8 Vector Analysis

1. Vectors

Given the set of mutually perpendicular unit vectors **i**, **j**, and **k** (Figure 8.1), then any vector in the space may be represented as $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, where a, b, and c are *components*.

- *Magnitude of* $\mathbf{F}$

$$|\mathbf{F}| = (a^2 + b^2 + c^2)^{\frac{1}{2}}$$

- *Product by scalar p*

$$p\mathbf{F} = pa\mathbf{i} + pb\mathbf{j} + pc\mathbf{k}.$$

- *Sum of* $\mathbf{F}_1$ *and* $\mathbf{F}_2$

$$\mathbf{F}_1 + \mathbf{F}_2 = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j} + (c_1 + c_2)\mathbf{k}$$

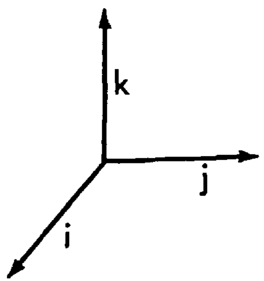

FIGURE 8.1. The unit vectors **i**, **j**, and **k**.

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- *Scalar Product*

$$\mathbf{F}_1 \bullet \mathbf{F}_2 = a_1 a_2 + b_1 b_2 + c_1 c_2$$

(Thus, $\mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{k} = 1$ and $\mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{i} = 0$.)
Also

$$\mathbf{F}_1 \bullet \mathbf{F}_2 = \mathbf{F}_2 \bullet \mathbf{F}_1$$

$$(\mathbf{F}_1 + \mathbf{F}_2) \bullet \mathbf{F}_3 = \mathbf{F}_1 \bullet \mathbf{F}_3 + \mathbf{F}_2 \bullet \mathbf{F}_3$$

- *Vector Product*

$$\mathbf{F}_1 \times \mathbf{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

(Thus, $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$, $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.)
Also,

$$\mathbf{F}_1 \times \mathbf{F}_2 = -\mathbf{F}_2 \times \mathbf{F}_1$$

$$(\mathbf{F}_1 + \mathbf{F}_2) \times \mathbf{F}_3 = \mathbf{F}_1 \times \mathbf{F}_3 + \mathbf{F}_2 \times \mathbf{F}_3$$

$$\mathbf{F}_1 \times (\mathbf{F}_2 + \mathbf{F}_3) = \mathbf{F}_1 \times \mathbf{F}_2 + \mathbf{F}_1 \times \mathbf{F}_3$$

$$\mathbf{F}_1 \times (\mathbf{F}_2 \times \mathbf{F}_3) = (\mathbf{F}_1 \bullet \mathbf{F}_3)\mathbf{F}_2 - (\mathbf{F}_1 \bullet \mathbf{F}_2)\mathbf{F}_3$$

$$\mathbf{F}_1 \bullet (\mathbf{F}_2 \times \mathbf{F}_3) = (\mathbf{F}_1 \times \mathbf{F}_2) \bullet \mathbf{F}_3$$

2. *Vector Differentiation*

If **V** is a vector function of a scalar variable t, then

$$\mathbf{V} = a(t)\mathbf{i} + b(t)\mathbf{j} + c(t)\mathbf{k}$$

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and

$$\frac{d\mathbf{V}}{dt} = \frac{da}{dt}\mathbf{i} + \frac{db}{dt}\mathbf{j} + \frac{dc}{dt}\mathbf{k}.$$

For several vector functions $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n$

$$\frac{d}{dt}(\mathbf{V}_1 + \mathbf{V}_2 + \ldots + \mathbf{V}_n) = \frac{d\mathbf{V}_1}{dt} + \frac{d\mathbf{V}_2}{dt} + \ldots + \frac{d\mathbf{V}_n}{dt},$$

$$\frac{d}{dt}(\mathbf{V}_1 \bullet \mathbf{V}_2) = \frac{d\mathbf{V}_1}{dt} \bullet \mathbf{V}_2 + \mathbf{V}_1 \bullet \frac{d\mathbf{V}_2}{dt},$$

$$\frac{d}{dt}(\mathbf{V}_1 \times \mathbf{V}_2) = \frac{d\mathbf{V}_1}{dt} \times \mathbf{V}_2 + \mathbf{V}_1 \times \frac{d\mathbf{V}_2}{dt}.$$

For a scalar valued function $g(x, y, z)$

(gradient) $$\operatorname{grad} g = \nabla g = \frac{\delta g}{\delta x}\mathbf{i} + \frac{\delta g}{\delta y}\mathbf{j} + \frac{\delta g}{\delta z}\mathbf{k}.$$

For a vector valued function $\mathbf{V}(a, b, c)$, where a, b, c are each a function of x, y, and z,

(divergence) $$\operatorname{div} \mathbf{V} = \nabla \bullet \mathbf{V} = \frac{\delta a}{\delta x} + \frac{\delta b}{\delta y} + \frac{\delta c}{\delta z}$$

(curl) $$\operatorname{curl} \mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\delta}{\delta x} & \dfrac{\delta}{\delta y} & \dfrac{\delta}{\delta z} \\ a & b & c \end{vmatrix}$$

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Also,

$$\operatorname{div}\operatorname{grad} g = \nabla^2 g = \frac{\delta^2 g}{\delta x^2} + \frac{\delta^2 g}{\delta y^2} + \frac{\delta^2 g}{\delta z^2}.$$

and

$$\operatorname{curl}\operatorname{grad} g = \mathbf{0}; \qquad \operatorname{div}\operatorname{curl}\mathbf{V} = 0;$$

$$\operatorname{curl}\operatorname{curl}\mathbf{V} = \operatorname{grad}\operatorname{div}\mathbf{V} - (\mathbf{i}\nabla^2 a + \mathbf{j}\nabla^2 b + \mathbf{k}\nabla^2 c).$$

3. *Divergence Theorem (Gauss)*

Given a vector function F with continuous partial derivatives in a region R bounded by a closed surface S, then

$$\iiint_R \mathit{div}\,\mathbf{F}\,dV = \iint_S \mathbf{n} \bullet \mathbf{F}\,dS,$$

where $\mathbf{n}$ is the (sectionally continuous) unit normal to S.

4. *Stokes' Theorem*

Given a vector function with continuous gradient over a surface S that consists of portions that are piecewise smooth and bounded by regular closed curves such as C, then

$$\iint_S \mathbf{n} \bullet \operatorname{curl}\mathbf{F}\,dS = \oint_C \mathbf{F} \bullet d\mathbf{r}$$

5. *Planar Motion in Polar Coordinates*

Motion in a plane may be expressed with regard to polar coordinates (r, θ). Denoting the position vector by $\mathbf{r}$ and its magnitude by r, we have $\mathbf{r} = r\mathbf{R}(\theta)$, where $\mathbf{R}$ is the unit vector. Also, $\mathrm{d}\mathbf{R}/\mathrm{d}\theta = \mathbf{P}$, a unit vector

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perpendicular to **R**. The velocity and acceleration are then

$$\mathbf{v} = \frac{dr}{dt}\mathbf{R} + r\frac{d\theta}{dt}\mathbf{P};$$

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{R} + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{P}.$$

Note that the component of acceleration in the **P** direction (transverse component) may also be written

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

so that in purely radial motion it is zero and

$$r^2\frac{d\theta}{dt} = C \text{ (constant)}$$

which means that the position vector sweeps out area at a constant rate (see Area in Polar Coordinates, Section 7.4).

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