7 Integral Calculus

1. *Indefinite Integral*

If $F(x)$ is differentiable for all values of x in the interval (a, b) and satisfies the equation $dy/dx = f(x)$, then $F(x)$ is an integral of $f(x)$ with respect to x. The notation is $F(x) = \int f(x)\,dx$ or, in differential form, $dF(x) = f(x)\,dx$.

For any function $F(x)$ that is an integral of $f(x)$ it follows that $F(x) + C$ is also an integral. We thus write

$$\int f(x)\,dx = F(x) + C.$$

(See Table of Integrals.)

2. *Definite Integral*

Let $f(x)$ be defined on the interval $[a, b]$ which is partitioned by points $x_1, x_2, \ldots, x_j, \ldots, x_{n-1}$ between $a = x_0$ and $b = x_n$. The jth interval has length $\Delta x_j = x_j - x_{j-1}$, which may vary with j. The sum $\sum_{j=1}^{n} f(v_j)\Delta x_j$, where v_j is arbitrarily chosen in the jth subinterval, depends on the numbers $x_0, \ldots, x_n$ and the choice of the v as well as f; but if such sums approach a common value as all Δx approach zero, then this value is the definite integral of f over the interval (a, b) and is denoted $\int_a^b f(x)\,dx$. The *fundamental theorem of integral calculus* states that

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is any continuous indefinite integral of f in the interval (a, b).

3. *Properties*

$$\int_a^b [f_1(x) + f_2(x) + \cdots + f_j(x)]\,dx = \int_a^b f_1(x)\,dx +$$

$$\int_a^b f_2(x)\,dx + \cdots + \int_a^b f_j(x)\,dx.$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx, \text{ if } c \text{ is a constant.}$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

4. *Common Applications of the Definite Integral*

- *Area (Rectangular Coordinates)*

Given the function $y = f(x)$ such that $y > 0$ for all x between a and b, the area bounded by the curve

$y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b f(x)\, dx.$$

- *Length of Arc* (*Rectangular Coordinates*)

Given the smooth curve $f(x, y) = 0$ from point (x_1, y_1) to point (x_2, y_2), the length between these points is

$$L = \int_{x_1}^{x_2} \sqrt{1 + (dy/dx)^2}\, dx,$$

$$L = \int_{y_1}^{y_2} \sqrt{1 + (dx/dy)^2}\, dy.$$

- *Mean Value of a Function*

The mean value of a function $f(x)$ continuous on $[a, b]$ is

$$\frac{1}{(b-a)} \int_a^b f(x)\, dx.$$

- *Area* (*Polar Coordinates*)

Given the curve $r = f(\theta)$, continuous and non-negative for $\theta_1 \leq \theta \leq \theta_2$, the area enclosed by this curve and the radial lines $\theta = \theta_1$ and $\theta = \theta_2$ is given by

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} [f(\theta)]^2\, d\theta.$$

- *Length of Arc* (*Polar Coordinates*)

Given the curve $r = f(\theta)$ with continuous derivative $f'(\theta)$ on $\theta_1 \leq \theta \leq \theta_2$, the length of arc from $\theta = \theta_1$ to $\theta = \theta_2$ is

$$L = \int_{\theta_1}^{\theta_2} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta.$$

- *Volume of Revolution*

Given a function $y = f(x)$ continuous and non-negative on the interval (a, b), when the region bounded by $f(x)$ between a and b is revolved about the x-axis the volume of revolution is

$$V = \pi \int_a^b [f(x)]^2\, dx.$$

- *Surface Area of Revolution*
(revolution about the x-axis, between a and b)

If the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ is revolved about the x-axis, the area A of the surface generated is given by the following:

$$A = \int_a^b 2\pi f(x) \{1 + [f'(x)]^2\}^{1/2}\, dx$$

- *Work*

If a variable force $f(x)$ is applied to an object in the direction of motion along the x-axis between $x = a$ and $x = b$, the work done is

$$W = \int_a^b f(x)\,dx.$$

5. *Cylindrical and Spherical Coordinates*

a. Cylindrical coordinates (Figure 7.1)

$$x = r\cos\theta$$

$$y = r\sin\theta$$

element of volume $dV = r\,dr\,d\theta\,dz$.

b. Spherical coordinates (Figure 7.2)

$$x = \rho\sin\phi\cos\theta$$

$$y = \rho\sin\phi\sin\theta$$

$$z = \rho\cos\phi$$

element of volume $dV = \rho^2\sin\phi\,d\rho, d\phi\,d\theta$.

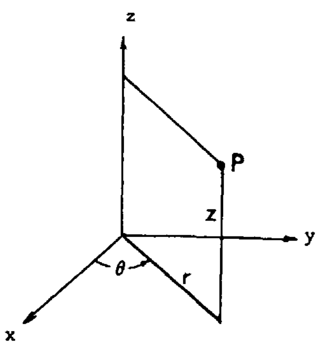


FIGURE 7.1. Cylindrical coordinates.

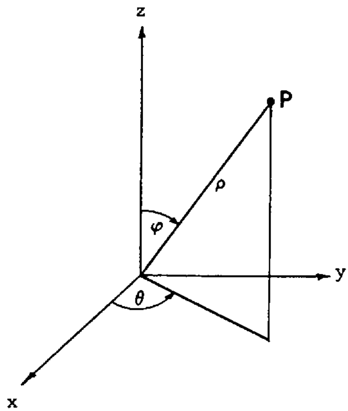


FIGURE 7.2. Spherical coordinates.

6. *Double Integration*

The evaluation of a double integral of $f(x, y)$ over a plane region R

$$\iint_R f(x,y)\, dA$$

is practically accomplished by iterated (repeated) integration. For example, suppose that a vertical straight line meets the boundary of R in at most two points so that there is an upper boundary, $y = y_2(x)$, and a lower boundary, $y = y_1(x)$. Also, it is assumed that these functions are continuous from a to b. (See Fig. 7.3). Then

$$\iint_R f(x,y)\, dA = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x,y)\, dy \right) dx$$

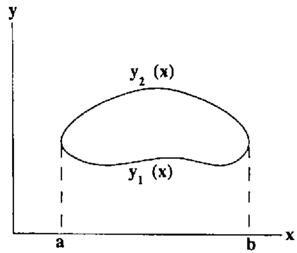


FIGURE 7.3. Region R bounded by $y_2(x)$ and $y_1(x)$.

If R has left-hand boundary, $x = x_1(y)$, and a right-hand boundary, $x = x_2(y)$, which are continuous from c to d (the extreme values of y in R) then

$$\iint_R f(x, y)\, dA = \int_c^d \left(\int_{x_1(y)}^{x_2(y)} f(x, y)\, dx \right) dy$$

Such integrations are sometimes more convenient in polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$; $dA = r\,dr\,d\theta$.

7. *Surface Area and Volume by Double Integration*

For the surface given by $z = f(x, y)$, which projects onto the closed region R of the x–y-plane, one may calculate the volume V bounded above by the surface and below by R, and the surface area S by the following:

$$V = \iint_R z\,dA = \iint_R f(x, y)\, dx\,dy$$

$$S = \iint_R [1 + (\delta z/\delta x)^2 + (\delta z/\delta y)^2]^{1/2}\, dx\,dy$$

[In polar coordinates, (r, θ), we replace dA by $r dr d\theta$].

8. *Centroid*

The centroid of a region R of the x–y-plane is a point (x', y') where

$$x' = \frac{1}{A}\iint_R x\,dA; \qquad y' = \frac{1}{A}\iint_R y\,dA$$

and A is the area of the region.

Example
For the circular sector of angle 2α and radius R, the area A is αR^2; the integral needed for x', expressed in polar coordinates is

$$\iint x\,dA = \int_{-\alpha}^{\alpha}\int_0^R (r\cos\theta) r\,dr\,d\theta$$

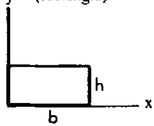
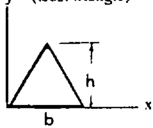
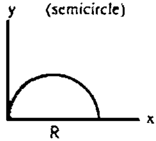
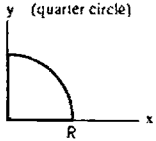
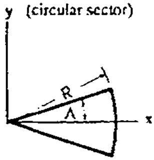
$$= \left[\frac{R^3}{3}\sin\theta\right]_{-\alpha}^{+\alpha} = \frac{2}{3}R^3\sin\alpha$$

and thus,

$$x' = \frac{\frac{2}{3}R^3\sin\alpha}{\alpha R^2} = \frac{2}{3}R\frac{\sin\alpha}{\alpha}.$$

Centroids of some common regions are shown below:

Centroids

	Area	*x'*	*y'*
(rectangle) 	bh	b/2	h/2
(isos. triangle)* 	bh/2	b/2	h/3
(semicircle) 	$\pi R^2/2$	R	$4R/3\pi$
(quarter circle) 	$\pi R^2/4$	$4R/3\pi$	$4R/3\pi$
(circular sector) 	R^2A	$2R \sin A / 3A$	0

* y' = h/3 for any triangle of altitude h.

FIGURE 7.4.