CHAPTER 2 DERIVATIVES

2.1 THE DERIVATIVE AS A FUNCTION

1. Step 1: $f(x) = 4 - x^2$ and $f(x+h) = 4 - (x+h)^2$

   Step 2: $\frac{f(x+h) - f(x)}{h} = \frac{[4-(x+h)^2]-(4-x^2)}{h} = \frac{(4-x^2-2xh-h^2)-4+x^2}{h} = \frac{-2xh-h^2}{h} = \frac{h(-2x-h)}{h}$
   $= -2x - h$ if $h \neq 0$

   Step 3: $f'(x) = \lim_{h\to 0} (-2x-h) = -2x$; $f'(-3) = 6$, $f'(0) = 0$

2. Step 1: $g(t) = \frac{1}{t^2}$ and $g(t+h) = \frac{1}{(t+h)^2}$

   Step 2: $\frac{g(t+h)-g(t)}{h} = \frac{\frac{1}{(t+h)^2} - \frac{1}{t^2}}{h} = \frac{\left(\frac{t^2-(t+h)^2}{(t+h)^2 \cdot t^2}\right)}{h} = \frac{t^2-(t^2+2th+h^2)}{(t+h)^2 \cdot t^2 \cdot h} = \frac{-2th-h^2}{(t+h)^2 t^2 h}$
   $= \frac{h(-2t-h)}{(t+h)^2 t^2 h} = \frac{-2t-h}{(t+h)^2 t^2}$ if $h \neq 0$

   Step 3: $g'(t) = \lim_{h\to 0} \frac{-2t-h}{(t+h)^2 t^2} = \frac{-2t}{t^2 \cdot t^2} = \frac{-2}{t^3}$; $g'(-1) = 2$, $g'(2) = -\frac{1}{4}$

3. Step 1: $s(t) = t^3 - t^2$ and $s(t+h) = (t+h)^3 - (t+h)^3$

   Step 2: $\frac{s(t+h)-s(t)}{h} = \frac{[(t+h)^3-(t+h)^2]-(t^3-t^2)}{h}$
   $= \frac{(t^3+3t^2h+3th^2+h^3)+(t^2+2th+h^2)-(t^3-t^2)}{h}$
   $= \frac{h(3t^2+3th+h^2-2t-h)}{h} = 3t^2 - 2t + (3t-1)h + h^2$ if $h \neq 0$

   Step 3: $\frac{ds}{dt} = \lim_{h\to 0} (3t^2-2t+(3t-1)h+h^2) = 3t^2 - 2t$; $\left.\frac{ds}{dt}\right|_{t=-1} = 5$

4. Step 1: $f(x) = x + \frac{9}{x}$ and $f(x+h) = (x+h) + \frac{9}{(x+h)}$

   Step 2: $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)+\frac{9}{(x+h)} - \left(x+\frac{9}{x}\right)}{h} = \frac{\frac{(x+h)^2+9}{(x+h)} - \left(\frac{x^2+9}{x}\right)}{h}$
   $= \frac{x(x^2+2xh+h^2+9)-(x+h)(x^2+9)}{xh(x+h)}$
   $= \frac{(x^3+2x^2h+xh^2+9x)-(x^3+x^2h+9x+9h)}{xh(x+h)} = \frac{h(x^2+xh-9)}{xh(x+h)}$

$$= \frac{x^2 + xh - 9}{x(x+h)} \text{ if } h \neq 0$$

Step 3: $f'(x) = \lim_{h\to 0} \frac{x^2 + xh - 9}{x(x+h)} = \frac{x^2 - 9}{x^2} = 1 - \frac{9}{x^2}$; $f'(-3) = 0$

5. Step 1: $p(\theta) = \sqrt{3\theta}$ and $p(\theta + h) = \sqrt{3(\theta + h)}$

Step 2: $\frac{p(\theta+h) - p(\theta)}{h} = \frac{\sqrt{3(\theta+h)} - \sqrt{3\theta}}{h} = \frac{\left(\sqrt{3\theta + 3h} - \sqrt{3\theta}\right)}{h} \cdot \frac{\left(\sqrt{3\theta + 3h} + \sqrt{3\theta}\right)}{\left(\sqrt{3\theta + 3h} + \sqrt{3\theta}\right)} = \frac{(3\theta + 3h) - 3\theta}{h\left(\sqrt{3\theta + 3h} + \sqrt{3\theta}\right)}$

$= \frac{3h}{h\left(\sqrt{3\theta + 3h} + \sqrt{3\theta}\right)} = \frac{3}{\sqrt{3\theta + 3h} + \sqrt{3\theta}}$

Step 3: $p'(\theta) = \lim_{h\to 0} \frac{3}{\sqrt{3\theta + 3h} + \sqrt{3\theta}} = \frac{3}{\sqrt{3\theta} + \sqrt{3\theta}} = \frac{3}{2\sqrt{3\theta}}$; $p'(0.25) = \sqrt{3}$

6. $r = f(\theta) = \frac{2}{\sqrt{4-\theta}}$ and $f(\theta + h) = \frac{2}{\sqrt{4 - (\theta + h)}} \Rightarrow \frac{dr}{d\theta} = \lim_{h\to 0} \frac{f(\theta+h) - f(\theta)}{h} = \lim_{h\to 0} \frac{\frac{2}{\sqrt{4-\theta-h}} - \frac{2}{\sqrt{4-\theta}}}{h}$

$= \lim_{h\to 0} \frac{2\sqrt{4-\theta} - 2\sqrt{4-\theta-h}}{h\sqrt{4-\theta}\sqrt{4-\theta-h}} = \lim_{h\to 0} \frac{2\sqrt{4-\theta} - 2\sqrt{4-\theta-h}}{h\sqrt{4-\theta}\sqrt{4-\theta-h}} \cdot \frac{\left(2\sqrt{4-\theta} + 2\sqrt{4-\theta-h}\right)}{\left(2\sqrt{4-\theta} + 2\sqrt{4-\theta-h}\right)}$

$= \lim_{h\to 0} \frac{4(4-\theta) - 4(4-\theta-h)}{2h\sqrt{4-\theta}\sqrt{4-\theta-h}\left(\sqrt{4-\theta} + \sqrt{4-\theta-h}\right)} = \lim_{h\to 0} \frac{2}{\sqrt{4-\theta}\sqrt{4-\theta-h}\left(\sqrt{4-\theta} + \sqrt{4-\theta-h}\right)}$

$= \frac{2}{(4-\theta)\left(2\sqrt{4-\theta}\right)} = \frac{1}{(4-\theta)\sqrt{4-\theta}} \Rightarrow \left.\frac{dr}{d\theta}\right|_{\theta=0} = \frac{1}{8}$

7. $y = x^2 + x + 8 \Rightarrow \frac{dy}{dx} = 2x + 1 + 0 = 2x + 1 \Rightarrow \frac{d^2y}{dx^2} = 2$

8. $s = 5t^3 - 3t^5 \Rightarrow \frac{ds}{dt} = \frac{d}{dt}(5t^3) - \frac{d}{dt}(3t^5) = 15t^2 - 15t^4 \Rightarrow \frac{d^2s}{dt^2} = \frac{d}{dt}(15t^2) - \frac{d}{dt}(15t^4) = 30t - 60t^3$

9. $y = \frac{4x^3}{3} - 4 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{4}{3}x^3\right) - \frac{d}{dx}(4) = 4x^2 \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(4x^2) = 8x$

10. $y = \frac{x^3 + 7}{x} = x^2 + 7x^{-1} \Rightarrow \frac{dy}{dx} = 2x - 7x^{-2} \Rightarrow \frac{d^2y}{dx^2} = 2 + 14x^{-3}$

11. $y = \frac{1}{2}x^4 - \frac{3}{2}x^2 - x \Rightarrow y' = 2x^3 - 3x - 1 \Rightarrow y'' = 6x^2 - 3 \Rightarrow y''' = 12x \Rightarrow y^{(4)} = 12 \Rightarrow y^{(n)} = 0$ for all $n \geq 5$

12. $y = \frac{1}{120}x^5 \Rightarrow y' = \frac{1}{24}x^4 \Rightarrow y'' = \frac{1}{6}x^3 \Rightarrow y''' = \frac{1}{2}x^2 \Rightarrow y^{(4)} = x \Rightarrow y^{(5)} = 1 \Rightarrow y^{(n)} = 0$ for all $n \geq 6$

13. (a) $\frac{dy}{dx} = 3x^2 - 4 \Rightarrow m = \left.\frac{dy}{dx}\right|_{x=2} = 3(2)^2 - 4 = 8$

Therefore, the equation of the line tangent to the curve at the point $(2, 1)$ is $y - 1 = 8(x - 2)$ or $y = 8x - 15$.

(b) Since $x^2 \geq 0$ for all real values of x, it follows that $3x^2 \geq 0$ and $3x^2 - 4 \geq -4$. In addition, $3x^2 - 4 \to +\infty$ as $x \to \pm\infty$. Therefore, the range of values of the curve's slope is $[-4, \infty)$. The graph of the derivative is a parabola that opens upward and its vertex is at the point $(0, -4)$.

(c) The equation of one such tangent line is found in part (a) when $x = 2$. Also, $\frac{dy}{dx} = 8 \Rightarrow 3x^2 - 4 = 8$ $\Rightarrow x^2 = 4 \Rightarrow x = 2$ or $x = -2$. At $x = -2$, $y = (-2)^3 - 4(-2) + 1 = 1$. Therefore, the equation of the line tangent to the curve at the point $(-2, 1)$ is $y - 1 = 8(x - (-2))$ or $y = 8x + 17$.

14. (a) Set $\frac{dy}{dx} = 0$ and solve for x: $\frac{dy}{dx} = 1 - \frac{3}{2\sqrt{x}} = 0 \Rightarrow \sqrt{x} = \frac{3}{2} \Rightarrow x = \frac{9}{4}$. At $x = \frac{9}{4}$, the curve has value $y = \frac{9}{4} - 3\sqrt{\frac{9}{4}} = \frac{9}{4} - 3\left(\frac{3}{2}\right) = -\frac{9}{4}$. Therefore, an equation for the horizontal tangent to the curve at the point $\left(\frac{9}{4}, -\frac{9}{4}\right)$ is $y = -\frac{9}{4}$.

(b) The domain of the function $y = x - 3\sqrt{x}$ is $[0, \infty)$. The derivative, however, is undefined at $x = 0$. Therefore, to determine the range of values for the curve's slopes, consider $0 < x < \infty$. As $x \to \infty$, $\frac{dy}{dx} = 1 - \frac{3}{2\sqrt{x}} \to 1$ and, as $x \downarrow 0$, $\frac{dy}{dx} = 1 - \frac{3}{2\sqrt{x}} \to -\infty$. For all values of x between 0 and ∞, the function $\frac{dy}{dx} = 1 - \frac{3}{2\sqrt{x}}$ is increasing toward 1 as x increases. Therefore, the curve's slopes range from $-\infty$ near $x = 0$, to 1 but never reaching 1, as $x \to \infty$. That is, $-\infty < \frac{dy}{dx} < 1$ for $0 < x < \infty$.

15. Note that as x increases, the slope of the tangent line to the curve is first negative, then zero (when $x = 0$), then positive $\Rightarrow$ the slope is always increasing which matches (b).

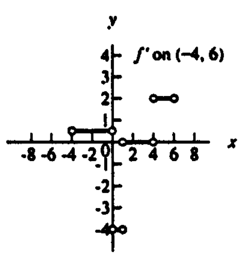
16. Note that the slope of the tangent line is never negative. For x negative, $f_2'(x)$ is positive but decreasing as x increases. When $x = 0$, the slope of the tangent line to x is 0. For $x > 0$, $f_2'(x)$ is positive and increasing. This graph matches (a).

17. $f_3(x)$ is an oscillating function like the cosine. Everywhere that the graph of f_3 has a horizontal tangent we expect f_3' to be zero, and (d) matches this condition.

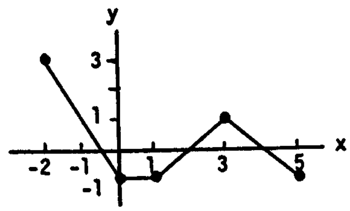
18. The graph matches with (c).

19. (a) f' is not defined at $x = 0, 1, 4$. At these points, the left-hand and right-hand derivatives do not agree. For example, $\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$ = slope of line joining $(-4, 0)$ and $(0, 2) = \frac{1}{2}$ but $\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$ = slope of line joining $(0, 2)$ and $(1, -2) = -4$. Since these values are not equal, $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist.

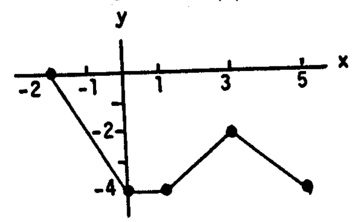
(b)



20. (a)



(b) Shift the graph in (a) down 3 units



21. Left-hand derivative: For $h < 0$, $f(0+h) = f(h) = h^2$ (using $y = x^2$ curve) $\Rightarrow \lim\limits_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$

$= \lim\limits_{h \to 0^-} \frac{h^2 - 0}{h} = \lim\limits_{h \to 0^-} h = 0;$

Right-hand derivative: For $h > 0$, $f(0+h) = f(h) = h$ (using $y = x$ curve) $\Rightarrow \lim\limits_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$

$= \lim\limits_{h \to 0^+} \frac{h - 0}{h} = \lim\limits_{h \to 0^+} 1 = 1;$

Then $\lim\limits_{h \to 0^-} \frac{f(0+h) - f(0)}{h} \neq \lim\limits_{h \to 0^+} \frac{f(0+h) - f(0)}{h} \Rightarrow$ the derivative $f'(0)$ does not exist.

22. Left-hand derivative: When $h < 0$, $1 + h < 1 \Rightarrow f(1+h) = \sqrt{1+h} \Rightarrow \lim\limits_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$

$= \lim\limits_{h \to 0^-} \frac{\sqrt{1+h} - 1}{h} = \lim\limits_{h \to 0^-} \frac{\left(\sqrt{1+h} - 1\right)}{h} \cdot \frac{\left(\sqrt{1+h} + 1\right)}{\left(\sqrt{1+h} + 1\right)} = \lim\limits_{h \to 0^-} \frac{(1+h) - 1}{h\left(\sqrt{1+h} + 1\right)} = \lim\limits_{h \to 0^-} \frac{1}{\sqrt{1+h} + 1} = \frac{1}{2};$

Right-hand derivative: When $h > 0$, $1 + h > 1 \Rightarrow f(1+h) = 2(1+h) - 1 = 2h + 1 \Rightarrow \lim\limits_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$

$= \lim\limits_{h \to 0^+} \frac{(2h+1) - 1}{h} = \lim\limits_{h \to 0^+} 2 = 2;$

Then $\lim\limits_{h \to 0^-} \frac{f(1+h) - f(1)}{h} \neq \lim\limits_{h \to 0^+} \frac{f(1+h) - f(1)}{h} \Rightarrow$ the derivative $f'(1)$ does not exist.

23. (a) The function is differentiable on its domain $-2 \le x \le 3$ (it is smooth)
(b) none
(c) none

24. (a) f is differentiable on $-2 \le x < -1$, $-1 < x < 0$, $0 < x < 2$, and $2 < x \le 3$
(b) f is continuous but not differentiable at $x = -1$: $\lim\limits_{x \to -1} f(x) = 0$ exists but there is a corner at $x = -1$ since

$\lim\limits_{h \to 0^-} \frac{f(-1+h) - f(-1)}{h} = -3$ and $\lim\limits_{h \to 0^+} \frac{f(-1+h) - f(-1)}{h} = 3 \Rightarrow f'(-1)$ does not exist

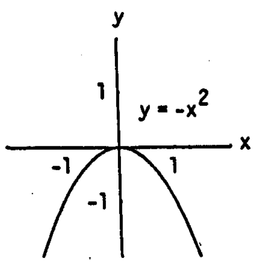
(c) f is neither continuous nor differentiable at $x = 0$ and $x = 2$:

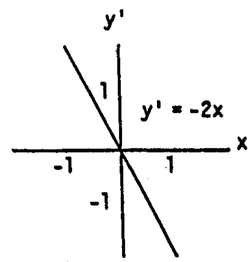
at $x = 0$, $\lim\limits_{x \to 0^-} f(x) = 3$ but $\lim\limits_{x \to 0^+} f(x) = 0 \Rightarrow \lim\limits_{x \to 0} f(x)$ does not exist;

at $x = 2$, $\lim\limits_{x \to 2} f(x)$ exists but $\lim\limits_{x \to 2} f(x) \neq f(2)$

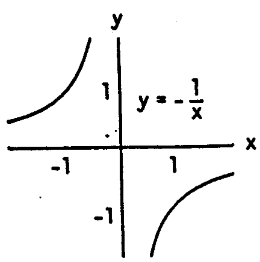
25. (a) f is differentiable on $-1 \le x < 0$ and $0 < x \le 2$
(b) f is continuous but not differentiable at $x = 0$: $\lim_{x \to 0} f(x) = 0$ exists but there is a cusp at $x = 0$, so
$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist
(c) none

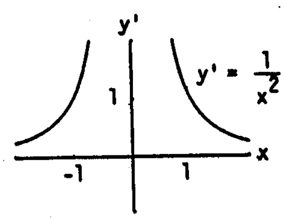
26. (a) f is differentiable on $-3 \le x < -2$, $-2 < x < 2$, and $2 < x \le 3$
(b) f is continuous but not differentiable at $x = -2$ and $x = 2$: there are corners at those points
(c) none

27. (a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-(x+h)^2 - (-x^2)}{h} = \lim_{h \to 0} \frac{-x^2 - 2xh - h^2 + x^2}{h} = \lim_{h \to 0} (-2x - h) = -2x$
(b)




(c) $y' = -2x$ is positive for $x < 0$, y' is zero when $x = 0$, y' is negative when $x > 0$
(d) $y = -x^2$ is increasing for $-\infty < x < 0$ and decreasing for $0 < x < \infty$; the function is increasing on intervals where $y' > 0$ and decreasing on intervals where $y' < 0$

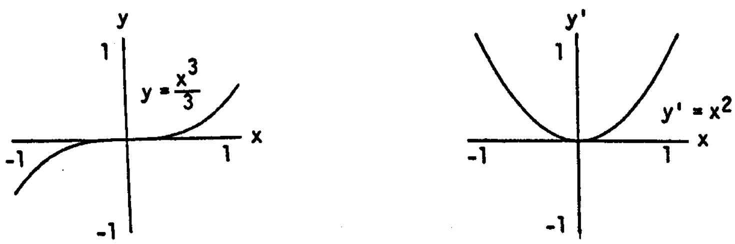
28. (a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left(\frac{-1}{x+h} - \frac{-1}{x}\right)}{h} = \lim_{h \to 0} \frac{-x + (x+h)}{x(x+h)h} = \lim_{h \to 0} \frac{1}{x(x+h)} = \frac{1}{x^2}$
(b)




(c) y' is positive for all $x \ne 0$, y' is never 0, y' is never negative
(d) $y = -\frac{1}{x}$ is increasing for $-\infty < x < 0$ and $0 < x < \infty$

29. (a) Using the alternate formula for calculating derivatives: $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{\left(\frac{x^3}{3} - \frac{c^3}{3}\right)}{x - c}$
$= \lim_{x \to c} \frac{x^3 - c^3}{3(x-c)} = \lim_{x \to c} \frac{(x-c)(x^2 + xc + c^2)}{3(x-c)} = \lim_{x \to c} \frac{x^2 + xc + c^2}{3} = c^2 \Rightarrow f'(x) = x^2$

(b)



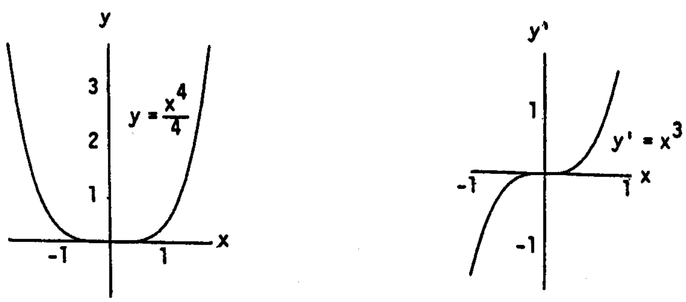
(c) y' is positive for all $x \neq 0$, and $y' = 0$ when $x = 0$; y' is never negative

(d) $y = \frac{x^3}{3}$ is increasing for all $x \neq 0$ (the graph is horizontal at $x = 0$) because y is increasing where $y' > 0$; y is never decreasing

30. (a) Using the alternate form for calculating derivatives: $f'(c) = \lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim\limits_{x \to c} \frac{\left(\frac{x^4}{4} - \frac{c^4}{4}\right)}{x - c}$

$$= \lim_{x \to c} \frac{x^4 - c^4}{4(x - c)} = \lim_{x \to c} \frac{(x - c)\left(x^3 + cx^2 + c^2x + c^3\right)}{4(x - c)} = \lim_{x \to c} \frac{x^3 + cx^2 + c^2x + c^3}{4} = c^3 \Rightarrow f'(x) = x^3$$

(b)



(c) y' is positive for $x > 0$, y' is zero for $x = 0$, y' is negative for $x < 0$

(d) $y = \frac{x^4}{4}$ is increasing on $0 < x < \infty$ and decreasing on $-\infty < x < 0$

31. $y' = \lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim\limits_{x \to c} \frac{x^3 - c^3}{x - c} = \lim\limits_{x \to c} \frac{(x - c)\left(x^2 + xc + c^2\right)}{x - c} = \lim\limits_{x \to c} \left(x^2 + xc + c^2\right) = 3c^2.$

The slope of the curve $y = x^3$ at $x = c$ is $y' = 3c^2$. Notice that $3c^2 \geq 0$ for all $c \Rightarrow y = x^3$ never has a negative slope.

32. Horizontal tangents occur where $y' = 0$. Thus, $y' = \lim\limits_{h \to 0} \frac{2\sqrt{x + h} - 2\sqrt{x}}{h}$

$$= \lim_{h \to 0} \frac{2\left(\sqrt{x + h} - \sqrt{x}\right)}{h} \cdot \frac{\left(\sqrt{x + h} + \sqrt{x}\right)}{\left(\sqrt{x + h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{2((x + h) - x))}{h\left(\sqrt{x + h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{2}{\sqrt{x + h} + \sqrt{x}} = \frac{1}{\sqrt{x}}.$$

Then $y' = 0$ when $\frac{1}{\sqrt{x}} = 0$ which is never true $\Rightarrow$ the curve has no horizontal tangents.

33. $y' = \lim_{h\to 0} \frac{(2(x+h)^2 - 13(x+h) + 5) - (2x^2 - 13x + 5)}{h} = \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - 13x - 13h + 5 - 2x^2 + 13x - 5}{h}$

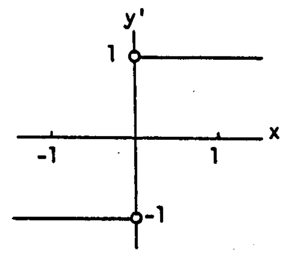
$= \lim_{h\to 0} \frac{4xh + 2h^2 - 13h}{h} = \lim_{h\to 0} (4x + 2h - 13) = 4x - 13$, slope at x. The slope is -1 when $4x - 13 = -1$ $\Rightarrow 4x = 12 \Rightarrow x = 3 \Rightarrow y = 2 \cdot 3^2 - 13 \cdot 3 + 5 = -16$. Thus the tangent line is $y + 16 = (-1)(x - 3)$ and the point of tangency is $(3, -16)$.

34. For the curve $y = \sqrt{x}$, we have $y' = \lim_{h\to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)}{h} \cdot \frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h\to 0} \frac{(x+h) - x}{\left(\sqrt{x+h} + \sqrt{x}\right)h}$

$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$. Suppose $(a, \sqrt{a})$ is the point of tangency of such a line and $(-1, 0)$ is the point on the line where it crosses the x-axis. Then the slope of the line is $\frac{\sqrt{a} - 0}{a - (-1)} = \frac{\sqrt{a}}{a+1}$ which must also equal $\frac{1}{2\sqrt{a}}$; using the derivative formula at $x = a \Rightarrow \frac{\sqrt{a}}{a+1} = \frac{1}{2\sqrt{a}} \Rightarrow 2a = a + 1 \Rightarrow a = 1$. Thus such a line does exist: its point of tangency is $(1, 1)$, its slope is $\frac{1}{2\sqrt{a}} = \frac{1}{2}$; and an equation of the line is $y - 1 = \frac{1}{2}(x - 1)$.

35. No. Derivatives of functions have the intermediate value property. The function $f(x) = \text{int } x$ satisfies $f(0) = 0$ and $f(1) = 1$ but does not take on the value $\frac{1}{2}$ anywhere in $[0, 1] \Rightarrow$ f does not have the intermediate value property. Thus f cannot be the derivative of any function on $[0, 1] \Rightarrow$ f cannot be the derivative of any function on $(-\infty, \infty)$.

36. The graphs are the same. So we know that for $f(x) = |x|$, we have $f'(x) = \frac{|x|}{x}$.



37. Yes; the derivative of $-f$ is $-f'$ so that $f'(x_0)$ exists $\Rightarrow -f'(x_0)$ exists as well.

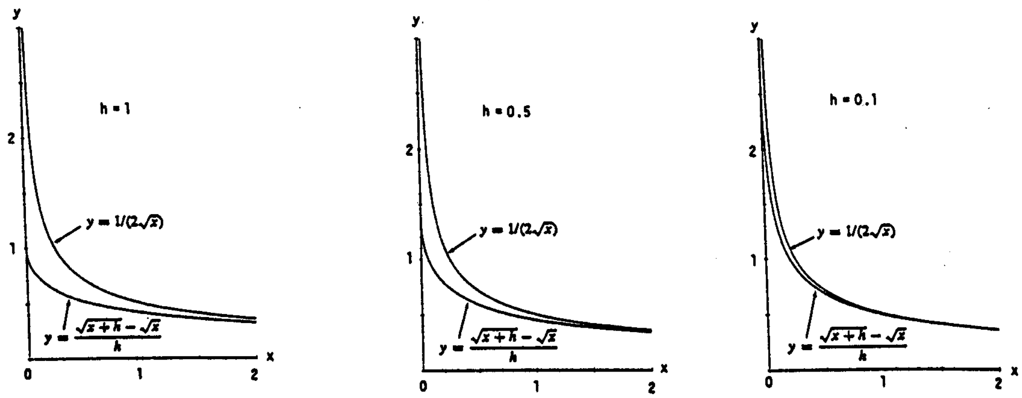
38. Yes; the derivative of 3g is $3g'$ so that $g'(7)$ exists $\Rightarrow 3g'(7)$ exists as well.

39. Yes, $\lim_{t\to 0} \frac{g(t)}{h(t)}$ can exist but it need not equal zero. For example, let $g(t) = mt$ and $h(t) = t$. Then $g(0) = h(0) = 0$, but $\lim_{t\to 0} \frac{g(t)}{h(t)} = \lim_{t\to 0} \frac{mt}{t} = \lim_{t\to 0} m = m$, which need not be zero.

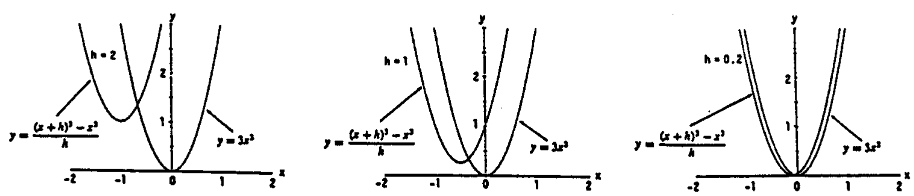
40. (a) Suppose $|f(x)| \le x^2$ for $-1 \le x \le 1$. Then $|f(0)| \le 0^2 \Rightarrow f(0) = 0$. Then $f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$ $= \lim_{h\to 0} \frac{f(h) - 0}{h} = \lim_{h\to 0} \frac{f(h)}{h}$. For $|h| \le 1$, $-h^2 \le f(h) \le h^2 \Rightarrow -h \le \frac{f(h)}{h} \le h \Rightarrow f'(0) = \lim_{h\to 0} \frac{f(h)}{h} = 0$ by the Sandwich Theorem for limits.

(b) Note that for $x \neq 0$, $|f(x)| = \left|x^2 \sin \frac{1}{x}\right| = |x^2||\sin x| \leq |x^2| \cdot 1 = x^2$ (since $-1 \leq \sin x \leq 1$). By part (a), f is differentiable at $x = 0$ and $f'(0) = 0$.

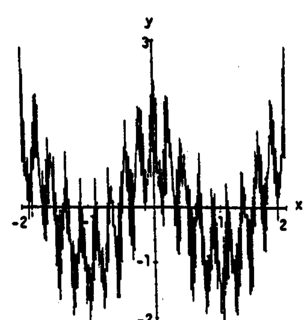
41. The graphs are shown below for h = 1, 0.5, 0.1. The function $y = \frac{1}{2\sqrt{x}}$ is the derivative of the function $y = \sqrt{x}$ so that $\frac{1}{2\sqrt{x}} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$. The graphs reveal that $y = \frac{\sqrt{x+h} - \sqrt{x}}{h}$ gets closer to $y = \frac{1}{2\sqrt{x}}$ as h gets smaller and smaller.



42. The graphs are shown below for h = 2, 1, 0.2. The function $y = 3x^2$ is the derivative of the function $y = x^3$ so that $3x^2 = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$. The graphs reveal that $y = \frac{(x+h)^3 - x^3}{h}$ gets closer to $y = 3x^2$ as h gets smaller and smaller.

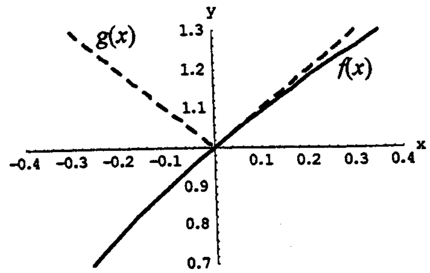


43. Weierstrass's nowhere differentiable continuous function.



$$g(x) = \cos(\pi x) + \left(\frac{2}{3}\right)^1 \cos(9\pi x) + \left(\frac{2}{3}\right)^2 \cos(9^2\pi x) + \left(\frac{2}{3}\right)^3 \cos(9^3\pi x) + \cdots + \left(\frac{2}{3}\right)^7 \cos(9^7\pi x)$$

44.



The function f(x) is differentiable at (0, 1) because the graph of f(x) is smooth at the point (0, 1). Tracing along the graph of f(x), from left to right, the value of the function continually increases through the point (0, 1) with no sudden change in the rate of increase. The function g(x) is not differentiable at (0, 1) because the graph of g(x) has a sharp corner there. Tracing along the graph of g(x), from left to right, there is an abrupt change at the point (0, 1). To the left of the point the values of g(x) decrease at a constant rate and to the right the values increase at a constant rate. There is no derivative at x = 0 because

$\lim_{h\to 0^-} \frac{(|0+h|+1)-(|0|+1)}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$ and $\lim_{h\to 0^+} \frac{(|0+h|+1)-(|0|+1)}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$. Consequently, $g'(0) = \lim_{h\to 0} \frac{g(0+h)-g(0)}{h}$ does not exist since the right- and left-hand limits are not equal.

45-50. Example CAS commands:

Maple:

```
f:=x -> x ∧ 2*cos(x);
q:=h -> (f(x + h) - f(x))/h;
slope:=limit(q(h),h=0);
fp:=unapply(%,x);
x0:=Pi/4;
L:=x -> f(x0) + fp(x0)*(x - x0);
plot({f(x),L(x)},x=x0 - 2..x0 + 1);
```

Mathematica:

```
Clear [f,m,x,y]
x0 = Pi/4; f[x_] = x ∧ 2 Cos[x]
Plot[ f[x], {x,x0 - 3,x0 + 3} ]
q[x_,h_] = (f[x + h] - f[x])/h
m[x_] = Limit[ q[x,h], h -> 0 ]
y = f[x0] + m[x0] (x - x0)
Plot[ {f[x],y}, {x,x0 - 3,x0 + 3} ]
m[x0 - 1]//N
m[x0 + 1]//N
Plot[ {f[x],m[x]}, {x,x0 - 3,x0 + 3} ]
```

In Exercise 63, you could define

x0 = 1; f[x_] = x ∧ (1/3) + x ∧ (2/3)

However, Mathematica 4.0 uses a complex branch for odd roots of negative numbers (as does Maple 6), so the above will only work for positive x. To get the real roots for all x, you could force it as below, but this form is not good for taking derivatives:

x0 = 1; f[x_] = Sign[x] Abs[x] ∧ (1/3) + Abs[x] ∧ (2/3)

2.2 THE DERIVATIVE AS A RATE OF CHANGE

1. $s = t^2 - 3t + 2,\ 0 \le t \le 2$

   (a) displacement $= \Delta s = s(2) - s(0) = -2\text{m},\ v_{av} = \frac{\Delta s}{\Delta t} = \frac{-2\text{ m}}{2\text{ sec}} = -1$ m/sec

   (b) $v = \frac{ds}{dt} = 2t - 3,\ |v(0)| = |-3| = 3$ m/sec, $|v(2)| = |1| = 1$ m/sec; $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 2$,

   $a(0) = a(2) = 2\text{ m/sec}^2$

   (c) $v = 0 \Rightarrow 2t - 3 = 0 \Rightarrow t = \frac{3}{2}$ sec. For $0 \le t < \frac{3}{2}$, v is negative and s is decreasing, whereas for $\frac{3}{2} < t \le 2$, v is positive and s is increasing. Therefore, the body changes direction at $t = \frac{3}{2}$.

2. $s = 6t - t^2,\ 0 \le t \le 6$

   (a) displacement $= \Delta s = s(6) - s(0) = 0 - 0 = 0,\ v_{av} = \frac{\Delta s}{\Delta t} = \frac{0\text{ m}}{6\text{ sec}} = 0$ m/sec

   (b) $v = \frac{ds}{dt} = 6 - 2t,\ |v(0)| = |6| = 6$ m/sec, $|v(6)| = |-6| = 6$ m/sec; $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -2$,

   $a(0) = a(6) = -2\text{ m/sec}^2$

   (c) $v = 0 \Rightarrow 6 - 2t = 0 \Rightarrow t = 3$ sec. For $0 \le t < 3$, v is positive and s is increasing, whereas for $3 < t \le 6$, v is negative and s is decreasing. Therefore, the body changes direction at $t = 3$.

3. $s = -t^3 + 3t^2 - 3t,\ 0 \le t \le 3$

   (a) displacement $= \Delta s = s(3) - s(0) = -9$ m, $v_{av} = \frac{\Delta s}{\Delta t} = \frac{-9}{3} = -3$ m/sec

   (b) $v = \frac{ds}{dt} = -3t^2 + 6t - 3 \Rightarrow |v(0)| = |-3| = 3$ m/sec and $|v(3)| = |-12| = 12$ m/sec; $a = \frac{d^2s}{dt^2} = -6t + 6$

   $\Rightarrow a(0) = 6\text{ m/sec}^2$ and $a(3) = -12\text{ m/sec}^2$

   (c) $v = 0 \Rightarrow -3t^2 + 6t - 3 = 0 \Rightarrow t^2 - 2t + 1 = 0 \Rightarrow (t-1)^2 = 0 \Rightarrow t = 1$. For all other values of t in the interval the velocity v is negative (the graph of $v = -3t^2 + 6t - 3$ is a parabola with vertex at $t = 1$ which opens downward $\Rightarrow$ the body never changes direction.

4. $s = \frac{t^4}{4} - t^3 + t^2,\ 0 \le t \le 2$

   (a) $\Delta s = s(2) - s(0) = 0$ m, $v_{av} = \frac{\Delta s}{\Delta t} = 0$ m/sec

   (b) $v = t^3 - 3t^2 + 2t \Rightarrow |v(0)| = 0$ m/sec and $|v(2)| = 0$ m/sec; $a = 3t^2 - 6t + 2 \Rightarrow a(0) = 2\text{ m/sec}^2$ and $a(2) = 2\text{ m/sec}^2$

(c) $v = 0 \Rightarrow t^3 - 3t^2 + 2t = 0 \Rightarrow t(t-2)(t-1) = 0 \Rightarrow t = 0, 1, 2 \Rightarrow v = t(t-2)(t-1)$ is positive in the interval for $0 < t < 1$ and v is negative for $1 < t < 2 \Rightarrow$ the body changes direction at $t = 1$.

5. $s = t^3 - 6t^2 + 9t$ and let the positive direction be to the right on the s-axis.
   (a) $v = 3t^2 - 12t + 9$ so that $v = 0 \Rightarrow t^2 - 4t + 3 = (t-3)(t-1) = 0 \Rightarrow t = 1$ or 3; $a = 6t - 12 \Rightarrow a(1) = -6$ m/sec^2 and $a(3) = 6$ m/sec^2. Thus the body is motionless but being accelerated left when $t = 1$, and motionless but being accelerated right when $t = 3$.
   (b) $a = 0 \Rightarrow 6t - 12 = 0 \Rightarrow t = 2$ with speed $|v(2)| = |12 - 24 + 9| = 3$ m/sec
   (c) The body moves to the right or forward on $0 \le t < 1$, and to the left or backward on $1 < t < 2$. The positions are $s(0) = 0$, $s(1) = 4$ and $s(2) = 2 \Rightarrow$ total distance $= |s(1) - s(0)| + |s(2) - s(1)| = |4| + |-2| = 6$ m.

6. $v = t^2 - 4t + 3 \Rightarrow a = 2t - 4$
   (a) $v = 0 \Rightarrow t^2 - 4t + 3 = 0 \Rightarrow t = 1$ or $3 \Rightarrow a(1) = -2$ m/sec^2 and $a(3) = 2$ m/sec^2
   (b) $v > 0 \Rightarrow (t-3)(t-1) > 0 \Rightarrow 0 < t < 1$ or $t > 3$ and the body is moving forward; $v < 0 \Rightarrow (t-3)(t-1) < 0 \Rightarrow 1 < t < 3$ and the body is moving backward
   (c) velocity increasing $\Rightarrow a > 0 \Rightarrow 2t - 4 > 0 \Rightarrow t > 2$; velocity decreasing $\Rightarrow a < 0 \Rightarrow 2t - 4 < 0 \Rightarrow t < 2$

7. $s_m = 1.86t^2 \Rightarrow v_m = 3.72t$ and solving $3.72t = 27.8 \Rightarrow t \approx 7.5$ sec on Mars; $s_j = 11.44t^2 \Rightarrow v_j = 22.88t$ and solving $22.88t = 27.8 \Rightarrow t \approx 1.2$ sec on Jupiter.

8. (a) $v(t) = s'(t) = 24 - 1.6t$ m/sec, and $a(t) = v'(t) = s''(t) = -1.6$ m/sec^2
   (b) Solve $v(t) = 0 \Rightarrow 24 - 1.6t = 0 \Rightarrow t = 15$ sec
   (c) $s(15) = 24(15) - .8(15)^2 = 180$ m
   (d) Solve $s(t) = 90 \Rightarrow 24t - .8t^2 = 90 \Rightarrow t = \frac{30 \pm 15\sqrt{2}}{2} \approx 4.39$ sec going up and 25.6 sec going down
   (e) Twice the time it took to reach its highest point or 30 sec

9. $s = 15t - \frac{1}{2}g_s t^2 \Rightarrow v = 15 - g_s t$ so that $v = 0 \Rightarrow 15 - g_s t = 0 \Rightarrow t = \frac{15}{g_s}$. Therefore $\frac{15}{g_s} = 20 \Rightarrow g_s = \frac{3}{4} = 0.75$ m/sec^2

10. Solving $s_m = 832t - 2.6t^2 = 0 \Rightarrow t(832 - 2.6t) = 0 \Rightarrow t = 0$ or $320 \Rightarrow 320$ sec on the moon; solving $s_e = 832t - 16t^2 = 0 \Rightarrow t(832 - 16t) = 0 \Rightarrow t = 0$ or $52 \Rightarrow 52$ sec on the earth. Also, $v_m = 832 - 5.2t = 0 \Rightarrow t = 160$ and $s_m(160) \approx 66{,}560$ ft, the height it reaches above the moon's surface; $v_e = 832 - 32t = 0 \Rightarrow t = 26$ and $s_e(26) \approx 10{,}816$ ft, the height it reaches above the earth's surface.

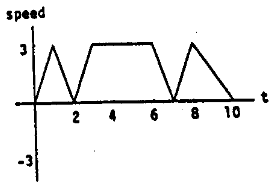
11. (a) $s = 179 - 16t^2 \Rightarrow v = -32t \Rightarrow$ speed $= |v| = 32t$ ft/sec and $a = -32$ ft/sec^2
    (b) $s = 0 \Rightarrow 179 - 16t^2 = 0 \Rightarrow t = \sqrt{\frac{179}{16}} \approx 3.3$ sec
    (c) When $t = \sqrt{\frac{179}{16}}$, $v = -32\sqrt{\frac{179}{16}} = -8\sqrt{179} \approx -107.0$ ft/sec

12. (a) $\lim_{\theta \to \frac{\pi}{2}} v = \lim_{\theta \to \frac{\pi}{2}} 9.8(\sin\theta)t = 9.8t$ so we expect $v = 9.8t$ m/sec in free fall
    (b) $a = \frac{dv}{dt} = 9.8$ m/sec^2

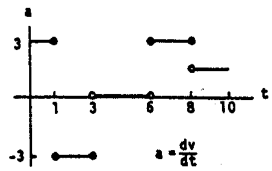
13. (a) at 2 and 7 seconds

(b) between 3 and 6 seconds: $3 \le t \le 6$

(c)

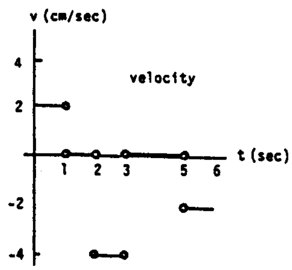


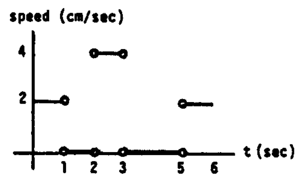
(d)



14. (a) P is moving to the left when $2 < t < 3$ or $5 < t < 6$; P is moving to the right when $0 < t < 1$; P is standing still when $1 < t < 2$ or $3 < t < 5$

(b)





15. (a) 190 ft/sec

(b) 2 sec

(c) at 8 sec, 0 ft/sec

(d) 10.8 sec, 90 ft/sec

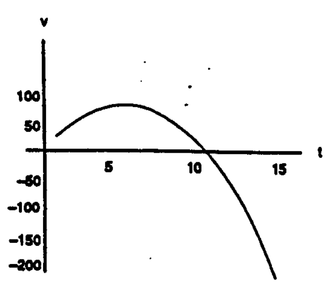
(e) From $t = 8$ until $t = 10.8$ sec, a total of 2.8 sec

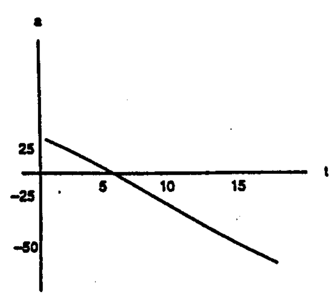
(f) Greatest acceleration happens 2 sec after launch

(g) From $t = 2$ to $t = 10.8$ sec; during this period, $a = \frac{v(10.8) - v(2)}{10.8 - 2} \approx -32$ ft/sec^2

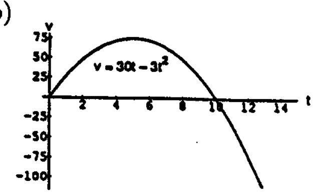
16. Answers will vary.

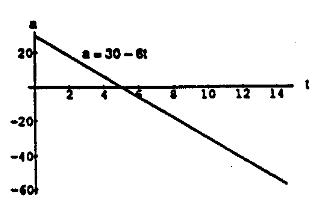
(a)





(b)





17. $s = 490t^2 \Rightarrow v = 980t \Rightarrow a = 980$

(a) Solving $160 = 490t^2 \Rightarrow t = \frac{4}{7}$ sec. The average velocity was $\frac{s(4/7) - s(0)}{4/7} = 280$ cm/sec.

(b) At the 160 cm mark the balls are falling at $v(4/7) = 560$ cm/sec. The acceleration at the 160 cm mark was 980 cm/sec^2.

(c) The light was flashing at a rate of $\frac{17}{4/7} = 29.75$ flashes per second.

18. C = position, A = velocity, and B = acceleration. Neither A nor C can be the derivative of B because B's derivative is constant. Graph C cannot be the derivative of A either, because A has some negative slopes while C has only positive values. So, C, being the derivative of neither A nor B must be the graph of position. Curve C has both positive and negative slopes, so its derivative, the velocity, must be A and not B. That leaves B for acceleration.

19. C = position, B = velocity, and A = acceleration. Curve C cannot be the derivative of either A or B because C has only negative values while both A and B have some positive slopes. So, C represents position. Curve C has no positive slopes, so its derivative, the velocity, must be B. That leaves A for acceleration. Indeed, A is negative where B has negative slopes and positive where B has positive slopes.

20. (a) $c(100) = 11{,}000 \Rightarrow c_{av} = \frac{11{,}000}{100} = \$110$; $c(x) = 2000 + 100x - .1x^2 \Rightarrow c'(x) = 100 - .2x$

(b) Marginal cost $= c'(x) \Rightarrow$ the marginal cost of producing 100 machines is $c'(100) = \$80$

(c) The cost of producing the 101st machine is $c(101) - c(100) = 100 - \frac{201}{10} = \$79.90$

21. (a) $r(x) = 20{,}000\left(1 - \frac{1}{x}\right) \Rightarrow r'(x) = \frac{20{,}000}{x^2} \Rightarrow r'(100) = \$2/\text{machine}$

(b) $\Delta r \approx r'(100) = \$2$

(c) $\lim\limits_{x\to\infty} r'(x) = \lim\limits_{x\to\infty} \frac{20{,}000}{x^2} = 0$. The increase in revenue as the number of items increases without bound will approach zero.

22. $b(t) = 10^6 + 10^4t - 10^3t^2 \Rightarrow b'(t) = 10^4 - (2)(10^3t) = 10^3(10 - 2t)$

(a) $b'(0) = 10^4$ bacteria/hr

(b) $b'(5) = 0$ bacteria/hr

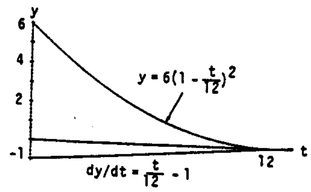
(c) $b'(10) = -10^4$ bacteria/hr

23. $Q(t) = 200(30 - t)^2 = 200(900 - 60t + t^2) \Rightarrow Q'(t) = 200(-60 + 2t) \Rightarrow Q'(10) = -8{,}000$ gallons/min is the rate the water is running at the end of 10 min. Then $\frac{Q(10) - Q(0)}{10} = -10{,}000$ gallons/min is the average rate the water flows during the first 10 min. The negative signs indicate water is leaving the tank.

24. (a) $y = 6\left(1 - \frac{t}{12}\right)^2 = 6\left(1 - \frac{t}{6} + \frac{t^2}{144}\right) \Rightarrow \frac{dy}{dt} = \frac{t}{12} - 1$

(b) The largest value of $\frac{dy}{dt}$ is 0 m/h when $t = 12$ and the fluid level is falling the slowest at that time. The smallest value of $\frac{dy}{dt}$ is -1 m/h, when $t = 0$, and the fluid level is falling the fastest at that time.

(c) In this situation, $\frac{dy}{dt} \le 0 \Rightarrow$ the graph of y is always decreasing. As $\frac{dy}{dt}$ increases in value, the slope of the graph of y increases from -1 to 0 over the interval $0 \le t \le 12$.



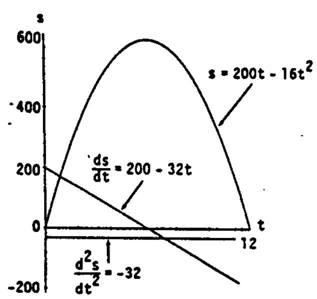
25. (a) $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2 \Rightarrow \left.\frac{dV}{dr}\right|_{r=2} = 4\pi(2)^2 = 16\pi\ ft^3/ft$

(b) When $r = 2$, $\frac{dV}{dr} = 16\pi$ so that when r changes by 1 unit, we expect V to change by approximately 16π. Therefore when r changes by 0.2 units V changes by approximately $(16\pi)(0.2) = 3.2\pi \approx 10.05\ ft^3$. Note that $V(2.2) - V(2) \approx 11.09\ ft^3$.

26. 200 km/hr $= 55\frac{5}{9} = \frac{500}{9}$ m/sec, and $D = \frac{10}{9}t^2 \Rightarrow V = \frac{20}{9}t$. Thus $V = \frac{500}{9} \Rightarrow \frac{20}{9}t = \frac{500}{9} \Rightarrow t = 25$ sec. When $t = 25$, $D = \frac{10}{9}(25)^2 = \frac{6250}{9}$ m

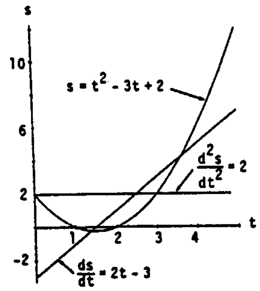
27. $s = v_0 t - 16t^2 \Rightarrow v = v_0 - 32t$; $v = 0 \Rightarrow t = \frac{v_0}{32}$; $1900 = v_0 t - 16t^2$ so that $t = \frac{v_0}{32} \Rightarrow 1900 = \frac{v_0^2}{32} - \frac{v_0^2}{64}$ $\Rightarrow v_0 = \sqrt{(64)(1900)} = 80\sqrt{19}$ ft/sec and, finally, $\frac{80\sqrt{19}\ ft}{sec} \cdot \frac{60\ sec}{1\ min} \cdot \frac{60\ min}{1\ hr} \cdot \frac{1\ mi}{5280\ ft} \approx 238$ mph.

28.



(a) $v = 0$ when $t = 6.25$ sec
(b) $v > 0$ when $0 \le t < 6.25 \Rightarrow$ body moves up; $v < 0$ when $6.25 < t \le 12.5 \Rightarrow$ body moves down
(c) body changes direction at $t = 6.25$ sec
(d) body speeds up on $(6.25, 12.5]$ and slows down on $[0, 6.25)$
(e) The body is moving fastest at the endpoints $t = 0$ and $t = 12.5$ when it is traveling 200 ft/sec. It's moving slowest at $t = 6.25$ when the speed is 0.
(f) When $t = 6.25$ the body is $s = 625$ m from the origin and farthest away.

29.



(a) $v = 0$ when $t = \frac{3}{2}$ sec

(b) $v < 0$ when $0 \le t < 1.5 \Rightarrow$ body moves left; $v > 0$ when $1.5 < t \le 5 \Rightarrow$ body moves right

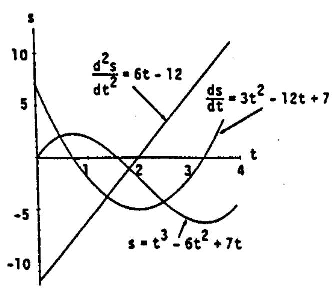
(c) body changes direction at $t = \frac{3}{2}$ sec

(d) body speeds up on $\left(\frac{3}{2}, 5\right]$ and slows down on $\left[0, \frac{3}{2}\right)$

(e) body is moving fastest at $t = 5$ when the speed $= |v(5)| = 7$ units/sec; it is moving slowest at $t = \frac{3}{2}$ when the speed is 0

(f) When $t = 5$ the body is $s = 10$ units from the origin and farthest away.

30.



(a) $v = 0$ when $t = \frac{6 \pm \sqrt{15}}{3}$ sec

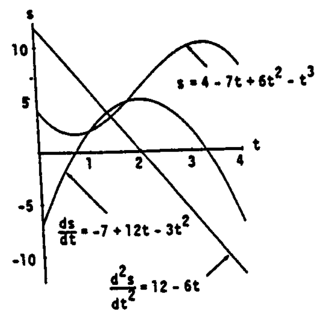
(b) $v < 0$ when $\frac{6 - \sqrt{15}}{3} < t < \frac{6 + \sqrt{15}}{3} \Rightarrow$ body moves left; $v > 0$ when $0 \le t < \frac{6 - \sqrt{15}}{3}$ or $\frac{6 + \sqrt{15}}{3} < t \le 4$ $\Rightarrow$ body moves right

(c) body changes direction at $t = \frac{6 \pm \sqrt{15}}{3}$ sec

(d) body speeds up on $\left(\frac{6 - \sqrt{15}}{3}, 2\right) \cup \left(\frac{6 + \sqrt{15}}{3}, 4\right]$ and slows down on $\left[0, \frac{6 - \sqrt{15}}{3}\right) \cup \left(2, \frac{6 + \sqrt{15}}{3}\right)$.

(e) The body is moving fastest at $t = 0$ and $t = 4$ when it is moving 7 units/sec and slowest at $t = \frac{6 \pm \sqrt{15}}{3}$ sec

(f) When $t = \frac{6 + \sqrt{15}}{3}$ the body is at position $s \approx -6.303$ units and farthest from the origin.

31. 

(a) $v = 0$ when $t = \frac{6 \pm \sqrt{15}}{3}$

(b) $v < 0$ when $0 \le t < \frac{6 - \sqrt{15}}{3}$ or $\frac{6 + \sqrt{15}}{3} < t \le 4 \Rightarrow$ body is moving left; $v > 0$ when $\frac{6 - \sqrt{15}}{3} < t < \frac{6 + \sqrt{15}}{3} \Rightarrow$ body is moving right

(c) body changes direction at $t = \frac{6 \pm \sqrt{15}}{3}$ sec

(d) body speeds up on $\left(\frac{6 - \sqrt{15}}{3}, 2\right) \cup \left(\frac{6 + \sqrt{15}}{3}, 4\right]$ and slows down on $\left[0, \frac{6 - \sqrt{15}}{3}\right) \cup \left(2, \frac{6 + \sqrt{15}}{3}\right)$

(e) The body is moving fastest at 7 units/sec when $t = 0$ and $t = 4$; it is moving slowest and stationary at $t = \frac{6 \pm \sqrt{15}}{3}$

(f) When $t = \frac{6 + \sqrt{15}}{3}$ the position is $s \approx 10.303$ units and the body is farthest from the origin.

32. (a) It takes 135 seconds.

(b) Average speed $= \frac{\Delta F}{\Delta t} = \frac{5 - 0}{73 - 0} = \frac{5}{73} \approx 0.068$ furlongs/sec.

(c) Using a symmetric difference quotient, the horse's speed is approximately $\frac{\Delta F}{\Delta t} = \frac{4 - 2}{59 - 33} = \frac{2}{26} = \frac{1}{13} \approx 0.077$ furlongs/sec.

(d) The horse is running the fastest during the last furlong (between 9th and 10th furlong markers). This furlong takes only 11 seconds to run, which is the least amount of time for a furlong.

(e) The horse accelerates the fastest during the first furlong (between markers 0 and 1).

2.3 DERIVATIVES OF PRODUCTS, QUOTIENTS, AND NEGATIVE POWERS

1. $y = 6x^2 - 10x - 5x^{-2} \Rightarrow \frac{dy}{dx} = 12x - 10 + 10x^{-3} \Rightarrow \frac{d^2y}{dx^2} = 12 - 0 - 30x^{-4} = 12 - 30x^{-4}$

2. $w = 3z^{-3} - z^{-1} \Rightarrow \frac{dw}{dz} = -9z^{-4} + z^{-2} = -9z^{-4} + \frac{1}{z^2} \Rightarrow \frac{d^2w}{dz^2} = 36z^{-5} - 2z^{-3} = 36z^{-5} - \frac{2}{z^3}$

3. $r = \frac{1}{3}s^{-2} - \frac{5}{2}s^{-1} \Rightarrow \frac{dr}{ds} = -\frac{2}{3}s^{-3} + \frac{5}{2}s^{-2} = \frac{-2}{3s^3} + \frac{5}{2s^2} \Rightarrow \frac{d^2r}{ds^2} = 2s^{-4} - 5s^{-3} = \frac{2}{s^4} - \frac{5}{s^3}$

4. $r = 12\theta^{-1} - 4\theta^{-3} + \theta^{-4} \Rightarrow \frac{dr}{d\theta} = -12\theta^{-2} + 12\theta^{-4} - 4\theta^{-5} = \frac{-12}{\theta^2} + \frac{12}{\theta^4} - \frac{4}{\theta^5} \Rightarrow \frac{d^2r}{d\theta^2} = 24\theta^{-3} - 48\theta^{-5} + 20\theta^{-6}$
$= \frac{24}{\theta^3} - \frac{48}{\theta^5} + \frac{20}{\theta^6}$

5. (a) $y = (3 - x^2)(x^3 - x + 1) \Rightarrow y' = (3 - x^2) \cdot \frac{d}{dx}(x^3 - x + 1) + (x^3 - x + 1) \cdot \frac{d}{dx}(3 - x^2)$
$= (3 - x^2)(3x^2 - 1) + (x^3 - x + 1)(-2x) = -5x^4 + 12x^2 - 2x - 3$
(b) $y = -x^5 + 4x^3 - x^2 - 3x + 3 \Rightarrow y' = -5x^4 + 12x^2 - 2x - 3$

6. $y = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x} + 1\right)$
(a) $y' = (x + x^{-1}) \cdot (1 + x^{-2}) + (x - x^{-1} + 1)(1 - x^{-2}) = 2x + 1 - \frac{1}{x^2} + \frac{2}{x^3}$
(b) $y = x^2 + x + \frac{1}{x} - \frac{1}{x^2} \Rightarrow y' = 2x + 1 - \frac{1}{x^2} + \frac{2}{x^3}$

7. $y = \frac{2x + 5}{3x - 2}$; use the quotient rule: $u = 2x + 5$ and $v = 3x - 2 \Rightarrow u' = 2$ and $v' = 3 \Rightarrow y' = \frac{vu' - uv'}{v^2}$
$= \frac{(3x - 2)(2) - (2x + 5)(3)}{(3x - 2)^2} = \frac{6x - 4 - 6x - 15}{(3x - 2)^2} = \frac{-19}{(3x - 2)^2}$

8. $g(x) = \frac{x^2 - 4}{x + 0.5}$; use the quotient rule: $u = x^2 - 4$ and $v = x + 0.5 \Rightarrow u' = 2x$ and $v' = 1 \Rightarrow g'(x) = \frac{vu' - uv'}{v^2}$
$= \frac{(x + 0.5)(2x) - (x^2 - 4)(1)}{(x + 0.5)^2} = \frac{2x^2 + x - x^2 + 4}{(x + 0.5)^2} = \frac{x^2 + x + 4}{(x + 0.5)^2}$

9. $f(t) = \frac{t^2 - 1}{t^2 + t - 2} \Rightarrow f'(t) = \frac{(t^2 + t - 2)(2t) - (t^2 - 1)(2t + 1)}{(t^2 + t - 2)^2} = \frac{(t - 1)(t + 2)(2t) - (t - 1)(t + 1)(2t + 1)}{(t - 1)^2(t + 2)^2}$
$= \frac{(t + 2)(2t) - (t + 1)(2t + 1)}{(t - 1)(t + 2)^2} = \frac{2t^2 + 4t - 2t^2 - 3t - 1}{(t - 1)(t + 2)^2} = \frac{t - 1}{(t - 1)(t + 2)^2} = \frac{1}{(t + 2)^2}$

10. $v = (1 - t)(1 + t^2)^{-1} = \frac{1 - t}{1 + t^2} \Rightarrow \frac{dv}{dt} = \frac{(1 + t^2)(-1) - (1 - t)(2t)}{(1 + t^2)^2} = \frac{-1 - t^2 - 2t + 2t^2}{(1 + t^2)^2} = \frac{t^2 - 2t - 1}{(1 + t^2)^2}$

11. $f(s) = \frac{\sqrt{s} - 1}{\sqrt{s} + 1} \Rightarrow f'(s) = \frac{(\sqrt{s} + 1)\left(\frac{1}{2\sqrt{s}}\right) - (\sqrt{s} - 1)\left(\frac{1}{2\sqrt{s}}\right)}{(\sqrt{s} + 1)^2} = \frac{(\sqrt{s} + 1) - (\sqrt{s} - 1)}{2\sqrt{s}(\sqrt{s} + 1)^2} = \frac{1}{\sqrt{s}(\sqrt{s} + 1)^2}$
NOTE: $\frac{d}{ds}(\sqrt{s}) = \frac{1}{2\sqrt{s}}$ from Example 1 in Section 2.1

12. $r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right) \Rightarrow r' = 2\left(\frac{\sqrt{\theta}(0) - 1\left(\frac{1}{2\sqrt{\theta}}\right)}{\theta} + \frac{1}{2\sqrt{\theta}}\right) = -\frac{1}{\theta^{3/2}} + \frac{1}{\theta^{1/2}}$

13. $y = \frac{1}{(x^2-1)(x^2+x+1)}$; use the quotient rule: $u = 1$ and $v = (x^2-1)(x^2+x+1) \Rightarrow u' = 0$ and

$v' = (x^2-1)(2x+1) + (x^2+x+1)(2x) = 2x^3 + x^2 - 2x - 1 + 2x^3 + 2x^2 + 2x = 4x^3 + 3x^2 - 1$

$\Rightarrow \frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{0 - 1(4x^3 + 3x^2 - 1)}{(x^2-1)^2(x^2+x+1)^2} = \frac{-4x^3 - 3x^2 + 1}{(x^2-1)^2(x^2+x+1)^2}$

14. $y = \frac{(x+1)(x+2)}{(x-1)(x-2)} = \frac{x^2+3x+2}{x^2-3x+2} \Rightarrow y' = \frac{(x^2-3x+2)(2x+3) - (x^2+3x+2)(2x-3)}{(x-1)^2(x-2)^2} = \frac{-6x^2+12}{(x-1)^2(x-2)^2}$

$= \frac{-6(x^2-2)}{(x-1)^2(x-2)^2}$

15. $s = \frac{t^2+5t-1}{t^2} = 1 + \frac{5}{t} - \frac{1}{t^2} = 1 + 5t^{-1} - t^{-2} \Rightarrow \frac{ds}{dt} = 0 - 5t^{-2} + 2t^{-3} = -5t^{-2} + 2t^{-3} \Rightarrow \frac{d^2s}{dt^2} = 10t^{-3} - 6t^{-4}$

16. $r = \frac{(\theta-1)(\theta^2+\theta+1)}{\theta^3} = \frac{\theta^3-1}{\theta^3} = 1 - \frac{1}{\theta^3} = 1 - \theta^{-3} \Rightarrow \frac{dr}{d\theta} = 0 + 3\theta^{-4} = 3\theta^{-4} \Rightarrow \frac{d^2r}{d\theta^2} = -12\theta^{-5}$

17. $w = \left(\frac{1+3z}{3z}\right)(3-z) = \left(\frac{1}{3}z^{-1} + 1\right)(3-z) = z^{-1} - \frac{1}{3} + 3 - z = z^{-1} + \frac{8}{3} - z \Rightarrow \frac{dw}{dz} = -z^{-2} + 0 - 1 = -z^{-2} - 1$

$\Rightarrow \frac{d^2w}{dz^2} = 2z^{-3} - 0 = 2z^{-3}$

18. $p = \left(\frac{q^2+3}{12q}\right)\left(\frac{q^4-1}{q^3}\right) = \frac{q^6 - q^2 + 3q^4 - 3}{12q^4} = \frac{1}{12}q^2 - \frac{1}{12}q^{-2} + \frac{1}{4} - \frac{1}{4}q^{-4} \Rightarrow \frac{dp}{dq} = \frac{1}{6}q + \frac{1}{6}q^{-3} + q^{-5}$

$\Rightarrow \frac{d^2p}{dq^2} = \frac{1}{6} - \frac{1}{2}q^{-4} - 5q^{-6}$

19. $u(0) = 5$, $u'(0) = 3$, $v(0) = -1$, $v'(0) = 2$

(a) $\frac{d}{dx}(uv) = uv' + vu' \Rightarrow \frac{d}{dx}(uv)\Big|_{x=0} = u(0)v'(0) + v(0)u'(0) = 5 \cdot 2 + (-1)(3) = 7$

(b) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} \Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right)\Big|_{x=0} = \frac{v(0)u'(0) - u(0)v'(0)}{(v(0))^2} = \frac{(-1)(3) - (5)(2)}{(-1)^2} = -13$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right) = \frac{uv' - vu'}{u^2} \Rightarrow \frac{d}{dx}\left(\frac{v}{u}\right)\Big|_{x=0} = \frac{u(0)v'(0) - v(0)u'(0)}{(u(0))^2} = \frac{(5)(2) - (-1)(3)}{(5)^2} = \frac{13}{25}$

(d) $\frac{d}{dx}(7v - 2u) = 7v' - 2u' \Rightarrow \frac{d}{dx}(7v - 2u)\Big|_{x=0} = 7v'(0) - 2u'(0) = 7 \cdot 2 - 2(3) = 8$

20. $u(1) = 2,\ u'(1) = 0,\ v(1) = 5,\ v'(1) = -1$

(a) $\frac{d}{dx}(uv)\Big|_{x=1} = u(1)v'(1) + v(1)u'(1) = 2\cdot(-1) + 5\cdot 0 = -2$

(b) $\frac{d}{dx}\left(\frac{u}{v}\right)\Big|_{x=1} = \frac{v(1)u'(1) - u(1)v'(1)}{(v(1))^2} = \frac{5\cdot 0 - 2\cdot(-1)}{(5)^2} = \frac{2}{25}$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right)\Big|_{x=1} = \frac{u(1)v'(1) - v(1)u'(1)}{(u(1))^2} = \frac{2\cdot(-1) - 5\cdot 0}{(2)^2} = -\frac{1}{2}$

(d) $\frac{d}{dx}(7v - 2u)\Big|_{x=1} = 7v'(1) - 2u'(1) = 7\cdot(-1) - 2\cdot 0 = -7$

21. $y = \frac{4x}{x^2+1} \Rightarrow \frac{dy}{dx} = \frac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2} = \frac{4x^2 + 4 - 8x^2}{(x^2+1)^2} = \frac{4(-x^2+1)}{(x^2+1)^2}$. When $x = 0$, $y = 0$ and $y' = \frac{4(0+1)}{1}$ $= 4$, so the tangent to the curve at $(0,0)$ is the line $y = 4x$. When $x = 1$, $y = 2 \Rightarrow y' = 0$, so the tangent to the curve at $(1,2)$ is the line $y = 2$.

22. $y = \frac{8}{x^2+4} \Rightarrow y' = \frac{(x^2+4)(0) - 8(2x)}{(x^2+4)^2} = \frac{-16x}{(x^2+4)^2}$. When $x = 2$, $y = 1$ and $y' = \frac{-16(2)}{(2^2+4)^2} = -\frac{1}{2}$, so the tangent line to the curve at $(2,1)$ has the equation $y - 1 = -\frac{1}{2}(x-2)$, or $y = -\frac{x}{2} + 2$.

23. $y = ax^2 + bx + c$ passes through $(0,0) \Rightarrow 0 = a(0) + b(0) + c \Rightarrow c = 0$; $y = ax^2 + bx$ passes through $(1,2)$ $\Rightarrow 2 = a + b$; $y' = 2ax + b$ and since the curve is tangent to $y = x$ at the origin, its slope is 1 at $x = 0$ $\Rightarrow y' = 1$ when $x = 0 \Rightarrow 1 = 2a(0) + b \Rightarrow b = 1$. Then $a + b = 2 \Rightarrow a = 1$. In summary $a = b = 1$ and $c = 0$ so the curve is $y = x^2 + x$.

24. $y = cx - x^2$ passes through $(1,0) \Rightarrow 0 = c(1) - 1 \Rightarrow c = 1 \Rightarrow$ the curve is $y = x - x^2$. For this curve, $y' = 1 - 2x$ and $x = 1 \Rightarrow y' = -1$. Since $y = x - x^2$ and $y = x^2 + ax + b$ have common tangents at $x = 0$, $y = x^2 + ax + b$ must also have slope -1 at $x = 1$. Thus $y' = 2x + a \Rightarrow -1 = 2\cdot 1 + a \Rightarrow a = -3$ $\Rightarrow y = x^2 - 3x + b$. Since this last curve passes through $(1,0)$, we have $0 = 1 - 3 + b \Rightarrow b = 2$. In summary, $a = -3$, $b = 2$ and $c = 1$ so the curves are $y = x^2 - 3x + 2$ and $y = x - x^2$.

25. Let c be a constant $\Rightarrow \frac{dc}{dx} = 0 \Rightarrow \frac{d}{dx}(u\cdot c) = u\cdot\frac{dc}{dx} + c\cdot\frac{du}{dx} = u\cdot 0 + c\,\frac{du}{dx} = c\,\frac{du}{dx}$. Thus when one of the functions is a constant, the Product Rule is just the Constant Multiple Rule $\Rightarrow$ the Constant Multiple Rule is a special case of the Product Rule.

26. (a) We use the Quotient rule to derive the Reciprocal Rule (with $u = 1$): $\frac{d}{dx}\left(\frac{1}{v}\right) = \frac{v\cdot 0 - 1\cdot\frac{dv}{dx}}{v^2} = \frac{-1\cdot\frac{dv}{dx}}{v^2}$ $= -\frac{1}{v^2}\cdot\frac{dv}{dx}$.

(b) Now, using the Reciprocal Rule and the Product Rule, we'll derive the Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{d}{dx}\left(u \cdot \frac{1}{v}\right)$

$= u \cdot \frac{d}{dx}\left(\frac{1}{v}\right) + \frac{1}{v} \cdot \frac{du}{dx}$ (Product Rule) $= u \cdot \left(\frac{-1}{v^2}\right)\frac{dv}{dx} + \frac{1}{v}\frac{du}{dx}$ (Reciprocal Rule) $\Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{-u\frac{dv}{dx} + v\frac{du}{dx}}{v^2}$

$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, the Quotient Rule.

27. (a) $\frac{d}{dx}(uvw) = \frac{d}{dx}((uv) \cdot w) = (uv)\frac{dw}{dx} + w \cdot \frac{d}{dx}(uv) = uv\frac{dw}{dx} + w\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) = uv\frac{dw}{dx} + wu\frac{dv}{dx} + wv\frac{du}{dx}$

$= uvw' + uv'w + u'vw$

(b) $\frac{d}{dx}(u_1u_2u_3u_4) = \frac{d}{dx}((u_1u_2u_3)u_4) = (u_1u_2u_3)\frac{du_4}{dx} + u_4\frac{d}{dx}(u_1u_2u_3) \Rightarrow \frac{d}{dx}(u_1u_2u_3u_4)$

$= u_1u_2u_3\frac{du_4}{dx} + u_4\left(u_1u_2\frac{du_3}{dx} + u_3u_1\frac{du_2}{dx} + u_3u_2\frac{du_1}{dx}\right)$ (using (a) above)

$\Rightarrow \frac{d}{dx}(u_1u_2u_3u_4) = u_1u_2u_3\frac{du_4}{dx} + u_1u_2u_4\frac{du_3}{dx} + u_1u_3u_4\frac{du_2}{dx} + u_2u_3u_4\frac{du_1}{dx}$

$= u_1u_2u_3u_4' + u_1u_2u_3'u_4 + u_1u_2'u_3u_4 + u_1'u_2u_3u_4$

(c) Generalizing (a) and (b) above, $\frac{d}{dx}(u_1 \cdots u_n) = u_1u_2 \cdots u_{n-1}u_n' + u_1u_2 \cdots u_{n-2}u_{n-1}'u_n + \ldots + u_1'u_2 \cdots u_n$

28. In this problem we don't know the Power Rule works with fractional powers so we can't use it. Remember $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$ (from Example 1 in Section 2.1)

(a) $\frac{d}{dx}(x^{3/2}) = \frac{d}{dx}(x \cdot x^{1/2}) = x \cdot \frac{d}{dx}(\sqrt{x}) + \sqrt{x}\frac{d}{dx}(x) = x \cdot \frac{1}{2\sqrt{x}} + \sqrt{x} \cdot 1 = \frac{\sqrt{x}}{2} + \sqrt{x} = \frac{3\sqrt{x}}{2} = \frac{3}{2}x^{1/2}$

(b) $\frac{d}{dx}(x^{5/2}) = \frac{d}{dx}(x^2 \cdot x^{1/2}) = x^2\frac{d}{dx}(\sqrt{x}) + \sqrt{x}\frac{d}{dx}(x^2) = x^2 \cdot \left(\frac{1}{2\sqrt{x}}\right) + \sqrt{x} \cdot 2x = \frac{1}{2}x^{3/2} + 2x^{3/2} = \frac{5}{2}x^{3/2}$

(c) $\frac{d}{dx}(x^{7/2}) = \frac{d}{dx}(x^3 \cdot x^{1/2}) = x^3\frac{d}{dx}(\sqrt{x}) + \sqrt{x}\frac{d}{dx}(x^3) = x^3 \cdot \left(\frac{1}{2\sqrt{x}}\right) + \sqrt{x} \cdot 3x^2 = \frac{1}{2}x^{5/2} + 3x^{5/2} = \frac{7}{2}x^{5/2}$

(d) We have $\frac{d}{dx}(x^{3/2}) = \frac{3}{2}x^{1/2}$, $\frac{d}{dx}(x^{5/2}) = \frac{5}{2}x^{3/2}$, $\frac{d}{dx}(x^{7/2}) = \frac{7}{2}x^{5/2}$ so it appears that $\frac{d}{dx}(x^{n/2}) = \frac{n}{2}x^{(n/2)-1}$ whenever n is an odd positive integer ≥ 3.

29. $P = \frac{nRT}{V - nb} - \frac{an^2}{V^2}$. We are holding T constant, and a, b, n, R are also constant so their derivatives are zero

$\Rightarrow \frac{dP}{dV} = \frac{(V - nb) \cdot 0 - (nRT)(1)}{(V - nb)^2} - \frac{V^2(0) - (an^2)(2V)}{(V^2)^2} = \frac{-nRT}{(V - nb)^2} + \frac{2an^2}{V^3}$

30. $A(q) = \frac{km}{q} + cm + \frac{hq}{2} = (km)q^{-1} + cm + \left(\frac{h}{2}\right)q$

$\frac{dA}{dq} = -(km)q^{-2} + \frac{h}{2} = -\frac{km}{q^2} + \frac{h}{2}$

$\frac{d^2A}{dq^2} = 2(km)q^{-3} = \frac{2km}{q^3}$

2.4 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

1. $y = -10x + 3\cos x \Rightarrow \frac{dy}{dx} = -10 + 3\frac{d}{dx}(\cos x) = -10 - 3\sin x$

2. $y = \frac{3}{x} + 5\sin x \Rightarrow \frac{dy}{dx} = \frac{-3}{x^2} + 5\frac{d}{dx}(\sin x) = \frac{-3}{x^2} + 5\cos x$

3. $y = \csc x - 4\sqrt{x} + 7 \Rightarrow \frac{dy}{dx} = -\csc x \cot x - \frac{4}{2\sqrt{x}} + 0 = -\csc x \cot x - \frac{2}{\sqrt{x}}$

4. $y = x^2 \cot x - \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = x^2 \frac{d}{dx}(\cot x) + \cot x \cdot \frac{d}{dx}(x^2) + \frac{2}{x^3} = -x^2 \csc^2 x + (\cot x)(2x) + \frac{2}{x^3}$
$= -x^2 \csc^2 x + 2x \cot x + \frac{2}{x^3}$

5. $y = (\sec x + \tan x)(\sec x - \tan x) \Rightarrow \frac{dy}{dx} = (\sec x + \tan x)\frac{d}{dx}(\sec x - \tan x) + (\sec x - \tan x)\frac{d}{dx}(\sec x + \tan x)$
$= (\sec x + \tan x)(\sec x \tan x - \sec^2 x) + (\sec x - \tan x)(\sec x \tan x + \sec^2 x)$
$= (\sec^2 x \tan x + \sec x \tan^2 x - \sec^3 x - \sec^2 x \tan x) + (\sec^2 x \tan x - \sec x \tan^2 x + \sec^3 x - \tan x \sec^2 x) = 0.$
$\left(\text{Note also that } y = \sec^2 x - \tan^2 x = (\tan^2 x + 1) - \tan^2 x = 1 \Rightarrow \frac{dy}{dx} = 0.\right)$

6. $y = (\sin x + \cos x)\sec x \Rightarrow \frac{dy}{dx} = (\sin x + \cos x)\frac{d}{dx}(\sec x) + \sec x \frac{d}{dx}(\sin x + \cos x)$
$= (\sin x + \cos x)(\sec x \tan x) + (\sec x)(\cos x - \sin x) = \frac{(\sin x + \cos x)\sin x}{\cos^2 x} + \frac{\cos x - \sin x}{\cos x}$
$= \frac{\sin^2 x + \cos x \sin x + \cos^2 x - \cos x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$
$\left(\text{Note also that } y = \sin x \sec x + \cos x \sec x = \tan x + 1 \Rightarrow \frac{dy}{dx} = \sec^2 x.\right)$

7. $y = \frac{\cot x}{1 + \cot x} \Rightarrow \frac{dy}{dx} = \frac{(1 + \cot x)\frac{d}{dx}(\cot x) - (\cot x)\frac{d}{dx}(1 + \cot x)}{(1 + \cot x)^2} = \frac{(1 + \cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1 + \cot x)^2}$
$= \frac{-\csc^2 x - \csc^2 x \cot x + \csc^2 x \cot x}{(1 + \cot x)^2} = \frac{-\csc^2 x}{(1 + \cot x)^2}$

8. $y = \frac{\cos x}{1 + \sin x} \Rightarrow \frac{dy}{dx} = \frac{(1 + \sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2} = \frac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2}$
$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} = \frac{-\sin x - 1}{(1 + \sin x)^2} = \frac{-(1 + \sin x)}{(1 + \sin x)^2} = \frac{-1}{1 + \sin x}$

9. $y = \frac{4}{\cos x} + \frac{1}{\tan x} = 4\sec x + \cot x \Rightarrow \frac{dy}{dx} = 4\sec x \tan x - \csc^2 x$

10. $y = \frac{\cos x}{x} + \frac{x}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{x(-\sin x) - (\cos x)(1)}{x^2} + \frac{(\cos x)(1) - x(-\sin x)}{\cos^2 x} = \frac{-x \sin x - \cos x}{x^2} + \frac{\cos x + x \sin x}{\cos^2 x}$

11. $y = x^2 \sin x + 2x \cos x - 2 \sin x \Rightarrow \frac{dy}{dx} = \left(x^2 \cos x + (\sin x)(2x)\right) + \left((2x)(-\sin x) + (\cos x)(2)\right) - 2 \cos x$
$= x^2 \cos x + 2x \sin x - 2x \sin x + 2 \cos x - 2 \cos x = x^2 \cos x$

12. $y = x^2 \cos x - 2x \sin x - 2 \cos x \Rightarrow \frac{dy}{dx} = \left(x^2(-\sin x) + (\cos x)(2x)\right) - \left(2x \cos x + (\sin x)(2)\right) - 2(-\sin x)$
$= -x^2 \sin x + 2x \cos x - 2x \cos x - 2 \sin x + 2 \sin x = -x^2 \sin x$

13. $s = \tan t - t \Rightarrow \frac{ds}{dt} = \frac{d}{dt}(\tan t) - 1 = \sec^2 t - 1$

14. $s = t^2 - \sec t + 1 \Rightarrow \frac{ds}{dt} = 2t - \frac{d}{dt}(\sec t) = 2t - \sec t \tan t$

15. $s = \frac{1 + \csc t}{1 - \csc t} \Rightarrow \frac{ds}{dt} = \frac{(1 - \csc t)(-\csc t \cot t) - (1 + \csc t)(\csc t \cot t)}{(1 - \csc t)^2}$
$= \frac{-\csc t \cot t + \csc^2 t \cot t - \csc t \cot t - \csc^2 t \cot t}{(1 - \csc t)^2} = \frac{-2 \csc t \cot t}{(1 - \csc t)^2}$

16. $s = \frac{\sin t}{1 - \cos t} \Rightarrow \frac{ds}{dt} = \frac{(1 - \cos t)(\cos t) - (\sin t)(\sin t)}{(1 - \cos t)^2} = \frac{\cos t - \cos^2 t - \sin^2 t}{(1 - \cos t)^2} = \frac{\cos t - 1}{(1 - \cos t)^2} = -\frac{1}{1 - \cos t}$
$= \frac{1}{\cos t - 1}$

17. $r = 4 - \theta^2 \sin \theta \Rightarrow \frac{dr}{d\theta} = -\left(\theta^2 \frac{d}{d\theta}(\sin \theta) + (\sin \theta)(2\theta)\right) = -\left(\theta^2 \cos \theta + 2\theta \sin \theta\right) = -\theta(\theta \cos \theta + 2 \sin \theta)$

18. $r = \theta \sin \theta + \cos \theta \Rightarrow \frac{dr}{d\theta} = (\theta \cos \theta + (\sin \theta)(1)) - \sin \theta = \theta \cos \theta$

19. $r = \sec \theta \csc \theta \Rightarrow \frac{dr}{d\theta} = (\sec \theta)(-\csc \theta \cot \theta) + (\csc \theta)(\sec \theta \tan \theta)$
$= \left(\frac{-1}{\cos \theta}\right)\left(\frac{1}{\sin \theta}\right)\left(\frac{\cos \theta}{\sin \theta}\right) + \left(\frac{1}{\sin \theta}\right)\left(\frac{1}{\cos \theta}\right)\left(\frac{\sin \theta}{\cos \theta}\right) = \frac{-1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta} = \sec^2 \theta - \csc^2 \theta$

20. $r = (1 + \sec \theta) \sin \theta \Rightarrow \frac{dr}{d\theta} = (1 + \sec \theta) \cos \theta + (\sin \theta)(\sec \theta \tan \theta) = (\cos \theta + 1) + \tan^2 \theta = \cos \theta + \sec^2 \theta$

21. $p = 5 + \frac{1}{\cot q} = 5 + \tan q \Rightarrow \frac{dp}{dq} = \sec^2 q$

22. $p = (1 + \csc q) \cos q \Rightarrow \frac{dp}{dq} = (1 + \csc q)(-\sin q) + (\cos q)(-\csc q \cot q) = (-\sin q - 1) - \cot^2 q = -\sin q - \csc^2 q$

23. $p = \frac{\sin q + \cos q}{\cos q} \Rightarrow \frac{dp}{dq} = \frac{(\cos q)(\cos q - \sin q) - (\sin q + \cos q)(-\sin q)}{\cos^2 q}$
$= \frac{\cos^2 q - \cos q \sin q + \sin^2 q + \cos q \sin q}{\cos^2 q} = \frac{1}{\cos^2 q} = \sec^2 q$

24. $p = \frac{\tan q}{1 + \tan q} \Rightarrow \frac{dp}{dq} = \frac{(1 + \tan q)\left(\sec^2 q\right) - (\tan q)\left(\sec^2 q\right)}{(1 + \tan q)^2} = \frac{\sec^2 q + \tan q \sec^2 q - \tan q \sec^2 q}{(1 + \tan q)^2} = \frac{\sec^2 q}{(1 + \tan q)^2}$

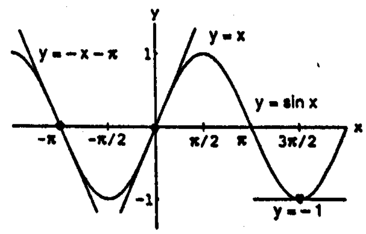
25. (a) $y = \csc x \Rightarrow y' = -\csc x \cot x \Rightarrow y'' = -\Big((\csc x)(-\csc^2 x) + (\cot x)(-\csc x \cot x)\Big) = \csc^3 x + \csc x \cot^2 x$
$= (\csc x)(\csc^2 x + \cot^2 x) = (\csc x)(\csc^2 x + \csc^2 x - 1) = 2\csc^3 x - \csc x$

(b) $y = \sec x \Rightarrow y' = \sec x \tan x \Rightarrow y'' = (\sec x)(\sec^2 x) + (\tan x)(\sec x \tan x) = \sec^3 x + \sec x \tan^2 x$
$= (\sec x)(\sec^2 x + \tan^2 x) = (\sec x)(\sec^2 x + \sec^2 x - 1) = 2\sec^3 x - \sec x$

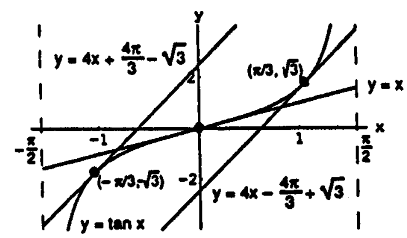
26. (a) $y = -2 \sin x \Rightarrow y' = -2 \cos x \Rightarrow y'' = -2(-\sin x) = 2 \sin x \Rightarrow y''' = 2 \cos x \Rightarrow y^{(4)} = -2 \sin x$

(b) $y = 9 \cos x \Rightarrow y' = -9 \sin x \Rightarrow y'' = -9 \cos x \Rightarrow y''' = -9(-\sin x) = 9 \sin x \Rightarrow y^{(4)} = 9 \cos x$

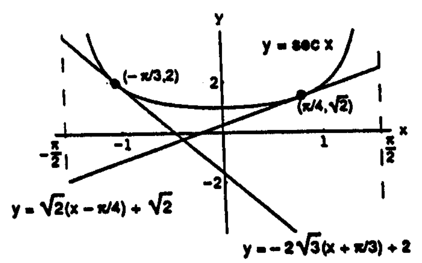
27. $y = \sin x \Rightarrow y' = \cos x \Rightarrow$ slope of tangent at $x = -\pi$ is $y'(-\pi) = \cos(-\pi) = -1$; slope of tangent at $x = 0$ is $y'(0) = \cos(0) = 1$; and slope of tangent at $x = \frac{3\pi}{2}$ is $y'\left(\frac{3\pi}{2}\right) = \cos\frac{3\pi}{2} = 0$. The tangent at $(-\pi, 0)$ is $y - 0 = -1(x + \pi)$, or $y = -x - \pi$; the tangent at $(0,0)$ is $y - 0 = 1(x - 0)$, or $y = x$; and the tangent at $\left(\frac{3\pi}{2}, -1\right)$ is $y = -1$.



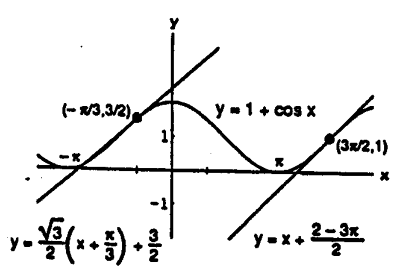
28. $y = \tan x \Rightarrow y' = \sec^2 x \Rightarrow$ slope of tangent at $x = -\frac{\pi}{3}$ is $\sec^2\left(-\frac{\pi}{3}\right) = 4$; slope of tangent at $x = 0$ is $\sec^2(0) = 1$; and slope of tangent at $x = \frac{\pi}{3}$ is $\sec^2\left(\frac{\pi}{3}\right) = 4$. The tangent at $\left(-\frac{\pi}{3}, \tan\left(-\frac{\pi}{3}\right)\right) = \left(-\frac{\pi}{3}, -\sqrt{3}\right)$ is $y + \sqrt{3} = 4\left(x + \frac{\pi}{3}\right)$; the tangent at $(0,0)$ is $y = x$; and the tangent at $\left(\frac{\pi}{3}, \tan\left(\frac{\pi}{3}\right)\right) = \left(\frac{\pi}{3}, \sqrt{3}\right)$ is $y - \sqrt{3} = 4\left(x - \frac{\pi}{3}\right)$.



29. $y = \sec x \Rightarrow y' = \sec x \tan x \Rightarrow$ slope of tangent at $x = -\frac{\pi}{3}$ is $\sec\left(-\frac{\pi}{3}\right)\tan\left(-\frac{\pi}{3}\right) = -2\sqrt{3}$; slope of tangent at $x = \frac{\pi}{4}$ is $\sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = \sqrt{2}$. The tangent at the point $\left(-\frac{\pi}{3}, \sec\left(-\frac{\pi}{3}\right)\right) = \left(-\frac{\pi}{3}, 2\right)$ is $y - 2 = -2\sqrt{3}\left(x + \frac{\pi}{3}\right)$; the tangent at the point $\left(\frac{\pi}{4}, \sec\left(\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{4}, \sqrt{2}\right)$ is $y - \sqrt{2} = \sqrt{2}\left(x - \frac{\pi}{4}\right)$.



30. $y = 1 + \cos x \Rightarrow y' = -\sin x \Rightarrow$ slope of tangent at $x = -\frac{\pi}{3}$ is $-\sin\left(-\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$; slope of tangent at $x = \frac{3\pi}{2}$ is $-\sin\left(\frac{3\pi}{2}\right) = 1$. The tangent at the point $\left(-\frac{\pi}{3}, 1 + \cos\left(-\frac{\pi}{3}\right)\right) = \left(-\frac{\pi}{3}, \frac{3}{2}\right)$ is $y - \frac{3}{2} = \frac{\sqrt{3}}{2}\left(x + \frac{\pi}{3}\right)$; the tangent at the point



$\left(\frac{3\pi}{2}, 1+\cos\left(\frac{3\pi}{2}\right)\right)=\left(\frac{3\pi}{2}, 1\right)$ is $y-1=x-\frac{3\pi}{2}$

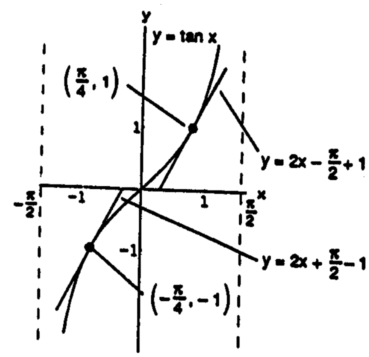
31. Yes, $y=x+\sin x \Rightarrow y'=1+\cos x$; horizontal tangent occurs where $1+\cos x=0 \Rightarrow \cos x=-1$ $\Rightarrow x=\pi$

32. No, $y=2x+\sin x \Rightarrow y'=2+\cos x$; horizontal tangent occurs where $2+\cos x=0 \Rightarrow \cos x=-2$. But there are no x-values for which $\cos x=-2$.

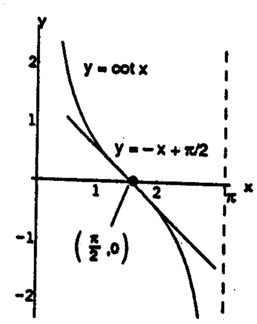
33. No, $y=x-\cot x \Rightarrow y'=1+\csc^2 x$; horizontal tangent occurs where $1+\csc^2 x=0 \Rightarrow \csc^2 x=-1$. But there are no x-values for which $\csc^2 x=-1$.

34. Yes, $y=x+2\cos x \Rightarrow y'=1-2\sin x$; horizontal tangent occurs where $1-2\sin x=0 \Rightarrow 1=2\sin x$ $\Rightarrow \frac{1}{2}=\sin x \Rightarrow x=\frac{\pi}{6}$ or $x=\frac{5\pi}{6}$

35. We want all points on the curve where the tangent line has slope 2. Thus, $y=\tan x \Rightarrow y'=\sec^2 x$ so that $y'=2 \Rightarrow \sec^2 x=2 \Rightarrow \sec x=\pm\sqrt{2}$ $\Rightarrow x=\pm\frac{\pi}{4}$. Then the tangent line at $\left(\frac{\pi}{4}, 1\right)$ has equation $y-1=2\left(x-\frac{\pi}{4}\right)$; the tangent line at $\left(-\frac{\pi}{4}, -1\right)$ has equation $y+1=2\left(x+\frac{\pi}{4}\right)$.



36. We want all points on the curve $y=\cot x$ where the tangent line has slope -1. Thus $y=\cot x$ $\Rightarrow y'=-\csc^2 x$ so that $y'=-1 \Rightarrow -\csc^2 x=-1$ $\Rightarrow \csc^2 x=1 \Rightarrow \csc x=\pm 1 \Rightarrow x=\frac{\pi}{2}$. The tangent line at $\left(\frac{\pi}{2}, 0\right)$ is $y=-x+\frac{\pi}{2}$.



37. $y=4+\cot x-2\csc x \Rightarrow y'=-\csc^2 x+2\csc x\cot x=-\left(\frac{1}{\sin x}\right)\left(\frac{1-2\cos x}{\sin x}\right)$

(a) When $x=\frac{\pi}{2}$, then $y'=-1$; the tangent line is $y=-x+\frac{\pi}{2}+2$.

(b) To find the location of the horizontal tangent set $y'=0 \Rightarrow 1-2\cos x=0 \Rightarrow x=\frac{\pi}{3}$ radians. When $x=\frac{\pi}{3}$, then $y=4-\sqrt{3}$ is the horizontal tangent.

38. $y=1+\sqrt{2}\csc x+\cot x \Rightarrow y'=-\sqrt{2}\csc x\cot x-\csc^2 x=-\left(\frac{1}{\sin x}\right)\left(\frac{\sqrt{2}\cos x+1}{\sin x}\right)$

(a) If $x=\frac{\pi}{4}$, then $y'=-4$; the tangent line is $y=-4x+\pi+4$.

(b) To find the location of the horizontal tangent set $y' = 0 \Rightarrow \sqrt{2}\cos x + 1 = 0 \Rightarrow x = \frac{3\pi}{4}$ radians. When $x = \frac{3\pi}{4}$, then $y = 2$ is the horizontal tangent.

39. $s = 2 - 2\sin t \Rightarrow v = \frac{ds}{dt} = -2\cos t \Rightarrow a = \frac{dv}{dt} = 2\sin t \Rightarrow j = \frac{da}{dt} = 2\cos t$. Therefore, velocity $= v\left(\frac{\pi}{4}\right)$ $= -\sqrt{2}$ m/sec; speed $= \left|v\left(\frac{\pi}{4}\right)\right| = \sqrt{2}$ m/sec; acceleration $= a\left(\frac{\pi}{4}\right) = \sqrt{2}$ m/sec^2; jerk $= j\left(\frac{\pi}{4}\right) = \sqrt{2}$ m/sec^3.

40. $s = \sin t + \cos t \Rightarrow v = \frac{ds}{dt} = \cos t - \sin t \Rightarrow a = \frac{dv}{dt} = -\sin t - \cos t \Rightarrow j = \frac{da}{dt} = -\cos t + \sin t$. Therefore velocity $= v\left(\frac{\pi}{4}\right) = 0$ m/sec; speed $= \left|v\left(\frac{\pi}{4}\right)\right| = 0$ m/sec; acceleration $= a\left(\frac{\pi}{4}\right) = -\sqrt{2}$ m/sec^2; jerk $= j\left(\frac{\pi}{4}\right) = 0$ m/sec^3.

41. $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \frac{\sin^2 3x}{x^2} = \lim\limits_{x\to 0} 9\left(\frac{\sin 3x}{3x}\right)\left(\frac{\sin 3x}{3x}\right) = 9$ so that f is continuous at $x = 0 \Rightarrow \lim\limits_{x\to 0} f(x) = f(0)$ $\Rightarrow c = 9$.

42. $\lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^-} (x + b) = b$ and $\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+} \cos x = 1$ so that g is continuous at $x = 0 \Rightarrow \lim\limits_{x\to 0^-} g(x)$ $= \lim\limits_{x\to 0^+} g(x) \Rightarrow b = 1$. Now g is not differentiable at $x = 0$: At $x = 0$, the left-hand derivative is $\left.\frac{d}{dx}(x + b)\right|_{x=0} = 1$, but the right-hand derivative is $\left.\frac{d}{dx}(\cos x)\right|_{x=0} = -\sin 0 = 0$. The left- and right-hand derivatives can never agree at $x = 0$, so g is not differentiable at $x = 0$ for any value of b (including $b = 1$).

43. $\frac{d^{999}}{dx^{999}}(\cos x) = \sin x$ because $\frac{d^4}{dx^4}(\cos x) = \cos x \Rightarrow$ the derivative of cos x any number of times that is a multiple of 4 is cos x. Thus, dividing 999 by 4 gives $999 = 249 \cdot 4 + 3 \Rightarrow \frac{d^{999}}{dx^{999}}(\cos x)$ $= \frac{d^3}{dx^3}\left[\frac{d^{249\cdot 4}}{dx^{249\cdot 4}}(\cos x)\right] = \frac{d^3}{dx^3}(\cos x) = \sin x$.

44. (a) $y = \sec x = \frac{1}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos^2 x} = \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) = \sec x \tan x$

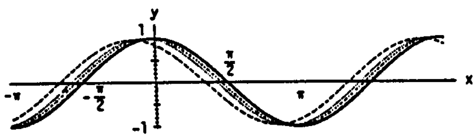
$\Rightarrow \frac{d}{dx}(\sec x) = \sec x \tan x$

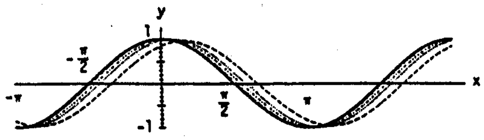
(b) $y = \csc x = \frac{1}{\sin x} \Rightarrow \frac{dy}{dx} = \frac{(\sin x)(0) - (1)(\cos x)}{(\sin x)^2} = \frac{-\cos x}{\sin^2 x} = \left(\frac{-1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) = -\csc x \cot x$

$\Rightarrow \frac{d}{dx}(\csc x) = -\csc x \cot x$

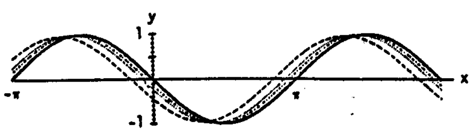
(c) $y = \cot x = \frac{\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$

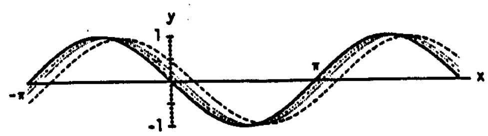
$\Rightarrow \frac{d}{dx}(\cot x) = -\csc^2 x$

45. 



As h takes on the values of 1, 0.5, 0.3 and 0.1 the corresponding dashed curves of $y = \frac{\sin(x+h) - \sin x}{h}$ get closer and closer to the black curve $y = \cos x$ because $\frac{d}{dx}(\sin x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \cos x$. The same is true as h takes on the values of -1, -0.5, -0.3 and -0.1.

46. 

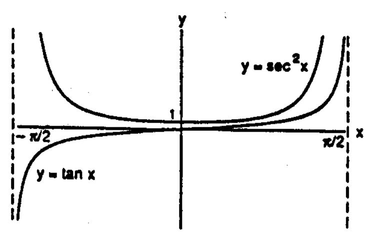


As h takes on the values of 1, 0.5, 0.3, and 0.1 the corresponding dashed curves of $y = \frac{\cos(x+h) - \cos x}{h}$ get closer and closer to the black curve $y = -\sin x$ because $\frac{d}{dx}(\cos x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = -\sin x$. The same is true as h takes on the values of -1, -0.5, -0.3, and -0.1.

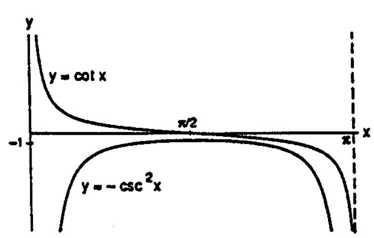
47. This is a grapher exercise. Compare your graphs with Exercises 45 and 46.

48. $\lim_{h\to 0} \frac{|0+h| - |0-h|}{2h} = \lim_{x\to 0} \frac{|h| - |h|}{2h} = \lim_{h\to 0} 0 = 0 \Rightarrow$ the limits of the centered difference quotient exists even though the derivative of $f(x) = |x|$ does not exist at $x = 0$.

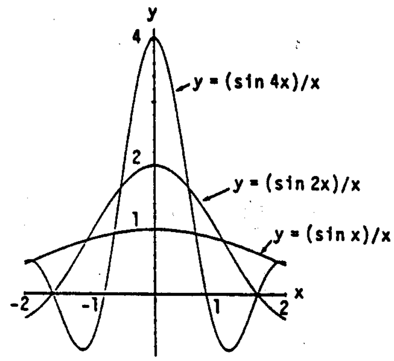
49. $y = \tan x \Rightarrow y' = \sec^2 x$, so the smallest value $y' = \sec^2 x$ takes on is $y' = 1$ when $x = 0$; y' has no maximum value since $\sec^2 x$ has no largest value on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; y' is never negative since $\sec^2 x \geq 1$.



50. $y = \cot x \Rightarrow y' = -\csc^2 x$ so y' has no smallest value since $-\csc^2 x$ has no minimum value on $(0, \pi)$; the largest value of y' is -1, when $x = \frac{\pi}{2}$; the slope is never positive since the largest value $y' = -\csc x$ takes on is the negative value -1.



51. $y = \frac{\sin x}{x}$ appears to cross the y-axis at $y = 1$, since $\lim_{x \to 0} \frac{\sin x}{x} = 1$; $y = \frac{\sin 2x}{x}$ appears to cross the y-axis at $y = 2$, since $\lim_{x \to 0} \frac{\sin 2x}{x} = 2$; $y = \frac{\sin 4x}{x}$ appears to cross the y-axis at $y = 4$, since $\lim_{x \to 0} \frac{\sin 4x}{x} = 4$. However, none of these graphs actually cross the y-axis since $x = 0$ is not in the domain of the functions. Also, $\lim_{x \to 0} \frac{\sin 5x}{x} = 5$, $\lim_{x \to 0} \frac{\sin(-3x)}{x} = -3$, and $\lim_{x \to 0} \frac{\sin kx}{x} = k \Rightarrow$ the graphs of $y = \frac{\sin 5x}{x}$, $y = \frac{\sin(-3x)}{x}$, and $y = \frac{\sin kx}{x}$ approach 5, −3, and k, respectively, as $x \to 0$. However, the graphs do not actually cross the y-axis.



52. (a)

h	$\frac{\sin h}{h}$	$\left(\frac{\sin h}{h}\right)\left(\frac{180}{\pi}\right)$
1	.017453283	.999999492
0.01	.017453292	1
0.001	.017453292	1
0.0001	.017453292	1

$$\lim_{h \to 0} \frac{\sin h^\circ}{h} = \lim_{x \to 0} \frac{\sin\left(h \cdot \frac{\pi}{180}\right)}{h} = \lim_{h \to 0} \frac{\frac{\pi}{180} \sin\left(h \cdot \frac{\pi}{180}\right)}{\frac{\pi}{180} \cdot h} = \lim_{\theta \to 0} \frac{\frac{\pi}{180} \sin \theta}{\theta} = \frac{\pi}{180} \qquad \left(\theta = h \cdot \frac{\pi}{180}\right)$$

(converting to radians)

(b)

h	$\frac{\cos h - 1}{h}$
1	−0.0001523
0.01	−0.0000015
0.001	−0.0000001
0.0001	0

$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$, whether h is measured in degrees or radians.

(c) In degrees, $\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} = \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$

$$= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h}\right) = (\sin x) \cdot \lim_{h \to 0} \left(\frac{\cos h - 1}{h}\right) + (\cos x) \cdot \lim_{h \to 0} \left(\frac{\sin h}{h}\right)$$

$$= (\sin x)(0) + (\cos x)\left(\frac{\pi}{180}\right) = \frac{\pi}{180} \cos x$$

(d) In degrees, $\frac{d}{dx}(\cos x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$

$= \lim_{h \to 0} \frac{(\cos x)(\cos h - 1) - \sin x \sin h}{h} = \lim_{h \to 0} \left(\cos x \cdot \frac{\cos h - 1}{h}\right) - \lim_{h \to 0} \left(\sin x \cdot \frac{\sin h}{h}\right)$

$= (\cos x) \lim_{h \to 0} \left(\frac{\cos h - 1}{h}\right) - (\sin x) \lim_{h \to 0} \left(\frac{\sin h}{h}\right) = (\cos x)(0) - (\sin x)\left(\frac{\pi}{180}\right) = -\frac{\pi}{180} \sin x$

(e) $\frac{d^2}{dx^2}(\sin x) = \frac{d}{dx}\left(\frac{\pi}{180} \cos x\right) = -\left(\frac{\pi}{180}\right)^2 \sin x$; $\frac{d^3}{dx^3}(\sin x) = \frac{d}{dx}\left(-\left(\frac{\pi}{180}\right)^2 \sin x\right) = -\left(\frac{\pi}{180}\right)^3 \cos x$;

$\frac{d^2}{dx^2}(\cos x) = \frac{d}{dx}\left(-\frac{\pi}{180} \sin x\right) = -\left(\frac{\pi}{180}\right)^2 \cos x$; $\frac{d^3}{dx^3}(\cos x) = \frac{d}{dx}\left(-\left(\frac{\pi}{180}\right)^2 \cos x\right) = \left(\frac{\pi}{180}\right)^3 \sin x$

2.5 THE CHAIN RULE

1. $f(u) = 6u - 9 \Rightarrow f'(u) = 6 \Rightarrow f'(g(x)) = 6$; $g(x) = \frac{1}{2}x^4 \Rightarrow g'(x) = 2x^3$; therefore $\frac{dy}{dx} = f'(g(x))g'(x)$
$= 6 \cdot 2x^3 = 12x^3$

2. $f(u) = 2u^3 \Rightarrow f'(u) = 6u^2 \Rightarrow f'(g(x)) = 6(8x-1)^2$; $g(x) = 8x - 1 \Rightarrow g'(x) = 8$; therefore $\frac{dy}{dx} = f'(g(x))g'(x)$
$= 6(8x-1)^2 \cdot 8 = 48(8x-1)^2$

3. $f(u) = \sin u \Rightarrow f'(u) = \cos u \Rightarrow f'(g(x)) = \cos(3x+1)$; $g(x) = 3x + 1 \Rightarrow g'(x) = 3$; therefore $\frac{dy}{dx} = f'(g(x))g'(x)$
$= (\cos(3x+1))(3) = 3\cos(3x+1)$

4. $f(u) = \cos u \Rightarrow f'(u) = -\sin u \Rightarrow f'(g(x)) = -\sin(\sin x)$; $g(x) = \sin x \Rightarrow g'(x) = \cos x$; therefore
$\frac{dy}{dx} = f'(g(x))g'(x) = -(\sin(\sin x))\cos x$

5. $f(u) = \tan u \Rightarrow f'(u) = \sec^2 u \Rightarrow f'(g(x)) = \sec^2(10x - 5)$; $g(x) = 10x - 5 \Rightarrow g'(x) = 10$; therefore
$\frac{dy}{dx} = f'(g(x))g'(x) = \left(\sec^2(10x-5)\right)(10) = 10\sec^2(10x-5)$

6. $f(u) = -\sec u \Rightarrow f'(u) = -\sec u \tan u \Rightarrow f'(g(x)) = -\sec(x^2 + 7x)\tan(x^2 + 7x)$; $g(x) = x^2 + 7x$
$\Rightarrow g'(x) = 2x + 7$; therefore $\frac{dy}{dx} = f'(g(x))g'(x) = -(2x+7)\sec(x^2+7x)\tan(x^2+7x)$

7. With $u = (4 - 3x)$, $y = u^9$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 9u^8 \cdot (-3) = -27(4-3x)^8$

8. With $u = \left(1 - \frac{x}{7}\right)$, $y = u^{-7}$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -7u^{-8} \cdot \left(-\frac{1}{7}\right) = \left(1 - \frac{x}{7}\right)^{-8}$

9. With $u = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)$, $y = u^4$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 4u^3 \cdot \left(\frac{x}{4} + 1 + \frac{1}{x^2}\right) = 4\left(\frac{x^2}{8} + x - \frac{1}{x}\right)^3 \left(\frac{x}{4} + 1 + \frac{1}{x^2}\right)$

10. With $u = \tan x$, $y = \sec u$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\sec u \tan u)(\sec^2 x) = (\sec(\tan x)\tan(\tan x))\sec^2 x$

11. With $u = \pi - \frac{1}{x}$, $y = \cot u$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (-\csc^2 u)\left(\frac{1}{x^2}\right) = -\frac{1}{x^2}\csc^2\left(\pi - \frac{1}{x}\right)$

12. With $u = \sin x$, $y = u^3$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2 \cos x = 3(\sin^2 x)(\cos x)$

13. $q = \sqrt{2r - r^2} = (2r - r^2)^{1/2} \Rightarrow \frac{dq}{dr} = \frac{1}{2}(2r - r^2)^{-1/2} \cdot \frac{d}{dr}(2r - r^2) = \frac{1}{2}(2r - r^2)^{-1/2}(2 - 2r) = \frac{1 - r}{\sqrt{2r - r^2}}$

14. $s = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right) \Rightarrow \frac{ds}{dt} = \cos\left(\frac{3\pi t}{2}\right) \cdot \frac{d}{dt}\left(\frac{3\pi t}{2}\right) - \sin\left(\frac{3\pi t}{2}\right) \cdot \frac{d}{dt}\left(\frac{3\pi t}{2}\right) = \frac{3\pi}{2} \cos\left(\frac{3\pi t}{2}\right) - \frac{3\pi}{2} \sin\left(\frac{3\pi t}{2}\right)$
$= \frac{3\pi}{2}\left(\cos \frac{3\pi t}{2} - \sin \frac{3\pi t}{2}\right)$

15. $r = (\csc\theta + \cot\theta)^{-1} \Rightarrow \frac{dr}{d\theta} = -(\csc\theta + \cot\theta)^{-2} \frac{d}{d\theta}(\csc\theta + \cot\theta) = \frac{\csc\theta \cot\theta + \csc^2\theta}{(\csc\theta + \cot\theta)^2} = \frac{\csc\theta(\cot\theta + \csc\theta)}{(\csc\theta + \cot\theta)^2}$
$= \frac{\csc\theta}{\csc\theta + \cot\theta}$

16. $r = -(\sec\theta + \tan\theta)^{-1} \Rightarrow \frac{dr}{d\theta} = (\sec\theta + \tan\theta)^{-2} \frac{d}{d\theta}(\sec\theta + \tan\theta) = \frac{\sec\theta \tan\theta + \sec^2\theta}{(\sec\theta + \tan\theta)^2} = \frac{\sec\theta(\tan\theta + \sec\theta)}{(\sec\theta + \tan\theta)^2}$
$= \frac{\sec\theta}{\sec\theta + \tan\theta}$

17. $y = x^2 \sin^4 x + x \cos^{-2} x \Rightarrow \frac{dy}{dx} = x^2 \frac{d}{dx}(\sin^4 x) + \sin^4 x \cdot \frac{d}{dx}(x^2) + x \frac{d}{dx}(\cos^{-2} x) + \cos^{-2} x \cdot \frac{d}{dx}(x)$
$= x^2\left(4 \sin^3 x \frac{d}{dx}(\sin x)\right) + 2x \sin^4 x + x\left(-2 \cos^{-3} x \cdot \frac{d}{dx}(\cos x)\right) + \cos^{-2} x$
$= x^2(4 \sin^3 x \cos x) + 2x \sin^4 x + x\left((-2 \cos^{-3} x)(-\sin x)\right) + \cos^{-2} x$
$= 4x^2 \sin^3 x \cos x + 2x \sin^4 x + 2x \sin x \cos^{-3} x + \cos^{-2} x$

18. $y = \frac{1}{x} \sin^{-5} x - \frac{x}{3} \cos^3 x \Rightarrow y' = \frac{1}{x} \frac{d}{dx}(\sin^{-5} x) + \sin^{-5} x \cdot \frac{d}{dx}\left(\frac{1}{x}\right) - \frac{x}{3} \frac{d}{dx}(\cos^3 x) - \cos^3 x \cdot \frac{d}{dx}\left(\frac{x}{3}\right)$
$= \frac{1}{x}(-5 \sin^{-6} x \cos x) + (\sin^{-5} x)\left(-\frac{1}{x^2}\right) - \frac{x}{3}\left((3 \cos^2 x)(-\sin x)\right) - (\cos^3 x)\left(\frac{1}{3}\right)$
$= -\frac{5}{x} \sin^{-6} x \cos x - \frac{1}{x^2} \sin^{-5} x + x \cos^2 x \sin x - \frac{1}{3} \cos^3 x$

19. $y = \frac{1}{21}(3x - 2)^7 + \left(4 - \frac{1}{2x^2}\right)^{-1} \Rightarrow \frac{dy}{dx} = \frac{7}{21}(3x - 2)^6 \cdot \frac{d}{dx}(3x - 2) + (-1)\left(4 - \frac{1}{2x^2}\right)^{-2} \cdot \frac{d}{dx}\left(4 - \frac{1}{2x^2}\right)$
$= \frac{7}{21}(3x - 2)^6 \cdot 3 + (-1)\left(4 - \frac{1}{2x^2}\right)^{-2}\left(\frac{1}{x^3}\right) = (3x - 2)^6 - \frac{1}{x^3\left(4 - \frac{1}{2x^2}\right)^2}$

20. $y = (4x + 3)^4(x + 1)^{-3} \Rightarrow \frac{dy}{dx} = (4x + 3)^4(-3)(x + 1)^{-4} \cdot \frac{d}{dx}(x + 1) + (x + 1)^{-3}(4)(4x + 3)^3 \cdot \frac{d}{dx}(4x + 3)$
$= (4x + 3)^4(-3)(x + 1)^{-4}(1) + (x + 1)^{-3}(4)(4x + 3)^3(4) = -3(4x + 3)^4(x + 1)^{-4} + 16(4x + 3)^3(x + 1)^{-3}$
$= \frac{(4x + 3)^3}{(x + 1)^4}[-3(4x + 3) + 16(x + 1)] = \frac{(4x + 3)^3(4x + 7)}{(x + 1)^4}$

21. $h(x) = x\tan(2\sqrt{x}) + 7 \Rightarrow h'(x) = x\frac{d}{dx}(\tan(2x^{1/2})) + \tan(2x^{1/2}) \cdot \frac{d}{dx}(x) + 0$
$= x\sec^2(2x^{1/2}) \cdot \frac{d}{dx}(2x^{1/2}) + \tan(2x^{1/2}) = x\sec^2(2\sqrt{x}) \cdot \frac{1}{\sqrt{x}} + \tan(2\sqrt{x}) = \sqrt{x}\sec^2(2\sqrt{x}) + \tan(2\sqrt{x})$

22. $k(x) = x^2\sec\left(\frac{1}{x}\right) \Rightarrow k'(x) = x^2\frac{d}{dx}\left(\sec\frac{1}{x}\right) + \sec\left(\frac{1}{x}\right) \cdot \frac{d}{dx}(x^2) = x^2\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right) \cdot \frac{d}{dx}\left(\frac{1}{x}\right) + 2x\sec\left(\frac{1}{x}\right)$
$= x^2\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) + 2x\sec\left(\frac{1}{x}\right) = 2x\sec\left(\frac{1}{x}\right) - \sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)$

23. $f(\theta) = \left(\frac{\sin\theta}{1+\cos\theta}\right)^2 \Rightarrow f'(\theta) = 2\left(\frac{\sin\theta}{1+\cos\theta}\right) \cdot \frac{d}{d\theta}\left(\frac{\sin\theta}{1+\cos\theta}\right) = \frac{2\sin\theta}{1+\cos\theta} \cdot \frac{(1+\cos\theta)(\cos\theta) - (\sin\theta)(-\sin\theta)}{(1+\cos\theta)^2}$
$= \frac{(2\sin\theta)(\cos\theta + \cos^2\theta + \sin^2\theta)}{(1+\cos\theta)^3} = \frac{(2\sin\theta)(\cos\theta+1)}{(1+\cos\theta)^3} = \frac{2\sin\theta}{(1+\cos\theta)^2}$

24. $r = \sin(\theta^2)\cos(2\theta) \Rightarrow \frac{dr}{d\theta} = \sin(\theta^2)(-\sin 2\theta)\frac{d}{d\theta}(2\theta) + \cos(2\theta)(\cos(\theta^2)) \cdot \frac{d}{d\theta}(\theta^2)$
$= \sin(\theta^2)(-\sin 2\theta)(2) + (\cos 2\theta)(\cos(\theta^2))(2\theta) = -2\sin(\theta^2)\sin(2\theta) + 2\theta\cos(2\theta)\cos(\theta^2)$

25. $r = (\sec\sqrt{\theta})\tan\left(\frac{1}{\theta}\right) \Rightarrow \frac{dr}{d\theta} = (\sec\sqrt{\theta})\left(\sec^2\frac{1}{\theta}\right)\left(-\frac{1}{\theta^2}\right) + \tan\left(\frac{1}{\theta}\right)(\sec\sqrt{\theta}\tan\sqrt{\theta})\left(\frac{1}{2\sqrt{\theta}}\right)$
$= -\frac{1}{\theta^2}\sec\sqrt{\theta}\sec^2\left(\frac{1}{\theta}\right) + \frac{1}{2\sqrt{\theta}}\tan\left(\frac{1}{\theta}\right)\sec\sqrt{\theta}\tan\sqrt{\theta} = (\sec\sqrt{\theta})\left[\frac{\tan\sqrt{\theta}\tan\left(\frac{1}{\theta}\right)}{2\sqrt{\theta}} - \frac{\sec^2\left(\frac{1}{\theta}\right)}{\theta^2}\right]$

26. $q = \sin\left(\frac{t}{\sqrt{t+1}}\right) \Rightarrow \frac{dq}{dt} = \cos\left(\frac{t}{\sqrt{t+1}}\right) \cdot \frac{d}{dt}\left(\frac{t}{\sqrt{t+1}}\right) = \cos\left(\frac{t}{\sqrt{t+1}}\right) \cdot \frac{\sqrt{t+1}(1) - t \cdot \frac{d}{dt}(\sqrt{t+1})}{(\sqrt{t+1})^2}$
$= \cos\left(\frac{t}{\sqrt{t+1}}\right) \cdot \frac{\sqrt{t+1} - \frac{t}{2\sqrt{t+1}}}{t+1} = \cos\left(\frac{t}{\sqrt{t+1}}\right)\left(\frac{2(t+1)-t}{2(t+1)^{3/2}}\right) = \left(\frac{t+2}{2(t+1)^{3/2}}\right)\cos\left(\frac{t}{\sqrt{t+1}}\right)$

27. $y = \sin^2(\pi t - 2) \Rightarrow \frac{dy}{dt} = 2\sin(\pi t - 2) \cdot \frac{d}{dt}\sin(\pi t - 2) = 2\sin(\pi t - 2) \cdot \cos(\pi t - 2) \cdot \frac{d}{dt}(\pi t - 2)$
$= 2\pi\sin(\pi t - 2)\cos(\pi t - 2)$

28. $y = (1+\cos 2t)^{-4} \Rightarrow \frac{dy}{dt} = -4(1+\cos 2t)^{-5} \cdot \frac{d}{dt}(1+\cos 2t) = -4(1+\cos 2t)^{-5}(-\sin 2t) \cdot \frac{d}{dt}(2t) = \frac{8\sin 2t}{(1+\cos 2t)^5}$

29. $y = \left(1+\cot\left(\frac{t}{2}\right)\right)^{-2} \Rightarrow \frac{dy}{dt} = -2\left(1+\cot\left(\frac{t}{2}\right)\right)^{-3} \cdot \frac{d}{dt}\left(1+\cot\left(\frac{t}{2}\right)\right) = -2\left(1+\cot\left(\frac{t}{2}\right)\right)^{-3} \cdot \left(-\csc^2\left(\frac{t}{2}\right)\right) \cdot \frac{d}{dt}\left(\frac{t}{2}\right)$
$= \frac{\csc^2\left(\frac{t}{2}\right)}{\left(1+\cot\left(\frac{t}{2}\right)\right)^3}$

30. $y = \sin(\cos(2t-5)) \Rightarrow \frac{dy}{dt} = \cos(\cos(2t-5)) \cdot \frac{d}{dt}\cos(2t-5) = \cos(\cos(2t-5)) \cdot (-\sin(2t-5)) \cdot \frac{d}{dt}(2t-5)$
$= -2\cos(\cos(2t-5))(\sin(2t-5))$

31. $y = \left[1+\tan^4\left(\frac{t}{12}\right)\right]^3 \Rightarrow \frac{dy}{dt} = 3\left[1+\tan^4\left(\frac{t}{12}\right)\right]^2 \cdot \frac{d}{dt}\left[1+\tan^4\left(\frac{t}{12}\right)\right] = 3\left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\left[4\tan^3\left(\frac{t}{12}\right) \cdot \frac{d}{dt}\tan\left(\frac{t}{12}\right)\right]$
$= 12\left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\left[\tan^3\left(\frac{t}{12}\right)\sec^2\left(\frac{t}{12}\right) \cdot \frac{1}{12}\right] = \left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\left[\tan^3\left(\frac{t}{12}\right)\sec^2\left(\frac{t}{12}\right)\right]$

32. $y = \left(1+\cos\left(t^2\right)\right)^{1/2} \Rightarrow \frac{dy}{dt} = \frac{1}{2}\left(1+\cos\left(t^2\right)\right)^{-1/2} \cdot \frac{d}{dt}\left(1+\cos\left(t^2\right)\right) = \frac{1}{2}\left(1+\cos\left(t^2\right)\right)^{-1/2}\left(-\sin\left(t^2\right) \cdot \frac{d}{dt}\left(t^2\right)\right)$
$= -\frac{1}{2}\left(1+\cos\left(t^2\right)\right)^{-1/2}\left(\sin\left(t^2\right)\right) \cdot 2t = -\frac{t\sin\left(t^2\right)}{\sqrt{1+\cos\left(t^2\right)}}$

33. $t = \frac{\pi}{4} \Rightarrow x = 2\cos\frac{\pi}{4} = \sqrt{2},\ y = 2\sin\frac{\pi}{4} = \sqrt{2};\ \frac{dx}{dt} = -2\sin t,\ \frac{dy}{dt} = 2\cos t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-2\sin t} = -\cot t$
$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = -\cot\frac{\pi}{4} = -1$; tangent line is $y - \sqrt{2} = -1\left(x - \sqrt{2}\right)$ or $y = -x + 2\sqrt{2}$; $\frac{dy'}{dt} = \csc^2 t$
$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\csc^2 t}{-2\sin t} = -\frac{1}{2\sin^3 t} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}} = -\sqrt{2}$

34. $t = \frac{2\pi}{3} \Rightarrow x = \cos\frac{2\pi}{3} = -\frac{1}{2},\ y = \sqrt{3}\cos\frac{2\pi}{3} = -\frac{\sqrt{3}}{2};\ \frac{dx}{dt} = -\sin t,\ \frac{dy}{dt} = -\sqrt{3}\sin t \Rightarrow \frac{dy}{dx} = \frac{-\sqrt{3}\sin t}{-\sin t} = \sqrt{3}$
$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{2\pi}{3}} = \sqrt{3}$; tangent line is $y - \left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}\left[x - \left(-\frac{1}{2}\right)\right]$ or $y = \sqrt{3}x$; $\frac{dy'}{dt} = 0 \Rightarrow \frac{d^2y}{dx^2} = \frac{0}{-\sin t} = 0$
$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{2\pi}{3}} = 0$

35. $t = \frac{1}{4} \Rightarrow x = \frac{1}{4},\ y = \frac{1}{2};\ \frac{dx}{dt} = 1,\ \frac{dy}{dt} = \frac{1}{2\sqrt{t}} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{2\sqrt{t}} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{1}{4}} = \frac{1}{2\sqrt{\frac{1}{4}}} = 1$; tangent line is
$y - \frac{1}{2} = 1 \cdot \left(x - \frac{1}{4}\right)$ or $y = x + \frac{1}{4}$; $\frac{dy'}{dt} = -\frac{1}{4}t^{-3/2} \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = -\frac{1}{4}t^{-3/2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{1}{4}} = -2$

36. $t = 3 \Rightarrow x = -\sqrt{3+1} = -2,\ y = \sqrt{3(3)} = 3;\ \frac{dx}{dt} = -\frac{1}{2}(t+1)^{-1/2},\ \frac{dy}{dt} = \frac{3}{2}(3t)^{-1/2} \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{3}{2}\right)(3t)^{-1/2}}{\left(-\frac{1}{2}\right)(t+1)^{-1/2}}$
$= -\frac{3\sqrt{t+1}}{\sqrt{3t}} = \left.\frac{dy}{dx}\right|_{t=3} = \frac{-3\sqrt{3+1}}{\sqrt{3(3)}} = -2$; tangent line is $y - 3 = -2[x - (-2)]$ or $y = -2x - 1$;
$\frac{dy'}{dt} = \frac{\sqrt{3t}\left[-\frac{3}{2}(t+1)^{-1/2}\right] + 3\sqrt{t+1}\left[\frac{3}{2}(3t)^{-1/2}\right]}{3t} = \frac{3}{2t\sqrt{3t}\,\sqrt{t+1}} \Rightarrow \frac{d^2y}{dx^2} = \frac{\left(\frac{3}{2t\sqrt{3t}\,\sqrt{t+1}}\right)}{\left(\frac{-1}{2\sqrt{t+1}}\right)} = -\frac{3}{t\sqrt{3t}}$

$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=3} = -\frac{1}{3}$

37. $t = -1 \Rightarrow x = 5,\ y = 1;\ \frac{dx}{dt} = 4t,\ \frac{dy}{dt} = 4t^3 \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3}{4t} = t^2 \Rightarrow \left.\frac{dy}{dx}\right|_{t=-1} = (-1)^2 = 1$; tangent line is $y - 1 = 1\cdot(x-5)$ or $y = x - 4;\ \frac{dy'}{dt} = 2t \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{2t}{4t} = \frac{1}{2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-1} = \frac{1}{2}$

38. $t = \frac{\pi}{3} \Rightarrow x = \frac{\pi}{3} - \sin\frac{\pi}{3} = \frac{\pi}{3} - \frac{\sqrt{3}}{2},\ y = 1 - \cos\frac{\pi}{3} = 1 - \frac{1}{2} = \frac{1}{2};\ \frac{dx}{dt} = 1 - \cos t,\ \frac{dy}{dt} = \sin t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ $= \frac{\sin t}{1-\cos t} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \frac{\sin\left(\frac{\pi}{3}\right)}{1-\cos\left(\frac{\pi}{3}\right)} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = \sqrt{3}$; tangent line is $y - \frac{1}{2} = \sqrt{3}\left(x - \frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$ $\Rightarrow y = \sqrt{3}x - \frac{\pi\sqrt{3}}{3} + 2;\ \frac{dy'}{dt} = \frac{(1-\cos t)(\cos t) - (\sin t)(\sin t)}{(1-\cos t)^2} = \frac{-1}{1-\cos t} \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\left(\frac{-1}{1-\cos t}\right)}{1-\cos t}$ $= \frac{-1}{(1-\cos t)^2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{3}} = -4$

39. $t = \frac{\pi}{2} \Rightarrow x = \cos\frac{\pi}{2} = 0,\ y = 1 + \sin\frac{\pi}{2} = 2;\ \frac{dx}{dt} = -\sin t,\ \frac{dy}{dt} = \cos t \Rightarrow \frac{dy}{dx} = \frac{\cos t}{-\sin t} = -\cot t$ $\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = -\cot\frac{\pi}{2} = 0$; tangent line is $y = 2;\ \frac{dy'}{dt} = \csc^2 t \Rightarrow \frac{d^2y}{dx^2} = \frac{\csc^2 t}{-\sin t} = -\csc^3 t \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{2}} = -1$

40. $t = -\frac{\pi}{4} \Rightarrow x = \sec^2\left(-\frac{\pi}{4}\right) - 1 = 1,\ y = \tan\left(-\frac{\pi}{4}\right) = -1;\ \frac{dx}{dt} = 2\sec^2 t\tan t,\ \frac{dy}{dt} = \sec^2 t$ $\Rightarrow \frac{dy}{dx} = \frac{\sec^2 t}{2\sec^2 t\tan t} = \frac{1}{2\tan t} = \frac{1}{2}\cot t \Rightarrow \left.\frac{dy}{dx}\right|_{t=-\frac{\pi}{4}} = \frac{1}{2}\cot\left(-\frac{\pi}{4}\right) = -\frac{1}{2}$; tangent line is $y - (-1) = -\frac{1}{2}(x-1)$ or $y = -\frac{1}{2}x - \frac{1}{2};\ \frac{dy'}{dt} = -\frac{1}{2}\csc^2 t \Rightarrow \frac{d^2y}{dx^2} = \frac{-\frac{1}{2}\csc^2 t}{2\sec^2 t\tan t} = -\frac{1}{4}\cot^3 t$ $\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-\frac{\pi}{4}} = \frac{1}{4}$

41. $y = \left(1+\frac{1}{x}\right)^3 \Rightarrow y' = 3\left(1+\frac{1}{x}\right)^2\left(-\frac{1}{x^2}\right) = -\frac{3}{x^2}\left(1+\frac{1}{x}\right)^2 \Rightarrow y'' = \left(-\frac{3}{x^2}\right)\cdot\frac{d}{dx}\left(1+\frac{1}{x}\right)^2 - \left(1+\frac{1}{x}\right)^2\cdot\frac{d}{dx}\left(\frac{3}{x^2}\right)$ $= \left(-\frac{3}{x^2}\right)\left(2\left(1+\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\right) + \left(\frac{6}{x^3}\right)\left(1+\frac{1}{x}\right)^2 = \frac{6}{x^4}\left(1+\frac{1}{x}\right) + \frac{6}{x^3}\left(1+\frac{1}{x}\right)^2 = \frac{6}{x^3}\left(1+\frac{1}{x}\right)\left(\frac{1}{x}+1+\frac{1}{x}\right)$ $= \frac{6}{x^3}\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x}\right)$

42. $y = (1-\sqrt{x})^{-1} \Rightarrow y' = -(1-\sqrt{x})^{-2}\left(-\frac{1}{2}x^{-1/2}\right) = \frac{1}{2}(1-\sqrt{x})^{-2}x^{-1/2}$ $\Rightarrow y'' = \frac{1}{2}\left[(1-\sqrt{x})^{-2}\left(-\frac{1}{2}x^{-3/2}\right) + x^{-1/2}(-2)(1-\sqrt{x})^{-3}\left(-\frac{1}{2}x^{-1/2}\right)\right]$

$= \frac{1}{2}\left[\frac{-1}{2}x^{-3/2}(1-\sqrt{x})^{-2} + x^{-1}(1-\sqrt{x})^{-3}\right] = \frac{1}{2}x^{-1}(1-\sqrt{x})^{-3}\left[-\frac{1}{2}x^{-1/2}(1-\sqrt{x})+1\right]$

$= \frac{1}{2x}(1-\sqrt{x})^{-3}\left(-\frac{1}{2\sqrt{x}}+\frac{1}{2}+1\right) = \frac{1}{2x}(1-\sqrt{x})^{-3}\left(\frac{3}{2}-\frac{1}{2\sqrt{x}}\right)$

43. $y = \frac{1}{9}\cot(3x-1) \Rightarrow y' = -\frac{1}{9}\csc^2(3x-1)(3) = -\frac{1}{3}\csc^2(3x-1) \Rightarrow y'' = \left(-\frac{2}{3}\right)\left(\csc(3x-1)\cdot\frac{d}{dx}\csc(3x-1)\right)$

$= -\frac{2}{3}\csc(3x-1)\left(-\csc(3x-1)\cot(3x-1)\cdot\frac{d}{dx}(3x-1)\right) = 2\csc^2(3x-1)\cot(3x-1)$

44. $y = 9\tan\left(\frac{x}{3}\right) \Rightarrow y' = 9\left(\sec^2\left(\frac{x}{3}\right)\right)\left(\frac{1}{3}\right) = 3\sec^2\left(\frac{x}{3}\right) \Rightarrow y'' = 3\cdot 2\sec\left(\frac{x}{3}\right)\left(\sec\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)\right)\left(\frac{1}{3}\right) = 2\sec^2\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)$

45. $g(x) = \sqrt{x} \Rightarrow g'(x) = \frac{1}{2\sqrt{x}} \Rightarrow g(1) = 1$ and $g'(1) = \frac{1}{2}$; $f(u) = u^5 + 1 \Rightarrow f'(u) = 5u^4 \Rightarrow f'(g(1)) = f'(1) = 5$;

therefore, $(f\circ g)'(1) = f'(g(1))\cdot g'(1) = 5\cdot\frac{1}{2} = \frac{5}{2}$

46. $g(x) = (1-x)^{-1} \Rightarrow g'(x) = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2} \Rightarrow g(-1) = \frac{1}{2}$ and $g'(-1) = \frac{1}{4}$; $f(u) = 1 - \frac{1}{u}$

$\Rightarrow f'(u) = \frac{1}{u^2} \Rightarrow f'(g(-1)) = f'\left(\frac{1}{2}\right) = 4$; therefore, $(f\circ g)'(-1) = f'(g(-1))g'(-1) = 4\cdot\frac{1}{4} = 1$

47. $g(x) = 5\sqrt{x} \Rightarrow g'(x) = \frac{5}{2\sqrt{x}} \Rightarrow g(1) = 5$ and $g'(1) = \frac{5}{2}$; $f(u) = \cot\left(\frac{\pi u}{10}\right) \Rightarrow f'(u) = -\csc^2\left(\frac{\pi u}{10}\right)\left(\frac{\pi}{10}\right)$

$= \frac{-\pi}{10}\csc^2\left(\frac{\pi u}{10}\right) \Rightarrow f'(g(1)) = f'(5) = -\frac{\pi}{10}\csc^2\left(\frac{\pi}{2}\right) = -\frac{\pi}{10}$; therefore, $(f\circ g)'(1) = f'(g(1))g'(1) = -\frac{\pi}{10}\cdot\frac{5}{2}$

$= -\frac{\pi}{4}$

48. $g(x) = \pi x \Rightarrow g'(x) = \pi \Rightarrow g\left(\frac{1}{4}\right) = \frac{\pi}{4}$ and $g'\left(\frac{1}{4}\right) = \pi$; $f(u) = u + \sec^2 u \Rightarrow f'(u) = 1 + 2\sec u\cdot\sec u\tan u$

$= 1 + 2\sec^2 u\tan u \Rightarrow f'\left(g\left(\frac{1}{4}\right)\right) = f'\left(\frac{\pi}{4}\right) = 1 + 2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4} = 5$; therefore, $(f\circ g)'\left(\frac{1}{4}\right) = f'\left(g\left(\frac{1}{4}\right)\right)g'\left(\frac{1}{4}\right) = 5\pi$

49. $g(x) = 10x^2 + x + 1 \Rightarrow g'(x) = 20x + 1 \Rightarrow g(0) = 1$ and $g'(0) = 1$; $f(u) = \frac{2u}{u^2+1} \Rightarrow f'(u) = \frac{(u^2+1)(2)-(2u)(2u)}{(u^2+1)^2}$

$= \frac{-2u^2+2}{(u^2+1)^2} \Rightarrow f'(g(0)) = f'(1) = 0$; therefore, $(f\circ g)'(0) = f'(g(0))g'(0) = 0\cdot 1 = 0$

50. $g(x) = \frac{1}{x^2} - 1 \Rightarrow g'(x) = -\frac{2}{x^3} \Rightarrow g(-1) = 0$ and $g'(-1) = 2$; $f(u) = \left(\frac{u-1}{u+1}\right)^2 \Rightarrow f'(u) = 2\left(\frac{u-1}{u+1}\right)\frac{d}{du}\left(\frac{u-1}{u+1}\right)$

$= 2\left(\frac{u-1}{u+1}\right)\cdot\frac{(u+1)(1)-(u-1)(1)}{(u+1)^2} = \frac{2(u-1)(2)}{(u+1)^3} = \frac{4(u-1)}{(u+1)^3} \Rightarrow f'(g(-1)) = f'(0) = -4$; therefore,

$(f\circ g)'(-1) = f'(g(-1))g'(-1) = (-4)(2) = -8$

51. (a) $y = 2f(x) \Rightarrow \frac{dy}{dx} = 2f'(x) \Rightarrow \frac{dy}{dx}\Big|_{x=2} = 2f'(2) = 2\left(\frac{1}{3}\right) = \frac{2}{3}$

(b) $y = f(x) + g(x) \Rightarrow \frac{dy}{dx} = f'(x) + g'(x) \Rightarrow \frac{dy}{dx}\Big|_{x=3} = f'(3) + g'(3) = 2\pi + 5$

(c) $y = f(x) \cdot g(x) \Rightarrow \frac{dy}{dx} = f(x)g'(x) + g(x)f'(x) \Rightarrow \frac{dy}{dx}\Big|_{x=3} = f(3)g'(3) + g(3)f'(3) = 3 \cdot 5 + (-4)(2\pi) = 15 - 8\pi$

(d) $y = \frac{f(x)}{g(x)} \Rightarrow \frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \Rightarrow \frac{dy}{dx}\Big|_{x=2} = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2} = \frac{(2)\left(\frac{1}{3}\right) - (8)(-3)}{2^2} = \frac{37}{6}$

(e) $y = f(g(x)) \Rightarrow \frac{dy}{dx} = f'(g(x))g'(x) \Rightarrow \frac{dy}{dx}\Big|_{x=2} = f'(g(2))g'(2) = f'(2)(-3) = \frac{1}{3}(-3) = -1$

(f) $y = (f(x))^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(f(x))^{-1/2} \cdot f'(x) = \frac{f'(x)}{2\sqrt{f(x)}} \Rightarrow \frac{dy}{dx}\Big|_{x=2} = \frac{f'(2)}{2\sqrt{f(2)}} = \frac{\left(\frac{1}{3}\right)}{2\sqrt{8}} = \frac{1}{6\sqrt{8}} = \frac{1}{12\sqrt{2}} = \frac{\sqrt{2}}{24}$

(g) $y = (g(x))^{-2} \Rightarrow \frac{dy}{dx} = -2(g(x))^{-3} \cdot g'(x) \Rightarrow \frac{dy}{dx}\Big|_{x=3} = -2(g(3))^{-3}g'(3) = -2(-4)^{-3} \cdot 5 = \frac{5}{32}$

(h) $y = \left((f(x))^2 + (g(x))^2\right)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left((f(x))^2 + (g(x))^2\right)^{-1/2}\left(2f(x) \cdot f'(x) + 2g(x) \cdot g'(x)\right)$

$\Rightarrow \frac{dy}{dx}\Big|_{x=2} = \frac{1}{2}\left((f(2))^2 + (g(2))^2\right)^{-1/2}\left(2f(2)f'(2) + 2g(2)g'(2)\right) = \frac{1}{2}\left(8^2 + 2^2\right)^{-1/2}\left(2 \cdot 8 \cdot \frac{1}{3} + 2 \cdot 2 \cdot (-3)\right)$

$= -\frac{5}{3\sqrt{17}}$

52. (a) $y = 5f(x) - g(x) \Rightarrow \frac{dy}{dx} = 5f'(x) - g'(x) \Rightarrow \frac{dy}{dx}\Big|_{x=1} = 5f'(1) - g'(1) = 5\left(-\frac{1}{3}\right) - \left(\frac{-8}{3}\right) = 1$

(b) $y = f(x)(g(x))^3 \Rightarrow \frac{dy}{dx} = f(x)\left(3(g(x))^2 g'(x)\right) + (g(x))^3 f'(x) \Rightarrow \frac{dy}{dx}\Big|_{x=0} = 3f(0)(g(0))^2 g'(0) + (g(0))^3 f'(0)$

$= 3(1)(1)^2\left(\frac{1}{3}\right) + (1)^3(5) = 6$

(c) $y = \frac{f(x)}{g(x) + 1} \Rightarrow \frac{dy}{dx} = \frac{(g(x)+1)f'(x) - f(x)\,g'(x)}{(g(x)+1)^2} \Rightarrow \frac{dy}{dx}\Big|_{x=1} = \frac{(g(1)+1)f'(1) - f(1)g'(1)}{(g(1)+1)^2}$

$= \frac{(-4+1)\left(-\frac{1}{3}\right) - (3)\left(-\frac{8}{3}\right)}{(-4+1)^2} = 1$

(d) $y = f(g(x)) \Rightarrow \frac{dy}{dx} = f'(g(x))g'(x) \Rightarrow \frac{dy}{dx}\Big|_{x=0} = f'(g(0))g'(0) = f'(1)\left(\frac{1}{3}\right) = \left(-\frac{1}{3}\right)\left(\frac{1}{3}\right) = -\frac{1}{9}$

(e) $y = g(f(x)) \Rightarrow \frac{dy}{dx} = g'(f(x))f'(x) \Rightarrow \frac{dy}{dx}\Big|_{x=0} = g'(f(0))f'(0) = g'(1)(5) = \left(-\frac{8}{3}\right)(5) = -\frac{40}{3}$

(f) $y = \left(x^{11} + f(x)\right)^{-2} \Rightarrow \frac{dy}{dx} = -2\left(x^{11} + f(x)\right)^{-3}\left(11x^{10} + f'(x)\right) \Rightarrow \frac{dy}{dx}\Big|_{x=1} = -2(1 + f(1))^{-3}(11 + f'(1))$

$= -2(1+3)^{-3}\left(11 - \frac{1}{3}\right) = \left(-\frac{2}{4^3}\right)\left(\frac{32}{3}\right) = -\frac{1}{3}$

(g) $y = f(x+g(x)) \Rightarrow \frac{dy}{dx} = f'(x+g(x))\left(1+g'(x)\right) \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = f'(0+g(0))\left(1+g'(0)\right) = f'(1)\left(1+\frac{1}{3}\right)$
$= \left(-\frac{1}{3}\right)\left(\frac{4}{3}\right) = -\frac{4}{9}$

53. $\frac{ds}{dt} = \frac{ds}{d\theta}\cdot\frac{d\theta}{dt}$: $s = \cos\theta \Rightarrow \frac{ds}{d\theta} = -\sin\theta \Rightarrow \left.\frac{ds}{d\theta}\right|_{\theta=\frac{3\pi}{2}} = -\sin\left(\frac{3\pi}{2}\right) = 1$ so that $\frac{ds}{dt} = \frac{ds}{d\theta}\cdot\frac{d\theta}{dt} = 1\cdot 5 = 5$

54. $\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$: $y = x^2+7x-5 \Rightarrow \frac{dy}{dx} = 2x+7 \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = 9$ so that $\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} = 9\cdot\frac{1}{3} = 3$

55. With $y = x$, we should get $\frac{dy}{dx} = 1$ for both (a) and (b):

(a) $y = \frac{u}{5}+7 \Rightarrow \frac{dy}{du} = \frac{1}{5}$; $u = 5x-35 \Rightarrow \frac{du}{dx} = 5$; therefore, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{5}\cdot 5 = 1$, as expected

(b) $y = 1+\frac{1}{u} \Rightarrow \frac{dy}{du} = -\frac{1}{u^2}$; $u = (x-1)^{-1} \Rightarrow \frac{du}{dx} = -(x-1)^{-2}(1) = \frac{-1}{(x-1)^2}$; therefore $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$
$= \frac{-1}{u^2}\cdot\frac{-1}{(x-1)^2} = \frac{-1}{\left((x-1)^{-1}\right)^2}\cdot\frac{-1}{(x-1)^2} = (x-1)^2\cdot\frac{1}{(x-1)^2} = 1$, again as expected

56. With $y = x^{3/2}$, we should get $\frac{dy}{dx} = \frac{3}{2}x^{1/2}$ for both (a) and (b):

(a) $y = u^3 \Rightarrow \frac{dy}{du} = 3u^2$; $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}}$; therefore, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 3u^2\cdot\frac{1}{2\sqrt{x}} = 3\left(\sqrt{x}\right)^2\cdot\frac{1}{2\sqrt{x}} = \frac{3}{2}\sqrt{x}$, as expected.

(b) $y = \sqrt{u} \Rightarrow \frac{dy}{du} = \frac{1}{2\sqrt{u}}$; $u = x^3 \Rightarrow \frac{du}{dx} = 3x^2$; therefore, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{2\sqrt{u}}\cdot 3x^2 = \frac{1}{2\sqrt{x^3}}\cdot 3x^2 = \frac{3}{2}x^{1/2}$, again as expected.

57. $y = 2\tan\left(\frac{\pi x}{4}\right) \Rightarrow \frac{dy}{dx} = \left(2\sec^2\frac{\pi x}{4}\right)\left(\frac{\pi}{4}\right) = \frac{\pi}{2}\sec^2\frac{\pi x}{4}$

(a) $\left.\frac{dy}{dx}\right|_{x=1} = \frac{\pi}{2}\sec^2\left(\frac{\pi}{4}\right) = \pi \Rightarrow$ slope of tangent is 2; thus, $y(1) = 2\tan\left(\frac{\pi}{4}\right) = 2$ and $y'(1) = \pi \Rightarrow$ tangent line is given by $y-2 = \pi(x-1) \Rightarrow y = \pi x + 2 - \pi$

(b) $y' = \frac{\pi}{2}\sec^2\left(\frac{\pi x}{4}\right)$ and the smallest value the secant function can have in $-2 < x < 2$ is $1 \Rightarrow$ the minimum value of y' is $\frac{\pi}{2}$ and that occurs when $\frac{\pi}{2} = \frac{\pi}{2}\sec^2\left(\frac{\pi x}{4}\right) \Rightarrow 1 = \sec^2\left(\frac{\pi x}{4}\right) \Rightarrow \pm 1 = \sec\left(\frac{\pi x}{4}\right) \Rightarrow x = 0$.

58. (a) $y = \sin 2x \Rightarrow y' = 2\cos 2x \Rightarrow y'(0) = 2\cos(0) = 2 \Rightarrow$ tangent to $y = \sin 2x$ at the origin is $y = 2x$; $y = -\sin\left(\frac{x}{2}\right) \Rightarrow y' = -\frac{1}{2}\cos\left(\frac{x}{2}\right) \Rightarrow y'(0) = -\frac{1}{2}\cos 0 = -\frac{1}{2} \Rightarrow$ tangent to $y = -\sin\left(\frac{x}{2}\right)$ at the origin is $y = -\frac{1}{2}x$. The tangents are perpendicular to each other at the origin since the product of their slopes is -1.

(b) $y = \sin(mx) \Rightarrow y' = m\cos(mx) \Rightarrow y'(0) = m\cos 0 = m$; $y = -\sin\left(\frac{x}{m}\right) \Rightarrow y' = -\frac{1}{m}\cos\left(\frac{x}{m}\right)$ $\Rightarrow y'(0) = -\frac{1}{m}\cos(0) = -\frac{1}{m}$. Since $m\cdot\left(-\frac{1}{m}\right) = -1$, the tangent lines are perpendicular at the origin.

(c) $y = \sin(mx) \Rightarrow y' = m\cos(mx)$. The largest value $\cos(mx)$ can attain is 1 at $x = 0 \Rightarrow$ the largest value y' can attain is $|m|$ because $|y'| = |m\cos(mx)| = |m||\cos mx| \le |m|\cdot 1 = |m|$. Also, $y = -\sin\left(\frac{x}{m}\right)$ $\Rightarrow y' = -\frac{1}{m}\cos\left(\frac{x}{m}\right) \Rightarrow |y'| = \left|\frac{-1}{m}\cos\left(\frac{x}{m}\right)\right| \le \left|\frac{1}{m}\right|\left|\cos\left(\frac{x}{m}\right)\right| \le \frac{1}{|m|} \Rightarrow$ the largest value y' can attain is $\left|\frac{1}{m}\right|$.

(d) $y = \sin(mx) \Rightarrow y' = m\cos(mx) \Rightarrow y'(0) = m \Rightarrow$ slope of curve at the origin is m. Also, $\sin(mx)$ completes m periods on $[0, 2\pi]$. Therefore the slope of the curve $y = \sin(mx)$ at the origin is the same as the number of periods it completes on $[0, 2\pi]$. In particular, for large m, we can think of "compressing" the graph of $y = \sin x$ horizontally which gives more periods completed on $[0, 2\pi]$, but also increases the slope of the graph at the origin.

59. $s = A\cos(2\pi bt) \Rightarrow v = \frac{ds}{dt} = -A\sin(2\pi bt)(2\pi b) = -2\pi bA\sin(2\pi bt)$. If we replace b with 2b to double the frequency, the velocity formula gives $v = -4\pi bA\sin(4\pi bt) \Rightarrow$ doubling the frequency causes the velocity to double. Also $v = -2\pi bA\sin(2\pi bt) \Rightarrow a = \frac{dv}{dt} = -4\pi^2b^2A\cos(2\pi bt)$. If we replace b with 2b in the acceleration formula, we get $a = -16\pi^2b^2A\cos(4\pi bt) \Rightarrow$ doubling the frequency causes the acceleration to quadruple. Finally, $a = -4\pi^2b^2A\cos(2\pi bt) \Rightarrow j = \frac{da}{dt} = 8\pi^3b^3A\sin(2\pi bt)$. If we replace b with 2b in the jerk formula, we get $j = 64\pi^3b^3A\sin(2\pi bt) \Rightarrow$ doubling the frequency multiplies the jerk by a factor of 8.

60. (a) $y = 37\sin\left[\frac{2\pi}{365}(x - 101)\right] + 25 \Rightarrow y' = 37\cos\left[\frac{2\pi}{365}(x - 101)\right]\left(\frac{2\pi}{365}\right) = \frac{74\pi}{365}\cos\left[\frac{2\pi}{365}(x - 101)\right]$.

The temperature is increasing the fastest when y' is as large as possible. The largest value of $\cos\left[\frac{2\pi}{365}(x - 101)\right]$ is 1 and occurs when $\frac{2\pi}{365}(x - 101) = 0 \Rightarrow x = 101 \Rightarrow$ on day 101 of the year ($\sim$ April 11), the temperature is increasing the fastest.

(b) $y'(101) = \frac{74\pi}{365}\cos\left[\frac{2\pi}{365}(101 - 101)\right] = \frac{74\pi}{365}\cos(0) = \frac{74\pi}{365} \approx 0.64$ °F/day

61. $s = (1 + 4t)^{1/2} \Rightarrow v = \frac{ds}{dt} = \frac{1}{2}(1 + 4t)^{-1/2}(4) = 2(1 + 4t)^{-1/2} \Rightarrow v(6) = 2(1 + 4\cdot 6)^{-1/2} = \frac{2}{5}$ m/sec; $v = 2(1 + 4t)^{-1/2} \Rightarrow a = \frac{dv}{dt} = -\frac{1}{2}\cdot 2(1 + 4t)^{-3/2}(4) = -4(1 + 4t)^{-3/2} \Rightarrow a(6) = -4(1 + 4\cdot 6)^{-3/2} = -\frac{4}{125}$ m/sec^2

62. We need to show $a = \frac{dv}{dt}$ is constant: $a = \frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt}$ and $\frac{dv}{ds} = \frac{d}{ds}(k\sqrt{s}) = \frac{k}{2\sqrt{s}} \Rightarrow a = \frac{dv}{ds}\cdot\frac{ds}{dt} = \frac{dv}{ds}\cdot v$ $= \frac{k}{2\sqrt{s}}\cdot k\sqrt{s} = \frac{k^2}{2}$ which is a constant.

63. v proportional to $\frac{1}{\sqrt{s}} \Rightarrow v = \frac{k}{\sqrt{s}}$ for some constant $k \Rightarrow \frac{dv}{ds} = -\frac{k}{2s^{3/2}}$. Thus, $a = \frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt} = \frac{dv}{ds}\cdot v$ $= -\frac{k}{2s^{3/2}}\cdot\frac{k}{\sqrt{s}} = -\frac{k^2}{2}\left(\frac{1}{s^2}\right) \Rightarrow$ acceleration is a constant times $\frac{1}{s^2}$ so a is proportional to $\frac{1}{s^2}$.

64. Let $\frac{dx}{dt} = f(x)$. Then, $a = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = \frac{dv}{dx}\cdot f(x) = \frac{d}{dx}\left(\frac{dx}{dt}\right)\cdot f(x) = \frac{d}{dx}(f(x))\cdot f(x) = f'(x)f(x)$, as required.

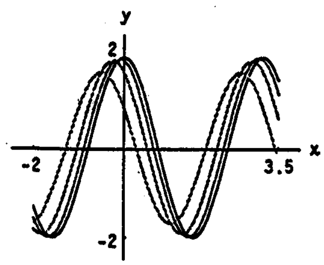
65. $T = 2\pi\sqrt{\frac{L}{g}} \Rightarrow \frac{dT}{dL} = 2\pi\cdot\frac{1}{2\sqrt{\frac{L}{g}}}\cdot\frac{1}{g} = \frac{\pi}{g\sqrt{\frac{L}{g}}} = \frac{\pi}{\sqrt{gL}}$. Therefore, $\frac{dT}{du} = \frac{dT}{dL}\cdot\frac{dL}{du} = \frac{\pi}{\sqrt{gL}}\cdot kL = \frac{\pi k\sqrt{L}}{\sqrt{g}} = \frac{1}{2}\cdot 2\pi k\sqrt{\frac{L}{g}}$ $= \frac{kT}{2}$, as required.

66. No. The chain rule says that when g is differentiable at 0 and f is differentiable at g(0), then $f \circ g$ is differentiable at 0. But the chain rule says nothing about what happens when g is not differentiable at 0 so there is no contradiction.

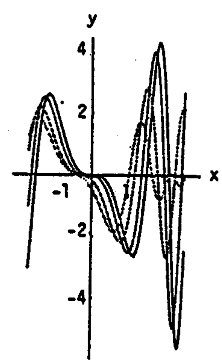
67. The graph of $y = (f \circ g)(x)$ has a horizontal tangent at $x = 1$ provided that $(f \circ g)'(1) = 0 \Rightarrow f'(g(1))g'(1) = 0$ $\Rightarrow$ either $f'(g(1)) = 0$ or $g'(1) = 0$ (or both) $\Rightarrow$ either the graph of f has a horizontal tangent at $u = g(1)$, or the graph of g has a horizontal tangent at $x = 1$ (or both).

68. $(f \circ g)'(-5) < 0 \Rightarrow f'(g(-5)) \cdot g'(-5) < 0 \Rightarrow f'(g(-5))$ and $g'(-5)$ are both nonzero and have opposite signs. That is, either $[f'(g(-5)) > 0$ and $g'(-5) < 0]$ or $[f'(g(-5)) < 0$ and $g'(-5) > 0]$.

69. As $h \to 0$, the graph of $y = \frac{\sin 2(x+h) - \sin 2x}{h}$ approaches the graph of $y = 2 \cos 2x$ because $\lim\limits_{h \to 0} \frac{\sin 2(x+h) - \sin 2x}{h} = \frac{d}{dx}(\sin 2x) = 2 \cos 2x.$



70. As $h \to 0$, the graph of $y = \frac{\cos[(x+h)^2] - \cos(x^2)}{h}$ approaches the graph of $y = -2x \sin(x^2)$ because $\lim\limits_{h \to 0} \frac{\cos[(x+h)^2] - \cos(x^2)}{h} = \frac{d}{dx}[\cos(x^2)] = -2x \sin(x^2).$



71. $\frac{dx}{dt} = \cos t$ and $\frac{dy}{dt} = 2 \cos 2t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2 \cos 2t}{\cos t} = \frac{2(2 \cos^2 t - 1)}{\cos t}$; then $\frac{dy}{dx} = 0 \Rightarrow \frac{2(2 \cos^2 t - 1)}{\cos t} = 0$ $\Rightarrow 2 \cos^2 t - 1 = 0 \Rightarrow \cos t = \pm \frac{1}{\sqrt{2}} \Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. In the 1st quadrant: $t = \frac{\pi}{4} \Rightarrow x = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $y = \sin 2\left(\frac{\pi}{4}\right) = 1 \Rightarrow \left(\frac{\sqrt{2}}{2}, 1\right)$ is the point where the tangent line is horizontal. At the origin: $x = 0$ and $y = 0$ $\Rightarrow \sin t = 0 \Rightarrow t = 0$ or $t = \pi$ and $\sin 2t = 0 \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$; thus $t = 0$ and $t = \pi$ give the tangent lines at the origin. Tangents at origin: $\left.\frac{dy}{dx}\right|_{t=0} = 2 \Rightarrow y = 2x$ and $\left.\frac{dy}{dx}\right|_{t=\pi} = -2 \Rightarrow y = -2x$

72. $\frac{dx}{dt} = 2 \cos 2t$ and $\frac{dy}{dt} = 3 \cos 3t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3 \cos 3t}{2 \cos 2t} = \frac{3(\cos 2t \cos t - \sin 2t \sin t)}{2(2 \cos^2 t - 1)}$

$= \frac{3[(2 \cos^2 t - 1)(\cos t) - 2 \sin t \cos t \sin t]}{2(2 \cos^2 t - 1)} = \frac{(3 \cos t)(2 \cos^2 t - 1 - 2 \sin^2 t)}{2(2 \cos^2 t - 1)} = \frac{(3 \cos t)(4 \cos^2 t - 3)}{2(2 \cos^2 t - 1)}$; then

$\frac{dy}{dx} = 0 \Rightarrow \frac{(3 \cos t)(4 \cos^2 t - 3)}{2(2 \cos^2 t - 1)} = 0 \Rightarrow 3 \cos t = 0$ or $4 \cos^2 t - 3 = 0$: $3 \cos t = 0 \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$ and

$4 \cos^2 t - 3 = 0 \Rightarrow \cos t = \pm\frac{\sqrt{3}}{2} \Rightarrow t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$. In the 1st quadrant: $t = \frac{\pi}{6} \Rightarrow x = \sin 2\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

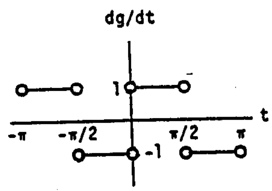
and $y = \sin 3\left(\frac{\pi}{6}\right) = 1 \Rightarrow \left(\frac{\sqrt{3}}{2}, 1\right)$ is the point where the graph has a horizontal tangent. At the origin: $x = 0$

and $y = 0 \Rightarrow \sin 2t = 0$ and $\sin 3t = 0 \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and $t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3} \Rightarrow t = 0$ and $t = \pi$ give

the tangent lines at the origin. Tangents at the origin: $\left.\frac{dy}{dx}\right|_{t=0} = \frac{3 \cos 0}{2 \cos 0} = \frac{3}{2} \Rightarrow y = \frac{3}{2}x$, and $\left.\frac{dy}{dx}\right|_{t=\pi}$

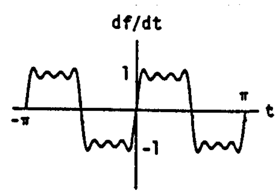
$= \frac{3 \cos(3\pi)}{2 \cos(2\pi)} = -\frac{3}{2} \Rightarrow y = -\frac{3}{2}x$

73. (a)

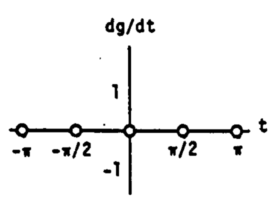


(b) $\frac{df}{dt} = 1.27324 \sin 2t + 0.42444 \sin 6t + 0.2546 \sin 10t + 0.18186 \sin 14t$

(c) The curve of $y = \frac{df}{dt}$ approximates $y = \frac{dg}{dt}$ the best when t is not $-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}$, nor π.

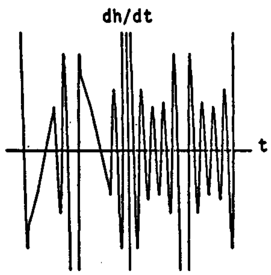


74. (a)



(b) $\frac{dh}{dt} = 2.5464 \cos(2t) + 2.5464 \cos(6t) + 2.5465 \cos(10t) + 2.54604 \cos(14t) + 2.54646 \cos(18t)$

(c)



75-80. Example CAS commands:

Maple:

```
x:= t -> exp(t) – t^2;
y:= t -> t + exp(–t);
plot([x(t), y(t), t = –1..2]);
diff(x(t),t);
dx:= unapply(%,t);
diff(y(t),t);
dy:= unapply(%,t);
dy(t)/dx(t);
dydx:= unapply(%,t);
diff(dydx(t),t);
simplify(%): dy1:= unapply(%,t);
dy1(t)/dx(t);
d2ydx2:= unapply(%,t);
t0:=1: evalf(d2ydx2(t0));
tanline:= t -> y(t0) + (dy(t0)/dx(t0))*(t – x(t(0));
plot({[x(t), y(t), t = –1..2], [t, tanline(t), t=t0–1..t0+2]});
```

Mathematica:

```
Clear[x,y,t]
{a,b} = {–Pi,Pi}; t0 = Pi/4;
x[t_] = t – Cos[t]
y[t_] = 1 + Sin[t]
p1 = ParametricPlot[ {x[t],y[t]}, {t,a,b} ]
yp[t_] = y'[t]/x'[t]
ypp[t_] = yp'[t]/x'[t]
yp[t0] // N
ypp[t0] // N
tanline[x_] = y[t0] + yp[t0]*(x–x[t0])
p2 = Plot[ tanline[x], {x,0,0.2} ]
Show[ {p1,p2} ]
```

2.6 IMPLICIT DIFFERENTIATION

1. $y = x^{9/4} \Rightarrow \frac{dy}{dx} = \frac{9}{4}x^{5/4}$

2. $y = \sqrt[3]{2x} = (2x)^{1/3} \Rightarrow \frac{dy}{dx} = \frac{1}{3}(2x)^{-2/3} \cdot 2 = \frac{2^{1/3}}{3x^{2/3}}$

3. $y = 7\sqrt{x+6} = 7(x+6)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{7}{2}(x+6)^{-1/2} = \frac{7}{2\sqrt{x+6}}$

4. $y = (1-6x)^{2/3} \Rightarrow \frac{dy}{dx} = \frac{2}{3}(1-6x)^{-1/3}(-6) = -4(1-6x)^{-1/3}$

5. $y = x\left(x^2+1\right)^{1/2} \Rightarrow y' = (1)\left(x^2+1\right)^{1/2} + \left(\frac{x}{2}\right)\left(x^2+1\right)^{-1/2}(2x) = \frac{2x^2+1}{\sqrt{x^2+1}}$

6. $y = x\left(x^2+1\right)^{-1/2} \Rightarrow y' = (1)\left(x^2+1\right)^{-1/2} + (x)\left(-\frac{1}{2}\right)\left(x^2+1\right)^{-3/2}(2x) = \left(x^2+1\right)^{-3/2}\left[\left(x^2+1\right) - x^2\right]$
$= \frac{1}{\left(x^2+1\right)^{3/2}}$

7. $s = \sqrt[7]{t^2} = t^{2/7} \Rightarrow \frac{ds}{dt} = \frac{2}{7}t^{-5/7}$

8. $r = \sqrt[4]{\theta^{-3}} = \theta^{-3/4} \Rightarrow \frac{dr}{d\theta} = -\frac{3}{4}\theta^{-7/4}$

9. $y = \sin\left((2t+5)^{-2/3}\right) \Rightarrow \frac{dy}{dt} = \cos\left((2t+5)^{-2/3}\right)\cdot\left(-\frac{2}{3}\right)(2t+5)^{-5/3}\cdot 2 = -\frac{4}{3}(2t+5)^{-5/3}\cos\left((2t+5)^{-2/3}\right)$

10. $f(x) = \sqrt{1-\sqrt{x}} = \left(1-x^{1/2}\right)^{1/2} \Rightarrow f'(x) = \frac{1}{2}\left(1-x^{1/2}\right)^{-1/2}\left(-\frac{1}{2}x^{-1/2}\right) = \frac{-1}{4\left(\sqrt{1-\sqrt{x}}\right)\sqrt{x}} = \frac{-1}{4\sqrt{x\left(1-\sqrt{x}\right)}}$

11. $g(x) = 2\left(2x^{-1/2}+1\right)^{-1/3} \Rightarrow g'(x) = -\frac{2}{3}\left(2x^{-1/2}+1\right)^{-4/3}\cdot(-1)x^{-3/2} = \frac{2}{3}\left(2x^{-1/2}+1\right)^{-4/3}x^{-3/2}$

12. $h(\theta) = \sqrt[3]{1+\cos(2\theta)} = (1+\cos 2\theta)^{1/3} \Rightarrow h'(\theta) = \frac{1}{3}(1+\cos 2\theta)^{-2/3}\cdot(-\sin 2\theta)\cdot 2 = -\frac{2}{3}(\sin 2\theta)(1+\cos 2\theta)^{-2/3}$

13. $x^2y + xy^2 = 6$:

Step 1: $\left(x^2\frac{dy}{dx} + y\cdot 2x\right) + \left(x\cdot 2y\frac{dy}{dx} + y^2\cdot 1\right) = 0$

Step 2: $x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2$

Step 3: $\frac{dy}{dx}\left(x^2 + 2xy\right) = -2xy - y^2$

Step 4: $\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$

14. $2xy + y^2 = x + y$:

Step 1: $\left(2x\frac{dy}{dx} + 2y\right) + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$

Step 2: $2x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 1 - 2y$

Step 3: $\frac{dy}{dx}(2x + 2y - 1) = 1 - 2y$

Step 4: $\frac{dy}{dx} = \frac{1-2y}{2x+2y-1}$

15. $x^3 - xy + y^3 = 1 \Rightarrow 3x^2 - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0 \Rightarrow (3y^2 - x)\frac{dy}{dx} = y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{y - 3x^2}{3y^2 - x}$

16. $x^2(x-y)^2 = x^2 - y^2$:

Step 1: $x^2\left[2(x-y)\left(1 - \frac{dy}{dx}\right)\right] + (x-y)^2(2x) = 2x - 2y\frac{dy}{dx}$

Step 2: $-2x^2(x-y)\frac{dy}{dx} + 2y\frac{dy}{dx} = 2x - 2x^2(x-y) - 2x(x-y)^2$

Step 3: $\frac{dy}{dx}\left[-2x^2(x-y) + 2y\right] = 2x\left[1 - x(x-y) - (x-y)^2\right]$

Step 4: $\frac{dy}{dx} = \frac{2x\left[1 - x(x-y) - (x-y)^2\right]}{-2x^2(x-y) + 2y} = \frac{x\left[1 - x(x-y) - (x-y)^2\right]}{y - x^2(x-y)} = \frac{x\left(1 - x^2 + xy - x^2 + 2xy - y^2\right)}{x^2y - x^3 + y}$

$= \frac{x - 2x^3 + 3x^2y - xy^2}{x^2y - x^3 + y}$

17. $y^2 = \frac{x-1}{x+1} \Rightarrow 2y\frac{dy}{dx} = \frac{(x+1) - (x-1)}{(x+1)^2} = \frac{2}{(x+1)^2} \Rightarrow \frac{dy}{dx} = \frac{1}{y(x+1)^2}$

18. $x^2 = \frac{x-y}{x+y} \Rightarrow x^3 + x^2y = x - y \Rightarrow 3x^2 + 2xy + x^2y' = 1 - y' \Rightarrow (x^2+1)y' = 1 - 3x^2 - 2xy \Rightarrow y' = \frac{1 - 3x^2 - 2xy}{x^2 + 1}$

19. $x = \tan y \Rightarrow 1 = (\sec^2 y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$

20. $x + \sin y = xy \Rightarrow 1 + (\cos y)\frac{dy}{dx} = y + x\frac{dy}{dx} \Rightarrow (\cos y - x)\frac{dy}{dx} = y - 1 \Rightarrow \frac{dy}{dx} = \frac{y-1}{\cos y - x}$

21. $y\sin\left(\frac{1}{y}\right) = 1 - xy \Rightarrow y\left[\cos\left(\frac{1}{y}\right)\cdot(-1)\frac{1}{y^2}\cdot\frac{dy}{dx}\right] + \sin\left(\frac{1}{y}\right)\cdot\frac{dy}{dx} = -x\frac{dy}{dx} - y \Rightarrow \frac{dy}{dx}\left[-\frac{1}{y}\cos\left(\frac{1}{y}\right) + \sin\left(\frac{1}{y}\right) + x\right] = -y$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{-\frac{1}{y}\cos\left(\frac{1}{y}\right) + \sin\left(\frac{1}{y}\right) + x} = \frac{-y^2}{y\sin\left(\frac{1}{y}\right) - \cos\left(\frac{1}{y}\right) + xy}$

22. $y^2\cos\left(\frac{1}{y}\right) = 2x + 2y \Rightarrow y^2\left[-\sin\left(\frac{1}{y}\right)\cdot(-1)\frac{1}{y^2}\cdot\frac{dy}{dx}\right] + \cos\left(\frac{1}{y}\right)\cdot 2y\frac{dy}{dx} = 2 + 2\frac{dy}{dx} \Rightarrow \frac{dy}{dx}\left[\sin\left(\frac{1}{y}\right) + 2y\cos\left(\frac{1}{y}\right) - 2\right] = 2$

$\Rightarrow \frac{dy}{dx} = \frac{2}{\sin\left(\frac{1}{y}\right) + 2y\cos\left(\frac{1}{y}\right) - 2}$

23. $\theta^{1/2} + r^{1/2} = 1 \Rightarrow \frac{1}{2}\theta^{-1/2} + \frac{1}{2}r^{-1/2}\cdot\frac{dr}{d\theta} = 0 \Rightarrow \frac{dr}{d\theta}\left[\frac{1}{2\sqrt{r}}\right] = \frac{-1}{2\sqrt{\theta}} \Rightarrow \frac{dr}{d\theta} = -\frac{2\sqrt{r}}{2\sqrt{\theta}} = -\frac{\sqrt{r}}{\sqrt{\theta}}$

24. $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4} \Rightarrow \frac{dr}{d\theta} - \theta^{-1/2} = \theta^{-1/3} + \theta^{-1/4} \Rightarrow \frac{dr}{d\theta} = \theta^{-1/2} + \theta^{-1/3} + \theta^{-1/4}$

25. $\sin(r\theta) = \frac{1}{2} \Rightarrow [\cos(r\theta)]\left(r + \theta \frac{dr}{d\theta}\right) = 0 \Rightarrow \frac{dr}{d\theta}[\theta \cos(r\theta)] = -r\cos(r\theta) \Rightarrow \frac{dr}{d\theta} = \frac{-r\cos(r\theta)}{\theta\cos(r\theta)} = -\frac{r}{\theta}$, $\cos(r\theta) \neq 0$

26. $\cos r + \cos\theta = r\theta \Rightarrow (-\sin r)\frac{dr}{d\theta} - \sin\theta = r + \theta\frac{dr}{d\theta} \Rightarrow \frac{dr}{d\theta}[-\theta - \sin r] = r + \sin\theta \Rightarrow \frac{dr}{d\theta} = \frac{-(r+\sin\theta)}{\theta + \sin r}$

27. $x^{2/3} + y^{2/3} = 1 \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx}\left[\frac{2}{3}y^{-1/3}\right] = -\frac{2}{3}x^{-1/3} \Rightarrow y' = \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$;

Differentiating again, $y'' = \frac{x^{1/3}\cdot\left(-\frac{1}{3}y^{-2/3}\right)y' + y^{1/3}\left(\frac{1}{3}x^{-2/3}\right)}{x^{2/3}} = \frac{x^{1/3}\cdot\left(-\frac{1}{3}y^{-2/3}\right)\left(-\frac{y^{1/3}}{x^{1/3}}\right) + y^{1/3}\left(\frac{1}{3}x^{-2/3}\right)}{x^{2/3}}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{3}x^{-2/3}y^{-1/3} + \frac{1}{3}y^{1/3}x^{-4/3} = \frac{y^{1/3}}{3x^{4/3}} + \frac{1}{3y^{1/3}x^{2/3}}$

28. $y^2 = x^2 + 2x \Rightarrow 2yy' = 2x + 2 \Rightarrow y' = \frac{2x+2}{2y} = \frac{x+1}{y}$; then $y'' = \frac{y - (x+1)y'}{y^2} = \frac{y - (x+1)\left(\frac{x+1}{y}\right)}{y^2}$

$\Rightarrow \frac{d^2y}{dx^2} = y'' = \frac{y^2 - (x+1)^2}{y^3}$

29. $2\sqrt{y} = x - y \Rightarrow y^{-1/2}y' = 1 - y' \Rightarrow y'\left(y^{-1/2} + 1\right) = 1 \Rightarrow \frac{dy}{dx} = y' = \frac{1}{y^{-1/2} + 1} = \frac{\sqrt{y}}{\sqrt{y} + 1}$; we can

differentiate the equation $y'\left(y^{-1/2} + 1\right) = 1$ again to find y'': $y'\left(-\frac{1}{2}y^{-3/2}y'\right) + \left(y^{-1/2} + 1\right)y'' = 0$

$\Rightarrow \left(y^{-1/2} + 1\right)y'' = \frac{1}{2}[y']^2 y^{-3/2} \Rightarrow \frac{d^2y}{dx^2} = y'' = \frac{\frac{1}{2}\left(\frac{1}{y^{-1/2}+1}\right)^2 y^{-3/2}}{\left(y^{-1/2}+1\right)} = \frac{1}{2y^{3/2}\left(y^{-1/2}+1\right)^3} = \frac{1}{2\left(1+\sqrt{y}\right)^3}$

30. $xy + y^2 = 1 \Rightarrow xy' + y + 2yy' = 0 \Rightarrow xy' + 2yy' = -y \Rightarrow y'(x+2y) = -y \Rightarrow y' = \frac{-y}{(x+2y)}$; $\frac{d^2y}{dx^2} = y''$

$= \frac{-(x+2y)y' + y(1+2y')}{(x+2y)^2} = \frac{-(x+2y)\left[\frac{-y}{(x+2y)}\right] + y\left[1 + 2\left(\frac{-y}{(x+2y)}\right)\right]}{(x+2y)^2} = \frac{\frac{1}{(x+2y)}[y(x+2y) + y(x+2y) - 2y^2]}{(x+2y)^2}$

$= \frac{2y(x+2y) - 2y^2}{(x+2y)^3} = \frac{2y^2 + 2xy}{(x+2y)^3} = \frac{2y(x+y)}{(x+2y)^3}$

31. $x^3 + y^3 = 16 \Rightarrow 3x^2 + 3y^2y' = 0 \Rightarrow 3y^2y' = -3x^2 \Rightarrow y' = -\frac{x^2}{y^2}$; we differentiate $y^2y' = -x^2$ to find y'':

$y^2y'' + y'[2y\cdot y'] = -2x \Rightarrow y^2y'' = -2x - 2y[y']^2 \Rightarrow y'' = \frac{-2x - 2y\left(-\frac{x^2}{y^2}\right)^2}{y^2} = \frac{-2x - \frac{2x^4}{y^3}}{y^2}$

$= \frac{-2xy^3 - 2x^4}{y^5} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(2,2)} = \frac{-32-32}{32} = -2$

32. $xy + y^2 = 1 \Rightarrow xy' + y + 2yy' = 0 \Rightarrow y'(x + 2y) = -y \Rightarrow y' = \frac{-y}{(x+2y)} \Rightarrow y'' = \frac{(x+2y)(-y') - (-y)(1+2y')}{(x+2y)^2}$;

since $y'\big|_{(0,-1)} = -\frac{1}{2}$ we obtain $y''\big|_{(0,-1)} = \frac{(-2)\left(\frac{1}{2}\right) - (1)(0)}{4} = -\frac{1}{4}$

33. $x^2 - 2tx + 2t^2 = 4 \Rightarrow 2x\frac{dx}{dt} - 2x - 2t\frac{dx}{dt} + 4t = 0 \Rightarrow (2x - 2t)\frac{dx}{dt} = 2x - 4t \Rightarrow \frac{dx}{dt} = \frac{2x-4t}{2x-2t} = \frac{x-2t}{x-t}$;

$2y^3 - 3t^2 = 4 \Rightarrow 6y^2\frac{dy}{dt} - 6t = 0 \Rightarrow \frac{dy}{dt} = \frac{6t}{6y^2} = \frac{t}{y^2}$; thus $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(\frac{t}{y^2}\right)}{\left(\frac{x-2t}{x-t}\right)} = \frac{t(x-t)}{y^2(x-2t)}$; $t = 2$

$\Rightarrow x^2 - 2(2)x + 2(2)^2 = 4 \Rightarrow x^2 - 4x + 4 = 0 \Rightarrow (x-2)^2 = 0 \Rightarrow x = 2$; $t = 2 \Rightarrow 2y^3 - 3(2)^2 = 4$

$\Rightarrow 2y^3 = 16 \Rightarrow y^3 = 8 \Rightarrow y = 2$; therefore $\frac{dy}{dx}\Big|_{t=2} = \frac{2(2-2)}{(2)^2(2-2(2))} = 0$

34. $x = \sqrt{5 - \sqrt{t}} \Rightarrow \frac{dx}{dt} = \frac{1}{2}\left(5 - \sqrt{t}\right)^{-1/2}\left(-\frac{1}{2}t^{-1/2}\right) = \frac{-1}{4\sqrt{t}\sqrt{5-\sqrt{t}}}$; $y(t-1) = \ln y \Rightarrow \frac{dy}{dt}(t-1) + y = \left(\frac{1}{y}\right)\frac{dy}{dt}$

$\Rightarrow \left(t - 1 - \frac{1}{y}\right)\frac{dy}{dt} = -y \Rightarrow \frac{dy}{dt} = \frac{-y}{\left(t-1-\frac{1}{y}\right)} = \frac{-y^2}{ty - y - 1}$; thus $\frac{dy}{dx} = \frac{\left(\frac{-y^2}{ty-y-1}\right)}{\left(\frac{-1}{4\sqrt{t}\sqrt{5-\sqrt{t}}}\right)} = \frac{4y^2\sqrt{t}\sqrt{5-\sqrt{t}}}{ty - y - 1}$;

$t = 1 \Rightarrow y(1-1) = \ln y \Rightarrow 0 = \ln y \Rightarrow y = 1$; therefore $\frac{dy}{dx}\Big|_{t=1} = \frac{4(1)^2\sqrt{1}\sqrt{5-\sqrt{1}}}{(1)(1)-1-1} = -8$

35. $x + 2x^{3/2} = t^2 + t \Rightarrow \frac{dx}{dt} + 3x^{1/2}\frac{dx}{dt} = 2t + 1 \Rightarrow \left(1 + 3x^{1/2}\right)\frac{dx}{dt} = 2t + 1 \Rightarrow \frac{dx}{dt} = \frac{2t+1}{1+3x^{1/2}}$; $y\sqrt{t+1} + 2t\sqrt{y} = 4$

$\Rightarrow \frac{dy}{dt}\sqrt{t+1} + y\left(\frac{1}{2}\right)(t+1)^{-1/2} + 2\sqrt{y} + 2t\left(\frac{1}{2}y^{-1/2}\right)\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt}\sqrt{t+1} + \frac{y}{2\sqrt{t+1}} + 2\sqrt{y} + \left(\frac{t}{\sqrt{y}}\right)\frac{dy}{dt} = 0$

$\Rightarrow \left(\sqrt{t+1} + \frac{t}{\sqrt{y}}\right)\frac{dy}{dt} = \frac{-y}{2\sqrt{t+1}} - 2\sqrt{y} \Rightarrow \frac{dy}{dt} = \frac{\left(\frac{-y}{2\sqrt{t+1}} - 2\sqrt{y}\right)}{\left(\sqrt{t+1} + \frac{t}{\sqrt{y}}\right)} = \frac{-y\sqrt{y} - 4y\sqrt{t+1}}{2\sqrt{y}(t+1) + 2t\sqrt{t+1}}$; thus

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(\frac{-y\sqrt{y} - 4y\sqrt{t+1}}{2\sqrt{y}(t+1) + 2t\sqrt{t+1}}\right)}{\left(\frac{2t+1}{1+3x^{1/2}}\right)}$; $t = 0 \Rightarrow x + 2x^{3/2} = 0 \Rightarrow x\left(1 + 2x^{1/2}\right) = 0 \Rightarrow x = 0$; $t = 0$

$\Rightarrow y\sqrt{0+1} + 2(0)\sqrt{y} = 4 \Rightarrow y = 4$; therefore $\frac{dy}{dx}\Big|_{t=0} = \frac{\left(\frac{-4\sqrt{4} - 4(4)\sqrt{0+1}}{2\sqrt{4}(0+1) + 2(0)\sqrt{0+1}}\right)}{\left(\frac{2(0)+1}{1+3(0)^{1/2}}\right)} = -6$

36. $x \sin t + 2x = t \Rightarrow \frac{dx}{dt} \sin t + x \cos t + 2\frac{dx}{dt} = 1 \Rightarrow (\sin t + 2)\frac{dx}{dt} = 1 - x \cos t \Rightarrow \frac{dx}{dt} = \frac{1 - x \cos t}{\sin t + 2}$;

$t \sin t - 2t = y \Rightarrow \sin t + t \cos t - 2 = \frac{dy}{dt}$; thus $\frac{dy}{dx} = \frac{\sin t + t \cos t - 2}{\left(\frac{1 - x \cos t}{\sin t + 2}\right)}$; $t = \pi \Rightarrow x \sin \pi + 2x = \pi$

$\Rightarrow x = \frac{\pi}{2}$; therefore $\left.\frac{dy}{dx}\right|_{t=\pi} = \frac{\sin \pi + \pi \cos \pi - 2}{\left[\frac{1 - \left(\frac{\pi}{2}\right)\cos \pi}{\sin \pi + 2}\right]} = \frac{-4\pi - 8}{2 + \pi} = -4$

37. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1) \Rightarrow 2y\frac{dy}{dx} + 2x = 4y^3\frac{dy}{dx} - 2 \Rightarrow 2y\frac{dy}{dx} - 4y^3\frac{dy}{dx} = -2 - 2x$

$\Rightarrow \frac{dy}{dx}(2y - 4y^3) = -2 - 2x \Rightarrow \frac{dy}{dx} = \frac{x + 1}{2y^3 - y} \Rightarrow \left.\frac{dy}{dx}\right|_{(-2,1)} = -1$ and $\left.\frac{dy}{dx}\right|_{(-2,-1)} = 1$

38. $(x^2 + y^2)^2 = (x - y)^2$ at$(1, 0)$ and $(1, -1) \Rightarrow 2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 2(x - y)\left(1 - \frac{dy}{dx}\right)$

$\Rightarrow \frac{dy}{dx}\left[2y(x^2 + y^2) + (x - y)\right] = -2x(x^2 + y^2) + (x - y) \Rightarrow \frac{dy}{dx} = \frac{-2x(x^2 + y^2) + (x - y)}{2y(x^2 + y^2) + (x - y)} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,0)} = -1$

and $\left.\frac{dy}{dx}\right|_{(1,-1)} = 1$

39. $x^2 + xy - y^2 = 1 \Rightarrow 2x + y + xy' - 2yy' = 0 \Rightarrow (x - 2y)y' = -2x - y \Rightarrow y' = \frac{2x + y}{2y - x}$;

(a) the slope of the tangent line $m = y'|_{(2,3)} = \frac{7}{4} \Rightarrow$ the tangent line is $y - 3 = \frac{7}{4}(x - 2) \Rightarrow y = \frac{7}{4}x - \frac{1}{2}$

(b) the normal line is $y - 3 = -\frac{4}{7}(x - 2) \Rightarrow y = -\frac{4}{7}x + \frac{29}{7}$

40. $x^2y^2 = 9 \Rightarrow 2xy^2 + 2x^2yy' = 0 \Rightarrow x^2yy' = -xy^2 \Rightarrow y' = -\frac{y}{x}$;

(a) the slope of the tangent line $m = y'|_{(-1,3)} = -\left.\frac{y}{x}\right|_{(-1,3)} = 3 \Rightarrow$ the tangent line is $y - 3 = 3(x + 1)$

$\Rightarrow y = 3x + 6$

(b) the normal line is $y - 3 = -\frac{1}{3}(x + 1) \Rightarrow y = -\frac{1}{3}x + \frac{8}{3}$

41. $y^2 - 2x - 4y - 1 = 0 \Rightarrow 2yy' - 2 - 4y' = 0 \Rightarrow 2(y - 2)y' = 2 \Rightarrow y' = \frac{1}{y - 2}$;

(a) the slope of the tangent line $m = y'|_{(-2,1)} = -1 \Rightarrow$ the tangent line is $y - 1 = -1(x + 2) \Rightarrow y = -x - 1$

(b) the normal line is $y - 1 = 1(x + 2) \Rightarrow y = x + 3$

42. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0 \Rightarrow 12x + 3y + 3xy' + 4yy' + 17y' = 0 \Rightarrow y'(3x + 4y + 17) = -12x - 3y$

$\Rightarrow y' = \frac{-12x - 3y}{3x + 4y + 17}$;

(a) the slope of the tangent line $m = y'|_{(-1,0)} = \left.\frac{-12x - 3y}{3x + 4y + 17}\right|_{(-1,0)} = \frac{6}{7} \Rightarrow$ the tangent line is $y - 0 = \frac{6}{7}(x + 1)$

$\Rightarrow y = \frac{6}{7}x + \frac{6}{7}$

(b) the normal line is $y - 0 = -\frac{7}{6}(x + 1) \Rightarrow y = -\frac{7}{6}x - \frac{7}{6}$