6 Differential Calculus

1. Notation

For the following equations, the symbols $f(x)$, $g(x)$, etc., represent functions of x. The value of a function $f(x)$ at $x=a$ is denoted $f(a)$. For the function $y=f(x)$ the derivative of y with respect to x is denoted by one of the following:

$$\frac{dy}{dx}, \quad f'(x), \quad D_x y, \quad y'.$$

Higher derivatives are as follows:

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}f'(x)=f''(x)$$

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d}{dx}f''(x)=f'''(x), \text{ etc.}$$

and values of these at $x=a$ are denoted $f''(a)$, $f'''(a)$, etc. (see Table of Derivatives).

2. Slope of a Curve

The tangent line at a point $P(x,y)$ of the curve $y=f(x)$ has a slope $f'(x)$ provided that $f'(x)$ exists at P. The slope at P is defined to be that of the tangent line at P. The tangent line at $P(x_1,y_1)$ is given by

$$y - y_1 = f'(x_1)(x - x_1).$$

The *normal line* to the curve at $P(x_1, y_1)$ has slope $-1/f'(x_1)$ and thus obeys the equation

$$y - y_1 = [-1/f'(x_1)](x - x_1)$$

(The slope of a vertical line is not defined.)

3. *Angle of Intersection of Two Curves*

Two curves, $y = f_1(x)$ and $y = f_2(x)$, that intersect at a point $P(X, Y)$ where derivatives $f_1'(X)$, $f_2'(X)$ exist, have an angle (α) of intersection given by

$$\tan \alpha = \frac{f_2'(X) - f_1'(X)}{1 + f_2'(X) \cdot f_1'(X)}.$$

If $\tan \alpha > 0$, then α is the acute angle; if $\tan \alpha < 0$, then α is the obtuse angle.

4. *Radius of Curvature*

The radius of curvature R of the curve $y = f(x)$ at point $P(x, y)$ is

$$R = \frac{\{1 + [f'(x)]^2\}^{3/2}}{f''(x)}$$

In polar coordinates (θ, r) the corresponding formula is

$$R = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

The *curvature* K is $1/R$.

5. *Relative Maxima and Minima*

The function f has a relative maximum at $x = a$ if $f(a) \geq f(a + c)$ for all values of c (positive or negative) that are sufficiently near zero. The function f has a relative minimum at $x = b$ if $f(b) \leq f(b + c)$ for all values of c that are sufficiently close to zero. If the function f is defined on the closed interval $x_1 \leq x \leq x_2$, and has a relative maximum or minimum at $x = a$, where $x_1 < a < x_2$, and if the derivative $f'(x)$ exists at $x = a$, then $f'(a) = 0$. It is noteworthy that a relative maximum or minimum may occur at a point where the derivative does not exist. Further, the derivative may vanish at a point that is neither a maximum or a minimum for the function. Values of x for which $f'(x) = 0$ are called "critical values." To determine whether a critical value of x, say x_c, is a relative maximum or minimum for the function at x_c, one may use the second derivative test

1. If $f''(x_c)$ is positive, $f(x_c)$ is a minimum
2. If $f''(x_c)$ is negative, $f(x_c)$ is a maximum
3. If $f''(x_c)$ is zero, no conclusion may be made

The sign of the derivative as x advances through x_c may also be used as a test. If $f'(x)$ changes from positive to zero to negative, then a maximum occurs at x_c, whereas a change in $f'(x)$ from negative to zero to positive indicates a minimum. If $f'(x)$ does not change sign as x advances through x_c, then the point is neither a maximum nor a minimum.

6. *Points of Inflection of a Curve*

The sign of the second derivative of f indicates whether the graph of $y=f(x)$ is concave upward or concave downward:

$$f''(x) > 0\text{: concave upward}$$

$$f''(x) < 0\text{: concave downward}$$

A point of the curve at which the direction of concavity changes is called a point of inflection (Figure 6.1). Such a point may occur where $f''(x)=0$ or where $f''(x)$ becomes infinite. More precisely, if the function $y=f(x)$ and its first derivative $y'=f'(x)$ are continuous in the interval $a \le x \le b$, and if $y''=f''(x)$ exists in $a<x<b$, then the graph of $y=f(x)$ for $a<x<b$ is concave

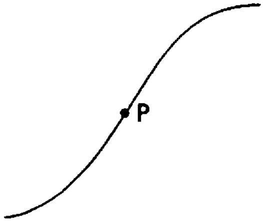

FIGURE 6.1. Point of inflection.

upward if $f''(x)$ is positive and concave downward if $f''(x)$ is negative.

7. *Taylor's Formula*

If f is a function that is continuous on an interval that contains a and x, and if its first $(n+1)$ derivatives are continuous on this interval, then

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$

$$+\frac{f'''(a)}{3!}(x-a)^3+\ldots$$

$$+\frac{f^{(n)}(a)}{n!}(x-a)^n+R,$$

where R is called the *remainder*. There are various common forms of the remainder:

Lagrange's form:

$$R=f^{(n+1)}(\beta)\cdot\frac{(x-a)^{n+1}}{(n+1)!};\ \beta \text{ between } a \text{ and } x.$$

Cauchy's form:

$$R=f^{(n+1)}(\beta)\cdot\frac{(x-\beta)^n(x-a)}{n!};$$

β between a and x.

Integral form:

$$R=\int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt.$$

8. *Indeterminant Forms*

If $f(x)$ and $g(x)$ are continuous in an interval that includes $x=a$ and if $f(a)=0$ and $g(a)=0$, the limit $\lim_{x\to a}(f(x)/g(x))$ takes the form "0/0", called an *indeterminant form*. *L'Hôpital's rule* is

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Similarly, it may be shown that if $f(x)\to\infty$ and $g(x)\to\infty$ as $x\to a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

(The above holds for $x\to\infty$.)

Examples

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1$$

$$\lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}=\lim_{x\to\infty}\frac{2}{e^x}=0$$

9. *Numerical Methods*

a. *Newton's method* for approximating roots of the equation $f(x)=0$: A first estimate x_1 of the root is made; then provided that $f'(x_1) \neq 0$, a better approximation is x_2

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

The process may be repeated to yield a third approximation x_3 to the root:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

provided $f'(x_2)$ exists. The process may be repeated. (In certain rare cases the process will not converge.)

b. *Trapezoidal rule for areas* (Figure 6.2): For the function $y = f(x)$ defined on the interval (a, b) and positive there, take n equal subintervals of width $\Delta x = (b-a)/n$. The area bounded by the curve between

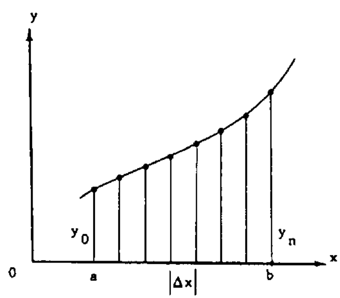

FIGURE 6.2. Trapezoidal rule for area.

$x = a$ and $x = b$ (or definite integral of $f(x)$) is approximately the sum of trapezoidal areas, or

$$A \sim \left(\frac{1}{2}y_0 + y_1 + y_2 + \ldots + y_{n-1} + \frac{1}{2}y_n\right)(\Delta x)$$

Estimation of the error (E) is possible if the second derivative can be obtained:

$$E = \frac{b-a}{12} f''(c)(\Delta x)^2,$$

where c is some number between a and b.

10. Functions of Two Variables

For the function of two variables, denoted $z = f(x, y)$, if y is held constant, say at $y = y_1$, then the resulting function is a function of x only. Similarly, x may be held constant at x_1, to give the resulting function of y.

- *The Gas Laws*

A familiar example is afforded by the ideal gas law that relates the pressure p, the volume V and the absolute temperature T of an ideal gas:

$$pV = nRT$$

where n is the number of moles and R is the gas constant per mole, 8.31 ($\mathrm{J \cdot {}^\circ K^{-1} \cdot mole^{-1}}$). By rearrangement, any one of the three variables may be expressed as a function of the other two. Further, either one of these two may be held constant. If T is held constant, then we get the form known as Boyle's law:

$$p = kV^{-1} \qquad \text{(Boyle's law)}$$

where we have denoted nRT by the constant k and, of course, $V > 0$. If the pressure remains constant, we have Charles' law:

$$V = bT \qquad \text{(Charles' law)}$$

where the constant b denotes nR/p. Similarly, volume may be kept constant:

$$p = aT$$

where now the constant, denoted a, is nR/V.

11. Partial Derivatives

The physical example afforded by the ideal gas law permits clear interpretations of processes in which one of the variables is held constant. More generally, we may consider a function $z = f(x, y)$ defined over some region of the x-y-plane in which we hold one of the two coordinates, say y, constant. If the resulting function of x is differentiable at a point (x, y) we denote this derivative by one of the notations

$$f_x, \qquad \delta f/\delta x, \qquad \delta z/\delta x$$

called the *partial derivative with respect to* x. Similarly, if x is held constant and the resulting function of y is differentiable, we get the *partial derivative with respect to* y, denoted by one of the following:

$$f_y \qquad \delta f/\delta y \qquad \delta z/\delta y$$

Example

$$\text{Given } z = x^4 y^3 - y \sin x + 4y, \text{ then}$$

$$\delta z / \delta x = 4(xy)^3 - y \cos x;$$

$$\delta z / \delta y = 3x^4 y^2 - \sin x + 4.$$