2 Determinants, Matrices, and Linear Systems of Equations

1. *Determinants*

Definition. The square array (matrix) A, with n rows and n columns, has associated with it the determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

a number equal to

$$\sum (\pm) a_{1i} a_{2j} a_{3k} \ldots a_{nl}$$

where $i, j, k, \ldots, l$ is a permutation of the n integers $1, 2, 3, \ldots, n$ in some order. The sign is plus if the permutation is *even* and is minus if the permutation is *odd* (see 1.12). The 2×2 determinant

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

has the value $a_{11}a_{22} - a_{12}a_{21}$ since the permutation $(1, 2)$ is even and $(2, 1)$ is odd. For 3×3 determinants, permutations are as follows:

1,	2,	3	even
1,	3,	2	odd
2,	1,	3	odd
2,	3,	1	even
3,	1,	2	even
3,	2,	1	odd

Thus,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{Bmatrix} +a_{11} \cdot a_{22} \cdot a_{33} \\ -a_{11} \cdot a_{23} \cdot a_{32} \\ -a_{12} \cdot a_{21} \cdot a_{33} \\ +a_{12} \cdot a_{23} \cdot a_{31} \\ +a_{13} \cdot a_{21} \cdot a_{32} \\ -a_{13} \cdot a_{22} \cdot a_{31} \end{Bmatrix}$$

A determinant of order n is seen to be the sum of $n!$ signed products.

2. *Evaluation by Cofactors*

Each element a_{ij} has a determinant of order $(n-1)$ called a *minor* (M_{ij}) obtained by suppressing all elements in row i and column j. For example, the minor of element a_{22} in the 3×3 determinant above is

$$\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

The cofactor of element a_{ij}, denoted A_{ij}, is defined as $\pm M_{ij}$, where the sign is determined from i and j:

$$A_{ij} = (-1)^{i+j} M_{ij}.$$

The value of the $n \times n$ determinant equals the sum of products of elements of any row (or column) and their respective cofactors. Thus, for the 3×3 determinant

$$\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \text{ (first row)}$$

or

$$= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \text{ (first column)}$$

etc.

3. *Properties of Determinants*

a. If the corresponding columns and rows of A are interchanged, $\det A$ is unchanged.

b. If any two rows (or columns) are interchanged, the sign of $\det A$ changes.

c. If any two rows (or columns) are identical, $\det A = 0$.

d. If A is triangular (all elements above the main diagonal equal to zero), $A = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$:

$$\begin{vmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

e. If to each element of a row or column there is added C times the corresponding element in another row (or column), the value of the determinant is unchanged.

4. *Matrices*

Definition. A matrix is a rectangular array of numbers and is represented by a symbol A or $[a_{ij}]$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

The numbers a_{ij} are termed *elements* of the matrix; subscripts i and j identify the element as the number in row i and column j. The order of the matrix is $m \times n$ ("m by n"). When $m = n$, the matrix is square and is said to be of order n. For a square matrix of order n the elements $a_{11}, a_{22}, \ldots, a_{nn}$ constitute the main diagonal.

5. *Operations*

Addition. Matrices A and B of the same order may be added by adding corresponding elements, i.e., $A + B = [(a_{ij} + b_{ij})]$.

Scalar multiplication. If $A = [a_{ij}]$ and c is a constant (scalar), then $cA = [ca_{ij}]$, that is, every element of A is multiplied by c. In particular, $(-1)A = -A = [-a_{ij}]$ and $A + (-A) = 0$, a matrix with all elements equal to zero.

Multiplication of matrices. Matrices A and B may be multiplied only when they are conformable, which means that the number of columns of A equals the number of rows of B. Thus, if A is $m \times k$ and B is $k \times n$, then the product $C = AB$ exists as an $m \times n$ matrix with elements c_{ij} equal to the sum of products of elements in row

i of A and corresponding elements of column j of B:

$$c_{ij} = \sum_{l=1}^{k} a_{il} b_{lj}$$

For example, if

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & \cdots & a_{mk} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{k1} & b_{k2} & \cdots & b_{kn} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

then element c_{21} is the sum of products $a_{21}b_{11} + a_{22}b_{21} + \ldots + a_{2k}b_{k1}$.

6. *Properties*

$$\begin{aligned} &A + B = B + A \\ &A + (B + C) = (A + B) + C \\ &(c_1 + c_2)A = c_1 A + c_2 A \\ &c(A + B) = cA + cB \\ &c_1(c_2 A) = (c_1 c_2)A \\ &(AB)(C) = A(BC) \\ &(A + B)(C) = AC + BC \\ &AB \neq BA \text{ (in general)} \end{aligned}$$

7. *Transpose*

If A is an $n \times m$ matrix, the matrix of order $m \times n$ obtained by interchanging the rows and columns of A is called the *transpose* and is denoted A^T. The following are properties of A, B, and their respective transposes:

$$\begin{aligned}
(A^T)^T &= A \\
(A+B)^T &= A^T + B^T \\
(cA)^T &= cA^T \\
(AB)^T &= B^T A^T
\end{aligned}$$

A *symmetric* matrix is a square matrix A with the property $A = A^T$.

8. *Identity Matrix*

A square matrix in which each element of the main diagonal is the same constant a and all other elements zero is called a *scalar* matrix.

$$\begin{bmatrix}
a & 0 & 0 & \cdots & 0 \\
0 & a & 0 & \cdots & 0 \\
0 & 0 & a & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \\
0 & 0 & 0 & \cdots & a
\end{bmatrix}$$

When a scalar matrix multiplies a conformable second matrix A, the product is aA; that is, the same as multiplying A by a scalar a. A scalar matrix with diagonal elements 1 is called the *identity*, or *unit* matrix and is denoted I. Thus, for any nth order matrix A, the identity matrix of order n has the property

$$AI = IA = A$$

9. Adjoint

If A is an n-order square matrix and A_{ij} the cofactor of element a_{ij}, the transpose of $[A_{ij}]$ is called the *adjoint* of A:

$$adj A = [A_{ij}]^T$$

10. Inverse Matrix

Given a square matrix A of order n, if there exists a matrix B such that $AB = BA = I$, then B is called the *inverse* of A. The inverse is denoted A^{-1}. A necessary and sufficient condition that the square matrix A have an inverse is det $A \neq 0$. Such a matrix is called *nonsingular*; its inverse is unique and it is given by

$$A^{-1} = \frac{adj A}{\det A}$$

Thus, to form the inverse of the nonsingular matrix A, form the adjoint of A and divide each element of the adjoint by det A. For example,

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 4 & 5 & 6 \end{bmatrix} \text{ has matrix of cofactors}$$

$$\begin{bmatrix} -11 & -14 & 19 \\ 10 & -2 & -5 \\ 2 & 5 & -1 \end{bmatrix},$$

$$\text{adjoint} = \begin{bmatrix} -11 & 10 & 2 \\ -14 & -2 & 5 \\ 19 & -5 & -1 \end{bmatrix} \text{ and determinant 27.}$$

Therefore,

$$A^{-1} = \begin{bmatrix} \frac{-11}{27} & \frac{10}{27} & \frac{2}{27} \\ \frac{-14}{27} & \frac{-2}{27} & \frac{5}{27} \\ \frac{19}{27} & \frac{-5}{27} & \frac{-1}{27} \end{bmatrix}.$$

11. Systems of Linear Equations

Given the system

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + \cdots + & a_{nn}x_n & = & b_n \end{array}$$

a unique solution exists if det $A \neq 0$, where A is the $n \times n$ matrix of coefficients $[a_{ij}]$.

- *Solution by Determinants (Cramer's Rule)*

$$x_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & & \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & & a_{nn} \end{vmatrix} \div \det A$$

$$x_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & & \cdots \\ \vdots & \vdots & & & \\ a_{n1} & b_n & a_{n3} & & a_{nn} \end{vmatrix} \div \det A$$

$$\vdots$$

$$x_k = \frac{\det A_k}{\det A},$$

where A_k is the matrix obtained from A by replacing the kth column of A by the column of b's.

12. Matrix Solution

The linear system may be written in matrix form $AX = B$ where A is the matrix of coefficients $[a_{ij}]$ and X and B are

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If a unique solution exists, $\det A \neq 0$; hence A^{-1} exists and

$$X = A^{-1}B.$$