43. $2xy + \pi \sin y = 2\pi \Rightarrow 2xy' + 2y + \pi(\cos y)y' = 0 \Rightarrow y'(2x + \pi \cos y) = -2y \Rightarrow y' = \frac{-2y}{2x + \pi \cos y}$;

(a) the slope of the tangent line $m = y'\big|_{\left(1,\frac{\pi}{2}\right)} = \frac{-2y}{2x + \pi \cos y}\Big|_{\left(1,\frac{\pi}{2}\right)} = -\frac{\pi}{2} \Rightarrow$ the tangent line is

$y - \frac{\pi}{2} = -\frac{\pi}{2}(x - 1) \Rightarrow y = -\frac{\pi}{2}x + \pi$

(b) the normal line is $y - \frac{\pi}{2} = \frac{2}{\pi}(x - 1) \Rightarrow y = \frac{2}{\pi}x - \frac{2}{\pi} + \frac{\pi}{2}$

44. $x \sin 2y = y \cos 2x \Rightarrow x(\cos 2y)2y' + \sin 2y = -2y \sin 2x + y' \cos 2x \Rightarrow y'(2x \cos 2y - \cos 2x)$

$= -\sin 2y - 2y \sin 2x \Rightarrow y' = \frac{\sin 2y + 2y \sin 2x}{\cos 2x - 2x \cos 2y}$;

(a) the slope of the tangent line $m = y'\big|_{\left(\frac{\pi}{4},\frac{\pi}{2}\right)} = \frac{\sin 2y + 2y \sin 2x}{\cos 2x - 2x \cos 2y}\Big|_{\left(\frac{\pi}{4},\frac{\pi}{2}\right)} = \frac{\pi}{\frac{\pi}{2}} = 2 \Rightarrow$ the tangent line is

$y - \frac{\pi}{2} = 2\left(x - \frac{\pi}{4}\right) \Rightarrow y = 2x$

(b) the normal line is $y - \frac{\pi}{2} = -\frac{1}{2}\left(x - \frac{\pi}{4}\right) \Rightarrow y = -\frac{1}{2}x + \frac{5\pi}{8}$

45. $y = 2 \sin(\pi x - y) \Rightarrow y' = 2[\cos(\pi x - y)] \cdot (\pi - y') \Rightarrow y'[1 + 2 \cos(\pi x - y)] = 2\pi \cos(\pi x - y)$

$\Rightarrow y' = \frac{2\pi \cos(\pi x - y)}{1 + 2 \cos(\pi x - y)}$;

(a) the slope of the tangent line $m = y'\big|_{(1,0)} = \frac{2\pi \cos(\pi x - y)}{1 + 2 \cos(\pi x - y)}\Big|_{(1,0)} = 2\pi \Rightarrow$ the tangent line is

$y - 0 = 2\pi(x - 1) \Rightarrow y = 2\pi x - 2\pi$

(b) the normal line is $y - 0 = -\frac{1}{2\pi}(x - 1) \Rightarrow y = -\frac{x}{2\pi} + \frac{1}{2\pi}$

46. $x^2 \cos^2 y - \sin y = 0 \Rightarrow x^2(2 \cos y)(-\sin y)y' + 2x \cos^2 y - y' \cos y = 0 \Rightarrow y'\left[-2x^2 \cos y \sin y - \cos y\right]$

$= -2x \cos^2 y \Rightarrow y' = \frac{2x \cos^2 y}{2x^2 \cos y \sin y + \cos y}$;

(a) the slope of the tangent line $m = y'\big|_{(0,\pi)} = \frac{2x \cos^2 y}{2x^2 \cos y \sin y + \cos y}\Big|_{(0,\pi)} = 0 \Rightarrow$ the tangent line is $y = \pi$

(b) the normal line is $x = 0$

47. Solving $x^2 + xy + y^2 = 7$ and $y = 0 \Rightarrow x^2 = 7 \Rightarrow x = \pm\sqrt{7} \Rightarrow \left(-\sqrt{7}, 0\right)$ and $\left(\sqrt{7}, 0\right)$ are the points where the curve crosses the x-axis. Now $x^2 + xy + y^2 = 7 \Rightarrow 2x + y + xy' + 2yy' = 0 \Rightarrow (x + 2y)y' = -2x - y$

$\Rightarrow y' = -\frac{2x + y}{x + 2y} \Rightarrow m = -\frac{2x + y}{x + 2y} \Rightarrow$ the slope at $\left(-\sqrt{7}, 0\right)$ is $m = -\frac{-2\sqrt{7}}{-\sqrt{7}} = -2$ and the slope at $\left(\sqrt{7}, 0\right)$ is

$m = -\frac{2\sqrt{7}}{\sqrt{7}} = -2$. Since the slope is -2 in each case, the corresponding tangents must be parallel.

48. $x^2 + xy + y^2 = 7 \Rightarrow 2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \Rightarrow (x + 2y)\frac{dy}{dx} = -2x - y \Rightarrow \frac{dy}{dx} = \frac{-2x - y}{x + 2y}$ and $\frac{dx}{dy} = \frac{x + 2y}{-2x - y}$;

(a) Solving $\frac{dy}{dx} = 0 \Rightarrow -2x - y = 0 \Rightarrow y = -2x$ and substitution into the original equation gives

$x^2 + x(-2x) + (-2x)^2 = 7 \Rightarrow 3x^2 = 7 \Rightarrow x = \pm\sqrt{\frac{7}{3}}$ and $y = \mp 2\sqrt{\frac{7}{3}}$ when the tangents are parallel to the x-axis.

(b) Solving $\frac{dx}{dy} = 0 \Rightarrow x + 2y = 0 \Rightarrow y = -\frac{x}{2}$ and substitution gives $x^2 + x\left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2 = 7 \Rightarrow \frac{3x^2}{4} = 7$

$\Rightarrow x = \pm 2\sqrt{\frac{7}{3}}$ and $y = \mp\sqrt{\frac{7}{3}}$ when the tangents are parallel to the y-axis.

49. $y^4 = y^2 - x^2 \Rightarrow 4y^3y' = 2yy' - 2x \Rightarrow 2(2y^3 - y)y' = -2x \Rightarrow y' = \frac{x}{y - 2y^3}$; the slope of the tangent line at

$\left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$ is $\left.\frac{x}{y - 2y^3}\right|_{\left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)} = \frac{\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2} - \frac{6\sqrt{3}}{8}} = \frac{\frac{1}{4}}{\frac{1}{2} - \frac{3}{4}} = \frac{1}{2 - 3} = -1$; the slope of the tangent line at $\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right)$

is $\left.\frac{x}{y - 2y^3}\right|_{\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right)} = \frac{\frac{\sqrt{3}}{4}}{\frac{1}{2} - \frac{2}{8}} = \frac{2\sqrt{3}}{4 - 2} = \sqrt{3}$

50. $y^2(2 - x) = x^3 \Rightarrow 2yy'(2 - x) + y^2(-1) = 3x^2 \Rightarrow y' = \frac{y^2 + 3x^2}{2y(2 - x)}$; the slope of the tangent line is

$m = \left.\frac{y^2 + 3x^2}{2y(2 - x)}\right|_{(1,1)} = \frac{4}{2} = 2 \Rightarrow$ the tangent line is $y - 1 = 2(x - 1) \Rightarrow y = 2x - 1$; the normal line is

$y - 1 = -\frac{1}{2}(x - 1) \Rightarrow y = -\frac{1}{2}x + \frac{3}{2}$

51. $y^4 - 4y^2 = x^4 - 9x^2 \Rightarrow 4y^3y' - 8yy' = 4x^3 - 18x \Rightarrow y'(4y^3 - 8y) = 4x^3 - 18x \Rightarrow y' = \frac{4x^3 - 18x}{4y^3 - 8y} = \frac{2x^3 - 9x}{2y^3 - 4y}$

$= \frac{x(2x^2 - 9)}{y(2y^2 - 4)} = m$; $(-3, 2)$: $m = \frac{(-3)(18 - 9)}{2(8 - 4)} = -\frac{27}{8}$; $(-3, -2)$: $m = \frac{27}{8}$; $(3, 2)$: $m = \frac{27}{8}$; $(3, -2)$: $m = -\frac{27}{8}$

52. $x^3 + y^3 - 9xy = 0 \Rightarrow 3x^2 + 3y^2y' - 9xy' - 9y = 0 \Rightarrow y'(3y^2 - 9x) = 9y - 3x^2 \Rightarrow y' = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$

(a) $y'|_{(4,2)} = \frac{5}{4}$ and $y'|_{(2,4)} = \frac{4}{5}$;

(b) $y' = 0 \Rightarrow \frac{3y - x^2}{y^2 - 3x} = 0 \Rightarrow 3y - x^2 = 0 \Rightarrow y = \frac{x^2}{3} \Rightarrow x^3 + \left(\frac{x^2}{3}\right)^3 - 9x\left(\frac{x^2}{3}\right) = 0 \Rightarrow x^6 - 54x^3 = 0$

$\Rightarrow x^3(x^3 - 54) = 0 \Rightarrow x = 0$ or $x = \sqrt[3]{54} = 3\sqrt[3]{2} \Rightarrow$ there is a horizontal tangent at $x = 3\sqrt[3]{2}$. To find the corresponding y-value, we will use part (c).

(c) $\frac{dx}{dy} = 0 \Rightarrow \frac{y^2 - 3x}{3y - x^2} = 0 \Rightarrow y^2 - 3x = 0 \Rightarrow y = \pm\sqrt{3x}$; $y = \sqrt{3x} \Rightarrow x^3 + \left(\sqrt{3x}\right)^3 - 9x\sqrt{3x} = 0$

$\Rightarrow x^3 - 6\sqrt{3}\,x^{3/2} = 0 \Rightarrow x^{3/2}\left(x^{3/2} - 6\sqrt{3}\right) = 0 \Rightarrow x^{3/2} = 0$ or $x^{3/2} = 6\sqrt{3} \Rightarrow x = 0$ or $x = \sqrt[3]{108} = 3\sqrt[3]{4}$.

Since the equation $x^3 + y^3 - 9xy = 0$ is symmetric in x and y, the graph is symmetric about the line $y = x$. That is, if (a, b) is a point on the folium, then so is (b, a). Moreover, if $y'|_{(a,b)} = m$, then $y'|_{(b,a)} = \frac{1}{m}$. Thus, if the folium has a horizontal tangent at (a, b), it has a vertical tangent at (b, a) so one might expect that with a horizontal tangent at $x = \sqrt[3]{54}$ and a vertical tangent at $x = 3\sqrt[3]{4}$, the points of tangency are $\left(\sqrt[3]{54}, 3\sqrt[3]{4}\right)$ and $\left(3\sqrt[3]{4}, \sqrt[3]{54}\right)$, respectively. One can check that these points do satisfy the equation $x^3 + y^3 - 9xy = 0$.

53. (a) if $f(x) = \frac{3}{2}x^{2/3} - 3$, then $f'(x) = x^{-1/3}$ and $f''(x) = -\frac{1}{3}x^{-4/3}$ so the claim $f''(x) = x^{-1/3}$ is false

(b) if $f(x) = \frac{9}{10}x^{5/3} - 7$, then $f'(x) = \frac{3}{2}x^{2/3}$ and $f''(x) = x^{-1/3}$ is true

(c) $f''(x) = x^{-1/3} \Rightarrow f'''(x) = -\frac{1}{3}x^{-4/3}$ is true

(d) if $f'(x) = \frac{3}{2}x^{2/3} + 6$, then $f''(x) = x^{-1/3}$ is true

54. $2x^2 + 3y^2 = 5 \Rightarrow 4x + 6yy' = 0 \Rightarrow y' = -\frac{2x}{3y} \Rightarrow y'\big|_{(1,1)} = -\frac{2x}{3y}\Big|_{(1,1)} = -\frac{2}{3}$ and $y'\big|_{(1,-1)} = -\frac{2x}{3y}\Big|_{(1,-1)} = \frac{2}{3}$;

also, $y^2 = x^3 \Rightarrow 2yy' = 3x^2 \Rightarrow y' = \frac{3x^2}{2y} \Rightarrow y'\big|_{(1,1)} = \frac{3x^2}{2y}\Big|_{(1,1)} = \frac{3}{2}$ and $y'\big|_{(1,-1)} = \frac{3x^2}{2y}\Big|_{(1,-1)} = -\frac{3}{2}$. Therefore the tangents to the curves are perpendicular at $(1,1)$ and $(1,-1)$ (i.e., the curves are orthogonal at these two points of intersection).

55. $x^2 + 2xy - 3y^2 = 0 \Rightarrow 2x + 2xy' + 2y - 6yy' = 0 \Rightarrow y'(2x - 6y) = -2x - 2y \Rightarrow y' = \frac{x+y}{3y-x} \Rightarrow$ the slope of the tangent line $m = y'\big|_{(1,1)} = \frac{x+y}{3y-x}\Big|_{(1,1)} = 1 \Rightarrow$ the equation of the normal line at $(1,1)$ is $y - 1 = -1(x-1)$ $\Rightarrow y = -x + 2$. To find where the normal line intersects the curve we substitute into its equation: $x^2 + 2x(2-x) - 3(2-x)^2 = 0 \Rightarrow x^2 + 4x - 2x^2 - 3\left(4 - 4x + x^2\right) = 0 \Rightarrow -4x^2 + 16x - 12 = 0 \Rightarrow x^2 - 4x + 3 = 0$ $\Rightarrow (x-3)(x-1) = 0 \Rightarrow x = 3$ and $y = -x + 2 = -1$. Therefore, the normal to the curve at $(1,1)$ intersects the curve at the point $(3,-1)$. Note that it also intersects the curve at $(1,1)$.

56. $xy + 2x - y = 0 \Rightarrow x\frac{dy}{dx} + y + 2 - \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y+2}{1-x}$; the slope of the line $2x + y = 0$ is -2. In order to be parallel, the normal lines must also have slope of -2. Since a normal is perpendicular to a tangent, the slope of the tangent is $\frac{1}{2}$. Therefore, $\frac{y+2}{1-x} = \frac{1}{2} \Rightarrow 2y + 4 = 1 - x \Rightarrow x = -3 - 2y$. Substituting in the original equation, $y(-3-2y) + 2(-3-2y) - y = 0 \Rightarrow y^2 + 4y + 3 = 0 \Rightarrow y = -3$ or $y = -1$. If $y = -3$, then $x = 3$ and $y + 3 = -2(x-3) \Rightarrow y = -2x + 3$. If $y = -1$, then $x = -1$ and $y + 1 = -2(x+1) \Rightarrow y = -2x - 3$.

57. $y^2 = x \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$. If a normal is drawn from $(a,0)$ to (x_1, y_1) on the curve its slope satisfies $\frac{y_1 - 0}{x_1 - a} = -2y_1$ $\Rightarrow y_1 = -2y_1(x_1 - a)$ or $a = x_1 + \frac{1}{2}$. Since $x_1 \ge 0$ on the curve, we must have that $a \ge \frac{1}{2}$. By symmetry, the two points on the parabola are $\left(x_1, \sqrt{x_1}\right)$ and $\left(x_1, -\sqrt{x_1}\right)$. For the normal to be perpendicular, $\left(\frac{\sqrt{x_1}}{x_1 - a}\right)\left(\frac{\sqrt{x_1}}{a - x_1}\right) = -1 \Rightarrow \frac{x_1}{(a-x_1)^2} = 1 \Rightarrow x_1 = (a - x_1)^2 \Rightarrow x_1 = \left(x_1 + \frac{1}{2} - x_1\right)^2 \Rightarrow x_1 = \frac{1}{4}$ and $y_1 = \pm\frac{1}{2}$. Therefore, $\left(\frac{1}{4}, \pm\frac{1}{2}\right)$ and $a = \frac{3}{4}$.

58. Ex. 5a.) $y = x^{1/2}$ has no derivative at $x = 0$ because the slope of the graph becomes vertical at $x = 0$.

Ex. 5b.) $y = x^{2/3}$ has no derivative at $x = 0$ because the slope of the graph becomes vertical at $x = 0$.

Ex. 6a.) $y = (1 - x^2)^{1/4}$ has a derivative only on $(-1, 1)$ because the function is defined only on $[-1, 1]$ and the slope of the tangent becomes vertical at both $x = -1$ and $x = 1$.

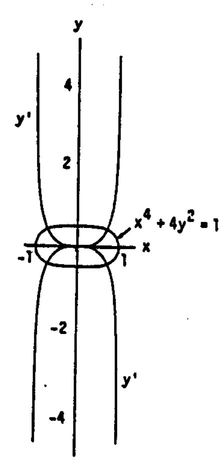
59. $xy^3 + x^2y = 6 \Rightarrow x\left(3y^2 \frac{dy}{dx}\right) + y^3 + x^2 \frac{dy}{dx} + 2xy = 0 \Rightarrow \frac{dy}{dx}(3xy^2 + x^2) = -y^3 - 2xy \Rightarrow \frac{dy}{dx} = \frac{-y^3 - 2xy}{3xy^2 + x^2}$ $= -\frac{y^3 + 2xy}{3xy^2 + x^2}$; also, $xy^3 + x^2y = 6 \Rightarrow x(3y^2) + y^3 \frac{dx}{dy} + x^2 + y\left(2x \frac{dx}{dy}\right) = 0 \Rightarrow \frac{dx}{dy}(y^3 + 2xy) = -3xy^2 - x^2$ $\Rightarrow \frac{dx}{dy} = -\frac{3xy^2 + x^2}{y^3 + 2xy}$; thus $\frac{dx}{dy}$ appears to equal $\frac{1}{\frac{dy}{dx}}$. The two different treatments view the graphs as functions symmetric across the line $y = x$, so their slopes are reciprocals of one another at the corresponding points (a, b) and (b, a).

60. $x^3 + y^2 = \sin^2 y \Rightarrow 3x^2 + 2y \frac{dy}{dx} = (2 \sin y)(\cos y) \frac{dy}{dx} \Rightarrow \frac{dy}{dx}(2y - 2 \sin y \cos y) = -3x^2 \Rightarrow \frac{dy}{dx} = \frac{-3x^2}{2y - 2 \sin y \cos y}$ $= \frac{3x^2}{2 \sin y \cos y - 2y}$; also, $x^3 + y^2 = \sin^2 y \Rightarrow 3x^2 \frac{dx}{dy} + 2y = 2 \sin y \cos y \Rightarrow \frac{dx}{dy} = \frac{2 \sin y \cos y - 2y}{3x^2}$; thus $\frac{dx}{dy}$ appears to equal $\frac{1}{\frac{dy}{dx}}$. The two different treatments view the graphs as functions symmetric across the line $y = x$ so their slopes are reciprocals of one another at the corresponding points (a, b) and (b, a).

61. $x^4 + 4y^2 = 1$:

(a) $y^2 = \frac{1 - x^4}{4} \Rightarrow y = \pm\frac{1}{2}\sqrt{1 - x^4} \Rightarrow \frac{dy}{dx} = \pm\frac{1}{4}(1 - x^4)^{-1/2}(-4x^3) = \frac{\pm x^3}{(1 - x^4)^{1/2}}$; differentiating implicitly, we find, $4x^3 + 8y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-4x^3}{8y} = \frac{-4x^3}{8\left(\pm\frac{1}{2}\sqrt{1 - x^4}\right)} = \frac{\pm x^3}{(1 - x^4)^{1/2}}$.

(b)

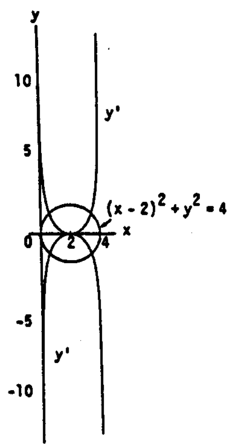


62. $(x-2)^2 + y^2 = 4$:

(a) $y = \pm\sqrt{4-(x-2)^2} \Rightarrow \frac{dy}{dx} = \pm\frac{1}{2}\left(4-(x-2)^2\right)^{-1/2}(-2(x-2)) = \frac{\pm(x-2)}{\left[4-(x-2)^2\right]^{1/2}}$; differentiating implicitly,

$$2(x-2) + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2(x-2)}{2y} = \frac{-(x-2)}{y} = \frac{-(x-2)}{\pm\left[4-(x-2)^2\right]^{1/2}} = \frac{\pm(x-2)}{\left[4-(x-2)^2\right]^{1/2}}.$$

(b)



63-70. Example CAS commands:

Maple:

```
with(plots):
eq1 := x + tan(y/x) = 2;
x0 := 1: y0 := Pi/4:
subs({x=x0, y=y0}, eq1);
implicitplot(eq1, x=x0 - 3..x0 + 3, y=y0 - 3..y0 + 3);
subs(y=y(x),eq1):
diff(%,x);
solve(%,diff(y(x),x));
m:=subs({x=x0,y(x)=y0},%);
tanline := y = y0 + m*(x-x0);
implicitplot({eq1,tanline}, x=x0 - 2..x0 + 2,y=y0 - 3..y0 + 2);
```

Mathematica:

```
Graphics 'ImplicitPlot'
Clear[x,y]
{x0,y0} = {1,Pi/4}; eqn = x + Tan[y/x] == 2
ImplicitPlot[ eqn, {x,x0 - 3,x0 + 3}, {y,y0 - 3,y0 + 3} ]
eqn /. {x -> x0, y -> y0}
eqn /. {y -> y[x]}
D[%,x]
Solve[%, y'[x] ]
slope = y'[x] /. First[%]
m = slope /. {x -> x0, y[x] -> y0}
tanline = y == y0 + m (x - x0)
ImplicitPlot[{eqn, tanline}, {x,x0 - 3,x0 + 3}, {y,y0 - 3,y0 + 3} ]
```

2.7 RELATED RATES

1. $A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

2. $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

3. (a) $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$
   (b) $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = 2\pi r h \frac{dr}{dt}$
   (c) $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt}$

4. (a) $V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt}$
   (b) $V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt}$
   (c) $\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{2}{3}\pi r h \frac{dr}{dt}$

5. (a) $\frac{dV}{dt} = 1$ volt/sec
   (b) $\frac{dI}{dt} = -\frac{1}{3}$ amp/sec
   (c) $\frac{dV}{dt} = R\left(\frac{dI}{dt}\right) + I\left(\frac{dR}{dt}\right) \Rightarrow \frac{dR}{dt} = \frac{1}{I}\left(\frac{dV}{dt} - R\frac{dI}{dt}\right) \Rightarrow \frac{dR}{dt} = \frac{1}{I}\left(\frac{dV}{dt} - \frac{V}{I}\frac{dI}{dt}\right)$
   (d) $\frac{dR}{dt} = \frac{1}{2}\left[1 - \frac{12}{2}\left(-\frac{1}{3}\right)\right] = \left(\frac{1}{2}\right)(3) = \frac{3}{2}$ ohms/sec, R is increasing

6. (a) $P = Ri^2 \Rightarrow \frac{dP}{dt} = i^2 \frac{dR}{dt} + 2Ri \frac{di}{dt}$
   (b) $P = Ri^2 \Rightarrow 0 = \frac{dP}{dt} = i^2 \frac{dR}{dt} + 2Ri \frac{di}{dt} \Rightarrow \frac{dR}{dt} = -\frac{2Ri}{i^2}\frac{di}{dt} = -\frac{2\left(\frac{P}{i}\right)}{i^2}\frac{di}{dt} = -\frac{2P}{i^3}\frac{di}{dt}$

7. (a) $s = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2} \Rightarrow \frac{ds}{dt} = \frac{x}{\sqrt{x^2+y^2}}\frac{dx}{dt}$
   (b) $s = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2} \Rightarrow \frac{ds}{dt} = \frac{x}{\sqrt{x^2+y^2}}\frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2}}\frac{dy}{dt}$
   (c) $s = \sqrt{x^2+y^2} \Rightarrow s^2 = x^2+y^2 \Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow 2s \cdot 0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow \frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$

8. (a) $s = \sqrt{x^2+y^2+z^2} \Rightarrow s^2 = x^2+y^2+z^2 \Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt}$
   $\Rightarrow \frac{ds}{dt} = \frac{x}{\sqrt{x^2+y^2+z^2}}\frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{dy}{dt} + \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{dz}{dt}$
   (b) From part (a) with $\frac{dx}{dt} = 0 \Rightarrow \frac{ds}{dt} = \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{dy}{dt} + \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{dz}{dt}$
   (c) From part (a) with $\frac{ds}{dt} = 0 \Rightarrow 0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} \Rightarrow \frac{dx}{dt} + \frac{y}{x}\frac{dy}{dt} + \frac{z}{x}\frac{dz}{dt} = 0$

9. (a) $A = \frac{1}{2}ab\sin\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}ab\cos\theta\frac{d\theta}{dt}$
   (b) $A = \frac{1}{2}ab\sin\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}ab\cos\theta\frac{d\theta}{dt} + \frac{1}{2}b\sin\theta\frac{da}{dt}$

(c) $A = \frac{1}{2}ab \sin\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}ab \cos\theta \frac{d\theta}{dt} + \frac{1}{2}b \sin\theta \frac{da}{dt} + \frac{1}{2}a \sin\theta \frac{db}{dt}$

10. Given $A = \pi r^2$ $\frac{dr}{dt} = 0.01$ cm/sec, and r = 50 cm. Since $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$, then $\left.\frac{dA}{dt}\right|_{r=50} = 2\pi(50)\left(\frac{1}{100}\right)$ $= \pi$ cm^2/min.

11. Given $\frac{d\ell}{dt} = -2$ cm/sec, $\frac{dw}{dt} = 2$ cm/sec, $\ell = 12$ cm and w = 5 cm.

(a) $A = \ell w \Rightarrow \frac{dA}{dt} = \ell \frac{dw}{dt} + w \frac{d\ell}{dt} \Rightarrow \frac{dA}{dt} = 12(2) + 5(-2) = 14$ cm^2/sec, increasing

(b) $P = 2\ell + 2w \Rightarrow \frac{dP}{dt} = 2\frac{d\ell}{dt} + 2\frac{dw}{dt} = 2(-2) + 2(2) = 0$ cm/sec, constant

(c) $D = \sqrt{w^2 + \ell^2} = (w^2 + \ell^2)^{1/2} \Rightarrow \frac{dD}{dt} = \frac{1}{2}(w^2 + \ell^2)^{-1/2}\left(2w \frac{dw}{dt} + 2\ell \frac{d\ell}{dt}\right) \Rightarrow \frac{dD}{dt} = \frac{w\frac{dw}{dt} + \ell\frac{d\ell}{dt}}{\sqrt{w^2 + \ell^2}}$

$= \frac{(5)(2) + (12)(-2)}{\sqrt{25 + 144}} = -\frac{14}{13}$ cm/sec, decreasing

12. (a) $V = xyz \Rightarrow \frac{dV}{dt} = yz\frac{dx}{dt} + xz\frac{dy}{dt} + xy\frac{dz}{dt} \Rightarrow \left.\frac{dV}{dt}\right|_{(4,3,2)} = (3)(2)(1) + (4)(2)(-2) + (4)(3)(1) = 2$ m^3/sec

(b) $S = 2xy + 2xz + 2yz \Rightarrow \frac{dS}{dt} = (2y + 2z)\frac{dx}{dt} + (2x + 2z)\frac{dy}{dt} + (2x + 2y)\frac{dz}{dt}$

$\Rightarrow \left.\frac{dS}{dt}\right|_{(4,3,2)} = (10)(1) + (12)(-2) + (14)(1) = 0$ m^2/sec

(c) $\ell = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{1/2} \Rightarrow \frac{d\ell}{dt} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\frac{dx}{dt} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\frac{dy}{dt} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\frac{dz}{dt}$

$\Rightarrow \left.\frac{d\ell}{dt}\right|_{(4,3,2)} = \left(\frac{4}{\sqrt{29}}\right)(1) + \left(\frac{3}{\sqrt{29}}\right)(-2) + \left(\frac{2}{\sqrt{29}}\right)(1) = 0$ m/sec

13. Given: $\frac{dx}{dt} = 5$ ft/sec, the ladder is 13 ft long, and x = 12, y = 5 at the instant of time

(a) Since $x^2 + y^2 = 169 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\left(\frac{12}{5}\right)(5) = -12$ ft/sec, the ladder is sliding down the wall

(b) The area of the triangle formed by the ladder and walls is $A = \frac{1}{2}xy \Rightarrow \frac{dA}{dt} = \left(\frac{1}{2}\right)\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$. The area is changing at $\frac{1}{2}[12(-12) + 5(5)] = -\frac{119}{2} = -59.5$ ft^2/sec.

(c) $\cos\theta = \frac{x}{13} \Rightarrow -\sin\theta\frac{d\theta}{dt} = \frac{1}{13}\cdot\frac{dx}{dt} \Rightarrow \frac{d\theta}{dt} = -\frac{1}{13\sin\theta}\cdot\frac{dx}{dt} = -\left(\frac{1}{5}\right)(5) = -1$ rad/sec

14. $s^2 = y^2 + x^2 \Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow \frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) \Rightarrow \frac{ds}{dt} = \frac{1}{\sqrt{169}}[5(-442) + 12(-481)]$

$= -614$ knots

15. Let s represent the distance between the girl and the kite and x represents the horizontal distance between the girl and kite $\Rightarrow s^2 = (300)^2 + x^2 \Rightarrow \frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt} = \frac{400(25)}{500} = 20$ ft/sec.

16. When the diameter is 3.8 in., the radius is 1.9 in. and $\frac{dr}{dt} = \frac{1}{3000}$ in/min. Also $V = 6\pi r^2 \Rightarrow \frac{dV}{dt} = 12\pi r \frac{dr}{dt}$ $\Rightarrow \frac{dV}{dt} = 12\pi(1.9)\left(\frac{1}{3000}\right) = 0.0076\pi$. The volume is changing at about 0.0239 in^3/min.

17. $V = \frac{1}{3}\pi r^2 h$, $h = \frac{3}{8}(2r) = \frac{3r}{4} \Rightarrow r = \frac{4h}{3} \Rightarrow V = \frac{1}{3}\pi\left(\frac{4h}{3}\right)^2 h = \frac{16\pi h^3}{27} \Rightarrow \frac{dV}{dt} = \frac{16\pi h^2}{9}\frac{dh}{dt}$

(a) $\left.\frac{dh}{dt}\right|_{h=4} = \left(\frac{9}{16\pi 4^2}\right)(10) = \frac{90}{256\pi} \approx 0.1119$ m/sec $= 11.19$ cm/sec

(b) $r = \frac{4h}{3} \Rightarrow \frac{dr}{dt} = \frac{4}{3}\frac{dh}{dt} = \frac{4}{3}\left(\frac{90}{256\pi}\right) = \frac{15}{32\pi} \approx 0.1492$ m/sec $= 14.92$ cm/sec

18. (a) $V = \frac{1}{3}\pi r^2 h$ and $r = \frac{15h}{2} \Rightarrow V = \frac{1}{3}\pi\left(\frac{15h}{2}\right)^2 h = \frac{75\pi h^3}{4} \Rightarrow \frac{dV}{dt} = \frac{225\pi h^2}{4}\frac{dh}{dt} \Rightarrow \left.\frac{dh}{dt}\right|_{h=5} = \frac{4(-50)}{225\pi(5)^2} = \frac{-8}{225\pi}$ ≈ -0.0113 m/min $= -1.13$ cm/min

(b) $r = \frac{15h}{2} \Rightarrow \frac{dr}{dt} = \frac{15}{2}\frac{dh}{dt} \Rightarrow \left.\frac{dr}{dt}\right|_{h=5} = \left(\frac{15}{2}\right)\left(\frac{-8}{225\pi}\right) = \frac{-4}{15\pi} \approx -0.0849$ m/sec $= -8.49$ cm/sec

19. (a) $V = \frac{\pi}{3}y^2(3R - y) \Rightarrow \frac{dV}{dt} = \frac{\pi}{3}[2y(3R - y) + y^2(-1)]\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \left[\frac{\pi}{3}(6Ry - 3y^2)\right]^{-1}\frac{dV}{dt} \Rightarrow$ at $R = 13$ and $y = 8$ we have $\frac{dy}{dt} = \frac{1}{144\pi}(-6) = \frac{-1}{24\pi}$ m/min

(b) The hemisphere is on the circle $r^2 + (13 - y)^2 = 169 \Rightarrow r = \sqrt{26y - y^2}$ m

(c) $r = (26y - y^2)^{1/2} \Rightarrow \frac{dr}{dt} = \frac{1}{2}(26y - y^2)^{-1/2}(26 - 2y)\frac{dy}{dt} \Rightarrow \frac{dr}{dt} = \frac{13 - y}{\sqrt{26y - y^2}}\frac{dy}{dt} \Rightarrow \left.\frac{dr}{dt}\right|_{y=8} = \frac{13 - 8}{\sqrt{26 \cdot 8 - 64}}\left(\frac{-1}{24\pi}\right)$ $= \frac{-5}{288\pi}$ m/min

20. If $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$, and $\frac{dV}{dt} = kS = 4k\pi r^2$, then $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow 4k\pi r^2 = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = k$, a constant. Therefore, the radius is increasing at a constant rate.

21. If $V = \frac{4}{3}\pi r^3$, $r = 5$, and $\frac{dV}{dt} = 100\pi$ ft^3/min, then $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = 1$ ft/min. Then $S = 4\pi r^2 \Rightarrow \frac{dS}{dt}$ $= 8\pi r \frac{dr}{dt} = 8\pi(5)(1) = 40\pi$ ft^2/min, the rate at which the area is increasing.

22. Let s represent the length of the rope and x the horizontal distance of the boat from the dock.

(a) We have $s^2 = x^2 + 36 \Rightarrow \frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt} = \frac{s}{\sqrt{s^2 - 36}}\frac{ds}{dt}$. Therefore, the boat is approaching the dock at $\left.\frac{dx}{dt}\right|_{s=10} = \frac{10}{\sqrt{10^2 - 36}}(2) = 2.5$ ft/sec.

(b) $\cos\theta = \frac{6}{r} \Rightarrow -\sin\theta\frac{d\theta}{dt} = -\frac{6}{r^2}\frac{dr}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{6}{r^2\sin\theta}\frac{dr}{dt}$. Thus, $r = 10$, $x = 8$, and $\sin\theta = \frac{8}{10}$ $\Rightarrow \frac{d\theta}{dt} = \frac{6}{10^2\left(\frac{8}{10}\right)} \cdot (-2) = -\frac{3}{20}$ rad/sec

23. Let s represent the distance between the bicycle and balloon, h the height of the balloon and x the horizontal distance between the balloon and the bicycle. The relationship between the variables is $s^2 = h^2 + x^2$ $\Rightarrow \frac{ds}{dt} = \frac{1}{s}\left(h\frac{dh}{dt} + x\frac{dx}{dt}\right) \Rightarrow \frac{ds}{dt} = \frac{1}{85}[68(1) + 51(17)] = 11$ ft/sec.

24. (a) Let h be the height of the coffee in the pot. Since the radius of the pot is 3, the volume of the coffee is $V = 9\pi h \Rightarrow \frac{dV}{dt} = 9\pi\frac{dh}{dt} \Rightarrow$ the rate the coffee is rising is $\frac{dh}{dt} = \frac{1}{9\pi}\frac{dV}{dt} = \frac{10}{9\pi}$ in/min.

(b) Let h be the height of the coffee in the pot. From the figure, the radius of the filter $r = \frac{h}{2} \Rightarrow V = \frac{1}{3}\pi r^2 h = \frac{\pi h^3}{12}$, the volume of the filter. The rate the coffee is falling is $\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} = \frac{4}{25\pi}(-10) = -\frac{8}{5\pi}$ in/min.

25. $y = QD^{-1} \Rightarrow \frac{dy}{dt} = D^{-1}\frac{dQ}{dt} - QD^{-2}\frac{dD}{dt} = \frac{1}{41}(0) - \frac{233}{(41)^2}(-2) = \frac{466}{1681}$ L/min $\Rightarrow$ increasing about 0.2772 L/min

26. (a) $\frac{dc}{dt} = (3x^2 - 12x + 15)\frac{dx}{dt} = (3(2)^2 - 12(2) + 15)(0.1) = 0.3$, $\frac{dr}{dt} = 9\frac{dx}{dt} = 9(0.1) = 0.9$, $\frac{dp}{dt} = 0.9 - 0.3 = 0.6$

(b) $\frac{dc}{dt} = (3x^2 - 12x - 45x^{-2})\frac{dx}{dt} = (3(1.5)^2 - 12(1.5) - 45(1.5)^{-2})(0.05) = -1.5625$, $\frac{dr}{dt} = 70\frac{dx}{dt} = 70(0.05) = 3.5$, $\frac{dp}{dt} = 3.5 - (-1.5625) = 5.0625$

27. Let P(x, y) represent a point on the curve $y = x^2$ and θ the angle of inclination of a line containing P and the origin. Consequently, $\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{x^2}{x} = x \Rightarrow \sec^2\theta\,\frac{d\theta}{dt} = \frac{dx}{dt} \Rightarrow \frac{d\theta}{dt} = \cos^2\theta\,\frac{dx}{dt}$. Since $\frac{dx}{dt} = 10$ m/sec and $\cos^2\theta\big|_{x=3} = \frac{x^2}{y^2 + x^2} = \frac{3^2}{9^2 + 3^2} = \frac{1}{10}$, we have $\frac{d\theta}{dt}\Big|_{x=3} = 1$ rad/sec.

28. $y = (-x)^{1/2}$ and $\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{(-x)^{1/2}}{x} \Rightarrow \sec^2\theta\,\frac{d\theta}{dt} = \frac{\left(\frac{1}{2}\right)(-x)^{-1/2}(-1)x - (-x)^{1/2}(1)}{x^2}\,\frac{dx}{dt}$

$\Rightarrow \frac{d\theta}{dt} = \left(\frac{\frac{-x}{2\sqrt{-x}} - \sqrt{-x}}{x^2}\right)(\cos^2\theta)\left(\frac{dx}{dt}\right)$. Now, $\tan\theta = \frac{2}{-4} = -\frac{1}{2} \Rightarrow \cos\theta = -\frac{2}{\sqrt{5}} \Rightarrow \cos^2\theta = \frac{4}{5}$. Then

$\frac{d\theta}{dt} = \left(\frac{\frac{4}{4} - 2}{16}\right)\left(\frac{4}{5}\right)(-8) = \frac{2}{5}$ rad/sec.

29. The distance from the origin is $s = \sqrt{x^2 + y^2}$ and we wish to find $\frac{ds}{dt}\Big|_{(5,12)}$

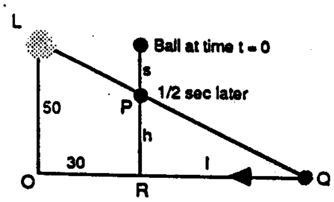
$= \frac{1}{2}(x^2 + y^2)^{-1/2}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)\Big|_{(5,12)} = \frac{(5)(-1) + (12)(-5)}{\sqrt{25 + 144}} = -5$ m/sec

30. When s represents the length of the shadow and x the distance of the man from the streetlight, then $s = \frac{3}{5}x$.

(a) If I represents the distance of the tip of the shadow from the streetlight, then $I = s + x \Rightarrow \frac{dI}{dt} = \frac{ds}{dt} + \frac{dx}{dt}$ (which is velocity not speed) $\Rightarrow \left|\frac{dI}{dt}\right| = \left|\frac{3}{5}\frac{dx}{dt} + \frac{dx}{dt}\right| = \left|\frac{8}{5}\right|\left|\frac{dx}{dt}\right| = \frac{8}{5}|-5| = 8$ ft/sec, the speed the tip of the shadow is moving along the ground.

(b) $\frac{ds}{dt} = \frac{3}{5}\frac{dx}{dt} = \frac{3}{5}(-5) = -3$ ft/sec, so the length of the shadow is decreasing at a rate of 3 ft/sec.

31. Let $s = 16t^2$ represent the distance the ball has fallen, h the distance between the ball and the ground, and I the distance between the shadow and the point directly beneath the ball. Accordingly, $s + h = 50$ and since the triangle LOQ and triangle PRQ are similar we have



$$I = \frac{30h}{50 - h} \Rightarrow h = 50 - 16t^2 \text{ and } I = \frac{30(50 - 16t^2)}{50 - (50 - 16t^2)}$$

$$= \frac{1500}{16t^2} - 30 \Rightarrow \frac{dI}{dt} = -\frac{1500}{8t^3} \Rightarrow \left.\frac{dI}{dt}\right|_{t=\frac{1}{2}} = -1500 \text{ ft/sec.}$$

32. Let s = distance of car from foot of perpendicular in the textbook diagram $\Rightarrow \tan\theta = \frac{s}{132} \Rightarrow \sec^2\theta \frac{d\theta}{dt} = \frac{1}{132}\frac{ds}{dt}$ $\Rightarrow \frac{d\theta}{dt} = \frac{\cos^2\theta}{132}\frac{ds}{dt}$; $\frac{ds}{dt} = -264$ and $\theta = 0 \Rightarrow \frac{d\theta}{dt} = -2$ rad/sec. A half second later the car has traveled 132 ft right of the perpendicular $\Rightarrow |\theta| = \frac{\pi}{4}$, $\cos^2\theta = \frac{1}{2}$, and $\frac{ds}{dt} = 264$ (since s increases) $\Rightarrow \frac{d\theta}{dt} = \frac{\left(\frac{1}{2}\right)}{132}(264) = 1$ rad/sec.

33. The volume of the ice is $V = \frac{4}{3}\pi r^3 - \frac{4}{3}\pi 4^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \left.\frac{dr}{dt}\right|_{r=6} = -\frac{5}{72\pi}$ in/min when $\frac{dV}{dt} = -10\ \text{in}^3/\text{min}$.

The surface area is $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt} \Rightarrow \left.\frac{dS}{dt}\right|_{r=6} = 48\pi\left(-\frac{5}{72\pi}\right) = -\frac{10}{3}\ \text{in}^2/\text{min}$.

34. Let s represent the horizontal distance between the car and plane while r is the line-of-sight distance between the car and plane $\Rightarrow 9 + s^2 = r^2 \Rightarrow \frac{ds}{dt} = \frac{r}{\sqrt{r^2 - 9}}\frac{dr}{dt} \Rightarrow \left.\frac{ds}{dt}\right|_{r=5} = \frac{5}{\sqrt{16}}(-160) = -200$ mph

$\Rightarrow$ speed of plane + speed of car = 200 mph $\Rightarrow$ the speed of the car is 80 mph.

35. When x represents the length of the shadow, then $\tan\theta = \frac{80}{x} \Rightarrow \sec^2\theta \frac{d\theta}{dt} = -\frac{80}{x^2}\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{-x^2\sec^2\theta}{80}\frac{d\theta}{dt}$.

We are given that $\frac{d\theta}{dt} = 0.27° = \frac{3\pi}{2000}$ rad/min. At $x = 60$, $\cos\theta = \frac{3}{5} \Rightarrow$

$$\left|\frac{dx}{dt}\right| = \left|\frac{-x^2\sec^2\theta}{80}\frac{d\theta}{dt}\right|\Bigg|_{\left(\frac{d\theta}{dt} = \frac{3\pi}{2000} \text{ and } \sec\theta = \frac{5}{3}\right)} = \frac{3\pi}{16} \text{ ft/min} \approx 0.589 \text{ ft/min} \approx 7.1 \text{ in/min.}$$

36. $\tan\theta = \frac{A}{B} \Rightarrow \sec^2\theta \frac{d\theta}{dt} = \frac{1}{B}\frac{dA}{dt} - \frac{A}{B^2}\frac{dB}{dt} \Rightarrow$ at A = 10 m and B = 20 m we have $\cos\theta = \frac{20}{10\sqrt{5}} = \frac{2}{\sqrt{5}}$ and

$$\frac{d\theta}{dt} = \left[\left(\frac{1}{20}\right)(-2) - \left(\frac{10}{400}(1)\right)\right]\left(\frac{4}{5}\right) = \left(\frac{-1}{10} - \frac{1}{40}\right)\left(\frac{4}{5}\right) = -\frac{1}{10} \text{ rad/sec} = -\frac{18°}{\pi}\text{/sec} \approx -6°\text{/sec}$$

37. Let x represent distance of the player from second base and s the distance to third base. Then $\frac{dx}{dt} = -16$ ft/sec

(a) $s^2 = x^2 + 8100 \Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt} \Rightarrow \frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$. When the player is 30 ft from first base, $x = 60$ $\Rightarrow s = 30\sqrt{13}$ and $\frac{ds}{dt} = \frac{60}{30\sqrt{13}}(-16) = \frac{-32}{\sqrt{13}} \approx -8.875$ ft/sec

(b) $\cos\theta_1 = \frac{90}{s} \Rightarrow -\sin\theta_1 \frac{d\theta_1}{dt} = -\frac{90}{s^2}\cdot\frac{ds}{dt} \Rightarrow \frac{d\theta_1}{dt} = \frac{90}{s^2\sin\theta_1}\cdot\frac{ds}{dt} = \frac{90}{sx}\cdot\frac{ds}{dt}$. Therefore, $x = 60$ and $s = 30\sqrt{13}$

$$\Rightarrow \frac{d\theta_1}{dt} = \frac{90}{(30\sqrt{13})(60)}\cdot\left(\frac{-32}{\sqrt{13}}\right) = \frac{-8}{65} \text{ rad/sec; } \sin\theta_2 = \frac{90}{s} \Rightarrow \cos\theta_2 \frac{d\theta_2}{dt} = -\frac{90}{s^2}\cdot\frac{ds}{dt} \Rightarrow \frac{d\theta_2}{dt} = \frac{-90}{s^2\cos\theta_2}\cdot\frac{ds}{dt}$$

$= \frac{-90}{sx} \cdot \frac{ds}{dt}$. Therefore, x = 60 and $s = 30\sqrt{13} \Rightarrow \frac{d\theta_2}{dt} = \frac{8}{65}$ rad/sec.

(c) $\frac{d\theta_1}{dt} = \frac{90}{s^2 \sin\theta_1} \cdot \frac{ds}{dt} = \frac{90}{\left(s^2 \cdot \frac{x}{s}\right)} \cdot \left(\frac{x}{s}\right) \cdot \left(\frac{dx}{dt}\right) = \left(\frac{90}{s^2}\right)\left(\frac{dx}{dt}\right) = \left(\frac{90}{x^2+8100}\right)\frac{dx}{dt} \Rightarrow \lim_{x\to 0} \frac{d\theta_1}{dt}$

$= \lim_{x\to 0}\left(\frac{90}{x^2+8100}\right)(-15) = -\frac{1}{6}$ rad/sec; $\frac{d\theta_2}{dt} = \frac{-90}{s^2 \cos\theta_2} \cdot \frac{ds}{dt} = \left(\frac{-90}{s^2 \cdot \frac{x}{s}}\right)\left(\frac{x}{s}\right)\left(\frac{dx}{dt}\right) = \left(\frac{-90}{s^2}\right)\left(\frac{dx}{dt}\right)$

$= \left(\frac{-90}{x^2+8100}\right)\frac{dx}{dt} \Rightarrow \lim_{x\to 0} \frac{d\theta_2}{dt} = \frac{1}{6}$ rad/sec

38. Let a represent the distance between point O and ship A, b the distance between point O and ship B, and D the distance between the ships. By the Law of Cosines, $D^2 = a^2 + b^2 - 2ab\cos 120°$ $\Rightarrow \frac{dD}{dt} = \frac{1}{2D}\left[2a\frac{da}{dt} + 2b\frac{db}{dt} + a\frac{db}{dt} + b\frac{da}{dt}\right]$. When a = 5, $\frac{da}{dt} = 14$, b = 3, and $\frac{db}{dt} = 21$, then $\frac{dD}{dt} = \frac{413}{2D}$ where D = 7. The ships are moving $\frac{dD}{dt} = 29.5$ knots apart.

2.8 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

1. $y = \cos^{-1}(x^2) \Rightarrow \frac{dy}{dx} = -\frac{2x}{\sqrt{1-(x^2)^2}} = \frac{-2x}{\sqrt{1-x^4}}$

2. $y = \cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

3. $y = \sin^{-1}\sqrt{2}t \Rightarrow \frac{dy}{dt} = \frac{\sqrt{2}}{\sqrt{1-(\sqrt{2}t)^2}} = \frac{\sqrt{2}}{\sqrt{1-2t^2}}$

4. $y = \sin^{-1}(1-t) \Rightarrow \frac{dy}{dt} = \frac{-1}{\sqrt{1-(1-t)^2}} = \frac{-1}{\sqrt{2t-t^2}}$

5. $y = \sec^{-1}(2s+1) \Rightarrow \frac{dy}{ds} = \frac{2}{|2s+1|\sqrt{(2s+1)^2-1}} = \frac{2}{|2s+1|\sqrt{4s^2+4s}} = \frac{1}{|2s+1|\sqrt{s^2+s}}$

6. $y = \sec^{-1} 5s \Rightarrow \frac{dy}{ds} = \frac{5}{|5s|\sqrt{(5s)^2-1}} = \frac{1}{|s|\sqrt{25s^2-1}}$

7. $y = \csc^{-1}(x^2+1) \Rightarrow \frac{dy}{dx} = -\frac{2x}{|x^2+1|\sqrt{(x^2+1)^2-1}} = \frac{-2x}{(x^2+1)\sqrt{x^4+2x^2}}$

8. $y = \csc^{-1}\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = -\frac{\left(\frac{1}{2}\right)}{\left|\frac{x}{2}\right|\sqrt{\left(\frac{x}{2}\right)^2-1}} = \frac{-1}{|x|\sqrt{\frac{x^2-4}{4}}} = \frac{-2}{|x|\sqrt{x^2-4}}$

9. $y = \sec^{-1}\left(\frac{1}{t}\right) = \cos^{-1} t \Rightarrow \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$

10. $y = \sin^{-1}\left(\frac{3}{t^2}\right) = \csc^{-1}\left(\frac{t^2}{3}\right) \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{2t}{3}\right)}{\left|\frac{t^2}{3}\right|\sqrt{\left(\frac{t^2}{3}\right)^2 - 1}} = \frac{-2t}{t^2\sqrt{\frac{t^4-9}{9}}} = \frac{-6}{t\sqrt{t^4-9}}$

11. $y = \cot^{-1}\sqrt{t} = \cot^{-1} t^{1/2} \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{1}{2}\right)t^{-1/2}}{1+\left(t^{1/2}\right)^2} = \frac{-1}{2\sqrt{t}(1+t)}$

12. $y = \cot^{-1}\sqrt{t-1} = \cot^{-1}(t-1)^{1/2} \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{1}{2}\right)(t-1)^{-1/2}}{1+\left[(t-1)^{1/2}\right]^2} = \frac{-1}{2\sqrt{t-1}\,(1+t-1)} = \frac{-1}{2t\sqrt{t-1}}$

13. $y = s\sqrt{1-s^2} + \cos^{-1} s = s\left(1-s^2\right)^{1/2} + \cos^{-1} s \Rightarrow \frac{dy}{ds} = \left(1-s^2\right)^{1/2} + s\left(\frac{1}{2}\right)\left(1-s^2\right)^{-1/2}(-2s) - \frac{1}{\sqrt{1-s^2}}$

$= \sqrt{1-s^2} - \frac{s^2}{\sqrt{1-s^2}} - \frac{1}{\sqrt{1-s^2}} = \sqrt{1-s^2} - \frac{s^2+1}{\sqrt{1-s^2}} = \frac{1-s^2-s^2-1}{\sqrt{1-s^2}} = \frac{-2s^2}{\sqrt{1-s^2}}$

14. $y = \sqrt{s^2-1} - \sec^{-1} s = \left(s^2-1\right)^{1/2} - \sec^{-1} s \Rightarrow \frac{dy}{ds} = \left(\frac{1}{2}\right)\left(s^2-1\right)^{-1/2}(2s) - \frac{1}{|s|\sqrt{s^2-1}} = \frac{s}{\sqrt{s^2-1}} - \frac{1}{|s|\sqrt{s^2-1}}$

$= \frac{s|s|-1}{|s|\sqrt{s^2-1}}$

15. $y = \tan^{-1}\sqrt{x^2-1} + \csc^{-1} x = \tan^{-1}\left(x^2-1\right)^{1/2} + \csc^{-1} x \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{2}\right)\left(x^2-1\right)^{-1/2}(2x)}{1+\left[\left(x^2-1\right)^{1/2}\right]^2} - \frac{1}{|x|\sqrt{x^2-1}}$

$= \frac{x}{\sqrt{x^2-1}\left(1+x^2-1\right)} - \frac{1}{|x|\sqrt{x^2-1}} = \frac{1}{x\sqrt{x^2-1}} - \frac{1}{|x|\sqrt{x^2-1}} = 0$, for $x > 1$

16. $y = \cos^{-1}\left(\frac{1}{x}\right) - \tan^{-1} x = \frac{\pi}{2} - \tan^{-1}\left(x^{-1}\right) - \tan^{-1} x \Rightarrow \frac{dy}{dx} = 0 - \frac{-x^{-2}}{1+\left(x^{-1}\right)^2} - \frac{1}{1+x^2} = \frac{1}{x^2+1} - \frac{1}{1+x^2} = 0$

17. $y = x\sin^{-1} x + \sqrt{1-x^2} = x\sin^{-1} x + \left(1-x^2\right)^{1/2} \Rightarrow \frac{dy}{dx} = \sin^{-1} x + x\left(\frac{1}{\sqrt{1-x^2}}\right) + \left(\frac{1}{2}\right)\left(1-x^2\right)^{-1/2}(-2x)$

$= \sin^{-1} x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \sin^{-1} x$

18. $y = \frac{1}{\sin^{-1}(2x)} = \left[\sin^{-1}(2x)\right]^{-1} \Rightarrow \frac{dy}{dx} = -\left[\sin^{-1}(2x)\right]^{-2}\frac{d}{dx}\sin^{-1}(2x) = -\left[\sin^{-1}(2x)\right]^{-2}\frac{1}{\sqrt{1-4x^2}}(2)$

$= -\frac{2}{\left[\sin^{-1}(2x)\right]^2\sqrt{1-4x^2}}$

19. (a) Since $\frac{dy}{dx} = \sec^2 x$, the slope at $\left(\frac{\pi}{4}, 1\right)$ is $\sec^2\left(\frac{\pi}{4}\right) = 2$. The tangent line is given by $y = 2\left(x - \frac{\pi}{4}\right) + 1$, or $y = 2x - \frac{\pi}{2} + 1$.

(b) Since $\frac{dy}{dx} = \frac{1}{1+x^2}$, the slope at $\left(1, \frac{\pi}{4}\right)$ is $\frac{1}{1+1^2} = \frac{1}{2}$. The tangent line is given by $y = \frac{1}{2}(x-1) + \frac{\pi}{4}$, or $y = \frac{1}{2}x - \frac{1}{2} + \frac{\pi}{4}$.

20. (a) Note that $f'(x) = 5x^4 + 6x^2 + 1$. Thus $f(1) = 3$ and $f'(1) = 12$.

(b) Since the graph of $y = f(x)$ includes the point $(1, 3)$ and the slope of the graph is 12 at this point, the graph of $y = f^{-1}(x)$ will include $(3, 1)$ and the slope will be $\frac{1}{12}$. Thus, $f^{-1}(3) = 1$ and $(f^{-1})'(3) = \frac{1}{12}$. (We have assumed that $f^{-1}(x)$ is defined and differentiable at $x = 3$. This is true by Theorem 5, because $f'(x) = 5x^4 + 6x^2 + 1$, which is never zero.)

21. (a) Note that $f'(x) = -\sin x + 3$, which is always between 2 and 4. Thus f is differentiable at every point on the interval $(-\infty, \infty)$ and $f'(x)$ is never zero on this interval, so f has a differentiable inverse by Theorem 5.

(b) $f(0) = \cos 0 + 3(0) = 1$;
$f'(0) = -\sin 0 + 3 = 3$

(c) Since the graph of $y = f(x)$ includes the point $(0, 1)$ and the slope of the graph is 3 at this point, the graph of $y = f^{-1}(x)$ will include $(1, 0)$ and the slope will be $\frac{1}{3}$. Thus, $f^{-1}(1) = 0$ and $(f^{-1})'(1) = \frac{1}{3}$.

22. (a) $v(t) = \frac{dx}{dt} = \frac{1}{1+t^2}$ which is always positive.

(b) $a(t) = \frac{dv}{dt} = -\frac{2t}{(1+t^2)^2}$ which is always negative.

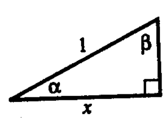
(c) $\frac{\pi}{2}$

23. $\frac{d}{dx}\cos^{-1}(x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1} x\right) = 0 - \frac{d}{dx}\sin^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$

24. $\frac{d}{dx}\cot^{-1}(x) = \frac{d}{dx}\left(\frac{\pi}{2} - \tan^{-1}(x)\right) = 0 - \frac{d}{dx}\tan^{-1}(x) = -\frac{1}{1+x^2}$

25. $\frac{d}{dx}\csc^{-1}(x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sec^{-1}(x)\right) = 0 - \frac{d}{dx}\sec^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$

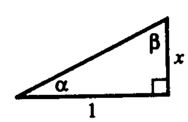
26. (a)



$\alpha = \cos^{-1} x$, $\beta = \sin^{-1} x$

So $\cos^{-1} x + \sin^{-1} x = \alpha + \beta = \frac{\pi}{2}$.

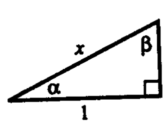
(b)



$\alpha = \tan^{-1} x$, $\beta = \cot^{-1} x$

So $\tan^{-1} x + \cot^{-1} x = \alpha + \beta = \frac{\pi}{2}$.

(c)



$\alpha = \sec^{-1} x$, $\beta = \csc^{-1} x$

So $\sec^{-1} x + \csc^{-1} x = \alpha + \beta = \frac{\pi}{2}$.

27. (a) $y = \frac{\pi}{2}$

(b) $y = -\frac{\pi}{2}$

(c) None, since $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} \neq 0$.

28. (a) $y = 0$

(b) $y = \pi$

(c) None, since $\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2} \neq 0$.

29. (a) $y = \frac{\pi}{2}$

(b) $y = \frac{\pi}{2}$

(c) None, since $\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}} \neq 0$.

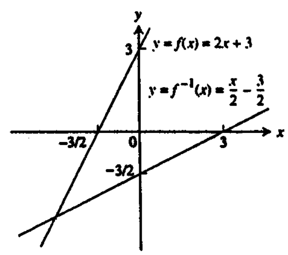
30. (a) $y = 0$

(b) $y = 0$

(c) None, since $\frac{d}{dx} \csc^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}} \neq 0$.

31. (a) $y = 2x + 3 \Rightarrow 2x = y - 3$
$\Rightarrow x = \frac{y}{2} - \frac{3}{2} \Rightarrow f^{-1}(x) = \frac{x}{2} - \frac{3}{2}$

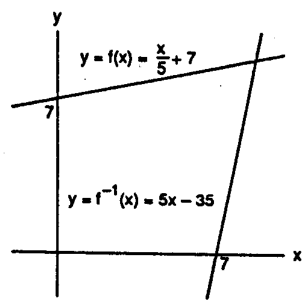
(b)



(c) $\left.\frac{df}{dx}\right|_{x=-1} = 2, \left.\frac{df^{-1}}{dx}\right|_{x=1} = \frac{1}{2}$

32. (a) $y = \frac{1}{5}x + 7 \Rightarrow \frac{1}{5}x = y - 7$
$\Rightarrow x = 5y - 35 \Rightarrow f^{-1}(x) = 5x - 35$

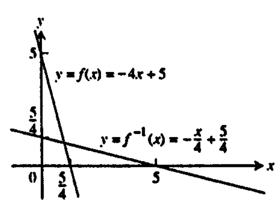
(b)



(c) $\left.\frac{df}{dx}\right|_{x=-1} = \frac{1}{5}, \left.\frac{df^{-1}}{dx}\right|_{x=34/5} = 5$

33. (a) $y = 5 - 4x \Rightarrow 4x = 5 - y$
$\Rightarrow x = \frac{5}{4} - \frac{y}{4} \Rightarrow f^{-1}(x) = \frac{5}{4} - \frac{x}{4}$

(b)

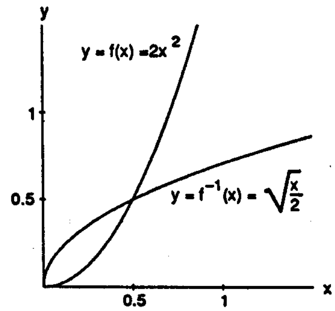


(c) $\left.\frac{df}{dx}\right|_{x=1/2} = -4, \left.\frac{df^{-1}}{dx}\right|_{x=3} = -\frac{1}{4}$

34. (a) $y = 2x^2 \Rightarrow x^2 = \frac{1}{2}y$

$\Rightarrow x = \frac{1}{\sqrt{2}}\sqrt{y} \Rightarrow f^{-1}(x) = \sqrt{\frac{x}{2}}$

(b)

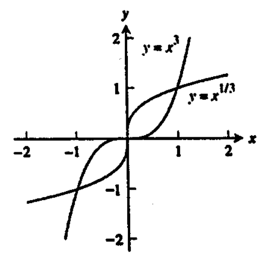


(c) $\left.\frac{df}{dx}\right|_{x=5} = 4x\,|_{x=5} = 20,$

$\left.\frac{df^{-1}}{dx}\right|_{x=50} = \left.\frac{1}{2\sqrt{2}}x^{-1/2}\right|_{x=50} = \frac{1}{20}$

35. (a) $f(g(x)) = \left(\sqrt[3]{x}\right)^3 = x,\ g(f(x)) = \sqrt[3]{x^3} = x$

(b)



(c) $f'(x) = 3x^2 \Rightarrow f'(1) = 3,\ f'(-1) = 3;$

$g'(x) = \frac{1}{3}x^{-2/3} \Rightarrow g'(1) = \frac{1}{3},\ g'(-1) = \frac{1}{3}$

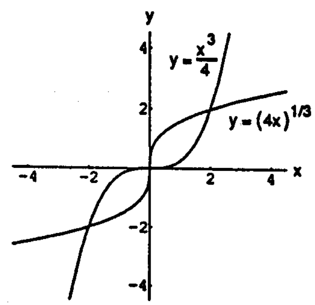
(d) The line $y = 0$ is tangent to $f(x) = x^3$ at $(0,0)$;

the line $x = 0$ is tangent to $g(x) = \sqrt[3]{x}$ at $(0,0)$

36. (a) $h(k(x)) = \frac{1}{4}\left((4x)^{1/3}\right)^3 = x,$

$k(h(x)) = \left(4\cdot\frac{x^3}{4}\right)^{1/3} = x$

(b)



(c) $h'(x) = \frac{3x^2}{4} \Rightarrow h'(2) = 3,\ h'(-2) = 3;$

$k'(x) = \frac{4}{3}(4x)^{-2/3} \Rightarrow k'(2) = \frac{1}{3},\ k'(-2) = \frac{1}{3}$

(d) The line $y = 0$ is tangent to $h(x) = \frac{x^3}{4}$ at $(0,0)$;

the line $x = 0$ is tangent to $k(x) = (4x)^{1/3}$ at $(0,0)$

37. (a) $y = mx \Rightarrow x = \frac{1}{m}y \Rightarrow f^{-1}(x) = \frac{1}{m}x$

(b) The graph of $y = f^{-1}(x)$ is a line through the origin with slope $\frac{1}{m}$.

38. $y = mx + b \Rightarrow x = \frac{y}{m} - \frac{b}{m} \Rightarrow f^{-1}(x) = \frac{1}{m}x - \frac{b}{m}$; the graph of $f^{-1}(x)$ is a line with slope $\frac{1}{m}$ and y-intercept $-\frac{b}{m}$.

39-46. Example CAS commands:

<u>Maple</u>:

```
identity:= z -> z;
eq:= y=(3*x + 2)/(2*x - 11);
solve(eq,y);
simplify(%): f:= unapply(%,x);
diff(f(x),x);
simplify(%): df:= unapply(% ,x);
```

```
plot({f,df}, −5..5, −5..5);
solve(eq,x);
g:= unapply(%,y);
finv:= y -> g(y);
plot({f,finv,identity}, −1..1, −2..1);
x0:= 1/2; y0:= f(x0):
ftan:= x -> f(x0) + df(x0)*(x − x0);
finvtan:= y -> x0 + (1/df)(x0)*(y − y0);
plot({f,finv,identity,ftan,finvtan,[x0,y0,y0,x0]}, −1..1, −1.5..1, scaling=constrained);
```

Mathematica

```
Clear[x,y]
{a,b} = {−2,2}; x0 = 1/2 ;
f[x_] = (3x + 2)/(2x − 11)
Plot[ {f[x],f'[x]}, {x,a,b} ]
Solve[ y == f[x], x ]
g[y_] = x /. First [%]
y0 = f[x0]
ftan[x_] = y0 + f'[x0] (x − x0)
gtan[y_] = x0 + (1/f'[x0]) (y − y0)
Plot[{f[x],ftan[x],g[x],gtan[x],Identity[x]}, {x,a,b},
Epilog -> {Line[{{x0,y0},{y0,x0}}]},
PlotRange -> {{a,b}, {a,b}},
AspectRatio -> Automatic]
```

Remark:

Other problems are similar to the example, except for adjusting plot ranges to see both source and inverse points. (Note: functions involving cube roots only show the positive branch.)

47-48. Example CAS commands:

Maple:

```
identity:= z -> z;
eq:= y^(1/3) − 1 = (x + 2)^3;
solve(eq,y);
f:= unapply(%,x);
diff(f(x),x);
df:= unapply(%,x);
plot({f,df}, −2..0,−5..5);
solve(eq,x);
g:= unapply(%[1],y);
finv:= y -> if (1<=y) then g(y) elif (0<=y) then −(1 − y^(1/3))^(1/3) − 2 elif (−1<y) then −(
1+(−y)^(1/3))^(1/3) − 2 else −((−y^(1/3) + 1)^(1/3) − 2 fi;
plot({f,finv,identity}, −2..2, −5..5);
x0:= −3/2: y0:= f(x0):
ftan:= x -> f(x0) + df(x0)*(x − x0);
finvtan:= y -> x0 + (1/df)(x0)*(y − y0);
plot({f,finv,identity,ftan,finvtan,[x0,y0,y0,x0]}, −5..5,−5..5, scaling = constrained);
```

Mathematica:

```
Clear[x,y]
{a,b} = {−5,5}; x0 = −3/2;
eqn = y^(1/3) − 1 == (x + 2)^3
Solve[ eqn, y ]
```

```
f[x_] = y /. First[%]
Plot[ {f[x],f'[x]}, {x,a,b} ]
Solve[ eqn, x ]
g[y_] = x /. First[%}
y0 = f[x0]
ftan[x_] = y0 + f'[x0] (x - x0)
gtan[y_] = x0 + (1/f'[x0]) (y - y0)
Plot[{f[x],ftan[x],g[x],gtan[x],Identity[x]},{x,a,b},
Epilog -> {Line[{{x0,y0},{y0,x0}}]},
PlotRange -> {{a,b}, {a,b}},
AspectRatio -> Automatic]
```

2.9 DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

1. $\frac{dy}{dx} = \frac{d}{dx}(2e^x) = 2e^x$

2. $\frac{dy}{dx} = \frac{d}{dx} e^{x+\sqrt{2}} = e^{x+\sqrt{2}} \cdot \frac{d}{dx}(x + \sqrt{2}) = e^{x+\sqrt{2}}$

3. $\frac{dy}{dx} = \frac{d}{dx} e^{-3x/2} = e^{-3x/2} \cdot \frac{d}{dx}\left(-\frac{3}{2}x\right) = -\frac{3}{2}e^{-3x/2}$

4. $\frac{dy}{dx} = \frac{d}{dx}e^{-5x} = e^{-5x} \frac{d}{dx}(-5x) = -5e^{-5x}$

5. $\frac{dy}{dx} = \frac{d}{dx}e^{2x/3} = e^{2x/3} \frac{d}{dx}\left(\frac{2x}{3}\right) = \frac{2}{3}e^{2x/3}$

6. $\frac{dy}{dx} = \frac{d}{dx}e^{-x/4} = e^{-x/4} \frac{d}{dx}\left(-\frac{x}{4}\right) = -\frac{1}{4}e^{-x/4}$

7. $\frac{dy}{dx} = \frac{d}{dx}(xe^x) - \frac{d}{dx}(e^x) = xe^x + e^x - e^x = xe^x$

8. $\frac{dy}{dx} = \frac{d}{dx}(x^2e^x) - \frac{d}{dx}(xe^x) = (x^2)(e^x) + (e^x)(2x) - \left[(x)(e^x) + (e^x)(1)\right] = x^2e^x + xe^x - e^x$

9. $\frac{dy}{dx} = \frac{d}{dx}e^{\sqrt{x}} = e^{\sqrt{x}} \frac{d}{dx}(\sqrt{x}) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

10. $\frac{dy}{dx} = \frac{d}{dx} e^{(x^3)} = e^{x^3} \cdot \frac{d}{dx}(x^3) = 3x^2e^{x^3}$

11. $\frac{dy}{dx} = \frac{d}{dx}(x^\pi) = \pi x^{\pi-1}$

12. $\frac{dy}{dx} = \frac{d}{dx}\left(x^{1+\sqrt{2}}\right) = (1 + \sqrt{2})x^{1+\sqrt{2}-1} = (1 + \sqrt{2})x^{\sqrt{2}}$

13. $\frac{dy}{dx} = \frac{d}{dx}x^{-\sqrt{2}} = -\sqrt{2}\,x^{-\sqrt{2}-1}$

14. $\frac{dy}{dx} = \frac{d}{dx}x^{1-e} = (1 - e)x^{1-e-1} = (1 - e)x^{-e}$

15. $\frac{dy}{dx} = \frac{d}{dx}8^x = 8^x \ln 8$

16. $\frac{dy}{dx} = \frac{d}{dx}9^{-x} = 9^{-x}(\ln 9)\frac{d}{dx}(-x) = -9^{-x}\ln 9$

17. $\frac{dy}{dx} = \frac{d}{dx}3^{\csc x} = 3^{\csc x}(\ln 3)\frac{d}{dx}(\csc x) = 3^{\csc x}(\ln 3)(-\csc x \cot x) = -3^{\csc x}(\ln 3)(\csc x \cot x)$

18. $\frac{dy}{dx} = \frac{d}{dx}3^{\cot x} = 3^{\cot x}(\ln 3)\frac{d}{dx}(\cot x) = 3^{\cot x}(\ln 3)(-\csc^2 x) = -3^{\cot x}(\ln 3)(\csc^2 x)$

19. $\frac{dy}{dx} = \frac{d}{dx}\frac{e^x}{e^{-x}+1} = \frac{(e^{-x}+1)e^x - e^x(-e^{-x})}{(e^{-x}+1)^2} = \frac{1+e^x+1}{e^{-2x}+2e^{-x}+1} = \frac{e^x+2}{e^{-2x}+2e^{-x}+1}$

20. $\frac{dy}{dx} = \frac{d}{dx}\frac{e^{-x}}{e^x+1} = \frac{(e^x+1)(-e^{-x}) - e^{-x}(e^x)}{(e^x+1)^2} = \frac{-1-e^{-x}-1}{e^{2x}+2e^x+1} = -\frac{e^{-x}+2}{e^{2x}+2e^x+1}$

21. $\frac{dy}{dx} = \frac{d}{dx}\ln(x^2) = \frac{1}{x^2}\frac{d}{dx}(x^2) = \frac{1}{x^2}(2x) = \frac{2}{x}$

22. $\frac{dy}{dx} = \frac{d}{dx}(\ln x)^2 = 2\ln x\,\frac{d}{dx}(\ln x) = \frac{2\ln x}{x}$

23. $\frac{dy}{dx} = \frac{d}{dx}\ln(x^{-1}) = \frac{d}{dx}(-\ln x) = -\frac{1}{x}$, $x > 0$

24. $\frac{dy}{dx} = \frac{d}{dx}\ln\left(\frac{10}{x}\right) = \frac{d}{dx}(\ln 10 - \ln x) = 0 - \frac{1}{x} = -\frac{1}{x}$, $x > 0$

25. $\frac{dy}{dx} = \frac{d}{dx}\ln(x+2) = \frac{1}{x+2}\frac{d}{dx}(x+2) = \frac{1}{x+2}$, $x > -2$

26. $\frac{dy}{dx} = \frac{d}{dx}\ln(2x+2) = \frac{1}{2x+2}\frac{d}{dx}(2x+2) = \frac{2}{2x+2} = \frac{1}{x+1}$, $x > -1$

27. $\frac{dy}{dx} = \frac{d}{dx}\ln(2-\cos x) = \frac{1}{2-\cos x}\frac{d}{dx}(2-\cos x) = \frac{\sin x}{2-\cos x}$

28. $\frac{dy}{dx} = \frac{d}{dx}\ln(x^2+1) = \frac{1}{x^2+1}\frac{d}{dx}(x^2+1) = \frac{2x}{x^2+1}$

29. $\frac{dy}{dx} = \frac{d}{dx}\ln(\ln x) = \frac{1}{\ln x}\frac{d}{dx}\ln x = \frac{1}{\ln x}\cdot\frac{1}{x} = \frac{1}{x\ln x}$

30. $\frac{dy}{dx} = \frac{d}{dx}(x\ln x - x) = (x)\left(\frac{1}{x}\right) + (\ln x)(1) - 1 = 1 + \ln x - 1 = \ln x$

31. $\frac{dy}{dx} = \frac{d}{dx}\left(\log_4 x^2\right) = \frac{d}{dx}\frac{\ln x^2}{\ln 4} = \frac{d}{dx}\left[\left(\frac{2}{\ln 4}\right)(\ln x)\right] = \frac{2}{\ln 4}\cdot\frac{1}{x} = \frac{2}{x\ln 4} = \frac{1}{x\ln 2}$

32. $\frac{dy}{dx} = \frac{d}{dx}(\log_5 \sqrt{x}) = \frac{d}{dx}\frac{\ln x^{1/2}}{\ln 5} = \frac{d}{dx}\frac{\frac{1}{2}\ln x}{\ln 5} = \frac{1}{2\ln 5}\frac{d}{dx}(\ln x) = \frac{1}{2\ln 5}\cdot\frac{1}{x} = \frac{1}{2x\ln 5}$, $x > 0$

33. $\frac{dy}{dx} = \frac{d}{dx}\log_2(3x+1) = \frac{1}{(3x+1)\ln 2}\frac{d}{dx}(3x+1) = \frac{3}{(3x+1)\ln 2}$, $x > -\frac{1}{3}$

34. $\frac{dy}{dx} = \frac{d}{dx}\log_{10}(x+1)^{1/2} = \frac{1}{2}\frac{d}{dx}\log_{10}(x+1) = \frac{1}{2}\frac{1}{(x+1)\ln 10}\frac{d}{dx}(x+1) = \frac{1}{2(x+1)\ln 10}$, $x > -1$

35. $\frac{dy}{dx} = \frac{d}{dx}\log_2\left(\frac{1}{x}\right) = \frac{d}{dx}(-\log_2 x) = -\frac{1}{x\ln 2}$, $x > 0$

36. $\frac{dy}{dx} = \frac{d}{dx}\frac{1}{\log_2 x} = -\frac{1}{(\log_2 x)^2}\frac{d}{dx}(\log_2 x) = -\frac{1}{(\log_2 x)^2}\frac{1}{x\ln 2} = -\frac{1}{x(\ln 2)(\log_2 x)^2}$ or $-\frac{\ln 2}{x(\ln x)^2}$

37. $\frac{dy}{dx} = \frac{d}{dx}(\ln 2\cdot\log_2 x) = (\ln 2)\frac{d}{dx}(\log_2 x) = (\ln 2)\left(\frac{1}{x\ln 2}\right) = \frac{1}{x}$, $x > 0$

38. $\frac{dy}{dx} = \frac{d}{dx}\log_3(1+x\ln 3) = \frac{1}{(1+x\ln 3)\ln 3}\frac{d}{dx}(1+x\ln 3) = \frac{\ln 3}{(1+x\ln 3)\ln 3} = \frac{1}{1+x\ln 3}$, $x > -\frac{1}{\ln 3}$

39. $\frac{dy}{dx} = \frac{d}{dx} \log_{10} e^x = \frac{d}{dx}(x \log_{10} e) = \log_{10} e = \frac{\ln e}{\ln 10} = \frac{1}{\ln 10}$

40. $\frac{dy}{dx} = \frac{d}{dx} \ln 10^x = \frac{d}{dx}(x \ln 10) = \ln 10$

41. $y = x^{\ln x}, x > 0 \Rightarrow \ln y = \ln\left(x^{\ln x}\right) \Rightarrow \ln y = (\ln x)^2 \Rightarrow \frac{1}{y}\frac{dy}{dx} = 2(\ln x)\left(\frac{1}{x}\right) \Rightarrow \frac{dy}{dx} = \left(x^{\ln x}\right)\left(\frac{\ln x^2}{x}\right)$

42. $y = x^{(1/\ln x)} \Rightarrow \ln y = \ln\left(x^{(1/\ln x)}\right) \Rightarrow \ln y = \frac{\ln x}{\ln x} = 1 \Rightarrow \frac{d}{dx}(\ln y) = 0 \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(1) \Rightarrow \frac{dy}{dx} = 0$

43. $y = (\sin x)^x \Rightarrow \ln y = \ln(\sin x)^x \Rightarrow \ln y = x \ln(\sin x) \Rightarrow \frac{d}{dx} \ln y = \frac{d}{dx}[x \ln(\sin x)]$
$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (x)\left(\frac{1}{\sin x}\right)(\cos x) + \ln(\sin x)(1) \Rightarrow \frac{dy}{dx} = y[x \cot x + \ln(\sin x)] \Rightarrow \frac{dy}{dx} = (\sin x)^x[x \cot x + \ln(\sin x)]$

44. $y = x^{\tan x} \Rightarrow \ln y = \ln(x^{\tan x}) \Rightarrow \ln y = (\tan x)(\ln x) \Rightarrow \frac{d}{dx} \ln y = \frac{d}{dx}[(\tan x)(\ln x)]$
$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (\tan x)\left(\frac{1}{x}\right) + (\ln x)(\sec^2 x) \Rightarrow \frac{dy}{dx} = y\left[\frac{\tan x}{x} + (\ln x)(\sec^2 x)\right] \Rightarrow \frac{dy}{dx} = x^{\tan x}\left[\frac{\tan x}{x} + (\ln x)(\sec^2 x)\right]$

45. $y = \sqrt[5]{\frac{(x-3)^4(x^2+1)}{(2x+5)^3}} = \left(\frac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5} \Rightarrow \ln y = \ln\left(\frac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5}$
$\Rightarrow \ln y = \frac{1}{5} \ln \frac{(x-3)^4(x^2+1)}{(2x+5)^3} \Rightarrow \ln y = \frac{1}{5}\left[4 \ln(x-3) + \ln(x^2+1) - 3 \ln(2x+5)\right]$
$\Rightarrow \frac{d}{dx}(\ln y) = \frac{4}{5}\frac{d}{dx} \ln(x-3) + \frac{1}{5}\frac{d}{dx}(x^2+1) - \frac{3}{5}\frac{d}{dx} \ln(2x+5) \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{4}{5}\frac{1}{x-3} + \frac{1}{5}\frac{1}{x^2+1}(2x) - \frac{3}{5}\frac{1}{2x+5}(2)$
$\Rightarrow \frac{dy}{dx} = y\left(\frac{4}{5(x-3)} + \frac{2x}{5(x^2+1)} - \frac{6}{5(2x+5)}\right) \Rightarrow \frac{dy}{dx} = \left(\frac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5} \cdot \left(\frac{4}{5(x-3)} + \frac{2x}{5(x^2+1)} - \frac{6}{5(2x+5)}\right)$

46. $y = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}} = \frac{x(x^2+1)^{1/2}}{(x+1)^{2/3}} \Rightarrow \ln y = \ln \frac{x(x^2+1)^{1/2}}{(x+1)^{2/3}} \Rightarrow \ln y = \ln x + \frac{1}{2}\ln(x^2+1) - \frac{2}{3}\ln(x+1)$
$\Rightarrow \frac{d}{dx} \ln y = \frac{d}{dx} \ln x + \frac{1}{2}\frac{d}{dx} \ln(x^2+1) - \frac{2}{3}\frac{d}{dx} \ln(x+1) \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\frac{1}{x^2+1}(2x) - \frac{2}{3}\frac{1}{x+1}(1)$
$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right) \Rightarrow \frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left(\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right)$

47. The line passes through (a, e^a) for some value of a and has slope $m = e^a$. Since the line also passes through the origin, the slope is also given by $m = \frac{e^a - 0}{a - 0}$ and we have $e^a = \frac{e^a}{a}$, so $a = 1$. Hence, the slope is e and the equation is $y = ex$.

48. For $y = xe^x$, we have $y' = (x)(e^x) + (e^x)(1) = (x+1)e^x$, so the normal line through the point (a, ae^a) has slope $m = -\frac{1}{(a+1)e^a}$ and its equation is $y = -\frac{1}{(a+1)e^a}(x-a) + ae^a$. The desired normal line includes the point

(0,0), so we have: $0 = -\frac{1}{(a+1)e^a}(0-a) + ae^a \Rightarrow 0 = \frac{a}{(a+1)e^a} + ae^a \Rightarrow 0 = a\left(\frac{1}{(a+1)e^a} + e^a\right)$

$\Rightarrow a = 0$ or $\frac{1}{(a+1)e^a} + e^a = 0$. The equation $\frac{1}{(a+1)e^a} + e^a = 0$ has no solution, so we need to use $a = 0$. The equation of the normal line is $y = -\frac{1}{(0+1)e^0}(x-0) + 0e^0$, or $y = -x$.

49. $\frac{dA}{dt} = 20\,\frac{d}{dt}\left(\frac{1}{2}\right)^{t/140} = 20\,\frac{d}{dt}\,2^{-t/140} = 20\left(2^{-t/140}\right)(\ln 2)\,\frac{d}{dt}\left(-\frac{t}{140}\right)$

$= 20\left(2^{-t/140}\right)(\ln 2)\left(-\frac{1}{140}\right) = -\frac{\left(2^{-t/140}\right)(\ln 2)}{7}$

At $t = 2$ days, we have $\frac{dA}{dt} = -\frac{\left(2^{-1/70}\right)(\ln 2)}{7} \approx -0.098$ grams/day. This means that the rate of *decay* is the positive rate of approximately 0.098 grams/day.

50. (a) $y = y_0e^{kt} \Rightarrow 0.99y_0 = y_0e^{1000k} \Rightarrow k = \frac{\ln 0.99}{1000} \approx -0.00001$

(b) $0.9 = e^{(-0.00001)t} \Rightarrow (-0.00001)t = \ln(0.9) \Rightarrow t = \frac{\ln(0.9)}{-0.00001} \approx 10{,}536$ years

(c) $y = y_0e^{(20,000)k} \approx y_0e^{-0.2} = y_0(0.82) \Rightarrow 82\%$

51. (a) There are $(60)(60)(24)(365) = 31{,}536{,}000$ seconds in a year. Thus, assuming exponential growth,

$P = 257{,}313{,}431e^{kt}$ and $257{,}313{,}432 = 257{,}313{,}431e^{(14k/31,536,000)} \Rightarrow \ln\left(\frac{257{,}313{,}432}{257{,}313{,}431}\right) = \frac{14k}{31{,}536{,}000}$

$\Rightarrow k \approx 0.008754168326$

(b) $P = 257{,}313{,}431e^{(0.008754168326)8} \approx 275{,}979{,}963$ (to the nearest integer). Answers will vary considerably with the number of decimal places retained.

52. $y = y_0e^{-kt} = y_0e^{-(k)(3/k)} = y_0e^{-3} = \frac{y_0}{e^3} < \frac{y_0}{20} = (0.05)(y_0) \Rightarrow$ after three mean lifetimes less than 5% remains

53. (a) $g'(0) = L$ because $g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \Rightarrow g'(0) = \lim_{h\to 0}\frac{g(0+h)-g(0)}{h} = \lim_{h\to 0}\frac{g(h)-g(0)}{h}$

$= \lim_{h\to 0}\frac{0.5^h - 1}{h} = L.$

(b)

h	0.1	0.01	0.001	0.0001	0.00001
$\frac{0.5^h-1}{h}$	−0.6697	−0.6908	−0.6929	−0.6931	−0.6931

$L \approx -0.6931$

54. $\frac{d}{dx}\left(-\frac{1}{2}x^2 + k\right) = -x$ and $\frac{d}{dx}(\ln x + c) = \frac{1}{x}$. Therefore, at any given value of x, these two curves will have perpendicular tangent lines.

55. Recall that a point (a, b) is on the graph of $y = e^x$ if and only if the point (b, a) is on the graph of $y = \ln x$. Since there are points (x, e^x) on the graph of $y = e^x$ with arbitrarily large x-coordinates, there will be points $(x, \ln x)$ on the graph of $y = \ln x$ with arbitrarily large y-coordinates.

56. The command solve $(x\hat{}2 = 2\hat{}x, x)$ on a TI-89 calculator gives three solutions. They are: $x = 4$, 2, and -0.766665. Another way to find the solutions is to graph $y = x^2 - 2^x$ and then use trace and zoom or use the zero function that is available on some calculators.

57. (a) Since the line passes through the origin and has slope $\frac{1}{e}$, its equation is $y = \frac{x}{e}$.

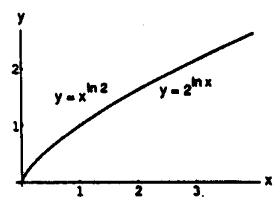
(b) The graph of $y = \ln x$ lies below the graph of the line $y = \frac{x}{e}$ for all positive $x \neq e$. Therefore, $\ln x < \frac{x}{e}$ for all positive $x \neq e$.

(c) Multiplying by e, $e \ln x < x$ or $\ln x^e < x$.

(d) Exponentiating both sides of $\ln x^e < x$, we have $e^{\ln x^e} < e^x$, or $x^e < e^x$ for all positive $x \neq e$.

(e) Let $x = \pi$ to see that $\pi^e < e^\pi$. Therefore, e^π is bigger.

58. The functions $f(x) = x^{\ln 2}$ and $g(x) = 2^{\ln x}$ appear to have identical graphs for $x > 0$. This is no accident, because $x^{\ln 2} = e^{\ln 2 \cdot \ln x} = \left(e^{\ln 2}\right)^{\ln x} = 2^{\ln x}$.



CHAPTER 2 PRACTICE EXERCISES

1. $y = x^5 - 0.125x^2 + 0.25x \Rightarrow \frac{dy}{dx} = 5x^4 - 0.25x + 0.25$

2. $y = x^3 - 3\left(x^2 + \pi^2\right) \Rightarrow \frac{dy}{dx} = 3x^2 - 3(2x + 0) = 3x^2 - 6x = 3x(x - 2)$

3. $y = x^7 + \sqrt{7}x - \frac{1}{\pi + 1} \Rightarrow \frac{dy}{dx} = 7x^6 + \sqrt{7}$

4. $y = (2x - 5)(4 - x)^{-1} \Rightarrow \frac{dy}{dx} = (2x - 5)(-1)(4 - x)^{-2}(-1) + (4 - x)^{-1}(2) = (4 - x)^{-2}[(2x - 5) + 2(4 - x)]$
$= 3(4 - x)^{-2}$

5. $y = \left(\theta^2 + \sec\theta + 1\right)^3 \Rightarrow \frac{dy}{d\theta} = 3\left(\theta^2 + \sec\theta + 1\right)^2(2\theta + \sec\theta\tan\theta)$

6. $s = \frac{\sqrt{t}}{1 + \sqrt{t}} \Rightarrow \frac{ds}{dt} = \frac{(1 + \sqrt{t})\cdot\frac{1}{2\sqrt{t}} - \sqrt{t}\left(\frac{1}{2\sqrt{t}}\right)}{(1 + \sqrt{t})^2} = \frac{(1 + \sqrt{t}) - \sqrt{t}}{2\sqrt{t}(1 + \sqrt{t})^2} = \frac{1}{2\sqrt{t}(1 + \sqrt{t})^2}$

7. $s = \frac{1}{\sqrt{t} - 1} \Rightarrow \frac{ds}{dt} = \frac{(\sqrt{t} - 1)(0) - 1\left(\frac{1}{2\sqrt{t}}\right)}{(\sqrt{t} - 1)^2} = \frac{-1}{2\sqrt{t}(\sqrt{t} - 1)^2}$

8. $y = 2\tan^2 x - \sec^2 x \Rightarrow \frac{dy}{dx} = (4\tan x)(\sec^2 x) - (2\sec x)(\sec x \tan x) = 2\sec^2 x \tan x$

9. $y = \frac{1}{\sin^2 x} - \frac{2}{\sin x} = \csc^2 x - 2\csc x \Rightarrow \frac{dy}{dx} = (2\csc x)(-\csc x \cot x) - 2(-\csc x \cot x) = (2\csc x \cot x)(1 - \csc x)$

10. $s = \cos^4(1-2t) \Rightarrow \frac{ds}{dt} = 4\cos^3(1-2t)(-\sin(1-2t))(-2) = 8\cos^3(1-2t)\sin(1-2t)$

11. $s = \cot^3\left(\frac{2}{t}\right) \Rightarrow \frac{ds}{dt} = 3\cot^2\left(\frac{2}{t}\right)\left(-\csc^2\left(\frac{2}{t}\right)\right)\left(\frac{-2}{t^2}\right) = \frac{6}{t^2}\cot^2\left(\frac{2}{t}\right)\csc^2\left(\frac{2}{t}\right)$

12. $s = (\sec t + \tan t)^5 \Rightarrow \frac{ds}{dt} = 5(\sec t + \tan t)^4(\sec t \tan t + \sec^2 t) = 5(\sec t)(\sec t + \tan t)^5$

13. $r = \sqrt{2\theta \sin\theta} = (2\theta\sin\theta)^{1/2} \Rightarrow \frac{dr}{d\theta} = \frac{1}{2}(2\theta\sin\theta)^{-1/2}(2\theta\cos\theta + 2\sin\theta) = \frac{\theta\cos\theta + \sin\theta}{\sqrt{2\theta\sin\theta}}$

14. $r = \sin(\theta + \sqrt{\theta+1}) \Rightarrow \frac{dr}{d\theta} = \cos(\theta + \sqrt{\theta+1})\left(1 + \frac{1}{2\sqrt{\theta+1}}\right) = \frac{2\sqrt{\theta+1}+1}{2\sqrt{\theta+1}}\cos(\theta + \sqrt{\theta+1})$

15. $y = \frac{1}{2}x^2 \csc\frac{2}{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^2\left(-\csc\frac{2}{x}\cot\frac{2}{x}\right)\left(\frac{-2}{x^2}\right) + \left(\csc\frac{2}{x}\right)\left(\frac{1}{2}\cdot 2x\right) = \csc\frac{2}{x}\cot\frac{2}{x} + x\csc\frac{2}{x}$

16. $y = x^{-1/2}\sec(2x)^2 \Rightarrow \frac{dy}{dx} = x^{-1/2}\sec(2x)^2\tan(2x)^2(2(2x)\cdot 2) + \sec(2x)^2\left(-\frac{1}{2}x^{-3/2}\right)$
$= 8x^{1/2}\sec(2x)^2\tan(2x)^2 - \frac{1}{2}x^{-3/2}\sec(2x)^2 = \frac{1}{2}x^{1/2}\sec(2x)^2[16\tan(2x)^2 - x^{-2}]$

17. $y = 5\cot x^2 \Rightarrow \frac{dy}{dx} = 5(-\csc^2 x^2)(2x) = -10x\csc^2(x^2)$

18. $y = x^2\sin^2(2x^2) \Rightarrow \frac{dy}{dx} = x^2(2\sin(2x^2))(\cos(2x^2))(4x) + \sin^2(2x^2)(2x) = 8x^3\sin(2x^2)\cos(2x^2) + 2x\sin^2(2x^2)$

19. $s = \left(\frac{4t}{t+1}\right)^{-2} \Rightarrow \frac{ds}{dt} = -2\left(\frac{4t}{t+1}\right)^{-3}\left(\frac{(t+1)(4) - (4t)(1)}{(t+1)^2}\right) = -2\left(\frac{4t}{t+1}\right)^{-3}\frac{4}{(t+1)^2} = -\frac{(t+1)}{8t^3}$

20. $y = \left(\frac{\sqrt{x}}{x+1}\right)^2 \Rightarrow \frac{dy}{dx} = 2\left(\frac{\sqrt{x}}{x+1}\right)\cdot\frac{(x+1)\left(\frac{1}{2\sqrt{x}}\right) - (\sqrt{x})(1)}{(x+1)^2} = \frac{(x+1)-2x}{(x+1)^3} = \frac{1-x}{(x+1)^3}$

21. $y = 4x\sqrt{x+\sqrt{x}} = 4x(x+x^{1/2})^{1/2} \Rightarrow \frac{dy}{dx} = 4x\left(\frac{1}{2}\right)(x+x^{1/2})^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right) + (x+x^{1/2})^{1/2}(4)$
$= (x+\sqrt{x})^{-1/2}\left[2x\left(1+\frac{1}{2\sqrt{x}}\right) + 4(x+\sqrt{x})\right] = (x+\sqrt{x})^{-1/2}(2x+\sqrt{x}+4x+4\sqrt{x}) = \frac{6x+5\sqrt{x}}{\sqrt{x+\sqrt{x}}}$

22. $r = \left(\frac{\sin\theta}{\cos\theta - 1}\right)^2 \Rightarrow \frac{dr}{d\theta} = 2\left(\frac{\sin\theta}{\cos\theta-1}\right)\left[\frac{(\cos\theta-1)(\cos\theta) - (\sin\theta)(-\sin\theta)}{(\cos\theta-1)^2}\right]$

$$= 2\left(\frac{\sin\theta}{\cos\theta - 1}\right)\left(\frac{\cos^2\theta - \cos\theta + \sin^2\theta}{(\cos\theta - 1)^2}\right) = \frac{(2\sin\theta)(1-\cos\theta)}{(\cos\theta - 1)^3} = \frac{-2\sin\theta}{(\cos\theta - 1)^2}$$

23. $y = 20(3x-4)^{1/4}(3x-4)^{-1/5} = 20(3x-4)^{1/20} \Rightarrow \frac{dy}{dx} = 20\left(\frac{1}{20}\right)(3x-4)^{-19/20}(3) = \frac{3}{(3x-4)^{19/20}}$

24. $y = 3\left(5x^2 + \sin 2x\right)^{-3/2} \Rightarrow \frac{dy}{dx} = 3\left(-\frac{3}{2}\right)\left(5x^2 + \sin 2x\right)^{-5/2}[10x + (\cos 2x)(2)] = \frac{-9(5x + \cos 2x)}{\left(5x^2 + \sin 2x\right)^{5/2}}$

25. $y = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} \Rightarrow \frac{dy}{dx} = \frac{1}{4}\left[x\left(4e^{4x}\right) + e^{4x}(1)\right] - \frac{1}{16}\left(4e^{4x}\right) = xe^{4x} + \frac{1}{4}e^{4x} - \frac{1}{4}e^{4x} = xe^{4x}$

26. $y = x^2e^{-2/x} = x^2e^{-2x^{-1}} \Rightarrow \frac{dy}{dx} = x^2\left[\left(2x^{-2}\right)e^{-2x^{-1}}\right] + e^{-2x^{-1}}(2x) = (2 + 2x)e^{-2x^{-1}} = 2e^{-2/x}(1 + x)$

27. $y = \ln\left(\sin^2\theta\right) \Rightarrow \frac{dy}{d\theta} = \frac{2(\sin\theta)(\cos\theta)}{\sin^2\theta} = \frac{2\cos\theta}{\sin\theta} = 2\cot\theta$

28. $y = \log_2\left(\frac{x^2}{2}\right) = \frac{\ln\left(\frac{x^2}{2}\right)}{\ln 2} \Rightarrow \frac{dy}{dx} = \frac{1}{\ln 2}\left(\frac{x}{\left(\frac{x^2}{2}\right)}\right) = \frac{2}{(\ln 2)x}$

29. $y = \log_5(3x-7) = \frac{\ln(3x-7)}{\ln 5} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{\ln 5}\right)\left(\frac{3}{3x-7}\right) = \frac{3}{(\ln 5)(3x-7)}$

30. $y = 8^{-t} \Rightarrow \frac{dy}{dt} = 8^{-t}(\ln 8)(-1) = -8^{-t}(\ln 8)$

31. $y = 5x^{3.6} \Rightarrow \frac{dy}{dx} = 5(3.6)x^{2.6} = 18x^{2.6}$

32. $y = \sqrt{2}\,x^{-\sqrt{2}} \Rightarrow \frac{dy}{dx} = (\sqrt{2})(-\sqrt{2})x^{\left(-\sqrt{2}-1\right)} = -2x^{\left(-\sqrt{2}-1\right)}$

33. $y = (x+2)^{x+2} \Rightarrow \ln y = \ln(x+2)^{x+2} = (x+2)\ln(x+2) \Rightarrow \frac{1}{y}\frac{dy}{dx} = (x+2)\left(\frac{1}{x+2}\right) + (1)\ln(x+2)$

$\Rightarrow \frac{dy}{dx} = (x+2)^{x+2}[\ln(x+2) + 1]$

34. $y = 2(\ln x)^{x/2} \Rightarrow \ln y = \ln\left[2(\ln x)^{x/2}\right] = \ln(2) + \left(\frac{x}{2}\right)\ln(\ln x) \Rightarrow \frac{1}{y}\frac{dy}{dx} = 0 + \left(\frac{x}{2}\right)\left[\frac{\left(\frac{1}{x}\right)}{\ln x}\right] + (\ln(\ln x))\left(\frac{1}{2}\right)$

$\Rightarrow \frac{dy}{dx} = \left[\frac{1}{2\ln x} + \left(\frac{1}{2}\right)\ln(\ln x)\right]2(\ln x)^{x/2} = (\ln x)^{x/2}\left[\ln(\ln x) + \frac{1}{\ln x}\right]$

35. $y = \sin^{-1}\sqrt{1-u^2} = \sin^{-1}\left(1-u^2\right)^{1/2} \Rightarrow \frac{dy}{du} = \frac{\frac{1}{2}\left(1-u^2\right)^{-1/2}(-2u)}{\sqrt{1-\left[\left(1-u^2\right)^{1/2}\right]^2}} = \frac{-u}{\sqrt{1-u^2}\sqrt{1-\left(1-u^2\right)}} = \frac{-u}{|u|\sqrt{1-u^2}}$

$= \frac{-u}{u\sqrt{1-u^2}} = \frac{-1}{\sqrt{1-u^2}},\ 0 < u < 1$

36. $y = \ln(\cos^{-1} x) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{-1}{\sqrt{1-x^2}}\right)}{\cos^{-1} x} = \frac{-1}{\sqrt{1-x^2}\cos^{-1} x}$

37. $y = z\cos^{-1} z - \sqrt{1-z^2} = z\cos^{-1} z - (1-z^2)^{1/2} \Rightarrow \frac{dy}{dz} = \cos^{-1} z - \frac{z}{\sqrt{1-z^2}} - \left(\frac{1}{2}\right)(1-z^2)^{-1/2}(-2z)$
$= \cos^{-1} z - \frac{z}{\sqrt{1-z^2}} + \frac{z}{\sqrt{1-z^2}} = \cos^{-1} z$

38. $y = t\tan^{-1} t - \left(\frac{1}{2}\right)\ln t \Rightarrow \frac{dy}{dt} = \tan^{-1} t + t\left(\frac{1}{1+t^2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{t}\right) = \tan^{-1} t + \frac{t}{1+t^2} - \frac{1}{2t}$

39. $y = (1+t^2)\cot^{-1} 2t \Rightarrow \frac{dy}{dt} = 2t\cot^{-1} 2t + (1+t^2)\left(\frac{-2}{1+4t^2}\right)$

40. $y = z\sec^{-1} z - \sqrt{z^2-1} = z\sec^{-1} z - (z^2-1)^{1/2} \Rightarrow \frac{dy}{dz} = z\left(\frac{1}{|z|\sqrt{z^2-1}}\right) + (\sec^{-1} z)(1) - \frac{1}{2}(z^2-1)^{-1/2}(2z)$
$= \frac{z}{|z|\sqrt{z^2-1}} - \frac{z}{\sqrt{z^2-1}} + \sec^{-1} z = \frac{1-z}{\sqrt{z^2-1}} + \sec^{-1} z,\ z > 1$

41. $y = \csc^{-1}(\sec\theta) \Rightarrow \frac{dy}{d\theta} = \frac{-\sec\theta\tan\theta}{|\sec\theta|\sqrt{\sec^2\theta - 1}} = -\frac{\tan\theta}{|\tan\theta|} = -1,\ 0 < \theta < \frac{\pi}{2}$

42. $y = (1+x^2)e^{\tan^{-1} x} \Rightarrow \frac{dy}{dx} = 2xe^{\tan^{-1} x} + (1+x^2)\left(\frac{e^{\tan^{-1} x}}{1+x^2}\right) = 2xe^{\tan^{-1} x} + e^{\tan^{-1} x}$

43. $xy + 2x + 3y = 1 \Rightarrow \left(x\frac{dy}{dx} + y\right) + 2 + 3\frac{dy}{dx} = 0 \Rightarrow x\frac{dy}{dx} + 3\frac{dy}{dx} = -2 - y \Rightarrow \frac{dy}{dx}(x+3) = -2 - y \Rightarrow \frac{dy}{dx} = -\frac{y+2}{x+3}$

44. $x^2 + xy + y^2 - 5x = 2 \Rightarrow 2x + \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} - 5 = 0 \Rightarrow x\frac{dy}{dx} + 2y\frac{dy}{dx} = 5 - 2x - y \Rightarrow \frac{dy}{dx}(x+2y)$
$= 5 - 2x - y \Rightarrow \frac{dy}{dx} = \frac{5-2x-y}{x+2y}$

45. $x^3 + 4xy - 3y^{4/3} = 2x \Rightarrow 3x^2 + \left(4x\frac{dy}{dx} + 4y\right) - 4y^{1/3}\frac{dy}{dx} = 2 \Rightarrow 4x\frac{dy}{dx} - 4y^{1/3}\frac{dy}{dx} = 2 - 3x^2 - 4y$
$\Rightarrow \frac{dy}{dx}\left(4x - 4y^{1/3}\right) = 2 - 3x^2 - 4y \Rightarrow \frac{dy}{dx} = \frac{2-3x^2-4y}{4x-4y^{1/3}}$

46. $5x^{4/5} + 10y^{6/5} = 15 \Rightarrow 4x^{-1/5} + 12y^{1/5}\frac{dy}{dx} = 0 \Rightarrow 12y^{1/5}\frac{dy}{dx} = -4x^{-1/5} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}x^{-1/5}y^{-1/5} = -\frac{1}{3(xy)^{1/5}}$

47. $(xy)^{1/2} = 1 \Rightarrow \frac{1}{2}(xy)^{-1/2}\left(x\frac{dy}{dx} + y\right) = 0 \Rightarrow x^{1/2}y^{-1/2}\frac{dy}{dx} = -x^{-1/2}y^{1/2} \Rightarrow \frac{dy}{dx} = -x^{-1}y \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

48. $x^2y^2 = 1 \Rightarrow x^2\left(2y\frac{dy}{dx}\right) + y^2(2x) = 0 \Rightarrow 2x^2y\frac{dy}{dx} = -2xy^2 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

49. $e^{x+2y} = 1 \Rightarrow e^{x+2y}\left(1 + 2\frac{dy}{dx}\right) = 0 \Rightarrow \frac{dy}{dx} = -\frac{1}{2}$

50. $y^2 = 2e^{-1/x} \Rightarrow 2y\frac{dy}{dx} = 2e^{-1/x}\frac{d}{dx}\left(-x^{-1}\right) = \frac{2e^{-1/x}}{x^2} \Rightarrow \frac{dy}{dx} = \frac{e^{-1/x}}{yx^2}$

51. $\ln\left(\frac{x}{y}\right) = 1 \Rightarrow \frac{1}{x/y}\frac{d}{dx}\left(\frac{x}{y}\right) = 0 \Rightarrow \frac{y(1) - x\frac{dy}{dx}}{y^2} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x}$

52. $x\sin^{-1}y = 1 + x^2 \Rightarrow y = \sin\left(x^{-1} + x\right) \Rightarrow \frac{dy}{dx} = \cos\left(x^{-1} + x\right)\frac{d}{dx}\left(x^{-1} + x\right) = \left(1 - x^{-2}\right)\cos\left(x^{-1} + x\right)$
$= \left(\frac{x^2 - 1}{x^2}\right)\cos\left(\frac{x^2 + 1}{x}\right)$

53. $ye^{\tan^{-1}x} = 2 \Rightarrow y = 2e^{-\tan^{-1}x} \Rightarrow \frac{dy}{dx} = 2e^{-\tan^{-1}x}\frac{d}{dx}\left(-\tan^{-1}x\right) = -2e^{-\tan^{-1}x}\left(\frac{1}{1+x^2}\right) = -\frac{2e^{-\tan^{-1}x}}{1+x^2}$

54. $x^y = \sqrt{2} \Rightarrow \ln\left(x^y\right) = \ln\left(2^{1/2}\right) \Rightarrow y\ln x = \frac{\ln 2}{2} \Rightarrow \frac{d}{dx}(y\ln x) = 0 \Rightarrow y\left(\frac{1}{x}\right) + \ln x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x\ln x}$
$= -\frac{\ln 2}{2x(\ln x)^2}$

55. $r\cos 2s + \sin^2 s = \pi \Rightarrow r(-\sin 2s)(2) + (\cos 2s)\left(\frac{dr}{ds}\right) + 2\sin s\cos s = 0 \Rightarrow \frac{dr}{ds}(\cos 2s) = 2r\sin 2s - 2\sin s\cos s$
$\Rightarrow \frac{dr}{ds} = \frac{2r\sin 2s - \sin 2s}{\cos 2s} = \frac{(2r-1)(\sin 2s)}{\cos 2s} = (2r-1)(\tan 2s)$

56. $2rs - r - s + s^2 = -3 \Rightarrow 2\left(r + s\frac{dr}{ds}\right) - \frac{dr}{ds} - 1 + 2s = 0 \Rightarrow \frac{dr}{ds}(2s - 1) = 1 - 2s - 2r \Rightarrow \frac{dr}{ds} = \frac{1 - 2s - 2r}{2s - 1}$

57. (a) $x^3 + y^3 = 1 \Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^2}{y^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{y^2(-2x) - \left(-x^2\right)\left(2y\frac{dy}{dx}\right)}{y^4}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2xy^2 + \left(2yx^2\right)\left(-\frac{x^2}{y^2}\right)}{y^4} = \frac{-2xy^2 - \frac{2x^4}{y}}{y^4} = \frac{-2xy^3 - 2x^4}{y^5}$

(b) $y^2 = 1 - \frac{2}{x} \Rightarrow 2y\frac{dy}{dx} = \frac{2}{x^2} \Rightarrow \frac{dy}{dx} = \frac{1}{yx^2} \Rightarrow \frac{dy}{dx} = \left(yx^2\right)^{-1} \Rightarrow \frac{d^2y}{dx^2} = -\left(yx^2\right)^{-2}\left[y(2x) + x^2\frac{dy}{dx}\right]$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2xy - x^2\left(\frac{1}{yx^2}\right)}{y^2x^4} = \frac{-2xy^2 - 1}{y^3x^4}$

58. (a) $x^2 - y^2 = 1 \Rightarrow 2x - 2y\frac{dy}{dx} = 0 \Rightarrow -2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

(b) $\frac{dy}{dx} = \frac{x}{y} \Rightarrow \frac{d^2y}{dx^2} = \frac{y(1) - x\frac{dy}{dx}}{y^2} = \frac{y - x\left(\frac{x}{y}\right)}{y^2} = \frac{y^2 - x^2}{y^3} = \frac{-1}{y^3}$ (since $y^2 - x^2 = -1$)

59. (a) Let $h(x) = 6f(x) - g(x) \Rightarrow h'(x) = 6f'(x) - g'(x) \Rightarrow h'(1) = 6f'(1) - g'(1) = 6\left(\frac{1}{2}\right) - (-4) = 7$

(b) Let $h(x) = f(x)g^2(x) \Rightarrow h'(x) = f(x)\left(2g(x)\right)g'(x) + g^2(x)f'(x) \Rightarrow h'(0) = 2f(0)g(0)g'(0) + g^2(0)f'(0)$
$= 2(1)(1)\left(\frac{1}{2}\right) + (1)^2(-3) = -2$

(c) Let $h(x) = \frac{f(x)}{g(x)+1} \Rightarrow h'(x) = \frac{(g(x)+1)f'(x) - f(x)g'(x)}{(g(x)+1)^2} \Rightarrow h'(1) = \frac{(g(1)+1)f'(1) - f(1)g'(1)}{(g(1)+1)^2}$
$= \frac{(5+1)\left(\frac{1}{2}\right) - 3(-4)}{(5+1)^2} = \frac{5}{12}$

(d) Let $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x))g'(x) \Rightarrow h'(0) = f'(g(0))g'(0) = f'(1)\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

(e) Let $h(x) = g(f(x)) \Rightarrow h'(x) = g'(f(x))f'(x) \Rightarrow h'(0) = g'(f(0))f'(0) = g'(1)f'(0) = (-4)(-3) = 12$

(f) Let $h(x) = (x + f(x))^{3/2} \Rightarrow h'(x) = \frac{3}{2}(x + f(x))^{1/2}\left(1 + f'(x)\right) \Rightarrow h'(1) = \frac{3}{2}(1 + f(1))^{1/2}\left(1 + f'(1)\right)$
$= \frac{3}{2}(1+3)^{1/2}\left(1 + \frac{1}{2}\right) = \frac{9}{2}$

(g) Let $h(x) = f(x + g(x)) \Rightarrow h'(x) = f'(x + g(x))\left(1 + g'(x)\right) \Rightarrow h'(0) = f'(g(0))\left(1 + g'(0)\right)$
$= f'(1)\left(1 + \frac{1}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = \frac{3}{4}$

60. (a) Let $h(x) = \sqrt{x}\,f(x) \Rightarrow h'(x) = \sqrt{x}\,f'(x) + f(x)\cdot\frac{1}{2\sqrt{x}} \Rightarrow h'(1) = \sqrt{1}\,f'(1) + f(1)\cdot\frac{1}{2\sqrt{1}} = \frac{1}{5} + (-3)\left(\frac{1}{2}\right) = -\frac{13}{10}$

(b) Let $h(x) = (f(x))^{1/2} \Rightarrow h'(x) = \frac{1}{2}(f(x))^{-1/2}\left(f'(x)\right) \Rightarrow h'(0) = \frac{1}{2}(f(0))^{-1/2}f'(0) = \frac{1}{2}(9)^{-1/2}(-2) = -\frac{1}{3}$

(c) Let $h(x) = f\left(\sqrt{x}\right) \Rightarrow h'(x) = f'\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}} \Rightarrow h'(1) = f'\left(\sqrt{1}\right)\cdot\frac{1}{2\sqrt{1}} = \frac{1}{5}\cdot\frac{1}{2} = \frac{1}{10}$

(d) Let $h(x) = f(1 - 5\tan x) \Rightarrow h'(x) = f'(1 - 5\tan x)\left(-5\sec^2 x\right) \Rightarrow h'(0) = f'(1 - 5\tan 0)\left(-5\sec^2 0\right)$
$= f'(1)(-5) = \frac{1}{5}(-5) = -1$

(e) Let $h(x) = \frac{f(x)}{2 + \cos x} \Rightarrow h'(x) = \frac{(2 + \cos x)f'(x) - f(x)(-\sin x)}{(2 + \cos x)^2} \Rightarrow h'(0) = \frac{(2+1)f'(0) - f(0)(0)}{(2+1)^2} = \frac{3(-2)}{9} = -\frac{2}{3}$

(f) Let $h(x) = 10\sin\left(\frac{\pi x}{2}\right)f^2(x) \Rightarrow h'(x) = 10\sin\left(\frac{\pi x}{2}\right)\left(2f(x)f'(x)\right) + f^2(x)\left(10\cos\left(\frac{\pi x}{2}\right)\right)\left(\frac{\pi}{2}\right)$
$\Rightarrow h'(1) = 10\sin\left(\frac{\pi}{2}\right)\left(2f(1)f'(1)\right) + f^2(1)\left(10\cos\left(\frac{\pi}{2}\right)\right)\left(\frac{\pi}{2}\right) = 20(-3)\left(\frac{1}{5}\right) + 0 = -12$

61. $x = t^2 + \pi \Rightarrow \frac{dx}{dt} = 2t$; $y = 3\sin 2x \Rightarrow \frac{dy}{dx} = 3(\cos 2x)(2) = 6\cos 2x = 6\cos\left(2t^2 + 2\pi\right) = 6\cos\left(2t^2\right)$; thus,
$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} = 6\cos\left(2t^2\right)\cdot 2t \Rightarrow \left.\frac{dy}{dt}\right|_{t=0} = 6\cos(0)\cdot 0 = 0$

62. $t = \left(u^2 + 2u\right)^{1/3} \Rightarrow \frac{dt}{du} = \frac{1}{3}\left(u^2 + 2u\right)^{-2/3}(2u + 2) = \frac{2}{3}\left(u^2 + 2u\right)^{-2/3}(u + 1)$; $s = t^2 + 5t \Rightarrow \frac{ds}{dt} = 2t + 5$
$= 2\left(u^2 + 2u\right)^{1/3} + 5$; thus $\frac{ds}{du} = \frac{ds}{dt}\cdot\frac{dt}{du} = \left[2\left(u^2 + 2u\right)^{1/3} + 5\right]\left(\frac{2}{3}\right)\left(u^2 + 2u\right)^{-2/3}(u + 1)$

$$\Rightarrow \left.\frac{ds}{du}\right|_{u=2} = \left[2\left(2^2+2(2)\right)^{1/3}+5\right]\left(\frac{2}{3}\right)\left(2^2+2(2)\right)^{-2/3}(2+1) = 2\left(2\cdot 8^{1/3}+5\right)\left(8^{-2/3}\right) = 2(2\cdot 2+5)\left(\frac{1}{4}\right) = \frac{9}{2}$$

63. $\frac{dw}{ds} = \frac{dw}{dr}\frac{dr}{ds} = \left[\cos\left(e^{\sqrt{r}}\right)\left(e^{\sqrt{r}}\,\frac{1}{2\sqrt{r}}\right)\right]\left[3\cos\left(s+\frac{\pi}{6}\right)\right]$ at $s=0$, $r = 3\sin\frac{\pi}{6} = \frac{3}{2}$

$$\Rightarrow \frac{dw}{ds} = \cos\left(e^{\sqrt{3/2}}\right)\left(\frac{e^{\sqrt{3/2}}}{2\sqrt{3/2}}\right)\left(3\cos\left(\frac{\pi}{6}\right)\right) = \frac{3\sqrt{3}\,e^{\sqrt{3/2}}}{4\sqrt{3/2}}\cos\left(e^{\sqrt{3/2}}\right) = \frac{3\sqrt{2}\,e^{\sqrt{3/2}}}{4}\cos\left(e^{\sqrt{3/2}}\right)$$

64. $\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}$; $\frac{dr}{d\theta} = \frac{1}{3}\left(\theta^2+7\right)^{-2/3}(2\theta)$; $\theta^2 e^t + \theta = 1 \Rightarrow \frac{d}{dt}\left(\theta^2 e^t + \theta\right) = \frac{d}{dt}(1) \Rightarrow \theta^2 e^t + 2\theta\,\frac{d\theta}{dt}e^t + \frac{d\theta}{dt} = 0$

$$\Rightarrow \left(1+2\theta e^t\right)\frac{d\theta}{dt} = -\theta^2 e^t \Rightarrow \frac{d\theta}{dt} = -\frac{\theta^2 e^t}{1+2\theta e^t} \Rightarrow \frac{dr}{dt} = \left[\frac{2\theta}{3\left(\theta^2+7\right)^{2/3}}\right]\left[-\frac{\theta^2 e^t}{1+2\theta e^t}\right] = -\frac{2\theta^3 e^t}{3\left(1+2\theta e^t\right)\left(\theta^2+7\right)^{2/3}}.$$

At $t=0$, $\theta^2+\theta-1=0 \Rightarrow \theta = \frac{-1\pm\sqrt{5}}{2} \Rightarrow \frac{dr}{dt} = -\frac{2\left(\frac{-1\pm\sqrt{5}}{2}\right)^3}{\left(3\left(1+(-1\pm\sqrt{5})\right)\left(\left(\frac{-1\pm\sqrt{5}}{2}\right)\right)^2+7\right)^{2/3}}$

65. $y^3+y = 2\cos x \Rightarrow 3y^2\frac{dy}{dx}+\frac{dy}{dx} = -2\sin x \Rightarrow \frac{dy}{dx}\left(3y^2+1\right) = -2\sin x \Rightarrow \frac{dy}{dx} = \frac{-2\sin x}{3y^2+1} \Rightarrow \left.\frac{dy}{dx}\right|_{(0,1)}$

$$= \frac{-2\sin(0)}{3+1} = 0;\ \frac{d^2y}{dx^2} = \frac{\left(3y^2+1\right)(-2\cos x) - (-2\sin x)\left(6y\,\frac{dy}{dx}\right)}{\left(3y^2+1\right)^2}$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(0,1)} = \frac{(3+1)(-2\cos 0) - (-2\sin 0)(6\cdot 0)}{(3+1)^2} = -\frac{1}{2}$$

66. $x^{1/3}+y^{1/3} = 4 \Rightarrow \frac{1}{3}x^{-2/3}+\frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y^{2/3}}{x^{2/3}} \Rightarrow \left.\frac{dy}{dx}\right|_{(8,8)} = -1$; $\frac{dy}{dx} = \frac{-y^{2/3}}{x^{2/3}}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\left(x^{2/3}\right)\left(-\frac{2}{3}y^{-1/3}\frac{dy}{dx}\right) - \left(-y^{2/3}\right)\left(\frac{2}{3}x^{-1/3}\right)}{\left(x^{2/3}\right)^2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(8,8)} = \frac{\left(8^{2/3}\right)\left[-\frac{2}{3}\cdot 8^{-1/3}\cdot(-1)\right] + \left(8^{2/3}\right)\left(\frac{2}{3}\cdot 8^{-1/3}\right)}{8^{4/3}}$$

$$= \frac{\frac{1}{3}+\frac{1}{3}}{8^{2/3}} = \frac{\frac{2}{3}}{4} = \frac{1}{6}$$

67. $f(t) = \frac{1}{2t+1}$ and $f(t+h) = \frac{1}{2(t+h)+1} \Rightarrow \frac{f(t+h)-f(t)}{h} = \frac{\frac{1}{2(t+h)+1} - \frac{1}{2t+1}}{h} = \frac{2t+1-(2t+2h+1)}{(2t+2h+1)(2t+1)h}$

$$= \frac{-2h}{(2t+2h+1)(2t+1)h} = \frac{-2}{(2t+2h+1)(2t+1)} \Rightarrow f'(t) = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h} = \lim_{h\to 0}\frac{-2}{(2t+2h+1)(2t+1)}$$

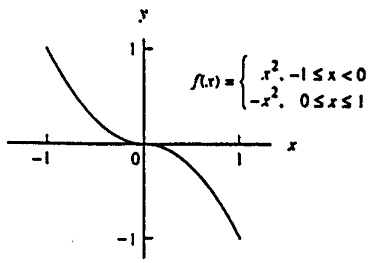
$$= \frac{-2}{(2t+1)^2}$$

68. $g(x) = 2x^2 + 1$ and $g(x+h) = 2(x+h)^2 + 1 = 2x^2 + 4xh + 2h^2 + 1 \Rightarrow \frac{g(x+h) - g(x)}{h}$

$= \frac{(2x^2 + 4xh + 2h^2 + 1) - (2x^2 + 1)}{h} = \frac{4xh + 2h^2}{h} = 4x + 2h \Rightarrow g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} (4x + 2h)$

$= 4x$

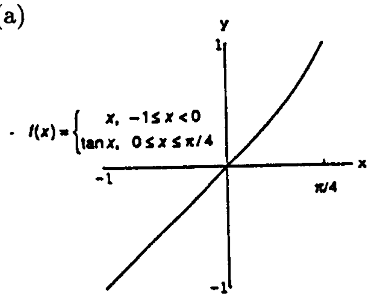
69. (a)



(b) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} -x^2 = 0 \Rightarrow \lim_{x \to 0} f(x) = 0$. Since $\lim_{x \to 0} f(x) = 0 = f(0)$ it follows that f is continuous at $x = 0$.

(c) $\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} (2x) = 0$ and $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} (-2x) = 0 \Rightarrow \lim_{x \to 0} f'(x) = 0$. Since this limit exists, it follows that f is differentiable at $x = 0$.

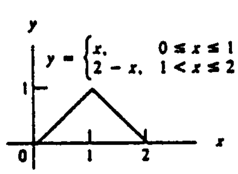
70. (a)



(b) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x = 0$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \tan x = 0 \Rightarrow \lim_{x \to 0} f(x) = 0$. Since $\lim_{x \to 0} f(x) = 0 = f(0)$, it follows that f is continuous at $x = 0$.

(c) $\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} 1 = 1$ and $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} \sec^2 x = 1 \Rightarrow \lim_{x \to 0} f'(x) = 1$. Since this limit exists it follows that f is differentiable at $x = 0$.

71. (a)



(b) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$ and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2 - x) = 1 \Rightarrow \lim_{x \to 1} f(x) = 1$. Since $\lim_{x \to 1} f(x) = 1 = f(1)$, it follows that f is continuous at $x = 1$.

(c) $\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} 1 = 1$ and $\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} -1 = -1 \Rightarrow \lim_{x\to 1^-} f'(x) \neq \lim_{x\to 1^+} f'(x)$, so $\lim_{x\to 1} f'(x)$ does not exist $\Rightarrow$ f is not differentiable at $x = 1$.

72. (a) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sin 2x = 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} mx = 0 \Rightarrow \lim_{x\to 0} f(x) = 0$, independent of m; since $f(0) = 0 = \lim_{x\to 0} f(x)$ it follows that f is continuous at $x = 0$ for all values of m.

(b) $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} (\sin 2x)' = \lim_{x\to 0^-} 2\cos 2x = 2$ and $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} (mx)' = \lim_{x\to 0^+} m = m \Rightarrow$ f is differentiable at $x = 0$ provided that $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) \Rightarrow m = 2$.

73. $y = \frac{x}{2} + \frac{1}{2x-4} = \frac{1}{2}x + (2x-4)^{-1} \Rightarrow \frac{dy}{dx} = \frac{1}{2} - 2(2x-4)^{-2}$; the slope of the tangent is $-\frac{3}{2} \Rightarrow -\frac{3}{2}$ $= \frac{1}{2} - 2(2x-4)^{-2} \Rightarrow -2 = -2(2x-4)^{-2} \Rightarrow 1 = \frac{1}{(2x-4)^2} \Rightarrow (2x-4)^2 = 1 \Rightarrow 4x^2 - 16x + 16 = 1$ $\Rightarrow 4x^2 - 16x + 15 = 0 \Rightarrow (2x-5)(2x-3) = 0 \Rightarrow x = \frac{5}{2}$ or $x = \frac{3}{2} \Rightarrow \left(\frac{5}{2}, \frac{9}{4}\right)$ and $\left(\frac{3}{2}, -\frac{1}{4}\right)$ are points on the curve where the slope is $-\frac{3}{2}$.

74. $y = x - e^{-x}$; $\frac{dy}{dx} = 1 + e^{-x} = 2 \Rightarrow e^{-x} = 1 \Rightarrow x = 0 \Rightarrow y = 0 - e^0 = -1$. Therefore, the curve has a tangent with a slope of 2 at the point $(0, -1)$.

75. $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2 \Rightarrow \left.\frac{dy}{dx}\right|_{(-2,-8)} = 12$; an equation of the tangent line at $(-2, -8)$ is $y + 8 = 12(x+2)$ $\Rightarrow y = 12x + 16$; x-intercept: $0 = 12x + 16 \Rightarrow x = -\frac{4}{3} \Rightarrow \left(-\frac{4}{3}, 0\right)$; y-intercept: $y = 12(0) + 16 = 16 \Rightarrow (0, 16)$

76. $y = 2x^3 - 3x^2 - 12x + 20 \Rightarrow \frac{dy}{dx} = 6x^2 - 6x - 12$

(a) The tangent is perpendicular to the line $y = 1 - \frac{x}{24}$ when $\frac{dy}{dx} = -\left(\frac{1}{-\left(\frac{1}{24}\right)}\right) = 24$; $6x^2 - 6x - 12 = 24$ $\Rightarrow x^2 - x - 2 = 4 \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x-3)(x+2) = 0 \Rightarrow x = -2$ or $x = 3 \Rightarrow (-2, 16)$ and $(3, 11)$ are points where the tangent is perpendicular to $y = 1 - \frac{x}{24}$.

(b) The tangent is parallel to the line $y = \sqrt{2} - 12x$ when $\frac{dy}{dx} = -12 \Rightarrow 6x^2 - 6x - 12 = -12 \Rightarrow x^2 - x = 0$ $\Rightarrow x(x-1) = 0 \Rightarrow x = 0$ or $x = 1 \Rightarrow (0, 20)$ and $(1, 7)$ are points where the tangent is parallel to $y = \sqrt{2} - 12x$.

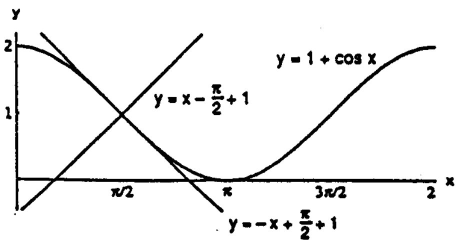
77. $y = \frac{\pi \sin x}{x} \Rightarrow \frac{dy}{dx} = \frac{x(\pi \cos x) - (\pi \sin x)(1)}{x^2} \Rightarrow m_1 = \left.\frac{dy}{dx}\right|_{x=\pi} = \frac{-\pi^2}{\pi^2} = -1$ and $m_2 = \left.\frac{dy}{dx}\right|_{x=-\pi} \frac{\pi^2}{\pi^2} = 1$. Since $m_1 = -\frac{1}{m_2}$ the tangents intersect at right angles.

78. $y = 1 + \cos x \Rightarrow \frac{dy}{dx} = -\sin x \Rightarrow \left.\frac{dy}{dx}\right|_{\left(\frac{\pi}{2}, 1\right)} = -1$

$\Rightarrow$ the tangent at $\left(\frac{\pi}{2}, 1\right)$ is the line $y - 1 = -\left(x - \frac{\pi}{2}\right)$

$\Rightarrow y = -x + \frac{\pi}{2} + 1$; the normal at $\left(\frac{\pi}{2}, 1\right)$ is

$y - 1 = (1)\left(x - \frac{\pi}{2}\right) \Rightarrow y = x - \frac{\pi}{2} + 1$



79. $y = x^2 + C \Rightarrow \frac{dy}{dx} = 2x$ and $y = x \Rightarrow \frac{dy}{dx} = 1$; the parabola is tangent to $y = x$ when $2x = 1 \Rightarrow x = \frac{1}{2} \Rightarrow y = \frac{1}{2}$;

thus, $\frac{1}{2} = \left(\frac{1}{2}\right)^2 + C \Rightarrow C = \frac{1}{4}$

80. Let $\left(b, \pm\sqrt{a^2 - b^2}\right)$ be a point on the circle $x^2 + y^2 = a^2$. Then $x^2 + y^2 = a^2 \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=b} = \frac{-b}{\pm\sqrt{a^2 - b^2}} \Rightarrow$ normal line through $\left(b, \pm\sqrt{a^2 - b^2}\right)$ has slope $\frac{\mp\sqrt{a^2 - b^2}}{b} \Rightarrow$ normal line is

$y - \left(\mp\sqrt{a^2 - b^2}\right) = \frac{\mp\sqrt{a^2 - b^2}}{b}(x - b) \Rightarrow y \pm \sqrt{a^2 - b^2} = \frac{\mp\sqrt{a^2 - b^2}}{b}x \pm \sqrt{a^2 - b^2} \Rightarrow y = \mp\frac{\sqrt{a^2 - b^2}}{b}x$

which passes through the origin.

81. $x^2 + 2y^2 = 9 \Rightarrow 2x + 4y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{2y} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,2)} = -\frac{1}{4} \Rightarrow$ the tangent line is $y = 2 - \frac{1}{4}(x - 1)$

$= -\frac{1}{4}x + \frac{9}{4}$ and the normal line is $y = 2 + 4(x - 1) = 4x - 2$.

82. $e^x + y^2 = 2 \Rightarrow \frac{d}{dx}(e^x + y^2) = \frac{d}{dx}(2) \Rightarrow e^x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{e^x}{2y} \Rightarrow m_{tan} = \left.\frac{dy}{dx}\right|_{(0,1)} = -\frac{e^0}{2(1)} = -\frac{1}{2}$;

$m_{\perp} = -\frac{1}{m_{tan}} = 2$; tangent line: $y - 1 = -\frac{1}{2}(x - 0) \Rightarrow y = 1 - \frac{x}{2}$; normal line: $y - 1 = 2(x - 0) \Rightarrow y = 2x + 1$

83. $xy + 2x - 5y = 2 \Rightarrow \left(x\frac{dy}{dx} + y\right) + 2 - 5\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx}(x - 5) = -y - 2 \Rightarrow \frac{dy}{dx} = \frac{-y-2}{x-5} \Rightarrow \left.\frac{dy}{dx}\right|_{(3,2)} = 2$

$\Rightarrow$ the tangent line is $y = 2 + 2(x - 3) = 2x - 4$ and the normal line is $y = 2 + \frac{-1}{2}(x - 3) = -\frac{1}{2}x + \frac{7}{2}$.

84. $(y - x)^2 = 2x + 4 \Rightarrow 2(y - x)\left(\frac{dy}{dx} - 1\right) = 2 \Rightarrow (y - x)\frac{dy}{dx} = 1 + (y - x) \Rightarrow \frac{dy}{dx} = \frac{1 + y - x}{y - x} \Rightarrow \left.\frac{dy}{dx}\right|_{(6,2)} = \frac{3}{4}$

$\Rightarrow$ the tangent line is $y = 2 + \frac{3}{4}(x - 6) = \frac{3}{4}x - \frac{5}{2}$ and the normal line is $y = 2 - \frac{4}{3}(x - 6) = -\frac{4}{3}x + 10$.

85. $x + \sqrt{xy} = 6 \Rightarrow 1 + \frac{1}{2\sqrt{xy}}\left(x\frac{dy}{dx} + y\right) = 0 \Rightarrow x\frac{dy}{dx} + y = -2\sqrt{xy} \Rightarrow \frac{dy}{dx} = \frac{-2\sqrt{xy} - y}{x} \Rightarrow \left.\frac{dy}{dx}\right|_{(4,1)} = \frac{-5}{4}$

$\Rightarrow$ the tangent line is $y = 1 - \frac{5}{4}(x - 4) = -\frac{5}{4}x + 6$ and the normal line is $y = 1 + \frac{4}{5}(x - 4) = \frac{4}{5}x - \frac{11}{5}$.

86. $x^{3/2} + 2y^{3/2} = 17 \Rightarrow \frac{3}{2}x^{1/2} + 3y^{1/2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x^{1/2}}{2y^{1/2}} \Rightarrow \frac{dy}{dx}\Big|_{(1,4)} = -\frac{1}{4} \Rightarrow$ the tangent line is $y = 4 - \frac{1}{4}(x-1) = -\frac{1}{4}x + \frac{17}{4}$ and the normal line is $y = 4 + 4(x-1) = 4x$.

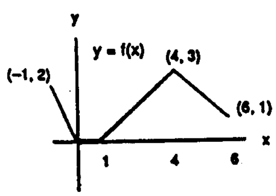
87. $x^3y^3 + y^2 = x + y \Rightarrow \left[x^3\left(3y^2\frac{dy}{dx}\right) + y^3(3x^2)\right] + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx} \Rightarrow 3x^3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 1 - 3x^2y^3$
$\Rightarrow \frac{dy}{dx}(3x^3y^2 + 2y - 1) = 1 - 3x^2y^3 \Rightarrow \frac{dy}{dx} = \frac{1-3x^2y^3}{3x^3y^2+2y-1} \Rightarrow \frac{dy}{dx}\Big|_{(1,1)} = -\frac{2}{4}$, but $\frac{dy}{dx}\Big|_{(1,-1)}$ is undefined.
Therefore, the curve has slope $-\frac{1}{2}$ at $(1,1)$ but the slope is undefined at $(1,-1)$.

88. $y = \sin(x - \sin x) \Rightarrow \frac{dy}{dx} = [\cos(x - \sin x)](1 - \cos x)$; $y = 0 \Rightarrow \sin(x - \sin x) = 0 \Rightarrow x - \sin x = k\pi$, $k = -2, -1, 0, 1, 2$ (for our interval) $\Rightarrow \cos(x - \sin x) = \cos(k\pi) = \pm 1$. Therefore, $\frac{dy}{dx} = 0$ and $y = 0$ when $1 - \cos x = 0$ and $x = k\pi$. For $-2\pi \le x \le 2\pi$, these equations hold when $k = -2, 0,$ and 2 (since $\cos(-\pi) = \cos\pi = -1$). Thus the curve has horizontal tangents at the x-axis for the x-values -2π, 0, and 2π (which are even integer multiples of π) $\Rightarrow$ the curve has an infinite number of horizontal tangents.

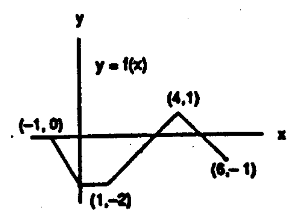
89. B = graph of f, A = graph of f′. Curve B cannot be the derivative of A because A has only negative slopes while some of B's values are positive.

90. A = graph of f, B = graph of f′. Curve A cannot be the derivative of B because B has only negative slopes while A has positive values for $x > 0$.

91.



92.



93. (a) 0, 0

(b) largest 1700, smallest about 1400

94. rabbits/day and foxes/day

95. (a) $S = 2\pi r^2 + 2\pi rh$ and h constant $\Rightarrow \frac{dS}{dt} = 4\pi r\frac{dr}{dt} + 2\pi h\frac{dr}{dt} = (4\pi r + 2\pi h)\frac{dr}{dt}$

(b) $S = 2\pi r^2 + 2\pi rh$ and r constant $\Rightarrow \frac{dS}{dt} = 2\pi r\frac{dh}{dt}$

(c) $S = 2\pi r^2 + 2\pi rh \Rightarrow \frac{dS}{dt} = 4\pi r\frac{dr}{dt} + 2\pi\left(r\frac{dh}{dt} + h\frac{dr}{dt}\right) = (4\pi r + 2\pi h)\frac{dr}{dt} + 2\pi r\frac{dh}{dt}$

(d) S constant $\Rightarrow \frac{dS}{dt} = 0 \Rightarrow 0 = (4\pi r + 2\pi h)\frac{dr}{dt} + 2\pi r\frac{dh}{dt} \Rightarrow (2r + h)\frac{dr}{dt} = -r\frac{dh}{dt} \Rightarrow \frac{dr}{dt} = \frac{-r}{2r+h}\frac{dh}{dt}$

96. $S = \pi r\sqrt{r^2+h^2} \Rightarrow \frac{dS}{dt} = \pi r \cdot \frac{\left(r\frac{dr}{dt} + h\frac{dh}{dt}\right)}{\sqrt{r^2+h^2}} + \pi\sqrt{r^2+h^2}\,\frac{dr}{dt}$;

(a) h constant $\Rightarrow \frac{dh}{dt} = 0 \Rightarrow \frac{dS}{dt} = \frac{\pi r^2 \frac{dr}{dt}}{\sqrt{r^2+h^2}} + \pi\sqrt{r^2+h^2}\,\frac{dr}{dt} = \left[\pi\sqrt{r^2+h^2} + \frac{\pi r^2}{\sqrt{r^2+h^2}}\right]\frac{dr}{dt}$

(b) r constant $\Rightarrow \frac{dr}{dt} = 0 \Rightarrow \frac{dS}{dt} = \frac{\pi rh}{\sqrt{r^2+h^2}}\,\frac{dh}{dt}$

(c) In general, $\frac{dS}{dt} = \left[\pi\sqrt{r^2+h^2} + \frac{\pi r^2}{\sqrt{r^2+h^2}}\right]\frac{dr}{dt} + \frac{\pi rh}{\sqrt{r^2+h^2}}\,\frac{dh}{dt}$

97. $A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r\,\frac{dr}{dt}$; so $r = 10$ and $\frac{dr}{dt} = -\frac{2}{\pi}$ m/sec $\Rightarrow \frac{dA}{dt} = (2\pi)(10)\left(-\frac{2}{\pi}\right) = -40\ m^2/sec$

98. $V = s^3 \Rightarrow \frac{dV}{dt} = 3s^2 \cdot \frac{ds}{dt} \Rightarrow \frac{ds}{dt} = \frac{1}{3s^2}\,\frac{dV}{dt}$; so $s = 20$ and $\frac{dV}{dt} = 1200\ cm^3/min \Rightarrow \frac{ds}{dt} = \frac{1}{3(20)^2}(1200) = 1$ cm/min

99. $\frac{dR_1}{dt} = -1$ ohm/sec, $\frac{dR_2}{dt} = 0.5$ ohm/sec; and $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow \frac{-1}{R^2}\,\frac{dR}{dt} = \frac{-1}{R_1^2}\,\frac{dR_1}{dt} - \frac{1}{R_2^2}\,\frac{dR_2}{dt}$. Also, $R_1 = 75$ ohms and $R_2 = 50$ ohms $\Rightarrow \frac{1}{R} = \frac{1}{75} + \frac{1}{50} \Rightarrow R = 30$ ohms. Therefore, from the derivative equation, $\frac{-1}{(30)^2}\,\frac{dR}{dt} = \frac{-1}{(75)^2}(-1) - \frac{1}{(50)^2}(0.5) = \left(\frac{1}{5625} - \frac{1}{5000}\right) \Rightarrow \frac{dR}{dt} = (-900)\left(\frac{5000-5625}{5625 \cdot 5000}\right) = \frac{9(625)}{50(5625)} = \frac{1}{50}$ $= 0.02$ ohm/sec.

100. $\frac{dR}{dt} = 3$ ohms/sec and $\frac{dX}{dt} = -2$ ohms/sec; $Z = \sqrt{R^2+X^2} \Rightarrow \frac{dZ}{dt} = \frac{R\frac{dR}{dt} + X\frac{dX}{dt}}{\sqrt{R^2+X^2}}$ so that $R = 10$ ohms and $X = 20$ ohms $\Rightarrow \frac{dZ}{dt} = \frac{(10)(3)+(20)(-2)}{\sqrt{10^2+20^2}} = \frac{-1}{\sqrt{5}} \approx -0.45$ ohm/sec.

101. Given $\frac{dx}{dt} = 10$ m/sec and $\frac{dy}{dt} = 5$ m/sec, let D be the distance from the origin $\Rightarrow D^2 = x^2 + y^2 \Rightarrow 2D\,\frac{dD}{dt}$ $= 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} \Rightarrow D\,\frac{dD}{dt} = x\,\frac{dx}{dt} + y\,\frac{dy}{dt}$. When $(x,y) = (3,-4)$, $D = \sqrt{3^2+(-4)^2} = 5$ and $5\,\frac{dD}{dt} = (5)(10) + (12)(5) \Rightarrow \frac{dD}{dt} = \frac{110}{5} = 22$. Therefore, the particle is moving away from the origin at 22 m/sec (because the distance D is increasing).

102. Let D be the distance from the origin. We are given that $\frac{dD}{dt} = 11$ units/sec. Then $D^2 = x^2 + y^2$ $= x^2 + \left(x^{3/2}\right)^2 = x^2 + x^3 \Rightarrow 2D\,\frac{dD}{dt} = 2x\,\frac{dx}{dt} + 3x^2\,\frac{dx}{dt} = x(2+3x)\,\frac{dx}{dt}$; $x = 3 \Rightarrow D = \sqrt{3^2+3^3} = 6$ and substitution in the derivative equation gives $(2)(6)(11) = (3)(2+9)\,\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 4$ units/sec.

103. (a) From the diagram we have $\frac{10}{h} = \frac{4}{r} \Rightarrow r = \frac{2}{5}h$.

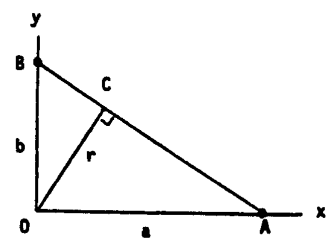
(b) $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{4\pi h^3}{75} \Rightarrow \frac{dV}{dt} = \frac{4\pi h^2}{25}\frac{dh}{dt}$, so $\frac{dV}{dt} = -5$ and $h = 6 \Rightarrow \frac{dh}{dt} = -\frac{125}{144\pi}$ ft/min.

104. From the sketch in the text, $s = r\theta \Rightarrow \frac{ds}{dt} = r\frac{d\theta}{dt} + \theta\frac{dr}{dt}$. Also $r = 1.2$ is constant $\Rightarrow \frac{dr}{dt} = 0$ $\Rightarrow \frac{ds}{dt} = r\frac{d\theta}{dt} = (1.2)\frac{d\theta}{dt}$. Therefore, $\frac{ds}{dt} = 6$ ft/sec and $r = 1.2$ ft $\Rightarrow \frac{d\theta}{dt} = 5$ rad/sec

105. (a) From the sketch in the text, $\frac{d\theta}{dt} = -0.6$ rad/sec and $x = \tan\theta$. Also $x = \tan\theta \Rightarrow \frac{dx}{dt} = \sec^2\theta\,\frac{d\theta}{dt}$; at point A, $x = 0 \Rightarrow \theta = 0 \Rightarrow \frac{dx}{dt} = (\sec^2 0)(-0.6) = -0.6$. Therefore the speed of the light is $0.6 = \frac{3}{5}$ km/sec when it reaches point A.

(b) $\frac{(3/5)\text{ rad}}{\text{sec}} \cdot \frac{1\text{ rev}}{2\pi\text{ rad}} \cdot \frac{60\text{ sec}}{\text{min}} = \frac{18}{\pi}$ revs/min

106. From the figure, $\frac{a}{r} = \frac{b}{BC} \Rightarrow \frac{a}{r} = \frac{b}{\sqrt{b^2 - r^2}}$. We are given that r is constant. Differentiation gives,



$$\frac{1}{r}\cdot\frac{da}{dt} = \frac{\left(\sqrt{b^2 - r^2}\right)\left(\frac{db}{dt}\right) - (b)\left(\frac{b}{\sqrt{b^2 - r^2}}\right)\left(\frac{db}{dt}\right)}{b^2 - r^2}.$$ Then,

$b = 2r$ and $\frac{db}{dt} = -0.3r$

$$\Rightarrow \frac{da}{dt} = r\left[\frac{\sqrt{(2r)^2 - r^2}\,(-0.3r) - (2r)\left(\frac{2r(-0.3r)}{\sqrt{(2r)^2 - r^2}}\right)}{(2r)^2 - r^2}\right]$$

$$= \frac{\sqrt{3r^2}\,(-0.3r) + \frac{4r^2(0.3r)}{\sqrt{3r^2}}}{3r} = \frac{(3r^2)(-0.3r) + (4r^2)(0.3r)}{3\sqrt{3}\,r^2} = \frac{0.3r}{3\sqrt{3}} = \frac{r}{10\sqrt{3}}$$ m/sec. Since $\frac{da}{dt}$ is positive,

the distance OA is increasing when OB = 2r, and B is moving toward 0 at the rate of 0.3r m/sec.

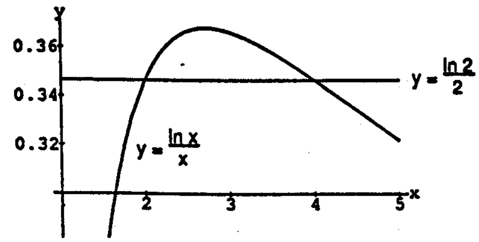
107. $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$; $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \left(\frac{1}{x}\right)\sqrt{x} = \frac{1}{\sqrt{x}} \Rightarrow \left.\frac{dy}{dt}\right|_{e^2} = \frac{1}{e}$ m/sec

108. $y = 9e^{-x/3} \Rightarrow \frac{dy}{dx} = -3e^{-x/3}$; $\frac{dx}{dt} = \frac{(dy/dt)}{(dy/dx)} \Rightarrow \frac{dx}{dt} = \frac{\left(-\frac{1}{4}\right)\sqrt{9-y}}{-3e^{-x/3}}$; $x = 9 \Rightarrow y = 9e^{-3}$

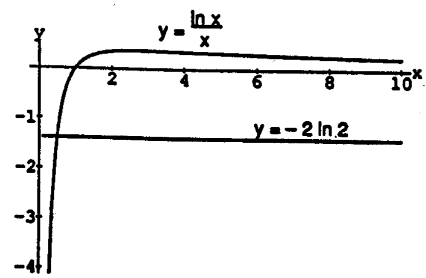
$$\Rightarrow \left.\frac{dx}{dt}\right|_{x=9} = \frac{\left(-\frac{1}{4}\right)\sqrt{9 - \frac{9}{e^3}}}{\left(-\frac{3}{e^3}\right)} = \frac{1}{4}\sqrt{e^3}\sqrt{e^3 - 1} \approx 4.873 \approx 5$$ ft/sec (taking e^3 to be 20)

109. The two functions differ by $\ln \frac{5}{3}$ because $K = \ln(5x) - \ln(3x) = \ln 5 + \ln x - \ln 3 - \ln x = \ln 5 - \ln 3 = \ln \frac{5}{3}$.

110. (a) No, there are two intersections: one at $x = 2$ and the other at $x = 4$.



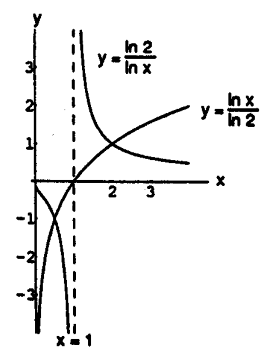
(b) Yes, because there is only one intersection.



111. $\frac{\log_4 x}{\log_2 x} = \frac{\left(\frac{\ln x}{\ln 4}\right)}{\left(\frac{\ln x}{\ln 2}\right)} = \frac{\ln x}{\ln 4} \cdot \frac{\ln 2}{\ln x} = \frac{\ln 2}{\ln 4} = \frac{\ln 2}{2 \ln 2} = \frac{1}{2}$

112. (a) $f(x) = \frac{\ln 2}{\ln x}$, $g(x) = \frac{\ln x}{\ln 2}$

(b) f is negative when g is negative, positive when g is positive, and undefined when $g = 0$; the values of f decrease as those of g increase



113. $y'(r) = \frac{d}{dr}\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi d}}\right) = \left(\frac{1}{2l}\sqrt{\frac{T}{\pi d}}\right)\frac{d}{dr}\left(\frac{1}{r}\right) = -\frac{1}{2r^2 l}\sqrt{\frac{T}{\pi d}}$

$y'(l) = \frac{d}{dl}\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi d}}\right) = \left(\frac{1}{2r}\sqrt{\frac{T}{\pi d}}\right)\frac{d}{dl}\left(\frac{1}{l}\right) = -\frac{1}{2r^2 l}\sqrt{\frac{T}{\pi d}}$

$y'(d) = \frac{d}{dd}\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi d}}\right) = \left(\frac{1}{2rl}\sqrt{\frac{T}{\pi}}\right)\frac{d}{dd}\left(d^{-1/2}\right) = \frac{1}{2rl}\sqrt{\frac{T}{\pi}}\left(-\frac{1}{2}d^{-3/2}\right) = -\frac{1}{4rl}\sqrt{\frac{T}{\pi d^3}}$

$y'(T) = \frac{d}{dT}\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi d}}\right) = \left(\frac{1}{2rl}\frac{1}{\sqrt{\pi d}}\right)\frac{d}{dT}\left(\sqrt{T}\right) = \frac{1}{2rl}\sqrt{\frac{1}{\pi d}}\left(\frac{1}{2\sqrt{T}}\right) = \frac{1}{4rl\sqrt{\pi dT}}$

Since $y'(r) < 0$, $y'(l) < 0$, and $y'(d) < 0$, increasing r, l, or d would decrease the frequency. Since $y'(T) > 0$, increasing T would increase the frequency.

114. (a) $P(0) = \frac{200}{1+e^5} \approx 1$ student

(b) $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{200}{1+e^{5-t}} = \frac{200}{1} = 200$ students

(c) $P'(t) = \frac{d}{dt} 200\left(1+e^{5-t}\right)^{-1} = -200\left(1+e^{5-t}\right)^{-2}\left(e^{5-t}\right)(-1) = \frac{200e^{5-t}}{\left(1+e^{5-t}\right)^2}$

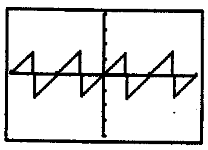
$$P''(t) = \frac{\left(1+e^{5-t}\right)^2\left(200e^{5-t}\right)(-1) - \left(200e^{5-t}\right)(2)\left(1+e^{5-t}\right)\left(e^{5-t}\right)(-1)}{\left(1+e^{5-t}\right)^4}$$

$$= \frac{\left(1+e^{5-t}\right)\left(-200e^{5-t}\right) + 400\left(e^{5-t}\right)^2}{\left(1+e^{5-t}\right)^3} = \frac{\left(200e^{5-t}\right)\left(e^{5-t}-1\right)}{\left(1+e^{5-t}\right)^3}$$

Since $P'' = 0$ when $t = 5$, the critical point of $y = P'(t)$ occurs at $t = 5$. To confirm that this corresponds to the maximum value of $P'(t)$, note that $P''(t) > 0$ for $t < 5$ and $P''(t) < 0$ for $t > 5$. The maximum rate occurs at $t = 5$, and this rate is $P'(5) = \frac{200e^0}{\left(1+e^0\right)^2} = \frac{200}{2^2} = 50$ students per day.

Note: This problem can also be solved graphically.

115.



$[-\pi, \pi]$ by $[-4, 4]$

(a) $x \neq k\frac{\pi}{4}$, where k is an odd integer

(b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) Where it's not defined, at $x = k\frac{\pi}{4}$, k an odd integer

(d) It has period $\frac{\pi}{2}$ and continues to repeat the pattern seen in this window.

116. Use implicit differentiation.

$x^2 - y^2 = 1 \Rightarrow \frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) = \frac{d}{dx}(1) \Rightarrow 2x - 2yy' = 0 \Rightarrow y' = \frac{2x}{2y} = \frac{x}{y} \Rightarrow y'' = \frac{d}{dx}\frac{x}{y}$

$= \frac{(y)(1) - (x)(y')}{y^2} = \frac{y - x\left(\frac{x}{y}\right)}{y^2} = \frac{y^2 - x^2}{y^3} = -\frac{1}{y^3}$ (since the given equation is $x^2 - y^2 = 1$)

At $(2, \sqrt{3})$, $\frac{d^2y}{dx^2} = -\frac{1}{y^3} = -\frac{1}{\left(\sqrt{3}\right)^3} = -\frac{1}{3\sqrt{3}}$

CHAPTER 2 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. (a) $\sin 2\theta = 2\sin\theta\cos\theta \Rightarrow \frac{d}{d\theta}(\sin 2\theta) = \frac{d}{d\theta}(2\sin\theta\cos\theta) \Rightarrow 2\cos 2\theta = 2[(\sin\theta)(-\sin\theta) + (\cos\theta)(\cos\theta)]$
$\Rightarrow \cos 2\theta = \cos^2\theta - \sin^2\theta$
   (b) $\cos 2\theta = \cos^2\theta - \sin^2\theta \Rightarrow \frac{d}{d\theta}(\cos 2\theta) = \frac{d}{d\theta}\left(\cos^2\theta - \sin^2\theta\right) \Rightarrow -2\sin 2\theta = (2\cos\theta)(-\sin\theta) - (2\sin\theta)(\cos\theta)$
$\Rightarrow \sin 2\theta = \cos\theta\sin\theta + \sin\theta\cos\theta \Rightarrow \sin 2\theta = 2\sin\theta\cos\theta$

2. The derivative of $\sin(x + a) = \sin x\cos a + \cos x\sin a$ with respect to x is $\cos(x + a) = \cos x\cos a - \sin x\sin a$, which is also an identity. This principle does not apply to the equation $x^2 - 2x - 8 = 0$, since $x^2 - 2x - 8 = 0$ is not an identity: it holds for 2 values of x (-2 and 4), but not for all x.

3. (a) $f(x) = \cos x \Rightarrow f'(x) = -\sin x \Rightarrow f''(x) = -\cos x$, and $g(x) = a + bx + cx^2 \Rightarrow g'(x) = b + 2cx \Rightarrow g''(x) = 2c$; also, $f(0) = g(0) \Rightarrow \cos(0) = a \Rightarrow a = 1$; $f'(0) = g'(0) \Rightarrow -\sin(0) = b \Rightarrow b = 0$; $f''(0) = g''(0)$
$\Rightarrow -\cos(0) = 2c \Rightarrow c = -\frac{1}{2}$. Therefore, $g(x) = 1 - \frac{1}{2}x^2$.
   (b) $f(x) = \sin(x + a) \Rightarrow f'(x) = \cos(x + a)$, and $g(x) = b\sin x + c\cos x \Rightarrow g'(x) = b\cos x - c\sin x$; also, $f(0) = g(0) \Rightarrow \sin(a) = b\sin(0) + c\cos(0) \Rightarrow c = \sin a$; $f'(0) = g'(0) \Rightarrow \cos(a) = b\cos(0) - c\sin(0)$
$\Rightarrow b = \cos a$. Therefore, $g(x) = \sin x\cos a + \cos x\sin a$.
   (c) When $f(x) = \cos x$, $f'''(x) = \sin x$ and $f^{(4)}(x) = \cos x$; when $g(x) = 1 - \frac{1}{2}x^2$, $g'''(x) = 0$ and $g^{(4)}(x) = 0$. Thus $f'''(0) = 0 = g'''(0)$ so the third derivatives agree at $x = 0$. However, the fourth derivatives do not agree since $f^{(4)}(0) = 1$ but $g^{(4)}(0) = 0$. In case (b), when $f(x) = \sin(x + a)$ and $g(x)$ $= \sin x\cos a + \cos x\sin a$, notice that $f(x) = g(x)$ for all x, not just $x = 0$. Since this is an identity, we have $f^{(n)}(x) = g^{(n)}(x)$ for any x and any positive integer n.

4. (a) $y = \sin x \Rightarrow y' = \cos x \Rightarrow y'' = -\sin x \Rightarrow y'' + y = -\sin x + \sin x = 0$; $y = \cos x \Rightarrow y' = -\sin x$
$\Rightarrow y'' = -\cos x \Rightarrow y'' + y = -\cos x + \cos x = 0$; $y = a\cos x + b\sin x \Rightarrow y' = -a\sin x + b\cos x$
$\Rightarrow y'' = -a\cos x - b\sin x \Rightarrow y'' + y = (-a\cos x - b\sin x) + (a\cos x + b\sin x) = 0$
   (b) $y = \sin(2x) \Rightarrow y' = 2\cos(2x) \Rightarrow y'' = -4\sin(2x) \Rightarrow y'' + 4y = -4\sin(2x) + 4\sin(2x) = 0$. Similarly, $y = \cos(2x)$ and $y = a\cos(2x) + b\sin(2x)$ satisfy the differential equation $y'' + 4y = 0$. In general, $y = \cos(mx)$, $y = \sin(mx)$ and $y = a\cos(mx) + b\sin(mx)$ satisfy the differential equation $y'' + m^2y = 0$.

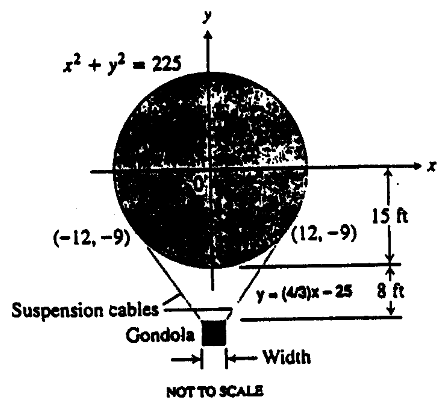
5. If the circle $(x - h)^2 + (y - k)^2 = a^2$ and $y = x^2 + 1$ are tangent at $(1, 2)$, then the slope of this tangent is $m = 2x|_{(1,2)} = 2$ and the tangent line is $y = 2x$. The line containing (h, k) and $(1, 2)$ is perpendicular to $y = 2x \Rightarrow \frac{k-2}{h-1} = -\frac{1}{2} \Rightarrow h = 5 - 2k \Rightarrow$ the location of the center is $(5 - 2k, k)$. Also, $(x - h)^2 + (y - k)^2 = a^2$
$\Rightarrow x - h + (y - k)y' = 0 \Rightarrow 1 + (y')^2 + (y - k)y'' = 0 \Rightarrow y'' = \frac{1 + (y')^2}{k - y}$. At the point $(1, 2)$ we know $y' = 2$ from the tangent line and that $y'' = 2$ from the parabola. Since the second derivatives are equal at $(1, 2)$ we obtain $2 = \frac{1 + (2)^2}{k - 2} \Rightarrow k = \frac{9}{2}$. Then $h = 5 - 2k = -4 \Rightarrow$ the circle is $(x + 4)^2 + \left(y - \frac{9}{2}\right)^2 = a^2$. Since $(1, 2)$ lies on the circle we have that $a = \frac{5\sqrt{5}}{2}$.

6. The total revenue is the number of people times the price of the fare: $r(x) = xp = x\left(3 - \frac{x}{40}\right)^2$, where $0 \le x \le 60$. The marginal revenue is $\frac{dr}{dx} = \left(3 - \frac{x}{40}\right)^2 + 2x\left(3 - \frac{x}{40}\right)\left(-\frac{1}{40}\right) \Rightarrow \frac{dr}{dx} = \left(3 - \frac{x}{40}\right)\left[\left(3 - \frac{x}{40}\right) - \frac{2x}{40}\right]$ $= 3\left(3 - \frac{x}{40}\right)\left(1 - \frac{x}{40}\right)$. Then $\frac{dr}{dx} = 0 \Rightarrow x = 40$ (since $x = 120$ does not belong to the domain). When 40 people are on the bus the marginal revenue is zero and the fare is $p(40) = \left(3 - \frac{x}{40}\right)^2\Big|_{x=40} = \$4.00$.

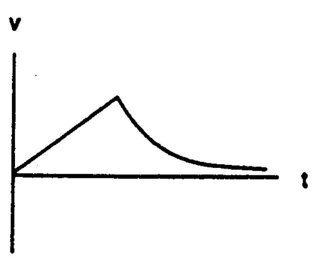
7. (a) $y = uv \Rightarrow \frac{dy}{dt} = \frac{du}{dt}v + u\frac{dv}{dt} = (0.04u)v + u(0.05v) = 0.09uv = 0.09y$

   (b) If $\frac{du}{dt} = -0.02u$ and $\frac{dv}{dt} = 0.03v$, then $\frac{dy}{dt} = (-0.02u)v + (0.03v)u = 0.01uv = 0.01y$, increasing at 1% per year.

8. When $x^2 + y^2 = 225$, then $y' = -\frac{x}{y}$. The tangent line to the balloon at $(12, -9)$ is $y + 9 = \frac{4}{3}(x - 12)$ $\Rightarrow y = \frac{4}{3}x - 25$. The top of the gondola is $15 + 8$ $= 23$ ft below the center of the balloon. The intersection of $y = -23$ and $y = \frac{4}{3}x - 25$ is at the far right edge of the gondola $\Rightarrow -23 = \frac{4}{3}x - 25$ $\Rightarrow x = \frac{3}{2}$. Thus the gondola is $2x = 3$ ft wide.



9. Answers will vary. Here is one possibility



10. $s(t) = 10\cos\left(t + \frac{\pi}{4}\right) \Rightarrow v(t) = \frac{ds}{dt} = -10\sin\left(t + \frac{\pi}{4}\right) \Rightarrow a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -10\cos\left(t + \frac{\pi}{4}\right)$

   (a) $s(0) = 10\cos\left(\frac{\pi}{4}\right) = \frac{10}{\sqrt{2}}$

   (b) Left: -10, Right: 10

(c) Solving $10\cos\left(t+\frac{\pi}{4}\right) = -10 \Rightarrow \cos\left(t+\frac{\pi}{4}\right) = -1 \Rightarrow t = \frac{3\pi}{4}$ when the particle is farthest to the left. Solving $10\cos\left(t+\frac{\pi}{4}\right) = 10 \Rightarrow \cos\left(t+\frac{\pi}{4}\right) = 1 \Rightarrow t = -\frac{\pi}{4}$, but $t \ge 0 \Rightarrow t = 2\pi + \frac{-\pi}{4} = \frac{7\pi}{4}$ when the particle is farthest to the right. Thus, $v\left(\frac{3\pi}{4}\right) = 0$, $v\left(\frac{7\pi}{4}\right) = 0$, $a\left(\frac{3\pi}{4}\right) = 10$, and $a\left(\frac{7\pi}{4}\right) = -10$.

(d) Solving $10\cos\left(t+\frac{\pi}{4}\right) = 0 \Rightarrow t = \frac{\pi}{4} \Rightarrow v\left(\frac{\pi}{4}\right) = -10$, $\left|v\left(\frac{\pi}{4}\right)\right| = 10$ and $a\left(\frac{\pi}{4}\right) = 0$.

11. (a) $s(t) = 64t - 16t^2 \Rightarrow v(t) = \frac{ds}{dt} = 64 - 32t = 32(2-t)$. The maximum height is reached when $v(t) = 0 \Rightarrow t = 2$ sec. The velocity when it leaves the hand is $v(0) = 64$ ft/sec.

(b) $s(t) = 64t - 2.6t^2 \Rightarrow v(t) = \frac{ds}{dt} = 64 - 5.2t$. The maximum height is reached when $v(t) = 0 \Rightarrow t \approx 12.31$ sec. The maximum height is about $s(12.31) = 393.85$ ft.

12. $s_1 = 3t^3 - 12t^2 + 18t + 5$ and $s_2 = -t^3 + 9t^2 - 12t \Rightarrow v_1 = 9t^2 - 24t + 18$ and $v_2 = -3t^2 + 18t - 12$; $v_1 = v_2 \Rightarrow 9t^2 - 24t + 18 = -3t^2 + 18t - 12 \Rightarrow 2t^2 - 7t + 5 = 0 \Rightarrow (t-1)(2t-5) = 0 \Rightarrow t = 1$ sec and $t = 2.5$ sec.

13. $m\left(v^2 - v_0^2\right) = k\left(x_0^2 - x^2\right) \Rightarrow m\left(2v\,\frac{dv}{dt}\right) = k\left(-2x\,\frac{dx}{dt}\right) \Rightarrow m\,\frac{dv}{dt} = k\left(-\frac{2x}{2v}\right)\frac{dx}{dt} \Rightarrow m\,\frac{dv}{dt} = -kx\left(\frac{1}{v}\right)\frac{dx}{dt}$. Then substituting $\frac{dx}{dt} = v \Rightarrow m\,\frac{dv}{dt} = -kx$, as claimed.

14. (a) $x = At^2 + Bt + C$ on $[t_1, t_2] \Rightarrow v = \frac{dx}{dt} = 2At + B \Rightarrow v\left(\frac{t_1+t_2}{2}\right) = 2A\left(\frac{t_1+t_2}{2}\right) + B = A(t_1+t_2) + B$ is the instantaneous velocity at the midpoint. The average velocity over the time interval is $v_{av} = \frac{\Delta x}{\Delta t}$

$$= \frac{\left(At_2^2 + Bt_2 + C\right) - \left(At_1^2 + Bt_1 + C\right)}{t_2 - t_1} = \frac{(t_2 - t_1)[A(t_2+t_1) + B]}{t_2 - t_1} = A(t_2 + t_1) + B.$$

(b) On the graph of the parabola $x = At^2 + Bt + C$, the slope of the curve at the midpoint of the interval $[t_1, t_2]$ is the same as the average slope of the curve over the interval.

15. (a) To be continuous at $x = \pi$ requires that $\lim_{x\to\pi^-} \sin x = \lim_{x\to\pi^+} (mx + b) \Rightarrow 0 = m\pi + b \Rightarrow m = -\frac{b}{\pi}$;

(b) If $y' = \begin{cases} \cos x, & x < \pi \\ m, & x \ge \pi \end{cases}$ is differentiable at $x = \pi$, then $\lim_{x\to\pi^-} \cos x = m \Rightarrow m = -1$ and $b = \pi$.

16. $f'(0) = \lim_{x\to0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to0} \frac{\frac{1-\cos x}{x} - 0}{x} = \lim_{x\to0} \left(\frac{1-\cos x}{x^2}\right)\left(\frac{1+\cos x}{1+\cos x}\right) = \lim_{x\to0} \left(\frac{\sin x}{x}\right)^2\left(\frac{1}{1+\cos x}\right) = \frac{1}{2}$. Therefore $f'(0)$ exists with value $\frac{1}{2}$.

17. (a) For all a, b and for all $x \ne 2$, f is differentiable at x. Next, f differentiable at $x = 2 \Rightarrow$ f continuous at $x = 2 \Rightarrow \lim_{x\to2^-} f(x) = f(2) \Rightarrow 2a = 4a - 2b + 3 \Rightarrow 2a - 2b + 3 = 0$. Also, f differentiable at $x \ne 2$

$\Rightarrow f'(x) = \begin{cases} a, & x < 2 \\ 2ax - b, & x > 2 \end{cases}$. In order that $f'(2)$ exist we must have $a = 2a(2) - b \Rightarrow a = 4a - b \Rightarrow 3a = b$.

Then $2a - 2b + 3 = 0$ and $3a = b \Rightarrow a = \frac{3}{4}$ and $b = \frac{9}{4}$.

(b) For $x < 2$, the graph of f is a straight line having a slope of $\frac{3}{4}$ and passing through the origin for $x \geq 2$, the graph of f is a parabola. At $x = 2$, the value of the y-coordinate on the parabola is $\frac{3}{2}$ which matches the y-coordinate of the point on the straight line at $x = 2$. In addition, the slope of the parabola at the match up point is $\frac{3}{4}$ which is equal to the slope of the straight line. Therefore, since the graph is differentiable at the match up point, the graph is smooth there.

18. (a) For any a, b and for any $x \neq -1$, g is differentiable at x. Next, g differentiable at $x = -1 \Rightarrow$ g continuous at $x = -1 \Rightarrow \lim\limits_{x \to -1^+} g(x) = g(-1) \Rightarrow -a - 1 + 2b = -a + b \Rightarrow b = 1$. Also, g differentiable at $x \neq -1$ $\Rightarrow g'(x) = \begin{cases} a, & x < -1 \\ 3ax^2 + 1, & x > -1 \end{cases}$. In order that $g'(-1)$ exist we must have $a = 3a(-1)^2 + 1 \Rightarrow a = 3a + 1$ $\Rightarrow a = -\frac{1}{2}$.

(b) For $x \leq -1$, the graph of f is a straight line having a slope of $-\frac{1}{2}$ and a y-intercept of 1. For $x > -1$, the graph of f is a parabola. At $x = -1$, the value of the y-coordinate on the parabola is $\frac{3}{2}$ which matches the y-coordinate of the point on the straight line at $x = -1$. In addition, the slope of the parabola at the up point is $-\frac{1}{2}$ which is equal to the slope of the straight line. Therefore, since the graph is differentiable at the match up point, the graph is smooth there.

19. f odd $\Rightarrow f(-x) = -f(x) \Rightarrow \frac{d}{dx}(f(-x)) = \frac{d}{dx}(-f(x)) \Rightarrow f'(-x)(-1) = -f'(x) \Rightarrow f'(-x) = f'(x) \Rightarrow f'$ is even.

20. f even $\Rightarrow f(-x) = f(x) \Rightarrow \frac{d}{dx}(f(-x)) = \frac{d}{dx}(f(x)) \Rightarrow f'(-x)(-1) = f'(x) \Rightarrow f'(-x) = -f'(x) \Rightarrow f'$ is odd.

21. Let $h(x) = (fg)(x) = f(x)\,g(x) \Rightarrow h'(x) = \lim\limits_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} = \lim\limits_{x \to x_0} \frac{f(x)\,g(x) - f(x_0)\,g(x_0)}{x - x_0}$

$$= \lim_{x \to x_0} \frac{f(x)\,g(x) - f(x)\,g(x_0) + f(x)\,g(x_0) - f(x_0)\,g(x_0)}{x - x_0} = \lim_{x \to x_0} \left[f(x) \left[\frac{g(x) - g(x_0)}{x - x_0} \right] \right] + \lim_{x \to x_0} \left[g(x_0) \left[\frac{f(x) - f(x_0)}{x - x_0} \right] \right]$$

$$= f(x_0) \lim_{x \to x_0} \left[\frac{g(x) - g(x_0)}{x - x_0} \right] + g(x_0)\,f'(x_0) = 0 \cdot \lim_{x \to x_0} \left[\frac{g(x) - g(x_0)}{x - x_0} \right] + g(x_0)\,f'(x_0) = g(x_0)\,f'(x_0),$$ if g is continuous at x_0. Therefore $(fg)(x)$ is differentiable at x_0 if $f(x_0) = 0$, and $(fg)'(x_0) = g(x_0)\,f'(x_0)$.

22. From Exercise 21 we have that fg is differentiable at 0 if f is differentiable at 0, $f(0) = 0$ and g is continuous at 0.

(a) If $f(x) = \sin x$ and $g(x) = |x|$, then $|x| \sin x$ is differentiable because $f'(0) = \cos(0) = 1$, $f(0) = \sin(0) = 0$ and $g(x) = |x|$ is continuous at $x = 0$.

(b) If $f(x) = \sin x$ and $g(x) = x^{2/3}$, then $x^{2/3} \sin x$ is differentiable because $f'(0) = \cos(0) = 1$, $f(0) = \sin(0) = 0$ and $g(x) = x^{2/3}$ is continuous at $x = 0$.

(c) If $f(x) = 1 - \cos x$ and $g(x) = \sqrt[3]{x}$, then $\sqrt[3]{x}(1 - \cos x)$ is differentiable because $f'(0) = \sin(0) = 0$, $f(0) = 1 - \cos(0) = 0$ and $g(x) = x^{1/3}$ is continuous at $x = 0$.

(d) If $f(x) = x$ and $g(x) = x \sin\left(\frac{1}{x}\right)$, then $x^2 \sin\left(\frac{1}{x}\right)$ is differentiable because $f'(0) = 1$, $f(0) = 0$ and

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = \lim_{x\to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{t\to\infty} \frac{\sin t}{t} = 0 \text{ (so g is continuous at } x = 0).$$

23. If $f(x) = x$ and $g(x) = x \sin\left(\frac{1}{x}\right)$, then $x^2 \sin\left(\frac{1}{x}\right)$ is differentiable at $x = 0$ because $f'(0) = 1$, $f(0) = 0$ and $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = \lim_{x\to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{t\to\infty} \frac{\sin t}{t} = 0$ (so g is continuous at $x = 0$). In fact, from Exercise 21, $h'(0) = g(0) f'(0) = 0$. However, for $x \neq 0$, $h'(x) = \left[x^2 \cos\left(\frac{1}{x}\right)\right]\left(-\frac{1}{x^2}\right) + 2x \sin\left(\frac{1}{x}\right)$. But $\lim_{x\to 0} h'(x) = \lim_{x\to 0}\left[-\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)\right]$ does not exist because $\cos\left(\frac{1}{x}\right)$ has no limit as $x \to 0$. Therefore, the derivative is not continuous at $x = 0$ because it has no limit there.

24. From the given conditions we have $f(x+h) = f(x) f(h)$, $f(h) - 1 = hg(h)$ and $\lim_{h\to 0} g(h) = 1$. Therefore,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x) f(h) - f(x)}{h} = \lim_{h\to 0} f(x)\left[\frac{f(h) - 1}{h}\right] = f(x)\left[\lim_{h\to 0} g(h)\right] = f(x) \cdot 1 = f(x)$$

$\Rightarrow f'(x) = f(x)$ exists.

25. Step 1: The formula holds for $n = 2$ (a single product) since $y = u_1 u_2 \Rightarrow \frac{dy}{dx} = \frac{du_1}{dx} u_2 + u_1 \frac{du_2}{dx}$.

Step 2: Assume the formula holds for $n = k$:

$$y = u_1 u_2 \cdots u_k \Rightarrow \frac{dy}{dx} = \frac{du_1}{dx} u_2 u_3 \cdots u_k + u_1 \frac{du_2}{dx} u_3 \cdots u_k + \ldots + u_1 u_2 \cdots u_{k-1} \frac{du_k}{dx}.$$

If $y = u_1 u_2 \cdots u_k u_{k+1} = (u_1 u_2 \cdots u_k) u_{k+1}$, then $\frac{dy}{dx} = \frac{d(u_1 u_2 \cdots u_k)}{dx} u_{k+1} + u_1 u_2 \cdots u_k \frac{du_{k+1}}{dx}$

$$= \left(\frac{du_1}{dx} u_2 u_3 \cdots u_k + u_1 \frac{du_2}{dx} u_3 \cdots u_k + \cdots + u_1 u_2 \cdots u_{k-1} \frac{du_k}{dx}\right) u_{k+1} + u_1 u_2 \cdots u_k \frac{du_{k+1}}{dx}$$

$$= \frac{du_1}{dx} u_2 u_3 \cdots u_{k+1} + u_1 \frac{du_2}{dx} u_3 \cdots u_{k+1} + \cdots + u_1 u_2 \cdots u_{k-1} \frac{du_k}{dx} u_{k+1} + u_1 u_2 \cdots u_k \frac{du_{k+1}}{dx}.$$

Thus the original formula holds for $n = (k+1)$ whenever it holds for $n = k$.

NOTES: