1,u^2\right) \equiv \int_0^{u^2} e\left(u^1,t\right)dt \tag{68.24}$$

where we have assumed that the origin $(0,0)$ is a point in the domain of the coordinate system $\left(u^\Gamma\right)$. From (68.23) and (68.24) we see that

$$\frac{\partial \bar{u}^1}{\partial u^1} = 1, \quad \frac{\partial \bar{u}^1}{\partial u^2} = 0, \quad \frac{\partial \bar{u}^2}{\partial u^2} = e\left(u^1,u^2\right) \tag{68.25}$$

As a result, the Jacobian of the coordinate transformation is

$$\det\left[\frac{\partial \bar{u}^\Gamma}{\partial u^\Delta}\right] = e\left(u^1,u^2\right) \tag{68.26}$$

which implies immediately the desired result:

$$\bar{e}\left(\bar{u}^1,\bar{u}^2\right) = 1 \tag{68.27}$$

by virtue of (68.17). From (68.26) the coordinate system $\left(u^\Gamma\right)$ and $\left(\bar{u}^\Gamma\right)$ are of the same orientation. Hence if $\left(u^\Gamma\right)$ is positively oriented, then $\left(\bar{u}^\Gamma\right)$ is a positive isochoric coordinate system on $\mathscr{S}$.

An isochoric coordinate system $\mathbf{x} = \bar{\boldsymbol{\zeta}}\left(\bar{u}^\Gamma\right)$ in corresponds to an isochoric mapping $\bar{\boldsymbol{\zeta}}$ from a domain in $\mathscr{R}^2$ onto the coordinate neighborhood of $\bar{\boldsymbol{\zeta}}$ in $\mathscr{S}$. In general, $\bar{\boldsymbol{\zeta}}$ is not isometric, so that the surface metric $a_{\Gamma\Delta}$ relative to $\left(\bar{u}^\Gamma\right)$ need not be a Euclidean metric. In fact, the surface metric is Euclidean if and only if $\mathscr{S}$ is developable. Hence an isometric coordinate system (i.e., a rectangular Cartesian coordinate system) generally does not exist on an arbitrary surface $\mathscr{S}$. But the preceding proof shows that isochoric coordinate systems exist on all $\mathscr{S}$.

So far, we have defined the surface area for any domain $\mathscr{U}$ which can be covered by a single surface coordinate system. Now suppose that $\mathscr{U}$ is not homeomorphic to a domain in $\mathscr{R}^2$. Then we decompose $\mathscr{U}$ into a collection of subdomains, say

$$\mathscr{U} = \mathscr{U}_1 \cup \mathscr{U}_2 \cup \cdots \cup \mathscr{U}_K \tag{68.28}$$

whee the interiors of $\mathscr{U}_1, \ldots, \mathscr{U}_K$ are mutually disjoint. We assume that each $\mathscr{U}_a$ can be covered by a surface coordinate system so that the surface area $\sigma(\mathscr{U}_a)$ is defined. Then we define $\sigma(\mathscr{U})$ naturally by

$$\sigma(\mathscr{U}) = \sigma(\mathscr{U}_1) + \cdots + \sigma(\mathscr{U}_K) \tag{68.29}$$

While the decomposition (68.28) is not unique, of course, by the additive property *(vi)* of the integral we can verify easily that $\sigma(\mathscr{U})$ is independent of the decomposition. Thus the surface area is well defined.

Having considered the concepts of arc length and surface area in detail, we can now extend theidea to hypersurfaces in general. Specifically, let $\mathscr{S}$ be a hupersurface of dimension *M*. Then locally $\mathscr{S}$ can be represented by

$$\mathbf{x} \in \mathscr{S} \Leftrightarrow \mathbf{x} = \boldsymbol{\zeta}\left(u^1, \ldots u^M\right) \tag{68.30}$$

where $\boldsymbol{\zeta}$ is a smooth mapping. We define

$$\mathbf{h}_\Gamma \equiv \partial \boldsymbol{\zeta} / \partial u^\Gamma, \qquad \Gamma = 1, \ldots, M \tag{68.31}$$

and

$$a_{\Gamma\Delta} \equiv \mathbf{h}_\Gamma \cdot \mathbf{h}_\Delta \tag{68.32}$$

Then the $\{\mathbf{h}_\Gamma\}$ span the tangent space $\mathscr{S}_{\mathbf{x}}$ and the $a_{\Gamma\Delta}$ define the induced metric on $\mathscr{S}_{\mathbf{x}}$. We define the surface area density $e$ by the same formula (68.16) except that $e$ is now a smooth function of the *M* variables $\left(u^1, \ldots u^M\right)$, and the matrix $\left[a_{\Gamma\Delta}\right]$ is also $M \times M$.

Now let $\mathscr{U}$ be a domain with piecewise smooth boundary in $\mathscr{S}$, and assume that $\mathscr{U}$ can be covered by a single surface coordinate system. Then we define the surface area of $\mathscr{U}$ by

$$\sigma = \left| \int_{\zeta^{-1}(\mathscr{U})} \cdots \int e\left(u^1,\ldots,u^M\right) du^1 \cdots du^M \right| \tag{68.33}$$

By the same argument as before, $\sigma$ has the following two properties.

*(vii)* The surface area is additive and independent of the choice of surface coordinate system.

*(viii)* The surface area of an *M*-dimensional cube with sides $\mathbf{h}_1,\ldots,\mathbf{h}_M$ is

$$\sigma = \|\mathbf{h}_1\| \cdots \|\mathbf{h}_M\| \tag{68.34}$$

More generally, when $\mathscr{U}$ cannot be covered by a single coordinate system, we decompose $\mathscr{U}$ by (68.28) and define $\sigma(\mathscr{U})$ by (68.29).

We can extend the notion of an isochoric coordinate system to a hypersurface in general. To construct an isochoric coordinate system $\left(\bar{u}^\Gamma\right)$, we begin with an arbitrary coordinate system $\left(u^\Gamma\right)$. Then we put

$$\bar{u}^\Gamma = \bar{u}^\Gamma\left(u^1,\ldots,u^M\right) = u^\Gamma, \qquad \Gamma = 1,\ldots,M-1 \tag{68.35}$$

$$\bar{u}^M = \bar{u}^M\left(u^1,\ldots,u^M\right) \equiv \int_0^{u^M} e\left(u^1,\ldots,u^{M-1},t\right) dt \tag{68.36}$$

From (68.35) and (68.36) we get

$$\det\left[\frac{\partial \bar{u}^\Gamma}{\partial u^\Delta}\right] = e\left(u^1,\ldots u^M\right) \tag{68.37}$$

As a result, the coordinate system $\left(\bar{u}^\Gamma\right)$ is isochoric since

$$\bar{e}\left(\bar{u}^1,\ldots,\bar{u}^M\right) = e\left(u^1,\ldots,u^M\right)\frac{1}{\left|\det\left[\partial \bar{u}^\Gamma / \partial u^\Delta\right]\right|} = 1 \tag{68.38}$$

Finally, when $M = N$, $\mathscr{S}$ is nothing but a domain in $\mathscr{E}$. In this case $\left(u^1,\ldots,u^N\right)$ becomes an arbitrary local coordinate system in $\mathscr{E}$, and $e\left(u^1,\ldots,u^N\right)$ is just the Euclidean volume relative to $\left(u^i\right)$. The integral

$$\upsilon \equiv \int_{\zeta^{-1}(\mathscr{U})} \cdots \int e\left(u^1,\ldots,u^M\right) du^1 \cdots du^M \tag{68.39}$$

now defines the Euclidean volume of the domain $\mathscr{U}$.

## Section 69. Integration of Vector Fields and Tensor Fields

In the preceding section we have defined the concept of surface area for an arbitrary $M$-dimensional hyper surface imbedded in an $N$-dimensional Euclidean manifold $\mathscr{E}$. When $M=1$, the surface reduces to a path and the surface area becomes the arc length, while in the case $M=N$ the surface corresponds to a domain in $\mathscr{E}$, and the surface area becomes the volume. In this section we shall define the integrals of various fields relative to the surface area of an arbitrary hyper surface $\mathscr{S}$. We begin with the integral of a continuous function $f$ defined on $\mathscr{S}$.

As before, we assume that $\mathscr{S}$ is oriented and $\mathscr{U}$ is a domain in $\mathscr{S}$ with piecewise smoothboundary. We consider first the simple case when $\mathscr{U}$ can be covered by a single surface coordinate system $\mathbf{x}=\boldsymbol{\zeta}\left(u^1,\ldots u^M\right)$. We choose $\left(u^\Gamma\right)$ to be positively oriented, of course. Under these assumptions, we define the *integral* of $f$ on $\mathscr{U}$ by

$$\int_{\mathscr{U}} f\,\mathrm{d}\sigma \equiv \int \underset{\zeta^{-1}(\mathscr{U})}{\cdots} \int fe\,du^1\cdots du^M \tag{69.1}$$

where the function $f$ on the right-hand side denotes the representation of $f$ in terms of the surface coordinates $\left(u^\Gamma\right)$:

$$f\left(\mathbf{x}\right)=f\left(u^1,\ldots,u^M\right) \tag{69.2}$$

where

$$\mathbf{x}=\boldsymbol{\zeta}\left(u^1,\ldots,u^M\right) \tag{69.3}$$

It is understood that the multiple integral in (69.1) is taken over the positive orientation on $\boldsymbol{\zeta}^{-1}\left(\mathscr{U}\right)$ in $\mathscr{R}^M$.

By the same argument as in the preceding section, we see that the integral possesses the following properties.

*(i)* When $f$ is identical to 1 the integral of $f$ is just the surface area of $\mathscr{U}$, namely

$$\sigma\left(\mathscr{U}\right)=\int_{\mathscr{U}} d\sigma \tag{69.4}$$

*(ii)* The integral of $f$ is independent on the choice of the coordinate system $\left(u^\Gamma\right)$ and is additive with respect to its domain.

*(iii)* The integral is a linear function of the integrand in the sense that

$$\int_{\mathscr{U}} (\alpha_1 f_1 + \alpha_2 f_2)\, d\sigma = \alpha_1 \int_{\mathscr{U}} f_1\, d\sigma + \alpha_2 \int_{\mathscr{U}} f_2\, d\sigma \tag{69.5}$$

for all constants $\alpha_1$ and $\alpha_2$. Also, the integral is bounded by

$$\sigma(\mathscr{U}) \min_{\mathscr{U}} f \le \int_{\mathscr{U}} f\, d\sigma \le \sigma(\mathscr{U}) \max_{\mathscr{U}} f \tag{69.6}$$

where the extrema of $f$ are taken over the domain $\mathscr{U}$.

Property *(iii)* is a standard result of multiple integrals in calculus, so by virtue of the definition (69.1) the same is valid for the integral of $f$.

As before, if $\mathscr{U}$ cannot be covered by a single coordinate system, then we decompose $\mathscr{U}$ by (68.28) and define the integral of $f$ over $\mathscr{U}$ by

$$\int_{\mathscr{U}} f\, d\sigma = \int_{\mathscr{U}_1} f\, d\sigma + \cdots + \int_{\mathscr{U}K} f\, d\sigma \tag{69.7}$$

By property *(ii)* we can verify easily that the integral is independent of the decomposition.

Having defined the integral of a scalar field, we define next the integral of a vector field. Let $\mathbf{v}$ be a continuous vector field on $\mathscr{S}$, i.e.,

$$\mathbf{v} : \mathscr{S} \to \mathscr{V} \tag{69.8}$$

where $\mathscr{V}$ is the translation space of the underlying Euclidean manifold $\mathscr{E}$. Generally the values of $\mathbf{v}$ may or may not be tangent to $\mathscr{S}$. We choose an arbitrary Cartesian coordinate system with natural basis $\{\mathbf{e}_i\}$. Then $\mathbf{v}$ can be represented by

$$\mathbf{v}(\mathbf{x}) = \upsilon^i(\mathbf{x})\mathbf{e}_i \tag{69.9}$$

where $\upsilon^1(\mathbf{x}), i = 1, \ldots, N,$ are continuous scalar fields on $\mathscr{S}$. We define the integral of $\mathbf{v}$ by

$$\int_{\mathscr{U}} \mathbf{v}\, d\sigma \equiv \left(\int_{\mathscr{U}} \upsilon^i\, d\sigma\right)\mathbf{e}_i \tag{69.10}$$

Clearly the integral is independent of the choice of the basis $\{\mathbf{e}_i\}$.

More generally if $\mathbf{A}$ is a tensor field on $\mathscr{S}$ having the representation

$$\mathbf{A}(\mathbf{x}) = A^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}(\mathbf{x})\,\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_r} \otimes \mathbf{e}^{j_1} \otimes \cdots \otimes \mathbf{e}^{j_s} \tag{69.11}$$

where $\{\mathbf{e}^i\}$ denotes the reciprocal basis of $\{\mathbf{e}_i\}$, then we define

$$\int_{\mathscr{U}} \mathbf{A}\, d\sigma \equiv \left(\int_{\mathscr{U}} A^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}\, d\sigma\right)\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}^{j_s} \tag{69.12}$$

Again the integral is independent of the choice of the basis $\{\mathbf{e}_i\}$.

The integrals defined by (69.10) and (69.12) possess the same tensorial order as the integrand. The fact that a Cartesian coordinate system is used in (69.10) and (69.12) reflects clearly the crucial dependence of the integral on the Euclidean parallelism of $\mathscr{E}$. Without the Euclidean parallelism it is generally impossible to add vectors or tensors at different points of the domain. Then an integral is also meaningless. For example, if we suppress the Euclidean parallelism on the underlying Euclidean manifold $\mathscr{E}$, then the tangential vectors or tensors at different points of a hyper surface $\mathscr{S}$ generally do not belong to the same tangent space or tensor space. As a result, it is generally impossible to "sum" the values of a tangential field to obtain an integral without the use of some kind of path-independent parallelism. The Euclidean parallelism is just one example of such parallelisms. Another example is the Cartan parallelism on a continuous group defined in the preceding chapter. We shall consider integrals relative to the Cartan parallelism in Section 72.

In view of (69.10) and (69.12) we see that the integral of a vector field or a tensor field possesses the following properties.

(*iv*) The integral is linear with respect to the integrand.
(*v*) The integral is bounded by

$$\left\|\int_{\mathscr{U}} \mathbf{v}\, d\sigma\right\| \le \int_{\mathscr{U}} \|\mathbf{v}\|\, d\sigma \tag{69.13}$$

and similarly

$$\left\|\int_{\mathscr{U}} \mathbf{A}\, d\sigma\right\| \le \int_{\mathscr{U}} \|\mathbf{A}\|\, d\sigma \tag{69.14}$$

where the norm of a vector or a tensor is defined as usual by the inner product of $\mathscr{V}$. Then it follows from (69.6) that

$$\left\|\int_{\mathscr{U}} \mathbf{v}\, d\sigma\right\| \le \sigma(\mathscr{U}) \max_{\mathscr{U}} \|\mathbf{v}\| \tag{69.15}$$

and

$$\left\| \int_{\mathscr{U}} \mathsf{A}\, d\sigma \right\| \le \sigma(\mathscr{U}) \max_{\mathscr{U}} \|\mathsf{A}\| \tag{69.16}$$

However, it does not follow from (69.6), and in fact it is not true, that $\sigma(\mathscr{U}) \min_{\mathscr{U}} \|\mathbf{v}\|$ is a lower bound for thenorm of the integral of $\mathbf{v}$.

## *Section 70. Integration of Differential Forms*

Integration with respect to the surface area density of a hyper surface is a special case of a more general integration of differential forms. As remarked in Section 68, the transformation rule (68.17) of the surface area density is the basic condition which implies the important property that the surface area integral is independent of the choice of the surface coordinate system. Since the transformation rule (68.17) is essentially the same as that of the strict components of certain differential forms, we can extend the operation of integration to those forms also. This extension is the main result of this section.

We begin with the simple notion of a differential *N*-form $\mathsf{Z}$ on $\mathscr{E}$. By definition, $\mathsf{Z}$ is a completely skew-symmetric covariant tensor field of order *N*. Thus relative to any coordinate system $\left(u^i\right)$, $\mathsf{Z}$ has the representation

$$\begin{aligned}\mathsf{Z} = Z_{i_1\ldots i_N}\mathbf{h}^{i_1}\otimes\cdots\otimes\mathbf{h}^{i_N} &= Z_{1\ldots N}\mathbf{h}^1\wedge\cdots\wedge\mathbf{h}^N\\ &= z\,\mathbf{h}^1\wedge\cdots\wedge\mathbf{h}^N\end{aligned} \tag{70.1}$$

where $z$ is called the *relative scalar* or the *density* of $\mathsf{Z}$. As we have shown in Section 39, the transformation rule for $z$ is

$$z = \overline{z}\det\left[\frac{\partial\overline{u}^i}{\partial u^j}\right] \tag{70.2}$$

This formula is comparable to (68.17). In fact if we require that $\left(u^i\right)$ and $\left(\overline{u}^i\right)$ both be positively oriented, then (70.2) can be regarded as a special case of (68.17) with $M = N$. As a result, we can define the integral of $\mathsf{Z}$ over a domain $\mathscr{U}$ in $\mathscr{E}$ by

$$\int_{\mathscr{U}}\mathsf{Z} \equiv \int\cdots_{\zeta^{-1}(\mathscr{U})}\int z\left(u^1,\ldots,u^N\right)du^1\cdots du^N \tag{70.3}$$

and the integral is independent of the choice of the (positive) coordinate system $\mathbf{x} = \zeta\left(u^i\right)$.

Notice that in this definition the Euclidean metric and the Euclidean volume density $e$ are not used at all. In fact, (68.39) can be regarded as a special case of (70.3) when $\mathsf{Z}$ reduces to the Euclidean volume tensor

$$\mathsf{E} = e\,\mathbf{h}^1\wedge\cdots\wedge\mathbf{h}^N \tag{70.4}$$

Here we have assumed that the coordinate system is positively oriented; otherwise, a negative sign should be inserted on the right hand side since the volume density $e$ as defined by (68.16) is always positive. Hence, unlike the volume integral, the integral of a differential $N$-form $\mathbf{Z}$ is defined only if the underlying space $\mathscr{E}$ is oriented. Other than this aspect, the integral of $\mathbf{Z}$ and the volume integral have essentially the same properties since they both are defined by an invariant $N$-tuple integral over the coordinates.

Now more generally let $\mathscr{S}$ be an oriented hypersurface in $\mathscr{E}$ of dimension $M$, and suppose $\mathbf{Z}$ is a tangential differential $M$-form on $\mathscr{S}$. As before, we choose a positive surface coordinate system $\left(u^\Gamma\right)$ on $\mathscr{S}$ and represent $\mathbf{Z}$ by

$$\mathbf{Z} = z\,\mathbf{h}^1 \wedge \cdots \wedge \mathbf{h}^M \tag{70.5}$$

where $z$ is a function of $\left(u^1,\ldots,u^M\right)$ where $\left\{\mathbf{h}^\Gamma\right\}$ is the natural basis reciprocal to $\left\{\mathbf{h}^\Gamma\right\}$. Then the transformation rule for $z$ is

$$z = \overline{z}\det\left[\frac{\partial \overline{u}^\Gamma}{\partial u^\Delta}\right] \tag{70.6}$$

As a result, we can define the integral of $\mathbf{Z}$ over a domain $\mathscr{U}$ in $\mathscr{S}$ by

$$\int_{\mathscr{U}} \mathbf{Z} \equiv \int \underset{\zeta^{-1}(\mathscr{U})}{\cdots} \int z\left(u^1,\ldots,u^M\right) du^1 \cdots du^M \tag{70.7}$$

and the integral is independentof the choice of the positive surface coordinate system $\left(u^\Gamma\right)$. By the same remark as before, we can regard (70.7) as a generalization of (68.33).

The definition (70.7) is valid for any tangential $M$-form $\mathbf{Z}$ on $\mathscr{S}$. In this definition the surface metric and thesurface area density are not used. The fact that $\mathbf{Z}$ is a tangential field on $\mathscr{S}$ is not essential in the definition. Indeed, if $\mathbf{Z}$ is an arbitrary skew-symmetric spatial covariant tensorof order $M$ on $\mathscr{S}$, then we define the density of $\mathbf{Z}$ on $\mathscr{S}$ relative to $\left(u^\Gamma\right)$ simply by

$$z = \mathbf{Z}\left(\mathbf{h}_1,\ldots,\mathbf{h}_M\right) \tag{70.8}$$

Using this density, we define the integral of $\mathbf{Z}$ again by (70.7). Of course, the formula (70.8) is valid for a tangential $M$-form $\mathbf{Z}$ also, since it merely represents the strict component of the tangential projection of $\mathbf{Z}$.

This remark can be further generalized in the following situation: Suppose that $\mathscr{S}$ is a hyper surface contained in another hypersurface $\mathscr{S}_0$ in $\mathscr{E}$, and let $\mathbf{Z}$ be a tangential $M$-form on

$\mathscr{S}_0$. Then $\mathsf{Z}$ gives rise to a density on $\mathscr{S}$ by the same formula (70.8), and the integral of $\mathsf{Z}$ over any domain $\mathscr{U}$ in $\mathscr{S}$ is defined by (70.7). Algebraically, this remark is a consequence of the simple fact that a skew-symmetric tensor over the tangent space of $\mathscr{S}_0$ gives rise to a unique skew-symmetric tensor over the tangent space of $\mathscr{S}$, since the latter tangent space is a subspace of the former one.

It should be noted, however, that the integral of $\mathsf{Z}$ is defined over $\mathscr{S}$ only if the order of $\mathsf{Z}$ coincides with the dimension of $\mathscr{S}$. Further, the value of theintegral is always a scalar, not a vector or a tensor as in the preceding section. We can regard the intetgral of a vector field or a tensor field as a special case of the integral of a differential form only when the fields are represented interms of their Cartesian components as shown in (69.9) and (69.11).

An important special case of the integral of a differential form is the *line integral* in classical vector analysis. In this case $\mathscr{S}$ reduces to an oriented path $\boldsymbol{\lambda}$, and $\mathsf{Z}$ is a 1-form $\mathbf{w}$. When the Euclidean metric on $\mathscr{E}$ is used, $\mathbf{w}$ corresponds simply to a (spatial or tangential) vector field on $\boldsymbol{\lambda}$. Now using any positive parameter $t$ on $\boldsymbol{\lambda}$, we obtain from (70.7)

$$\int_{\boldsymbol{\lambda}} \mathbf{w} = \int_a^b \mathbf{w}(t) \cdot \dot{\boldsymbol{\lambda}}(t)\, dt \tag{70.9}$$

Here we have used the fact that for an inner product space the isomorphism of a vector and a covector is given by

$$\langle \mathbf{w}, \dot{\boldsymbol{\lambda}} \rangle = \mathbf{w} \cdot \dot{\boldsymbol{\lambda}} \tag{70.10}$$

The tangent vector $\dot{\boldsymbol{\lambda}}$ playes the role of the natural basis vector $\mathbf{h}_1$ associated with the parameter *t*, and (70.10) is just the special case of (70.8) when $M = 1$.

The reader should verify directly that the right-hand side of (70.9) is independent of the choice of the (positive) parameterization $t$ on $\boldsymbol{\lambda}$. By virtue of this remark, (70.9) is also written as

$$\int_{\boldsymbol{\lambda}} \mathbf{w} = \int_{\boldsymbol{\lambda}} \mathbf{w} \cdot d\,\boldsymbol{\lambda} \tag{70.11}$$

in the classical theory.

Similarly when $N = 3$ and $M = 2$, a 2-form $\mathsf{Z}$ is also representable by a vector field $\mathbf{w}$, namely

$$\mathsf{Z}(\mathbf{h}_1, \mathbf{h}_2) = \mathbf{w} \cdot (\mathbf{h}_1 \times \mathbf{h}_2) \tag{70.12}$$

Then the integral of $\mathbf{Z}$ over a domain $\mathscr{U}$ in a two-dimensional oriented surface $\mathscr{S}$ is given by

$$\int_{\mathscr{U}} \mathbf{Z} = \iint_{\zeta^{-1}(\mathscr{U})} \mathbf{w} \cdot \left(\mathbf{h}_1 \times \mathbf{h}_2\right) du^1 du^2 \equiv \int_{\mathscr{U}} \mathbf{w} \cdot d\sigma \tag{70.13}$$

where $d\boldsymbol{\sigma}$ is the positive area element of $\mathscr{S}$ defined by

$$d\boldsymbol{\sigma} \equiv \left(\mathbf{h}_1 \times \mathbf{h}_2\right) du^1 du^2 \tag{70.14}$$

The reader will verify easily that the right-hand side of (70.14) can be rewritten as

$$d\boldsymbol{\sigma} = e\mathbf{n}\, du^1\, du^2 \tag{70.15}$$

where $e$ is the surface area density on $\mathscr{S}$ defined by (68.16), and where $\mathbf{n}$ is the positive unit normal of $\mathscr{S}$ defined by

$$\mathbf{n} = \frac{\mathbf{h}_1 \times \mathbf{h}_2}{\left\|\mathbf{h}_1 \times \mathbf{h}_2\right\|} = \frac{1}{e}\mathbf{h}_1 \times \mathbf{h}_2 \tag{70.16}$$

Substituting (70.15) into (70.13), we see that the integral of $\mathbf{Z}$ can be represented by

$$\int_{\mathscr{U}} \mathbf{Z} = \iint_{\zeta^{-1}(\mathscr{U})} \left(\mathbf{w} \cdot \mathbf{n}\right) e\, du^1 du^2 \tag{70.17}$$

which shows clearly that the integral is independent of the choice of the (positive) surface coordinate system $\left(u^\Gamma\right)$.

Since the multipleof an $M$-form $\mathbf{Z}$ by a scalar field $f$ remains an $M$-form, we can define the integral of $f$ with respect to $\mathbf{Z}$ simply as the integral of $f\mathbf{Z}$. Using a Cartesian coordinate representation, we can extend this operation to integrals of a vector field or a tensor field relative to a differential form. The integrals defined in the preceding section are special cases of this general operation when the differential forms are the Euclidean surface area densities induced bythe Euclidean metric on the underlying space $\mathscr{E}$.

## *Section 71. Generalized Stokes' Theorem*

In Section 51 we have defined the operation of exterior derivative on differential forms on $\mathscr{E}$. Since this operation does not depend on the Euclidean metric and the Euclidean parallelism, it can be defined also for tangential differential forms on a hyper surface, as we have remarked in Section 55. Specifically, if $\mathbf{Z}$ is a *K*-form on an *M*-dimensional hypersurface $\mathscr{S}$, we choose a surface coordinate system $\left(u^\Gamma\right)$ and represent $\mathbf{Z}$ by

$$\mathbf{Z} = \sum_{\Gamma_1 < \cdots < \Gamma_K} Z_{\Gamma_1 \cdots \Gamma_K} \mathbf{h}^{\Gamma_1} \wedge \cdots \wedge \mathbf{h}^{\Gamma_K} \tag{71.1}$$

then the exterior derivative $d\mathbf{Z}$ of $\mathbf{Z}$ is a $(K+1)-$form given by

$$d\mathbf{Z} = \sum_{\Gamma_1 < \cdots < \Gamma_K} dZ_{\Gamma_1 \cdots \Gamma_K} \wedge \mathbf{h}^{\Gamma_1} \wedge \cdots \wedge \mathbf{h}^{\Gamma_K} \tag{71.2}$$

where $dZ_{\Gamma_1 \cdots \Gamma_K}$ is defined by

$$dZ_{\Gamma_1 \cdots \Gamma_K} \equiv \frac{\partial Z_{\Gamma_1 \cdots \Gamma_K}}{\partial u^\Delta} \mathbf{h}^\Delta \tag{71.3}$$

In this section we shall establish a general result which connects the integral of $d\mathbf{Z}$ over a $(K+1)-$dimensional domain $\mathscr{U}$ in $\mathscr{S}$ with the integral of $\mathbf{Z}$ over the *K*-dimensional boundary surface $\partial\mathscr{U}$ of $\mathscr{U}$. We begin with a preliminary lemma about a basic property of the exterior derivative.

**Lemma**. Let $\mathscr{S}_0$ be a *P*-dimensional hypersurface in $\mathscr{E}$ with local coordinate system $\mathbf{x} = \boldsymbol{\eta}\left(y^\alpha\right)$ and suppose that $\mathscr{S}$ is an *M*-dimensional hypersurface contained in $\mathscr{S}_0$ and characterized by the representation

$$\boldsymbol{\eta}^{-1}\left(y^\alpha\right) = \mathbf{x} \in \mathscr{S} \Leftrightarrow y^\alpha = y^\alpha\left(u^\Gamma\right), \quad \alpha = 1, \ldots, P \tag{71.4}$$

Let $\mathbf{W}$ be a *K*-form on $\mathscr{S}_0$ with the component form

$$\mathbf{W} = \sum_{\alpha_1 < \cdots < \alpha_K} W_{\alpha_1 \cdots \alpha_K} \mathbf{g}^{\alpha_1} \wedge \cdots \wedge \mathbf{g}^{\alpha_K} \tag{71.5}$$

where $\{\mathbf{g}^\alpha, \alpha = 1,\ldots,P\}$ denotes the natural basis of $(y^\alpha)$ on $\mathscr{S}_0$, and suppose that $\mathbf{Z}$ is the tangential projection of $\mathbf{W}$ on $\mathscr{S}$, i.e.,

$$\mathbf{Z} = \sum_{\alpha_1<\cdots<\alpha_K} W_{\alpha_1\cdots\alpha_K} \frac{\partial y^{\alpha_1}}{\partial u^{\Gamma_1}} \cdots \frac{\partial y^{\alpha_K}}{\partial u^{\Gamma_K}} \mathbf{h}^{\Gamma_1} \wedge \cdots \wedge \mathbf{h}^{\Gamma_K} \tag{71.6}$$

where $\{\mathbf{h}^\Gamma, \Gamma = 1,\ldots,M\}$ denotes thenatural basis of $(u^\Gamma)$ on $\mathscr{S}$. Then the exterior derivatives of $\mathbf{Z}$ coincides with the tangential projection of the exterior derivative of $\mathbf{W}$. In other words, theoperation of exterior derivative commutes with the operation of tangentialprojection.

We can prove this lemma by direct calculation of $d\mathbf{W}$ and $d\mathbf{Z}$. From (71.5) and (71.2) $d\mathbf{W}$ is given by

$$d\mathbf{W} = \sum_{\alpha_1<\cdots<\alpha_K} \frac{\partial W_{\alpha_1\cdots\alpha_K}}{\partial y^\beta} \mathbf{g}^\beta \wedge \mathbf{g}^{\alpha_1} \wedge \cdots \wedge \mathbf{g}^{\alpha_K} \tag{71.7}$$

Hence its tangential projection on $\mathscr{S}$ is

$$\sum_{\alpha_1<\cdots<\alpha_K} \frac{\partial W_{\alpha_1\cdots\alpha_K}}{\partial y^\beta} \frac{\partial y^\beta}{\partial u^\Delta} \frac{\partial y^{\alpha_1}}{\partial u^{\Gamma_1}} \cdots \frac{\partial y^{\alpha_K}}{\partial u^{\Gamma_K}} \mathbf{h}^\Delta \wedge \mathbf{h}^{\Gamma_1} \wedge \cdots \wedge \mathbf{h}^{\Gamma_K} \tag{71.8}$$

Similarly, from (71.6) and (71.7), $d\mathbf{Z}$ is given by

$$\begin{aligned} d\mathbf{Z} &= \sum_{\alpha_1<\cdots<\alpha_K} \frac{\partial}{\partial u^\Delta} \left( \mathbf{W}_{\alpha_1<\cdots<\alpha_K} \frac{\partial y^{\alpha_1}}{\partial u^{\Gamma_1}} \cdots \frac{\partial y^{\alpha_K}}{\partial u^{\Gamma_K}} \right) \mathbf{h}^\Delta \wedge \mathbf{h}^{\Gamma_1} \wedge \cdots \wedge \mathbf{h}^{\Gamma_K} \\ &= \sum_{\alpha_1<\cdots<\alpha_K} \frac{\partial \mathbf{W}_{\alpha_1\ldots\alpha_K}}{\partial y^\beta} \frac{\partial y^\beta}{\partial u^\Delta} \frac{\partial y^{\alpha_1}}{\partial u^{\Gamma_1}} \cdots \frac{\partial y^{\alpha_K}}{\partial u^{\Gamma_K}} \mathbf{h}^\Delta \wedge \mathbf{h}^{\Gamma_1} \wedge \cdots \wedge \mathbf{h}^{\Gamma_K} \end{aligned} \tag{71.9}$$

where we have used the skew symmetry of the exterior product and the symmetry of the second derivative $\partial^2 y^\alpha / \partial u^\Gamma \partial u^\Delta$ with respect to $\Gamma$ and $\Delta$. Comparing (71.9) with (71.8), we have completed the proof of the lemma.

Now we are ready to present the main result of this section.

**Generalized Stokes' Theorem.** Let $\mathscr{S}$ and $\mathscr{S}_0$ be hypersurfaces as defined in the preceding lemma and suppose that $\mathscr{U}$ is an oriented domain in $\mathscr{S}$ with piecewise smooth boundary $\partial\mathscr{U}$. (We orient the boundary $\partial\mathscr{U}$ as usual by requiring the outward normal of $\partial\mathscr{U}$ be the positive normal.) Then for any tangential $(M-1)$-form $\mathbf{Z}$ on $\mathscr{S}_0$ we have

$$\int_{\partial \mathscr{U}} \mathbf{Z} = \int_{\mathscr{U}} d\mathbf{Z} \tag{71.10}$$

Before proving this theorem, we remark first that other than the condition

$$\mathscr{S} \subset \mathscr{S}_0 \tag{71.11}$$

The hypersurface $\mathscr{S}_0$ is entirely arbitrary. In application we often take $\mathscr{S}_0 = \mathscr{E}$ but this choice is not necessary. Second, by virtue of the preceding lemma it suffices to prove (71.10) for tangential $(M-1)-$forms $\mathbf{Z}$ on $\mathscr{S}$ only. In other words, we can choose $\mathscr{S}_0$ to be the same as $\mathscr{S}$ without loss of generality. Indeed, as explained in the preceding section, the integrals in (71.10) are equal to those of tangential projection of the forms $\mathbf{Z}$ and $d\mathbf{Z}$ on $\mathscr{S}$. Then by virtue of the lemma the formula (71.10) amounts to nothing but a formula for tangential forms on $\mathscr{S}$.

Before proving the formula (71.10) in general, we consider first the simplest special case when $\mathbf{Z}$ is a 1-form and $\mathscr{S}$ is a two-dimensional surface. This case corresponds to the Stokes formula in classical vector analysis. As usual, we denote a 1-form by $\mathbf{w}$ since it is merely a covariant vector field on $\mathscr{S}$. Let the component form of $\mathbf{w}$ in $(u^\Gamma)$ be

$$\mathbf{w} = w_\Gamma \mathbf{h}^\Gamma \tag{71.12}$$

where $\Gamma$ is summed from 1 to 2. From (71.2) the exterior derivative of $\mathbf{w}$ is a 2-form

$$d\mathbf{w} = \frac{\partial w_\Gamma}{\partial u^\Delta}\, \mathbf{h}^\Delta \wedge \mathbf{h}^\Gamma = \left( \frac{\partial w_2}{\partial u^1} - \frac{\partial w_1}{\partial u^2} \right) \mathbf{h}^1 \wedge \mathbf{h}^2 \tag{71.13}$$

Thus (71.10) reduces to

$$\int_{\zeta^{-1}(\partial \mathscr{U})} w_\Gamma \dot{\lambda}^\Gamma dt = \iint\limits_{\zeta^{-1}(\mathscr{U})} \left( \frac{\partial w_2}{\partial u^1} - \frac{\partial w_1}{\partial u^2} \right) du^1\, du^2 \tag{71.14}$$

where $\left(\lambda^1(t), \lambda^2(t)\right)$ denotes the coordinates of the oriented boundary curve $\partial \mathscr{U}$ in the coordinate system $\left(u^1, u^2\right)$, the parameterization $t$ on $\partial \mathscr{U}$ being positively oriented but otherwise entirely arbitrary.

Now since the integrals in (71.14) are independent of the choice of positive coordinate system, for simplicity we consider first thecase when $\boldsymbol{\zeta}^{-1}(\mathscr{U})$ is the square $[0,1]\times[0,1]$ in $\mathscr{R}^2$. Naturally, we use the parameters $u^1$, $u^2$, $1-u^1$, and $1-u^2$ on the boundary segments

$(0,0)\to(1,0),(1,0)\to(1,1),(1,1)\to(0,1)$, and $(0,1)\to(0,0)$, respectively. Relative to this parameterization on $\partial\mathscr{U}$ the left-hand side of (71.14) reduces to

$$\int_0^1 w_1\left(u^1,0\right)du^1+\int_0^1 w_2\left(1,u^2\right)du^2 \\ -\int_0^1 w_1\left(u^1,1\right)du^1-\int_0^1 w_2\left(0,u^2\right)du^2 \tag{71.15}$$

Similarly, relative to the surface coordinate system $\left(u^1,u^2\right)$ the right-hand side of (71.14) reduces to

$$\int_0^1\int_0^1\left(\frac{\partial w_2}{\partial u^1}-\frac{\partial w_1}{\partial u^2}\right)du^1\,du^2 \tag{71.16}$$

which may be integrated by parts once with respect to one of the two variables $\left(u^1,u^2\right)$ and the result is precisely the sameas (71.15). Thus (71.14) is proved in this simple case.

In general $\mathscr{U}$ may not be homeomorphic to the square $[0,1]\times[0,1]$, of course. Then we decompose $\mathscr{U}$ as before by (68.28) and we assume that each $\mathscr{U}_a, a=1,\ldots,K$, can be represented by the range

$$\mathscr{U}_a=\zeta_a\left([0,1]\times[0,1]\right) \tag{71.17}$$

By the result for the simple case we then have

$$\int_{\partial\mathscr{U}_a}\mathbf{w}=\int_{\partial\mathscr{U}_a}d\mathbf{w},\qquad a=1,\ldots,K \tag{71.18}$$

Now adding (71.18) with respect to $a$ and observing the fact that all common boundaries of pairs of $\mathscr{U}_1,\ldots,\mathscr{U}_K$ are oriented oppositely as shown in Figure 10, we obtain

$$\int_{\partial\mathscr{U}}\mathbf{w}=\int_{\mathscr{U}}d\mathbf{w} \tag{71.19}$$

which is the special case of (71.10) when $M=2$.

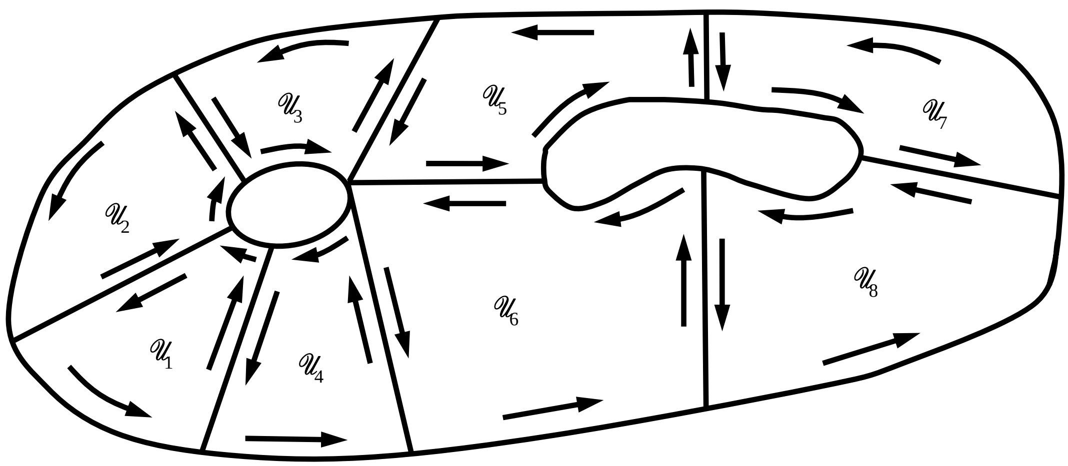


Figure 10

The proof of (71.10) for the general case with an arbitrary $M$ is essentially the same as the proof of the preceding special case. To illustrate the similarity of the proofs we consider next the case $M = 3$. In this case $\mathbf{Z}$ is a 2-form on a 3-dimensional hypersurface $\mathscr{S}$. Let $\left(u^{\Gamma}, \Gamma = 1, 2, 3\right)$ be a positive surface coordinate system on $\mathscr{S}$ as usual. Then $\mathbf{Z}$ can be represented by

$$\begin{aligned}\mathbf{Z} &= \sum_{\Gamma<\Delta} Z_{\Gamma\Delta}\mathbf{h}^{\Gamma} \wedge \mathbf{h}^{\Delta} \\ &= Z_{12}\mathbf{h}^{1} \wedge \mathbf{h}^{2} + Z_{13}\mathbf{h}^{1} \wedge \mathbf{h}^{3} + Z_{23}\mathbf{h}^{2} \wedge \mathbf{h}^{3}\end{aligned} \tag{71.20}$$

From (71.2) the exterior derivative of $\mathbf{Z}$ is given by

$$d\mathbf{Z} = \left(\frac{\partial Z_{12}}{\partial u^{3}} - \frac{\partial Z_{13}}{\partial u^{2}} + \frac{\partial Z_{23}}{\partial u^{1}}\right)\mathbf{h}^{1} \wedge \mathbf{h}^{2} \wedge \mathbf{h}^{3} \tag{71.21}$$

As before, we now assume that $\mathscr{U}$ can be represented by the range of a cube relative to a certain $\left(u^{\Gamma}\right)$, namely

$$\mathscr{U} = \zeta\left([0,1]\times[0,1]\times[0,1]\right) \tag{71.22}$$

Then the right-hand side of (71.10) reduces to the triple integral

$$\int_0^1 \int_0^1 \int_0^1 \left( \frac{\partial Z_{12}}{\partial u^3} - \frac{\partial Z_{13}}{\partial u^2} + \frac{\partial Z_{23}}{\partial u^1} \right) du^1\, du^2\, du^3 \tag{71.23}$$

which may be integrated by parts once with respect to one of the three variables $\left(u^1, u^2, u^3\right)$. The result consists of six terms:

$$\begin{aligned} &\int_0^1 \int_0^1 Z_{12}\left(u^1, u^2, 1\right) du^1\, du^2 - \int_0^1 \int_0^1 Z_{12}\left(u^1, u^2, 0\right) du^1\, du^2 \\ &\quad - \int_0^1 \int_0^1 Z_{13}\left(u^1, 1, u^3\right) du^1\, du^3 + \int_0^1 \int_0^1 Z_{13}\left(u^1, 0, u^3\right) du^1\, du^3 \\ &\quad + \int_0^1 \int_0^1 Z_{23}\left(1, u^2, u^3\right) du^2\, du^3 - \int_0^1 \int_0^1 Z_{23}\left(0, u^2, u^3\right) du^2\, du^3 \end{aligned} \tag{71.24}$$

which are precisely the representations of the left-hand side of (71.10) on the six faces of the cube with an appropriate orientation on each face. Thus (71.10) is proved when (71.22) holds.

In general, if $\mathscr{U}$ cannot be represented by (71.22), then we decompose $\mathscr{U}$ as before by (68.28), and we assume that each $\mathscr{U}_a$ may be represented by

$$\mathscr{U}_a = \boldsymbol{\zeta}_a \left( [0,1] \times [0,1] \times [0,1] \right) \tag{71.25}$$

for an appropriate $\boldsymbol{\zeta}_a$. By the preceding result we then have

$$\int_{\partial \mathscr{U}_a} \mathbf{Z} = \int_{\mathscr{U}_a} d\mathbf{Z}, \qquad a = 1, \ldots, K \tag{71.26}$$

Thus (71.10) follows by summing (71.26) with respect to $a$.

Following exactly the same pattern, the formula (71.10) can be proved by an arbitrary $M = 4, 5, 6, \ldots$. Thus the theorem is proved.

The formula (71.10) reduces to two important special cases in classical vector analysis when the underlying Euclidean manifold $\mathscr{E}$ is three-dimensional. First, when $M = 2$ and $\mathscr{U}$ is two-dimensional, the formula takes the form

$$\int_{\partial \mathscr{U}} \mathbf{w} = \int_{\mathscr{U}} d\mathbf{w} \tag{71.27}$$

which can be rewritten as

$$\int_{\partial \mathscr{U}} \mathbf{w} \cdot d\boldsymbol{\lambda} = \int_{\mathscr{U}} \operatorname{curl}\ \mathbf{w} \cdot d\boldsymbol{\sigma} \tag{71.28}$$

where $d\boldsymbol{\sigma}$ is given by (70.14) and where $\operatorname{curl}\mathbf{w}$ is the axial vector corresponding to $d\mathbf{w}$, i.e.,

$$d\mathbf{w}(\mathbf{u},\mathbf{v}) = \operatorname{curl}\ \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) \tag{71.29}$$

for any $\mathbf{u},\mathbf{v}$ in $\mathscr{V}$. Second, when $M=3$ and $\mathscr{U}$ is three-dimensional the formula takes the form

$$\iint_{\partial\mathscr{U}} \mathbf{w}\cdot d\boldsymbol{\sigma} = \iiint_{\mathscr{U}} \operatorname{div}\mathbf{w}\, d\upsilon \tag{71.30}$$

where $d\upsilon$ is the Euclidean volume element defined by [see (68.39)]

$$d\upsilon \equiv e\, du^1\, du^2\, du^3 \tag{71.31}$$

relative to any positive coordinate system $(u^i)$, or simply

$$d\upsilon = dx^1\, dx^2\, dx^3 \tag{71.32}$$

relative to a right-handed rectangular Cartesian coordinate system $(x^i)$. In (71.30), $\mathbf{w}$ is the axial vector field corresponding to the 2-form $\mathsf{Z}$, i.e., relative to $(x^i)$

$$\mathsf{Z} = w_3\mathbf{e}^1\wedge\mathbf{e}^2 - w_2\mathbf{e}^1\wedge\mathbf{e}^3 + w_1\mathbf{e}^2\wedge\mathbf{e}^3 \tag{71.33}$$

which is equivalent to

$$\mathsf{Z}(\mathbf{u},\mathbf{v}) = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}), \quad \mathbf{u},\mathbf{v}\in\mathscr{V} \tag{71.34}$$

From (71.33), $d\mathsf{Z}$ is given by

$$d\mathsf{Z} = \left(\frac{\partial w_1}{\partial x^1} + \frac{\partial w_2}{\partial x^2} + \frac{\partial w_3}{\partial x^3}\right)\mathbf{e}^1\wedge\mathbf{e}^2\wedge\mathbf{e}^3 = (\operatorname{div}\mathbf{w})\mathbf{E} \tag{71.35}$$

As a result, we have

$$\int_{\partial\mathscr{U}}\mathsf{Z} = \int_{\partial\mathscr{U}}\mathbf{w}\cdot d\boldsymbol{\sigma} \tag{71.36}$$

and

$$\int_{\mathscr{U}} d\mathbf{Z} = \iiint_{\mathscr{U}} \operatorname{div} \mathbf{w}\, d\upsilon \tag{71.37}$$

The formulas (71.28) and (71.30) are called *Stokes' theorem* and *Gauss' divergence theorem*, respectively, in classical vector analysis.

## *Section 72. Invariant Integrals on Continuous Groups*

In the preceding chapter we have considered various groups which are also hypersurfaces contained in the Euclidean space $\mathscr{L}(\mathscr{V};\mathscr{V})$. Since the integrations defined so far in this chapter can be applied to any hypersurface in a Euclidean manifold, in particular they can be applied to the continuous groups. However, these integrations are generally unrelated to the group structure and thus their applications are limited. This situation is similar to that about parallelism. While the induced metric and its Levi-Civita parallelism certainly exist on the underlying hypersurface of the group, they are not significant mathematically because they do not reflect the group structure. This remark has led us to consider the Cartan parallelism which is defined by the left-invariant fields on the group.

Now as far as integrationis concerned, the natural choice for a continuous group is the integration based on a left-invarianct volume density. Specifically, if $\{\mathbf{e}_\Gamma\}$ is a basis for the Lie algebra of the group, then a volume tensor field $\mathbf{Z}$ is a left-invariant if and only if it can be represented by

$$\mathbf{Z} = c\mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^M \tag{72.1}$$

where $c$ is a constant. The integral of $\mathbf{Z}$ over any domain $\mathscr{U}$ obeys the condition

$$\int_{\mathscr{U}} \mathbf{Z} = \int_{L_{\mathbf{X}}(\mathscr{U})} \mathbf{Z} \tag{72.2}$$

for all elements $\mathbf{X}$ belonging to the group.

A left-invariant volume tensor field $\mathbf{Z}$ is also right-invariant if and only if it is invariant under the inversion operation $\boldsymbol{J}$ when the dimension $M$ of the group is even or it is mapped into $-\mathbf{Z}$ by $\boldsymbol{J}$ $J$ when $M$ is odd. This fact is more or less obvious since in general $\boldsymbol{J}$ maps any left-invariant field into a right-invariant field. Also, the gradient of $\boldsymbol{J}$ at the identity element coincides with the negation operation. Consequently, when $M$ is even, $\mathbf{Z}(\mathbf{I})$ is invaritant under $\boldsymbol{J}$ , while if $M$ is odd, $\mathbf{Z}(\mathbf{I})$ is mapped into $-\mathbf{Z}(\mathbf{I})$ by $\boldsymbol{J}$ . Naturally we call $\mathbf{Z}$ an invariant volume tensor field if it is both left-invariant and right-invariant. Relative to an invariant $\mathbf{Z}$ the integral obeys the condition (72.2) as well as the conditions

$$\int_{\mathscr{U}} \mathbf{Z} = \int_{R_{\mathbf{X}}(\mathscr{U})} \mathbf{Z}, \qquad \int_{\mathscr{U}} \mathbf{Z} = \int_{J(\mathscr{U})} \mathbf{Z} \tag{72.3}$$

Here we have used the fact that $\boldsymbol{J}$ preserves the orientatin of the group when $M$ is even, while $\boldsymbol{J}$ reverses the orientation of the group when $M$ is odd.

By virtue of the representation (72.1) all left-invariant volume tensor fields differ from one another by a constant multiple only and thus they are either all right-invariant or all not right-invariant. As we shall see, the left-invariant volume tensor fields on $\mathscr{GL}(\mathscr{V})$, $\mathscr{SL}(\mathscr{V})$, $\mathscr{O}(\mathscr{V})$, and all continuous subgroups of $\mathscr{O}(\mathscr{V})$ are right-invariant. Hence invariant integrals exist on these groups. We consider first the general linear group $\mathscr{GL}(\mathscr{V})$.

To prove that the left-invariant volume tensor fields on the $\mathscr{GL}(\mathscr{V})$ are also right-invariant, we recall first from exterior algebra the transformation rule for a volume tensor under a linear map of the underlying vector space. Let $\mathscr{W}$ be an arbitrary vector space of dimension $M$, and suppose that $\mathbf{A}$ is a linear transformation of $\mathscr{W}$

$$\mathbf{A}: \quad \mathscr{W} \rightarrow \mathscr{W} \tag{72.4}$$

Then $\mathbf{A}$ maps any volume tensor $\mathsf{E}$ on $\mathscr{W}$ to $(\det \mathbf{A})\mathsf{E}$,

$$\mathbf{A}^*(\mathsf{E}) = (\det \mathbf{A})\mathsf{E} \tag{72.5}$$

since by the skew symmetry of $\mathsf{E}$ we have

$$\mathsf{E}(\mathbf{Ae}_1, ..., \mathbf{Ae}_M) = (\det \mathbf{A})\mathsf{E}(\mathbf{e}_1, ..., \mathbf{e}_M) \tag{72.6}$$

for any $\{\mathbf{e}_1, \ldots, \mathbf{e}_M\}$ in $\mathscr{W}$. From the representation of a left-invariant field on $\mathscr{GL}(\mathscr{V})$ the value of the field at any point $\mathbf{X}$ is obtained from the value at the identity $\mathbf{I}$ by the linear map

$$\nabla L_{\mathbf{X}} : \mathscr{L}(\mathscr{V};\mathscr{V}) \rightarrow \mathscr{L}(\mathscr{V};\mathscr{V}) \tag{72.7}$$

which is defined by

$$\nabla L_{\mathbf{X}}(\mathbf{K}) = \mathbf{XK}, \quad \mathbf{K} \in \mathscr{L}(\mathscr{V};\mathscr{V}) \tag{72.8}$$

Similiarly, a right-invariant field is obtained by the linear map $\nabla R_{\mathbf{X}}$ defined by

$$\nabla R_{\mathbf{X}}(\mathbf{K}) = \mathbf{KX}, \quad \mathbf{K} \in \mathscr{L}(\mathscr{V};\mathscr{V}) \tag{72.9}$$

Then by virtue of (72.5) a left-invariant field volume tensor field is also right-invariant if and only if

$$\det \nabla R_{\mathbf{X}} = \det \nabla L_{\mathbf{X}} \tag{72.10}$$

Since the dimension of $\mathscr{L}(\mathscr{V};\mathscr{V})$ is $N^2$, the matrices of $\nabla R_{\mathbf{X}}$ and $\nabla L_{\mathbf{X}}$ are $N^2 \times N^2$. For simplicity we use the product basis $\{\mathbf{e}_i \otimes \mathbf{e}^j\}$ for the space $\mathscr{L}(\mathscr{V};\mathscr{V})$; then (72.8) and (72.9) can be represented by

$$(\mathbf{XK})^i_{\ j} = C^{ik}_{jl} K^l_k = X^i_{\ l} K^l_k = X^i_{\ l} \delta^k_j K^l_k \tag{72.11}$$

and

$$(\mathbf{KX})^i_{\ j} = D^{ik}_{jl} K^l_k = K^l_k X^k_{\ j} = \delta^i_l X^k_{\ j} K^l_k \tag{72.12}$$

To prove (72.10), we have to show that the $N^2 \times N^2$ matrices $\left[C^{ik}_{jl}\right]$ and $\left[D^{ik}_{jl}\right]$ have the same determinant. But this fact is obvious since from (72.11) and (72.12), $\left[C^{ik}_{jl}\right]$ is simply the transpose of $\left[D^{ik}_{jl}\right]$, i.e.,

$$D^{ik}_{jl} = C^{ki}_{lj} \tag{72.13}$$

Thus invariant integrals exist on $\mathscr{GL}(\mathscr{V})$.

The situation with the subgroup $\mathscr{SL}(\mathscr{V})$ or $\mathscr{UM}(\mathscr{V})$ is somewhat more complicated, however, because the tangent planes at distinct points of the underlying group generally are not the same subspace of $\mathscr{L}(\mathscr{V};\mathscr{V})$. We recall first that the tangent plane of $\mathscr{SL}(\mathscr{V})$ at the identity $\mathbf{I}$ is the hyperplane consisting of all tensors $\mathbf{K} \in \mathscr{L}(\mathscr{V};\mathscr{V})$ such that

$$tr(\mathbf{K}) = 0 \tag{72.14}$$

This result means that the orthogonal complement of $\mathscr{SL}(\mathscr{V})_{\mathbf{I}}$ relative to the inner product on $\mathscr{L}(\mathscr{V};\mathscr{V})$ is the one-dimensional subspace

$$l = \{\alpha \mathbf{I}, \alpha \in \mathscr{R}\} \tag{72.15}$$

Now from (72.8) and (72.9) the linear maps $\nabla R_{\mathbf{X}}$ and $\nabla L_{\mathbf{X}}$ coincide on $l$, namely

$$\nabla R_{\mathbf{X}}(\alpha \mathbf{I}) = \alpha \mathbf{X} = \nabla L_{\mathbf{X}}(\alpha \mathbf{I}) \tag{72.16}$$

By virtue of (72.16) and (72.10) we see that the restrictions of $\nabla R_{\mathbf{X}}$ and $\nabla L_{\mathbf{X}}$ on $\mathscr{SL}(\mathscr{V})_{\mathbf{I}}$ give rise to the same volume tensor at $\mathbf{X}$ from any volume tensor at $\mathbf{I}$. As a result every left-invariant volume tensor field on $\mathscr{SL}(\mathscr{V})$ or $\mathscr{UM}(\mathscr{V})$ is also a right-invariant, and thus invariant integrals exist on $\mathscr{SL}(\mathscr{V})$ and $\mathscr{UM}(\mathscr{V})$.

Finally, we show that invariant integrals exists on $\mathscr{O}(\mathscr{V})$ and on all continuous subgroups of $\mathscr{O}(\mathscr{V})$. This result is entirely obvious because both $\nabla L_{\mathbf{X}}$ and $\nabla R_{\mathbf{X}}$ preserve the inner product on $\mathscr{L}(\mathscr{V};\mathscr{V})$ for any $\mathbf{X}\in\mathscr{O}(\mathscr{V})$. Indeed, if $\mathbf{K}$ and $\mathbf{H}$ are any elements of $\mathscr{L}(\mathscr{V};\mathscr{V})$, then

$$\begin{aligned}\mathbf{XK}\cdot\mathbf{XH} &= tr\left(\mathbf{XKH}^T\mathbf{X}^T\right) = tr\left(\mathbf{XKH}^T\mathbf{X}^{-1}\right)\\ &= tr\left(\mathbf{KH}^T\right) = \mathbf{K}\cdot\mathbf{H}\end{aligned} \tag{72.17}$$

and similiarly

$$\mathbf{KX}\cdot\mathbf{HX} = \mathbf{K}\cdot\mathbf{H} \tag{72.18}$$

for any $\mathbf{X}\in\mathscr{O}(\mathscr{V})$. As a result, the Euclidean volume tensor field $\mathsf{E}$ is invariant on $\mathscr{O}(\mathscr{V})$ and on all continuous subgroups of $\mathscr{O}(\mathscr{V})$.

It should be noted that $\mathscr{O}(\mathscr{V})$ is a bounded closed hypersurface in $\mathscr{L}(\mathscr{V};\mathscr{V})$. Hence the integral of $\mathsf{E}$ over the whole group is finite

$$0 < \int_{\mathscr{O}(\mathscr{V})} \mathsf{E} < \infty \tag{72.19}$$

Such is not the case for $\mathscr{GL}(\mathscr{V})$ or $\mathscr{SL}(\mathscr{V})$, since they are both unbounded in $\mathscr{L}(\mathscr{V};\mathscr{V})$. By virtue of (72.19) any continuous function $f$ on $\mathscr{O}(\mathscr{V})$ can be integrated with respect to $\mathsf{E}$ over the entire group $\mathscr{O}(\mathscr{V})$. From (72.2) and (72.3) the integral possesses the following properties:

$$\begin{aligned}\int_{\mathscr{O}(\mathscr{V})} f(\mathbf{Q})\mathsf{E}(\mathbf{Q}) &= \int_{\mathscr{O}(\mathscr{V})} f(\mathbf{Q}_0\mathbf{Q})\mathsf{E}(\mathbf{Q})\\ \int_{\mathscr{O}(\mathscr{V})} f(\mathbf{Q})\mathsf{E}(\mathbf{Q}) &= \int_{\mathscr{O}(\mathscr{V})} f(\mathbf{Q}\mathbf{Q}_0)\mathsf{E}(\mathbf{Q})\end{aligned} \tag{72.20}$$

for any $\mathbf{Q}_0\in\mathscr{O}(\mathscr{V})$ and

$$\int_{\mathscr{O}(\mathscr{V})} f(\mathbf{Q})\mathbf{E}(\mathbf{Q}) = \int_{\mathscr{O}(\mathscr{V})} f\left(\mathbf{Q}^{-1}\right)\mathbf{E}(\mathbf{Q}) \tag{72.21}$$

in addition to the standard properties of the integral relative to a differential form obtained in the preceding section. For the unbounded groups $\mathscr{GL}(\mathscr{V})$, $\mathscr{SL}(\mathscr{V})$, and $\mathscr{UM}(\mathscr{V})$, some continuous functions, such as functions which vanish identically outside some bounded domain, can be integrated over the whole group. If the integral of $f$ with respect to an invariant volume tensor field exists, it also possesses properties similar to (72.20) and (72.21).

Now, by using the invariant integral on $\mathscr{O}(\mathscr{V})$, we can find a representation for continuous isotropic functions

$$f : \mathscr{L}(\mathscr{V};\mathscr{V}) \times \cdots \times \mathscr{L}(\mathscr{V};\mathscr{V}) \to \mathscr{R} \tag{72.22}$$

which verify the condition of isotropy :

$$f\left(\mathbf{QK}_1\mathbf{Q}^T,\ldots,\mathbf{QK}_P\mathbf{Q}^T\right) = f\left(\mathbf{K}_1,\ldots,\mathbf{K}_P\right) \tag{72.23}$$

for all $\mathbf{Q} \in \mathscr{O}(\mathscr{V})$, the number of variables $P$ being arbitrary. The representation is

$$f\left(\mathbf{K}_1,\ldots,\mathbf{K}_P\right) = \frac{\int_{\mathscr{O}(\mathscr{V})} g\left(\mathbf{QK}_1\mathbf{Q}^T,\ldots,\mathbf{QK}_P\mathbf{Q}^T\right)\mathbf{E}(\mathbf{Q})}{\int_{\mathscr{O}(\mathscr{V})} \mathbf{E}(\mathbf{Q})} \tag{72.24}$$

where $g$ is an arbitrary continuous function

$$g : \mathscr{L}(\mathscr{V};\mathscr{V}) \times \cdots \times \mathscr{L}(\mathscr{V};\mathscr{V}) \to \mathscr{R} \tag{72.25}$$

and where $\mathbf{E}$ is an arbitrary invariant volume tensor field on $\mathscr{O}(\mathscr{V})$. By a similar argument we can also find representations for continuous functions $f$ satisfying the condition

$$f\left(\mathbf{QK}_1,\ldots,\mathbf{QK}_P\right) = f\left(\mathbf{K}_1,\ldots,\mathbf{K}_P\right) \tag{72.26}$$

for all $\mathbf{Q} \in \mathscr{O}(\mathscr{V})$. The representation is

$$f\left(\mathbf{K}_1,\ldots,\mathbf{K}_P\right) = \frac{\int_{\mathscr{O}(\mathscr{V})} g\left(\mathbf{QK}_1,\ldots,\mathbf{QK}_P\right)\mathbf{E}(\mathbf{Q})}{\int_{\mathscr{O}(\mathscr{V})} \mathbf{E}(\mathbf{Q})} \tag{72.27}$$

Similiarly, a representation for functions $f$ satisfying the condition

$$f\left(\mathbf{K}_1\mathbf{Q},\ldots,\mathbf{K}_P\mathbf{Q}\right)=f\left(\mathbf{K}_1,\ldots,\mathbf{K}_P\right) \tag{72.28}$$

for all $\mathbf{Q}\in\mathscr{O}\left(\mathscr{V}\right)$ is

$$f\left(\mathbf{K}_1,\ldots,\mathbf{K}_P\right)=\frac{\int_{\mathscr{O}(\mathscr{V})} g\left(\mathbf{K}_1\mathbf{Q},\ldots,\mathbf{K}_P\mathbf{Q}\right)\mathsf{E}\left(\mathbf{Q}\right)}{\int_{\mathscr{O}(\mathscr{V})}\mathsf{E}\left(\mathbf{Q}\right)} \tag{72.29}$$

We leave the proof of these representations as exercises.

If the condition (72.23), (72.26), or (72.28) is required to hold for all $\mathbf{Q}$ belonging to a continuous subgroup $\mathscr{G}$ of $\mathscr{O}\left(\mathscr{V}\right)$, the representation (72.24), (72.27), or (72.29), respectively, remains valid except that the integrals in the representations are taken over the group $\mathscr{G}$.

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