9 Special Functions

1. Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \operatorname{ctnh} x = \frac{1}{\tanh x}$$

$$\sinh(-x) = -\sinh x \qquad \operatorname{ctnh}(-x) = -\operatorname{ctnh} x$$

$$\cosh(-x) = \cosh x \qquad \operatorname{sech}(-x) = \operatorname{sech} x$$

$$\tanh(-x) = -\tanh x \qquad \operatorname{csch}(-x) = -\operatorname{csch} x$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \operatorname{ctnh} x = \frac{\cosh x}{\sinh x}$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad \cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1) \qquad \operatorname{ctnh}^2 x - \operatorname{csch}^2 x = 1$$

$$\operatorname{csch}^2 x - \operatorname{sech}^2 x = \operatorname{csch}^2 x \operatorname{sech}^2 x \qquad \tanh^2 x + \operatorname{sech}^2 x = 1$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$$

$$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$$

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

$$\tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$$

2. Gamma Function (Generalized Factorial Function)

The gamma function, denoted $\Gamma(x)$, is defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt, \qquad x > 0$$

- *Properties*

$$\Gamma(x+1) = x\Gamma(x), \qquad x > 0$$

$$\Gamma(1) = 1$$

$$\Gamma(n+1) = n\Gamma(n) = n!, \qquad (n = 1, 2, 3, \ldots)$$

$$\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$2^{2x-1}\Gamma(x)\Gamma\left(x + \frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2x)$$

3. Laplace Transforms

The Laplace transform of the function $f(t)$, denoted by $F(s)$ or $L\{f(t)\}$, is defined

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt$$

provided that the integration may be validly performed. A sufficient condition for the existence of $F(s)$ is that $f(t)$ be of exponential order as $t \to \infty$ and that it is sectionally continuous over every finite interval in the range $t \geq 0$. The Laplace transform of $g(t)$ is denoted by $L\{g(t)\}$ or $G(s)$.

- *Operations*

$f(t)$	$F(s) = \int_0^\infty f(t) e^{-st}\, dt$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$e^{at} f(t)$	$F(s-a)$

$\int_0^t f(t-\beta)\cdot g(\beta)\,d\beta$	$F(s)\cdot G(s)$
$f(t-a)$	$e^{-as}F(s)$
$f\left(\frac{t}{a}\right)$	$aF(as)$
$\int_0^t g(\beta)\,d\beta$	$\frac{1}{s}G(s)$
$f(t-c)\delta(t-c)$	$e^{-cs}F(s), c>0$

where

$$\delta(t-c)=0 \text{ if } 0\le t<c$$
$$=1 \text{ if } t\ge c$$

$f(t)=f(t+\omega)$ (periodic)	$\dfrac{\int_0^{\omega} e^{-s\tau}f(\tau)\,d\tau}{1-e^{-s\omega}}$

- *Table of Laplace Transforms*

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
$\dfrac{t^{n-1}}{(n-1)!}$	$1/s^n \qquad (n=1,2,3,\ldots)$
$\sqrt{t}$	$\dfrac{1}{2s}\sqrt{\dfrac{\pi}{s}}$

$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$	
e^{at}	$\frac{1}{s-a}$	
te^{at}	$\frac{1}{(s-a)^2}$	
$\frac{t^{n-1}e^{at}}{(n-1)!}$	$\frac{1}{(s-a)^n}$	$(n=1,2,3,\ldots)$
$\frac{t^x}{\Gamma(x+1)}$	$\frac{1}{s^{x+1}}$,	$x>-1$
$\sin at$	$\frac{a}{s^2+a^2}$	
$\cos at$	$\frac{s}{s^2+a^2}$	
$\sinh at$	$\frac{a}{s^2-a^2}$	
$\cosh at$	$\frac{s}{s^2-a^2}$	
$e^{at}-e^{bt}$	$\frac{a-b}{(s-a)(s-b)}$,	$(a\neq b)$
$ae^{at}-be^{bt}$	$\frac{s(a-b)}{(s-a)(s-b)}$,	$(a\neq b)$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$	
$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$	

$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}$
$\dfrac{\sin at}{t}$	$Arc\tan\dfrac{a}{s}$
$\dfrac{\sinh at}{t}$	$\dfrac{1}{2}\log_e\left(\dfrac{s+a}{s-a}\right)$

4. *Z-Transform*

For the real-valued sequence $\{f(k)\}$ and complex variable z, the z-transform, $F(z)=Z\{f(k)\}$ is defined by

$$Z\{f(k)\}=F(z)=\sum_{k=0}^{\infty} f(k)z^{-k}$$

For example, the sequence $f(k)=1,\ k=0,1,2,\ldots,$ has the z-transform

$$F(z)=1+z^{-1}+z^{-2}+z^{-3}\ldots+z^{-k}+\ldots.$$

• *z-Transform and the Laplace Transform*

For function $U(t)$ the output of the ideal sampler $U^*(t)$ is a set of values $U(kT),\ k=0,1,2,\ldots,$ that is,

$$U^*(t)=\sum_{k=0}^{\infty} U(t)\,\delta(t-kT)$$

The Laplace transform of the output is

$$\mathscr{L}\{U^*(t)\} = \int_0^\infty e^{-st} U^*(t)\, dt = \int_0^\infty e^{-st} \sum_{k=0}^{\infty} U(t)\delta(t-kT)\, dt$$

$$= \sum_{k=0}^{\infty} e^{-skT} U(kT)$$

Defining $z = e^{sT}$ gives

$$\mathscr{L}\{U^*(t)\} = \sum_{k=0}^{\infty} U(kT) z^{-k}$$

which is the z-transform of the sampled signal $U(kT)$.

- *Properties*

Linearity: $Z\{af_1(k) + bf_2(k)\} = aZ\{f_1(k)\} + bZ\{f_2(k)\}$
$= aF_1(z) + bF_2(z)$

Right-shifting property: $Z\{f(k-n)\} = z^{-n}F(z)$

Left-shifting property: $Z\{f(k+n)\} = z^n F(z) - \sum_{k=0}^{n-1} f(k) z^{n-k}$

Time scaling: $Z\{a^k f(k)\} = F(z/a)$

Multiplication by k: $Z\{kf(k)\} = -z\,dF(z)/dz$

Initial value: $f(0) = \lim_{z \to \infty} (1 - z^{-1})F(z) = F(\infty)$

Final value: $\lim_{k \to \infty} f(k) = \lim_{z \to 1} (1 - z^{-1})F(z)$

Convolution: $Z\{f_1(k) * f_2(k)\} = F_1(z)F_2(z)$

- *z-Transforms of Sampled Functions*

$f(k)$	$Z\{f(kT)\} = F(z)$
1 at k; else 0	z^{-k}
1	$\frac{z}{z-1}$
kT	$\frac{Tz}{(z-1)^2}$
$(kT)^2$	$\frac{T^2 z(z+1)}{(z-1)^3}$
$\sin \omega kT$	$\frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$
$\cos \omega T$	$\frac{z(z - \cos \omega T)}{z^2 - 2z \cos \omega T + 1}$
e^{-akT}	$\frac{z}{z - e^{-aT}}$
kTe^{-akT}	$\frac{zTe^{-aT}}{(z - e^{-aT})^2}$

$(kT)^2 e^{-akT}$	$\dfrac{T^2 e^{-aT} z(z+e^{-aT})}{(z-e^{-aT})^3}$
$e^{-akT}\sin\omega kT$	$\dfrac{ze^{-aT}\sin\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$
$e^{-akT}\cos\omega kT$	$\dfrac{z(z-e^{-aT}\cos\omega T)}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$
$a^k\sin\omega kT$	$\dfrac{az\sin\omega T}{z^2-2az\cos\omega T+a^2}$
$a^k\cos\omega kT$	$\dfrac{z(z-a\cos\omega T)}{z^2-2az\cos\omega T+a^2}$

5. *Fourier Series*

The periodic function $f(t)$, with period 2π may be represented by the trigonometric series

$$a_0+\sum_1^\infty (a_n\cos nt+b_n\sin nt)$$

where the coefficients are determined from

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \qquad (n=1,2,3,\ldots)$$

Such a trigonometric series is called the Fourier series corresponding to $f(t)$ and the coefficients are termed Fourier coefficients of $f(t)$. If the function is piecewise continuous in the interval $-\pi \le t \le \pi$, and has left- and right-hand derivatives at each point in that interval, then the series is convergent with sum $f(t)$ except at points t_i at which $f(t)$ is discontinuous. At such points of discontinuity, the sum of the series is the arithmetic mean of the right- and left-hand limits of $f(t)$ at t_i. The integrals in the formulas for the Fourier coefficients can have limits of integration that span a length of 2π, for example, 0 to 2π (because of the periodicity of the integrands).

6. *Functions with Period Other Than 2π*

If $f(t)$ has period P the Fourier series is

$$f(t) \sim a_0 + \sum_1^{\infty} \left(a_n \cos \frac{2\pi n}{P} t + b_n \sin \frac{2\pi n}{P} t \right),$$

where

$$a_0 = \frac{1}{P} \int_{-P/2}^{P/2} f(t)\, dt$$

$$a_n = \frac{2}{P} \int_{-P/2}^{P/2} f(t) \cos \frac{2\pi n}{P} t\, dt$$

$$b_n = \frac{2}{P} \int_{-P/2}^{P/2} f(t) \sin \frac{2\pi n}{P} t\, dt.$$

Again, the interval of integration in these formulas may be replaced by an interval of length P, for example, 0 to P.

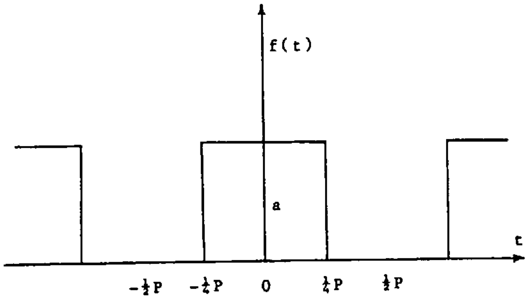

Figure 9.1. Square wave:

$$f(t) \sim \frac{a}{2} + \frac{2a}{\pi}\left(\cos\frac{2\pi t}{P} - \tfrac{1}{3}\cos\frac{6\pi t}{P} + \tfrac{1}{5}\cos\frac{10\pi t}{P} + \dots\right).$$

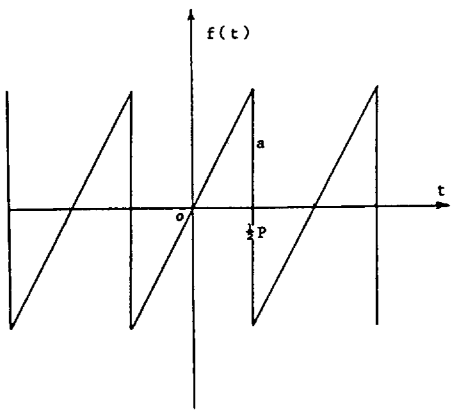

FIGURE 9.2. Sawtooth wave:

$$f(t) \sim \frac{2a}{\pi}\left(\sin\frac{2\pi t}{P} - \tfrac{1}{2}\sin\frac{4\pi t}{P} + \tfrac{1}{3}\sin\frac{6\pi t}{P} - \dots\right).$$

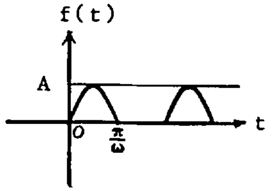

FIGURE 9.3. Half-wave rectifier:

$$f(t) \sim \frac{A}{\pi} + \frac{A}{2}\sin\,\omega t - \frac{2A}{\pi}\left(\frac{1}{(1)(3)}\cos 2\omega t + \frac{1}{(3)(5)}\cos 4\omega t + \dots\right).$$

7. *Bessel Functions*

Bessel functions, also called cylindrical functions, arise in many physical problems as solutions of the differential equation

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

which is known as Bessel's equation. Certain solutions of the above, known as *Bessel functions of the first kind of order n*, are given by

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(-n+k+1)}\left(\frac{x}{2}\right)^{-n+2k}$$

In the above it is noteworthy that the gamma function must be defined for the negative argument q: $\Gamma(q) = \Gamma(q+1)/q$, provided that q is not a negative integer. When q is a negative integer, $1/\Gamma(q)$ is defined to be zero. The functions $J_{-n}(x)$ and $J_n(x)$ are solutions of Bessel's equation for all real n. It is seen, for $n = 1, 2, 3, \ldots$ that

$$J_{-n}(x) = (-1)^n J_n(x)$$

and, therefore, these are not independent; hence, a linear combination of these is not a general solution. When, however, n is not a positive integer, a negative integer, nor zero, the linear combination with arbitrary constants c_1 and c_2

$$y = c_1 J_n(x) + c_2 J_{-n}(x)$$

is the general solution of the Bessel differential equation.

The zero order function is especially important as it arises in the solution of the heat equation (for a "long" cylinder):

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \ldots$$

while the following relations show a connection to the trigonometric functions:

$$J_{\frac{1}{2}}(x) = \left[\frac{2}{\pi x}\right]^{1/2} \sin x$$

$$J_{-\frac{1}{2}}(x) = \left[\frac{2}{\pi x}\right]^{1/2} \cos x$$

The following recursion formula gives $J_{n+1}(x)$ for any order in terms of lower order functions:

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

8. Legendre Polynomials

If Laplace's equation, $\nabla^2 V = 0$, is expressed in spherical coordinates, it is

$$r^2 \sin\theta \frac{\delta^2 V}{\delta r^2} + 2r \sin\theta \frac{\delta V}{\delta r} + \sin\theta \frac{\delta^2 V}{\delta\theta^2} + \cos\theta \frac{\delta V}{\delta\theta}$$

$$+ \frac{1}{\sin\theta} \frac{\delta^2 V}{\delta\phi^2} = 0$$

and any of its solutions, $V(r, \theta, \phi)$, are known as *spherical harmonics*. The solution as a product

$$V(r, \theta, \phi) = R(r)\Theta(\theta)$$

which is independent of ϕ, leads to

$$\sin^2\theta\Theta'' + \sin\theta\cos\theta\Theta' + [n(n+1)\sin^2\theta]\Theta = 0$$

Rearrangement and substitution of $x = \cos\theta$ leads to

$$(1 - x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + n(n+1)\Theta = 0$$

known as *Legendre's equation*. Important special cases are those in which n is zero or a positive integer, and, for such cases, Legendre's equation is satisfied by polynomials called Legendre polynomials, $P_n(x)$. A short list of Legendre polynomials, expressed in terms of x and $\cos\theta$, is given below. These are given by the following general formula:

$$P_n(x) = \sum_{j=0}^{L} \frac{(-1)^j (2n-2j)!}{2^n j!(n-j)!(n-2j)!} x^{n-2j}$$

where $L = n/2$ if n is even and $L = (n-1)/2$ if n is odd. Some are given below:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_0(\cos\theta) = 1$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \frac{1}{4}(3\cos 2\theta + 1)$$

$$P_3(\cos\theta) = \frac{1}{8}(5\cos 3\theta + 3\cos\theta)$$

$$P_4(\cos\theta) = \frac{1}{64}(35\cos 4\theta + 20\cos 2\theta + 9)$$

Additional Legendre polynomials may be determined from the *recursion formula*

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \qquad (n = 1, 2, \ldots)$$

or the *Rodrigues formula*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

9. *Laguerre Polynomials*

Laguerre polynomials, denoted $L_n(x)$, are solutions of the differential equation

$$xy'' + (1-x)y' + ny = 0$$

and are given by

$$L_n(x) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} C_{(n,j)} x^j \qquad (n = 0, 1, 2, \ldots)$$

Thus,

$$L_0(x) = 1$$

$$L_1(x) = 1 - x$$

$$L_2(x) = 1 - 2x + \frac{1}{2}x^2$$

$$L_3(x) = 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3$$

Additional Laguerre polynomials may be obtained from the recursion formula

$$(n+1)L_{n+1}(x) - (2n+1-x)L_n(x)$$
$$+ nL_{n-1}(x) = 0$$

10. Hermite Polynomials

The Hermite polynomials, denoted $H_n(x)$, are given by

$$H_0 = 1, \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n},$$

$$(n = 1, 2, \ldots)$$

and are solutions of the differential equation

$$y'' - 2xy' + 2ny = 0 \qquad (n = 0, 1, 2, \ldots)$$

The first few Hermite polynomials are

$$H_0 = 1 \qquad H_1(x) = 2x$$
$$H_2(x) = 4x^2 - 2 \qquad H_3(x) = 8x^3 - 12x$$
$$H_4(x) = 16x^4 - 48x^2 + 12$$

Additional Hermite polynomials may be obtained from the relation

$$H_{n+1}(x) = 2xH_n(x) - H_n'(x),$$

where prime denotes differentiation with respect to x.

11. *Orthogonality*

A set of functions $\{f_n(x)\}$ $(n=1,2,\ldots)$ is orthogonal in an interval (a,b) with respect to a given weight function $w(x)$ if

$$\int_a^b w(x) f_m(x) f_n(x)\,dx = 0 \qquad \text{when } m \neq n$$

The following polynomials are orthogonal on the given interval for the given $w(x)$:

Legendre polynomials:	$P_n(x)$	$w(x)=1$ $a=-1,\ b=1$
Laguerre polynomials:	$L_n(x)$	$w(x)=\exp(-x)$ $a=0,\ b=\infty$
Hermite polynomials:	$H_n(x)$	$w(x)=\exp(-x^2)$ $a=-\infty,\ b=\infty$

The Bessel functions *of order n*, $J_n(\lambda_1 x)$, $J_n(\lambda_2 x),\ldots$, are orthogonal with respect to $w(x)=x$ over the interval $(0,c)$ provided that the λ_i are the positive roots of $J_n(\lambda c)=0$:

$$\int_0^c x J_n(\lambda_j x) J_n(\lambda_k x)\,dx = 0 \qquad (j \neq k)$$

where n is fixed and $n \geq 0$.