10 Differential Equations

1. *First Order-First Degree Equations*

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

a. If the equation can be put in the form $A(x)\,dx + B(y)\,dy = 0$, it is *separable* and the solution follows by integration: $\int A(x)\,dx + \int B(y)\,dy = C$; thus, $x(1+y^2)\,dx + y\,dy = 0$ is separable since it is equivalent to $x\,dx + y\,dy/(1+y^2) = 0$, and integration yields $x^2/2 + \frac{1}{2}\log(1+y^2) + C = 0$.

b. If $M(x,y)$ and $N(x,y)$ are *homogeneous* and of the *same degree* in x and y, then substitution of vx for y (thus, $dy = v\ dx + x\ dv$) will yield a separable equation in the variables x and y. [A function such as $M(x,y)$ is homogeneous of degree n in x and y if $M(cx,cy) = c^n M(x,y)$.] For example, $(y-2x)dx + (2y+x)dy$ has M and N each homogeneous and of degree one so that substitution of $y = vx$ yields the separable equation

$$\frac{2}{x}\,dx + \frac{2v+1}{v^2+v-1}\,dv = 0.$$

c. If $M(x,y)\,dx + N(x,y)\,dy$ is the differential of some function $F(x,y)$, then the given equation is said to be *exact*. A necessary and sufficient condition for exactness is $\partial M/\partial y = \partial N/\partial x$. When the equation is exact, F is found from the relations $\partial F/\partial x = M$ and $\partial F/\partial y = N$, and the solution is $F(x, y) = C$ (constant). For example, $(x^2 + y)\, dy + (2xy - 3x^2)\, dx$ is exact since $\partial M/\partial y = 2x$ and $\partial N/\partial x = 2x$. F is found from $\partial F/\partial x = 2xy - 3x^2$ and $\partial F/\partial y = x^2 + y$. From the first of these, $F = x^2 y - x^3 + \phi(y)$; from the second, $F = x^2 y + y^2/2 + \Psi(x)$. It follows that $F = x^2 y - x^3 + y^2/2$, and $F = C$ is the solution.

d. Linear, order one in y: Such an equation has the form $dy + P(x) y\, dx = Q(x)\, dx$. Multiplication by $\exp[\int P(x)\, dx]$ yields

$$d\left[y \exp\left(\int P\, dx\right)\right] = Q(x) \exp\left(\int P\, dx\right) dx.$$

For example, $dy + (2/x) y\, dx = x^2\, dx$ is linear in y. $P(x) = 2/x$, so $\int P\, dx = 2 \ln x = \ln x^2$, and $\exp(\int P\, dx) = x^2$. Multiplication by x^2 yields $d(x^2 y) = x^4\, dx$, and integration gives the solution $x^2 y = x^5/5 + C$.

2. *Second Order Linear Equations (With Constant Coefficients)*

$$(b_0 D^2 + b_1 D + b_2) y = f(x), \qquad D = \frac{d}{dx}.$$

a. Right-hand side $= 0$ (homogeneous case)

$$(b_0 D^2 + b_1 D + b_2) y = 0.$$

The *auxiliary equation* associated with the above is

$$b_0 m^2 + b_1 m + b_2 = 0.$$

If the roots of the auxiliary equation are *real and distinct*, say m_1 and m_2, then the solution is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

where the C's are arbitrary constants.

If the roots of the auxiliary equation are *real and repeated*, say $m_1 = m_2 = p$, then the solution is

$$y = C_1 e^{px} + C_2 x e^{px}.$$

If the roots of the auxiliary equation are *complex* $a + ib$ and $a - ib$, then the solution is

$$y = C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx.$$

b. Right-hand side $\neq 0$ (nonhomogeneous case)

$$(b_0 D^2 + b_1 D + b_2) y = f(x)$$

The general solution is $y = C_1 y_1(x) + C_2 y_2(x) + y_p(x)$ where y_1 and y_2 are solutions of the corresponding homogeneous equation and y_p is a solution of the given nonhomogeneous differential equation. y_p has the form $y_p(x) = A(x) y_1(x) + B(x) y_2(x)$ and A and B are found from simultaneous solution of $A' y_1 + B' y_2 = 0$ and $A' y_1' + B' y_2' = f(x)/b_0$. A solution exists if the determinant

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

does not equal zero. The simultaneous equations yield A' and B' from which A and B follow by integration. For example,

$$(D^2 + D - 2)y = e^{-3x}.$$

The auxiliary equation has the distinct roots 1 and -2; hence $y_1 = e^x$ and $y_2 = e^{-2x}$, so that $y_p = Ae^x + Be^{-2x}$. The simultaneous equations are

$$A'e^x - 2B'e^{-2x} = e^{-3x}$$

$$A'e^x + B'e^{-2x} = 0$$

and give $A' = (1/3)e^{-4x}$ and $B' = (-1/3)e^{-x}$. Thus, $A = (-1/12)e^{-4x}$ and $B = (1/3)e^{-x}$ so that

$$y_p = (-1/12)e^{-3x} + (1/3)e^{-3x}$$

$$= \tfrac{1}{4}e^{-3x}.$$

$$\therefore y = C_1 e^x + C_2 e^{-2x} + \tfrac{1}{4}e^{-3x}.$$