1 Elementary Algebra and Geometry

Algebra

1. Fundamental Properties (Real Numbers)

$a+b=b+a$	Commutative Law for Addition
$(a+b)+c=a+(b+c)$	Associative Law for Addition
$a+0=0+a$	Identity Law for Addition
$a+(-a)=(-a)+a=0$	Inverse Law for Addition
$a(bc)=(ab)c$	Associative Law for Multiplication
$a\left(\frac{1}{a}\right)=\left(\frac{1}{a}\right)a=1,\ a\neq 0$	Inverse Law for Multiplication
$(a)(1)=(1)(a)=a$	Identity Law for Multiplication
$ab=ba$	Commutative Law for Multiplication
$a(b+c)=ab+ac$	Distributive Law

DIVISION BY ZERO IS NOT DEFINED

2. *Exponents*

For integers m and n

$$a^n a^m = a^{n+m}$$

$$a^n / a^m = a^{n-m}$$

$$(a^n)^m = a^{nm}$$

$$(ab)^m = a^m b^m$$

$$(a/b)^m = a^m / b^m$$

3. *Fractional Exponents*

$$a^{p/q} = (a^{1/q})^p$$

where $a^{1/q}$ is the positive qth root of a if $a > 0$ and the negative qth root of a if a is negative and q is odd. Accordingly, the five rules of exponents given above (for integers) are also valid if m and n are fractions, provided a and b are positive.

4. *Irrational Exponents*

If an exponent is irrational, e.g., $\sqrt{2}$, the quantity, such as $a^{\sqrt{2}}$ is the limit of the sequence, $a^{1.4}, a^{1.41}, a^{1.414}, \dots$.

- *Operations with Zero*

$$0^m = 0; \; a^0 = 1$$

5. *Logarithms*

If x, y, and b are positive and $b \neq 1$

$$
\begin{aligned}
\log_b(xy) &= \log_b x + \log_b y \\
\log_b(x/y) &= \log_b x - \log_b y \\
\log_b x^p &= p \log_b x \\
\log_b(1/x) &= -\log_b x \\
\log_b b &= 1 \\
\log_b 1 &= 0 \qquad \textit{Note}: b^{\log_b x} = x.
\end{aligned}
$$

- *Change of Base* $(a \neq 1)$

$$\log_b x = \log_a x \log_b a$$

6. *Factorials*

The factorial of a positive integer n is the product of all the positive integers less than or equal to the integer n and is denoted $n!$. Thus,

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n.$$

Factorial 0 is defined: $0! = 1$.

- *Stirling's Approximation*

$$\lim_{n \to \infty} (n/e)^n \sqrt{2\pi n} = n!$$

(See also 9.2.)

7. *Binomial Theorem*

For positive integer n

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2$$

$$+\frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots$$

$$+nxy^{n-1} + y^n.$$

8. *Factors and Expansion*

$$\begin{aligned}
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a-b)^2 &= a^2 - 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a-b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3 \\
(a^2 - b^2) &= (a-b)(a+b) \\
(a^3 - b^3) &= (a-b)(a^2 + ab + b^2) \\
(a^3 + b^3) &= (a+b)(a^2 - ab + b^2)
\end{aligned}$$

9. *Progression*

An *arithmetic progression* is a sequence in which the difference between any term and the preceding term is a constant (d):

$$a, a+d, a+2d, \ldots, a+(n-1)d.$$

If the last term is denoted l [$=a+(n-1)d$], then the sum is

$$s=\frac{n}{2}(a+l).$$

A *geometric progression* is a sequence in which the ratio of any term to the preceding term is a constant r. Thus, for n terms

$$a, ar, ar^2, \ldots, ar^{n-1}$$

The sum is

$$S=\frac{a-ar^n}{1-r}$$

10. Complex Numbers

A complex number is an ordered pair of real numbers (a, b).

Equality: $(a, b)=(c, d)$ if and only if $a=c$ and $b=d$
Addition: $(a, b)+(c, d)=(a+c, b+d)$
Multiplication: $(a, b)(c, d)=(ac-bd, ad+bc)$

The first element (a, b) is called the *real* part; the second the *imaginary* part. An alternate notation for (a, b) is $a+bi$, where $i^2=(-1,0)$, and $i=(0,1)$ or $0+1i$ is written for this complex number as a convenience. With this understanding, i behaves as a number, i.e., $(2-3i)(4+i)=8-12i+2i-3i^2=11-10i$. The conjugate of $a+bi$ is $a-bi$ and the product of a complex number and its conjugate is a^2+b^2. Thus, *quotients* are computed by multiplying numerator and denominator by the conjugate of the denominator, as illustrated below:

$$\frac{2+3i}{4+2i}=\frac{(4-2i)(2+3i)}{(4-2i)(4+2i)}=\frac{14+8i}{20}=\frac{7+4i}{10}$$

11. Polar Form

The complex number $x+iy$ may be represented by a plane vector with components x and y

$$x+iy=r(\cos\theta+i\sin\theta)$$

(see Figure 1.1). Then, given two complex numbers $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$, the product and quotient are

product: $z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)]$

quotient: $z_1/z_2=(r_1/r_2)[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$

powers: $z^n = [r(\cos\theta+i\sin\theta)]^n = r^n[\cos n\theta+i\sin n\theta]$

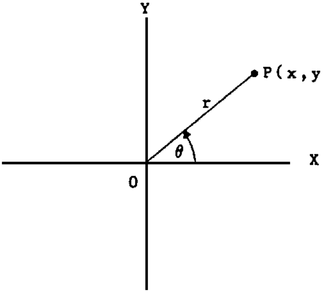

FIGURE 1.1. Polar form of complex number.

roots: $$z^{1/n} = [r(\cos\theta + i\sin\theta)]^{1/n}$$
$$= r^{1/n}\left[\cos\frac{\theta + k.360}{n} + i\sin\frac{\theta + k.360}{n}\right],$$
$$k = 0, 1, 2, \ldots, n-1$$

12. Permutations

A permutation is an ordered arrangement (sequence) of all or part of a set of objects. The number of permutations of n objects taken r at a time is

$$p(n, r) = n(n-1)(n-2)\ldots(n-r+1)$$

$$= \frac{n!}{(n-r)!}$$

A permutation of positive integers is "even" or "odd" if the total number of inversions is an even integer or an odd integer, respectively. Inversions are counted relative to each integer j in the permutation by counting the number of integers that follow j and are less than j. These are summed to give the total number of inversions. For example, the permutation 4132 has four inversions: three relative to 4 and one relative to 3. This permutation is therefore even.

13. Combinations

A combination is a selection of one or more objects from among a set of objects regardless of order. The number of combinations of n different objects taken r at a time is

$$C(n,r)=\frac{P(n,r)}{r!}=\frac{n!}{r!(n-r)!}$$

14. Algebraic Equations

- *Quadratic*

If $ax^2+bx+c=0$, and $a\neq 0$, then roots are

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

- *Cubic*

To solve $x^3+bx^2+cx+d=0$, let $x=y-b/3$. Then the *reduced cubic* is obtained:

$$y^3+py+q=0$$

where $p=c-(1/3)b^2$ and $q=d-(1/3)bc+(2/27)b^3$. Solutions of the original cubic are then in terms of the reduced cubic roots y_1, y_2, y_3:

$$x_1=y_1-(1/3)b \qquad x_2=y_2-(1/3)b$$

$$x_3=y_3-(1/3)b$$

The three roots of the reduced cubic are

$$y_1=(A)^{1/3}+(B)^{1/3}$$

$$y_2=W(A)^{1/3}+W^2(B)^{1/3}$$

$$y_3 = W^2(A)^{1/3} + W(B)^{1/3}$$

where

$$A = -\frac{1}{2}q + \sqrt{(1/27)p^3 + \frac{1}{4}q^2},$$

$$B = -\frac{1}{2}q - \sqrt{(1/27)p^3 + \frac{1}{4}q^2},$$

$$W = \frac{-1 + i\sqrt{3}}{2}, \quad W^2 = \frac{-1 - i\sqrt{3}}{2}.$$

When $(1/27)p^3 + (1/4)q^2$ is negative, A is complex; in this case A should be expressed in trigonometric form: $A = r(\cos\theta + i\sin\theta)$ where θ is a first or second quadrant angle, as q is negative or positive. The three roots of the reduced cubic are

$$y_1 = 2(r)^{1/3}\cos(\theta/3)$$

$$y_2 = 2(r)^{1/3}\cos\left(\frac{\theta}{3} + 120°\right)$$

$$y_3 = 2(r)^{1/3}\cos\left(\frac{\theta}{3} + 240°\right)$$

15. Geometry

The following is a collection of common geometric figures. Area (A), volume (V), and other measurable features are indicated.

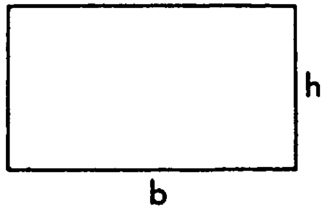

FIGURE 1.2. Rectangle. $A = bh$.

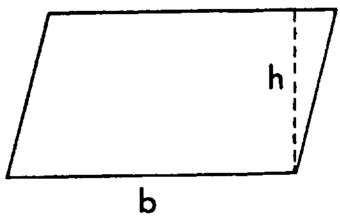

FIGURE 1.3. Parallelogram. $A = bh$.

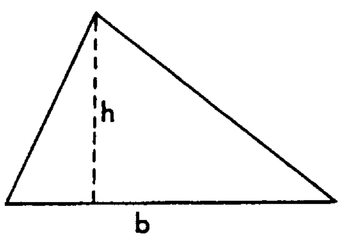

FIGURE 1.4. Triangle. $A = \frac{1}{2}bh$.

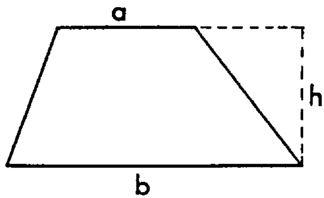

FIGURE 1.5. Trapezoid. $A = \frac{1}{2}(a+b)h$.

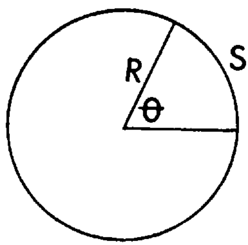

FIGURE 1.6. Circle. $A = \pi R^2$; circumference $= 2\pi R$; arc length $S = R\theta$ (θ in radians).

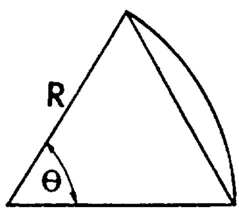

FIGURE 1.7. Sector of circle. $A_{\text{sector}} = \frac{1}{2}R^2\theta$; $A_{\text{segment}} = \frac{1}{2}R^2(\theta - \sin\theta)$.

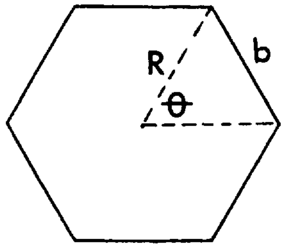

FIGURE 1.8. Regular polygon of n sides. $A = \frac{n}{4} b^2 \operatorname{ctn} \frac{\pi}{n}$; $R = \frac{b}{2} \csc \frac{\pi}{n}$.

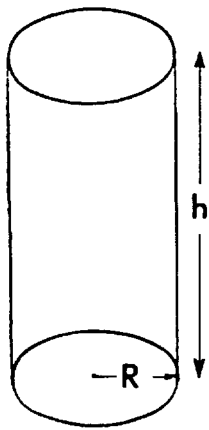

FIGURE 1.9. Right circular cylinder. $V = \pi R^2 h$; lateral surface area $= 2\pi R h$.

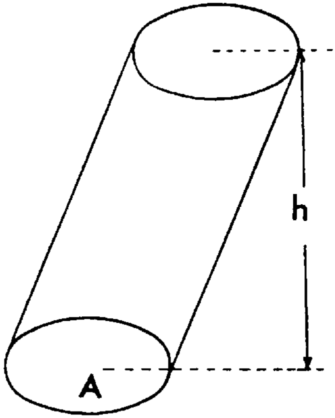

FIGURE 1.10. Cylinder (or prism) with parallel bases. $V = Ah$.

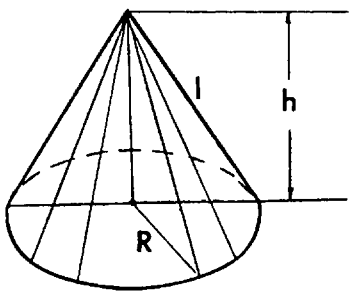

FIGURE 1.11. Right circular cone. $V = \frac{1}{3}\pi R^2 h$; lateral surface area $= \pi R l = \pi R \sqrt{R^2 + h^2}$.

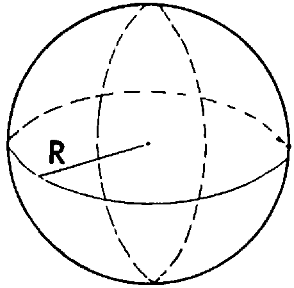

FIGURE 1.12. Sphere. $V = \frac{4}{3}\pi R^3$; surface area = $4\pi R^2$.