Problem Books in Mathematics

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Problems and Theorems in Classical Set Theory

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Dedicated to András Hajnal
and to the memory of
Paul Erdős and Géza Fodor

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Preface

Although the first decades of the 20th century saw some strong debates on set theory and the foundation of mathematics, afterwards set theory has turned into a solid branch of mathematics, indeed, so solid, that it serves as the foundation of the whole building of mathematics. Later generations, honest to Hilbert's dictum, "No one can chase us out of the paradise that Cantor has created for us" proved countless deep and interesting theorems and also applied the methods of set theory to various problems in algebra, topology, infinitary combinatorics, and real analysis.

The invention of forcing produced a powerful, technically sophisticated tool for solving unsolvable problems. Still, most results of the pre-Cohen era can be digested with just the knowledge of a commonsense introduction to the topic. And it is a worthy effort, here we refer not just to usefulness, but, first and foremost, to mathematical beauty.

In this volume we offer a collection of various problems in set theory. Most of classical set theory is covered, classical in the sense that independence methods are not used, but classical also in the sense that most results come from the period, say, 1920–1970. Many problems are also related to other fields of mathematics such as algebra, combinatorics, topology, and real analysis.

We do not concentrate on the axiomatic framework, although some aspects, such as the axiom of foundation or the rôle of the axiom of choice, are elaborated.

There are no drill exercises, and only a handful can be solved with just understanding the definitions. Most problems require work, wit, and inspiration. Some problems are definitely challenging, actually, several of them are published results.

We have tried to compose the sequence of problems in a way that earlier problems help in the solution of later ones. The same applies to the sequence of chapters. There are a few exceptions (using transfinite methods before their discussion)—those problems are separated at the end of the individual chapters by a line of asterisks.

We have tried to trace the origin of the problems and then to give proper reference at the end of the solution. However, as is the case with any other mathematical discipline, many problems are folklore and tracing their origin was impossible.

The reference to a problem is of the form "Problem x.y" where x denotes the chapter number and y the problem number within Chapter x. However, within Chapter x we omit the chapter number, so in that case the reference is simply "Problem y".

For the convenience of the reader we have collected into an appendix all the basic concepts and notations used throughout the book.

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We hope the readers will find as much enjoyment in solving some of the problems as we have found in writing them up.

Péter Komjáth and Vilmos Totik
Budapest and Szeged-Tampa, July 2005

Part I

Problems

1

Operations on sets

Basic operations among sets are union, intersection, and exponentiation. This chapter contains problems related to these basic operations and their relations.

If we are given a family of sets, then (two-term) intersection acts like multiplication. However, from many point of view, the analogue of addition is not union, but forming divided difference: $A\Delta B = (A \setminus B) \cup (B \setminus A)$, and several problems are on this Δ operation.

An interesting feature is that families of sets with appropriate set operations can serve as *canonical models* for structures from other areas of mathematics. In this chapter we shall see that graphs, partially ordered sets, distributive lattices, idempotent rings, and Boolean algebras can be modelled by (i.e., are isomorphic to) families of sets with appropriate operations on them.

1. For finite sets A_i we have

$$|A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots,$$

and

$$|A_1 \cap \cdots \cap A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cup A_j| + \sum_{i<j<k} |A_i \cup A_j \cup A_k| - \cdots.$$

2. Define the symmetric difference of the sets A and B as

$$A\Delta B = (A \setminus B) \cup (B \setminus A).$$

This is a commutative and associative operation such that $\cap$ is distributive with respect to Δ.

3. The set $A_1 \Delta A_2 \Delta \cdots \Delta A_n$ consists of those elements that belong to an odd number of the A_i's.

4. For finite sets A_i we have

$$|A_1 \Delta A_2 \cdots \Delta A_n| = \sum_i |A_i| - 2\sum_{i<j} |A_i \cap A_j| + 4 \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots.$$

5. Let our sets be subsets of a ground set X, and define the complement of A as $A^c = X \setminus A$. All the operations $\cap$, $\cup$ and $\setminus$ can be expressed by the operation $A \downarrow B = (A \cup B)^c$. The same is also true of $A \mid B = (A \cap B)^c$.
6. For any sets

   a)

$$\bigcup_{i\in I} \bigcap_{j\in J_i} A_{i,j} = \bigcap_{f\in\prod_{i\in I} J_i} \bigcup_{i\in I} A_{i,f(i)}$$

   b)

$$\bigcap_{i\in I} \bigcup_{j\in J_i} A_{i,j} = \bigcup_{f\in\prod_{i\in I} J_i} \bigcap_{i\in I} A_{i,f(i)}$$

   c)

$$\prod_{i\in I} \left(\bigcup_{j\in J_i} A_{i,j} \right) = \bigcup_{f\in\prod_{i\in I} J_i} \left(\prod_{i\in I} A_{i,f(i)} \right)$$

   d)

$$\prod_{i\in I} \left(\bigcap_{j\in J_i} A_{i,j} \right) = \bigcap_{f\in\prod_{i\in I} J_i} \left(\prod_{i\in I} A_{i,f(i)} \right)$$

   (general distributive laws).
7. Let X be a set and $A_1, A_2, \ldots, A_n \subseteq X$. Using the operations $\cap$, $\cup$ and $\cdot^c$ (complementation relative to X), one can construct at most 2^{2^n} different sets from $A_1, A_2, \ldots, A_n$.
8. Let

$$X = \{(x_1, \ldots, x_n) \ : \ 0 \le x_i < 1, 1 \le i \le n\}$$

   be the unit cube of $\mathbf{R}^n$, and set

$$A_k = \{(x_1, \ldots, x_n) \in X \ : \ 1/2 \le x_k < 1\}.$$

   Using the operations $\cap$, $\cup$, and $\cdot^c$ (complementation with respect to X), one can construct 2^{2^n} different sets from $A_1, A_2, \ldots, A_n$.
9. Using the operations $\setminus$, $\cap$ and $\cup$ one can construct at most 2^{2^n-1} different sets from a given family $A_1, A_2, \ldots, A_n$ of n sets. This 2^{2^n-1} bound can be achieved for some appropriately chosen $A_1, A_2, \ldots, A_n$.

10. For given A_i, B_i, $i \in I$ solve the system of equations
    (a) $A_i \cap X = B_i, \qquad i \in I,$
    (b) $A_i \cup X = B_i, \qquad i \in I,$
    (c) $A_i \setminus X = B_i, \qquad i \in I,$
    (d) $X \setminus A_i = B_i, \qquad i \in I.$
    What are the necessary and sufficient conditions for the existence and uniqueness of the solutions?
11. If $A_0, A_1, \ldots$ is an arbitrary sequence of sets, then there are pairwise disjoint sets $B_i \subseteq A_i$ such that $\cup A_i = \cup B_i$.
12. Let $A_0, A_1, \ldots$ and $B_0, B_1, \ldots$ be sequences of sets. Then the intersection $A_i \cap B_j$ is finite for all i, j if and only if there are disjoint sets C and D such that for all i the sets $A_i \setminus C$ and $B_i \setminus D$ are finite.
13. Let X be a ground set and $\mathcal{A} \subseteq \mathcal{P}(X)$ such that for every $A \in \mathcal{A}$ the complement $X \setminus A$ can be written as a countable intersection of elements of $\mathcal{A}$. Then the σ-algebra generated by $\mathcal{A}$ coincides with the smallest family of sets including $\mathcal{A}$ and closed under countable intersection and countable disjoint union.
14. Define

$$\liminf_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m,$$

$$\limsup_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m,$$

and we say that the sequence $\{A_n\}$ is convergent if these two sets are the same, say A, in which case we say that the limit of the sets $\{A_n\}$ is A. Then

a) $\liminf_n A_n \subseteq \limsup_n A_n$,
b) $\liminf_n A_n$ consists of those elements that belong to all, but finitely many of the A_n's.
c) $\limsup_n A_n$ consists of those elements that belong to infinitely many A_n's.

15. Let X be a set and for a subset A of X consider its characteristic function

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \in X \setminus A. \end{cases}$$

The mapping $A \to \chi_A$ is a bijection between $\mathcal{P}(X)$ and ${}^X\{0,1\}$. Furthermore, if $B = \liminf_{n\to\infty} A_n$, then

$$\chi_B = \liminf_{n\to\infty} \chi_{A_n},$$

and if $C = \limsup_{n\to\infty} A_n$, then

$$\chi_C = \limsup_{n\to\infty} \chi_{A_n}.$$

16. A sequence $\{A_n\}_{n=1}^{\infty}$ of sets is convergent if and only if for every sequences $\{m_i\}$ and $\{n_i\}$ with $\lim_{i\to\infty} m_i = \lim_{i\to\infty} n_i = \infty$ we have

$$\bigcap_i (A_{m_i} \Delta A_{n_i}) = \emptyset.$$

17. A sequence $\{A_n\}_{n=1}^{\infty}$ of sets converges if and only if for every sequences $\{m_i\}$ and $\{n_i\}$ with $\lim_{i\to\infty} m_i = \lim_{i\to\infty} n_i = \infty$ we have

$$\lim_{i\to\infty} (A_{m_i} \Delta A_{n_i}) = \emptyset$$

(if we regard Δ as subtraction, then this says that for convergence of sets "Cauchy's criterion" holds).

18. If A_n, $n = 0, 1, \ldots$ are subsets of the set of natural numbers, then one can select a convergent subsequence from $\{A_n\}_{n=0}^{\infty}$.
19. Construct a sequence $\{A_n\}_{n=0}^{\infty}$ of sets which does not include a convergent subsequence.
20. If $\mathcal{H}$ is any family of sets, then with the inclusion relation $\mathcal{H}$ is a partially ordered set. Every partially ordered set is isomorphic with a family of sets partially ordered by inclusion.
21. Every graph is isomorphic with a graph where the set of vertices is a family of sets, and two such vertices are connected precisely if their intersection is not empty.
22. Let $\mathcal{H}$ be a set that is closed for two-term intersection, union and symmetric difference. Then $\mathcal{H}$ is a ring with Δ as addition and $\cap$ as multiplication, in which every element is idempotent: $A \cap A = A$.
23. If $(A, +, \cdot, 0)$ is a ring in which every element is idempotent $(a \cdot a = a)$, then $(A, +, \cdot, 0)$ is isomorphic with a ring of sets defined in the preceding problem.
24. With the notation of Problem 22 let $\mathcal{H}$ be the set of all subsets of an infinite set X, and let $\mathcal{I}$ be the set of finite subsets of X. Then $\mathcal{I}$ is an ideal in $\mathcal{H}$. If $a \neq 0$ is any element in the quotient ring $\mathcal{H}/\mathcal{I}$, then there is a $b \neq 0, a$ such that $b \cdot a = b$ (in other words, in the quotient ring there are no atoms).
25. If $\mathcal{H}$ is a family of subsets of a given ground set X which is closed for two-term intersection and union, then $\mathcal{H}$ is a distributive lattice with the operations $H \wedge K = H \cap K$, $H \vee K = H \cup K$.
26. Every distributive lattice is isomorphic to one from the preceding problem.
27. If $\mathcal{H}$ is a family of subsets of a given ground set X which is closed under complementation (relative to X) and under two-term union, then $\mathcal{H}$ is a Boolean algebra with the operations $H \cdot K = H \cap K$, $H + K = H \cup K$, $H' = X \setminus H$ and with $1 = X$, $0 = \emptyset$.

28. Every Boolean algebra is isomorphic to one from the preceding problem.

29. $\mathcal{P}(X)$, the family of all subsets of a given set X, is a complete and completely distributive Boolean algebra with the operations $H \cdot K = H \cap K$, $H + K = H \cup K$, $H' = X \setminus H$ and with $1 = X$, $0 = \emptyset$ (in the Boolean algebra set $a \preceq b$ if $a \cdot b = a$, and completeness means that for any set K in the Boolean algebra there is a smallest upper majorant $\sup K$ and a largest lower minorant $\inf K$, and complete distributivity means that

$$\inf_{i\in I} \sup_{j\in J_i} a_{i,j} = \sup_{f\in\prod_{i\in I} J_i} \inf_i a_{i,f(i)}$$

for any elements in the algebra).

30. Every complete and completely distributive Boolean algebra is isomorphic with one from the preceding problem.

31. Let $\mathcal{H}$ be a family of sets such that if $\mathcal{H}^* \subset \mathcal{H}$ is any subfamily, then there is a smallest (with respect to inclusion) set in $\mathcal{H}$ that includes all the sets in $\mathcal{H}^*$, and there is a largest set in $\mathcal{H}$ that is included in all elements of $\mathcal{H}^*$. Then every mapping $f : \mathcal{H} \to \mathcal{H}$ that preserves the relation $\subseteq$ (i.e., for which $f(H) \subseteq f(K)$ whenever $H \subseteq K$) there is a fixed point, i.e., a set $F \in \mathcal{H}$ with $f(F) = F$.

* * *

32. The converse of Problem 31 is also true in the following sense. Suppose that $\mathcal{H}$ is a family of sets closed for two-term union and intersection such that for every mapping $f : \mathcal{H} \to \mathcal{H}$ that preserves $\subseteq$ there is a fixed point. Then if $\mathcal{H}^* \subset \mathcal{H}$ is any subfamily, then there is a smallest set in $\mathcal{H}$ that includes all the sets in $\mathcal{H}^*$, and there is a largest set in $\mathcal{H}$ that is included in all elements of $\mathcal{H}^*$.

33. With the notation of Problem 24 for each $a \neq 0$ there are at least continuum many different $b \neq 0$ such that $b \cdot a = b$.

34. With the notation of Problem 24 let $\mathcal{H}$ be the set of all subsets of a set X of cardinality κ, and let $\mathcal{I}$ be the ideal of subsets of X which have cardinality smaller than κ. Then the quotient ring $\mathcal{H}/\mathcal{I}$ is of cardinality 2^κ.

2

Countability

A set is called *countable* if its elements can be arranged into a finite or infinite sequence. Otherwise it is called *uncountable*. This notion reflects the fact that the set is "small" from the point of view of set theory; sometimes it is negligible. For example, the set Q of rational numbers is countable (Problem 9) while the set $\mathbf{R}$ of real numbers is not (Problem 7), hence "most" reals are irrational. On the other hand, a claim that a certain set is not countable usually means that the set has many elements.

If in an uncountable set A a certain property holds with the exception of elements in a countable subset B, then the property holds for "most" elements of A (in particular $A \setminus B$ is not empty). In this section many problems are related to this principle; in particular many problems claim that a certain set in $\mathbf{R}$ (or $\mathbf{R}^n$) is countable. Actually, the very first "sensational" achievement of set theory was of this sort when G. Cantor proved in 1874 that "most" real numbers are transcendental (and hence there are transcendental numbers), for the algebraic numbers form a countable subset of $\mathbf{R}$ (see Problems 6–8). Other examples when the notion of countability appears in real analysis will be given in Chapters 5 and 13.

The cardinality of countably infinite sets is denoted by ω or $\aleph_0$.

1. The union of countably many countable sets is countable.
2. The (Cartesian) product of finitely many countable sets is countable.
3. The set of k element sequences formed from a countable sets is countable.
4. The set of finite sequences formed from a countable set is countable.
5. The set of polynomials with integer coefficients is countable.
6. The set of algebraic numbers is countable.
7. $\mathbf{R}$ is not countable.
8. There are transcendental real numbers.
9. The following sets are countable:

a) $\mathbf{Q}$;

b) set of those functions that map a finite subset of a given countable set A into a given countable set B;

c) set of convergent sequences of natural numbers.

10. If $A_i \subseteq \mathbf{N}$, $i \in I$ is an arbitrary family of subsets of $\mathbf{N}$, then there is a countable subfamily A_i, $i \in J \subset I$ such that $\cap_{i\in J} A_i = \cap_{i\in I} A_i$ and $\cup_{i\in J} A_i = \cup_{i\in I} A_i$.
11. If A is an uncountable subset of the real line, then there is an $a \in A$ such that each of the sets $A \cap (-\infty, a)$ and $A \cap (a, \infty)$ is uncountable.
12. If k and K are positive integers and $\mathcal{H}$ is a family of subsets of $\mathbf{N}$ with the property that the intersection of every k members of $\mathcal{H}$ has at most K elements, then $\mathcal{H}$ is countable.
13. The set of subintervals of $\mathbf{R}$ with rational endpoints is countable.
14. Any disjoint collection of open intervals (open sets) on $\mathbf{R}$ (in $\mathbf{R}^n$) is countable.
15. Any discrete set in $\mathbf{R}$ (in $\mathbf{R}^n$) is countable.
16. Any open subset of $\mathbf{R}$ is a disjoint union of countably many open intervals.
17. The set of open disks (balls) in $\mathbf{R}^2$ ($\mathbf{R}^n$) with rational radius and rational center, is countable (rational center means that each coordinate of the center is rational).
18. Any open subset of $\mathbf{R}^2$ ($\mathbf{R}^n$) is a union of countably many open disks (balls) with rational radius and rational center.
19. If $\mathcal{H}$ is a family of circles such that for every $x \in \mathbf{R}$ there is a circle in $\mathcal{H}$ that touches the real line at the point x, then there are two intersecting circles in $\mathcal{H}$.
20. Is it true that if $\mathcal{H}$ is a family of circles such that for every $x \in \mathbf{R}$ there is a circle containing x, then there are two intersecting circles in H?
21. Let $\mathcal{C}$ be a family of circles on the plane such that no two cross each other. Then the points where two circles from $\mathcal{C}$ touch each other form a countable set.
22. One can place only countably many disjoint letters of the shape T on the plane.
23. In the plane call a union of three segments with a common endpoint a Y-set. Any disjoint family of Y-sets is countable.
24. If A is a countable set on the plane, then it can be decomposed as $A = B \cup C$ such that B, resp. C has only a finite number of points on every vertical, resp. horizontal line.
25. A is countable if and only if $A \times A$ can be decomposed as $B \cup C$ such that B intersects every "vertical" line $\{(x, y) \ : \ x = x_0\}$ in at most finitely many points, and C intersects every "horizontal" line $\{(x, y) \ : \ y = y_0\}$ in at most finitely many points.

26. If $A \subset \mathbf{R}$ is countable, then there is a real number a such that $(a+A) \cap A = \emptyset$.

27. If $A \subset \mathbf{R}^2$ is such that all the distances between the points of A are rational, then A is countable. Is there such an infinite bounded set not lying on a straight line?

28. Call a sequence $a_n \to \infty$ faster increasing than $b_n \to \infty$ if $a_n/b_n \to \infty$. If $\{b_n^{(i)}\}$, $i = 0, 1, \ldots$ is a countable family of sequences tending to ∞, then there is a sequence that increases faster than any $\{b_n^{(i)}\}$.

29. If there are given countably many sequences $\{s_n^{(i)}\}_{n=0}^{\infty}$, $i = 0, 1, \ldots$ of natural numbers, then construct a sequence $\{s_n\}_{n=0}^{\infty}$ of natural numbers such that for every i the equality $s_n = s_n^{(i)}$ holds only for finitely many n's.

30. Construct countably many sequences $\{s_n^{(i)}\}_{n=0}^{\infty}$, $i = 0, 1, \ldots$ of natural numbers, with the property that if $\{s_n\}_{n=0}^{\infty}$ is an arbitrary sequence of natural numbers, then the number those n's for which $s_n = s_n^{(i)}$ holds is unbounded as $i \to \infty$.

31. Are there countably many sequences $\{s_n^{(i)}\}_{n=0}^{\infty}$, $i = 0, 1, \ldots$ of natural numbers, with the property that if $\{s_n\}_{n=0}^{\infty}$ is an arbitrary sequence of natural numbers, then the number those n's for which $s_n = s_n^{(i)}$ holds tends to infinity as $i \to \infty$?

32. Let $\{r_k\}$ be a 1–1 enumeration of the rational numbers. Then if $\{x_n\}$ is an arbitrary sequence consisting of rational numbers, there are three permutations π_i, $i = 1, 2, 3$ of the natural numbers for which $x_n = r_{\pi_1(n)} + r_{\pi_2(n)} + r_{\pi_3(n)}$ holds for all n.

33. With the notation of the preceding problem give a sequence $\{x_n\}$ consisting of rational numbers for which there are no permutations π_i, $i = 1, 2$, of the natural numbers for which $x_n = r_{\pi_1(n)} + r_{\pi_2(n)}$ holds for all n.

34. Any two countably infinite Boolean algebras without atoms (i.e., without elements $a \neq 0$ such that $a \cdot b = a$ or $a \cdot b = 0$ for all b) are isomorphic.

35. Let $\mathcal{A} = (A, \ldots)$ be an arbitrary algebraic structure on the countable set A (i.e., $\mathcal{A}$ may have an arbitrary number of finitary operations and relations). Then the following are equivalent:

    a) $\mathcal{A}$ has uncountably many automorphisms;

    b) if B is a finite subset of A then there is a non-identity automorphism of $\mathcal{A}$ which is the identity when restricted to B.

36. Suppose we know that a rabbit is moving along a straight line on the lattice points of the plane by making identical jumps every minute (but we do not know where it is and what kind of jump it is making). If we can place a trap every hour to an arbitrary lattice point of the plane that captures the rabbit if it is there at that moment, then we can capture the rabbit.

37. Let $A \subset [0,1]$ be a set, and two players I and II play the following game: they alternatively select digits (i.e., numbers 0–9) $x_0, x_1, \ldots$ and $y_0, y_1, \ldots$, and I wins if the number $0.x_1y_1x_2y_2\ldots$ is in A, otherwise II wins. In this game if A is countable, then II has a winning strategy.

38. Let $A \subset [0,1]$ be a set, and two players I and II play the following game: I selects infinitely many digits $x_1, x_2, \ldots$ and II makes a permutation $y_1, y_2, \ldots$ of them. I wins if the number $0.y_1y_2\ldots$ is in A, otherwise II wins. For what countable closed sets A does I have a winning strategy?

39. Two players alternately choose uncountable subsets $K_0 \supset K_1 \supset \cdots$ of the real line. Then no matter how the first player plays, the second one can always achieve $\cap_{n=0}^{\infty} K_n = \emptyset$.

* * *

40. Let κ be an infinite cardinal. Then H is of cardinality at most κ if and only if $H \times H$ can be decomposed as $B \cup C$ such that B intersects every "vertical" line $\{(x,y) \ : \ x = x_0\}$ in less than κ points, and C intersects every "horizontal" line $\{(x,y) \ : \ y = y_0\}$ in less than κ points.

3

Equivalence

Equivalence of sets is the mathematical notion of "being of the same size". Two sets A and B are *equivalent* (in symbol $A \sim B$) if there is a one-to-one correspondence between their elements, i.e., a one-to-one mapping $f : A \to B$ of A onto B. In this case we also say that A and B are of the same *cardinality* without telling what "cardinality" means.

A finite set cannot be equivalent to its proper subset, but things change for infinite sets: any infinite set is equivalent to one of its proper subsets. In fact, quite often seemingly "larger" sets (like a plane) may turn out to be equivalent to much "smaller" sets (like a line on the plane).

The notion of infinity is one of the most intriguing concepts that has been created by mankind. It is with the aid of equivalence that in mathematics we can distinguish between different sorts of infinity, and this makes the theory of infinite sets extremely rich.

This chapter contains some simple exercises on equivalence of sets often encountered in algebra, analysis, and topology. To establish the equivalence of two sets can be quite a challenge, but things are tremendously simplified by the *equivalence theorem* (Problem 2): if each of A and B is equivalent to a subset of the other one, then they are equivalent. The reason for the efficiency of the equivalence theorem lies in the fact that usually it is much easier to find a one-to-one mapping of a set A into B than onto B.

1. Let $f : A \to B$ and $g : B \to A$ be 1-to-1 mappings. Then there is a decomposition $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ of A and B into disjoint sets such that f maps A_1 onto B_1 and g maps B_2 onto A_2.
2. (Equivalence theorem) If two sets are both equivalent to a subset of the other one, then the two sets are equivalent.
3. There is a 1-to-1 mapping from $A(\neq \emptyset)$ to B if and only if there is a mapping from B onto A.
4. If A is infinite and B is countable, then $A \cup B \sim A$.
5. If A is uncountable and B is countable, then $A \setminus B \sim A$.

6. The set of irrational numbers is equivalent to the set of real numbers.
7. The Cantor set is equivalent to the set of infinite 0–1 sequences.
8. Give a 1-to-1 mapping from the first set into the second one:
   a) $\mathbf{N} \times \mathbf{N}$; $\mathbf{N}$
   b) $(-\infty, \infty)$; $(0, 1)$
   c) $\mathbf{R}$; the set of infinite 0–1 sequences
   d) the set of infinite 0–1 sequences; $[0, 1]$
   e) the infinite sequences of the natural numbers; the set of infinite 0–1 sequences
   f) the set of infinite sequences of the real numbers; the set of infinite 0–1 sequences

   In each of the above cases **a)–f)** the two sets are actually equivalent.
9. Give a mapping from the first set onto the second one:
   a) $\mathbf{N}$; $\mathbf{N} \times \mathbf{N}$
   b) $\mathbf{N}$; $\mathbf{Q}$
   c) Cantor set; $[0, 1]$
   d) set of infinite 0–1 sequences; $[0, 1]$

   In each of the above cases **a)–d)** the two sets are actually equivalent.
10. Give a 1-to-1 correspondence between these pairs of sets:
    a) (a, b); (c, d) (where $a < b$ and $c < d$, and any of these numbers can be $\pm\infty$ as well)
    b) $\mathbf{N}$; $\mathbf{N} \times \mathbf{N}$
    c) $\mathcal{P}(X)$; ${}^{X}\{0, 1\}$ (X is an arbitrary set)
    d) set of infinite sequences of the numbers $0, 1, 2$; set of infinite 0–1 sequences
    e) $[0, 1)$; $[0, 1) \times [0, 1)$
11. There is a 1-to-1 correspondence between these pairs of sets:
    a) set of infinite 0–1 sequences; $\mathbf{R}$
    b) $\mathbf{R}$; $\mathbf{R}^n$
    c) $\mathbf{R}$; set of infinite real sequences
12. We have
    a) ${}^{B\cup C}A \sim {}^{B}A \times {}^{C}A$ provided $B \cap C = \emptyset$,
    b) ${}^{C}\left({}^{B}A\right) \sim {}^{C\times B}A$,
    c) ${}^{C}(A \times B) \sim {}^{C}A \times {}^{C}B$.
13. Let X be an arbitrary set.
    a) X is similar to a subset of $\mathcal{P}(X)$.
    b) $X \not\sim \mathcal{P}(X)$.

4

Continuum

A set is called *of power continuum* ($\mathbf{c}$) if it is equivalent with $\mathbf{R}$. Many sets arising in mathematical analysis and topology are of power continuum, and the present chapter lists several of them. For example, the set of Borel subsets of $\mathbf{R}^n$, the set of right continuous real functions, or a Hausdorff topological space with countable basis are all of power continuum.

The continuum is also the cardinality of the set of subsets of $\mathbf{N}$, and there are many examples of families of power continuum (i.e., families of maximal cardinality) of subsets of $\mathbf{N}$ or of a given countable set with a certain prescribed property. In particular, several problems in this chapter deal with almost disjoint sets and their variants: there are continuum many subsets of $\mathbf{N}$ with pairwise finite intersection (cf. Problems 29–43).

The problem if there is an uncountable subset of $\mathbf{R}$ which is not of power continuum arose very early during the development of set theory, and the "NO" answer has become known as the continuum hypothesis (CH). Thus, CH means that if $A \subseteq \mathbf{R}$ is infinite, then either $A \sim \mathbf{N}$ or $A \sim \mathbf{R}$ (other formulations are: there is no cardinality κ with $\aleph_0 < \kappa < \mathbf{c}$; $\aleph_1 = 2^{\aleph_0}$). This was the very first problem on Hilbert's famous list on the 1900 Paris congress, and finding the solution had a profound influence on set theory as well as on all of mathematics. Eventually it has turned out that it does not lead to a contradiction if we assume CH (K. Gödel, 1947) and neither leads to a contradiction if we assume CH to be false (P. Cohen, 1963). Therefore, CH is independent of the other standard axioms of set theory.

1. The plane cannot be covered with less than continuum many lines.
2. The set of infinite 0–1 sequences is of power continuum.
3. The set of infinite real sequences is of power continuum.
4. The Cantor set is of power continuum.
5. An infinite countable set has continuum many subsets.
6. An infinite set of cardinality at most continuum has continuum many countable subsets.

7. There are continuum many open (closed) sets in $\mathbf{R}^n$.
8. A Hausdorff topological space with countable base is of cardinality at most continuum.
9. In an infinite Hausdorff topological space there are at least continuum many open sets.
10. If A is countable and B is of cardinality at most continuum, then the set of functions $f : A \to B$ is of cardinality at most continuum.
11. The set of continuous real functions is of power continuum.
12. The product of countably many sets of cardinality at most continuum is of cardinality at most continuum.
13. The union of at most continuum many sets of cardinality at most continuum is of cardinality at most continuum.
14. The following sets are of power continuum.
    a) $\mathbf{R}^n$, $n = 1, 2, \ldots$
    b) $\mathbf{R}^\infty$ (which is the set of infinite real sequences)
    c) the set of continuous curves on the plane
    d) the set of monotone real functions
    e) the set of right-continuous real functions
    f) the set of those real functions that are continuous except for a countable set
    g) the set of lower semi-continuous real functions
    h) the set of permutations of the natural numbers
    i) the set of the (well) orderings of the natural numbers
    j) the set of closed additive subgroups of $\mathbf{R}$ (i.e., the set of additive subgroups of $\mathbf{R}$ that are at the same time closed sets in $\mathbf{R}$)
    k) the set of closed subspaces of $C[0, 1]$
    l) the set of bounded linear transformations of $L^2[0, 1]$
15. $\mathbf{R}$ cannot be represented as the union of countably many sets none of which is equivalent to $\mathbf{R}$.
16. If $A \subset \mathbf{R}^2$ is such that each horizontal line intersects A in finitely many points, then there is a vertical line that intersects the complement $\mathbf{R}^2 \setminus A$ of A in continuum many points.
17. If A is a subset of the real line of power continuum, then there is an $a \in A$ such that each of the sets $A \cap (-\infty, a)$ and $A \cap (a, \infty)$ is of power continuum.
18. Let $\mathcal{A} = (A, \ldots)$ be an arbitrary algebraic structure on the countable set A (i.e., $\mathcal{A}$ may have an arbitrary number of finitary operations and relations). Then the following are equivalent:
    a) $\mathcal{A}$ has uncountably many automorphisms,

b) $\mathcal{A}$ has continuum many automorphisms.

19. A σ-algebra is either finite or of cardinality at least continuum.
20. A σ-algebra generated by a set of cardinality at most continuum is of cardinality at most continuum.
21. There are continuum many Borel sets and Borel functions on the real line (in $\mathbf{R}^n$).
22. There are continuum many Baire functions on $[0,1]$.
23. The power set $\mathcal{P}(X)$ of X is of bigger cardinality than X.
24. If A has at least two elements, then the set ${}^X A$ of mappings from X to A is of bigger cardinality than X.
25. The following sets are of cardinality bigger than continuum.
    a) set of real functions
    b) set of the 1-to-1 correspondences between $\mathbf{R}$ and $\mathbf{R}^2$
    c) set of bases of $\mathbf{R}$ considered as a linear space over $\mathbf{Q}$ (Hamel bases)
    d) set of Riemann integrable functions
    e) set of Jordan measurable subsets of $\mathbf{R}$
    f) set of the additive subgroups of $\mathbf{R}$
    g) set of linear subspaces of $C[0,1]$
    h) set of linear functionals of $L^2[0,1]$
26. Which of the following sets are of power continuum?
    a) the set of real functions that are continuous at every rational point
    b) the set of real functions that are continuous at every irrational point
    c) the set of real functions f that satisfy the Cauchy equation

$$f(x+y) = f(x) + f(y)$$

27. If A is a set of cardinality continuum, then there are countably many functions $f_k : A \to \mathbf{N}$, $k = 0, 1, \ldots$ such that for an arbitrary function $f : A \to \mathbf{N}$ and for an arbitrary finite set $A' \subset A$ there is a k such that f_k agrees with f on A'.
28. The topological product of continuum many separable spaces is separable.
29. There are continuum many sets $A_\gamma \subseteq \mathbf{N}$ such that if $\gamma_1 \neq \gamma_2$, then $A_{\gamma_1} \cap A_{\gamma_2}$ is a finite set (such a collection is called almost disjoint).
30. Let k be a natural number, and suppose that A_γ, $\gamma \in \Gamma$ is a family of subsets of $\mathbf{N}$ such that if $\gamma_1 \neq \gamma_2$, then $A_{\gamma_1} \cap A_{\gamma_2}$ has at most k elements. Then Γ is countable.
31. To every $x \in \mathbf{R}$ one can assign a sequence $\{s_n^{(x)}\}$ of natural numbers such that if $x < y$, then $s_n^{(y)} - s_n^{(x)} \to \infty$ as $n \to \infty$.

32. There are continuum many sequences $\{s^\gamma\}_{n=0}^\infty$ of natural numbers such that if $\gamma_1 \neq \gamma_2$, then $|s_n^{\gamma_1} - s_{k_n}^{\gamma_2}|$ tends to infinity as $n \to \infty$, no matter how we choose the sequence $\{k_n\}$.
33. Let k be a positive integer, and suppose that $\{s_n^\gamma\}_{n=0}^\infty$, $\gamma \in \mathit{\Gamma}$ is a family of sequences of natural numbers such that if $\gamma_1 \neq \gamma_2$ then $s_n^{\gamma_1} = s_n^{\gamma_2}$ holds for at most k indices n. Then $\mathit{\Gamma}$ is countable.
34. There is an almost disjoint family of cardinality continuum of subsets of $\mathbf{N}$ each with upper density 1.
35. Let $k \geq 2$ be an integer. Then there is a family of cardinality continuum of subsets of $\mathbf{N}$ such that the intersection of any k members of the family is infinite, but the intersection of any $k+1$ members is finite.
36. If $\mathcal{H}$ is an uncountable family of subsets of $\mathbf{N}$ such that the intersection of any finitely many members of the family is infinite, then the intersection of some infinite subfamily of $\mathcal{H}$ is also infinite.
37. There is a family of cardinality continuum of subsets of $\mathbf{N}$ such that the intersection of any finitely many members of the family has positive upper density, but the intersection of any infinitely many members is of density zero.
38. If $\mathcal{H}$ is a family of subsets of $\mathbf{R}$ such that the intersection of any two sets in $\mathcal{H}$ is finite, then $\mathcal{H}$ is of cardinality at most continuum.
39. There is a family $\mathcal{H}$ of cardinality bigger than continuum of subsets of $\mathbf{R}$ such that the intersection of any two sets in $\mathcal{H}$ is of cardinality smaller than continuum.
40. The are continuum many sets $A_\gamma \subset \mathbf{N}$ such that if $\gamma_1 \neq \gamma_2$, then either $A_{\gamma_1} \subset A_{\gamma_2}$ or $A_{\gamma_2} \subset A_{\gamma_1}$.
41. There are continuum many sets $A_\gamma \subset \mathbf{N}$ such that if $\gamma_1 \neq \gamma_2$, then each of the sets $A_{\gamma_1} \setminus A_{\gamma_2}$, $A_{\gamma_2} \setminus A_{\gamma_1}$, and $A_{\gamma_1} \cap A_{\gamma_2}$ is infinite.
42. For every real number x give sets $A_x, B_x \subseteq \mathbf{N}$ such that $A_x \cap B_x = \emptyset$, but for different x and y the set $A_x \cap B_y$ is infinite.
43. There is a family A_x, $x \in \mathbf{R}$ of subsets of the natural numbers such that if $x_1, \ldots, x_n$ are different reals and $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$, then the density of the set $A_{x_1}^{\epsilon_1} \cap \cdots \cap A_{x_n}^{\epsilon_n}$ is 2^{-n} (here $A^1 = A$ and $A^0 = \mathbf{N} \setminus A$).
44. There is a function $f : \mathbf{R}^2 \to \mathbf{N}$ such that $f(x, y) = f(y, z)$ implies $x = y = z$.

5

Sets of reals and real functions

This chapter contains various problems from analysis and from the topology of Euclidean spaces that are connected with the notions of “countability” and “continuum”. They include problems on exceptional sets (like a monotone real function can have only countably many discontinuities), Lindelöf-type covering theorems and their consequences, Baire properties, Borel sets, and Peano curves.

1. If $A \subset \mathbf{R}$ is such that for every $a \in A$ there is a $\delta_a > 0$ such that either $(a, a + \delta_a) \cap A = \emptyset$ or $(a - \delta_a, a) \cap A = \emptyset$, then A is countable.
2. Any uncountable subset A of the real numbers includes a strictly decreasing sequence converging to a point in A.
3. Every discrete set on $\mathbf{R}$ (in $\mathbf{R}^n$) is countable.
4. A right-continuous real function can have only countably many discontinuities.
5. Let f be a real function such that at every point f is continuous either from the right or from the left. Then f can have only countably many discontinuities.
6. A monotone real function can have only countably many discontinuities.
7. If a real function has right and left derivatives at every point, then it is differentiable at every point with the exception of a countable set.
8. A convex function is differentiable at every point with the exception of a countable set.
9. The set of local maximum *values* of any real function is countable.
10. The set of strict local maximum points of a real function is countable.
11. If every point is a local extremal point for a continuous real function f, then f is constant.
12. If a collection G_γ, $\gamma \in \Gamma$ of open sets in $\mathbf{R}^n$ covers a set E, then there is a countable subcollection G_{γ_i}, $i = 0, 1, \ldots$, that also covers E (this property of subsets of $\mathbf{R}^n$ is called the Lindelöf property).

It is customary to rephrase the problem by saying that in $\mathbf{R}^n$ every open cover of a set includes a countable subcover.

13. If a collection G_γ, $\gamma \in \Gamma$ of semi-open intervals in $\mathbf{R}$ covers a set E, then there is a countable subcollection G_{γ_i}, $i = 0, 1, \ldots$, that also covers E. The same is true if the G_γ's are arbitrary nondegenerated intervals.
14. If a collection G_γ, $\gamma \in \Gamma$, nondegenerated intervals in $\mathbf{R}$ covers a set E, then there is a countable subcollection G_{γ_i}, $i = 0, 1, \ldots$, that also covers E.
15. Let the real function f be differentiable at every point of the set $H \subset \mathbf{R}$. Then the set of those y for which $f^{-1}\{y\} \cap H$ is uncountable is of measure zero.
16. Call a rectangle almost closed if its sides are parallel with the coordinate axes, and it is obtained from a closed rectangle by omitting the four vertices. Show that any union of a family of almost closed rectangles is already a union of a countable subfamily. Is the same true if the rectangles are closed?
17. Call x an accumulation point of a set $A \subset \mathbf{R}$ ($A \subset \mathbf{R}^n$) if every neighborhood of x contains uncountably many points of A. An uncountable set A has an accumulation point that lies in A.
18. For an uncountable $A \subset \mathbf{R}$ let A^* be the set of those $a \in A$ that are accumulation points of both $A \cap (-\infty, a)$ and of $A \cap (a, \infty)$. Then $A \setminus A^*$ is countable, and A^* is densely ordered.
19. The set of accumulation points of any set A is either empty or perfect.
20. Any closed set in $\mathbf{R}$ ($\mathbf{R}^n$) is the union of a perfect and a countable set.
21. A nonempty perfect set in $\mathbf{R}^n$ is of power continuum.
22. A closed set in $\mathbf{R}$ ($\mathbf{R}^n$) is either countable, or of power continuum.
23. Define the distance between two real sequences $\{a_j\}_{j=0}^\infty$ and $\{b_j\}_{j=0}^\infty$ by the formula

$$d\left(\{a_j\}_{j=0}^\infty, \{b_j\}_{j=0}^\infty\right) = \sum_{j=0}^\infty \frac{1}{2^j} \frac{|a_j - b_j|}{1 + |a_j - b_j|}.$$

With this $\mathbf{R}^\infty$ becomes a complete separable metric space.

24. Every closed set in $\mathbf{R}^\infty$ is the union of a perfect and a countable set.
25. Every closed set in $\mathbf{R}^\infty$ is either countable or of cardinality continuum.
26. Every Borel set in $\mathbf{R}^n$ is a (continuous and) one-to-one image of a closed subset of $\mathbf{R}^\infty$.
27. In $\mathbf{R}^n$ every Borel set is either countable or of cardinality continuum.
28. If $a < b$ and $[a, b] = \cup_{i=0}^\infty A_i$, then there is an interval $I \subset [a, b]$ and an i such that the set A_i is dense in I (Baire's theorem).

29. If $a < b$ and $[a,b] = \cup_{i=0}^{\infty} A_i$, then there is an interval $I \subset [a,b]$ and an i such that for any subinterval J of I the intersection $A_i \cap J$ is of power continuum.

30. If $A \subset \mathbf{R}^n$ is a set with nonempty interior, then A cannot be represented as a countable union of nowhere dense sets (Baire's theorem).

31. If $A \subset \mathbf{R}^n$ is a set with nonempty interior and $A = \cup_{i=0}^{\infty} A_i$, then there is a ball $B \subset A$ and an i such that for any ball $B' \subset B$ the intersection $A_i \cap B'$ is of power continuum.

32. There are pairwise disjoint sets $A_x \subset \mathbf{R}$, $x \in \mathbf{R}$ such that for any $x \in \mathbf{R}$ and any open interval $I \subset \mathbf{R}$ the set $I \cap A_x$ is of power continuum.

33. There is a real function that assumes every value in every interval continuum many times.

34. There is a continuous function $f : [0,1] \to [0,1]$ that assumes every value $y \in [0,1]$ continuum many times.

35. There exists a continuous mapping from $[0,1]$ onto $[0,1] \times [0,1]$ (such "curves" are called area filling or Peano curves).

36. There are continuous functions $f_n : [0,1] \to [0,1]$, $n = 0,1,2,\ldots$ with the property that if $x_0, x_1, \ldots$ is an arbitrary sequence from $[0,1]$, then there is a $t \in [0,1]$ such for all n we have $f_n(t) = x_n$ (thus, $F(t) = (f_0(t), f_1(t), \ldots)$ is a continuous mapping from $[0,1]$ onto the so-called Hilbert cube $[0,1]^\infty \equiv {}^{\mathbf{N}}[0,1]$).

* * *

37. If $\{a_\xi\}_{\xi<\omega_1}$ is a transfinite sequence of real numbers which is convergent (i.e., there is an $A \in \mathbf{R}$ such that for every $\epsilon > 0$ there is a $\nu < \omega_1$ for which $\xi > \nu$ implies $|a_\xi - A| \le \epsilon$), then there is a $\tau < \omega_1$ such that $a_\xi = a_\zeta$ for $\xi, \zeta > \tau$.

38. If $\{a_\xi\}_{\xi<\alpha}$ is a (strictly) monotone transfinite sequence of real numbers, then α is countable.

39. For every limit ordinal $\alpha < \omega_1$ there is a convergent, strictly increasing transfinite sequence $\{a_\xi\}_{\xi<\alpha}$ of real numbers (convergence means that there is an $A \in \mathbf{R}$ such that for every $\epsilon > 0$ there is a $\nu < \alpha$ for which $\xi > \nu$ implies have $|a_\xi - A| \le \epsilon$).

6

Ordered sets

Now we equip our sets with a structure by telling which element is larger than the other one. The theory of ordered sets is extremely rich, in fact, this list of problems is the longest one in the book.

This chapter contains problems on ordered sets and mappings between them. The types of ordered sets and arithmetic with types will be discussed in the next chapter. Occasionally later chapters will also discuss problems on ordered sets if the solution requires the methods of those chapters.

Particularly important are the well-ordered sets (see below), for they provide the infinite analogues of natural numbers. Well orderings offer enumeration of the elements of a given set in a transfinite sequence and thereby the possibility of proving results by transfinite induction.

Let A be a set and $\prec$ a binary relation on A. If $a \prec b$ does not hold, then we write $a \not\prec b$. $\langle A, \prec\rangle$ is called an *ordered set* (sometimes called linearly ordered) if

- $\prec$ *irreflexive*: $a \not\prec a$ for any $a \in A$,
- $\prec$ *transitive*: $a \prec b$ and $b \prec c$ imply $a \prec c$,
- $\prec$ *trichotomous*: for every $a, b \in A$ one of $a \prec b$, $a = b$, $b \prec a$ holds.

With every such "smaller than" relation $\prec$ we associate the corresponding "smaller than or equal" relation $\preceq$: $a \preceq b$ if either $a \prec b$ or $a = b$. This $\preceq$ has the following properties:

- *antisymmetric*: $a \preceq b$ and $b \preceq a$ imply $a = b$,
- *transitive*: $a \preceq b$ and $b \preceq c$ imply $a \preceq c$,
- *dichotomous*: for every $a, b \in A$ either of $a \preceq b$ or $b \preceq a$ holds.

If $\langle A, \prec\rangle$ is an ordered set and $B \subset A$ is a subset of A, then for notational simplicity we shall continue to denote the restriction of $\prec$ to $B \times B$ by $\prec$, so $\langle B, \prec\rangle$ is the ordered set with ground set B and with the ordering inherited from $\langle A, \prec\rangle$.

The ordered set $\langle A, \prec\rangle$ is called *well ordered* if every nonempty subset contains a smallest element, i.e., if for every $X \subseteq A$, $X \neq \emptyset$ there is an $a \in X$ such that for every $b \in X$ we have $a \preceq b$.

If $\langle A, \prec\rangle$ is an ordered set, then $X \subseteq A$ is an *initial segment* if $a \in X$ and $b \prec a$ imply $b \in X$ (intuitively, X consists of a starting section of $\langle A, \prec\rangle$), and in a similar fashion $X \subseteq A$ is called an *end segment* if $a \in X$ and $a \prec b$ imply $b \in X$. An initial segment that is not the whole set is called a proper initial segment. The intervals of $\langle A, \prec\rangle$ are its "convex" (or "connected") subsets, i.e., $X \subseteq A$ is an *interval* if $a, b \in X$ and $a \prec c \prec b$ implies $c \in X$. The intervals generate the so-called *interval topology* (also called *order topology*) on A. This is also the topology that is generated by the initial and end segments of $\langle A, \prec\rangle$.

Ordered sets are special algebraic structures (with no operations, and a single binary relation). Isomorphism among ordered sets is called similarity: $\langle A_1, \prec_1\rangle$ and $\langle A_2, \prec_2\rangle$ are *similar* if there is an $f : A_1 \to A_2$ 1-to-1 correspondence between the ground sets A_1 and A_2 that also preserves the ordering, i.e., $a \prec_1 b$ implies $f(a) \prec_2 f(b)$. In particular, similarity implies the equivalence of the ground sets. A mapping f from $\langle A_1, \prec_1\rangle$ into $\langle A_2, \prec_2\rangle$ (not necessarily onto) is called *monotone* if $a <_1 b$ implies $f(a) <_2 f(b)$. This is just the same as the notion of homomorphism from $\langle A_1, \prec_1\rangle$ into $\langle A_2, \prec_2\rangle$.

The *lexicographic product* of $\langle A_1, \prec_1\rangle$ and $\langle A_2, \prec_2\rangle$ is the ordered set $\langle A_1 \times A_2, \prec\rangle$ where $(a_1, a_2) \prec (a_1', a_2')$ precisely if $a_1 \prec_1 a_1'$ or if $a_1 = a_1'$ and $a_2 \prec_2 a_2'$ (i.e., in this ordering the first coordinate is decisive). On the other hand, in *antilexicographic ordering* first we compare the second coordinates and only when equality occurs compare the first coordinates. One can define in a similar manner the lexicographic or antilexicographic product of more than two sets. Lexicographic (antilexicographic) ordering is sometimes called ordering according to the first (last) difference.

Let $\langle A_i, <_i\rangle$, $i \in I$ be ordered sets with pairwise disjoint ground sets A_i and let the index set I be also ordered by the relation $<$. The *ordered union* of $\langle A_i, <_i\rangle$, $i \in I$ with respect to the ordered set $\langle I, <\rangle$ is the ordered set $\langle B, \prec\rangle$ in which $B = \cup_{i\in I} A_i$, and for $a \in A_i$ and $b \in A_j$ the relation $a \prec b$ holds if and only if $i < j$ or $i = j$ and $a <_i b$. The antilexicographic product of $\langle A_1, \prec_1\rangle$ and $\langle A_2, \prec_2\rangle$ is nothing else than the ordered union of the sets $\langle A_1 \times \{a\}, \prec_a\rangle$, $a \in A_2$ (where $(p, a) \prec_a (q, a)$ if and only if $p \prec_1 q$) with respect to $\langle A_2, \prec_2\rangle$.

Unless otherwise stated, if A is a subset of the real line, then we regard A to be ordered with respect to the standard $<$ relation between the reals. In this chapter we mean strict monotonicity if we say that a real-valued function on a subset of the reals is monotone.

An important concept related to ordered sets is their cofinality, which will be used many times in later chapters. A theorem of Hausdorff (Problem 44) says that in every ordered set $\langle A, \prec\rangle$ there is a well-ordered *cofinal subset*, i.e., a subset $B \subseteq A$ such that $\langle B, \prec\rangle$ is well ordered and for every $a \in A$ there is a $b \in B$ with $a \preceq b$. Now the *cofinality* $\mathrm{cf}(\langle A, \prec\rangle)$ is defined as the smallest possible order type of such cofinal $\langle B, \prec\rangle$'s.

The solutions of some problems require the following important result of R. Laver (see On Fraïssé's order type conjecture, *Ann. Math.,* **93**(1971), 89–111): If $\langle A_i, <_i \rangle$, $i = 0, 1, 2, \ldots$, are ordered sets such that neither of them includes a densely ordered subset, then there are $i < j$ such that $\langle A_i, <_i \rangle$ is similar to a subset of $\langle A_j, <_j \rangle$. The proof is considerably more complicated than it could be given in this book.

1. Any infinite sequence of different elements in an ordered set includes an infinite monotone subsequence.
2. Any two open subintervals of $\mathbf{R}$ are similar.
3. Give an ordered set with a smallest element, in which every element has a successor and every element but the least has a predecessor, yet the set is not similar to $\mathbf{N}$.
4. Give an ordering on the reals for which every element has a successor, as well as a predecessor.
5. An infinite ordered set $\langle A, \prec \rangle$ is similar to $\mathbf{N}$ if and only if for every $a \in A$ there are only finitely many elements $b \in A$ with $b \prec a$.
6. What are those infinite ordered sets $\langle A, \prec \rangle$ for which it is true that every infinite subset of A is similar to $\langle A, \prec \rangle$?
7. An infinite ordered set $\langle A, \prec \rangle$ is similar to $\mathbf{Z}$ if and only if it has no smallest or largest element, and every interval $\{c : a \prec c \prec b\}$, $a, b \in A$ is finite.
8. What are the infinite ordered sets $\langle A, \prec \rangle$ for which every interval $\{c : a \prec c \prec b\}$, $a, b \in A$ is finite?
9. There is a countable ordered set that has continuum many initial segments.
10. There is an ordered set of cardinality continuum that has more than continuum many initial segments.
11. There are infinitely many pairwise nonsimilar ordered sets such that every one of them is similar to an initial segment of any other one.
12. Let $\langle A, \prec \rangle$ and $\langle A', \prec' \rangle$ be ordered sets such that each of them is similar to a subset of the other one. Then there are disjoint decompositions $A = A_1 \cup A_2$ and $A' = A'_1 \cup A'_2$ such that $\langle A_i, \prec \rangle$ is similar to $\langle A'_i, \prec' \rangle$ for $i = 1, 2$.
13. If $\langle A, < \rangle$ and $\langle B, \prec \rangle$ are ordered sets such that $\langle A, < \rangle$ is similar to an initial segment of $\langle B, \prec \rangle$ and $\langle B, \prec \rangle$ is similar to an end segment of $\langle A, < \rangle$, then $\langle A, < \rangle$ and $\langle B, \prec \rangle$ are similar.
14. If $\langle A, < \rangle$ and $\langle B, \prec \rangle$ are ordered sets such that $\langle A, < \rangle$ is similar to an initial segment and to an end segment of $\langle B, \prec \rangle$ and $\langle B, \prec \rangle$ is similar an interval of $\langle A, < \rangle$, then $\langle A, < \rangle$ and $\langle B, \prec \rangle$ are similar.
15. There are continuum many subsets of $\mathbf{Q}$ no two of them similar.
16. How many subsets A does $\mathbf{R}$ have for which A is similar to $\mathbf{R}$?

17. There are continuum many pairwise disjoint subsets of $\mathbf{R}$ each similar to $\mathbf{R}$.
18. If $A \subseteq \mathbf{R}$, $A \neq \emptyset$, then $\mathbf{R}$ has continuum many subsets similar to A.
19. $\mathbf{R}$ has $2^{\mathfrak{c}}$ subsets of cardinality continuum no two of which are similar.
20. If we omit a countable set from the set of irrational numbers, then the set obtained is similar to the set of the irrational numbers.
21. If $\langle A, \prec \rangle$ has a countable subset B that is dense in A (i.e., for every $a_1, a_2 \in A$, $a_1 \prec a_2$ there is $b \in B$ such that $a_1 \preceq b \preceq a_2$), then $\langle A, \prec \rangle$ is similar to a subset of $\mathbf{R}$.
22. Suppose $A, B \subseteq \mathbf{R}$ are two similar subsets of $\mathbf{R}$. Is it true that then their complements $\mathbf{R} \setminus A$ and $\mathbf{R} \setminus B$ are also similar? What if A and B are countable dense subsets of $\mathbf{R}$?
23. Let $\mathcal{M}$ be a set of open subsets of $\mathbf{R}$ ordered with respect to inclusion "$\subset$". Then $\langle \mathcal{M}, \subset \rangle$ is similar to a subset of the reals.
24. There is a family $\mathcal{F}$ of closed and measure zero subsets of $\mathbf{R}$ such that $\langle \mathcal{F}, \subset \rangle$ is similar to $\mathbf{R}$.
25. There is a family of cardinality bigger than continuum of subsets of $\mathbf{R}$ that is ordered with respect to inclusion.
26. Any countable ordered set is similar to a subset of $\mathbf{Q} \cap (0, 1)$.
27. Any countable densely ordered set without smallest and largest elements is similar to $\mathbf{Q}$.
28. Any countable densely ordered set is similar to one of the sets $\mathbf{Q} \cap (0, 1)$, $\mathbf{Q} \cap [0, 1)$, $\mathbf{Q} \cap (0, 1]$, $\mathbf{Q} \cap [0, 1]$ (depending if it has a first or last element).
29. There is an uncountable ordered set such that all of its proper initial segments are similar to $\mathbf{Q}$ or to $\mathbf{Q} \cap (0, 1]$.
30. There is an uncountable ordered set which is similar to each of its uncountable subsets.
31. The antilexicographically ordered set of infinite 0–1 sequences that contain only a finite number of 1's is similar to $\mathbf{N}$.
32. The lexicographically ordered set of infinite 0–1 sequences that contain only a finite number of 1's is similar to $\mathbf{Q} \cap [0, 1)$.
33. The lexicographically ordered set of infinite 0–1 sequences is similar to the Cantor set.
34. The lexicographically ordered set of sequences of natural numbers is similar to $[0, 1)$.
35. Consider the set A of all sequences $n_0, -n_1, n_2, -n_3, \ldots$ where n_i are natural numbers. Then A, with the lexicographic ordering, is similar to the set of irrational numbers.
36. An ordered set is well ordered if and only if it does not include an infinite decreasing sequence.

37. If $A \subseteq \mathbf{R}$ is well ordered, then it is countable.
38. If $\mathcal{U}$ is a family of open (closed) subsets of $\mathbf{R}$ that is well ordered with respect to inclusion, then $\mathcal{U}$ is countable.
39. If $\langle A, \prec \rangle$ is well ordered, then for any $f : A \to A$ monotone mapping and for any $a \in A$ we have $a \preceq f(a)$.
40. There is at most one similarity mapping between two well-ordered sets.
41. A well-ordered set cannot be similar to a subset of one of its proper initial segments.
42. Given two well-ordered sets, one of them is similar to an initial segment of the other.
43. Two well-ordered sets, each of which is similar to a subset of the other one, are similar.
44. (Hausdorff's theorem) For every ordered set $\langle A, \prec \rangle$ there is a subset $B \subseteq A$ such that $\langle B, \prec \rangle$ is well ordered and cofinal (if $a \in A$ is arbitrary, then there is a $b \in B$ with $a \preceq b$). Furthermore, $B \subseteq A$ can also be selected in such a way that the order type of $\langle B, \prec \rangle$ does not exceed $|A|$ (the ordinal, with which the cardinal $|A|$ is identified).
45. If every proper initial segment of an ordered set is the union of countably many well-ordered sets, then so is the whole set itself.
46. If $\langle A, \prec \rangle$ is a nonempty countable well-ordered set, then $A \times [1, 0)$ with the lexicographic ordering is similar to $[0, 1)$.
47. There is an ordered set that is not similar to a subset of $\mathbf{R}$, but all of its proper initial segments are similar to $(0, 1)$ or to $(0, 1]$. Furthermore, this set is unique up to similarity.
48. Call a point $x \in A$ in an ordered set $\langle A, \prec \rangle$ a fixed point if $f(x) = x$ holds for every monotone $f : A \to A$. A point $x \in A$ is not a fixed point of $\langle A, \prec \rangle$ if and only if there is a monotone mapping from $\langle A, \prec \rangle$ into $\langle A \setminus \{x\}, \prec \rangle$.
49. If $x \neq y$ are fixed points of $\langle A, \prec \rangle$, then y is a fixed point of $\langle A \setminus \{x\}, \prec \rangle$.
50. Every countable ordered set has only finitely many fixed points.
51. For each $n < \infty$ give a countably infinite ordered set with exactly n fixed points.
52. If $\langle A, \prec \rangle$ has infinitely many fixed points, then it includes a subset similar to $\mathbf{Q}$.
53. Every ordered set is similar to a set of sets ordered with respect to inclusion.
54. Let $\mathcal{M}$ be a family of subsets of a set X that is ordered with respect to inclusion and which is a maximal family with this property. Define $\prec$ on X as follows: let $x \prec y$ be exactly if there is an $E \in \mathcal{M}$, such that $x \in E$ but $y \notin E$. Then $\langle X, \prec \rangle$ is an ordered set. What are the initial segments in this ordered set?

55. Every ordered set is similar to some $\langle X, \prec\rangle$ constructed in the preceding problem.
56. If $\langle A, \prec\rangle$ is an ordered set, then there is an ordered set $\langle A^*, \prec^*\rangle$ such that if $A^* = B \cup C$ is an arbitrary decomposition, then either B or C includes a subset similar to $\langle A, \prec\rangle$.
57. To every infinite ordered set there is another such that neither one is similar to a subset of the other.
58. To every countably infinite ordered set $\langle A, \prec\rangle$ there is another countably infinite ordered set that does not include a subset similar to $\langle A, \prec\rangle$.
59. For every n show n countable ordered sets such that neither of them is similar to a subset of another one.
60. If $\langle A_i, \prec_i\rangle$, $i = 0, 1, \ldots$, are countable ordered sets, then there are $i < j$ such that $\langle A_i, \prec_i\rangle$ is similar to a subset of $\langle A_j, \prec_j\rangle$.
61. Every countably infinite ordered set is similar to one of its proper subsets.
62. There is an infinite ordered set that is not similar to any one of its proper subsets.
63. In every infinite ordered set the position of one element can be changed in such a way that we get an ordered set that is not similar to the original one.
64. One can add to any ordered set one element so that the ordered set so obtained is not similar to the original one. Is the same true for removing one element?
65. Every ordered set is a subset of a densely ordered set.
66. Every densely ordered set is a dense subset of a continuously ordered set.
67. Any two continuously ordered sets without smallest and largest elements that include similar dense sets are similar.
68. A continuously ordered set containing at least two points includes a subset similar to $\mathbf{R}$.
69. If $\langle A, \prec\rangle$ is continuously ordered and $A_n = \{c : a_n \preceq c \preceq b_n\}$ is a sequence of nested closed intervals, i.e., $A_{n+1} \subseteq A_n$ for all $n = 0, 1, \ldots$, then $\cap_{n=0}^{\infty} A_n \neq \emptyset$.
70. There is an infinite ordered set $\langle A, \prec\rangle$ that is not continuously ordered but for every sequence $\{A_n\}_{n=0}^{\infty}$ of nested closed intervals $\cap_{n=0}^{\infty} A_n \neq \emptyset$.
71. Call a subset of an ordered set scattered, if it does not include a subset that is densely ordered. The union of finitely many scattered subsets of an ordered set is scattered.
72. A subset of the real line is scattered if and only if it has a countable closure.
73. A bounded subset A of the real line is scattered if and only if for any sequence $\epsilon_0, \epsilon_1, \ldots$ of positive numbers there exists a natural number N

such that A can be covered with some intervals $I_0, I_1, \dots, I_N$ of length $|I_i| = \epsilon_i$.

74. If α is an ordinal then let $H(\alpha)$ be the set of all functions $f : \alpha \to \{-1, 0, 1\}$ for which $D(f) = \{\beta < \alpha \ : \ f(\beta) \neq 0\}$ is finite. Order $H(\alpha)$ according to last difference, i.e., for $f, g \in H(\alpha)$ set $f \prec g$ if $f(\beta) \prec g(\beta)$ holds for the largest $\beta < \alpha$ with $f(\beta) \neq g(\beta)$. Then $\langle H(\alpha), \prec \rangle$ is scattered.

75. The product of two scattered ordered sets is scattered.

76. The ordered union of scattered ordered sets with respect to a scattered ordered set is scattered.

77. Every nonempty ordered set is either scattered, or is similar to the ordered union of nonempty scattered sets with respect to a densely ordered set.

78. Let $\mathcal{F}$ be a family of ordered sets with the following properties:
    - if $\langle S, \prec \rangle \in \mathcal{F}$ and $\langle S', \prec' \rangle$ is similar to $\langle S, \prec \rangle$, then $\langle S', \prec' \rangle \in \mathcal{F}$,
    - if $\langle S, \prec \rangle \in \mathcal{F}$ and S' is a subset of S then $\langle S', \prec \rangle \in \mathcal{F}$,
    - $\mathcal{F}$ is closed for well-ordered and reversely well-ordered unions,
    - there is a nonempty $\langle S, \prec \rangle$ in $\mathcal{F}$.

    Then every ordered set is either in $\mathcal{F}$, or it is similar to an ordered union of nonempty sets in $\mathcal{F}$ with respect to a densely ordered set.

79. Let $\mathcal{O}$ be the smallest family of ordered sets that contains $\emptyset$, 1 and is closed for well-ordered and reversely well-ordered unions as well as for similarity. Then $\mathcal{O}$ is precisely the family of scattered sets.

80. An ordered set is scattered if and only it can be embedded into one of the $\langle H(\alpha), \prec \rangle$ defined in Problem 74.

81. We say that an ordered set $\langle A, \prec \rangle$ has countable intervals if for every $a, b \in A$, $a \prec b$ the set $\{c \in A \ : \ a \prec c \prec b\}$ is countable. There is a maximal ordered set $\langle A, \prec \rangle$ with countable intervals in the sense that every ordered set with countable intervals is similar to a subset of $\langle A, \prec \rangle$.

82. Pick a natural number n_1, and for each $i = 1, 2, \dots$ perform the following two operations to define n_{2i} and n_{2i+1}:

    (i) write n_{2i-1} in base $i + 1$, and while keeping the coefficients, replace the base by $i + 2$. This gives a number that we call n_{2i};

    (ii) set $n_{2i+1} = n_{2i} - 1$.

    If $n_{2i+1} = 0$ then we stop, otherwise repeat this process. For example, if $n_1 = 23 = 2^4 + 2^2 + 2^1 + 1$, then $n_2 = 3^4 + 3^2 + 3^1 + 1 = 94$, $n_3 = 93$, $n_4 = 4^4 + 4^2 + 4^1 = 276$, $n_5 = 275$, then, since $275 = 4^4 + 4^2 + 3$, we have $n_6 = 5^4 + 5^2 + 3 = 3253$, etc.

    (a) No matter what n_1 is, there is an i such that $n_i = 0$.

    (b) The same conclusion holds if in **(i)** the actual base is changed to any larger base (i.e., when the bases are not $2, 3, \dots$ but some numbers $b_1 < b_2 < \dots$).

* * *

83. In every densely ordered set there are two disjoint dense subsets.
84. The elements of any ordered set can be colored by two colors in such a way that in between any two elements of the same color there is another one with a different color.
85. There is an ordered set which is not well ordered, yet no two different initial segments of it are similar.
86. There exists an ordered set that cannot be represented as a countable union of its well-ordered subsets, but in which every uncountable subset includes an uncountable well-ordered subset.
87. There are two subsets $A, B \subset \mathbf{R}$ of power continuum such that any subset of A that is similar to a subset of B is of cardinality smaller than continuum.
88. There is an infinite subset X of $\mathbf{R}$ such that if $f : X \to X$ is any monotone mapping, then f is the identity.
89. To every ordered set $\langle A, \prec \rangle$ of cardinality $\kappa \geq \aleph_0$ there is another ordered set of cardinality κ that does not include a subset similar to $\langle A, \prec \rangle$.
90. For every infinite cardinal κ there is an ordered set of cardinality κ that has more than κ initial segments.
91. In a set of cardinality κ there is a family of subsets of cardinality bigger than κ that is ordered with respect to inclusion.
92. If $\mathcal{H}$ is a family of subsets of an infinite set of cardinality κ that is well ordered with respect to inclusion, then $\mathcal{H}$ is of cardinality at most κ.
93. If κ is an infinite cardinal, then in the lexicographically ordered set ${}^{\kappa}\kappa$ (which is the set of transfinite sequences of type κ of ordinals smaller than κ ordered with respect to first difference) every well-ordered subset is of cardinality at most κ.
94. Let κ be an infinite cardinal and let T be the set ${}^{\kappa}\{0, 1\}$ of 0–1 sequences of type κ ordered with the lexicographic ordering. Then
    a) every nonempty subset of T has a least upper bound,
    b) every subset of T has cofinality at most κ,
    c) every well-ordered subset of T is of cardinality at most κ.
95. Every ordered set of cardinality κ is similar to a subset of the lexicographically ordered ${}^{\kappa}\{0, 1\}$.
96. Let κ be an infinite cardinal and $\mathcal{F}_\kappa$ the set of those $f : \kappa \to \{0, 1\}$ for which there is a last 1, i.e., there is an $\alpha < \kappa$ such that $f(\alpha) = 1$ but for all $\alpha < \beta < \kappa$ we have $f(\beta) = 0$. Every ordered set of cardinality κ is similar to a subset of the lexicographically ordered $\mathcal{F}_\kappa$.

97. If $\langle A, \prec \rangle$ is an ordered set and κ is a cardinal, then there is an ordered set $\langle B, < \rangle$ such that if $B = \cup_{\xi<\kappa} B_\xi$ is an arbitrary decomposition of B into at most κ subsets, then there is a $\xi < \kappa$ such that $\langle B_\xi, < \rangle$ includes a subset similar to $\langle A, \prec \rangle$.

11

Partially ordered sets

Let A be a set and $\prec$ a binary relation on A. $\langle A, \prec \rangle$ is called a *partially ordered set* if

- $\prec$ *irreflexive*: $a \not\prec a$ for any $a \in A$,
- $\prec$ *transitive*: $a \prec b$ and $b \prec c$ imply $a \prec c$.

Thus, the difference with ordered sets is that here we do not assume trichotomy (comparability of elements).

In a partially ordered set $\langle A, \prec \rangle$ two elements a, b are called *comparable* if (exactly) one of $a = b$, $a \prec b$ or $b \prec a$ holds, otherwise they are *incomparable*. An ordered subset of a partially ordered set is called a *chain* and a set of pairwise incomparable elements an *antichain*.

The main problem that we treat in this chapter is how information on the size of chains and antichains can be related to the structure of the set in question.

1. In an infinite partially ordered set there is an infinite chain or an infinite antichain.
2. If in a partially ordered set all chains have at most $l < \infty$ elements and all antichains have at most $k < \infty$ elements, where k, l are finite numbers, then the set has at most kl elements.
3. If in a partially ordered set all chains have at most $k < \infty$ elements, then the set is the union of k antichains.
4. If in a partially ordered set all antichains have at most $k < \infty$ elements, then the set is the union of k chains.
5. There is a partially ordered set in which all chains are finite, still the set is not the union of countably many antichains.
6. There is a partially ordered set in which all antichains are finite, still the set is not the union of countably many chains.

7. If in a partially ordered set all chains are finite and all antichains are countable, then the set is countable.
8. If in a partially ordered set all antichains are finite and all chains are countable, then the set is countable.
9. There is a partially ordered set of cardinality continuum in which all chains and all antichains are countable.
10. If in a partially ordered set all chains and all antichains have at most κ elements, then the set is of cardinality at most 2^κ.
11. If κ is an infinite cardinal, then there is a partially ordered set of cardinality 2^κ in which all chains and all antichains have at most κ elements.
12. For every cardinal κ there is a partially ordered set $\langle P, \prec \rangle$ in which every interval $[x, y] = \{z : x \preceq z \preceq y\}$ is finite, yet P is not the union of κ antichains.
13. If $\langle P, \prec \rangle$ is a partially ordered set, call two elements strongly incompatible if they have no common lower bound. Let $c(P, \prec)$ be the supremum of $|S|$ where $S \subseteq P$ is a strong antichain, that is, a set of pairwise strongly incompatible elements.
    (a) If $c(P, \prec)$ is an infinite cardinal that is not weakly inaccessible, i.e., it is not a regular limit cardinal, then $c(P, \prec)$ is actually a maximum.
    (b) If κ is a regular limit cardinal, then there is a partially ordered set $\langle P, \prec \rangle$ such that $c(P, \prec) = \kappa$ yet there is no strong antichain of cardinality κ.
14. If $\langle A, \prec \rangle$ is a partially ordered set, then there exists a cofinal subset $B \subseteq A$ such that $\langle B, \prec \rangle$ is well founded (i.e., in every nonempty subset there is a minimal element).
15. If there is no maximal element in the partially ordered set $\langle P, \prec \rangle$, then there are two disjoint cofinal subsets of $\langle P, \prec \rangle$.
16. There is a partially ordered set $\langle P, \prec \rangle$ which is the union of countably many centered sets but not the union of countably many filters. (A subset $Q \subseteq P$ is centered if for any $p_1, \ldots, p_n \in Q$ there is some $q \preceq p_1, \ldots, p_n$ in P. A subset $F \subseteq P$ is a filter, if for any $p_1, \ldots, p_n \in F$ there is some $q \preceq p_1, \ldots, p_n$ with $q \in F$.)
17. For two real functions $f \neq g$ let $f \prec g$ if $f(x) \leq g(x)$ for all $x \in \mathbf{R}$. In this partially ordered set there is an ordered subset of cardinality bigger than continuum. No such subset can be well ordered by $\prec$.

The following problems use two orderings on the set ${}^\omega\omega$ of all functions $f : \omega \to \omega$: let $f \ll g$ if $f(n) < g(n)$ for all large n, and $f \prec g$ if $g(n) - f(n) \to \infty$ as $n \to \infty$.

18. Each of $\langle {}^\omega\omega, \ll \rangle$ and $\langle {}^\omega\omega, \prec \rangle$ has an order-preserving mapping into the other, but they are not isomorphic.

19. For any countable subset $\{f_k\}_k$ of ${}^\omega\omega$ there is an f larger than any f_k with respect to $\prec$.
20. $\langle {}^\omega\omega, \prec\rangle$ includes a subset of order type ω_1.
21. $\langle {}^\omega\omega, \prec\rangle$ includes a subset of order type λ^m for each $m = 1, 2, \ldots$.
22. If θ is an order type and $\langle {}^\omega\omega, \prec\rangle$ includes a subset similar to θ, then it includes such a subset consisting of functions that are smaller than the identity function.
23. If θ_1, θ_2 are order types and $\langle {}^\omega\omega, \prec\rangle$ includes subsets similar to θ_1 and θ_2, respectively, then it includes subsets similar to $\theta_1 + \theta_2$ and $\theta_1 \cdot \theta_2$, respectively. It also includes a subset similar to θ_1^*, where θ_1^* is the reverse type to θ_1.
24. If θ_i, $i \in I$ are order types where $\langle I, <\rangle$ is an ordered set, and $\langle {}^\omega\omega, \prec\rangle$ includes subsets similar θ_i and also a subset similar to $\langle I, <\rangle$, then it includes subsets similar to $\sum_{i\in I(<)} \theta_i$. In particular, $\langle {}^\omega\omega, \prec\rangle$ includes a set of order type α for every $\alpha < \omega_2$.
25. If $\varphi < \omega_1$ is a limit ordinal and

$$f_0 \prec f_1 \prec \cdots \prec f_\alpha \prec \cdots \prec g_\alpha \prec \cdots g_1 \prec g_0, \qquad \alpha < \varphi,$$

then there is an f with $f_\alpha \prec f \prec g_\alpha$ for every $\alpha < \varphi$.

26. There exist functions

$$f_0 \prec f_1 \prec \cdots \prec f_\alpha \prec \cdots \prec g_\alpha \prec \cdots g_1 \prec g_0, \qquad \alpha < \omega_1,$$

such that there is no function f with $f_\alpha \prec f \prec g_\alpha$ for every $\alpha < \omega_1$.

8

Ordinals

Ordinals are the order types of well-ordered sets. They are the infinite analogues of the natural numbers, and in many respect they behave like the latter ones. In fact, the finite ordinals are the natural numbers, and hence the transfinite class of ordinals can be considered as an endless continuation of the sequence of natural numbers.

This chapter contains various problems on ordinals and on operations on them. The problems specifically related to ordinal arithmetic will be the content of the next chapter.

The von Neumann definition of ordinals is as follows (see below): a set α is called an ordinal if it is transitive and well ordered with respect to $\in$. When we talk about such an α we shall always assume that it is equipped with the $\in$ relation. It can be shown that every well-ordered set $\langle A, \prec\rangle$ is similar to such a unique α. Therefore, we can set α as the order type of $\langle A, \prec\rangle$. In particular, the order type of α is α.

We set $\beta < \alpha$ if $\beta \in \alpha$. It follows that

α is the set of ordinals smaller than α, and among ordinals the relation $\beta < \alpha$ is the same as $\beta \in \alpha$, and $\beta \le \alpha$ is the same as $\beta \subseteq \alpha$.

We shall not explicitly use von Neumann's definition, but we shall use the just-listed boldfaced convention.

In this chapter $\alpha, \beta, \ldots$ always denote ordinals. As always, ω, the smallest infinite ordinal, is the set of natural numbers, i.e., the set of finite ordinals. An ordinal α is called a *successor ordinal* if it is of the form $\beta+1$. The positive ordinals that are not successors are called *limit ordinals.* Thus, α is a limit ordinal if and only if $\beta < \alpha$ implies $\beta + 1 < \alpha$. The first ordinal 0 is neither limit, nor successor.

The first problem deals with the von Neumann definition of ordinals. A set x is called *transitive* if $y \in x$ and $z \in y$ imply $z \in x$ (or equivalently $y \in x \Longrightarrow y \subset x$). We say that $\in$ is a well-ordering on the set x if its restriction to x is a well-ordering on x. Call a set *N-set* (N for Neumann) if it is transitive and well ordered by $\in$. We always consider an N-set with the well-ordering $\in$, and for notational convenience sometimes we write $<_\in$ for $\in$. Part (h) shows that for a well-ordered set $\langle A, \prec\rangle$ we could define its order type as the unique N-set similar to it, and this is exactly the von Neumann definition of ordinals.

1. (a) Every element of an N-set is an N-set.
   (b) If x is an N-set, then $y = x \cup \{x\}$ is an N-set, and if z is an N-set containing x, then $y \subset z$.
   (c) If x is an N-set, $y \in x$, then y is an initial segment of x.
   (d) If x is an N-set and $Y \subset x$ is one of its initial segments, then Y is an N-set, and either $Y = x$ or $Y \in x$.
   (e) If x, y are N-sets, then $x = y$ or $x \in y$ or $y \in x$.
   (f) For N-sets x, y define $x < y$ if $x \in y$. Then this is irreflexive, transitive and trichotomous. Furthermore, if B is a nonempty set of N-sets, then there is a smallest element of B with respect to $<$ ("well order").
   (g) If x, y are different N-sets, then they are not similar.
   (h) Every well-ordered set is similar to a unique N-set.
2. There is no infinite decreasing sequence of ordinals.
3. Arbitrary infinite sequence of ordinals includes an infinite nondecreasing subsequence.
4. The following relations are true:
   a) $1 + \omega = \omega, \quad \omega + 1 \neq \omega,$
   b) $2 \cdot \omega = \omega, \quad \omega \cdot 2 \neq \omega.$
5. If a and b are natural numbers, then what is $(\omega + a) \cdot (\omega + b)$?
6. Solve the following equations for the ordinals ξ and ζ:
   (a) $\omega + \xi = \omega$
   (b) $\xi + \omega = \omega$
   (c) $\xi \cdot \omega = \omega$
   (d) $\omega \cdot \xi = \omega$
   (e) $\xi + \zeta = \omega$
   (f) $\xi \cdot \zeta = \omega$
7. Solve the equation $\xi + \zeta = \omega^2 + 1$ for the ordinals ξ and ζ.
8. Which one is bigger?
   a) $\omega + k$ or $k + \omega$ (k is a positive integer)
   b) $k \cdot \omega$ or $\omega \cdot k$ ($k \geq$ is an integer)
   c) $\omega + \omega_1$ or $\omega_1 + \omega$
   d) $P(\omega) = \omega^n \cdot a_n + \cdots + \omega \cdot a_1 + a_0$ or ω^{n+1}, where $n \geq 1$ and $a_0, \ldots, a_n$ are natural numbers

e) $P(\omega) = \omega^n \cdot a_n + \cdots + \omega \cdot a_1 + a_0$ or $Q(\omega) = \omega^m \cdot a'_m + \cdots + \omega \cdot a'_1 + a'_0$, where $n, m, a_0, a'_0 \ldots, a_n, a'_n$ are natural numbers

9. Addition among ordinals is monotonic in both arguments, and strictly monotonic in the second argument. The same is true of multiplication provided the first factor is nonzero.

10. a) $\gamma + \alpha = \gamma + \beta$ implies $\alpha = \beta$,

    b) $\alpha + \gamma = \beta + \gamma$ does not imply $\alpha = \beta$,

    c) $\gamma \cdot \alpha = \gamma \cdot \beta$, $\gamma > 0$ imply $\alpha = \beta$,

    d) $\alpha \cdot \gamma = \beta \cdot \gamma$, $\gamma > 0$ do not imply $\alpha = \beta$.

    Does the answer change in b) or d) if γ is a natural number?

11. If $\alpha \cdot \gamma = \beta \cdot \gamma$ and γ is a successor ordinal, then $\alpha = \beta$.

12. If k is a positive integer and $\alpha^k = \beta^k$, then $\alpha = \beta$.

13. If ξ is a limit ordinal, then

    a) $\sup_{\eta<\xi}(\alpha + \eta) = \alpha + \xi$,

    b) $\sup_{\eta<\xi}(\alpha \cdot \eta) = \alpha \cdot \xi$.

    Are the analogous relations true if we change the order of the terms in the sums and products?

14. If $\alpha \leq \beta$, then the equation $\alpha + \xi = \beta$ is uniquely solvable for ξ. Is the same true for the equation $\xi + \alpha = \beta$?

15. If $0 < \alpha$, then for any β there are unique ζ and $\xi < \alpha$ such that $\beta = \alpha \cdot \zeta + \xi$.

16. If $\alpha > 0$ is an arbitrary ordinal and β is sufficiently large, then $\alpha + \beta = \beta$.

17. If $\alpha + \beta = \beta + \alpha$ for all ordinals β, then $\alpha = 0$.

18. Every ordinal can be written in a unique manner in the form $\beta + n$ where β is a limit ordinal or zero and n is a natural number.

19. The limit ordinals are the ones that have the form $\omega \cdot \beta$, $\beta \geq 1$.

20. A positive ordinal α is a limit ordinal if and only if $n \cdot \alpha = \alpha$ for all positive integer n.

21. Let n be finite and α a limit ordinal. Then $(\alpha + n) \cdot \beta = \alpha \cdot \beta + n$ if β is a successor ordinal, and $(\alpha + n) \cdot \beta = \alpha \cdot \beta$ if β is 0 or a limit ordinal.

22. If $k \geq 1, n$ are natural numbers and α is a limit ordinal, then $(\alpha \cdot n)^k = \alpha^k \cdot n$.

23. Given $\alpha > 0$, what are those natural numbers n such that α can be written as $\alpha = n \cdot \beta$ for some ordinal β?

24. In each case find all ordinals α that satisfy the given equation.

    a) $\alpha + 1 = 1 + \alpha$

    b) $\alpha + \omega = \omega + \alpha$

    c) $\alpha \cdot \omega = \omega \cdot \alpha$

    d) $\alpha + (\omega + 1) = (\omega + 1) + \alpha$

e) $\alpha \cdot (\omega + 1) = (\omega + 1) \cdot \alpha$

25. If n is a positive integer, then $\sum_{\xi<\omega^n} \xi = \omega^{2n-1}$.
26. For every α there are only finitely many distinct γ such that $\alpha = \xi + \gamma$ with some ξ. Is the analogous statement true for the representation $\alpha = \gamma + \xi$?
27. For every $\alpha \neq 0$ there are only finitely many γ such that $\alpha = \xi \cdot \gamma$ with some ξ. Is the analogous statement true for the representation $\alpha = \gamma \cdot \xi$?
28. Let m be a positive integer. A successor ordinal can be represented as a product with m factors only in finitely many ways.
29. The equation $\xi^2 + \omega = \zeta^2$ has no solution for ξ and ζ.
30. Give infinitely many ξ and ζ such that ξ is infinite, and $\xi^2 + \omega^2 = \zeta^2$.
31. Solve $\alpha^2 \cdot 2 = \beta^2$ for α and β.
32. For every natural number k there is an infinite sequence of ordinals that form an arithmetic progression and in which each term is a kth power.
33. Give ordinals α and β with the property that for no $n = 2, 3, \ldots$ is $\alpha^n \cdot \beta^n$ or $\beta^n \cdot \alpha^n$ an nth power.
34. The sum $\omega + 1 + 2 + \cdots$ does not change if we alter the position of finitely many terms in it.
35. One can get infinitely many different ordinals from the sum $1+2+3+\cdots+\omega$ by changing the position of finitely many terms in it.
36. For every $n \geq 1$ give a sum $\alpha_0 + \alpha_1 + \cdots$ of positive ordinals from which one can get exactly n different sums by taking a permutation of the terms (possibly infinitely many) in the sum.
37. The sum of the $n+1$ ordinals $1, 2, \ldots, 2^{n-1}, \omega$ in all possible orders take 2^n different values.
38. Let $g(n)$ be the maximum number of different ordinals that can be obtained from n ordinals by taking their sums in all possible $n!$ different orders. Then
$$\lim_{n\to\infty} g(n)/n! = 0.$$
39. For every n give n ordinals such that all products of them taken in all possible $n!$ orders are different.
40. Let α be a limit ordinal, and call a set $A \subseteq \alpha$ of ordinals closed in α if the least upper bound of any increasing transfinite subsequence of A is in A or is equal to α. Then A is closed in α if and only if it is a closed subset of the topological space $(\alpha, \mathcal{T})$, where the topology $\mathcal{T}$ is generated by the intervals $\{\xi \ : \ \xi < \tau\}$, $\{\xi \ : \ \tau < \xi < \alpha\}$, $\tau < \alpha$ (this topology is called the interval topology on α).
It is also true that A is closed in α if and only if the supremum of every subset $B \subset A$ is in A or is equal to α.

41. With the notation of the preceding problem a function $f : \alpha \to \alpha$ is continuous in the interval topology if and only if $f(\sup A) = \sup_{\xi \in A} f(\xi)$ for any set $A \subset \alpha$ with $\sup A < \alpha$.

42. If $A \subseteq \alpha$ is of cardinality κ, then its closure in the interval topology is also of cardinality κ.

43. If $\{a_\xi\}_{\xi<\omega_1}$ is a transfinite sequence of countable ordinals converging in the topology on ω_1 to a $\sigma \in \omega_1$, then there is a $\nu < \omega_1$ such that $a_\xi = a_\zeta$ for all $\xi, \zeta > \nu$.

44. Assume that $f : \omega_1 \times \omega_1 \to \omega$ has the property that for $\alpha < \omega_1$, $n < \omega$ the set $\{\beta < \alpha \ : \ f(\beta, \alpha) \leq n\}$ is finite. Then all the sets

$$Z_f(\alpha, n) = \big\{\beta < \alpha : \text{there are } \beta = \beta_0 < \beta_1 < \cdots < \beta_k = \alpha, \\ \text{with } f(\beta_i, \beta_{i+1}) \leq n\big\}$$

are also finite.

45. There is a function $f : \omega_1 \times \omega_1 \to \omega$ such that for $\alpha < \omega_1$, $n < \omega$ the set $\{\beta < \alpha : f(\beta, \alpha) \leq n\}$ is finite and for any $\alpha_0 < \alpha_1 < \cdots$ we have $\sup_{k<\omega} f(\alpha_k, \alpha_{k+1}) = \omega$.

46. Two players, I and II, play the following game of length ω. At round i first I chooses a countable ordinal α_i at least as large as the previous ordinal chosen by him, then II selects a finite subset S_i of α_i. After ω many steps II wins if $S_0 \cup S_1 \cup \cdots = \sup\big(\{\alpha_i : i < \omega\}\big)$.
    (a) II has a winning strategy.
    (b) II even has a winning strategy that chooses S_i only depending on i, α_{i-1}, and α_i.

47. Two players, I and II, alternatively select countable ordinals. After ω steps they consider the set of all selected ordinals, and II wins if it is an initial segment, otherwise I wins.
    (a) There is a winning strategy for II.
    (b) There is no such winning strategy if the choice of II depends only on the set of ordinals selected before (by the two players).
    (c) Even such a strategy exists if II is allowed to select finitely many ordinals in every step.

* * *

48. Let κ be an infinite cardinal and let two players alternately choose sets $K_0 \supset K_1 \supset \cdots$ of cardinality κ. Then no matter how the first player plays, the second one can always achieve $\cap_{n=0}^{\infty} K_n = \emptyset$.

9

Ordinal arithmetic

This chapter can be regarded as the "infinite analogue" of classical number theory. It contains problems on the arithmetic properties of ordinals such as divisibility, representation in a base, decomposition, primeness, etc.

A special role is played by the so-called normal representation (Problem 16) which is representation in base ω. In fact, many problems simplify considerably if the ordinals are written in normal form.

In this chapter $\alpha, \beta, \dots$ always denote ordinals.

If $\alpha \cdot \beta = \gamma$, then we say that α (β) is a left (right) divisor of γ, and also that γ is a right (left) multiple of α (β).

1. If A is any set of nonzero ordinals, then there is a largest ordinal γ that divides every element of A from the left (this γ is called the greatest common left divisor of A). Every ordinal that divides every element of A from the left also divides γ from the left.
2. α is a limit ordinal if and only if ω divides α from the left.
3. α is divisible from the left by $\omega+2$ and by $\omega+3$ if and only if it is divisible from the left by ω^2.
4. α is divisible from the right by 2 and 3 if and only if it is divisible from the right by 6. Is the same true for divisibility from the left?
5. α is divisible from the right by $\omega + 2$ and by $\omega + 3$ if and only if it is divisible from the right by $\omega + 6$.
6. Every ordinal α has only a finite number of right divisors. Is the same true of left divisors? What if α is a successor ordinal?
7. If α and β are right divisors of $\gamma \geq 1$, then either
   a) α divides β from the right, or
   b) β divides α from the right, or
   c) $\alpha = \xi+p$, $\beta = \xi+q$, where ξ is a limit ordinal or 0, and p, q are positive natural numbers.

In case c) if $[p, q]$ is the smallest common multiple of p and q, then $\xi + [p, q]$ is the smallest common left multiple of α and β, and $\xi + [p, q]$ also divides γ from the right.

8. Any set of positive ordinals has a greatest common right divisor, and this greatest common right divisor is divisible from the right by any common right divisor.
9. Any set of positive ordinals has a least common (positive) right multiple, and this least common right multiple divides every common right multiple from the left.
10. Exhibit two ordinals that do not have a common (nonzero) left multiple.
11. Define ordinal exponentiation by transfinite recursion in the following way: $\gamma^0 = 1$, $\gamma^{\alpha+1} = \gamma^\alpha \cdot \gamma$, and for limit ordinal α let γ^α be the supremum of the ordinals γ^η, $\eta < \alpha$. For $\gamma > 1$ the following are true:
    (i) $\gamma^\alpha \cdot \gamma^\beta = \gamma^{\alpha+\beta}$,
    (ii) $(\gamma^\alpha)^\beta = \gamma^{\alpha \cdot \beta}$,
    (iii) if $\alpha < \beta$ then $\gamma^\alpha < \gamma^\beta$,
    (iv) $\alpha \le \gamma^\alpha$.
12. Consider the set $\Phi_{\alpha,\gamma}$ of all mappings $f : \alpha \to \gamma$ for which all but finitely many elements are mapped to 0, and for $f, g \in \Phi_{\alpha,\gamma}$, $f \ne g$ let $f \prec g$ if $f(\xi) < g(\xi)$ for the largest $\xi < \alpha$ for which $f(\xi) \ne g(\xi)$. Then $\langle \Phi_{\alpha,\gamma}, \prec \rangle$ is well ordered, and its order type is γ^α.
13. For any integer $n > 1$ we have
    a) $n^{\omega^{\omega^\omega}} = \omega^{\omega^\omega}$,
    b) $(\omega + n)^\omega = \omega^\omega$.
14. If α is a limit ordinal, then $1^\alpha + 2^\alpha = 3^\alpha$.
15. The following are true:
    a) $2^\omega = \omega$,
    b) if α is countable, then so is 2^α,
    c) for any cardinal $\kappa = \omega_\sigma$ we have $2^{\omega_\sigma} = \kappa$,
    d) if α is infinite, then the cardinality of 2^α is equal to the cardinality of α,
    e) every ordinal can be written in a unique manner in the form

$$2^{\xi_n} + 2^{\xi_{n-1}} + \cdots + 2^{\xi_0}, \tag{9.1}$$

    where $\xi_0 < \xi_1 \ldots < \xi_n$.

    What is the form (9.1) of the ordinal $\omega^4 \cdot 6 + \omega^2 \cdot 7 + \omega + 9$?
16. If $\gamma \ge 2$, then every ordinal can be written in a unique way in the form

$$\gamma^{\xi_n} \cdot \eta_n + \cdots + \gamma^{\xi_0} \cdot \eta_0,$$

where $\xi_0 < \xi_1 < \ldots < \xi_n$, and $1 \le \eta_j < \gamma$ for all $1 \le j \le n$.
This form is called the representation of the given ordinal in base γ. The representation of an ordinal α in base ω is called the normal form of α.

17. If
$$\alpha = \omega^{\xi_n} \cdot a_n + \cdots + \omega^{\xi_0} \cdot a_0, \tag{9.2}$$
$\xi_0 < \xi_1 < \cdots < \xi_n$, $a_0, a_1 \ldots, a_n \in \mathbf{N}$ is the normal expansion of α, then $\alpha < \omega^{\xi_n+1}$, and for any $\omega^{\xi_n+1} \le \beta$ we have $\alpha + \beta = \beta$.
18. Find the normal form of the sum and product of two ordinals given in normal form.
19. If the normal form (9.2) of α has $(n+1)$ components, then for $m = 1, 2, \ldots$ the normal form of α^m has $(n+1)$ components if α is a limit ordinal and it has $mn + 1$ components if α is a successor ordinal.
20. If the normal form of α is (9.2), then every $0 < \beta < \omega^{\xi_0}$ is a left divisor of α, and besides these there are only finitely many left divisors of α.
21. Given $\alpha > 0$, what are those natural numbers k such that α can be written as $\alpha = \beta \cdot k$ for some ordinal β?
22. Given an ordinal α, what is $\sum_{\beta<\omega^\alpha} \beta$?
23. If $\omega^\alpha = A \cup B$, then either A or B is of order type ω^α.
24. For every α there is a natural number N such that if α is decomposed as $\alpha = A_0 \cup \cdots \cup A_N$ into $N + 1$ disjoint sets, then there is a j such that $\cup_{i \ne j} A_i$ has order type α.
25. If κ is an infinite cardinal, then every ordinal α of cardinality at most κ can be decomposed as $\alpha = A_0 \cup A_1 \cup \cdots$ such that every A_n is of order type smaller than κ^ω.
26. Call an ordinal $\alpha > 0$ (additively) indecomposable if it cannot be written as a sum of two smaller ordinals. Give the first three infinite indecomposable ordinals.
27. For every ordinal there is a bigger indecomposable ordinal. Also, for every countable ordinal there is a bigger indecomposable countable ordinal.
28. If α is arbitrary, and γ is the smallest ordinal for which there is a β such that $\alpha = \beta + \gamma$, then γ is indecomposable.
29. α is indecomposable if and only if it does not have a right divisor that is a successor ordinal bigger than 1.
30. α is indecomposable if and only if $\xi + \alpha = \alpha$ for every $\xi < \alpha$.
31. The supremum of indecomposable ordinals is indecomposable.
32. If α is indecomposable, then so is every $\beta \cdot \alpha$, $\beta > 0$.
33. If α is indecomposable, then α is divisible from the left by all $1 \le \beta < \alpha$.
34. The smallest indecomposable ordinal bigger than $\alpha \ge 1$ is $\alpha \cdot \omega$.
35. Every positive ordinal can be represented in a unique manner as a sum of a finite sequence of nonincreasing indecomposable ordinals.

36. Let $\alpha = \beta_1 + \beta_2 + \cdots + \beta_n$ be the decomposition of α from the preceding problem. Then $\alpha = \beta + \gamma$ for some $\beta, \gamma \neq 0$ if and only if there are a $1 \leq m \leq n$ such that $\gamma = \beta_m + \beta_{m+1} + \cdots + \beta_n$ and $\beta = \beta_1 + \beta_2 + \cdots + \beta_{m-1} + \delta$, where δ is an arbitrary ordinal smaller than β_m.
37. The indecomposable ordinals are precisely the ordinals of the form ω^α.
38. Call an ordinal $\alpha > 1$ prime if it cannot be written as the product of two smaller ordinals. Give the first three infinite prime ordinals.
39. $\alpha > 1$ is prime if and only if $\alpha = \beta \cdot \gamma$, $\gamma > 1$ imply $\gamma = \alpha$.
40. If α is an indecomposable ordinal, then $\alpha + 1$ is prime.
41. An infinite successor ordinal is prime if and only if it is of the form $\omega^\xi + 1$.
42. A limit ordinal is prime if and only if it is of the form ω^{ω^ξ}.
43. Every ordinal has at most one infinite right divisor that is prime.
44. Every successor ordinal has at most one infinite left divisor that is prime. However, a limit ordinal may have infinitely many infinite left prime divisors.
45. Every ordinal $\alpha > 1$ is the product of finitely many prime ordinals. In general, this representation is not unique even if we require that no factor can be omitted without changing the product.
46. Every $\alpha > 1$ has a unique representation

$$\alpha = a_1 \cdots a_m \cdot b_1 \cdot c_1 \cdot b_2 \cdots b_s \cdot c_s \cdot b_{s+1},$$

where $a_1 \geq \ldots \geq a_m$ are limit primes, $c_1, \ldots, c_s$ are infinite successor primes, and $b_1, \ldots, b_{s+1} > 1$ are natural numbers (some of the terms may be missing).
47. Call two positive ordinals α and β additively commutative if $\alpha + \beta = \beta + \alpha$. If α is additively commutative with both β and γ, then β and γ are also additively commutative.
48. For every positive ordinal α there are only countably many ordinals with which α is additively commutative.
49. Let n, m be given positive integers. Two ordinals α and β are additively commutative if and only if $\alpha \cdot n$ and $\beta \cdot m$ are additively commutative.
50. Two ordinals α and β are additively commutative if and only if there are positive integers n, m such that $\alpha \cdot n = \beta \cdot m$.
51. Two ordinals α and β are additively commutative if and only if there are natural numbers n, m and an ordinal ξ such that $\alpha = \xi \cdot n$, $\beta = \xi \cdot m$.
52. For any α the ordinals that additively commute with α are of the form $\beta \cdot n$, $n = 1, 2, \ldots$, where β is the smallest ordinal additively commutative with α.

53. If the normal form of $\alpha > 0$ is (9.2), then the ordinals additively commutative with α are the ones with normal form

$$\omega^{\xi_n} \cdot c + \omega^{\xi_{n-1}} \cdot a_{n-1} \cdots + \omega^{\xi_0} \cdot a_0$$

where c is an arbitrary positive natural number.

54. The sum of n nonzero ordinals $\alpha_1, \ldots, \alpha_n$ is independent of their order if and only if there are positive integers $m_1, \ldots, m_n$ and an ordinal ξ such that $\alpha_1 = \xi \cdot m_1$, $\alpha_2 = \xi \cdot m_2$, ..., $\alpha_n = \xi \cdot m_n$.

55. Let $g(n)$ be the maximum number of different ordinals that can be obtained from n ordinals by taking their sums in all possible $n!$ different orders.

 (a) For each n

$$g(n) = \max_{1 \le k \le n-1} (k2^{k-1} + 1) g(n-k).$$

 (b) $g(1) = 1$, $g(2) = 2$, $g(3) = 5$, $g(4) = 13$, $g(5) = 33$, $g(6) = 81$, $g(7) = 193$, $g(8) = 449$, $g(9) = 33^2$, $g(10) = 33 \cdot 81$, $g(11) = 81^2$, $g(12) = 81 \cdot 193$, $g(13) = 193^2$, $g(14) = 33^2 \cdot 81$, $g(15) = 33 \cdot 81^2$.

 (c) For $m \ge 3$ we have $g(5m) = 33 \cdot 81^{m-1}$ $g(5m+1) = 81^m$, $g(5m+2) = 193 \cdot 81^{m-1}$, $g(5m+3) = 193^2 \cdot 81^{m-2}$ and $g(5m+4) = 193^3 \cdot 81^{m-3}$.

 (d) For $n \ge 21$ we have $g(n) = 81 g(n-5)$.

56. Call two ordinals $\alpha > 1$ and $\beta > 1$ multiplicatively commutative if $\alpha \cdot \beta = \beta \cdot \alpha$. If $\gamma > 1$ is multiplicatively commutative with the ordinals β and γ, then β and γ are also multiplicatively commutative.

57. No successor ordinal bigger than 1 is multiplicatively commutative with any limit ordinal, and no finite ordinal bigger than 1 is multiplicatively commutative with any infinite ordinal.

58. For every ordinal $\alpha > 1$ there are only countably many ordinals that are multiplicatively commutative with α.

59. Let m, n be positive integers. Two ordinals α and β are multiplicatively commutative if and only if α^n and β^m are multiplicatively commutative.

60. Two infinite ordinals α, β are multiplicatively commutative if and only if there are natural numbers n, m such that $\alpha^n = \beta^m$.

61. Two limit ordinals $\alpha < \beta$ are multiplicatively commutative if and only if there is a θ and positive integers p, r such that $\beta = \omega^{\theta \cdot r} \cdot \alpha$, and the highest power of ω in the normal representations of α is $\omega^{\theta \cdot p}$.

62. If α is an infinite successor ordinal and $\xi > 1$ is the smallest ordinal multiplicatively commutative with α, then every ordinal that is multiplicatively commutative with α is of the form ξ^n with $n = 0, 1 \ldots$.

63. Two infinite successor ordinals α and β are multiplicatively commutative if and only if there is an ordinal ξ and natural numbers n, m with which $\alpha = \xi^n$ and $\beta = \xi^m$.

64. The ordinals $\omega^2+\omega$ and $\omega^3+\omega^2$ are multiplicatively commutative, but there is no ordinal ξ and natural numbers n, m with which $\alpha=\xi^n$ and $\beta=\xi^m$ would be true.

65. The product of n ordinals $\alpha_1, \ldots, \alpha_n$, $\alpha_i \geq 2$ is independent of their order if and only if there are positive integers $m_1, \ldots, m_n$ for which $\alpha_1^{m_1} = \alpha_2^{m_2} = \cdots = \alpha_n^{m_n}$.

66. For every n give n ordinals such that all products of them taken in all possible $n!$ orders are different.

67. There are no different infinite ordinals that are simultaneously additively and multiplicatively commutative.

68. For infinite α the following statements are pairwise equivalent:
    a) if $\xi<\alpha$ and $\theta<\alpha$, then $\xi \cdot \theta<\alpha$,
    b) if $1 \leq \xi<\alpha$ then $\xi \cdot \alpha=\alpha$,
    c) $\alpha=\omega^{\omega^\beta}$ for some β.

69. Call an ordinal α epsilon-ordinal, if $\omega^\alpha=\alpha$. Find the smallest epsilon-ordinal.

70. For every ordinal there is a larger epsilon-ordinal and for every countable ordinal there is a larger countable epsilon-ordinal.

71. If α is an epsilon-ordinal, then
    (i) $\xi+\alpha=\alpha$ for $\xi<\alpha$,
    (ii) $\xi \cdot \alpha=\alpha$ for $1 \leq \xi<\alpha$,
    (iii) $\xi^\alpha=\alpha$ for $2 \leq \xi<\alpha$.

72. If $\beta \geq \omega$ and $\beta^\alpha=\alpha$, then α is an epsilon-ordinal.

73. α is an epsilon-ordinal if and only if $\omega<\alpha$ and $\beta^\gamma<\alpha$ whenever $\beta, \gamma<\alpha$.

74. For infinite ordinals $\alpha<\beta$ we have $\alpha^\beta=\beta^\alpha$ if and only if α is a limit ordinal and $\beta=\gamma \cdot \alpha$, where $\gamma>\alpha$ is an epsilon ordinal.

75. Define the product $\prod_{\xi<\theta} \alpha_\xi$ of a transfinite sequence $\{\alpha_\xi\}_{\xi<\theta}$ of ordinals, and discuss its properties!

76. If $\alpha_0+\alpha_1+\cdots$ is a sum of a sequence of ordinals of type ω, then by taking a permutation of (possibly infinitely many of) the terms in the sum, one can get only finitely many different ordinals.

77. If $\alpha_0+\alpha_1+\cdots$ is a sum of a sequence of ordinals of type ω, then by deleting finitely many terms and taking a permutation of (possibly infinitely many of) the remaining terms in the sum, one can get only finitely many different ordinals.

78. Given a positive integer n give a sum $\alpha_0+\alpha_1+\cdots$ of a sequence of infinite ordinals of type ω such that one can get exactly n different values by taking a permutation of the terms in the sum.

79. If $\alpha_0 \cdot \alpha_1 \cdots$ is a product of a sequence of ordinals of type ω, then by taking a permutation of (possibly infinitely many of) the terms in the product, one can get only finitely many different ordinals.

80. If $\alpha_0 \cdot \alpha_1 \cdots$ is a product of a sequence of ordinals of type ω, then by deleting finitely many terms and taking a permutation of (possibly infinitely many of) the remaining terms in the product, one can get only finitely many different ordinals.

81. Given a positive integer n give a product $\alpha_0 \cdot \alpha_1 \cdots$ of a sequence of infinite ordinals of type ω such that one can get exactly n different values by taking a permutation of the terms in the product.

82. Permuting finitely many terms in a sum $\sum_{\beta \leq \omega} \alpha_\beta$ (but keeping the permuted sum of type $\omega + 1$), one may get infinitely many different ordinals.

83. If γ is a countable ordinal and $\{\alpha_\beta\}_{\beta<\gamma}$ is a sequence of ordinals, then there are only countably many different sums of the form $\sum_{\beta<\gamma} \alpha_{\pi(\beta)}$, where $\pi : \gamma \to \gamma$ is any mapping.

84. Permuting finitely many terms in a product $\prod_{\beta \leq \omega} \alpha_\beta$ (but keeping the permuted sum of type $\omega + 1$), one may get infinitely many different ordinals.

85. If γ is a countable ordinal and $\{\alpha_\beta\}_{\beta<\gamma}$ is a sequence of ordinals, then there are only countably many different products of the form $\prod_{\beta<\gamma} \alpha_{\pi(\beta)}$, where $\pi : \gamma \to \gamma$ is any mapping.

86. Write $\Gamma(\alpha) = \prod_{\xi<\alpha} \xi$. Calculate $\Gamma(\omega)$, $\Gamma(\omega + 1)$, $\Gamma(\omega \cdot 2)$, and $\Gamma(\omega^2)$.

87. Find all operations $\mathcal{F}$ from the ordinals to the ordinals that are continuous in the interval topology and that satisfy the equation $\mathcal{F}(\alpha + \beta) = \mathcal{F}(\alpha) + \mathcal{F}(\beta)$ for all α and β.

88. Is there a not identically zero operation $\mathcal{F}$ from the ordinals to the ordinals that is continuous in the interval topology and that satisfies the equation $\mathcal{F}(\alpha + \beta) = \mathcal{F}(\beta) + \mathcal{F}(\alpha)$ for all α and β?

89. Find all operations $\mathcal{F}$ from the ordinals to the ordinals that are continuous in the interval topology and that satisfy the equation $\mathcal{F}(\alpha + \beta) = \mathcal{F}(\alpha) \cdot \mathcal{F}(\beta)$ for all α and β.

90. Is there a not identically zero and not identically 1 operation $\mathcal{F}$ from the ordinals to the ordinals that is continuous in the interval topology and that satisfies the equation $\mathcal{F}(\alpha + \beta) = \mathcal{F}(\beta) \cdot \mathcal{F}(\alpha)$ for all α and β?

91. Define the Hessenberg sum (or natural sum) $\alpha \oplus \beta$ of ordinals α, β with normal form

$$\alpha = \omega^{\delta_n} \cdot a_n + \cdots + \omega^{\delta_0} \cdot a_0, \qquad \beta = \omega^{\delta_n} \cdot b_n + \cdots + \omega^{\delta_0} \cdot b_0 \tag{9.3}$$

(with possibly $a_i = 0$ or $b_i = 0$) as

$$\alpha \oplus \beta = \omega^{\delta_n} \cdot (a_n + b_n) + \cdots \omega^{\delta_0} \cdot (a_0 + b_0).$$

(a) $\oplus$ is an associative and commutative operation.

(b) If $\beta < \gamma$, then $\alpha \oplus \beta < \alpha \oplus \gamma$.

(c) For a given α how many solutions does the equation $x \oplus y = \alpha$ have?

(d) Is $\mathcal{F}_\alpha(x) = \alpha \oplus x$ continuous?

(e) $\alpha_1 + \cdots + \alpha_n \leq \alpha_1 \oplus \cdots \oplus \alpha_n$. When does the equality hold?

(f) $\alpha_1 \oplus \cdots \oplus \alpha_n \leq \max\{\alpha_1, \ldots, \alpha_n\} \cdot (n+1)$.

92. $\alpha_1 \oplus \cdots \oplus \alpha_n$ is the largest ordinal that occurs as the order type of $A_1 \cup \cdots \cup A_n$, where $A_1, \ldots, A_n$ are subsets of some ordered set of order types $\alpha_1, \ldots, \alpha_n$, respectively.

93. If $\mathcal{F}(\alpha, \beta)$ is a commutative operation on the ordinals which is strictly increasing in either variable, then $\mathcal{F}(\alpha, \beta) \geq \alpha \oplus \beta$ holds for all α, β.

The "superbase" form of a natural number in base b is obtained by writing the number in base b, and all exponents and exponents of exponents, etc., in base b. For example, if $b = 2$, then $141 = 2^7 + 2^3 + 2^2 + 1 = 2^{2^2+2+1} + 2^{2+1} + 2^2 + 1$, and the latter form is its "superbase" 2 form.

94. Pick a natural number n_1, and for each $i = 1, 2, \ldots$ perform the following two operations to define the numbers n_{2i} and n_{2i-1}:

(i) write n_{2i-1} in "superbase" form in base $i+1$, and while keeping all coefficients, replace the base by $i+2$. This gives a number that we call n_{2i}.

(ii) set $n_{2i+1} = n_{2i} - 1$.

If $n_{2i+1} = 0$, then we stop, otherwise repeat these operations. For example, if $n_1 = 23$, then its "superbase" 2 form is $23 = 2^{2^2} + 2^2 + 2 + 1$, so $n_2 = 3^{3^3} + 3^3 + 3 + 1 = 7625597485018$, $n_3 = 7625597485017$. Since $n_3 = 3^{3^3} + 3^3 + 3$, and here we change the base 3 to base 4, we have $n_4 = 4^{4^4} + 4^4 + 4$, which is the following 155-digit number:

$$
\begin{gathered}
13407807929942597099574024998205846127479365820592 39\\
33777235614437217640300735469768018742981669034276 90\\
0318581864860508537538828119465699464336490060843 56.
\end{gathered}
$$

(a) No matter what n_1 is, there is an i such that $n_i = 0$.

(b) The same conclusion holds if in **(i)** the actual base is changed to any larger base (i.e., when the bases are not $2, 3, \ldots$ but some numbers $b_1 < b_2 < \ldots$).

10

Cardinals

Cardinals express the size of sets. Saying that two sets are equivalent (are of equal size) is the same as saying that their cardinality is the same. The cardinality of the set A is denoted by $|A|$, and it can be defined as the smallest ordinal equivalent to A: $|A| = \min\{\alpha \ : \ \alpha \sim A\}$.

We set $|A| < |B|$ if A is equivalent to a subset of B but not vice versa. It is easy to see that this is the same as $|A|$ being smaller than $|B|$ in the "smaller" relation (i.e., in $\in$) among ordinals. If κ_i, $i \in I$ are cardinals, then their sum $\sum_{i\in I} \kappa_i$ is defined as the cardinality of $\cup_{i\in I} A_i$, where A_i are disjoint sets of cardinality κ_i, and their product $\prod_{i\in I} \kappa_i$ is defined as the cardinality of the product set $\prod_{i\in I} A_i$ (recall that this is the same as the set of choice functions $f : I \to \cup_{i\in I} A_i$, $f(i) \in A_i$ for all i). Finally, we set $|A|^{|B|}$ as the cardinality of the set ${}^B A$ (which is the set of functions $f : B \to A$ from B into A).

This chapter contains problems related to cardinal operations. The fundamental theorem of cardinal arithmetic (Problem 2) says that for infinite cardinals κ, λ we have $\kappa + \lambda = \kappa\lambda = \max\{\kappa, \lambda\}$. Quite often this makes questions on cardinal addition and multiplication trivial. The situation is completely different with cardinal exponentiation; it is not trivial at all, and is one of the subtlest question of set theory with problems leading quite often to independence results. For this reason we shall barely touch upon cardinal exponentiation in this book.

An important property of some cardinals is their *regularity*: $\kappa = \mathrm{cf}(\kappa)$. It is equivalent to the fact that κ cannot be reached by (i.e., not the supremum of) less than κ smaller ordinals. Another equivalent formulation is that a set of cardinality κ is not the union of fewer than κ sets of cardinality smaller than κ (see Problems 9, 10). Some properties hold only for regular cardinals, and quite frequently proofs are simpler for regular cardinals than for singular (=nonregular) ones.

The finite cardinals are just the natural numbers. Infinite cardinals are listed in an endless "transfinite sequence" $\omega_0, \omega_1, \ldots, \omega_\alpha, \ldots$, numbered by ordinals α. Here $\omega_0 = \omega$ is the smallest infinite cardinal, and this numbering is done so that $\beta < \alpha$ implies $\omega_\beta < \omega_\alpha$. If $\kappa = \omega_\alpha$, then $\omega_{\alpha+1}$ is the successor cardinal to κ (i.e., the smallest cardinal larger than κ), and is denoted by κ^+. It is always a regular cardinal.

For historical reasons we also write $\aleph_\alpha$ instead of ω_a (note that ω_α has two faces; it is an ordinal and also a cardinal, and we use the aleph notation when we emphasize the cardinal aspect).

CH, the continuum hypothesis (i.e., that there is no cardinal between ω and $\mathbf{c}$) can be expressed as $\mathbf{c} = \aleph_1$ or as $2^{\aleph_0} = \aleph_1$. The generalized continuum hypothesis (GCH) stipulates that for all α we have $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. This is also independent of the axioms of set theory (cf. the introduction to Chapter 4).

1. What is the cardinal $a_0 \cdot a_1 \cdots$ if the a_i's are positive integers?
2. (Fundamental theorem of cardinal arithmetic) For every infinite cardinal κ we have $\kappa^2 = \kappa$.
3. If at least one of $\kappa > 0$ and $\lambda > 0$ is infinite, then
$$\kappa + \lambda = \kappa\lambda = \max\{\kappa, \lambda\}.$$
4. If X is of cardinality $\kappa \geq \aleph_0$, then the following sets are of cardinality κ:
   a) set of finite sequences of elements of X,
   b) set of those functions that map a finite subset of X into X.
5. Let X be a set of infinite cardinality κ, and call a set $Y \subset X$ "small" if there is a decomposition of X into subsets of cardinality κ each of which intersects Y in at most one point. Then X is the union of two of its "small" subsets.
6. The supremum of any set of cardinals (considered as a set of ordinals) is again a cardinal.
7. If $\rho_1 + \rho_2 = \sum_{\xi<\alpha} \lambda_\xi$, then there are cardinals $\lambda_\xi^{(i)}$, $i = 1, 2$, $\xi < \alpha$ such that $\rho_i = \sum_{\xi<\alpha} \lambda_\xi^{(i)}$, $i = 1, 2$, and for all ξ we have $\lambda_\xi = \lambda_\xi^{(1)} + \lambda_\xi^{(2)}$.
8. If α is the cofinality of an ordered set, then α is a regular cardinal.
9. If κ is an infinite cardinal, then $\mathrm{cf}(\kappa)$ coincides with the smallest ordinal α for which there is a transfinite sequence $\{\kappa_\xi\}_{\xi<\alpha}$ of cardinals smaller than κ with the property $\kappa = \sum_{\xi<\alpha} \kappa_\xi$.
10. An infinite cardinal is regular if and only if κ is not the sum of fewer than κ cardinals each of which is less than κ.
11. A successor cardinal is regular.
12. Which are the smallest three singular (i.e., not regular) infinite cardinals?
13. $\mathrm{cf}(\aleph_\alpha) = \aleph_\alpha$ if α is a successor ordinal, and $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha)$ if α is a limit ordinal.

14. Let n be a natural number. The cardinality of a set H is at most $\aleph_n$ if and only if ${}^{n+2}H (\equiv H^{n+2})$ can be represented in the form $A_1 \cup \cdots \cup A_{n+2}$, where A_k is finite "in the direction of the kth coordinate", i.e., if $h_1, \ldots, h_{k-1}, h_{k+1}, \ldots h_{n+2}$ are arbitrary elements from H, then there are only finitely many $h \in H$ such that $(h_1, \ldots, h_{k-1}, h, h_{k+1}, \ldots h_{n+2}) \in A_k$.
15. The cardinality of a set H is at most $\aleph_{\alpha+n}$ if and only if ${}^{n+2}H (\equiv H^{n+2})$ can be represented in the form $A_1 \cup \cdots \cup A_{n+2}$, where the cardinality of A_k "in the direction of the kth coordinate" is smaller than $\aleph_\alpha$, i.e., if $h_1, \ldots, h_{k-1}, h_{k+1}, \ldots h_{n+2}$ are arbitrary elements from H, then there are fewer than $\aleph_\alpha$ elements $h \in H$ such that

$$(h_1, \ldots, h_{k-1}, h, h_{k+1}, \ldots h_{n+2}) \in H_k.$$

16. (Cantor's inequality) For any κ we have $2^\kappa > \kappa$.
17. (König's inequality) If $\rho_i < \kappa_i$ for all $i \in I$, then

$$\sum_{i \in I} \rho_i < \prod_{i \in I} \kappa_i.$$

18. If the set of cardinals $\{\kappa_\xi\}_{\xi<\theta}$, $0 < \kappa_\xi < \kappa$ is cofinal with κ, then $\prod_{\xi<\theta} \kappa_\xi > \kappa$.
19. If κ is infinite, $\kappa = \sum_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi$ where $\kappa > \kappa_\xi > 1$, then

$$\prod_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi = \kappa^{\mathrm{cf}(\kappa)}.$$

20. If κ is infinite, then $\kappa^{\mathrm{cf}(\kappa)} > \kappa$.
21. If $\lambda \geq 2$ and κ is infinite, then $\mathrm{cf}(\lambda^\kappa) > \kappa$.
22. (Bernstein–Hausdorff–Tarski equality) Let κ be an infinite cardinal and λ a cardinal with $0 < \lambda < \mathrm{cf}(\kappa)$. Then

$$\kappa^\lambda = \left(\sum_{\rho<\kappa} \rho^\lambda \right) \kappa.$$

23. If α is a limit ordinal, $\{\kappa_\xi\}_{\xi<\alpha}$ is a strictly increasing sequence of cardinals and $\kappa = \sum_{\xi<\alpha} \kappa_\xi$, then for all $0 < \lambda < \mathrm{cf}(\alpha)$ we have $\kappa^\lambda = \sum_{\xi<\alpha} \kappa_\xi^\lambda$.
24. If λ is singular and there is a cardinal κ such that for some $\mu < \lambda$ for every cardinal τ between μ and λ we have $2^\tau = \kappa$, then $2^\lambda = \kappa$, as well.
25. If there is an ordinal γ such that $2^{\aleph_\alpha} = \aleph_{\alpha+\gamma}$ holds for every infinite cardinal $\aleph_\alpha$, then γ is finite.
26. The operation $\kappa \mapsto \kappa^{\mathrm{cf}(\kappa)}$ on cardinals determines
    (a) the operation $\kappa \mapsto 2^\kappa$,

(b) the operation $(\kappa, \lambda) \mapsto \kappa^\lambda$.

27. If n is finite, then for $\lambda \geq 1$
    (a) $\aleph_{\alpha+n}^\lambda = \aleph_\alpha^\lambda \, \aleph_{\alpha+n}$.
    (b) $\aleph_n^\lambda = 2^\lambda \aleph_n$.
28. When does
$$\prod_{n<\omega} \aleph_n = 2^{\aleph_0}$$
hold?
29.
$$\prod_{n<\omega} \aleph_n = \aleph_\omega^{\aleph_0}.$$
30. If for all $n < \omega$ we have $2^{\aleph_n} < \aleph_\omega$, then $2^{\aleph_\omega} = \aleph_\omega^{\aleph_0}$.
31. If $\rho \geq \omega$ is a given cardinal, then there are infinitely many cardinals κ for which $\kappa^\rho = \kappa$, and there are infinitely many for which $\kappa^\rho > \kappa$.
32. There are arbitrarily large cardinals λ with $\lambda^{\aleph_0} < \lambda^{\aleph_1}$.
33. For an infinite cardinal κ let μ be the minimal cardinal with $2^\mu > \kappa$. Then $\{\kappa^\lambda : \lambda < \mu\}$ is finite.
34. For an infinite cardinal κ let $\rho = \rho_\kappa$ be the smallest cardinal such that $\kappa^\rho > \kappa$. Then ρ_κ is a regular cardinal. What is ρ_ω? And ρ_{ω_ω}?
35. The smallest κ for which $2^\kappa > \mathbf{c}$ holds is regular.
36. Let $\kappa_0 = \aleph_0$, and for every natural number n let $\kappa_{n+1} = \aleph_{\kappa_n}$. Then $\kappa = \sup_n \kappa_n$ is the smallest cardinal with the property $\kappa = \aleph_\kappa$.
37. There are infinitely many cardinals κ such that the set of cardinals smaller than κ is of cardinality κ (i.e., $\kappa = \aleph_\kappa$). If we call such cardinals κ "large", then are there cardinals κ such that the set of "large" cardinals smaller than κ is of cardinality κ?
38. Under GCH (generalized continuum hypothesis) find all cardinals κ for which $\kappa^{\aleph_0} < \kappa^{\aleph_1} < \kappa^{\aleph_2}$ hold.
39. Assuming GCH evaluate $\prod_{\beta<\alpha} \aleph_\beta$.
40. Under GCH determine κ^λ.

11

Partially ordered sets

Let A be a set and $\prec$ a binary relation on A. $\langle A, \prec\rangle$ is called a *partially ordered set* if

- $\prec$ *irreflexive*: $a \not\prec a$ for any $a \in A$,
- $\prec$ *transitive*: $a \prec b$ and $b \prec c$ imply $a \prec c$.

Thus, the difference with ordered sets is that here we do not assume trichotomy (comparability of elements).

In a partially ordered set $\langle A, \prec\rangle$ two elements a, b are called *comparable* if (exactly) one of $a = b$, $a \prec b$ or $b \prec a$ holds, otherwise they are *incomparable*. An ordered subset of a partially ordered set is called a *chain* and a set of pairwise incomparable elements an *antichain*.

The main problem that we treat in this chapter is how information on the size of chains and antichains can be related to the structure of the set in question.

1. In an infinite partially ordered set there is an infinite chain or an infinite antichain.
2. If in a partially ordered set all chains have at most $l < \infty$ elements and all antichains have at most $k < \infty$ elements, where k, l are finite numbers, then the set has at most kl elements.
3. If in a partially ordered set all chains have at most $k < \infty$ elements, then the set is the union of k antichains.
4. If in a partially ordered set all antichains have at most $k < \infty$ elements, then the set is the union of k chains.
5. There is a partially ordered set in which all chains are finite, still the set is not the union of countably many antichains.
6. There is a partially ordered set in which all antichains are finite, still the set is not the union of countably many chains.

7. If in a partially ordered set all chains are finite and all antichains are countable, then the set is countable.
8. If in a partially ordered set all antichains are finite and all chains are countable, then the set is countable.
9. There is a partially ordered set of cardinality continuum in which all chains and all antichains are countable.
10. If in a partially ordered set all chains and all antichains have at most κ elements, then the set is of cardinality at most 2^κ.
11. If κ is an infinite cardinal, then there is a partially ordered set of cardinality 2^κ in which all chains and all antichains have at most κ elements.
12. For every cardinal κ there is a partially ordered set $\langle P, \prec\rangle$ in which every interval $[x, y] = \{z : x \preceq z \preceq y\}$ is finite, yet P is not the union of κ antichains.
13. If $\langle P, \prec\rangle$ is a partially ordered set, call two elements strongly incompatible if they have no common lower bound. Let $c(P, \prec)$ be the supremum of $|S|$ where $S \subseteq P$ is a strong antichain, that is, a set of pairwise strongly incompatible elements.
    (a) If $c(P, \prec)$ is an infinite cardinal that is not weakly inaccessible, i.e., it is not a regular limit cardinal, then $c(P, \prec)$ is actually a maximum.
    (b) If κ is a regular limit cardinal, then there is a partially ordered set $\langle P, \prec\rangle$ such that $c(P, \prec) = \kappa$ yet there is no strong antichain of cardinality κ.
14. If $\langle A, \prec\rangle$ is a partially ordered set, then there exists a cofinal subset $B \subseteq A$ such that $\langle B, \prec\rangle$ is well founded (i.e., in every nonempty subset there is a minimal element).
15. If there is no maximal element in the partially ordered set $\langle P, \prec\rangle$, then there are two disjoint cofinal subsets of $\langle P, \prec\rangle$.
16. There is a partially ordered set $\langle P, \prec\rangle$ which is the union of countably many centered sets but not the union of countably many filters. (A subset $Q \subseteq P$ is centered if for any $p_1, \ldots, p_n \in Q$ there is some $q \preceq p_1, \ldots, p_n$ in P. A subset $F \subseteq P$ is a filter, if for any $p_1, \ldots, p_n \in F$ there is some $q \preceq p_1, \ldots, p_n$ with $q \in F$.)
17. For two real functions $f \neq g$ let $f \prec g$ if $f(x) \leq g(x)$ for all $x \in \mathbf{R}$. In this partially ordered set there is an ordered subset of cardinality bigger than continuum. No such subset can be well ordered by $\prec$.

The following problems use two orderings on the set ${}^\omega\omega$ of all functions $f : \omega \to \omega$: let $f \ll g$ if $f(n) < g(n)$ for all large n, and $f \prec g$ if $g(n) - f(n) \to \infty$ as $n \to \infty$.

18. Each of $\langle {}^\omega\omega, \ll\rangle$ and $\langle {}^\omega\omega, \prec\rangle$ has an order-preserving mapping into the other, but they are not isomorphic.

19. For any countable subset $\{f_k\}_k$ of ${}^\omega\omega$ there is an f larger than any f_k with respect to $\prec$.
20. $\langle {}^\omega\omega, \prec \rangle$ includes a subset of order type ω_1.
21. $\langle {}^\omega\omega, \prec \rangle$ includes a subset of order type λ^m for each $m = 1, 2, \ldots$.
22. If θ is an order type and $\langle {}^\omega\omega, \prec \rangle$ includes a subset similar to θ, then it includes such a subset consisting of functions that are smaller than the identity function.
23. If θ_1, θ_2 are order types and $\langle {}^\omega\omega, \prec \rangle$ includes subsets similar to θ_1 and θ_2, respectively, then it includes subsets similar to $\theta_1 + \theta_2$ and $\theta_1 \cdot \theta_2$, respectively. It also includes a subset similar to θ_1^*, where θ_1^* is the reverse type to θ_1.
24. If θ_i, $i \in I$ are order types where $\langle I, < \rangle$ is an ordered set, and $\langle {}^\omega\omega, \prec \rangle$ includes subsets similar θ_i and also a subset similar to $\langle I, < \rangle$, then it includes subsets similar to $\sum_{i \in I(<)} \theta_i$. In particular, $\langle {}^\omega\omega, \prec \rangle$ includes a set of order type α for every $\alpha < \omega_2$.
25. If $\varphi < \omega_1$ is a limit ordinal and

$$f_0 \prec f_1 \prec \cdots \prec f_\alpha \prec \cdots \prec g_\alpha \prec \cdots g_1 \prec g_0, \qquad \alpha < \varphi,$$

then there is an f with $f_\alpha \prec f \prec g_\alpha$ for every $\alpha < \varphi$.

26. There exist functions

$$f_0 \prec f_1 \prec \cdots \prec f_\alpha \prec \cdots \prec g_\alpha \prec \cdots g_1 \prec g_0, \qquad \alpha < \omega_1,$$

such that there is no function f with $f_\alpha \prec f \prec g_\alpha$ for every $\alpha < \omega_1$.

12

Transfinite enumeration

This chapter deals with a fundamental technique based on the well-ordering theorem. Most of the problems in this chapter require the construction of some objects sometimes with quite surprising properties (like Problem 7: there is a set $A \subset \mathbf{R}^2$ intersecting every line in exactly two points). The objects cannot be given at once, but are obtained by a transfinite recursive process. The idea is to have a well ordering of the underlying structure (in the aforementioned example a well-ordering of the lines on $\mathbf{R}^2$ into a transfinite sequence $\{\ell_\alpha\}_{\alpha<\mathbf{c}}$ of type $\mathbf{c}$) and based on that the object is constructed one by one (in the example constructing an increasing sequence $\{A_\alpha\}_{\alpha<\mathbf{c}}$ of sets such that A_α has at most two points on any line, and it has exactly two points on ℓ_α).

Of similar spirit is the transfinite construction of some closure sets such as the set of Borel sets, the set of Baire functions, or the algebraic closures of fields.

This transfinite enumeration technique will be routinely used in later chapters.

1. If A_i, $i \in I$ is an arbitrary family of sets, then there are pairwise disjoint sets $B_i \subset A_i$ such that $\cup_{i\in I} B_i = \cup_{i\in I} A_i$.
2. If there are given $\kappa \geq \aleph_0$ sets X_ξ each of cardinality κ, then there are pairwise disjoint subsets $Y_\xi \subseteq X_\xi$ each of cardinality κ. Further, we can even have $|X_\xi \setminus Y_\xi| = \kappa$ for all $\xi < \kappa$.
3. If there are given $\kappa \geq \aleph_0$ sets X_ξ, $\xi < \kappa$ each of cardinality κ, then there are pairwise disjoint sets Y_α, $\alpha < \kappa$ such that for all $\alpha, \xi < \kappa$ the intersection $Y_\alpha \cap X_\xi$ is of cardinality κ.
4. Let κ be an infinite cardinal, X a set of cardinality κ, and $\mathcal{F}$ a family of cardinality at most κ of mappings with domain X. Then there is a family $\mathcal{H}$ of cardinality 2^κ of subsets of X with the property that if $H_1, H_2 \in \mathcal{H}$ are two different sets and $f \in \mathcal{F}$ is arbitrary, then $f[H_1] \neq H_2$.
5. If X is an infinite set of cardinality κ, then there is an almost disjoint family $\mathcal{H}$ of cardinality bigger than κ of subsets of X each of cardinality

κ (the intersection of any two members of $\mathcal{H}$ is of cardinality smaller than κ).

6. There is a family $\{N_\alpha\}_{\alpha<\omega_1}$ of subsets of $\mathbf{N}$ such that for $\alpha < \beta < \omega_1$ the set $N_\beta \setminus N_\alpha$ is finite, but the set $N_\alpha \setminus N_\beta$ is infinite.
7. There is a subset A of $\mathbf{R}^2$, that has exactly two points on every line.
8. Suppose that to every line ℓ on the plane a cardinal $2 \le m_\ell \le \mathbf{c}$ is assigned. Then there is a subset A of the plane such that $|A \cap \ell| = m_\ell$ holds for every ℓ.
9. If L_1 and L_2 are two disjoint sets of lines lying on the plane, then the plane can be divided into two sets $A_1 \cup A_2$ in such a way that every line in L_1 resp. L_2 intersects A_1 resp. A_2 in fewer than continuum many points.
10. $\mathbf{R}$ can be decomposed into continuum many pairwise disjoint sets of power continuum, such that each of these sets intersects every nonempty perfect set.
11. $\mathbf{R}$ can be decomposed into continuum many pairwise disjoint and non-measurable sets.
12. $\mathbf{R}$ can be decomposed into continuum many pairwise disjoint sets each of the second category.
13. There is a subset A of $\mathbf{R}^2$ that has at most two points on every line, but A is not of measure zero (with respect to two-dimensional Lebesgue measure).
14. There is a second category subset A of $\mathbf{R}^2$ that has at most two points on every line.
15. There is a set $A \subset \mathbf{R}$ such that every $x \in \mathbf{R}$ has exactly one representation $x = a + b$ with $a, b \in A$.
16. If $A \subset \mathbf{R}$ is an arbitrary set, then there is a function $f : A \to A$ that assumes every value only countably many times and for which $f(a) < a$ for all $a \in A$, except for the smallest element of A (if there is one).
17. Every real function is the sum of two 1-to-1 functions.
18. There is a real function that is not monotone on any set of cardinality continuum.
19. There is a real function F such that for all continuous real functions f the sum $F + f$ assumes all values $y \in \mathbf{R}$ in every interval.
20. There is a real function f such that if $\{x_n\}_{n=0}^\infty$ is an arbitrary sequence of distinct real numbers and $\{y_n\}_{n=0}^\infty$ is an arbitrary real sequence, then there is an $x \in \mathbf{R}$ such that for all n we have $f(x + x_n) = y_n$.
21. For $X \subseteq \mathbf{R}^n$ let X^L be the set of all limit points of X, and starting from $X_0 = X$ form the sets

$$X_\alpha = \begin{cases} X_\beta^L & \text{if } \alpha = \beta + 1, \\ \cap_{\xi<\alpha} X_\xi & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Then there is a countable ordinal θ such that $X_\alpha = X_\theta$ for all $\alpha > \theta$, and the set $X \setminus X_\theta$ is countable. Furthermore X_θ is empty or it is perfect.

22. Every closed set in $\mathbf{R}^n$ is the union of a perfect set and a countable set.
23. Starting from an arbitrary set X and a family $\mathcal{H}$ of subsets of X form the families $\mathcal{H}_\alpha$ of sets in the following way: $\mathcal{H}_0 = \mathcal{H}$; for every ordinal α let $\mathcal{H}_{\alpha+1}$ be the family of sets that can be obtained as a countable union of sets in $\mathcal{H}_\alpha$ or that are the complements (with respect to X) of some sets in $\mathcal{H}_\alpha$; and for a limit ordinal α set $\mathcal{H}_\alpha = \cup_{\beta<\alpha}\mathcal{H}_\beta$. Then $\mathcal{H}_{\omega_1} = \mathcal{H}_\alpha$ for every $\alpha > \omega_1$, and $\mathcal{H}_{\omega_1}$ is the σ-algebra generated by $\mathcal{H}$ (this is the intersection of all σ-algebras including $\mathcal{H}$, and is the smallest σ-algebra including $\mathcal{H}$).
24. The σ-algebra generated by at most continuum many sets is of power at most continuum.
25. The family of Borel sets in $\mathbf{R}^n$ is the smallest family of sets containing the open sets and closed under countable intersection and countable disjoint union.
26. Starting from the set $C[0,1]$ of continuous functions on the interval $[0,1]$ form the following families $\mathcal{B}_\alpha$ of functions: $\mathcal{B}_0 = C[0,1]$; for every α let $\mathcal{B}_{\alpha+1}$ be the set of those functions that can be obtained as pointwise limits of a sequence of functions from $\mathcal{B}_\alpha$; and for a limit ordinal α let $\mathcal{B}_\alpha = \cup_{\beta<\alpha}\mathcal{B}_\beta$. Then $\mathcal{B}_{\omega_1} = \mathcal{B}_\alpha$ for all $\alpha > \omega_1$, and $\mathcal{B}_{\omega_1}$ is the smallest set of functions that is closed for pointwise limits and that includes $C[0,1]$ (this is the set of so-called Baire functions on $[0,1]$).
27. Let $\langle \mathcal{A}, \cdots \rangle$ be an algebraic structure with at most ρ finitary operations. Then the subalgebra in $\mathcal{A}$ generated by a subset of $\kappa (\neq 0)$ elements has cardinality at most $\max\{\kappa, \rho, \aleph_0\}$ (the subalgebra generated by a set X of elements is the intersection of all subalgebras that include X).
28. If $\mathcal{F}$ is any field of cardinality κ, then there is an algebraically closed field $\mathcal{F} \subset \mathcal{F}^*$ of cardinality at most $\max\{\kappa, \aleph_0\}$ (a field

$$\mathcal{F}^* = \langle F^*, +, \cdot, 0, 1 \rangle$$

is called algebraically closed if for any polynomial $a_n \cdot x^n + \cdots + a_1 \cdot x + a_0$ with $a_i \in F^*$ there is an $a \in F^*$ such that $a_n \cdot a^n + \cdots + a_1 \cdot a + a_0 = 0$).
29. Every ordered set of cardinality κ is similar to a subset of the lexicographically ordered set ${}^\kappa\{0,1\}$.
30. Every ordered set is a subset of an ordered set no two different initial segments of which are similar.

13

Euclidean spaces

The problems in this section exhibit some interesting sets or interesting properties of sets in Euclidean n space or in their Hilbert space generalizations. Sometimes the set is given by an explicit construction, at other times by the transfinite enumeration technique of the preceding chapter.

1. If $\mathcal{U}$ is a family of open subsets of $\mathbf{R}^n$ that is well ordered with respect to inclusion, then $\mathcal{U}$ is countable.
2. Call a set $A \subset \mathbf{R}^n$ an algebraic variety if there is a non-identically zero polynomial $P(x_1, \dots, x_n)$ of n variables such that A is its zero set: $A = \{(a_1, \dots, a_n) \ : \ P(a_1, \dots, a_n) = 0\}$. Then $\mathbf{R}^n$ cannot be covered by less than continuum many algebraic varieties.
3. There is a set $A \subset \mathbf{R}^3$ of power continuum such that if we connect the different points of A by a segment, then all these segments are disjoint (except perhaps for their endpoints).
4. From any uncountable subset of $\mathbf{R}^n$ $(n = 1, 2, \dots)$ one can select uncountably many points such that all the distances between these points are different.
5. In ℓ_2 there are continuum many points such that all distances between them are rational (hence from this set one cannot select uncountably many points such that all the distances between the selected points are different).
6. If all the distances between the points of a set $H \subset \ell_2$ are the same, then H is countable.
7. If ℓ_2 is decomposed into countably many sets, then one of them includes an infinite subset A such that all the distances between the points in A are the same.
8. There are continuum many points in ℓ_2 of which every triangle is acute.
9. The plane can be colored with countably many colors such that no two points in rational distance get the same color.

10. $\mathbf{R}^n$ can be colored with countably many colors such that no two points in rational distance get the same color.
11. The plane can be decomposed into countably many pieces none containing the three nodes of an equilateral triangle.
12. Call a set $A \subset \mathbf{R}^2$ a "circle" if there is a point $P \in \mathbf{R}^2$ such that each half-line emanating from P intersects A in one point. The plane can be written as a countable union of "circles".
13. $\mathbf{R}^3$ can be decomposed into a disjoint union of circles of radius 1.
14. $\mathbf{R}^3$ can be decomposed into a disjoint union of lines no two of which are parallel.
15. If A, B are any two intervals on the real line (of positive length), then there are disjoint decompositions $A = \bigcup\{A_i : i = 0, 1, \ldots\}$ and $B = \bigcup\{B_i : i = 0, 1, \ldots\}$ such that B_i is a translated copy of A_i.

14

Zorn's lemma

In this chapter we investigate Zorn's lemma, a powerful tool to prove results for infinite structures. Assume $(\mathcal{P}, \leq)$ is a partially ordered set. A chain $L \subseteq \mathcal{P}$ is a subset in which any two elements are comparable, i.e., for $x, y \in L$ either $x \leq y$ or $y \leq x$ holds. Zorn's lemma states that, if in a partially ordered set $(\mathcal{P}, \leq)$ every chain L has an upper bound (an element $p \in \mathcal{P}$ such that $x \leq p$ holds for $x \in L$), then $(\mathcal{P}, \leq)$ has a maximal element, that is, some element $p \in \mathcal{P}$ with the property that for no $x \in \mathcal{P}$ does $p < x$ hold.

Zorn's lemma is equivalent to the axiom of choice as well as to the well-ordering theorem (see Problem 5), in particular it is independent of the other standard axioms of set theory. Still, as is the case with the axiom of choice, in everyday mathematics it is accepted, and it provides a convenient way to establish certain maximal objects. This chapter contains ample examples for that.

1. Deduce Zorn's lemma from the well-ordering theorem.
2. Prove that Zorn's lemma implies the axiom of choice.
3. Give a direct deduction of the well-ordering theorem from Zorn's lemma.
4. Give a direct deduction of Zorn's lemma from the axiom of choice.
5. The axiom of choice, the well-ordering theorem, and Zorn's lemma are pairwise equivalent.
6. With the help of Zorn's lemma, prove the following.
   (a) The set $\mathbf{R}^+$ of positive real numbers is the disjoint union of two nonempty sets, each closed under addition.
   (b) In a ring with unity, every proper ideal can be extended to a maximal ideal.
   (c) Every filter can be extended to an ultrafilter.
   (d) Every vector space has a basis. In fact, every linearly independent system of vectors can be extended to a basis.

(e) Every vector space has a basis. In fact, every generating system of vectors includes a basis.

(f) For Abelian groups the group $D \supseteq A$ is called the *divisible hull* of A if it is divisible and for every $x \in D$ there is some natural number n that $nx \in A$. If D_1, D_2 are divisible hulls of A, then they are isomorphic over A: there is an isomorphism $\varphi : D_1 \to D_2$ which is the identity on A.

(g) Every field can be embedded into an algebraically closed field.

(h) Every algebraically closed field has a transcendence basis.

(i) Assume F is a field in which 0 is not the sum of nonzero square elements. Then F is orderable, that is, there is an ordering $<$ on F in which $x < y$ implies that $x + z < y + z$ holds for every z, and $x < y$, $z > 0$ imply that $xz < yz$.

(k) If G is an Abelian group and A is a divisible subgroup, then A is a direct summand of G.

(l) Every connected graph includes a spanning tree.

(m) If (V, X) is a graph with chromatic number κ then there is a decomposition of V into κ independent (=edgeless) sets such that between any two there goes an edge.

(n) If X is a compact topological space and $+$ is an associative operation on X which is right semi-continuous (i.e., the mapping $x \mapsto p + x$ is continuous for every $p \in X$), then $+$ has a fixed point, that is, an element $p \in X$, that $p + p = p$.

7. Let S be a set, $\mathcal{F} \subseteq \mathcal{P}(S)$ a family of subsets such that every $x \in S$ is contained in only finitely many elements of $\mathcal{F}$ and for every finite $X \subseteq S$ some $\mathcal{G} \subseteq \mathcal{F}$ constitutes an exact cover of X (i.e., every $x \in X$ is contained in one and only one element of $\mathcal{G}$). Then there is an exact cover $\mathcal{G} \subseteq \mathcal{F}$ of S.

8. (a) For any partially ordered set $(P, <)$ there is an ordered set $(P, <')$ on the same ground set that extends $(P, <)$, i.e., $x < y$ implies $x <' y$.

   (b) Prove that actually $x < y$ holds if and only if $x <' y$ for every such extension.

   (c) If, in part (a), $(\mathcal{P}, <)$ is well-founded, then $(\mathcal{P}, <')$ can be made well ordered.

   (d) Why does part (b) imply part (a) ?

9. (Alexander subbase theorem) Assume that X is a topological space with a subbase $\mathcal{S}$ with the finite cover property, i.e., if the union of some subfamily $\mathcal{S}' \subseteq \mathcal{S}$ covers X, then some finitely many members of $\mathcal{S}'$ cover X, as well. Then X is compact.

10. (Tychonoff's theorem) The topological product of compact spaces is compact.

15

Hamel bases

In this chapter we consider *Hamel bases*, i.e., bases of the vector space of the reals ($\mathbf{R}$) over the field of the rationals ($\mathbf{Q}$). To elaborate, such a basis is a set $B = \{b_i : i \in I\}$ such that every real x can be uniquely written in the form $x = \lambda_0 b_0 + \cdots + \lambda_n b_n$ where $\lambda_0, \ldots, \lambda_n$ are nonzero rationals and $b_0, \ldots, b_n$ are distinct elements of B.

Hamel bases can be used in many intriguing constructions involving the reals. This chapter lists some problems on Hamel bases, as well as on their applications.

Let us call a set $H \subset \mathbf{R}$ *rationally independent* if it is an independent set in the vector space $\mathbf{R}$ over the field $\mathbf{Q}$, and let us call H a *generating subset* if the linear hull of H (over $\mathbf{Q}$) is the whole $\mathbf{R}$.

1. If $H \subset \mathbf{R}$ is rationally independent, then there is a Hamel basis including H.
2. If $H \subset \mathbf{R}$ is a generating set, then it includes a Hamel basis.
3. Every Hamel basis has cardinality $\mathbf{c}$.
4. There are $2^{\mathbf{c}}$ distinct Hamel bases.
5. There is an everywhere-dense Hamel basis.
6. There is a nowhere-dense, measure zero Hamel basis.
7. There is a Hamel basis of full outer measure.
8. A Hamel basis, if measurable, is of measure zero.
9. A Hamel basis cannot be an analytic set.
10. If the continuum hypothesis is true, then $\mathbf{R} \setminus \{0\}$ is the union of countably many Hamel bases.
11. (Cont'd) If $\mathbf{R} \setminus \{0\}$ is the union of countably many Hamel bases, then the continuum hypothesis holds.

12. If the continuum hypothesis is true, then there is a Hamel basis $B = \{b_i : i \in I\}$ such that the set B^+ of real numbers x written in the form $x = \sum\{\lambda_i b_i : i \in I\}$ with nonnegative coefficients is a measure zero set.
13. Describe, in terms of Hamel bases, all solutions of the functional equations
    (a) $f(x+y) = f(x) + f(y)$ (additive functions, Cauchy functions);
    (b) $f(x+y) = f(x)f(y)$;
    (c) $f(xy) = f(x)f(y)$;
    (d) $f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$;
    (e) $f(x+y) = f(x) + f(y) + c$ with some fixed constant c;
    (f) $f(x+y) = g(x) + h(y)$;
    (g) $f(x+y) = af(x) + bf(y)$ with some fixed constants a, b.
14. If the real numbers α, β are not commensurable, then for any $A, B \in \mathbf{R}$ there is a function $f : \mathbf{R} \to \mathbf{R}$ for which $f(x+y) = f(x) + f(y)$ always holds and $f(\alpha) = A$, $f(\beta) = B$.
15. The function $F(x) = x$ (for $x \in \mathbf{R}$) is the sum of two periodic functions.
16. (Cont'd) The function $F(x) = x^2$ (for $x \in \mathbf{R}$) is the sum of three periodic functions but not of two.
17. (Cont'd) Let $k \geq 1$ be a natural number. The function $F(x) = x^k$ (for $x \in \mathbf{R}$) is the sum of $(k+1)$ periodic functions but not the sum of k periodic functions.
18. There exists $A \subset \mathbf{R}$ such that there are countably infinitely many subsets of $\mathbf{R}$ congruent to A.
19. There is a set $A \subset \mathbf{R}$ different from $\emptyset$ and $\mathbf{R}$ such that for all $x \in \mathbf{R}$ only finitely many of the sets A, $A + x$, $A + 2x$, $A + 3x, \ldots$ are different.
20. There exists a set $A \subset \mathbf{R}$ with both A, $\mathbf{R} \setminus A$ everywhere dense, which has the property that if a is a real number, then either $A \subseteq A + a$ or $A + a \subseteq A$.
21. There exists a partition of the set $\mathbf{R} \setminus \mathbf{Q}$ of irrational numbers into two sets, both closed under addition.
22. There exists a partition of the set $\mathbf{R}^+ = \{x \in \mathbf{R} : x > 0\}$ of positive real numbers into two nonempty sets, both closed under addition.
23. We are given 17 real numbers with the property that if we remove any one of them then the remaining 16 numbers can be rearranged into two 8-element groups with equal sums. Prove that the numbers are equal.
24. $\mathbf{R}$ is the union of countably many sets, none of which including a (non-trivial) 3-element arithmetic progression.
25. If a rectangle can be decomposed into the union of finitely many rectangles each having commeasurable sides, then the original rectangle also has commeasurable sides.

26. The set of reals carries an ordering $\prec$ such that there are no elements $x \prec y \prec z$, forming a 3-element arithmetic progression (that is, $y = \frac{x+z}{2}$).
27. There is an addition preserving bijection between $\mathbf{R}$ and $\mathbf{C}$.

16

The continuum hypothesis

The continuum hypothesis (CH) claims that every infinite subset of the reals is equivalent either to **N** or to **R**. It is independent of the standard axioms of set theory (see the introduction to Chapter 4), and in general it is not assumed when one deals with set theory or problems related to set theory.

Since the continuum hypothesis says something about the set of the reals, it is no wonder that it has many equivalent formulations involving real functions or sets in Euclidean spaces. This chapter lists several of these reformulations. Also, in the presence of CH the set of reals "looks differently" than otherwise, and this is reflected in the existence of sets (such as Lusin sets or Sierpinski sets) with various properties. The problems below contain several examples of this phenomenon. CH coupled with the enumeration technique of Chapter 12 is particularly powerful, for in a construction only countably many previously constructed objects have to be taken care of.

1. (Sierpinski's decomposition) CH is equivalent to the statement that the plane is the union of two sets, A and B, such that A intersects every horizontal line and B intersects every vertical line in a countable set.
2. CH holds if and only if the plane is the union of the graphs of countably many $x \mapsto y$ and $y \mapsto x$ functions.
3. CH is equivalent to the existence of a decomposition $\mathbf{R}^3 = A_1 \cup A_2 \cup A_3$ such that if L is a line in the direction of the x_i-axis then $A_i \cap L$ is finite.
4. For no natural number m exists a decomposition $\mathbf{R}^3 = A_1 \cup A_2 \cup A_3$ such that if L is a line in the direction of the x_i-axis then $|A_i \cap L| \leq m$.
5. $c \leq \aleph_n$ if and only if there is a decomposition $\mathbf{R}^{n+2} = A_1 \cup A_2 \cup \cdots \cup A_{n+2}$ such that if L is a line in the direction of the x_i-axis then $A_i \cap L$ is finite.
6. CH holds if and only if there is a surjection from $\mathbf{R}$ onto $\mathbf{R} \times \mathbf{R}$ of the form $x \mapsto (f_1(x), f_2(x))$ with the property that for every $x \in \mathbf{R}$ either $f_1'(x)$ or $f_2'(x)$ exists.
7. CH holds if and only if $\mathbf{R}$ is the union of an increasing chain of countable sets.

8. CH holds if and only if there is a function $f : \mathbf{R} \to \mathcal{P}(\mathbf{R})$ with $f(x)$ countable for every $x \in \mathbf{R}$ and such that $f[X] = \mathbf{R}$ holds for every uncountable set $X \subseteq \mathbf{R}$.
9. CH holds if and only if there exist functions $f_0, f_1, \ldots : \mathbf{R} \to \mathbf{R}$ such that if $a \in \mathbf{R}$ then for all but countably many $x \in \mathbf{R}$ the set $A_{x,a} = \{n < \omega : f_n(x) = a\}$ is infinite.
10. CH holds if and only if there exist functions $f_0, f_1, \ldots : \mathbf{R} \to \mathbf{R}$ such that if $\underline{a} = \{a_0, a_1, \ldots\}$ is an arbitrary real sequence then for all but countably many $x \in \mathbf{R}$ the set $A_{x,\underline{a}} = \{n < \omega : f_n(x) = a_n\}$ is infinite.
11. CH holds if and only if there exist an uncountable family $\mathcal{F}$ of real sequences with the property that if $\{a_0, a_1, \ldots\}$ is an arbitrary real sequence then for all but countably many $\{y_n\} \in \mathcal{F}$ there are infinitely many n with $y_n = a_n$.
12. CH holds if and only if there exist functions $f_0, f_1, \ldots : \mathbf{R} \to \mathbf{R}$ with the property that if $X \subseteq \mathbf{R}$ is uncountable then $f_n[X] = \mathbf{R}$ holds for all but finitely many $n < \omega$.
13. CH holds if and only if there is a family $\{A_\alpha : \alpha < \omega_1\}$ of infinite subsets of ω such that if $X \subseteq \omega$ is infinite then there is some $\alpha < \omega_1$ with $A_\alpha \setminus X$ finite.
14. CH holds if and only if there is a family $\mathcal{H} = \{A_i : i \in I\}$ of subsets of $\mathbf{R}$ with $|I| = c$, $|A_i| = \aleph_0$ such that if $B \subseteq \mathbf{R}$ is infinite then for all but countably many i we have $A_i \cap B \neq \emptyset$.
15. CH holds if and only if $\mathbf{R}$ can be decomposed as $\mathbf{R} = A \cup B$ into uncountable sets in such a way that for every real a the intersection $(A + a) \cap B$ is countable.
16. CH holds if and only if the plane can be decomposed into countably many parts none containing 4 distinct points a, b, c, and d such that $\text{dist}(a, b) = \text{dist}(c, d)$ ("dist" is the Euclidean distance).
17. CH holds if and only if $\mathbf{R}$ can be colored by countably many colors such that the equation $x + y = u + v$ has no solution with different x, y, u, v of the same color.
18. If the continuum hypothesis holds then there is a function $f : \mathbf{R} \to \mathbf{R}$ such that for every $x \in \mathbf{R}$ we have

$$\lim_{h \to 0} \max\left(f(x-h), f(x+h)\right) = \infty.$$

19. CH holds if and only if there exists an uncountable family $\mathcal{F}$ of entire functions (on the complex plane $\mathbf{C}$) such that for every $a \in \mathbf{C}$ the set $\{f(a) : f \in \mathcal{F}\}$ is countable.
20. (a) If CH holds, then there is a set A of reals of cardinality continuum such that A intersects every set of first category in a countable set (such a set is called a Lusin set).

(b) Every Lusin set is of measure zero.

21. CH is equivalent to the statement that there is a Lusin set and every subset of $\mathbf{R}$ of cardinality $< \mathbf{c}$ is of first category.

22. (a) If CH holds, then there is a set A of reals of cardinality continuum such that A intersects every set of measure zero in a countable set (such a set is called a Sierpinski set).

    (b) Every Sierpinski set is of first category.

23. CH is equivalent to the statement that there is a Sierpinski set and every subset of $\mathbf{R}$ of cardinality $< \mathbf{c}$ is of measure zero.

24. If CH holds and $A \subseteq [0,1]^2$ is a measurable set of measure one, then there exist sets $B, C \subseteq [0,1]$ of outer measure one with $B \times C \subseteq A$. (Note that there is an $A \subseteq [0,1]^2$ of measure one such that if $B, C \subseteq [0,1]$ are measurable sets with $B \times C \subseteq A$, then they are of measure zero.)

25. If CH holds, then there is an uncountable set $A \subseteq \mathbf{R}$ such that if $G \supseteq \mathbf{Q}$ is an open set then $A \setminus G$ is countable (A is concentrated around $\mathbf{Q}$).

26. If CH holds, then there is an uncountable $A \subset \mathbf{R}$ such that any uncountable $B \subset A$ is dense in some open interval.

27. If CH holds, then there is an uncountable densely ordered set $\langle A, \prec \rangle$ such that any nowhere dense set (in the interval topology) in $\langle A, \prec \rangle$ is countable.

28. If CH holds, then there is an uncountable set $A \subseteq \mathbf{R}$ such that if $\varepsilon_0, \varepsilon_1, \ldots$ are arbitrary positive reals then there is a cover $I_0 \cup I_1 \cup \cdots$ of A such that I_n is an interval of length ε_n.

29. If CH holds, then there is a permutation $\pi : \mathbf{R} \to \mathbf{R}$ of the reals such that $A \subseteq \mathbf{R}$ is of first category if and only if $\pi[A]$ is of measure zero.

17

Ultrafilters on ω

If X is a ground set, then a family $\mathcal{F}$ of subsets of X is called a *filter* if

- $\emptyset \notin \mathcal{F}$,
- $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
- $A \in \mathcal{F}$ and $A \subseteq B$ imply $B \in \mathcal{F}$.

A filter $\mathcal{F}$ is called *principal* or *trivial* if $\mathcal{F} = \{A \subset X \ : \ A_0 \subset A\}$ for some $A_0 \subset X$.

A filter that is not a proper subset of another filter is called an *ultrafilter.*

The elements of an ultrafilter $\mathcal{F}$ can be considered as "large" subsets of X, and if the set of elements of X for which a property holds belongs to $\mathcal{F}$, then we consider the property to hold for almost all elements of X.

Ultrafilters play important roles in algebra and logic; in particular, the ultraproduct construction is based on them. They also appear in several solutions in this book.

A dual concept to filter is the concept of an *ideal.* If X is a ground set, then a family $\mathcal{I}$ of subsets of X is called an ideal if

- $X \notin \mathcal{I}$,
- $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$
- $A \in \mathcal{I}$ and $B \subseteq A$ imply $B \in \mathcal{I}$.

An ideal that is not a proper subset of another ideal is called a *prime ideal.*

It is clear that $\mathcal{F}$ is a filter (ultrafilter) if and only if $\{X \setminus F \ : \ F \in \mathcal{F}\}$ is an ideal (prime ideal).

This chapter contains various problems on, and properties of ultrafilters on the set of natural numbers. Problem 19 gives an application in analysis, it verifies the existence of Banach limits—a limit concept that extends the standard notion of limit to all real sequences.

1. A filter $\mathcal{F}$ on ω is an ultrafilter if and only if for every $A \subset \omega$ exactly one of A or $X \setminus A$ belongs to $\mathcal{F}$.

2. Every filter on ω is included in an ultrafilter.
3. There are $2^{\mathbf{c}}$ ultrafilters on ω.
4. If $\mathcal{U}_1, \ldots, \mathcal{U}_n$ are nonprincipal ultrafilters on ω, then there is some infinite, co-infinite $A \in \mathcal{U}_1 \cap \ldots \cap \mathcal{U}_n$.
5. If $\mathcal{U}$ is an ultrafilter on ω and $0 = n_0 < n_1 < \cdots$ are arbitrary natural numbers, then there exists an $A \in \mathcal{U}$ with $A \cap [n_i, n_{i+1}) = \emptyset$ for infinitely many $i < \omega$.
6. If $\mathcal{U}$ is an ultrafilter on ω, then $\mathcal{U}$ contains a set $A \subset \omega$ of lower density zero.
7. There is an ultrafilter $\mathcal{U}$ on ω such that every $A \in \mathcal{U}$ has positive upper density.
8. Is there a translation invariant ultrafilter on ω? Is there a translation invariant ultrafilter on $\mathbf{Q}$?
9. Let $\mathcal{U}$ be a nonprincipal ultrafilter on ω. Two players consecutively say natural numbers $0 < n_0 < n_1 < \cdots$ with player I beginning. Player I wins if and only if the set $[0, n_0) \cup [n_1, n_2) \cup \cdots$ is in $\mathcal{U}$. Show that neither player has a winning strategy.
10. (CH) There is nonprincipal ultrafilter $\mathcal{U}$ on ω such that if $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ are elements of $\mathcal{U}$, then there is an element B of $\mathcal{U}$ such that $B \setminus A_n$ is finite for every n. (Such an ultrafilter is called a p-point.)
11. (CH) There is a nonprincipal ultrafilter $\mathcal{U}$ on ω such that if $f : [\omega]^r \to \{1, 2, \ldots, n\}$ is a coloring of all r-element subsets of ω with finitely many colors, then there is a monochromatic element of $\mathcal{U}$. (Such an ultrafilter is called Ramsey ultrafilter).
12. Assume that $(A, \prec)$ is a countable ordered set and $\mathcal{U}$ is a Ramsey ultrafilter on A. Then there is an element $B \in \mathcal{U}$ which is a set of type either ω or ω^*.
13. Let $\mathcal{U}$ be a Ramsey ultrafilter on ω and let $f : \omega \to \omega$ be arbitrary. Then either f is essentially constant (i.e., $\{n < \omega : f(n) = k\} \in \mathcal{U}$ for some $k < \omega$), or f is essentially one-to-one (i.e., $f|_A$ is one-to-one on a set $A \in \mathcal{U}$).
14. Let $\mathcal{U}$ be a Ramsey ultrafilter on ω and $n_0 < n_1 < \cdots$ arbitrary numbers. Then there is a set $A \in \mathcal{U}$ with $|A \cap [n_i, n_{i+1})| = 1$ for all $i = 0, 1, \ldots$.
15. Let $\mathcal{U}$ be a Ramsey ultrafilter on ω, $\{a_n\}$ a positive sequence converging to 0 and $\epsilon > 0$ arbitrary. Then there is an $A \in \mathcal{U}$ with
$$\sum_{n \in A} a_n < \epsilon.$$
16. There are an ultrafilter $\mathcal{U}$ on ω and a positive sequence $\{a_n\}$ converging to 0, such that if $A \in \mathcal{U}$ then $\sum_{n \in A} a_n = \infty$.

17. There is an ultrafilter $\mathcal{U}$ on ω that is not generated by less than continuum many elements, i.e., if $\mathcal{F}$ is a family of subsets of $\mathcal{U}$ of cardinality smaller than continuum, then there is an element $A \in \mathcal{U}$ such that $F \not\subset A$ for $F \in \mathcal{F}$.

18. Associate with every $A \subseteq \omega$ the real number $x_A = 0, \alpha_0\alpha_1 \ldots$ where $\alpha_i = 1$ if and only if $i \in A$. This way to every subset $\mathcal{U}$ of $\mathcal{P}(\omega)$ we associate a subset $X_{\mathcal{U}}$ of $[0, 1]$. Show that if $\mathcal{U}$ is a nonprincipal ultrafilter on ω, then $X_{\mathcal{U}}$ cannot be a Lebesgue measurable set.

19. If D is a nonprincipal ultrafilter and $\{x_n : n < \omega\}$ is a sequence of reals, then set $\lim_D x_n = r$ if and only if $\{n : p < x_n < q\} \in D$ holds whenever $p < r < q$. If this is the case we say that $\{x_n\}$ has a D-limit.
    (a) Every bounded sequence has a unique D-limit.
    (b) The D-limit of a convergent sequence coincides with its ordinary limit.
    (c) $\lim_D cx_n = c \lim_D x_n$.
    (d) $\lim_D(x_n + y_n) = \lim_D x_n + \lim_D y_n$.
    (e) $|\limsup_D x_n| \le \sup_n |x_n|$.
    (f) If the sequences $\{x_n\}$ and $\{y_n\}$ have the property that $x_n - y_n \to 0$, then $\lim_D x_n = \lim_D y_n$.
    (g) If $\lim_D x_n = a$ and f is a real function continuous at the point a, then $\lim_D f(x_n) = f(a)$.
    (h) If $r \in \mathbf{R}$ is a limit point of the set $\{x_n : n < \omega\}$ then there exists a nonprincipal ultrafilter D such that $\lim_D x_n = r$.
    (i) Set $\lim_D x_n = \infty$ if and only if $\{n : p < x_n\} \in D$ holds whenever $p < \infty$, and define $\lim_D x_n = -\infty$ analogously. Then every real sequence has a (possibly infinite) D-limit.

20. Show that there is a function $f : \mathcal{P}(\mathbf{N}) \to [0, 1]$ such that $f(A) = d(A)$ whenever the set $A \subseteq \mathbf{N}$ has density $d(A)$, and f is finitely additive, i.e., $f(A \cup B) = f(A) + f(B)$ when A, B are disjoint.

21. Let there be an infinite sequence of switches, $S_0, S_1, \ldots$ each having three positions $\{0, 1, 2\}$, and a light also with three states $\{L_0, L_1, L_2\}$. They are connected in such a way that if the positions of all switches are simultaneously changed then the state of the light also changes. Let us also suppose that if all the switches are in the ith position then the light is also in the L_i state. Show that there is a (possibly principal) ultrafilter $\mathcal{U}$ that determines the state of the light in the sense that it is L_i precisely when
$$\{j \ : \ S_j \text{ is in the } i\text{th position}\} \in \mathcal{U}.$$

22. Suppose that in an election there are $n \ge 3$ candidates and a set of voters I, each of whom makes a ranking of the candidates. There are two rules for the outcome:

- if all the voters enter the same ranking, then this is the outcome,
- if a candidate a precedes candidate b in the outcome depends only on their order on the different ranking lists of the individual voters (and it does not depend on where a and b are on those lists, i.e., on how the voters ranked other candidates).

Then there is an ultrafilter $\mathcal{F}$ on I such that the outcome is an order π if and only if the set F_π of those voters $i \in I$ whose ranking is π belongs to $\mathcal{F}$. In particular, if I is finite, then in every such voting scheme there is a dictator whose ranking gives the outcome.

18

Families of sets

The problems in this chapter discuss various combinatorial properties of families of sets and functions.

1. For every cardinal $\kappa \geq \omega$ there is a family $A_{\xi,\eta}$, $\xi < \kappa$, $\eta < \kappa^+$ of subsets of κ^+ such that for fixed ξ the sets $A_{\xi,\eta}$, $\eta < \kappa^+$ are disjoint, and for each $\eta < \kappa^+$ the set $\kappa^+ \setminus \cup_{\xi<\kappa} A_{\xi,\eta}$ is of cardinality $< \kappa^+$. (Such a family is called an Ulam matrix. The matrix is of size $\kappa \times \kappa^+$, the κ^+ elements in a row are disjoint, and yet the union of the κ elements in every column is κ^+ save a set of size $< \kappa^+$).
2. For every cardinal $\kappa \geq \omega$ there is a family $\mathcal{F}$ of κ^+ *almost disjoint* subsets of κ of cardinality κ, that is, for A, $B \in \mathcal{F}$, $A \neq B$ we have $|A| = |B| = \kappa$ but $|A \cap B| < \kappa$.
3. If X is an infinite set of cardinality κ, then there are 2^κ subsets $A_\gamma \subset X$ such that if $\gamma_1 \neq \gamma_2$, then each of the sets $A_{\gamma_1} \setminus A_{\gamma_2}$, $A_{\gamma_2} \setminus A_{\gamma_1}$, and $A_{\gamma_1} \cap A_{\gamma_2}$ is of cardinality κ.
4. For every cardinal $\kappa \geq \omega$ there are κ^+ subsets of κ so that selecting any two of them, one includes the other.
5. If X is an infinite set of cardinality κ, then there is a family $\mathcal{F}$ of cardinality 2^κ of subsets of A such that no member of $\mathcal{F}$ is a proper subset of another member of $\mathcal{F}$ (such a family is called an *antichain*).
6. Let $\kappa \geq \omega$ be a cardinal. For every S, the set $[S]^\kappa$ is the union of 2^κ antichains.
7. If κ is an infinite cardinal, then there are 2^κ sets A_α, B_α, $\alpha < 2^\kappa$ of cardinality κ such that $A_\alpha \cap B_\beta \neq \emptyset$ if and only if $\alpha \neq \beta$.
8. Let κ be an infinite cardinal and A_i, B_i, $i \in I$ a family of sets with the property $|A_i|, |B_i| \leq \kappa$ and $A_i \cap B_j \neq \emptyset$ if and only if $i \neq j$. Then $|I| \leq 2^\kappa$.
9. There are two disjoint families $\mathcal{F}, \mathcal{G} \subset \mathcal{P}(\mathbf{N})$ of subsets of $\mathbf{N}$ such that every infinite subset $A \subseteq \mathbf{N}$ includes an element of $\mathcal{F}$ and of $\mathcal{G}$.

10. For any infinite set X there are two disjoint families $\mathcal{F}, \mathcal{G} \subset \mathcal{P}(X)$ of countably infinite subsets of X such that every infinite subset $A \subseteq X$ includes an element of $\mathcal{F}$ and of $\mathcal{G}$.

Call a family $\mathcal{F}$ of subsets of a set S *independent* if the following statement is true: if $X_1, \ldots, X_n$ are different members of $\mathcal{F}$, $\varepsilon_1, \ldots, \varepsilon_n < 2$, then

$$X_1^{\varepsilon_1} \cap \cdots \cap X_n^{\varepsilon_n} \neq \emptyset$$

where for a set X we put $X^1 = X$, $X^0 = S \setminus X$.

11. For every $\kappa \geq \omega$ there is an independent family of cardinality 2^κ of subsets of κ.
12. For every $\kappa \geq \omega$ there are 2^{2^κ} ultrafilters on κ.
13. Let A be an infinite set of cardinality κ. Then there is a family $\mathcal{F}$ of cardinality 2^κ of functions $f : A \to \omega$ with the property that if $f_1, \ldots, f_n \in \mathcal{F}$ are finitely many different functions from $\mathcal{F}$, then there is an $a \in A$ where the functions $f_1, \ldots, f_n$ take different values: $f_i(a) \neq f_j(a)$ if $1 \leq i < j \leq n$.
14. Let A be an infinite set of cardinality 2^κ. Then there is a family $\mathcal{G}$ of cardinality κ of functions $f_k : A \to \kappa$ such that for an arbitrary function $f : A \to \kappa$ and for an arbitrary finite set $A' \subset A$ there is a $g \in \mathcal{G}$ such that g agrees with f on A'.
15. Let κ be infinite. If $\mathcal{T}_i$, $i < 2^\kappa$ are 2^κ topological spaces each of which has a dense subset of cardinality at most κ, then the same is true of their product.
16. Let $\mathcal{F}$ be a countable family of infinite sets with $|A \cap B| = 1$ for $A, B \in \mathcal{F}$, $A \neq B$. Then there is a set X with $1 \leq |X \cap A| \leq 2$ for every $A \in \mathcal{F}$.
17. Let $\mathcal{F}$ be a countable family of infinite sets with $|A \cap B| \leq 2$ for $A, B \in \mathcal{F}$, $A \neq B$. Then there are two sets X, Y such that for every $A \in \mathcal{F}$ either $|A \cap X| = 1$ or $|A \cap Y| = 1$.
18. Prove that for every $\aleph_1 \leq \kappa < \aleph_\omega$ there is a family $\mathcal{F} \subseteq [\kappa]^{\aleph_0}$ of cardinality κ such that for every $X \in [\kappa]^{\aleph_0}$ there is some $Y \in \mathcal{F}$ with $X \subseteq Y$. Prove that no such family exists for $\kappa = \aleph_\omega$.
19. If κ, μ are infinite cardinals, then there is an almost disjoint family of μ-element sets which is not κ-colorable. That is, there is $\mathcal{H} \subseteq [V]^\mu$ for some set V with $|H \cap H'| < \mu$ for $H, H' \in \mathcal{H}$, $H \neq H'$, such that if $F : V \to \kappa$ is a coloring then some member of $\mathcal{H}$ is monocolored.

19

The Banach–Tarski paradox

This chapter deals with a surprising consequence of the axiom of choice, namely the so-called Banach–Tarski paradox claiming that any two balls (with possibly different radii) in the space can be decomposed into each other, i.e., if B_1 and B_2 are such balls then there are disjoint decompositions $B_1 = E_1 \cup \cdots \cup E_n$, $B_2 = F_1 \cup \cdots \cup F_n$ such that each E_i is congruent to F_i. Actually, any two bounded sets in $\mathbf{R}^3$ with nonempty interior can be decomposed into each other.

A "common sense" argument against such a decomposition runs as follows: take a nontrivial finitely additive and isometry invariant measure μ on all subsets of $\mathbf{R}^3$ (think of μ as a "volume" associated with each set). Then the μ-measure of B_1 is different from the μ-measure of B_2 if their radii are different, hence the aforementioned decomposition of B_1 into B_2 is impossible, since measure is preserved under isometry. Of course, this argument fails if there is no such measure, and the Banach–Tarski paradox shows precisely that such a measure does not exist in $\mathbf{R}^3$. Hidden behind the Banach–Tarski paradox is the axiom of choice appearing, for example, in the solution of Problem 17,(c).

Let us also note that in $\mathbf{R}$ and $\mathbf{R}^2$ there are finitely additive isometry invariant measures (see Chapter 28), so in $\mathbf{R}$ and $\mathbf{R}^2$ a Banach–Tarski type paradox cannot be established. The difference between $\mathbf{R}, \mathbf{R}^2$, and $\mathbf{R}^3$ (and of course every $\mathbf{R}^n$ with $n \geq 3$) is that the isometry groups of $\mathbf{R}$ and $\mathbf{R}^2$ are relatively simple, while that of $\mathbf{R}^3$ includes a free subgroup generated by two appropriate rotations.

This chapter contains various problems regarding decompositions (via different kinds of transformations on the parts) culminating in Problem 17 containing the Banach–Tarski paradox. We consider the equidecomposability of subsets of some set X, where sets are decomposed into the union of finitely many subsets and are transformed by the elements of Φ, a family of $X \to X$ bijections, containing the identity, closed under composition and taking inverse (i.e., Φ is a group with respect to composition). If $A, B \subseteq X$, then A *is equidecomposable to* B *via* Φ, in symbol $A \sim_\Phi B$, if there are partitions $A = A_1 \cup \cdots \cup A_n$, $B = B_1 \cup \cdots \cup B_n$, such that $B_i = f_i[A_i]$ for some $f_i \in \Phi$.

If there is no danger of confusion we simply write $A \sim B$ instead of $A \sim_{\Phi} B$. $A \preceq B$ if $A \sim B'$ holds for some $B' \subseteq B$. $\{A_1, \ldots, A_t\}$ is a *p-cover of A* (is a $\leq$ *p-cover of A*) if $A_1, \ldots, A_t \subseteq A$ and every element of A is in exactly p of the A_i's (and every element of A is in $\leq p$ of the A_i's). If $A, B \subseteq X$, then $pA \sim qB$ denotes that there is a p-cover $\{A_1, \ldots, A_t\}$ of A such that for appropriate $f_1, \ldots, f_t \in \Phi$, the sets $f_1[A_1], \ldots, f_t[A_t]$ constitute a q-cover of B. If, on the other hand, $f_1[A_1], \ldots, f_t[A_t]$ is just a $\leq q$-cover of B, then we write $pA \preceq qB$. $A \subseteq X$ is *paradoxical* if $A \sim 2A$. Usually it is "obvious" what Φ is, still, in most cases, we indicate it. If $X = \mathbf{S}^n$ (the n-dimensional unit sphere) then Φ is the set of rotations around its center; if $X = \mathbf{R}^n$, then Φ is the set of the congruences; if X is a group, then Φ is the set of left multiplications: $\Phi = \{f_x : x \in X\}$ where $f_x(y) = xy$.

1. $\sim$ is an equivalence relation.
2. If $A \preceq B$ and $B \preceq A$, then $A \sim B$.
3. If $pA \preceq qB$ and $qB \preceq rC$, then $pA \preceq rC$ holds as well (p, q, r are nonzero natural numbers).
4. If $pA \preceq qB$, $qB \preceq pA$ hold for some natural numbers p, q, then $pA \sim qB$.
5. If $pA \sim qB$ and $qB \sim rC$, then $pA \sim rC$ holds as well (p, q, r are nonzero natural numbers).
6. If $kpA \preceq kqB$ holds for some natural numbers k, p, q, $k \geq 1$, then $pA \preceq qB$. Therefore, $kpA \sim kqB$ implies $pA \sim qB$.
7. The following are equivalent.
   (a) $(n+1)A \preceq nA$ for some natural number n;
   (b) A is paradoxical;
   (c) A can be decomposed as $A = A' \cup A''$ with $A' \sim A'' \sim A$;
   (d) For every $k \geq 2$, A can be decomposed as $A = A_1 \cup \cdots \cup A_k$ with $A_1 \sim A_2 \sim \cdots \sim A_k \sim A$;
   (e) $pA \sim qA$ holds whenever p, q are positive natural numbers.
8. If A is paradoxical and $A \preceq B \preceq nA$ holds for some natural number n, then B is paradoxical as well.
9. (a) There exists a countable, paradoxical planar set.
   (b) There exists a *bounded* paradoxical set on the plane.
10. If $A \subseteq \mathbf{S}^2$, $|A| < \mathbf{c}$ then $\mathbf{S}^2 \sim \mathbf{S}^2 \setminus A$ (via rotations).
11. $[0, 1] \sim (0, 1]$ (with translations).
12. $Q \sim Q \setminus I$, where Q is the unit square, I is one of its (closed) sides (via translations).
13. If P is a (closed) planar polygon, F is its boundary, then $P \sim P \setminus F$ (via translations).
14. If P, Q are planar polygons, equidecomposable in the geometrical sense, then they are equidecomposable (via planar congruences).

15. Assume that $E \sim \mathbf{Z}$ holds (via translations) for some $E \subseteq \mathbf{Z}$. What is E?
16. (a) No nonempty subset of $\mathbf{Z}^n$ is paradoxical (via translations).
    (b) No nonempty subset of an Abelian group is paradoxical (via multiplication by group elements).
    (c) No nonempty subset of $\mathbf{R}$ is paradoxical (via congruences).
17. (a) For some $A \subseteq F_2$, natural number n, $\aleph_0 A \preceq F_2 = nA$. ($F_2$ is the free group generated by 2 elements.) Notice that this gives that A, therefore F_2 is paradoxical.
    (b) There are two independent rotations around the center of $\mathbf{S}^2$.
    (c) $\mathbf{S}^2$ is paradoxical (via rotations).
    (d) If $A, B \subseteq \mathbf{S}^2$ both have inner points, then $A \sim B$ (via rotations).
    (e) $\mathbf{B}^3$, the unit ball of $\mathbf{R}^3$ is paradoxical (via congruences).
    (f) **(Banach–Tarski paradox)** If $A, B \subseteq \mathbf{R}^3$ are bounded sets with inner points, then $A \sim B$ (via congruences).
18. If $A, B \subseteq \mathbf{R}^2$ are bounded sets with inner points and $\epsilon > 0$, then A is equidecomposable into B via ϵ-contractions, that is, there are partitions $A = A_1 \cup \cdots \cup A_n$ and $B = B_1 \cup \cdots \cup B_n$ and bijections $f_i : A_i \to B_i$ such that for x, $y \in A_i$ one has $d(f_i(x), f_i(y)) \leq \epsilon d(x, y)$ ($d(x, y)$ is the distance of x and y).

20

Stationary sets in ω_1

This chapter deals with two basic notions of infinite combinatorics, namely with the club (closed and unbounded) sets and with stationary sets in ω_1.

First some definitions. We say that a sequence $\{\alpha_n\}_{n=0}^{\infty}$ of ordinals from ω_1 *converges* to α if $\alpha_n \le \alpha$ for all n and for every $\beta < \alpha$ there is an N such that $\alpha_n > \beta$ for $n > N$. Note that then necessarily $\alpha < \omega_1$. It is easy to see that this is the same as convergence in the *order topology* on ω_1 (generated by sets of the form $\{\alpha : \alpha < \beta\}$ and $\{\alpha : \alpha > \beta\}$). A subset $A \subseteq \omega_1$ is called

- *closed* if $\alpha_n \to \alpha$ and each α_n is in A then $\alpha \in A$,
- *unbounded* if given any $\beta < \omega_1$ there is a $\beta < \alpha \in A$,
- *club set* if it is closed and unbounded.

A set is closed precisely if it is closed in the order topology, and a closed set is unbounded precisely if it is not compact in this topology.

A set $S \subseteq \omega_1$ is *stationary* if it has a nonempty intersection with every club set. Otherwise, it is a *nonstationary* set.

Closed sets play the role of "full measure" sets among subsets of ω_1, while stationary sets play the role of "sets of positive measure". Club sets are very "thick", the intersection of any countable family of club sets is still a club set, while stationary sets are still sufficiently "thick" in the sense that if some property holds for the elements of a stationary set then we consider it to hold for many elements (like elements in a set of positive measure). The analogy with measure theory stops here: there is an uncountable family of disjoint stationary sets.

A function $f : A \to \omega_1$ is a *regressive function* if $f(x) < x$ holds for every $x \in A \setminus \{0\}$. The basic connection between stationary sets and regressive functions is Fodor's theorem (Problem 9): if f is regressive function on a stationary set, then it is constant on a stationary subset.

1. When is a cofinite subset of ω_1 a club?
2. Assume that $A \subseteq B \subseteq \omega_1$.

(a) Does the stationarity of A imply the stationarity of B?

(b) Does the clubness of A imply the clubness of B?

(c) Does the nonstationarity of B imply the nonstationarity of A?

3. The intersection of countably many club sets is a club set again.
4. The union of countably many nonstationary sets is nonstationary.
5. If S is stationary, C is closed, unbounded, then $S \cap C$ is stationary.
6. If C_α are club sets for $\alpha < \omega_1$, then their *diagonal intersection*
$$\bigtriangledown\{C_\alpha : \alpha < \omega_1\} = \{\alpha < \omega_1 : \beta < \alpha \longrightarrow \alpha \in C_\beta\}$$
is also a club set.
7. If $f : [\omega_1]^{<\omega} \to \omega_1$ is a function, then the set
$$C(f) = \{\alpha < \omega_1 : \text{if } \beta_1, \ldots, \beta_n < \alpha \text{ then } f(\beta_1, \ldots, \beta_n) < \alpha\}$$
is a closed, unbounded set.
8. If $C \subseteq \omega_1$ is a club set, then there is a function $f : [\omega_1]^{<\omega} \to \omega_1$ such that $C(f) \setminus \{0\} \subseteq C$.
9. A set is closed, unbounded if and only if it is the range of a strictly increasing, continuous $\omega_1 \to \omega_1$ function.
10. If $f, g : \omega_1 \to \omega_1$ are strictly increasing continuous functions, then for club many $\alpha < \omega_1$, $f(\alpha) = g(\alpha)$ holds.
11. The set of countable epsilon numbers, i.e.,
$$\{\epsilon < \omega_1 : \epsilon = \omega^\epsilon\}$$
is a club set.
12. Assume that $f : \omega_1 \to \omega_1$ is a regressive function. Then some value is assumed uncountably many times.
13. Assume $S \subseteq \omega_1$ is a stationary set and $f : S \to \omega_1$ is a regressive function. Then some value is assumed uncountably many times.
14. If $N \subseteq \omega_1$ is nonstationary, then there is a regressive function $f : N \to \omega_1$ that assumes every value countably many times.
15. If $N \subseteq \omega_1$ is nonstationary, then there is a regressive function $f : N \to \omega_1$ that assumes every value at most twice.
16. (Fodor's theorem) If $S \subseteq \omega_1$ is a stationary set and $f : S \to \omega_1$ is a regressive function, then some value is assumed on a stationary set.
17. If $S \subseteq \omega_1$ is a stationary set and $F(\alpha) \subseteq \alpha$ is a finite set for $\alpha \in S$, then for some finite set s the set $\{\alpha \in S : F(\alpha) = s\}$ is stationary.
18. A slot machine returns $\aleph_0$ quarters when a quarter is inserted. Still, no matter what strategy she follows, if somebody starts with a single coin (and plays through a transfinite series of steps), after countably many steps she loses all her money.

19. There are two disjoint stationary sets.
20. If $f : \omega_1 \to \mathbf{R}$ is monotonic, then it is constant from a point onward.
21. If $f : \omega_1 \to \mathbf{R}$ is continuous, then it is constant from a point onward.
22. ω_1, endowed with the order topology, is not metrizable.
23. (a) If $\alpha < \omega_1$, then $\alpha \times \omega_1$ is a normal topological space.
    (b) $\omega_1 \times \omega_1$ is a normal topological space.
24. $(\omega_1 + 1) \times \omega_1$ is not a normal topological space.
25. Assume that we are given $\aleph_1$ disjoint nonstationary sets. Prove that there are $\aleph_1$ of them with nonstationary union.
26. Two players, I and II, play by alternatively selecting elements of a decreasing sequence $A_0 \supseteq A_1 \supseteq \cdots$ of stationary subsets of ω_1. Player II wins if and only if $\bigcap\{A_i : i < \omega\}$ has at most one element. Show that II has a winning strategy.
27. Assume that there are $\aleph_2$ stationary sets with pairwise nonstationary intersection. Show that there are $\aleph_2$ stationary sets with pairwise countable intersection.
28. (CH) Assume that we are given $\aleph_2$ closed, unbounded subsets of ω_1. Prove that the intersection of some $\aleph_1$ of them is a closed, unbounded set.
29. If there are $\aleph_2$ functions from ω_1 into ω such that any two differ on a closed, unbounded set then there are $\aleph_2$ such functions such that any two are eventually different.
30. There exists a regressive function $f : \omega_1 \to \omega_1$ such that for every limit ordinal $\alpha < \omega_1$ there is an increasing sequence α_n, $n < \omega$, converging to α with $f(\alpha_{n+1}) = \alpha_n$ for all n.

21

Stationary sets in larger cardinals

Now we consider the analogues of questions discussed in the preceding chapter but for larger cardinals. In general, the discussion will be given in a regular cardinal (instead of ω_1), but we shall also indicate how everything works in any ordinal of cofinality larger then ω. We shall copy the treatment for ω_1 only to the extent that is necessary; several new features will emerge in the problems. For example, Problem 20 proves the deep result of Solovay: any stationary set in κ can be decomposed into κ disjoint stationary sets.

In this chapter, unless otherwise stated, κ is always an uncountable regular cardinal.

We say that a transfinite sequence $\{\alpha_\tau : \tau < \mu\}$ of elements of κ *converges* to some $\alpha < \kappa$ if $\alpha_\tau \leq \alpha$ for all $\tau < \mu$ and for every $\beta < \alpha$ there is a $\nu < \mu$ such that $\alpha_\tau > \beta$ whenever $\tau > \nu$. A set $C \subseteq \kappa$ is called

- *closed* if whenever a transfinite sequence $\{\alpha_\tau : \tau < \mu\}$ of elements of C converges to some $\alpha < \kappa$ then $\alpha \in C$,
- *unbounded* if for any $\beta < \kappa$ there is an $\alpha \in C$ with $\beta < \alpha < \kappa$,
- a *club set* if it is closed and unbounded.

It is true again that a set $C \subset \kappa$ is closed if and only if it is closed in the order topology on κ, and a closed set is unbounded precisely if it is not compact in this topology.

If something holds for every element of a club set, we sometimes use the lingo *almost everywhere*, or *for almost every*, in short, a.e.

A set $S \subseteq \kappa$ is *stationary* if it has a nonempty intersection with every closed, unbounded set. Otherwise, it is *nonstationary*. For $A \subseteq \kappa$ a function $f : A \to \kappa$ is *regressive* if $f(x) < x$ holds for every $x \in A$, $x \neq 0$.

1. The intersection of less than κ many club sets is a club set again.
2. If $C \subseteq \kappa$ is a club set, then for a.e. α the intersection $C \cap \alpha$ is a cofinal set in α of order type α

3. If $f : [\kappa]^{<\omega} \to [\kappa]^{<\kappa}$ is a function then the set

$$C(f) = \{\alpha < \kappa : \text{if } \beta_1, \dots, \beta_n < \alpha \text{ then } f(\beta_1, \dots, \beta_n) \subseteq \alpha\}$$

is a closed, unbounded set. In the other direction, if $C \subseteq \kappa$ is a club set then there is a function $f : \kappa \to \kappa$ such that $C(f) \setminus \{0\} \subseteq C$.

4. Let $\mathcal{A}$ be an algebraic structure on the set A of cardinality κ, with fewer than κ finitary operations, and let $\{a_\gamma : \gamma < \kappa\}$ be an enumeration of A. Then for almost all $\alpha < \kappa$ the set $\{a_\gamma : \gamma < \alpha\}$ is a substructure of $\mathcal{A}$.

5. If C_α are club sets for $\alpha < \kappa$ then their *diagonal intersection*

$$\triangledown\{C_\alpha : \alpha < \kappa\} = \{\alpha < \kappa : \beta < \alpha \longrightarrow \alpha \in C_\beta\}$$

is also a club set.

6. The union of less than κ many nonstationary sets is nonstationary.

7. If S is stationary, C is closed, unbounded, then $S \cap C$ is stationary.

8. If $\mu < \kappa$ is regular, then $S = \big\{\alpha < \kappa : \mathrm{cf}\,(\alpha) = \mu\big\}$ is stationary. Is it a club set? What if the condition $\mathrm{cf}\,(\alpha) = \mu$ is relaxed to $\mathrm{cf}\,(\alpha) \le \mu$ or to $\mathrm{cf}\,(\alpha) \ge \mu$?

9. (Fodor's theorem, pressing down lemma) If $S \subseteq \kappa$ is a stationary set and $f : S \to \kappa$ is a regressive function, then some value is assumed on a stationary set.

10. Assume that $\mu < \kappa$ is such that if $\tau < \kappa$ then $\tau^\mu < \kappa$ (for example, if $\kappa = (2^\mu)^+$). Let $S \subseteq \big\{\kappa : \mathrm{cf}\,(\alpha) = \mu^+\big\}$ be a stationary set and $f(\alpha) \in [\alpha]^{\le\mu}$ for $\alpha \in S$. Then f is constant on a stationary set.

11. If A_α $(\alpha < \kappa)$ are nonstationary, then so is $\bigcup\big\{A_\alpha \setminus (\alpha + 1) : \alpha < \kappa\big\}$.

12. Let $\{A_\alpha : \alpha < \kappa\}$ be disjoint nonstationary sets in κ. Then $A = \bigcup\{A_\alpha : \alpha < \kappa\}$ is stationary if and only if $B = \{\min(A_\alpha) : \alpha < \kappa\}$ is.

13. Out of κ disjoint nonstationary sets the union of some κ is nonstationary.

14. If A, B are subsets of κ define $A \le B$ if $A \setminus B$ is nonstationary. Set $A < B$ if $A \le B$ but $B \le A$ is not true. (This gives a Boolean algebra if we identify two sets when their symmetric difference is nonstationary.) Prove that every family of at most κ sets has a least upper bound.

In Problems 15–19 we extend these notions to subsets of limit ordinals. If α is a limit ordinal, $X \subseteq \alpha$ is *unbounded* if it contains arbitrarily large elements below α. It is *closed* if it contains its limit points smaller than α. For $\mathrm{cf}\,(\alpha) > \omega$, $S \subseteq \alpha$ is *stationary* if it intersects every closed, unbounded subset of α. If $\mathrm{cf}\,(\alpha) = \omega$, then we declare α (and all subsets thereof) nonstationary.

15. (a) Every stationary set is unbounded.
    (b) $\mathrm{cf}\,(\alpha)$ is the minimal cardinality/ordinal of the closed, unbounded sets in α.

(c) If $\mathrm{cf}(\alpha) = \omega$ then there are two disjoint closed, unbounded sets in α.

(d) If $\mathrm{cf}(\alpha) > \omega$ then the intersection of less than $\mathrm{cf}(\alpha)$ closed, unbounded sets is a closed, unbounded set.

(e) If $\mathrm{cf}(\alpha) = \omega$ then $X \subseteq \alpha$ intersects every closed, unbounded set if and only if X includes some end segment of α.

16. Assume that $\kappa = \mathrm{cf}(\alpha) > \omega$. Let $C \subseteq \alpha$ be a closed, unbounded set of order type κ with increasing enumeration $C = \{c_\gamma : \gamma < \kappa\}$.

 (a) If D is closed, unbounded in κ then $\{c_\gamma : \gamma \in D\}$ is closed, unbounded in α.

 (b) If D is closed, unbounded in α then $\{\gamma : c_\gamma \in D\}$ is closed, unbounded in κ.

 (c) $X \subseteq \alpha$ is stationary if and only if $\{\gamma : c_\gamma \in X\}$ is stationary in κ.

17. (a) If $\mathrm{cf}(\alpha) < \alpha$, then there exists a regressive $f : \alpha \setminus \{0\} \to \alpha$ such that $f^{-1}(\xi)$ is bounded for every $\xi < \alpha$.

 (b) If $S \subseteq \alpha$ is stationary, $f : S \to \alpha$ is regressive, then there is a stationary $S' \subseteq S$ such that f is bounded on S'.

18. If $C \subseteq \kappa$ is closed, unbounded, then for a.e. $\alpha < \kappa$ the set $C \cap \alpha$ is a club set in α.

19. If S, $T \subseteq \kappa$ are stationary sets, define $S < T$ if for almost every $\alpha \in T$, $S \cap \alpha$ is stationary in α. Then

 (a) $S < S$ never holds;

 (b) $<$ is transitive;

 (c) $<$ is well founded.

20. (Solovay's theorem) If $S \subseteq \kappa$ is a stationary set, then it is the union of κ disjoint stationary sets. Prove this theorem through the following steps. Assume that S is a counterexample.

 (a) Every stationary subset of S is also a counterexample.

 (b) If $f : S \to \kappa$ is regressive, then it is essentially bounded, i.e., there are an ordinal $\gamma < \kappa$ and a closed, unbounded set $C \subseteq \kappa$ such that $f(\alpha) < \gamma$ holds for $\alpha \in C \cap S$.

 (c) Almost every element of S is a regular cardinal.

 (d) There is a closed, unbounded set $D \subseteq \kappa$ such that if $\alpha \in D \cap S$ then α is an uncountable, regular cardinal and $S \cap \alpha$ is stationary in α.

 (e) Conclude by showing that no set D as in (d) exists.

21. There is a function $f : \kappa \to \kappa$ such that if $X \subseteq \kappa$ has a club subset, then $f[X] = \kappa$.

22. If $S \subseteq \kappa$ is stationary, then there is a family $\mathcal{F}$ of 2^κ stationary subsets of S such that $A \setminus B$, $B \setminus A$ are stationary if A, B are distinct elements of $\mathcal{F}$.

23. Assume that κ, μ are regular cardinals, $\kappa > \mu^+$, $\mu > \omega$. There exists a family $\{C_\alpha : \alpha < \kappa, \mathrm{cf}\,(\alpha) = \mu\}$ such that C_α is closed, unbounded in α and for every closed unbounded subset $E \subseteq \kappa$, there is some $C_\alpha \subseteq E$.

24. Assume that $\kappa \geq \omega_2$ is a regular cardinal. Then there exists a family $\{C_\alpha : \alpha < \kappa, \mathrm{cf}\,(\alpha) = \omega\}$ such that C_α is a cofinal subset of α of type ω and for every closed, unbounded subset E of κ, there is some $C_\alpha \subseteq E$.

22

Canonical functions

In this chapter for a regular uncountable cardinal κ we introduce a family of κ^+ functions that possess various canonicity properties. In some sense they are the first κ^+ functions from κ into the ordinals, this makes it possible to use them for various diverse results in set theory.

For $\kappa > \omega$ regular we construct the *canonical functions* $h_\alpha : \kappa \to \kappa$ for $\alpha < \kappa^+$ as follows. $h_0(\gamma) = 0$ for $\gamma < \kappa$. $h_{\alpha+1}(\gamma) = h_\alpha(\gamma) + 1$ $(\gamma < \kappa)$. If $\alpha < \kappa^+$ is limit with $\mu = \mathrm{cf}\,(\alpha) < \kappa$ then fix a sequence $\{\alpha_\tau : \tau < \mu\}$ converging to α and set

$$h_\alpha(\gamma) = \sup\{h_{\alpha_\tau}(\gamma) : \tau < \mu\}$$

for $\gamma < \kappa$.

Finally, if $\mathrm{cf}\,(\alpha) = \kappa$ and $\{\alpha_\tau : \tau < \kappa\}$ converges to α, then let

$$h_\alpha(\gamma) = \sup\{h_{\alpha_\tau}(\gamma) : \tau < \gamma\}.$$

Notice that the values of the functions $h_\alpha(\gamma)$ depend on the above sequences converging to α, as well.

1. Describe h_α for $\alpha \leq \kappa \cdot 2$.
2. If $\beta < \alpha < \kappa^+$, then $h_\beta(\gamma) < h_\alpha(\gamma)$ holds for a.e. γ.
3. If $\{f_\alpha : \alpha < \kappa^+\}$ is a system of $\kappa \to \kappa$ functions such that for $\beta < \alpha < \kappa^+$, $f_\beta(\gamma) < f_\alpha(\gamma)$ holds for a.e. γ, then for every $\alpha < \kappa^+$, $f_\alpha(\gamma) \geq h_\alpha(\gamma)$ holds almost everywhere.
4. If $f(\gamma) < h_\alpha(\gamma)$ holds on a stationary set for some function $f : \kappa \to \kappa$, then there is a $\beta < \alpha$ such that $f(\gamma) \leq h_\beta(\gamma)$ holds for stationary many γ.
5. If $f(\gamma) < h_\alpha(\gamma)$ holds on a stationary set, then $f(\gamma) = h_\beta(\gamma)$ holds on a stationary set for some $\beta < \alpha$.

6. Assume that $\{f_\alpha : \alpha < \kappa^+\}$ is a family of $\kappa \to \kappa$ functions that
   a) $f_0(\gamma) = 0$ a.e.;
   b) $f_\beta(\gamma) < f_\alpha(\gamma)$ for a.e. γ $(\beta < \alpha < \kappa^+)$;
   c) if $f(\gamma) < f_\alpha(\gamma)$ for stationarily many γ then $f(\gamma) \le f_\beta(\gamma)$ for stationarily many γ, for some $\beta < \alpha$.

   Then $f_\alpha(\gamma) = h_\alpha(\gamma)$ holds for a.e. γ.
7. For every $\alpha < \kappa^+$, $h_\alpha(\gamma) < |\gamma|^+$ holds for a.e. γ. (Here $|\gamma|^+$ is the cardinal successor of $|\gamma|$.)

   In Problems 8–13 we describe an alternative construction of canonical functions. Fix, for every $0 < \alpha < \kappa^+$, a surjection $g_\alpha : \kappa \to \alpha$. Let $f_\alpha(\gamma)$ be the order type of the set $g_\alpha[\gamma]$ (a subset of α). For $\alpha = 0$ set $f_0(\gamma) = 0$ $(\gamma < \kappa)$.
8. If $g_\alpha, g'_\alpha : \kappa \to \alpha$ are surjections, then the above derived functions f_α, f'_α agree almost everywhere.
9. If $0 < \beta < \alpha < \kappa^+$ then for a.e. $\gamma < \kappa$, $g_\beta[\gamma] = g_\alpha[\gamma] \cap \beta$ holds.
10. If $\beta < \alpha$ then $f_\beta(\gamma) < f_\alpha(\gamma)$ holds a.e.
11. If $f(\gamma) < f_\alpha(\gamma)$ holds on a stationary set for some function $f : \kappa \to \kappa$, then there is a $\beta < \alpha$ such that $f(\gamma) = f_\beta(\gamma)$ holds for stationary many γ.
12. $f_\alpha(\gamma) = h_\alpha(\gamma)$ almost everywhere.
13. $f_\alpha(\gamma) < |\gamma|^+$ holds for every γ.

23

Infinite graphs

It frequently occurs in mathematics that a relation is visualized by drawing a graph. If the underlying set is infinite, then we get an infinite graph. Formally, a *graph* is a pair $G = (V, X)$ where V is a set (the *vertex set*) and $X \subseteq [V]^2$, i.e., it is a subset of the two element sets of V (the *edge set*). Sometimes we just speak of X, therefore identifying the graph with its edge set. We say that x and y are joined if $\{x, y\} \in X$. The *complement* $(V, \overline{X})$ of a graph (V, X) is $\big(V, [V]^2 \setminus X\big)$, that is, it has the same set of vertices and two vertices are joined in $(V, \overline{X})$ if and only if they are not joined in (V, X). The *degree* of a vertex v is the number of edges emanating from v.

We call (V', X') a *subgraph* of (V, X) if $V' \subseteq V$ and $X' \subseteq X$. It is an *induced subgraph* if

$$X' = \{\{x, y\} \,:\, x, y \in V',\ \{x, y\} \in X\},$$

i.e., if two elements in V' are connected precisely if they are connected in (V, X).

A subset $A \subseteq V$ is *independent* if it contains no edges: $X \cap [A]^2 = \emptyset$.

A subset $X' \subseteq X$ is a *matching* if every vertex is an endpoint of precisely one edge in X'.

A *path* in a graph is a (finite, one-way or two-way infinite) sequence $\{\dots, v_n, v_{n+1}, \dots\}$ of consecutively joined points (i.e., $\{v_n, v_{n+1}\} \in V$ for all n). A *circuit* is such a finite sequence with the same starting and ending point.

A *forest* is a graph with no circuits.

If (V, X), (W, Y) are graphs, a *topological* (V, X) is given by an injection $f : V \to W$ and a function g that sends every edge $e = \{x, y\}$ in X into a path in (W, Y) connecting $f(x)$ and $f(y)$, the paths $\{g(e) : e \in X\}$ being vertex disjoint except at their extremities.

A *good coloring* or sometimes a *coloring* of a graph (V, X) with a color set C is a mapping $f : V \to C$ such that $f(x) \neq f(y)$ for $\{x, y\} \in X$ (i.e., the vertices are colored in such a way that vertices that are joined get different colors). The *chromatic number* $\mathrm{Chr}(X)$ of a graph (V, X) is the smallest cardinal κ

for which the graph can be colored by κ colors. Therefore, a graph (V, X) has a good coloring with κ colors if and only if $\mathrm{Chr}(X) \le \kappa$.

More generally, if $\mathcal{F}$ is a set system over a ground set S, then a good coloring of $\mathcal{F}$ is a coloring of S in such a way that for no $F \in \mathcal{F}$ get all points of F the same color (there is no monochromatic F).

One would expect that the chromatic number of a graph is large only if the graph includes a large complete subgraph. Problem 24 shows it otherwise: the chromatic number can be arbitrarily large even if the graph does not contain three pairwise connected points. Still, a large chromatic number does imply the existence of certain types of subgraphs, e.g., every uncountably chromatic graph must include an infinite path, all circuits of even length and all odd circuits of sufficiently large length (Problems 29, 30).

Let K_κ denote the complete graph (i.e., any two different points are joined) on a vertex set of cardinality κ. A graph (V, X) is called *bipartite* if the vertex set can be decomposed as $V = V_1 \cup V_2$ such that all edges go between V_1 and V_2 (in this case V_1 and V_2 are called the *bipartition classes*). $K_{\kappa,\lambda}$ denotes the complete bipartite graph with bipartition classes of cardinality κ and λ, respectively.

We also make the following definition. Given a class $\mathcal{F}$ of graphs, a *universal graph* in $\mathcal{F}$ is a graph $X_0 \in \mathcal{F}$ such that every graph $X \in \mathcal{F}$ is (isomorphic to) a subgraph of X_0. If $X_0 \in \mathcal{F}$ is such that every $X \in \mathcal{F}$ appears as an induced subgraph in X_0 then it is a *strongly universal graph.*

Many problems from this section are used elsewhere in the book. Problem 8 is particularly useful if one wants to deduce a conclusion for infinite sets provided one knows it for all finite subsets. It states the compactness property for graph coloring.

There are some more problems on infinite graphs in Chapter 24.

1. An infinite graph or its complement includes an infinite complete subgraph.
2. The pairs of ω are colored with $k < \omega$ colors. Then there is a partition of ω into k parts such that the ith part is a finite or one-way infinite path in color i.
3. If X is a graph on $\kappa \ge \omega$ vertices then either X or its complement includes a topological K_κ.
4. If the degree of every vertex in a graph is at most $n < \omega$, then the graph can be colored with $n + 1$ colors.
5. If the degree of every vertex in a graph is at most $\kappa \ge \omega$, then the graph can be colored with κ colors.
6. If the vertex set of a graph has a well-ordering in which every vertex is joined to fewer than κ smaller vertices, then the graph is κ-colorable.
7. Let $\kappa \ge \omega$. If the vertex set of a graph has an ordering in which every vertex is joined to fewer than κ smaller vertices, then the graph is κ-colorable.

8. (de Bruijn–Erdős theorem) If, for some $n < \omega$, every finite subgraph of a graph X is n-colorable, then so is X.
9. A graph is finitely chromatic if and only if every countable subgraph is finitely chromatic.
10. Let X be a graph on some well-ordered set. Then X is finitely chromatic if and only if every subset of order type ω is finitely chromatic.
11. Construct a graph X on ω_1^2 such that every subgraph of order type ω_1 is countably chromatic yet X is uncountably chromatic.
12. Given the graphs (V, X) and (W, Y) form their product $X \times Y$ as follows. The vertex set is $V \times W$, and $\langle x, y\rangle$ is joined to $\langle x', y'\rangle$ if and only if $\{x, x'\} \in X$ and $\{y, y'\} \in Y$. If the chromatic number of (V, X) is the finite k and the chromatic number of (W, Y) is infinite, then the chromatic number of $(V \times W, X \times Y)$ is k.
13. **(a)** If the vertices of a graph (V, X) are partitioned as $\{V_i : i \in I\}$ and X_i is the subgraph induced by V_i then $\mathrm{Chr}(X) \leq \sum \mathrm{Chr}(X_i)$.
    (b) If the edges of a graph (V, X) are decomposed into the subgraphs $\{X_i : i \in I\}$, then $\mathrm{Chr}(X) \leq \prod \mathrm{Chr}(X_i)$.
14. Assume that X is a bipartite graph with bipartition classes A and B and for every $x \in A$ the set $\Gamma(x)$ of the neighbors of x is finite. Then there is a matching of A into B in X if and only if for any finite subset $\{x_1, \ldots, x_k\}$ of A the set $\Gamma(x_1) \cup \cdots \cup \Gamma(x_k)$ has at least k elements.
15. Assume that $p, q \geq 1$ are natural numbers and X is a graph as in the preceding problem. There is a function $f : E \to \{0, 1, \ldots, p\}$ on the edge set E such that
$$\sum_{e:\ x \in e} f(e) = p \quad (x \in A),$$
$$\sum_{e:\ y \in e} f(e) \leq q \quad (y \in B)$$
    if and only if the following condition holds: for any k-element finite subset $\{x_1, \ldots, x_k\}$ of A, the set $\Gamma(x_1) \cup \cdots \cup \Gamma(x_k)$ has at least pk/q elements.
16. A graph X is planar if and only if
    (a) X includes no topological K_5 or $K_{3,3}$;
    (b) X has only countably many vertices with degree at least 3;
    (c) X has at most continuum many vertices.
    (A graph is planar if it can be drawn in the plane where the vertices are represented by distinct points, the edges by noncrossing Jordan curves.)
17. A graph is spatial (it can be represented as in the previous problem but in the 3-space) if and only if it has at most continuum many vertices.
18. For an infinite cardinal κ the complete graph on κ^+ vertices is the union of κ forests but the complete graph on $(\kappa^+)^+$ vertices is not.

19. The edge set of a graph can be decomposed into countably many bipartite graphs if and only if the chromatic number of the graph is at most $\mathbf{c}$.
20. There exists a strongly universal countable graph.
21. There is no universal countable K_ω-free graph.
22. There is no universal countable locally finite graph (that is, in which every degree is finite).
23. There is no universal $K_{\aleph_1}$-free graph of cardinality $\mathbf{c}$.
24. For every infinite cardinal κ there is a κ-chromatic, triangle-free graph.
25. Define a graph (ω_1^3, X) on the set ω_1^3 in such a way that (α, β, γ) and $(\alpha', \beta', \gamma')$ are connected if and only if $\alpha < \beta < \alpha' < \gamma < \beta' < \gamma'$ or $\alpha' < \beta' < \alpha < \gamma' < \beta < \gamma$. Then $A \subseteq \omega_1^3$ spans a countable chromatic subgraph if and only if its order type (in the lexicographic ordering) is $< \omega_1^3$.
26. If (V, X) is a graph on the ordered set $(V, <)$ we define the following graph (V', X'). The vertex set is $V' = X$. We create the edges X' as follows. The edge $\{x, y\}$ with $x < y$ is joined to the edge $\{z, t\}$ with $z < t$ if and only if either $y = z$ or $x = t$ holds.
    (a) $\mathrm{Chr}(X') \leq \kappa$ if and only if $\mathrm{Chr}(X) \leq 2^\kappa$.
    (b) If (V, X) does not include odd circuits of length $3, 5, \ldots, 2n-1$ then (V', X') does not include odd circuits of length $3, 5, \ldots, 2n+1$.
    (c) For every natural number n and cardinal κ there is a graph with chromatic number greater than κ, and not including odd circuits of length $3, \ldots, 2n+1$.
27. There is an uncountably chromatic graph all whose subgraphs of cardinality at most $\mathbf{c}$ are countably chromatic.
28. If $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$, $2^{\aleph_2} = \aleph_3$, then there is a graph with chromatic number $\aleph_2$ with no induced subgraph of chromatic number $\aleph_1$.
29. Every uncountably chromatic graph includes $K_{n,\aleph_1}$ for all finite n, the complete bipartite graph with bipartition classes of size n, $\aleph_1$, respectively. In particular, it includes circuits of length $4, 6, \ldots$.
30. Every uncountably chromatic graph includes every sufficiently long odd circuit.
31. Every uncountably chromatic graph includes an infinite path.
32. Assume that X is an $\aleph_1$-chromatic graph on the vertex set V. Then V can be decomposed into the union of $\aleph_1$ disjoint subsets each spanning a subgraph of chromatic number $\aleph_1$.
33. Assume that X is an uncountably chromatic graph on the vertex set V. Then V can be decomposed into the union of two (or even $\aleph_0$) disjoint subsets each spanning a subgraph of uncountable chromatic number.

34. The following graph (V, X) is uncountably chromatic. The vertex set is

$$V = \{f : \alpha \to \omega \text{ injective}, \ \alpha < \omega_1\},$$

and two functions are joined if one of them extends the other.

35. If the set system $\mathcal{H}$ consists of finite sets with at least two elements and $|A \cap B| \neq 1$ holds for $A, B \in \mathcal{H}$ then $\mathcal{H}$ is 2-chromatic.

36. Assume that the set system $\mathcal{H}$ consists of countably infinite sets such that $|A \cap B| \neq 1$ holds for $A, B \in \mathcal{H}$. Then $\mathcal{H}$ is ω-chromatic but not necessarily finitely chromatic.

37. Assume that $\mathcal{H}$ is a system of $\aleph_1$ three-element sets no two intersecting in two elements. Then $\mathcal{H}$ is ω-colorable.

38. Consider the graph $G_{n,\alpha}$ with vertex set S^n (the unit sphere of $\mathbf{R}^{n+1}$) and two points are connected if their distance is bigger than α. Then $\mathrm{Chr}(G_{n,\alpha}) \geq n+2$ for all $\alpha < 2$, and $\mathrm{Chr}(G_{n,\alpha}) = n+2$ for $\alpha < 2$ sufficiently close to 2.

39. For $\alpha < 1/2$ let the vertices of the graph G be those measurable subsets $E \subset [0,1]$ which have measure α, and let two such subsets be connected if they are disjoint. Then the chromatic number of G is $\aleph_0$.

24

Partition relations

In partition calculus transfinite generalizations are obtained for the (infinite) Ramsey theorem: if $2 \leq k, r < \omega$ and the r-tuples of some infinite set are colored with k colors, then there is an infinite subset, all whose r-element subsets get the same color (Problem 2).

If X is a set and $f : [X]^r \to I$ is a coloring (partition) of its r-tuples, then $Y \subseteq X$ is called *homogeneous* or *monochromatic* with respect to f if there is an $i \in I$ such that $f\big(\{y_1, \dots, y_r\}\big) = i$ holds for all $\{y_1, \dots, y_r\} \in [Y]^r$. We usually contract the notation $f\big(\{y_1, \dots, y_r\}\big)$ to $f(y_1, \dots, y_r)$. The *partition relation* $\kappa \to (\lambda)^r_\rho$ expresses that if the r-tuples of a set of cardinality κ are colored with ρ colors then there is a monochromatic subset of cardinality λ (Rado's notation). If this statement fails, then we write $\kappa \not\to (\lambda)^r_\rho$. With this notation the infinite Ramsey theorem reads as $\omega \to (\omega)^r_k$ for r, k finite.

This branch of combinatorial set theory investigates how large homogeneous set can be guaranteed for a given coloring. The most important result is the Erdős–Rado theorem stating that $\exp_r(\kappa)^+ \to (\kappa^+)^{r+1}_\kappa$ holds when κ is an infinite cardinal and $1 \leq r < \omega$ (Problem 25). Here $\exp_r$ denotes the r-fold iterated exponential function, i.e., $\exp_0(\kappa) = \kappa, \exp_1(\kappa) = 2^\kappa, \exp_2(\kappa) = 2^{2^\kappa}, \dots$, etc. These values are sharp.

In this chapter we consider this basic result and various generalizations and variants. We present applications to point set topology, some problems of this chapter will also be used elsewhere in the book.

A *tournament* is a directed graph in which between any two vertices there is an edge in one and only one direction.

1. If $2 \leq k < \omega$, then $\omega \to (\omega)^2_k$; i.e., if we color the edges of an infinite complete graph with finitely many colors, then there is an infinite monochromatic subgraph.
2. (Ramsey's theorem) If $1 \leq r < \omega$, $2 \leq k < \omega$, then $\omega \to (\omega)^r_k$. That is, if we color the r-tuples of an infinite set by finitely many colors, then there is an infinite monochromatic set.

3. Every infinite partially ordered set includes either an infinite chain or an infinite antichain (i.e., either an infinite ordered set or an infinite set of pairwise incomparable elements).
4. Every infinite ordered set includes either an infinite increasing or infinite decreasing sequence.
5. If X is an infinite planar set, then there is an infinite convex subset $Y \subseteq X$, that is, no point in Y lies in the interior of a triangle formed by three other elements of Y.
6. Every infinite tournament includes an infinite transitive subtournament.
7. If X is an infinite directed graph with at most one edge between any two vertices, then either there is an infinite independent set, or there is an infinite, transitively directed subgraph.
8. The edges of a complete directed graph of cardinality continuum can be colored by ω colors so that there are no connected edges of the same color (two edges are connected if the endpoint of one is the starting point of the other).
9. If $f : [\omega]^2 \to \omega$ is a coloring such that for every $i < \omega$ there is a finite set A_i with $f(i,j) \in A_i$ $(i < j < \omega)$, then there is an infinite set $A \subseteq \omega$ which is *endhomogeneous*, that is, in A, $f(i,j)$ only depends on i.
10. If f is a coloring of $[\omega]^2$ with no restriction on the colors, then there is an infinite $H \subseteq$ such that either
    (a) H is homogeneous for f, or
    (b) if $x < y$, $x' < y'$ are from H, then $f(x,y) = f(x',y')$ if and only if $x = x'$, or
    (c) if $x < y$, $x' < y'$ are from H, then $f(x,y) = f(x',y')$ if and only if $y = y'$, or
    (d) the values $\{f(x,y) : \{x,y\} \in [H]^2\}$ are different.
11. Let $f : \omega \to \omega$ be a function with $f(r) \to \infty$ $(r \to \infty)$. Assume that for every $1 \leq r < \infty$ H_r colors $[\omega]^r$ with finitely many colors. Then there is an infinite $X \subseteq \omega$ such that H_r on $[X]^r$ assumes at most $f(r)$ values. The statement fails if $f(r) \not\to \infty$.
12. There is a constant c with the following property. If $f : [\omega]^2 \to 3$ is a coloring, then there is an infinite sequence $a_0 < a_1 < \cdots$ with $a_n < c^n$ for infinitely many n such that f assumes only two values on this sequence.
13. If κ is an uncountable cardinal, then $\kappa \to (\kappa, \aleph_0)^2$. That is, if $f : [\kappa]^2 \to \{0,1\}$, then either there is a set of cardinality κ monochromatic in color 0 or else there is an infinite set monochromatic in color 1. Show this when κ is
    (a) regular,
    (b) singular.

14. For cardinals $\lambda \geq 2$, $\kappa \geq \omega$ order the $\kappa \to \lambda$ functions lexicographically. There is no decreasing sequence of length κ^+. There is no increasing sequence of length $\max(\kappa, \lambda)^+$.
15. If $\langle A, < \rangle$ is an ordered set, $|A| \leq 2^\kappa$, then there is some $f : [A]^2 \to \kappa$ with no $x < y < z$ such that $f(x, y) = f(y, z)$.
16. There is an uncountable tournament with no uncountable transitive subtournament.
17. (Todorcevic) There is a function $F : [\omega_1]^2 \to \omega_1$ such that for every uncountable $X \subseteq \omega_1$ F assumes every element of ω_1 on $[X]^2$.
18. If $\kappa \geq \aleph_0$ is a cardinal, $r \geq 1$ a natural number and f is a coloring of the $(r+1)$-tuples of $(2^\kappa)^+$ with κ colors, then there is a set $X \subseteq (2^\kappa)^+$, $|X| = \kappa^+$ on which f is endhomogeneous, that is, for $x_1 < \cdots < x_r < y < y'$ from X, $f(x_1, \ldots, x_r, y) = f(x_1, \ldots, x_r, y')$ holds.
19. If $\kappa \geq \aleph_0$ is a cardinal, then $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$. That is, if the pairs of $(2^\kappa)^+$ are colored with κ colors, then there is a homogeneous subset of cardinality κ^+.
20. If $\kappa \geq \aleph_0$ is a cardinal, then $(2^\kappa)^+ \to \left((2^\kappa)^+, (\kappa^+)_\kappa\right)^2$. That is, if $f : (2^\kappa)^+ \to \kappa$, then either there is a homogeneous subset in color 0 of cardinality $(2^\kappa)^+$ or else there is a homogeneous subset in some color $0 < \alpha < \kappa$ of cardinality κ^+.
21. If κ is an infinite cardinal and $\left\{f_\alpha : \alpha < \left(2^\kappa\right)^+\right\}$ is a sequence of ordinal-valued functions defined on κ, then there is a pointwise increasing subsequence of cardinality $\left(2^\kappa\right)^+$, that is, there is a set $Z \subseteq \left(2^\kappa\right)^+$, $|Z| = \left(2^\kappa\right)^+$, such that $f_\alpha(\xi) \leq f_\beta(\xi)$ holds for $\alpha < \beta$, $\alpha, \beta \in Z$, $\xi < \kappa$.
22. If X is a set then $|X| \leq \mathbf{c}$ if and only if there is an "antimetric" on X, i.e., a function $d : X \times X \to [0, \infty)$ which is symmetric, $d(x, y) = 0$ exactly when $x = y$, and for distinct $x, y, z \in X$ for some permutation x', y', z' of them $d(x', z') > d(x', y') + d(y', z')$ holds.
23. $2^\kappa \not\to (\kappa^+)^2_2$. That is, if $|S| = 2^\kappa$, then there is $f : [S]^2 \to \{0, 1\}$ with no monochromatic set of size κ^+.
24. $2^\kappa \not\to (3)^2_\kappa$. That is, if $|S| = 2^\kappa$, then there is $f : [S]^2 \to \kappa$ with no monochromatic triangle.
25. (Erdős–Rado theorem) If κ is an infinite cardinal, set $\exp_0(\kappa) = \kappa$ and then by induction $\exp_{r+1}(\kappa) = 2^{\exp_r(\kappa)}$. If $\kappa \geq \aleph_0$ is a cardinal, then $\exp_r(\kappa)^+ \to (\kappa^+)^{r+1}_\kappa$. That is, if the $(r+1)$-tuples of $\exp_r(\kappa)^+$ are colored with κ colors, then there is a homogeneous subset of cardinality κ^+.
26. If κ is an infinite cardinal, $r < \omega$, there is a function $f : \left[\exp_r(\kappa)\right]^{r+1} \to \kappa$ such that if $x_0 < x_1 < \cdots < x_{r+1}$, then $f(x_0, \ldots, x_{r+1}) \neq f(x_1, \ldots, x_{r+2})$, specifically, $\exp_r(\kappa) \not\to (r+2)^{r+1}_\kappa$.

27. Let κ be an infinite cardinal, $|A| = \kappa^+$, $|B| = (\kappa^+)^+$, and k finite. If $f : A \times B \to \kappa$, then there exist $A' \subseteq A$, $B' \subseteq B$, $|A'| = |B'| = k$ such that $A' \times B'$ is monochromatic.
28. If $|A| = \aleph_1$, $|B| = \aleph_0$, k is finite, $f : A \times B \to k$, then there exist $A' \subseteq A$, $B' \subseteq B$, $|A'| = |B'| = \aleph_0$ such that $A' \times B'$ is monochromatic.
29. (Canonization) Assume λ is a strong limit singular cardinal and S, a set of cardinality λ is partitioned as $S = \bigcup\{S_\alpha : \alpha < \mu\}$ where $\mu = \mathrm{cf}(\lambda)$ and each S_α is of cardinality $< \lambda$. Assume that $f : [S]^2 \to \kappa$ with $\kappa < \lambda$. Then there is a set $X \subseteq S$, $|X| = \lambda$, on which f is canonical in the sense that if $x, y \in X$ then $f(x, y)$ is fully determined by α, β where α, $\beta < \mu$ are those ordinals with $x \in S_\alpha$, $y \in S_\beta$.
30. If λ is a strong limit singular cardinal with $\mathrm{cf}(\lambda) = \omega$, $3 \leq k < \omega$, then $\lambda \to [\lambda]^2_k$ holds, that is, if $f : [\lambda]^2 \to k$ then on some subset of cardinality λ f assumes at most two values.
31. For a set I of indices let the sets $\{A_i, B_i : i \in I\}$ be given with $|A_i|$, $|B_i| \leq \kappa$ and $A_i \cap B_j = \emptyset$ if and only if $i = j$. Then $|I| \leq 2^\kappa$.
32. If $\kappa > \omega$ is regular, then $\kappa \to (\kappa, \omega + 1)^2$. That is, if $f : [\kappa]^2 \to \{0, 1\}$, then either there is a set of order type κ monochromatic in color 0 or else there is a set of order type $\omega + 1$ monochromatic in color 1.
33. For $k < \omega$, $\omega_1 \to (\omega + 1)^2_k$. That is, if we color $[\omega_1]^2$ with k colors, then there is a monochromatic set of order type $\omega + 1$.
34. If $k < \omega$ and λ denotes the order type of the reals, then $\lambda \to (\omega + 1)^2_k$ holds. That is, if $f : [\mathbf{R}]^2 \to k$, then there is a monochromatic set of order type $\omega + 1$.
35. Assume that $\kappa > \omega$ is a cardinal for which $\kappa \to (\kappa)^2_2$ holds. Then κ is
    (a) regular,
    (b) strong limit (i.e., if $\lambda < \kappa$ then $2^\lambda < \kappa$),
    (c) not the least cardinal with (a) and (b).
36. Define, for $k < \omega$, by transfinite recursion on $\alpha < \omega_1$, the notion of semihomogeneous coloring $f : [S]^2 \to k$ for every $\langle S, < \rangle$ of order type ω^α. For $\alpha = 0$, no condition is imposed. For $\alpha = \beta + 1$, f is semihomogeneous if and only if there is a decomposition $S = S_0 \cup S_1 \cup \cdots$ with $S_0 < S_1 < \cdots$, each S_i having order type ω^β, f is semihomogeneous on every S_i, and gets the same value on all pairs between distinct S_i's. For α limit, f is semihomogeneous if and only if there is a decomposition $S = S_0 \cup S_1 \cup \cdots$ where $S_0 < S_1 < \cdots$, with S_i of order type ω^{α_i} where $\alpha_0 < \alpha_1 < \cdots$ converges to α, f is semihomogeneous on every S_i, and gets the same value on all pairs between distinct S_i's. Then given $\beta < \omega_1$, $k < \omega$, there exists $\alpha < \omega_1$, such that every semihomogeneous coloring of $[\omega^\alpha]^2$ with k colors includes a homogeneous set of type β.

37. If V, a vector space over $\mathbf{Q}$ with $|V| \geq \aleph_2$, is colored with countably many colors, then there is a monochromatic solution of $x+y=z+u$ with pairwise distinct x, y, z, u.
38. If V, a vector space over $\mathbf{Q}$ with $|V| \geq \mathbf{c}^+$ is colored with countably many colors, then there is a monochromatic solution of $x+y=z$ with x, y, z different from zero and each other. This is not true for $|V| \leq \mathbf{c}$.
39. If $\langle X, \mathcal{T} \rangle$ is a Hausdorff topological space with a dense set of cardinality κ, then $|X| \leq 2^{2^\kappa}$.
40. If $\langle X, \mathcal{T} \rangle$ is a Hausdorff topological space with $|X| > 2^{2^\kappa}$, then there is a discrete subspace of cardinality κ^+.
41. If $\langle X, \mathcal{T} \rangle$ is a hereditarily Lindelöf Hausdorff topological space, then $|X| \leq \mathbf{c}$ ("hereditarily Lindelöf" means that every open cover of any subspace includes a countable subcover).
42. If $\langle X, \mathcal{T} \rangle$ is a first countable Hausdorff topological space with no uncountable system of pairwise disjoint, nonempty open sets, then $|X| \leq \mathbf{c}$ ("first countable" means that for every point in the space there is a countable family $\{U_i\}_{i<\omega}$ of neighborhoods of x such that every neighborhood of x includes a U_i).
43. If the elements of $\mathcal{P}(\omega)$ are colored with countably many colors, then there is a monocolored nontrivial solution of $X \cup Y = Z$.
44. There is a set S such that if the elements of $\mathcal{P}(S)$ are colored with countably many colors, then there is a monocolored nontrivial solution of $X \cup Y = Z$ with X, Y disjoint.
45. For every set S there is a coloring of $\mathcal{P}(S)$ with countably many colors such that there do not exist pairwise disjoint $X_0, X_1, \ldots \subseteq S$ with all nonempty, finite subunions in the same color class.
46. For every infinite set S there is a coloring $f : [S]^{\aleph_0} \to \{0,1\}$ of the countably infinite subsets of S with two colors that admits no infinite homogeneous subset, i.e., $\kappa \not\to (\aleph_0)_2^{\aleph_0}$ holds for any κ.

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Δ-systems

Regarding the inclusion relation the simplest possible family is a family of pairwise disjoint sets. Often, from a family of sets one would like to select a subfamily with such a simple structure, however, with pairwise disjoint sets this is not always possible. A possible remedy is the selection of a Δ-system, where $\{A_i : i \in I\}$ is called a Δ-*system* (or a Δ-family) if the pairwise intersections of the members is the same; $A_i \cap A_j = S$ for some set S (for $i \neq j$ in I). Thus, a Δ-system has a simple structure: all sets in it have a common core, and outside this common core the sets are disjoint.

In this chapter we consider the problem how large Δ-systems can be selected from a given family of sets. As an application we shall obtain in Problem 5 that in no power of **R** (regarded as a topological space) can one find an uncountable system of pairwise disjoint open sets.

1. An infinite family of n-element sets ($n < \omega$) includes an infinite Δ-subfamily.
2. An uncountable family of finite sets includes an uncountable Δ-subfamily.
3. Let $\mathcal{F}$ be a family of finite sets, $\kappa = |\mathcal{F}|$ a regular cardinal. Then $\mathcal{F}$ has a Δ-subfamily of cardinality κ. This is not true if κ is singular.
4. Is it true that every family $\mathcal{F}$ of finite sets with $|\mathcal{F}| = \aleph_1$ is the union of countably many Δ-subfamilies ?
5. Let A, B be arbitrary sets, let B be countable, and let $F(A, B)$ be the set of all functions from a finite subset of A into B. Then among uncountably many elements of $F(A, B)$ there are two which possess a common extension.
6. Consider the topological product of an arbitrary number of copies of **R**, regarded as a topological space. In this space there are no uncountably many pairwise disjoint nonempty open subsets.
7. If $\{A_\alpha : \alpha < \omega_1\}$ is a family of finite sets, then $\{A_\alpha : \alpha \in S\}$ is a Δ-subsystem for some stationary set S.

8. (a) Let $\mathcal{F}$ be a family of countable sets, $|\mathcal{F}| = \mathbf{c}^+$. Then $\mathcal{F}$ has a Δ-subfamily $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| = \mathbf{c}^+$.

   (b) Let $\mathcal{F}$ be a family of sets of cardinality $\leq \mu$, with $\lambda = |\mathcal{F}|$ regular and with the property that $\kappa < \lambda$ implies $\kappa^\mu < \lambda$ (for example, $\lambda = (2^\mu)^+$). Then $\mathcal{F}$ has a Δ-subfamily of cardinality λ.

9. For μ infinite, there is a set system of cardinality 2^μ, consisting of sets of cardinality μ, with no 3-element Δ-subsystem.

10. For a set I of indices the sets $\{A_i, B_i : i \in I\}$ are given with $|A_i|$, $|B_i| \leq \mu$ and $A_i \cap B_j = \emptyset$ holds if and only if $i = j$. Then $|I| \leq 2^\mu$.

11. Assume that $\lambda > \kappa \geq \omega$ and $\mathcal{F}$ is a family of cardinality λ of sets of cardinality $< \kappa$. Then there is a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of cardinality λ such that

$$\left| \bigcup_{A \neq B \in \mathcal{F}'} (A \cap B) \right| < \lambda$$

assuming that either

   (a) λ is regular or

   (b) GCH holds.

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Set mappings

In the following problems a *set mapping* is a function $f : S \to \mathcal{P}(S)$ for some set S (or, in some cases, $f : [S]^n \to \mathcal{P}(S)$ for some set S and some finite $n \geq 2$) usually with some restriction on the images. We shall always assume, even if we do not explicitly mention it, that $x \notin f(x)$ (or, in the other case, $x_1, \dots, x_n \notin f(x_1, \dots, x_n)$). Given a set mapping $f : S \to \mathcal{P}(S)$ a *free set* is some set $X \subseteq S$ with $x \notin f(y)$ for $x, y \in X$. (If $f : [S]^n \to \mathcal{P}(S)$ then the condition is that $y \notin f(x_1, \dots, x_n)$ for $y, x_1, \dots, x_n \in X$).

A basic problem for set mappings is how large free set can be guaranteed under a set mapping. In what follows we shall consider both positive and negative results on this problem.

1. Assume that $f : \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set mapping with $x \notin \overline{f(x)}$. Then there is a free set that is
   (a) of the second category,
   (b) of cardinality continuum.
2. There is a set mapping $f : \mathbf{R} \to \mathcal{P}(\mathbf{R})$ with $f(x)$ bounded, but with no 2-element free set.
3. There is a set mapping $f : \mathbf{R} \to \mathcal{P}(\mathbf{R})$ with $|f(x)| < \mathbf{c}$ and with no 2-element free sets.
4. If $f : \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set mapping with $f(x)$ nowhere dense, then there is always an everywhere dense free set.
5. Assume that $f : \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set mapping such that $|f(x)| < \mathbf{c}$, $f(x)$ not everywhere dense in $\mathbf{R}$. Then there is a 2-element free set. Is there a 3-element free set?
6. Assume that $f : \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set mapping such that $f(x)$ is always a bounded set with outer measure at most 1. Then for every finite n there is an n-element free set.

7. (CH) There is a set mapping $f : \mathbf{R} \to \mathcal{P}(\mathbf{R})$ such that for every real number $x \in \mathbf{R}$ the image $f(x)$ is a sequence converging to x, yet there is no uncountable free set.
8. Assume $\mu < \kappa$ are infinite cardinals with κ regular. Let $f : \kappa \to [\kappa]^{<\mu}$ be a set mapping. There is a free set of cardinality κ if κ is
   (a) regular (S. Piccard),
   (b) singular (A. Hajnal).
9. Assume that $f : S \to \mathcal{P}(S)$ is a set mapping with $|f(x)| \leq k$ for some natural number k. Then S is the union of at most $2k+1$ free sets.
10. Assume that $f : S \to \mathcal{P}(S)$ is a set mapping with $|f(x)| < \mu$ for some infinite cardinal μ. Then S is the union of at most μ free sets.
11. Assume that $f : \omega_1 \to \mathcal{P}(\omega_1)$ is a set mapping such that $f(x) \cap f(y)$ is finite for $x \neq y$. Then for every $\alpha < \omega_1$ there is a free subset of type α.
12. Assume that $f : [S]^k \to [S]^{<\omega}$ is a set mapping for some set S where k is finite. If $|S| \geq \aleph_k$ then there is a free set of size $k+1$, but this is not true if $|S| < \aleph_k$.
13. If $f : [S]^2 \to [S]^{<\omega}$ is a set mapping on a set S of cardinality $\aleph_2$, then for every $n < \omega$ there is a free set of size n.

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Trees

In this chapter we consider the somewhat technical but important notion of tree. We start with König's lemma, whose easy yet powerful statement can be formulated as: if there will be infinitely many generations, then there is an infinite dynasty. Then we proceed to higher equivalents, that is, to Aronszajn trees and variants.

A *tree* $\langle T, \prec\rangle$ is a partially ordered set in which the set $T_{<x} = \{y : y \prec x\}$ of the elements smaller than x is well ordered for every $x \in T$. The order type $o(x)$ of $T_{<x}$ denotes *how high* the element x is in the tree: those elements with $o(x) = \alpha$ form the *αth level* T_α of T. In order to be reader-friendly, we will occasionally use the nonstandard but self-explanatory notation $T_{>x} = \{y : x \prec y\}$, $T_{<\alpha} = \bigcup\{T_\beta : \beta < \alpha\}$, $T_{>\alpha} = \bigcup\{T_\beta : \alpha < \beta\}$, etc. The *height*, $h(T)$ of T, is the least α with $T_\alpha = \emptyset$. An *α-branch* of a tree $\langle T, \prec\rangle$ is an ordered subset $b \subseteq T_{<\alpha}$ that intersects every level T_β $(\beta < \alpha)$ (in exactly one point).

A tree $\langle T, \prec\rangle$ is *normal* if

(A) for every $x \in T$, $T_{>x}$ contains elements arbitrary high below $h(T)$;

(B) if $x \in T$, then there exist distinct y, y' with $x \prec y$, $x \prec y'$, $o(y) = o(y') = o(x) + 1$;

(C) if $\alpha < h(T)$ is a limit ordinal, $x \neq x' \in T_\alpha$, then $T_{<x} \neq T_{<x'}$.

If $s \prec t$, then we call t a *successor* of s, s a *predecessor* of t. If $s \prec t$ or $t \prec s$ holds, then we call s, t *comparable.* If neither $s \prec t$ nor $t \prec s$ holds, then s, t are *incomparable.* If $s \prec t$ and there are no further elements between s and t (i.e., they are on consecutive levels of the tree), then t is an *immediate successor* of s, s is an *immediate predecessor* of t.

If κ is a cardinal, a tree $\langle T, \prec\rangle$ is a *κ-tree* if $h(T) = \kappa$ and $|T_\alpha| < \kappa$ holds for every $\alpha < \kappa$.

An *Aronszajn tree* is an ω_1-tree with no ω_1-branches, and in general, a *κ-Aronszajn tree* is a κ-tree with no κ-branches. If every κ-tree has a κ-branch, that is, there are no κ-Aronszajn trees, then κ is said to have the *tree property.*

In a tree $\langle T, \prec\rangle$ a subset $A \subseteq T$ is an *antichain* if it consists of pairwise incomparable elements. An ω_1-tree is *special* if it is the union of countably many antichains.

A subset $D \subseteq T$ of a tree is *dense* if for every $x \in T$ there is a $y \in D$ with $x \preceq y$. A subset $D \subseteq T$ of a tree is *open* if $x \prec y$, $x \in D$ imply that $y \in D$.

An ω_1-tree is a *Suslin tree* if there is no ω_1-branch or uncountable antichain in it.

Squashing a tree: if $\langle T, \prec \rangle$ is a tree, then we can transform it into an ordered set as follows. Let $<_\alpha$ be an ordering on T_α. If x, y are distinct elements of T, then set $x <_{\text{lex}} y$ if and only if either $x \prec y$ or $T_{\leq x}$ is "lexicographically smaller" than $T_{\leq y}$. That is, if $T_{\leq x} = \{p_\alpha(x) : \alpha \leq o(x)\}$ where $p_\alpha(x)$ is the only element of $T_{\leq x}$ on T_α, and $T_{\leq y} = \{p_\alpha(y) : \alpha \leq o(y)\}$ is the corresponding set for y, then $p_\alpha(x) <_\alpha p_\alpha(y)$ holds for the least α where $p_\alpha(x) \neq p_\alpha(y)$. Notice that if $\langle T, \prec \rangle$ is normal then it suffices to define $<_\alpha$ on T_0 and for every element s of T on the set of immediate successors of s.

A *Specker type* is the order type of an ordered set that does not embed ω_1, ω_1^*, or an uncountable subset of the reals.

A *Countryman type* is the order type of an ordered set $\langle S, \prec \rangle$ if $S \times S$ is the union of countably many chains under the partial order $\langle x, y \rangle \preceq \langle x', y' \rangle$ if and only if $x \preceq x'$ and $y \preceq y'$.

A *Suslin line* is a nonseparable ordered set that is ccc, that is, it does not include a countable dense subset and every family of pairwise disjoint nonempty open intervals is countable.

There are two more notions of trees: in Chapter 31 what we call trees are certain trees of height ω and of course in graph theory the connected, circuitless graphs are called trees.

1. (König's lemma) ω has the tree property, that is, if every level of an infinite tree is finite, then there is an infinite branch.
2. There is a tree T of height ω, with $|T_n| = \aleph_0$ for every $n < \omega$ such that T has no infinite branch.
3. If an infinite connected graph is locally finite (every vertex has finite degree), then it includes an infinite path.
4. Suppose that $\mathcal{H}$ is an infinite set of finite 0–1 sequences closed under restriction, that is, if $a_1 \cdots a_n \in \mathcal{H}$, then $a_1 \cdots a_m \in \mathcal{H}$ holds for every $m < n$. Then there is an infinite 0–1 sequence all whose (finite) initial segments belong to $\mathcal{H}$.
5. Let A_i, $i < \omega$ be finite sets and let $f_k \in \prod_{i<k} A_i$ for $k = 0, 1, \ldots$. Then there is an $f \in \prod_{i<\omega} A_i$ such that on any finite set $S \subseteq \omega$ the function f agrees with one of the f_k's (i.e., $f|_S = f_k|_S$).
6. An infinite bounded set of reals has a limit point.
7. Given the natural numbers r, k, and s there is a natural number n such that if all r-tuples of $\{0, 1, \ldots, n-1\}$ are colored with k colors, then there is a homogeneous subset increasingly enumerated as $\{a_1, \ldots, a_p\}$ with $p \geq s$ and also with $p \geq a_1$.

8. A domino is a one-by-one square, where the four sides are colored. Given a collection D of dominoes with finitely many different color types, we want to tile the plane with them, i.e., to place a domino on each lattice point with its center on the lattice point, in a horizontal-vertical position such that the common sides of neighboring dominoes have the same color.
   (a) If for every $n < \omega$ an $n \times n$ square has a tiling from D, then so has the plane.
   (b) If the plane has a tiling from D, then it has from D', where D' is obtained from D by omitting those types that contain only finitely many pieces.
9. The vertex set of a locally finite graph can be partitioned into two sets, A and B such that if for v, a vertex, $d_A(v)$, $d_B(v)$ denote the number of vertices joined to v in A, B, respectively, then $d_A(v) \leq d_B(v)$ if $v \in A$ and $d_A(v) \geq d_B(v)$ if $v \in B$.
10. (a) If $a_1 + \cdots + a_n$ is a sum of positive reals, then there are indices $0 = k(0) < k(1) < \cdots < k(r) = n$ such that $S_1 \geq \cdots \geq S_r$ holds for the subsums $S_i = a_{k(i-1)+1} + \cdots + a_{k(i)}$ and $S_1 < 2\sqrt{a_1^2 + \cdots + a_n^2}$.
    (b) If $\sum_1^\infty a_i$ is a divergent series of positive terms and $\sum a_i^2 < \infty$, then there are indices $0 = k(0) < k(1) < \cdots$ such that $S_1 \geq \cdots \geq S_r$ holds for the subsums $S_i = a_{k(i-1)+1} + \cdots + a_{k(i)}$.
11. There is an Aronszajn-tree.
12. There is a special Aronszajn-tree.
13. Every special ω_1-tree is Aronszajn.
14. If $\langle T, \prec\rangle$ is a tree, then $\langle T, <_{\text{lex}}\rangle$ is an ordered set.
15. If $\langle T, \prec\rangle$ is an Aronszajn-tree, then the order type of $\langle T, <_{\text{lex}}\rangle$ is a Specker type.
16. There exist functions $\{e_\alpha : \alpha < \omega_1\}$ such that each $e_\alpha : \alpha \to \omega$ is injective and for $\beta < \alpha$ the functions e_β and $e_\alpha|_\beta$ are identical at all but finitely many points.
17. The tree $T = \{e_\alpha|_\beta : \beta \leq \alpha < \omega_1\}$ (with the functions of the previous problem) is an Aronszajn-tree, where $g \prec g'$ if and only if g' properly extends g.
18. Let e_α from Problem 16, and set $S = \{e_\alpha : \alpha < \omega_1\}$, where $\prec$ is the lexicographic ordering. Then the order type of S is a Countryman type.
19. Every Countryman type is a Specker type.
20. An ω_1-tree $\langle T, \prec\rangle$ is special if and only if there is an order preserving $f : \langle T, \prec\rangle \to \langle \mathbf{Q}, <\rangle$.
21. Assume that $\langle T, \prec\rangle$ is an ω_1-tree with a function $f : T \setminus T_0 \to T$ such that $f(t) \prec t$ and for every t and for every element $s \in T$ the set $f^{-1}(s)$ is the union of countably many antichains. Then $\langle T, \prec\rangle$ is special.

22. If a normal ω_1-tree $\langle T, \prec\rangle$ has no uncountable antichain, then it is a Suslin tree.
23. If $\langle T, \prec\rangle$ is a Suslin tree then for all but countably many $x \in T$, the set $T_{\geq x}$ is uncountable.
24. If there is a Suslin tree, then there is a normal Suslin tree.
25. There is a Suslin tree if and only if there is a Suslin line.
26. If $\langle T, \prec\rangle$ is a Suslin tree, $D \subseteq T$ is dense, open then D is co-countable in T.
27. If $\langle T, \prec\rangle$ is a normal Suslin tree, $D_0, D_1, \ldots \subseteq T$ are dense, open sets, then $D_0 \cap D_1 \cap \cdots$ is also a dense, open set.
28. If $\langle T, \prec\rangle$ is a Suslin tree, $A \subseteq T$ is uncountable then A is somewhere dense, i.e., there is some $t \in T$ such that for every $x \succeq t$ there is $y \succeq x$, $y \in A$.
29. If $\langle T, \prec\rangle$ is a normal Suslin tree, $f : T \to \mathbf{R}$ preserves $\preceq$, then f has countable range. There is no such f that preserves $\prec$.

In Problems 30–31 we consider the topology of the tree $\langle T, \prec\rangle$ generated by the open intervals, i.e., of the sets of the form $(p, q) = \{t \in T : p \prec t \prec q\}$. This amounts to declaring $t \in T_\alpha$ isolated if $\alpha = 0$ or successor, and if α is limit then the sets of the form $(s, t]$ $(s \prec t)$ give a neighborhood base of t.

30. If $\langle T, \prec\rangle$ is a normal Suslin tree, $f : T \to \mathbf{R}$ is continuous, then f has countable range.
31. If $\langle T, \prec\rangle$ is a normal Suslin tree, then it is a normal topological space.
32. On a normal ω_1-tree $\langle T, \prec\rangle$ two players, I and II alternatively pick the successive elements of the sequence $t_0 \prec t_1 \prec \cdots$ with I choosing t_0. I wins if and only if there is an element of T above all of $t_0, t_1, \ldots$.
    (a) I has no winning strategy.
    (b) If $\langle T, \prec\rangle$ is special, II has winning strategy.
    (c) If $\langle T, \prec\rangle$ is Suslin, II has no winning strategy.
33. If κ is regular, $\lambda < \kappa$, $\langle T, \prec\rangle$ is a κ-tree with $|T_\alpha| < \lambda$ for $\alpha < \kappa$ then $\langle T, \prec\rangle$ has a κ-branch. This is not true if κ is singular.
34. If, for some regular $\kappa \geq \omega$, there is a κ-Aronszajn tree, then there is a normal one.
35. If $\langle T, \prec\rangle$ is a κ-tree for some regular cardinal κ, then the following are equivalent.
    (a) $\langle T, \prec\rangle$ has a κ-branch.
    (b) $\langle T, <_{\text{lex}}\rangle$ includes a subset of order type κ or κ^*.

36. There exists a κ^+-Aronszajn tree if $\square_\kappa$ holds, that is, for every limit $\alpha < \kappa^+$ there is a closed, unbounded subset $C_\alpha \subseteq \alpha$ of order type $\leq \kappa$ such that if $\beta < \alpha$ is a limit point of C_α, then $C_\beta = C_\alpha \cap \beta$.
37. There exists a κ^+-Aronszajn tree if κ is regular and $2^\mu \leq \kappa$ holds for $\mu < \kappa$.
38. κ has the tree property if κ is real measurable (see Chapter 28).
39. Assume that κ is a singular cardinal such that for every $\lambda < \kappa$ there is an ultrafilter D_λ on the subsets of κ^+ such that if $A \in D_\lambda$ then $|A| = \kappa^+$ and if $A_\alpha \in D_\lambda$ $(\alpha < \lambda)$ then $\bigcap_{\alpha<\lambda} A_\alpha \in D_\lambda$. Then κ^+ has the tree property.
40. If $\kappa \to (\kappa)^2_2$ then every ordered set of cardinality κ includes either a well-ordered or a reversely well-ordered subset of cardinality κ.
41. If every ordered set of cardinality κ includes either a subset of order type κ or a subset of order type κ^*, then κ is strongly inaccessible.
42. If κ has the tree property, then κ is regular.
43. If κ is the smallest strong limit regular cardinal bigger than ω, then κ does not have the tree property.
44. For an infinite cardinal κ the following are equivalent.
    (a) $\kappa \to (\kappa)^2_2$,
    (b) $\kappa \to (\kappa)^n_\sigma$ for any $\sigma < \kappa$ and $n < \omega$,
    (c) κ is strongly inaccessible and has the tree property,
    (d) in any ordered set of cardinality κ there is either a well-ordered or a reversely well-ordered subset of cardinality κ.

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The measure problem

It has always been an important problem to measure length, area, volume, etc. In the 19th and 20th centuries various measure and integral concepts (like Riemann and Lebesgue measures and integrals) were developed for these purposes and they have proved adequate in most situations. However, it is natural to ask what their limitations are, e.g., to what larger classes of sets can the notion of Lebesgue measure be extended by preserving its well-known properties. The standard proof for the existence of not Lebesgue measurable set in $\mathbf{R}$ (using the axiom of choice!) shows that there is no nontrivial translation invariant σ-additive measure on all subsets of $\mathbf{R}$. It was S. Banach who proved that in $\mathbf{R}$ and $\mathbf{R}^2$ there is a finitely additive nontrivial isometry invariant measure. If we go to $\mathbf{R}^3$, then the situation changes: by the Banach–Tarski paradox (Chapter 19) a ball can be decomposed into two balls of the same size; therefore, there is no nontrivial finitely additive isometry invariant measure on all subsets of $\mathbf{R}^n$ with $n \geq 3$.

In this chapter we discuss the problem when we do not care for translation invariance, but want to keep σ-additivity or some kind of higher-order additivity. Let X be an infinite set. By the phrase "μ is a measure on X" we mean a measure $\mu : \mathcal{P}(X) \to [0,1]$ on all subsets of X. Such a measure is called nontrivial if $\mu(X) = 1$ and $\mu(\{x\}) = 0$ for each $x \in X$. Since we shall only be interested in nontrivial measures, in what follows we shall always assume that the measures in question are nontrivial (hence we exclude discrete measures, which are completely additive). μ is called κ-additive if for any disjoint family Y_i, $i \in I$ of fewer than κ sets (i.e., $|I| < \kappa$) we have $\mu(\cup_{i\in I} Y_i) = \sum_{i\in I} \mu(Y_i)$. The right-hand side is defined as the supremum of its finite partial sums, and, as a consequence, on the right-hand side only countably many $\mu(Y_i)$ can be positive. Instead of ω-additivity we shall keep saying "finite additivity" and instead of ω_1-additivity we say "σ-additivity".

It turns out (see Problems 8, 9) that the first cardinal κ on which there is a σ-additive measure has also the stronger property that it carries a κ-additive measure as well. A cardinal $\kappa > \omega$ is called real measurable if there is a κ-additive $[0,1]$-valued measure on κ. It is called measurable if there is such a measure taking only the values 0 and 1. Real measurable but not measurable cardinals are at most as large as the continuum (Problem 7), but measurable cardinals are very large, their existence cannot be proven in ZFC (Zermelo–Fraenkel axiom system with the axiom of choice). On the other hand, R. Solovay proved in 1966 that if ZFC is consistent with the existence of a real measurable cardinal, then

- ZFC is consistent with the existence of a measurable cardinal,
- ZFC is consistent with **c** being real measurable,
- ZF is consistent with the statement that all subsets of **R** are Lebesgue-measurable.

In the present chapter we discuss a few properties of measurable cardinals. One of the main results in this subject is the existence of a normal ultrafilter on any measurable cardinal (Problem 14), which has the easy consequence that all measurable cardinals are weakly compact, that is, $\kappa \to (\kappa)^2_2$ holds for them. A stronger Ramsey property will be established in Problem 16.

In analogy with κ-additivity of measures let us call an ideal κ-complete if it is closed for $< \kappa$ unions and a filter κ-complete if it is closed for $< \kappa$ intersections. Recall that an ideal/filter on a ground set X is called a prime ideal/ultrafilter if for all $Y \subset X$ either Y or $X \setminus Y$ belongs to it (and this is equivalent to the maximality of the ideal/filter). A prime ideal $\mathcal{I} \subset \mathcal{P}(X)$ is called nontrivial if it contains all singletons $\{x\}$, $x \in X$, and an ultrafilter $\mathcal{F} \subset \mathcal{P}(X)$ is called nontrivial if it does not contain any of the $\{x\}$, $x \in X$.

In the problems below all measures, prime ideals, and ultrafilters will be assumed to be nontrivial.

1. On any infinite set there is a finitely additive nontrivial 0–1-valued measure.
2. Let X be an infinite set and $\kappa \geq \omega$ a cardinal. The following are equivalent:
   - there is a κ-additive 0–1-valued measure on X;
   - there is a κ-complete prime ideal on X;
   - there is a κ-complete ultrafilter on X.
3. There is no σ-additive $[0,1]$-valued measure on ω_1 (i.e., $\aleph_1$ is not real measurable).
4. If **R** is decomposed into a disjoint union of $\aleph_1$ sets of Lebesgue measure zero, then some of these sets have nonmeasurable union.
5. If κ is real measurable, then it is a regular limit cardinal.
6. If there is a $[0,1]$-valued σ-additive measure μ on $[0,1]$ then there is such a $\overline{\mu}$ extending the Lebesgue measure. Furthermore, if μ is κ-additive for some κ, then so is $\overline{\mu}$.

7. If $\kappa > \mathbf{c}$ is real measurable, then it is measurable.
8. If κ is the smallest cardinal on which there is a σ-additive $[0,1]$-valued measure, then κ is real measurable.
9. If κ is the smallest cardinal on which there is a σ-additive 0–1-valued measure, then κ is measurable.
10. There is no σ-additive 0–1-valued measure on $\mathbf{R}$.
11. If κ is measurable, then it is a strong limit regular cardinal.

    If $\kappa > 0$ is a regular cardinal, then a filter $\mathcal{F}$ on κ is called a normal filter if for every $F \in \mathcal{F}$ and every $f : F \to \kappa$ regressive function f there is an $\alpha < \kappa$ such that $f^{-1}(\alpha) \in \mathcal{F}$.

12. Let κ be regular. An ultrafilter $\mathcal{F}$ on κ is normal if and only if it is closed for diagonal intersection (see Problem 21.5).
13. Let κ be regular and $\mathcal{F}$ a normal ultrafilter on κ. Then $\mathcal{F}$ is κ-complete if and only if no element of $\mathcal{F}$ is of cardinality smaller than κ.
14. If κ is measurable, then on κ there is a κ-complete normal ultrafilter. Prove this via the following outline.
    (a) Let μ be a κ-additive measure on κ, and for $f, g \in {}^{\kappa}\kappa$ set $f \equiv g$ if $f(\alpha) = g(\alpha)$ for a.e. α (i.e., the μ-measure of the set of the exceptional α is 0). Then this is an equivalence relation, and between the equivalence classes $\overline{f}$ and $\overline{g}$ of f and g set $\overline{f} \prec \overline{g}$ if $f(\alpha) < g(\alpha)$ a.e. This is a well-ordering on the set of equivalence classes ${}^{\kappa}\kappa/_{\equiv}$.
    (b) Let Y be the set of those functions $f \in {}^{\kappa}\kappa$ for which $f^{-1}(\alpha)$ is of measure 0 for all $\alpha \in \kappa$, and let $f_0 \in Y$ be such that its equivalence class is minimal in $Y/_{\equiv}$. Then $\mathcal{F} = \{F : f_0^{-1}[F] \text{ is of measure } 1\}$ is a κ-complete normal ultrafilter on κ.
15. If κ is measurable, then $\kappa \to (\kappa)^r_\sigma$ for any $r < \omega$ and $\sigma < \kappa$.
16. If κ is measurable, then $\kappa \to (\kappa)^{<\omega}_\sigma$ for any $\sigma < \kappa$, i.e., if we color the finite subsets of κ by $\sigma < \kappa$ colors then there is a set A of cardinality κ that is homogeneous in the sense that for every fixed $r < \omega$ all the r tuples of A have the same color (cardinals with the property $\kappa \to (\kappa)^{<\omega}_\sigma$ for $\sigma < \kappa$ are called Ramsey cardinals).

The following problems lead to the existence of finitely additive isometry invariant measures on all subsets of $\mathbf{R}$ and $\mathbf{R}^2$. First we deal with the case when the whole space has measure 1, and then with the case that extends Jordan measure (in this case the measure necessarily is extended-valued, i.e., it is infinite on the whole space). Such measures are called Banach measures. Note that by the Banach–Tarski paradox (see Chapter 19) in $\mathbf{R}^3$ (and in $\mathbf{R}^n$ with $n \geq 3$) there is no such measure.

The construction of finitely additive isometry invariant measures on all subsets runs parallel with the construction of additive positive linear functionals on the space of bounded functions, which is the analogue of integration. We shall also construct these so-called Banach integrals in $\mathbf{R}$ and $\mathbf{R}^2$ both in the normalized case (when the identically 1 function has integral 1) and also in the case which extends the Riemann integral. Actually, Banach measures are obtained by taking the Banach integral of characteristic functions.

Let $\mathcal{B}_A$ denote the set of all bounded real-valued functions on the set A equipped with the supremum norm $\|f\| = \sup_{a\in A} |f(a)|$. We call a function $I : \mathcal{B}_A \to \mathbf{R}$

- *linear* if for any $f_1, f_2 \in \mathcal{B}_A$, $c_1, c_2 \in \mathbf{R}$ we have $I(c_1 f_1 + c_2 f_2) = c_1 I(f_1) + c_2 I(f_2)$,
- *nontrivial* if $I(1) = 1$, where 1 denotes the identically 1 function,
- *normed* if it is nontrivial and $|I(f)| \le \|f\|$ for all $f \in \mathcal{B}_A$,
- *positive* if it is nonnegative for nonnegative functions: $I(f) \ge 0$ if $f \ge 0$.

Positivity is clearly equivalent to monotonicity: if $f \le g$, then $I(f) \le I(g)$. In what follows in statements **(a)–(k)** the adjective "normed" can be replaced everywhere by "positive", since a linear functional I for which $I(1) = 1$ is positive if and only if $|I(f)| \le \|f\|$.

If Φ is a family of automorphisms of A, then we say that I is *Φ-invariant* if $I(f) = I(f_\varphi)$ for all $f \in \mathcal{B}_A$ and $\varphi \in \Phi$, where $f_\varphi(x) = f(\varphi(x))$.

17. (a) There is a normed linear functional on $\mathcal{B}_{\mathbf{N}}$.
    (b) There is a translation invariant normed linear functional I on $\mathcal{B}_{\mathbf{N}}$, i.e., if $g(n) = f(n+1)$, $n \in \mathbf{N}$, then $I(f) = I(g)$ (such a functional is called a Banach limit).
    (c) There is a translation invariant normed linear functional on $\mathcal{B}_{\mathbf{Z}}$.
    (d) For any finite n there is a translation invariant normed linear functional on $\mathcal{B}_{\mathbf{Z}^n}$.
    (e) If A is an Abelian group and $s_1, \ldots, s_n \in A$ are finitely many elements, then there is a normed linear functional I on $\mathcal{B}_A$ that is invariant for translation with any s_j (i.e., if $f_j(x) = f(s_j + x)$, then $I(f_j) = I(f)$ for all $1 \le j \le n$).
    (f) If A is an Abelian group, then there is a translation invariant normed linear functional on $\mathcal{B}_A$.
    (g) If A is an Abelian group, then there is a finitely additive translation invariant measure μ on all subsets of A such that $\mu(A) = 1$. In particular, there is a finitely additive translation invariant measure μ on all subsets of $\mathbf{R}^n$ such that $\mu(\mathbf{R}^n) = 1$.
    (h) There is an isometry invariant normed linear functional on $\mathcal{B}_{\mathbf{R}}$.

(i) There is a finitely additive isometry invariant measure μ on all subsets of $\mathbf{R}$ such that $\mu(\mathbf{R}) = 1$.

(j) There is an isometry invariant normed linear functional on $\mathcal{B}_{\mathbf{R}^2}$.

(k) There is a finitely additive isometry invariant measure μ on all subsets of $\mathbf{R}^2$ such that $\mu(\mathbf{R}^2) = 1$.

In statements **(l)–(p)** we allow the measure to take infinite values, and in these statements $\mathcal{B}^b_{\mathbf{R}^n}$ denotes the set of bounded functions on $\mathbf{R}^n$ with bounded support.

(l) There is a translation invariant positive linear functional on $\mathcal{B}^b_{\mathbf{R}}$ that extends the Riemann integral.

(m) For every n there is a translation invariant positive linear functional on $\mathcal{B}^b_{\mathbf{R}^n}$ that extends the Riemann integral.

(n) There is a translation invariant finitely additive measure on all subsets of $\mathbf{R}^n$ that extends the Jordan measure.

(o) For $n = 1, 2$ there is an isometry invariant positive linear functional on $\mathcal{B}^b_{\mathbf{R}^n}$ that extends the Riemann integral (Banach integral).

(p) For $n = 1, 2$ there is a finitely additive isometry invariant measure on all subsets of $\mathbf{R}^n$ that extends the Jordan measure (Banach measure).

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Stationary sets in $[\lambda]^{<\kappa}$

In this chapter we consider subsets of $[\lambda]^{<\kappa}$ where $\kappa > \omega$ is regular and $\lambda > \kappa$. $X \subseteq [\lambda]^{<\kappa}$ is called

- *unbounded* if for every $P \in [\lambda]^{<\kappa}$ there exists some $Q \in X$ with $P \subseteq Q$,
- *closed* if whenever $\alpha < \kappa$ and $\{P_\beta : \beta < \alpha\}$ is an increasing transfinite sequence of elements of X then $\bigcup\{P_\beta : \beta < \alpha\} \in X$,
- a *club set* when it is both closed and unbounded.

If something is true for the elements of a closed, unbounded set, then we say that it holds for *almost every* $P \in [\lambda]^{<\kappa}$ (a.e. P). Similarly, if $X \subseteq [\lambda]^{<\kappa}$, then some property holds for *almost every element of* X if there is a closed, unbounded set C such that it holds for the elements of $C \cap X$. $S \subseteq [\lambda]^{<\kappa}$ is *stationary* if it intersects every closed, unbounded set. Otherwise, it is *nonstationary*.

As we shall see these notions extend the classical notion of club sets and stationary sets. Most of the classical results from Chapters 20–21 have an analogue in this setting, and the present generalization opens space for some other questions as well.

We define $\kappa(P) = P \cap \kappa$ whenever it is $< \kappa$, i.e., when P intersects κ in an initial segment.

1. $[\lambda]^{<\kappa}$ is the union of κ bounded sets.
2. The union of $< \kappa$ bounded sets is bounded again.
3. For every $\alpha < \lambda$ the cone $\{P \in [\lambda]^{<\kappa} : \alpha \in P\}$ is a closed, unbounded set. In general, if $Q \in [\lambda]^{<\kappa}$, then $\{P \in [\lambda]^{<\kappa} : Q \subseteq P\}$ is a closed, unbounded set.
4. Every stationary set is unbounded.
5. As all ordinals, specifically all ordinals $< \kappa$, are identified with the initial segment determined by them, $\kappa \subseteq [\kappa]^{<\kappa}$ holds. A set $A \subseteq \kappa$ is stationary, (or closed, unbounded) in the sense of κ exactly if it is in the sense of $[\kappa]^{<\kappa}$.

6. $X \subseteq [\lambda]^{<\kappa}$ is closed if and only if for every directed set $Y \subseteq X$ of cardinality $< \kappa$, $\bigcup Y \in X$ holds (Y is called directed if for any $P_1, P_2 \in Y$ there is a $P \in Y$ such that $P_1 \cup P_2 \subseteq P$).
7. If $f : [\lambda]^{<\omega} \to [\lambda]^{<\kappa}$, then define $C(f) = \{P \in [\lambda]^{<\kappa} : P \text{ is closed under } f\}$.
   (a) $C(f)$ is a closed, unbounded set.
   (b) If C is a closed, unbounded set, then $C(f) \setminus \{\emptyset\} \subseteq C$ holds for an appropriate f.
8. The intersection of $< \kappa$ closed, unbounded sets is a closed, unbounded set again.
9. For a.e. P, $\kappa \cap P < \kappa$ holds (that is, P intersects the interval κ in an initial segment).
10. Given an algebraic structure with countably many operations (group, ring, etc.) on λ, a.e. $P \in [\lambda]^{<\kappa}$ is a substructure.
11. Almost every $P \in [\lambda]^{<\kappa}$ is the disjoint union of intervals of the type $[\kappa \cdot \alpha, \kappa \cdot \alpha + \beta)$ with $\beta = \kappa(P)$.
12. If $\{C_\alpha : \alpha < \lambda\}$ are closed, unbounded sets, then so is their diagonal intersection

$$\nabla_{\alpha<\lambda} C_\alpha = \{P \in [\lambda]^{<\kappa} : \alpha \in P \longrightarrow P \in C_\alpha\}.$$

13. Assume that $S \subseteq [\lambda]^{<\kappa}$ is stationary, $f(P) \in P$ holds for every $P \in S$, $P \neq \emptyset$. Then for some $\alpha < \lambda$, $f^{-1}(\alpha)$ is stationary.
14. Assume that $S \subseteq [\lambda]^{<\kappa}$ is stationary, $f(P) \in [P]^{<\omega}$ holds for every $P \in S$. Then for some s, $f^{-1}(s)$ is stationary.
15. If $X \subseteq [\lambda]^{<\kappa}$ is a nonstationary set, then there exists a function f with $f(P) \in [P]^{<\omega}$ for every $P \in X$ such that $f^{-1}(s)$ is bounded for every finite set s.
16. If $C \subseteq \kappa$ is a closed, unbounded set, then so is $\{P \in [\lambda]^{<\kappa} : \kappa(P) \in C\}$.
17. If λ is regular, $C \subseteq \lambda$ is a closed, unbounded set, then

$$A = \left\{P \in [\lambda]^{<\kappa} : \sup(P) \in C\right\}$$

    is again a closed, unbounded set.
18. If $S \subseteq \kappa$ is a stationary set, then so is $\{P \in [\lambda]^{<\kappa} : \kappa(P) \in S\}$.
19. There is a stationary set in $[\omega_2]^{<\aleph_1}$ of cardinality $\aleph_2$.
20. Every closed, unbounded set in $[\omega_2]^{<\aleph_1}$ is of maximal cardinality $\aleph_2^{\aleph_0}$.
21. Set $Z = \{P \in [\lambda]^{<\kappa} : \kappa(P) = |P|\}$. (Remember the identification of cardinals with ordinals!)
   (a) Z is stationary.
   (b) If $S \subseteq Z$ is a stationary set, then it is the disjoint union of λ stationary sets.

22. Every stationary set in $[\lambda]^{<\kappa}$ is the union of κ disjoint stationary sets. Prove this via the following steps. Let S be a counterexample.

   (a) Every stationary $S' \subseteq S$ is also a counterexample.

   (b) For almost every $P \in S$, $\kappa(P) < |P|$ holds.

   (c) Assume that $f(P) \in P$ holds for every $P \in S$, $P \neq \emptyset$. Then there is some $Q \in [\lambda]^{<\kappa}$ such that $f(P) \in Q$ holds for a. e. $P \in S$.

   (d) κ is weakly inaccessible (a regular limit cardinal).

   (e) If $S' \subseteq S$ is stationary, $f(P) \subseteq P$, $|f(P)| < \kappa(P)$ holds for $P \in S'$ then there is some $Q \in [\lambda]^{<\kappa}$ such that $f(P) \in Q$ holds for a. e. $P \in S'$.

   (f) For a. e. $P \in S$, $\kappa(P)$ is weakly inaccessible.

   (g) For a. e. $P \in S$, $S \cap [P]^{<\kappa(P)}$ is stationary in $[P]^{<\kappa(P)}$.

   (h) Get the desired contradiction.

23. (GCH) Set $\lambda = \aleph_\omega$, $\kappa = \aleph_2$. There is a stationary set $S \subseteq [\lambda]^{<\kappa}$ such that every unbounded subset of S is stationary.

24. For any nonempty set A call $S \subseteq \mathcal{P}(A)$ A-stationary if for every function $f : [A]^{<\omega} \to [A]^{\leq\aleph_0}$ there is some $B \in S$, $B \neq \emptyset$ which is closed under f.

   (a) $S = \{A\}$ is A-stationary on A.

   (b) If S is A-stationary on A, then $A = \bigcup S$.

   (c) If $A = \lambda \geq \omega_1$ is a cardinal, $S \subseteq [\lambda]^{<\aleph_1}$ then S is λ-stationary on λ if and only if it is stationary.

   (d) If S is A-stationary, $\emptyset \neq B \subseteq A$, then $T = \{P \cap B : P \in S\}$ is B-stationary.

   (e) If S is A-stationary, $B \supseteq A$, then $T = \{P \subseteq B : P \cap A \in S\}$ is B-stationary.

   (f) If S is A-stationary, $F(P) \in P$ holds for every $P \in S$, $P \neq \emptyset$, then for some x, the set $F^{-1}(x)$ is A-stationary.

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The axiom of choice

In this chapter we do not assume the axiom of choice.

We now enter a strange and interesting world. Strange, as our everyday tools cannot be used; we no longer have the trivial rule for addition and multiplication of two cardinals, and as some sets may not be well orderable, we cannot always apply transfinite induction or recursion. Interesting, as we are still able to prove some statements similar to the corresponding statements under the axiom of choice, only it requires delicate arguments, and in some cases we discover phenomena that can only hold if AC fails.

We can use the notion of a cardinal, in the naive sense, that is, without the von Neumann identification of cardinals with ordinals. That is, we can speak of the equality, sum, etc., of two cardinals.

AC_ω is the axiom of choice for countably many nonempty sets.

1. For no cardinal κ does $2^\kappa = \aleph_0$ hold.
2. If φ is an ordinal, then there is a sequence $\langle f_\alpha : \omega \leq \alpha < \varphi \rangle$ such that $f_\alpha : \alpha \times \alpha \to \alpha$ is an injection.
3. If $0 < \alpha < \omega_2$, then there is a surjection $\mathbf{R} \to \alpha$.
4. There is a mapping from the set of reals onto a set of cardinality *greater* than continuum if either
   (a) every uncountable set of reals has a perfect subset, or
   (b) every set of reals is measurable, or else
   (c) (AC_ω) there are no two disjoint stationary subsets of ω_1.
5. Let C_n denote the axiom of choice for n-element sets. Then C_m implies C_n if m is a multiple of n.
6. C_2 implies C_4.
7. C_2 and C_3 imply C_6.
8. If every set carries an ordering then $\mathrm{C}_{<\omega}$ (the axiom of choice for families of finite sets) holds.

9. Let κ, λ be cardinals, n a natural number, and assume that $\kappa+n=\lambda+n$ holds. Then $\kappa=\lambda$.
10. If $\kappa \geq \aleph_0$, then $\kappa+\aleph_0=\kappa$.
11. If $\kappa > 1$, then $\kappa+1 < 2^\kappa$.
12. If $\kappa \geq \aleph_0$, then $\kappa+2^\kappa=2^\kappa$.
13. Set $\kappa \ll \lambda$ if and only if $\kappa+\lambda=\lambda$. This $\ll$ is transitive. Furthermore, $\kappa \ll \lambda$ holds if and only if $\aleph_0\kappa \leq \lambda$.
14. If κ is of the form either $\kappa=\aleph_0\lambda$ for some cardinal λ or $\kappa=2^\lambda$ for some cardinal $\lambda \geq \aleph_0$, then $\kappa+\kappa=\kappa$.
15. If a, b are cardinals and $2a=2b$, then $a=b$.
16. If κ is an infinite cardinal then $\aleph_0 \leq 2^{2^\kappa}$.
17. $\aleph_1 \leq 2^{2^{\aleph_0}}$.
18. $\kappa \cdot \kappa \leq 2^{2^\kappa}$ holds for every cardinal κ.
19. (Hartogs' lemma) If κ is a cardinal then there is an ordinal $H(\kappa)$ with $|H(\kappa)| \leq 2^{2^{2^\kappa}}$ such that $|H(\kappa)| \not\leq \kappa$.
20. If $\kappa^2=\kappa$ holds for every infinite cardinal κ then the axiom of choice is true.
21. The generalized continuum hypothesis implies the axiom of choice. That is, if for no infinite κ exists a cardinal λ with $\kappa < \lambda < 2^\kappa$ then the AC holds.
22. AC is implied by the following statement: if $\{A_i : i \in I\}$ is a set of nonempty sets, then there is a function that selects a nonempty finite subset of each.
23. If every vector space has a basis, then the axiom of choice holds.

In the following problem, the *chromatic number* of graph $G=(V,E)$ is the minimal cardinality (if it exists) of the form $|A|$ for which there is a surjection $f\colon V \to A$ which is a good coloring, i.e., if x, $y \in V$ are joined, then $f(x) \neq f(y)$.

24. The axiom of choice is equivalent to the statement that every graph has a chromatic number.
25. Hajnal's set mapping theorem (Problem 26.8) implies the axiom of choice.
26. If $\mathbf{R}$ is the union of countably many countable sets, then so is ω_1 and $\mathrm{cf}\,(\omega_1)=\omega$.
27. ω_2 is not the union of countably many countable sets.

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Well-founded sets and the axiom of foundation

In this chapter we investigate well-founded sets. These are partially ordered sets where every nonempty subset has a least element (one with no predecessor in the subset). These sets share many properties with the well-ordered sets. We can, therefore, use some techniques developed for well-ordered sets, as transfinite induction. In applications, e.g., in descriptive set theory, important facts can be transformed into the existence (or nonexistence) of an infinite decreasing chain in some specific partially ordered sets, which we call trees. That these two properties are equivalent for any given partially ordered set follows from the axiom of dependent choice (a weakening of the axiom of choice), which says that if A is a nonempty set, R is a binary relation on A with the property that for every element $x \in A$ there is some $y \in A$ such that $R(x, y)$ holds, then there is an infinite sequence $x_0, x_1, \ldots$ of elements of A such that $R(x_0, x_1), R(x_1, x_2), \ldots$ hold.

The axiom of foundation (or regularity) says that if A is a nonempty set, then there is some element x of it with $x \cap A = \emptyset$. This claims that the universe is well founded under $\in$ and that implies that it is possible to create every set from the empty set by iterating the power set operation (cumulative hierarchy).

In this chapter, we assume the axioms of choice and regularity, unless indicated otherwise.

A class is a defined part of the universe which is not necessarily a set. If a class is indeed not a set, then we call it a proper class. An operation is a well-defined mapping on some part of the universe which is possibly not a function, that is, it does not necessarily go between sets.

1. The following statements are equivalent:
   (a) DC, the axiom of dependent choice;
   (b) If the nonempty partially ordered set $\langle P, < \rangle$ has no minimal element, then there is an infinite descending chain in $\langle P, < \rangle$,

(c) A partially ordered set is well founded iff there is no infinite descending chain in it.

2. If $\langle P, < \rangle$ is a partially ordered set, then there is an order-preserving ordinal-valued function f on P, that is, $x < y$ implies $f(x) < f(y)$ if and only if $\langle P, < \rangle$ is well founded.
3. If $\langle P, < \rangle$ is a partially ordered set, then there exists a cofinal subset $Q \subseteq P$ such that $\langle Q, < \rangle$ is well founded.
4. Let $\langle P, < \rangle$ be a partially ordered set that does not include an infinite increasing or decreasing sequence. Is it true that P is the union of countably many antichains (an antichain is a set of pairwise incomparable elements)?
5. If $\langle P, < \rangle$ is a well-founded set, then there is a unique ordinal-valued function r (the rank function of $\langle P, < \rangle$) with the properties

   (a) if $x < y$, then $r(x) < r(y)$,

   (b) if $\alpha = r(x)$ and $\beta < \alpha$, then there exists some $y < x$ with $r(y) = \beta$.

For κ a cardinal let $\mathrm{FS}(\kappa)$ be the set of all finite strings of ordinals less than κ. We think the elements of $\mathrm{FS}(\kappa)$ as finite functions from n to κ for some $n < \omega$ and simply write $s = s(0)s(1)\cdots s(n-1)$ (rather than using e.g., the ordered sequence notation). If $s, t \in \mathrm{FS}(\kappa)$ we set $s < t$ if t properly extends s, and $s \lhd t$ if t is a one-step extension of s. $s\hat{\ }t$ is the *juxtaposition* of s and t; that is, if $s = s(0)s(1)\cdots s(n-1)$ $t = t(0)t(1)\cdots t(m-1)$, then $s\hat{\ }t = s(0)s(1)\cdots s(n-1)t(0)t(1)\cdots t(m-1)$.

For Problems 6–10 we define a set $T \subseteq \mathrm{FS}(\kappa)$ a *tree* if it is closed under restriction, i.e., $s < t \in T$ implies that $s \in T$. The nth level of T is formed by those elements of length n. T is well founded if it does not include an infinite branch, that is, if $(T, >)$ is well founded in the original sense. In this case, let $R(T)$ be the ordinal assigned to the root (the empty sequence) by Problem 5. (Notice that these trees are trees in the sense of Chapter 27, only turned upside down.)

6. If $T \subseteq \mathrm{FS}(\kappa)$ is a well-founded tree, then $R(T) < \kappa^+$. For every ordinal $\alpha < \kappa^+$ there is a well-founded tree $T \subseteq \mathrm{FS}(\kappa)$ with $R(T) = \alpha$.
7. If T, T' are well-founded trees and $R(T) \le R(T')$ then $T \preceq T'$, i.e., there is a level and extension preserving (but not necessaily one–one) map from T into T'.
8. For any two trees, T and T' either $T \preceq T'$ or $T' \preceq T$ holds.
9. Define the Kleene–Brouwer ordering $<_{\mathrm{KB}}$ on $\mathrm{FS}(\kappa)$ as follows. If $s = s(0)s(1)\cdots s(n)$ and $t = t(0)\cdots t(m)$, then $s <_{\mathrm{KB}} t$ if and only if either s properly extends t or $s(i) < t(i)$ holds for the least i where they differ. This is an ordering on $\mathrm{FS}(\kappa)$. A tree $T \subseteq \mathrm{FS}(\kappa)$ is well founded if and only if it is well ordered by $<_{\mathrm{KB}}$.

10. (Galvin's tree game) Two players, W and B, play the following game. They play on the isomorphic well-founded trees, T_W and T_B. At the beginning both players have a pawn at the root of his/her own tree. At every round first W makes a move with either pawn, i.e., moves it to one of the immediate extensions of its current position, then B does the same with one of the pawns. B may pass but W may not. The winner is whose pawn first reaches a leaf (that is, queens).
    (a) One of the players has a winning strategy.
    (b) W has a winning strategy.
11. Exhibit two well-founded sets such that neither has an order-preserving (not necessarily injective) mapping into the other.
    A set (or possibly a class) A is *transitive* if $x \in A$, $y \in x$ imply that $y \in A$.
12. There is no set x with $x \in x$.
13. There are no sets x, y with $x \in y$ and $y \in x$.
14. For every natural number n, there is an n-element set A with the following properties: if $x, y \in A$, then either $x \in y$, or $x = y$, or $y \in x$, and if $x \in A$, $y \in x$, then $y \in A$. For a given n, can there be more than one such sets?
15. What are the transitive singletons?
16. The intersection and union of transitive sets are transitive.
17. Let A be a set. Define $A_0 = \{A\}$, $A_{n+1} = \bigcup A_n$ for $n = 0, 1, \ldots$, $\mathrm{TC}(A) = A_0 \cup A_1 \cup \cdots$ (the transitive closure of A). $\mathrm{TC}(A)$ is transitive and if $A \in B$, B is transitive, then $\mathrm{TC}(A) \subseteq B$.
18. (Cumulative hierarchy) Construct, by transfinite recursion, the following sets. $V_0 = \emptyset$. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$. If α is a limit ordinal, then $V_\alpha = \bigcup\{V_\beta : \beta < \alpha\}$.
    If a set x is an element of some V_α then x is a *ranked set*, and $\mathrm{rk}(x)$ (the rank of x) is the least α with $x \in V_\alpha$.
    (a) Every V_α is a transitive set.
    (b) $V_\beta \subseteq V_\alpha$ holds for $\beta < \alpha$.
    (c) $\mathrm{rk}(x)$ is always a successor ordinal.
    (d) If x is ranked and $y \in x$, then y is also ranked and $\mathrm{rk}(y) < \mathrm{rk}(x)$.
    (e) If every element of x is ranked, then so is x.
    (f) The axiom of foundation holds if and only if every set is ranked.
19. Solve the equation $X \times Y = X$ in sets X, Y.
20. If $\mathcal{C}$ is a proper class, then there is a surjection from $\mathcal{C}$ onto the class of ordinals such that the inverse image of every ordinal is a
    (a) set,
    (b) proper class.

21. Assume that $\mathcal{C}$ is a class, $\sim$ is an equivalence relation on it. Then there is an operation $\mathcal{F}$ defined on $\mathcal{C}$ such that $\mathcal{F}(x) = \mathcal{F}(y)$ holds iff $x \sim y$ is true.
22. The axiom of choice is equivalent to the statement that every set can be embedded into every proper class.
23. The following are equivalent.
    (a) (The axiom of global choice) There is an operation $\mathcal{F}$ defined on all nonempty sets, such that $\mathcal{F}(X) \in X$ holds for every such set X.
    (b) The universe has a well-ordering, that is, a relation $<$ such that every nonempty class has a $<$-least minimal element.
    (c) Moreover, $<$ is set-like, that is, the predecessors of every set form a set.
    (d) If $\mathcal{A}$, $\mathcal{B}$ are proper classes, then there is an injection of $\mathcal{A}$ into $\mathcal{B}$.
    (e) If $\mathcal{A}$, $\mathcal{B}$ are proper classes, then there is a bijection between $\mathcal{A}$ and $\mathcal{B}$.
24. If κ is an infinite cardinal, then $H_\kappa = \{x : |\mathrm{TC}(x)| < \kappa\}$ is a set (here $\mathrm{TC}(x)$ is the transitive closure of x; see Problem 17).
25. (Mostowski's collapsing lemma) Assume that M is a class, E is a binary relation on M which is
    (a) irreflexive, that is, xEx holds for no $x \in M$;
    (b) extensional: if $\{z : zEx\} = \{z : zEy\}$, then $x = y$;
    (c) well founded: there is no infinite E-decreasing chain, i.e., a sequence $\{x_n : n < \omega\}$ with $x_{n+1}Ex_n$ for $n = 0, 1, \ldots$.
    (d) set-like: for every $x \in M$, $\{y : yEx\}$ is a set.

    Then there are a unique transitive class N, and a unique isomorphism $\pi : (M, E) \to (N, \in)$.

Part II

Solutions