CHAPTER 3 APPLICATIONS OF DERIVATIVES

3.1 EXTREME VALUES OF FUNCTIONS

1. An absolute minimum at $x = c_2$, an absolute maximum at $x = b$. Theorem 1 guarantees the existence of such extreme values because h is continuous on $[a, b]$.

2. An absolute minimum at $x = b$, an absolute maximum at $x = c$. Theorem 1 guarantees the existence of such extreme values because f is continuous on $[a, b]$.

3. No absolute minimum. An absolute maximum at $x = c$. Since the function's domain is an open interval, the function does not satisfy the hypotheses of Theorem 1 and need not have absolute extreme values.

4. No absolute extrema. The function is neither continuous nor defined on a closed interval, so it need not fulfill the conclusions of Theorem 1.

5. An absolute minimum at $x = a$ and an absolute maximum at $x = c$. Note that $y = g(x)$ is not continuous but still has extrema. When the hypothesis of Theorem 1 is satisfied then extrema are guaranteed, but when the hypothesis is not satisfied, absolute extrema may or may not occur.

6. Absolute minimum at $x = c$ and an absolute maximum at $x = a$. Note that $y = g(x)$ is not continuous but still has absolute extrema. When the hypothesis of Theorem 1 is satisfied then extrema are guaranteed, but when the hypothesis is not satisfied, absolute extrema may or may not occur.

7. Local minimum at $(-1, 0)$, local maximum at $(1, 0)$

8. Minima at $(-2, 0)$ and $(2, 0)$, maximum at $(0, 2)$

9. Maximum at $(0, 5)$. Note that there is no minimum since the endpoint $(2, 0)$ is excluded from the graph.

10. Local maximum at $(-3, 0)$, local minimum at $(2, 0)$, maximum at $(1, 2)$, minimum at $(0, -1)$

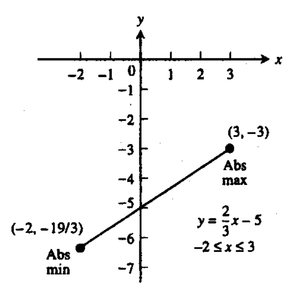
11. Graph (c), since this is the only graph that has positive slope at c.

12. Graph (b), since this is the only graph that represents a differentiable function at a and b and has negative slope at c.

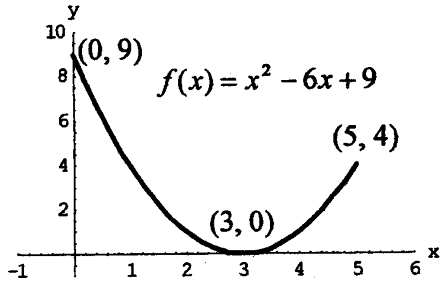
13. Graph (d), since this is the only graph representing a function that is differentiable at b but not at a.

14. Graph (a), since this is the only graph that represents a function that is not differentiable at a or b.

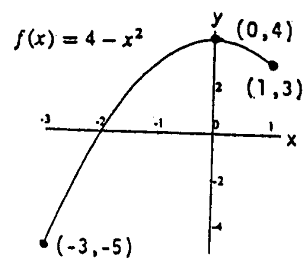
15. $f(x) = \frac{2}{3}x - 5 \Rightarrow f'(x) = \frac{2}{3} \Rightarrow$ no critical points; $f(-2) = -\frac{19}{3}$, $f(3) = -3 \Rightarrow$ the absolute maximum is -3 at $x = 3$ and the absolute minimum is $-\frac{19}{3}$ at $x = -2$



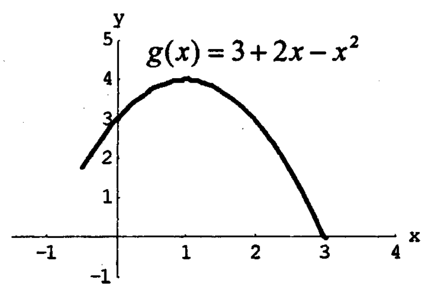
16. $f(x) = x^2 - 6x + 9 \Rightarrow f'(x) = 2x - 6 \Rightarrow$ a critical point at $x = 3$; $f(0) = 9$, $f(3) = 0$, and $f(5) = 4$ $\Rightarrow$ the absolute maximum is 9 at $x = 0$ and the absolute minimum is 0 at $x = 3$.

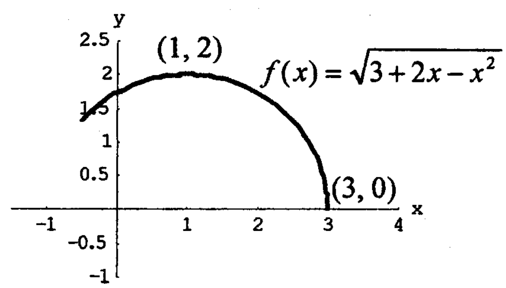


17. $f(x) = 4 - x^2 \Rightarrow f'(x) = -2x \Rightarrow$ a critical point at $x = 0$; $f(-3) = -5$, $f(0) = 4$, $f(1) = 3 \Rightarrow$ the absolute maximum is 4 at $x = 0$ and the absolute minimum is -5 at $x = -3$



18.

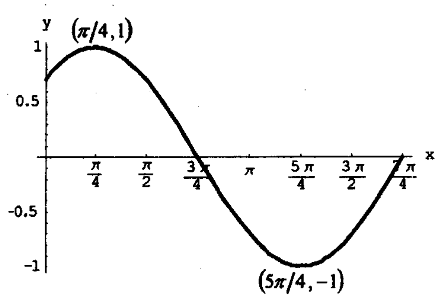




The extreme values of $f(x) = \sqrt{3 + 2x - x^2}$ occur at the extreme values of $g(x) = 3 + 2x - x^2$. Therefore, $g'(x) = 2 - 2x \Rightarrow x = 1$ is a critical value; $f(-0.5) = \sqrt{1.75} \approx 1.32288$, $f(1) = 2$, $f(3) = 0 \Rightarrow$ the absolute maximum is 2 at $x = 1$ and the absolute minimum is 0 at $x = 3$.

19. The first derivative of $f'(x) = \cos\left(x + \frac{\pi}{4}\right)$, has zeros at $x = \frac{\pi}{4}$, $x = \frac{5\pi}{4}$.

Critical point values:	$x = \frac{\pi}{4}$	$f(x) = 1$
	$x = \frac{5\pi}{4}$	$f(x) = -1$
Endpoint values:	$x = 0$	$f(x) = \frac{1}{\sqrt{2}}$
	$x = \frac{7\pi}{4}$	$f(x) = 0$



Maximum value is 1 at $x = \frac{\pi}{4}$;

minimum value is -1 at $x = \frac{5\pi}{4}$;

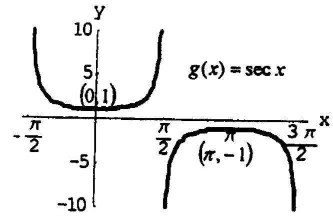
local minimum at $\left(0, \frac{1}{\sqrt{2}}\right)$;

local maximum at $\left(\frac{7\pi}{4}, 0\right)$

20. The first derivative $g'(x) = \sec x \tan x$ has zeros at $x = 0$ and $x = \pi$ and is undefined at $x = \frac{\pi}{2}$. Since $g(x) = \sec x$ is also undefined at $x = \frac{\pi}{2}$, the critical points occur only at $x = 0$ and $x = \pi$.

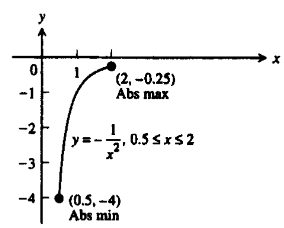
Critical point values: $x = 0$ $\quad g(x) = 1$

$x = \pi$ $\quad g(x) = -1$

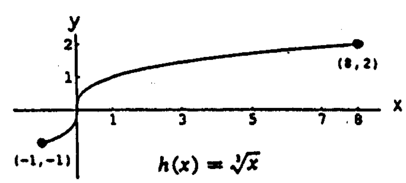


Since the range of $g(x)$ is $(-\infty, -1] \cup [1, \infty)$, these values must be a local minimum and local maximum, respectively. Local minimum at $(0, 1)$; local maximum at $(\pi, -1)$. There are no absolute extrema on the interval $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

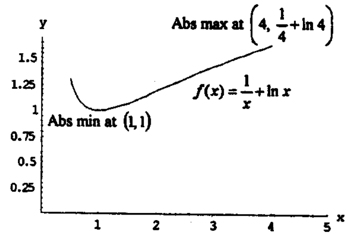
21. $F(x) = -\frac{1}{x^2} = -x^{-2} \Rightarrow F'(x) = 2x^{-3} = \frac{2}{x^3}$, however $x = 0$ is not a critical point since 0 is not in the domain; $F(0.5) = -4$, $F(2) = -0.25 \Rightarrow$ the absolute maximum is -0.25 at $x = 2$ and the absolute minimum is -4 at $x = 0.5$



22. $h(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow h'(x) = \frac{1}{3}x^{-2/3} \Rightarrow$ a critical point at $x = 0$; $h(-1) = -1$, $h(0) = 0$, $h(8) = 2 \Rightarrow$ the absolute maximum is 2 at $x = 8$ and the absolute minimum is -1 at $x = -1$



23. The first derivative $f'(x) = -\frac{1}{x^2} + \frac{1}{x}$ has a zero at $x = 1$.



Critical point value: $f(1) = 1 + \ln 1 = 1$

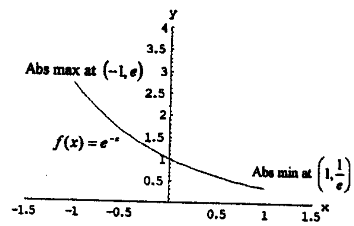
Endpoint values: $f(0.5) = 2 + \ln 0.5 \approx 1.307$;

$f(4) = \frac{1}{4} + \ln 4 \approx 1.636$

Absolute maximum value is $\frac{1}{4} + \ln 4$ at $x = 4$;

absolute minimum value is 1 at $x = 1$; local maximum at $\left(\frac{1}{2}, 2 - \ln 2\right)$

24. The first derivative $g'(x) = -e^{-x}$ has no zeros, so we need only consider the endpoints.

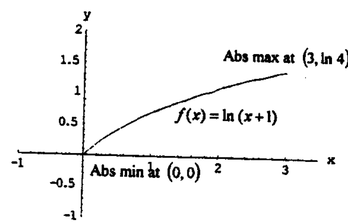


$g(-1) = e^{-(-1)} = e$; $g(1) = e^{-1} = \frac{1}{e}$

Maximum value is e at $x = -1$;

minimum value is $\frac{1}{e}$ at $x = 1$.

25. The first derivative $h'(x) = \frac{1}{x+1}$ has no zeros,



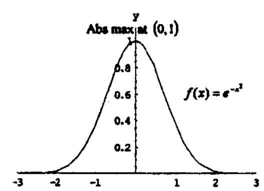
so we need only consider the endpoints.

$h(0) = \ln 1 = 0$; $h(3) = \ln 4$

Maximum value is $\ln 4$ at $x = 3$;

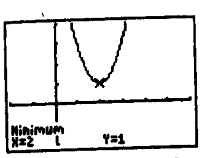
minimum value is 0 at $x = 0$.

26. The first derivative $k'(x) = -2xe^{-x^2}$ has a zero



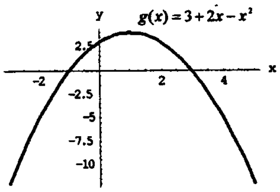
at $x = 0$. Since the domain has no endpoints, any extreme value must occur at $x = 0$. Since $k(0) = e^{-0^2} = 1$ and $\lim_{x \to \pm\infty} k(x) = 0$, the maximum value is 1 at $x = 0$.

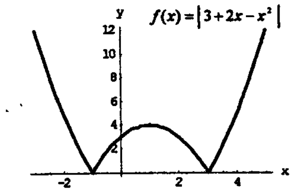
27.



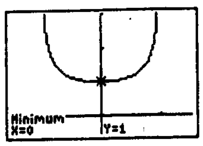
[−2, 6] by [−2, 4]

$y' = 4x - 8 = 0 \Rightarrow$ critical value at $x = 2$ and $y'' = 4 \Rightarrow$ minimum value is 1 at $x = 2$.

28. 

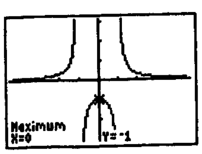


The minimum values of f(x) occur wherever $g(x) = 3 + 2x - x^2 = 0 \Rightarrow x = -1$ and $x = 3$. There is a relative maximum at the point where g(x) has a relative maximum $\Rightarrow g'(x) = 2 - 2x \Rightarrow$ there is a critical value at $x = 1$. There is no absolute maximum value of f(x), the absolute minimum value is 0 at $x = -1$ and $x = 3$. There is a relative maximum of 4 at $x = 1$. Note that $f'(x)$ is undefined at $x = -1$ and $x = 3$, and so these are critical points of f.

29. 

[−1.5, 1.5] by [−0.5, 3]

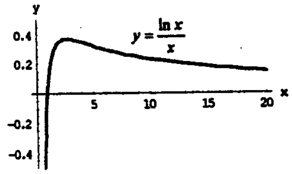
$y' = \dfrac{x}{(1 - x^2)^{3/2}} \Rightarrow$ critical value at $x = 0$; $y'' = \dfrac{2x^2 + 1}{(1 - x^2)^{5/2}}$; at $x = 0$, $y'' = 1 \Rightarrow$ minimum value is 1 at $x = 0$.

30. 

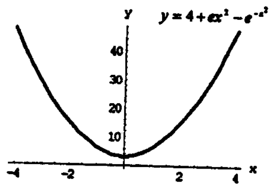
[−4.7, 4.7] by [−3.1, 3.1]

To confirm that there are no "hidden" extrema, note that $y' = -(x^2 - 1)^{-2}(2x) = \dfrac{-2x}{(x^2 - 1)^2}$ which is zero only at $x = 0$ and is undefined only where y is undefined. There is a local maximum at $(0, -1)$.

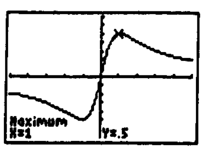
31. $y = \frac{\ln x}{x} \Rightarrow \dfrac{dy}{dx} = \dfrac{x\left(\frac{1}{x}\right) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2} \Rightarrow$ there is a critical point where $\ln x = 1 \Rightarrow x = e$. The graph of the function shows a relative and absolute maximum near $x = e$ and the "maximum" function on the TI-89 calculator gives a maximum of $y = 0.36789 = \frac{1}{e}$ at $x = 2.71828 \approx e$. There is no absolute minimum.



32. $y = 4 + ex^2 - e^{-x^2} \Rightarrow \frac{dy}{dx} = 2ex + 2xe^{-x^2} = 2x\left(e + e^{-x^2}\right) \Rightarrow$ the only critical point is at $x = 0$, since $e + e^{-x^2} > 0$ for all real x. The graph of the function shows an absolute minimum value at $x = 0$, and the "minimum" function on the TI-89 calculator gives a minimum of $y = 3$ at $x = 0$, as expected. There is no absolute maximum.



33.

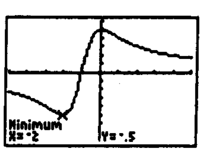


[−5, 5] by [−0.7, 0.7]

$y' = \frac{1 - x^2}{(x^2 + 1)^2} \Rightarrow$ critical values at $x = \pm 1$; $y'' = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}$; at $x = -1$, $y'' = \frac{1}{2}$ and at $x = 1$, $y'' = -\frac{1}{2}$

$\Rightarrow$ maximum value is $\frac{1}{2}$ at $x = 1$; minimum value is $-\frac{1}{2}$ at $x = -1$.

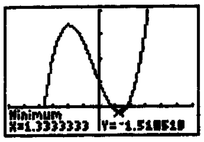
34.



[−5, 5] by [−0.8, 0.6]

$y' = \frac{-x(x+2)}{(x^2+2x+2)^2} \Rightarrow$ critical values at $x = 0$ and $x = -2$; $y'' = \frac{2(x+1)(x^2+2x-2)}{(x^2+2x+2)^3}$; at $x = 0$, $y'' = -\frac{1}{2}$ and at $x = -2$, $y'' = \frac{1}{2} \Rightarrow$ maximum value is $\frac{1}{2}$ at $x = 0$; minimum value is $-\frac{1}{2}$ at $x = -2$.

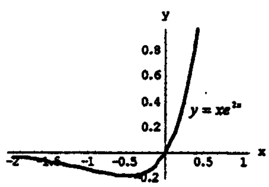
35.



[−6, 6] by [−5, 20]

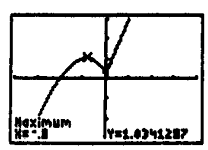
To find the exact values, note that $y' = 3x^2 + 2x - 8 = (3x-4)(x+2)$, which is zero when $x = -2$ or $x = \frac{4}{3}$. Local maximum at $(-2, 17)$; local minimum at $\left(\frac{4}{3}, -\frac{41}{27}\right)$

36. $y = xe^{2x} \Rightarrow \frac{dy}{dx} = (1+2x)e^{2x} \Rightarrow$ the only critical point is at $x = -\frac{1}{2}$, since $e^{2x} > 0$ for all real x. The graph of the function shows an absolute minimum value near $x = -\frac{1}{2}$, and the "minimum" function on the TI-89 calculator gives a minimum of $y = -0.18394 \approx \frac{e^{-1}}{2}$ at $x = -0.5$, as expected. There is no absolute maximum.



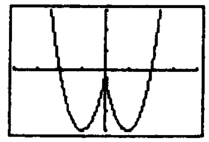
37. $y' = x^{2/3}(1) + \frac{2}{3}x^{-1/3}(x+2) = \frac{5x+4}{3\sqrt[3]{x}}$

crit. pt.	derivative	extremum	value
$x = -\frac{4}{5}$	0	local max	$\frac{12}{25}10^{1/3} = 1.034$
$x = 0$	undefined	local min	0



[−4, 4] by [−3, 3]

38. $y' = x^{2/3}(2x) + \frac{2}{3}x^{-1/3}(x^2-4) = \frac{8x^2-8}{3\sqrt[3]{x}}$

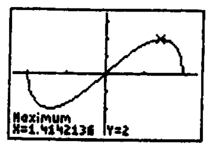


[−4, 4] by [−3, 3]

crit. pt.	derivative	extremum	value
$x=-1$	0	minimum	−3
$x=0$	undefined	local max	0
$x=1$	0	minimum	−3

39. $y' = x \cdot \frac{1}{2\sqrt{4-x^2}}(-2x) + (1)\sqrt{4-x^2}$

$$= \frac{-x^2+(4-x^2)}{\sqrt{4-x^2}} = \frac{4-2x^2}{\sqrt{4-x^2}}$$

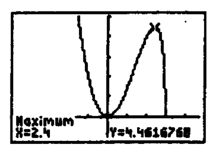


[−2.35, 2.35] by [−3.5, 3.5]

crit. pt.	derivative	extremum	value
$x=-2$	undefined	local max	0
$x=-\sqrt{2}$	0	minimum	−2
$x=\sqrt{2}$	0	maximum	2
$x=2$	undefined	local min	0

40. $y' = x^2 \cdot \frac{1}{2\sqrt{3-x}}(-1) + 2x\sqrt{3-x}$

$$= \frac{-x^2+(4x)(3-x)}{2\sqrt{3-x}} = \frac{-5x^2+12x}{2\sqrt{3-x}}$$

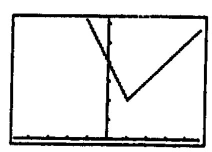


[−4.7, 4.7] by [−1, 5]

crit. pt.	derivative	extremum	value
$x=0$	0	minimum	0
$x=\frac{12}{5}$	0	local max	$\frac{144}{125}15^{1/2} \approx 4.462$
$x=3$	undefined	minimum	0

41.

$$y' = \begin{cases} -2, & x<1 \\ 1, & x>1 \end{cases}$$

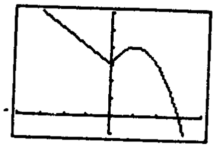


[−4.7, 4.7] by [0, 6.2]

crit. pt.	derivative	extremum	value
$x=1$	undefined	minimum	2

42.

$$y' = \begin{cases} -1, & x < 0 \\ 2 - 2x, & x > 0 \end{cases}$$

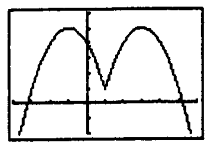


[−4, 4] by [−1, 6]

crit. pt.	derivative	extremum	value
$x = 0$	undefined	local min	3
$x = 1$	0	local max	4

43.

$$y' = \begin{cases} -2x - 2, & x < 1 \\ -2x + 6, & x > 1 \end{cases}$$

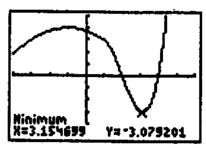


[−4, 6] by [−2, 6]

crit. pt.	derivative	extremum	value
$x = -1$	0	maximum	5
$x = 1$	undefined	local min	1
$x = 3$	0	maximum	5

44. We begin by determining whether $f'(x)$ is defined at $x = 1$, where

$$f(x) = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$$



[−4, 6] by [−5, 5]

Left-hand derivative:

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0^-} \frac{-\frac{1}{4}(1+h)^2 - \frac{1}{2}(1+h) + \frac{15}{4} - 3}{h} = \lim_{h \to 0^-} \frac{-h^2 - 4h}{4h} = \lim_{h \to 0^-} \frac{1}{4}(-h - 4) = -1$$

Right-hand derivative

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h)^3 - 6(1+h)^2 + 8(1+h) - 3}{h} = \lim_{h \to 0^+} \frac{h^3 - 3h^2 - h}{4h} = \lim_{h \to 0^+} (h^2 - 3h - 1)$$

$$= -1$$

$$\text{Thus } f'(x) = \begin{cases} -\frac{1}{2}x - \frac{1}{2}, & x \le 1 \\ 3x^2 - 12x + 8, & x > 1 \end{cases}$$

Note that $-\frac{1}{2}x - \frac{1}{2} = 0$ when $x = -1$, and $3x^2 - 12x + 8 = 0$ when $x = \dfrac{12 \pm \sqrt{12^2 - 4(3)(8)}}{2(3)}$

$= \frac{12 \pm \sqrt{48}}{6} = 2 \pm \frac{2\sqrt{3}}{3}$. But $2 - \frac{2\sqrt{3}}{3} \approx 0.845 < 1$, so the only critical points occur at $x = -1$ and $x = 2 + \frac{2\sqrt{3}}{3} \approx 3.155$.

crit. pt.	derivative	extremum	value
$x = -1$	0	local max	4
$x \approx 3.155$	0	local max	≈ -3.079

45. (a) No, since $f'(x) = \frac{2}{3}(x-2)^{-1/3}$, which is undefined at $x = 2$.

(b) The derivative is defined and nonzero for all $x \neq 2$. Also, $f(2) = 0$ and $f(x) > 0$ for all $x \neq 2$.

(c) No, f(x) need not have a global maximum because its domain is all real numbers. Any restriction of f to a closed interval of the form [a, b] would have both a maximum value and a minimum value on the interval.

(d) The answers are the same as (a) and (b) with 2 replaced by a.

46. Note that $f(x) = \begin{cases} -x^3 + 9x, & x \leq -3 \text{ or } 0 \leq x < 3 \\ x^3 - 9x, & -3 < x < 0 \text{ or } x \geq 3 \end{cases}$

Therefore, $f'(x) = \begin{cases} -3x^2 + 9, & x < -3 \text{ or } 0 < x < 3 \\ 3x^2 - 9, & -3 < x < 0 \text{ or } x > 3 \end{cases}$

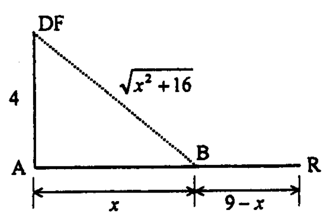
(a) No, since the left- and right-hand derivatives at $x = 0$ are -9 and 9, respectively.

(b) No, since the left- and right-hand derivatives at $x = 3$ are -18 and 18, respectively.

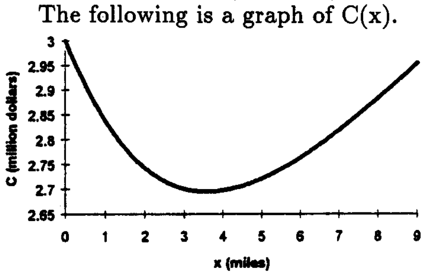
(c) No, since the left- and right-hand derivatives at $x = -3$ are -18 and 18, respectively.

(d) The critical points occur when $f'(x) = 0$ (at $x = \pm\sqrt{3}$) and when $f'(x)$ is undefined (at $x = 0$ and $x = \pm 3$). The minimum value is 0 at $x = -3$, at $x = 0$, and at $x = 3$; local maxima occur at $(-\sqrt{3}, 6\sqrt{3})$ and $(\sqrt{3}, 6\sqrt{3})$.

47.



(a) The construction cost is $C(x) = 0.3\sqrt{16 + x^2} + 0.2(9 - x)$ million dollars, where $0 \leq x \leq 9$ miles. The following is a graph of C(x).

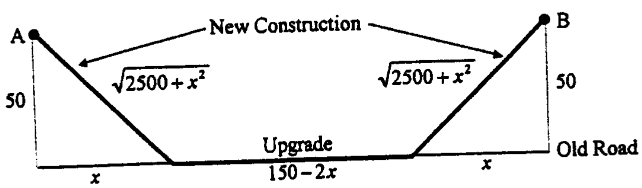


Solving $C'(x) = \frac{0.3x}{\sqrt{16+x^2}} - 0.2 = 0$ gives $x = \pm\frac{8\sqrt{5}}{5} \approx \pm 3.58$ miles, but only $x = 3.58$ miles is a critical point in the specified domain. Evaluating the costs at the critical and endpoints gives $C(0) = \$3$ million, $C(8\sqrt{5}/5) \approx \$2.694$ million, and $C(9) \approx \$2.955$ million. Therefore, to minimize the cost of construction, the pipeline should be placed from the docking facility to point B, 3.58 miles along the shore from point A, and then along the shore from B to the refinery.

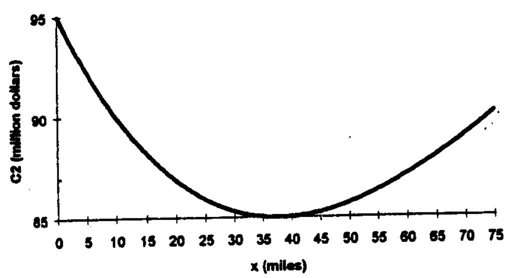
(b) If the per mile cost of underwater construction is p, then $C(x) = p\sqrt{16+x^2} + 0.2(9-x)$ and $C'(x) = \frac{px}{\sqrt{16+x^2}} - 0.2 = 0$ gives $x_c = \frac{0.8}{\sqrt{p^2 - 0.04}}$, which minimizes the construction cost provided $x_c \le 9$. The value of p that gives $x_c = 9$ miles is 0.218864. Consequently, if the underwater construction costs \$218,864 per mile or less, then running the pipeline along a straight line directly from the docking facility to the refinery will minimize the cost of construction.

In theory, p would have to be infinite to justify running the pipe directly from the docking facility to point A (i.e., for x_c to be zero). For all values of $p > 0.218864$ there is always an $x_c \in (0,9)$ that will give a minimum value for C. This is proved by looking at $C''(x_c) = \frac{16p}{\left(16 + x_c^2\right)^{3/2}}$ which is always positive for $p > 0$.

48. There are two options to consider. The first is to build a new road straight from Village A to Village B. The second is to build a new highway segment from Village A to the Old Road, reconstruct a segment of Old Road, and build a new highway segment from Old Road to Village B, as shown in the figure. The cost of the first option is $C_1 = 0.5(150) = \$75$ million.

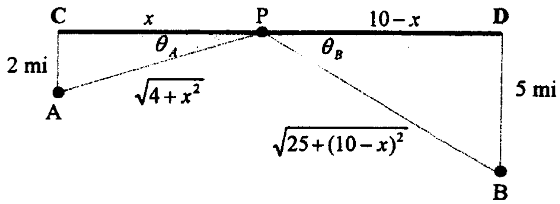


The construction cost for the second option is $C_2(x) = 0.5(2\sqrt{2500+x^2}) + 0.3(150-2x)$ million dollars for $0 \le x \le 75$ miles. The following is a graph of $C_2(x)$.

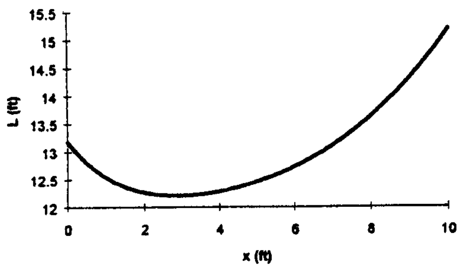


Solving $C_2'(x) = \frac{x}{\sqrt{2500+x^2}} - 0.6 = 0$ gives $x = \pm 37.5$ miles, but only $x = 37.5$ miles is in the specified domain. In summary, $C_1 = \$75$ million, $C_2(0) = \$95$ million, $C_2(37.5) = \$85$ million, and $C_2(75) = \$90.139$ million. Consequently, a new road straight from Village A to Village B is the least expensive option.

49.



The length of pipeline is $L(x) = \sqrt{4 + x^2} + \sqrt{25 + (10 - x)^2}$ for $0 \le x \le 10$. The following is a graph of L(x).

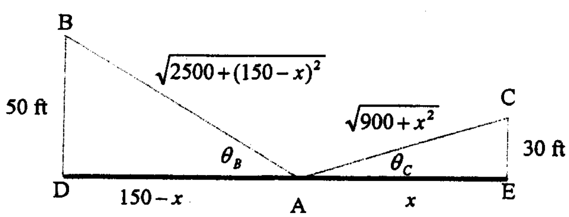


Setting the derivative of L(x) equal to zero gives $L'(x) = \frac{x}{\sqrt{4+x^2}} - \frac{(10-x)}{\sqrt{25+(10-x)^2}} = 0$. Note that $\frac{x}{\sqrt{4+x^2}} = \cos\theta_A$ and $\frac{10-x}{\sqrt{25-(10-x)^2}} = \cos\theta_B$, therefore, $L'(x) = 0$ when $\cos\theta_A = \cos\theta_B$, or $\theta_A = \theta_B$ and ΔACP is similar to ΔBDP. Use simple proportions to determine x as follows:

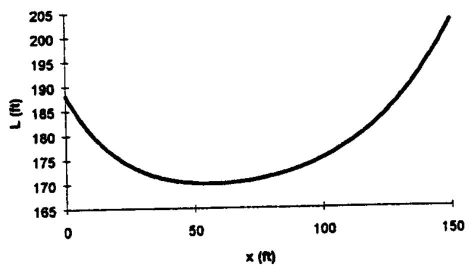
$\frac{x}{2} = \frac{10-x}{5} \Rightarrow x = \frac{20}{7} \approx 2.857$ miles along the coast from town A to town B.

If the two towns were on opposite sides of the river, the obvious solution would be to place the pump station on a straight line (the shortest distance) between the two towns, again forcing $\theta_A = \theta_B$. The shortest length of pipe is the same regardless of whether the towns are on the same or opposite sides of the river.

50.



(a) The length of guy wire is $L(x) = \sqrt{900 + x^2} + \sqrt{2500 + (150 - x)^2}$ for $0 \le x \le 150$. The following is a graph of L(x):



Setting L'(x) equal to zero gives $L'(x) = \frac{x}{\sqrt{900 + x^2}} - \frac{(150 - x)}{\sqrt{2500 + (150 - x)^2}} = 0$. Note that $\frac{x}{\sqrt{900 + x^2}} = \cos\theta_C$ and $\frac{150 - x}{\sqrt{2500 - (150 - x)^2}} = \cos\theta_B$. Therefore, $L'(x) = 0$ when $\cos\theta_C = \cos\theta_B$, or $\theta_C = \theta_B$ and ΔACE is similar to ΔABD. Use simple proportions to determine x: $\frac{x}{30} = \frac{150 - x}{50}$ $\Rightarrow x = \frac{225}{4} = 56.25$ feet.

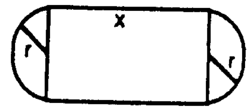
(b) If the heights of the towers are h_B and h_C, and the horizontal distance between them is s, then $L(x) = \sqrt{h_C^2 + x^2} + \sqrt{h_B^2 + (s - x)^2}$ and $L'(x) = \frac{x}{\sqrt{h_C^2 + x^2}} - \frac{(s - x)}{\sqrt{h_B^2 + (s - x)^2}}$. However, $\frac{x}{\sqrt{h_C^2 + x^2}} = \cos\theta_C$ and $\frac{(s - x)}{\sqrt{h_B^2 + (s - x)^2}} = \cos\theta_B$. Therefore, $L'(x) = 0$ when $\cos\theta_C = \cos\theta_B$, or $\theta_C = \theta_B$, and ΔACE is similar to ΔABD. Simple proportions can again be used to determine the optimum x: $\frac{x}{h_C} = \frac{s - x}{h_B}$ $\Rightarrow x = \left(\frac{h_C}{h_B + h_C}\right)s$.

51. (a) $V(x) = 160x - 52x^2 + 4x^3$
$V'(x) = 160 - 104x + 12x^2 = 4(x - 2)(3x - 20)$
The only critical point in the interval (0, 5) is at x = 2. The maximum value of V(x) is 144 at x = 2.
(b) The largest possible volume of the box is 144 cubic units, and it occurs when x = 2.

52. (a) $P'(x) = 2 - 200x^{-2}$
The only critical point in the interval $(0, \infty)$ is at x = 10. The minimum value of P(x) is 40 at x = 10.
(b) The smallest possible perimeter of the rectangle is 40 units and it occurs at x = 10, which makes the rectangle a 10 by 10 square.

53. Let x represent the length of the base and $\sqrt{25 - x^2}$ the height of the triangle. The area of the triangle is represented by $A(x) = \frac{x}{2}\sqrt{25 - x^2}$ where $0 \le x \le 5$. Consequently, solving $A'(x) = 0 \Rightarrow \frac{25 - 2x^2}{2\sqrt{25 - x^2}} = 0$ $\Rightarrow x = \frac{5}{\sqrt{2}}$. Since $A(0) = A(5) = 0$, A(x) is maximized at $x = \frac{5}{\sqrt{2}}$. The largest possible area is $A\left(\frac{5}{\sqrt{2}}\right) = \frac{25}{4}$ cm^2.

54. (a) From the diagram the perimeter $P = 2x + 2\pi r = 400$ $\Rightarrow x = 200 - \pi r$. We wish to maximize the area $A = 2rx$ $\Rightarrow A(r) = 400r - 2\pi r^2$



(b) $A'(r) = 400 - 4\pi r$ and $A''(r) = -4\pi$. The critical point is $r = \frac{100}{\pi}$ and $A''\left(\frac{100}{\pi}\right) = -4\pi < 0$. There is a maximum at $r = \frac{100}{\pi}$. The values $x = 100$ m and $r = \frac{100}{\pi} \approx 31.83$ m maximize the area of the rectangle.

55. $s = -\frac{1}{2}gt^2 + v_0t + s_0 \Rightarrow \frac{ds}{dt} = -gt + v_0 = 0 \Rightarrow t = \frac{v_0}{g}$. Then $s\left(\frac{v_0}{g}\right) = -\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\left(\frac{v_0}{g}\right) + s_0$ $= \frac{v_0^2}{2g} + s_0$ is the maximum height since $\frac{d^2s}{dt^2} = -g < 0$.

56. $\frac{di}{dt} = -2 \sin t + 2 \cos t$, solving $\frac{di}{dt} = 0 \Rightarrow \tan t = 1 \Rightarrow t = \frac{\pi}{4} + n\pi$ where n is a nonnegative integer (in this exercise t is never negative) $\Rightarrow$ the peak current is $2\sqrt{2}$ amps

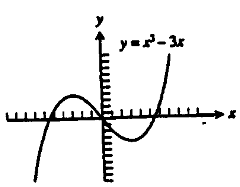
57. Yes, since $f(x) = |x| = \sqrt{x^2} = (x^2)^{1/2} \Rightarrow f'(x) = \frac{1}{2}(x^2)^{-1/2}(2x) = \frac{x}{(x^2)^{1/2}} = \frac{x}{|x|}$ is not defined at $x = 0$. Thus it is not required that f' be zero at a local extreme point since f' may be undefined there.

58. If f(c) is a local maximum value of f, then $f(x) \le f(c)$ for all x in some open interval (a, b) containing c. Since f is even, $f(-x) = f(x) \le f(c) = f(-c)$ for all $-x$ in the open interval $(-b, -a)$ containing $-c$. That is, f assumes a local maximum at the point $-c$. This is also clear from the graph of f because the graph of an even function is symmetric about the y-axis.

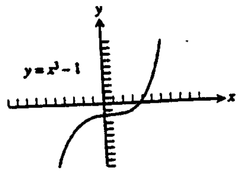
59. If g(c) is a local minimum value of g, then $g(x) \ge g(c)$ for all x in some open interval (a, b) containing c. Since g is odd, $g(-x) = -g(x) \le -g(c) = g(-c)$ for all $-x$ in the open interval $(-b, -a)$ containing $-c$. That is, g assumes a local maximum at the point $-c$. This is also clear from the graph of g because the graph of an odd function is symmetric about the origin.

60. If there are no boundary points or critical points the function will have no extreme values in its domain. Such functions do indeed exist, for example $f(x) = x$ for $-\infty < x < \infty$. (Any other linear function $f(x) = mx + b$ with $m \ne 0$ will do as well.)

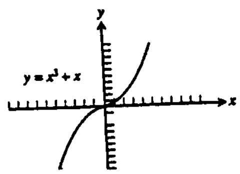
61. (a) $f'(x) = 3ax^2 + 2bx + c$ is a quadratic, so it can have 0, 1, or 2 zeros, which would be the critical points of f. Examples:



The function $f(x) = x^3 - 3x$ has two critical points at $x = -1$ and $x = 1$.



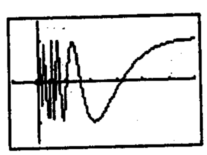
The function $f(x) = x^3 - 1$ has one critical point at $x = 0$.



The function $f(x) = x^3 + x$ has no critical points.

(b) The function can have either two local extreme values or no extreme values. (If there is only one critical point, the cubic function has no extreme values.)

62. (a)



[−0.1, 0.6] by [−1.5, 1.5]

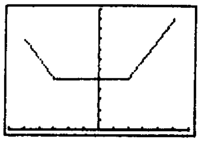
$f(0) = 0$ is not a local extreme value because in any open interval containing $x = 0$, there are infinitely many points where $f(x) = 1$ and where $f(x) = -1$.

(b) One possible answer, on the interval $[0, 1]$:

$$f(x) = \begin{cases} (1 - x)\cos\dfrac{1}{1 - x}, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

This function has no local extreme value at $x = 1$. Note that it is continuous on $[0, 1]$.

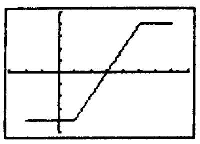
63.



[−6, 6] by [0, 12]

Maximum value is 11 at x = 5; minimum value is 5 on the interval [−3, 2]; local maximum is at (−5, 9)

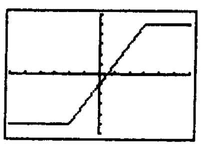
64.



[−3, 8] by [−5, 5]

Maximum value is 4 on the interval [5, 7]; minimum value is −4 on the interval [−2, 1].

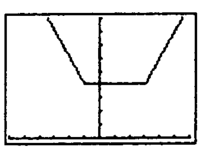
65.



[−6, 6] by [−6, 6]

Maximum value is 5 on the interval [3, ∞); minimum value is −5 on the interval (−∞, −2].

66.



[−6, 6] by [0, 9]

Minimum value is 4 on the interval [−1, 3]

67-74. Example CAS commands:

Maple:

```
f:=x -> 2 + 2*x - 3*(x ∧ 2) ∧ (1/3);
plot(f(x), x=-1..10/3);
fp:=diff(f(x),x);
solve(fp=0,x);
simplify(fp);
```

```
den:=denom(%);
solve(denom(fp)=0,x);
evalf([f(−1),f(0),f(1),f(10/3)]);
```

Mathematica:

```
Note: Here, use (x ∧ 2) ∧ (1/3) instead of x ∧ (2/3), to avoid complex roots for negative x
a = −1; b = 10/3; f[x_] = 2 + 2 x − 3 (x ∧ 2) ∧ (1/3)
f'[x]
Plot[ {f[x], f'[x]}, {x,a,b} ]
NSolve[f'[x]==0]
Note: include critical point x=0
{f[a], f[0], f[x] /. %, f[b]} // N
```

3.2 THE MEAN VALUE THEOREM AND DIFFERENTIAL EQUATIONS

1. (a) f is continuous on $[0,1]$ and differentiable on $(0,1)$.

   (b) $f'(c) = \frac{f(1)-f(0)}{1-0} \Rightarrow 2c + 2 = \frac{2-(-1)}{1} \Rightarrow 2c = 1 \Rightarrow c = \frac{1}{2}$

2. (a) f is continuous on $[0,1]$ and differentiable on $(0,1)$.

   (b) $f'(c) = \frac{f(1)-f(0)}{1-0} \Rightarrow \frac{2}{3}c^{-1/3} = \frac{1-0}{1} \Rightarrow c^{-1/3} = \frac{3}{2} \Rightarrow c = \left(\frac{3}{2}\right)^{-3} \Rightarrow c = \frac{8}{27}$

3. (a) f is continuous on $[-1,1]$ and differentiable on $(-1,1)$.

   (b) $f'(c) = \frac{f(1)-f(-1)}{1-(-1)} \Rightarrow \frac{1}{\sqrt{1-c^2}} = \frac{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)}{2} \Rightarrow \sqrt{1-c^2} = \frac{2}{\pi} \Rightarrow 1-c^2 = \frac{4}{\pi^2} \Rightarrow c^2 = 1-\frac{4}{\pi^2}$

   $\Rightarrow c = \pm\sqrt{1-\frac{4}{\pi^2}} \approx \pm 0.771$

4. (a) f is continuous on $[2,4]$ and differentiable on $(2,4)$.

   (b) $f'(c) = \frac{f(4)-f(2)}{4-2} \Rightarrow \frac{1}{c-1} = \frac{\ln 3 - \ln 1}{2} \Rightarrow c - 1 = \frac{2}{\ln 3} \Rightarrow c = 1 + \frac{2}{\ln 3} \approx 2.820$

5. Since f(x) is not continuous on $0 \le x \le 1$, Rolle's Theorem does not apply because $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} x = 1$ $\ne 0 = f(1)$ and f(x) is not continuous at $x = 1$.

6. Since f(x) must be continuous at $x = 0$ and $x = 1$ we have $\lim_{x\to 0^+} f(x) = a = f(0) \Rightarrow a = 3$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) \Rightarrow -1 + 3 + a = m + b \Rightarrow 5 = m + b$. Since f(x) must also be differentiable at $x = 1$ we have $\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^+} f'(x) \Rightarrow -2x + 3\big|_{x=1} = m\big|_{x=1} \Rightarrow 1 = m$. Therefore, $a = 3$, $m = 1$ and $b = 4$.

7. By Corollary 1, $f'(x) = 0$ for all $x \Rightarrow f(x) = C$, where C is a constant. Since $f(-1) = 3$ we have $C = 3$ $\Rightarrow f(x) = 3$ for all x.

8. $g(t) = 2t + 5 \Rightarrow g'(t) = 2 = f'(t)$ for all t. By Corollary 2, $f(t) = g(t) + C$ for some constant C. Then $f(0) = g(0) + C \Rightarrow 5 = 5 + C \Rightarrow C = 0 \Rightarrow f(t) = g(t) = 2t + 5$ for all t.

9. (a) $y = \frac{x^2}{2} + C$ (b) $y = \frac{x^3}{3} + C$ (c) $y = \frac{x^4}{4} + C$

10. (a) $y = x^2 + C$ (b) $y = x^2 - x + C$ (c) $y = x^3 + x^2 - x + C$

11. (a) $y = \ln \theta + C$ if $\theta > 0$ and $y = \ln(-\theta) + C$ if $\theta < 0$, where C is a constant. (These functions can be combined as $y = \ln|\theta| + C$.)
    (b) $y = \theta - \ln \theta + C$ if $\theta > 0$ and $y = \theta - \ln(-\theta) + C$ if $\theta < 0$, where C is a constant. (These functions can be combined as $y = \theta - \ln|\theta| + C$.)
    (c) $y = 5\theta + \ln \theta + C$ if $\theta > 0$ and $y = 5\theta + \ln(-\theta) + C$ if $\theta < 0$, where C is a constant. (These functions can be combined as $y = 5\theta + \ln|\theta| + C$.)

12. (a) $y' = \frac{1}{2}t^{-1/2} \Rightarrow y = t^{1/2} + C \Rightarrow y = \sqrt{t} + C$ (b) $y = 2\sqrt{t} + C$
    (c) $y = 2t^2 - 2\sqrt{t} + C$

13. $f(x) = x^2 - x + C$; $0 = f(0) = 0^2 - 0 + C \Rightarrow C = 0 \Rightarrow f(x) = x^2 - x$

14. $g(x) = \left\{ \begin{array}{ll} x^2 + \ln x + C & \text{if } x > 0 \\ x^2 + \ln(-x) + C & \text{if } x < 0 \end{array} \right\} = x^2 + \ln|x| + C$; $g(1) = -1 \Rightarrow 1^2 + \ln 1 + C = -1 \Rightarrow C = -2$
    $\Rightarrow g(x) = x^2 + \ln|x| - 2$

15. $f(x) = \frac{e^{2x}}{2} + C$; $f(0) = \frac{3}{2} \Rightarrow \frac{e^{2(0)}}{2} + C = \frac{3}{2} \Rightarrow C = 1 \Rightarrow f(x) = 1 + \frac{e^{2x}}{2}$

16. $r(t) = \sec t - t + C$; $0 = r(0) = \sec(0) - 0 + C \Rightarrow C = -1 \Rightarrow r(t) = \sec t - t - 1$

17. $v = \frac{ds}{dt} = 9.8t + 5 \Rightarrow s = 4.9t^2 + 5t + C$; at $s = 10$ and $t = 0$ we have $C = 10 \Rightarrow s = 4.9t^2 + 5t + 10$

18. $v = \frac{ds}{dt} = 32t - 2 \Rightarrow s = 16t^2 - 2t + C$; at $s = 4$ and $t = \frac{1}{2}$ we have $C = 1 \Rightarrow s = 16t^2 - 2t + 1$

19. $v = \frac{ds}{dt} = \sin(\pi t) \Rightarrow s = -\frac{1}{\pi}\cos(\pi t) + C$; at $s = 0$ and $t = 0$ we have $C = \frac{1}{\pi} \Rightarrow s = \frac{1 - \cos(\pi t)}{\pi}$

20. $v = \frac{ds}{dt} = \frac{1}{t+2} \Rightarrow s = \ln(t+2) + C$; at $s = \frac{1}{2}$ and $t = -1$ we have $C = \frac{1}{2} \Rightarrow s = \frac{1}{2} + \ln(t+2)$

21. $a = \frac{dv}{dt} = e^t \Rightarrow v = e^t + C$; at $v = 20$ and $t = 0$ we have $C = 19 \Rightarrow v = e^t + 19$
    $v = \frac{ds}{dt} = e^t + 19 \Rightarrow s = e^t + 19t + C$; at $s = 5$ and $t = 0$ we have $C = 4 \Rightarrow s = e^t + 19t + 4$

22. $a = 9.8 \Rightarrow v = 9.8t + C_1$; at $v = -3$ and $t = 0$ we have $C_1 = -3 \Rightarrow v = 9.8t - 3 \Rightarrow s = 4.9t^2 - 3t + C_2$; at $s = 0$ and $t = 0$ we have $C_2 = 0 \Rightarrow s = 4.9t^2 - 3t$

23. $a = -4\sin(2t) \Rightarrow v = 2\cos(2t) + C_1$; at $v = 2$ and $t = 0$ we have $C_1 = 0 \Rightarrow v = 2\cos(2t)$
    $\Rightarrow s = \sin(2t) + C_2$; at $s = -3$ and $t = 0$ we have $C_2 = -3 \Rightarrow s = \sin(2t) - 3$

24. $a = \frac{9}{\pi^2}\cos\left(\frac{3t}{\pi}\right) \Rightarrow v = \frac{3}{\pi}\sin\left(\frac{3t}{\pi}\right) + C_1$; at $v = 0$ and $t = 0$ we have $C_1 = 0 \Rightarrow v = \frac{3}{\pi}\sin\left(\frac{3t}{\pi}\right)$
$\Rightarrow s = -\cos\left(\frac{3t}{\pi}\right) + C_2$; at $s = -1$ and $t = 0$ we have $C_2 = 0 \Rightarrow s = -\cos\left(\frac{3t}{\pi}\right)$

25. $a(t) = v'(t) = 1.6 \Rightarrow v(t) = 1.6t + C$; at $(0,0)$ we have $C = 0 \Rightarrow v(t) = 1.6t$. When $t = 30$, then $v(30) = 48$ m/sec.

26. $a(t) = v'(t) = 20 \Rightarrow v(t) = 20t + C$; at $(0,0)$ we have $C = 0 \Rightarrow v(t) = 20t$. When $t = 60$, then $v(60) = 20(60)$ $= 1200$ m/sec.

27. $a(t) = v'(t) = 9.8 \Rightarrow v(t) = 9.8t + C_1$; at $(0,0)$ we have $C_1 = 0 \Rightarrow s'(t) = v(t) = 9.8t \Rightarrow s(t) = 4.9t^2 + C_2$; at $(0,0)$ we have $C_2 = 0 \Rightarrow s(t) = 4.9t^2$. Then $s(t) = 10 \Rightarrow t^2 = \frac{10}{4.9} \Rightarrow t = \sqrt{\frac{10}{4.9}}$, and $v\left(\sqrt{\frac{10}{4.9}}\right) = 9.8\sqrt{\frac{10}{4.9}}$
$= \frac{2(4.9)\sqrt{10}}{\sqrt{4.9}} = (2)\sqrt{4.9}\sqrt{10} = 14$ m/sec.

28. $a(t) = v'(t) = -3.72 \Rightarrow v(t) = -3.72t + C_1$; at $(0,93)$ we have $C_1 = 93 \Rightarrow s'(t) = v(t) = -3.72t + 93$
$\Rightarrow s(t) = -1.86t^2 + 93t + C_2$; at $(0,0)$ we have $C_2 = 0 \Rightarrow s(t) = -1.86t^2 + 93t$. Then $v(t) = 0 \Rightarrow t = \frac{93}{3.72} = 25$ so the maximum height of the rock is $s(25) = 1162.5$ m.

29. (a) $v = \int a\,dt = \int \left(15t^{1/2} - 3t^{-1/2}\right) dt = 10t^{3/2} - 6t^{1/2} + C$; $\frac{ds}{dt}(1) = 4 \Rightarrow 4 = 10(1)^{3/2} - 6(1)^{1/2} + C \Rightarrow C = 0$
$\Rightarrow v = 10t^{3/2} - 6t^{1/2}$

(b) $s = \int v\,dt = \int \left(10t^{3/2} - 6t^{1/2}\right) dt = 4t^{5/2} - 4t^{3/2} + C$; $s(1) = 0 \Rightarrow 0 = 4(1)^{5/2} - 4(1)^{3/2} + C \Rightarrow C = 0$
$\Rightarrow s = 4t^{5/2} - 4t^{3/2}$

30. (a) $\frac{ds}{dt} = 9.8t - 3 \Rightarrow s = 4.9t^2 - 3t + C$; i) at $s = 5$ and $t = 0$ we have $C = 5 \Rightarrow s = 4.9t^2 - 3t + 5$; displacement $= s(3) - s(1) = [(4.9)(9) - 9 + 5] - (4.9 - 3 + 5) = 33.2$ units; ii) at $s = -2$ and $t = 0$ we have $C = -2 \Rightarrow s = 4.9t^2 - 3t - 2$; displacement $= s(3) - s(1) = ((4.9)(9) - 9 - 2) - (4.9 - 3 - 2) = 33.2$ units; iii) at $s = s_0$ and $t = 0$ we have $C = s_0 \Rightarrow s = 4.9t^2 - 3t + s_0$; displacement $= s(3) - s(1)$
$= ((4.9)(9) - 9 + s_0) - (4.9 - 3 + s_0) = 33.2$ units

(b) True. Given an antiderivative $f(t)$ of the velocity function, we know that the body's position function is $s = f(t) + C$ for some constant C. Therefore, the displacement from $t = a$ to $t = b$ is $(f(b) + C) - (f(a) + C)$ $= f(b) - f(a)$. Thus we can find the displacement from any antiderivative f as the numerical difference $f(b) - f(a)$ without knowing the exact values of C and s.

31. If $T(t)$ is the temperature of the thermometer at time t, then $T(0) = -19°$ C and $T(14) = 100°$ C. From the Mean Value Theorem there exists a $0 < t_0 < 14$ such that $\frac{T(14) - T(0)}{14 - 0} = 8.5°$ C/sec $= T'(t_0)$, the rate at which the temperature was changing at $t = t_0$ as measured by the rising mercury on the thermometer.

32. Because the trucker's average speed was 79.5 mph, and by the Mean Value Theorem, the trucker must have been going that speed at least once during the trip.

33. Because its average speed was approximately 7.667 knots, and by the Mean Value Theorem, it must have been going that speed at least once during the trip.

34. The runner's average speed for the marathon was approximately 11.909 mph. Therefore, by the Mean Value Theorem, the runner must have been going that speed at least once during the marathon. Since the initial speed and final speed are both 0 mph and the runner's speed is continuous, by the Intermediate Value Theorem, the runner's speed must have been 11 mph at least twice.

35. The conclusion of the Mean Value Theorem yields $\frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{c^2} \Rightarrow c^2\left(\frac{a-b}{ab}\right) = a-b \Rightarrow c = \sqrt{ab}$.

36. The conclusion of the Mean Value Theorem yields $\frac{b^2-a^2}{b-a} = 2c \Rightarrow c = \frac{a+b}{2}$.

37. $f'(x) = [\cos x \sin(x+2) + \sin x \cos(x+2)] - 2\sin(x+1)\cos(x+1) = \sin(x+x+2) - \sin 2(x+1)$ $= \sin(2x+2) - \sin(2x+2) = 0$. Therefore, the function has the constant value $f(0) = -\sin^2 1 \approx -0.7081$ which explains why the graph is a horizontal line.

38. Example CAS commands:

<u>Maple</u>:

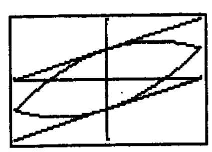
```
(x+2)*(x+1)*x*(x-1)*(x-2);
expand(%);
f:=unapply(%,x);
plot({f(x),diff(f(x),x)},x=-2..2);
```

<u>Mathematica</u>:

```
(x+2) (x+1)x(x-1) (x-2)
Expand[%]
f[x_] = %
Plot[ {f[x],f'[x]}, {x,-2,2} ]
```

39. f(x) must be zero at least once between a and b by the Intermediate Value Theorem. Now suppose that f(x) is zero twice between a and b. Then by the Mean Value Theorem, f'(x) would have to be zero at least once between the two zeros of f(x), but this can't be true since we are given that $f'(x) \neq 0$ on this interval. Therefore, f(x) is zero once and only once between a and b.

40. Consider the function $k(x) = f(x) - g(x)$. $k(x)$ is continuous and differentiable on $[a, b]$, and since $k(a) = f(a) - g(a) = 0$ and $k(b) = f(b) - g(b) = 0$, by the Mean Value Theorem, there must be a point c in (a, b) where $k'(c) = 0$. But since $k'(c) = f'(c) - g'(c)$, this means that $f'(c) = g'(c)$, and c is a point where the graphs of f and g have parallel or identical tangent lines.


[−1, 1] by [−2, 2]

41. Yes. By Corollary 2 we have $f(x) = g(x) + C$ since $f'(x) = g'(x)$. If the graphs start at the same point $x = a$, then $f(a) = g(a) \Rightarrow C = 0 \Rightarrow f(x) = g(x)$.

42. Let $f(x) = \sin x$ for $a \le x \le b$. From the Mean Value Theorem there exists a c between a and b such that $\frac{\sin b - \sin a}{b - a} = \cos c \Rightarrow -1 \le \frac{\sin b - \sin a}{b - a} \le 1 \Rightarrow \left|\frac{\sin b - \sin a}{b - a}\right| \le 1 \Rightarrow |\sin b - \sin a| \le |b - a|$.

43. By the Mean Value Theorem, $\frac{f(b) - f(a)}{b - a} = f'(c)$ for some point c between a and b. Since $b - a > 0$ and $f(b) < f(a)$, we have $f(b) - f(a) < 0 \Rightarrow f'(c) < 0$.

44. The condition is that f' should be continuous over $[a, b]$. The Mean Value Theorem then guarantees the existence of a point c in (a, b) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$. If f' is continuous, then it has a minimum and maximum value on $[a, b]$, and $\min f' \le f'(c) \le \max f'$, as required.

45. $f'(x) = \left(1 + x^4 \cos x\right)^{-1} \Rightarrow f''(x) = -\left(1 + x^4 \cos x\right)^{-2}\left(4x^3 \cos x - x^4 \sin x\right)$
$= -x^3\left(1 + x^4 \cos x\right)^{-2}(4 \cos x - x \sin x) < 0$ for $0 \le x \le 0.1 \Rightarrow f'(x)$ is decreasing when $0 \le x \le 0.1$
$\Rightarrow \min f' \approx 0.9999$ and $\max f' = 1$. Now we have $0.9999 \le \frac{f(0.1) - 1}{0.1} \le 1 \Rightarrow 0.09999 \le f(0.1) - 1 \le 0.1$
$\Rightarrow 1.09999 \le f(0.1) \le 1.1$.

46. $f'(x) = \left(1 - x^4\right)^{-1} \Rightarrow f''(x) = -\left(1 - x^4\right)^{-2}\left(-4x^3\right) = \frac{4x^3}{\left(1 - x^4\right)^2} > 0$ for $0 < x \le 0.1 \Rightarrow f'(x)$ is increasing when $0 \le x \le 0.1 \Rightarrow \min f' = 1$ and $\max f' = 1.0001$. Now we have $1 \le \frac{f(0.1) - 2}{0.1} \le 1.0001$
$\Rightarrow 0.1 \le f(0.1) - 2 \le 0.10001 \Rightarrow 2.1 \le f(0.1) \le 2.10001$.

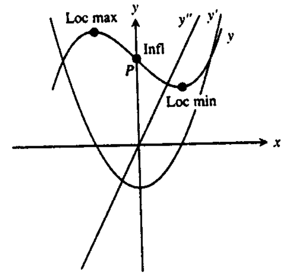
47-50. Example CAS commands

Maple:
```
with(plots): with(DEtools):
a:=0;b:=1;
eq:= D(y) (x)=x*sqrt(1−x);
sol:= dsolve({eq},y(x));
tograph:={seq(subs(_C1=i,sol),i={−2,−1,−,1,2})};
plot1:= implicitplot(tograph,x=a..b,y=−6..6):
display({plot1});
partsol:=dsolve({eq,y(1/2)=1},y(x));
implicitplot(partsol,x=a..b,y=−6..6,scaling=CONSTRAINED);
```

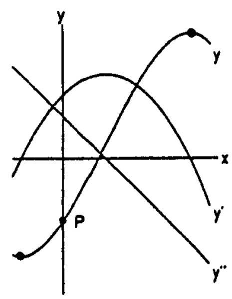
Mathematica:
```
a=0;b=1;
eq=D[y[x],x] = x*Sqrt[1−x]
sol=Flatten[DSolve[eq,y[x],x]]
cvals={−2,−1,1,2};
tograph=Table[y[x] /. (sol /. C[1] → cvals[[i]]), {i,1,4}]
Plot[Evaluate[tograph],{x,a,b}];
partsol = DSolve[{eq,y[1/2]=1},y[x],x]//Flatten
Plot[y[x] /. partsol,{x,a,b}]
```

3.3 THE SHAPE OF A GRAPH

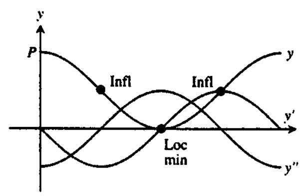
1. The graph of $y = f''(x) \Rightarrow$ the graph of $y = f(x)$ is concave up on $(0, \infty)$, concave down on $(-\infty, 0) \Rightarrow$ a point of inflection at $x = 0$; the graph of $y = f'(x) \Rightarrow y' = +++ \mid --- \mid +++ \Rightarrow$ the graph $y = f(x)$ has both a local maximum and a local minimum



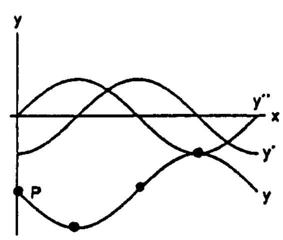
2. The graph of $y = f''(x) \Rightarrow y'' = +++ \mid --- \Rightarrow$ the graph of $y = f(x)$ has a point of inflection, the graph of $y = f'(x)$ $\Rightarrow y' = --- \mid +++ \mid --- \Rightarrow$ the graph of $y = f(x)$ has both a local maximum and a local minimum

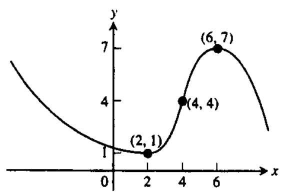


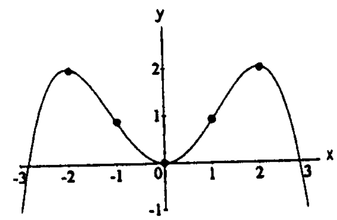
3. The graph of $y = f''(x) \Rightarrow y'' = --- \mid +++ \mid --- \Rightarrow$ the graph of $y = f(x)$ has two points of inflection, the graph of $y = f'(x) \Rightarrow y' = --- \mid +++ \Rightarrow$ the graph of $y = f(x)$ has a local minimum



4. The graph of $y = f''(x) \Rightarrow y'' = +++ \mid --- \Rightarrow$ the graph of $y = f(x)$ has a point of inflection; the graph of $y = f'(x)$ $\Rightarrow y' = --- \mid +++ \mid --- \Rightarrow$ the graph of $y = f(x)$ has both a local maximum and a local minimum



5. 

6. 

7. (a) Zero: $x = \pm 1$;
positive: $(-\infty, -1)$ and $(1, \infty)$;
negative: $(-1, 1)$
(b) Zero: $x = 0$;
positive: $(0, \infty)$;
negative: $(-\infty, 0)$

8. (a) Zero: $x \approx 0, \ \pm 1.25$;
positive: $(-1.25, 0)$ and $(1.25, \infty)$;
negative: $(-\infty, -1.25)$ and $(0, 1.25)$
(b) Zero: $x \approx \pm 0.7$;
positive: $(-\infty, -0.7)$ and $(0.7, \infty)$;
negative: $(-0.7, 0.7)$

9. (a) $(-\infty, -2]$ and $[0, 2]$
(b) $[-2, 0]$ and $[2, \infty)$
(c) Local maxima: $x = -2$ and $x = 2$;
local minimum: $x = 0$

10. (a) $[-2, 2]$
(b) $(-\infty, -2]$ and $[2, \infty)$
(c) Local maximum: $x = 2$;
local minimum: $x = -2$

11. (a) $[0, 1]$, $[3, 4]$, and $[5.5, 6]$
(b) $[1, 3]$ and $[4, 5.5]$
(c) Local maxima: $x = 1$, $x = 4$ (if f is continuous at $x = 4$), and $x = 6$;
local minima: $x = 0$, $x = 3$, and $x = 5.5$

12. If f is continuous on the interval $[0, 3]$;
(a) $[0, 3]$
(b) Nowhere
(c) Local maximum: $x = 3$;
local minimum: $x = 0$

13. (a) $f'(x) = (x-1)(x+2) \Rightarrow$ critical points at -2 and 1
(b) $f' = +++\underset{-2}{|}---\underset{1}{|}+++ \Rightarrow$ increasing on $(-\infty, -2]$ and $[1, \infty)$, decreasing on $[-2, 1]$
(c) Local maximum at $x = -2$ and a local minimum at $x = 1$

14. (a) $f'(x) = (x-1)^2(x+2) \Rightarrow$ critical points at -2 and 1
(b) $f' = ---\underset{-2}{|}+++\underset{1}{|}+++ \Rightarrow$ increasing on $[-2, 1]$ and $[1, \infty)$, decreasing on $(-\infty, -2]$
(c) No local maximum and a local minimum at $x = -2$

15. (a) $f'(x) = (x-1)e^{-x} \Rightarrow$ critical point at $x = 1$
(b) $f' = ----\underset{1}{|}++++ \Rightarrow$ decreasing on $(-\infty, 1]$, increasing on $[1, \infty)$
(c) Local (and absolute) minimum at $x = 1$

16. (a) $f'(x) = x^{-1/3}(x+2) \Rightarrow$ critical points at -2 and 0
(b) $f' = +++\underset{-2}{|}---\underset{0}{)(}+++ \Rightarrow$ increasing on $(-\infty, -2]$ and $[0, \infty)$, decreasing on $[-2, 0]$

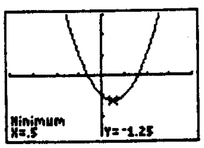
(c) Local maximum at $x = -2$, local minimum at $x = 0$

17. $y' = 2x - 1$

Intervals	$x < \frac{1}{2}$	$x > \frac{1}{2}$
Sign of y'	$-$	$+$
Behavior of y	Decreasing	Increasing

$y'' = 2$ (always positive: concave up)

Graphical support:



[−4, 4] by [−3, 3]

(a) $\left[\frac{1}{2}, \infty\right)$

(b) $\left(-\infty, \frac{1}{2}\right]$

(c) $(-\infty, \infty)$

(d) Nowhere

(e) Local (and absolute) minimum at $\left(\frac{1}{2}, -\frac{5}{4}\right)$

(f) None

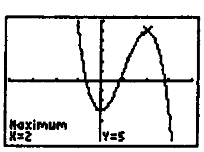
18. $y' = -6x^2 + 12x = -6x(x-2)$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of y'	$-$	$+$	$-$
Behavior of y	Decreasing	Increasing	Decreasing

$y'' = -12x + 12 = -12(x-1)$

Intervals	$x < 1$	$x > 1$
Sign of y''	$+$	$-$
Behavior of y	Concave up	Concave down

Graphical support:



[−4, 4] by [−6, 6]

(a) $[0, 2]$

(b) $(-\infty, 0]$ and $[2, \infty)$

(c) $(-\infty, 1)$

(d) $(1, \infty)$

(e) Local maximum: $(2,5)$;
local minimum: $(0,-3)$

(f) At $(1,1)$

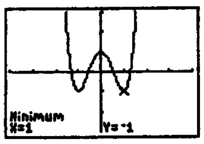
19. $y' = 8x^3 - 8x = 8x(x-1)(x+1)$

Intervals	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
Sign of y'	$-$	$+$	$-$	$+$
Behavior of y	Decreasing	Increasing	Decreasing	Increasing

$y'' = 24x^2 - 8 = 8(\sqrt{3}x - 1)(\sqrt{3}x + 1)$

Intervals	$x < -\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}} < x$
Sign of y''	$+$	$-$	$+$
Behavior of y	Concave up	Concave down	Concave up

Graphical support:



$[-4, 4]$ by $[-3, 3]$

(a) $[-1,0]$ and $[1,\infty)$

(b) $(-\infty,1]$ and $[0,1]$

(c) $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$

(d) $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

(e) Local maximum: $(0,1)$;
local (and absolute) minima: $(-1,-1)$ and $(1,-1)$

(f) $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{9}\right)$

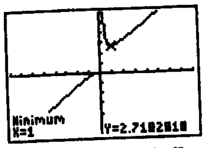
20. $y' = xe^{1/x}(-x^{-2}) + e^{1/x} = e^{1/x}\left(1 - \frac{1}{x}\right)$

Intervals	$x < 0$	$0 < x < 1$	$1 < x$
Sign of y'	$+$	$-$	$+$
Behavior of y	Increasing	Decreasing	Increasing

$y'' = e^{1/x}(x^{-2}) + \left(1 - \frac{1}{x}\right)e^{1/x}(-x^{-2}) = \frac{e^{1/x}}{x^3}$

Intervals	$x < 0$	$x > 0$
Sign of y''	$-$	$+$
Behavior of y	Concave down	Concave down

Graphical support:



[−8, 8] by [−6, 6]

(a) $(-\infty, 0)$ and $[1, \infty)$
(b) $(0, 1]$
(c) $(0, \infty)$
(d) $(-\infty, 0)$
(e) Local minimum: $(1, e)$
(f) None

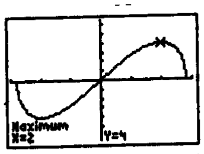
21. $y' = x\dfrac{1}{2\sqrt{8-x^2}}(-2x) + \left(\sqrt{8-x^2}\right)(1) = \dfrac{8-2x^2}{\sqrt{8-x^2}}$

Intervals	$-\sqrt{8} < x < -2$	$-2 < x < 2$	$2 < x < \sqrt{8}$
Sign of y′	−	+	−
Behavior of y	Decreasing	Increasing	Decreasing

$$y'' = \frac{\left(\sqrt{8-x^2}\right)(-4x) - (8-2x^2)\dfrac{1}{2\sqrt{8-x^2}}(-2x)}{\left(\sqrt{8-x^2}\right)^2} = \frac{2x^3-24x}{(8-x^2)^{3/2}} = \frac{2x(x^2-12)}{(8-x^2)^{3/2}}$$

Intervals	$-\sqrt{8} < x < 0$	$0 < x < \sqrt{8}$
Sign of y′	+	−
Behavior of y	Concave up	Concave down

Graphical support:



[−3.02, 3.02] by [−6.5, 6.5]

(a) $[-2, 2]$
(b) $\left[-\sqrt{8}, -2\right]$ and $\left[2, \sqrt{8}\right]$
(c) $\left(-\sqrt{8}, 0\right)$
(d) $\left(0, \sqrt{8}\right)$
(e) Local maxima: $\left(-\sqrt{8}, 0\right)$ and $(2, 4)$;
local minima: $(-2, -4)$ and $\left(\sqrt{8}, 0\right)$
Note that the local extrema at $x = \pm 2$ are also absolute extrema
(f) $(0, 0)$

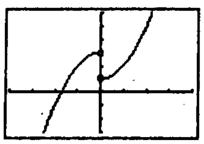
22. $y' = \begin{cases} -2x, & x < 0 \\ 2x, & x > 0 \end{cases}$

Intervals	$x < 0$	$x > 0$
Sign of y'	$+$	$+$
Behavior of y	Increasing	Increasing

$y'' = \begin{cases} -2, & x < 0 \\ 2, & x > 0 \end{cases}$

Intervals	$x < 0$	$x > 0$
Sign of y''	$-$	$+$
Behavior of y	Concave down	Concave up

Graphical support:



[−4, 4] by [−3, 6]

(a) $(-\infty, \infty)$

(b) None

(c) $(0, \infty)$

(d) $(-\infty, 0)$

(e) Local minimum: $(0, 1)$

(f) Note that $(0, 1)$ is not an inflection point because the graph has no tangent line at this point. There are no inflection points.

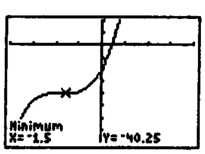
23. $y' = 12x^2 + 42x + 36 = 6(x+2)(2x+3)$

Intervals	$x < -2$	$-2 < x < -\frac{3}{2}$	$-\frac{3}{2} < x$
Sign of y'	$+$	$-$	$+$
Behavior of y	Increasing	Decreasing	Increasing

$y'' = 24x + 42 = 6(4x + 7)$

Intervals	$x < -\frac{7}{4}$	$-\frac{7}{4} < x$
Sign of y''	$-$	$+$
Behavior of y	Concave down	Concave up

Graphical support:



[−4, 4] by [−80, 20]

(a) $(-\infty, -2]$ and $\left[-\frac{3}{2}, \infty\right)$

(b) $\left[-2, -\frac{3}{2}\right]$

(c) $\left(-\frac{7}{4}, \infty\right)$

(d) $\left(-\infty, -\frac{7}{4}\right)$

(e) Local maximum: $(-2, -40)$; local minimum: $\left(-\frac{3}{2}, -\frac{161}{4}\right)$

(f) $\left(-\frac{7}{4}, -\frac{321}{8}\right)$

24. $y' = -4x^3 + 12x^2 - 4$

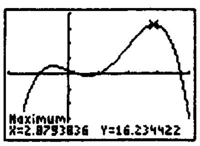
Using grapher techniques, the zeros of y' are $x \approx -0.53$, $x \approx 0.65$, and $x \approx 2.88$.

Intervals	$x < -0.53$	$-0.53 < x < 0.65$	$0.65 < x < 2.88$	$2.88 < x$
Sign of y'	+	−	+	−
Behavior of y	Increasing	Decreasing	Increasing	Decreasing

$y'' = -12x^2 + 24x = -12x(x-2)$

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of y''	−	+	−
Behavior of y	Concave down	Concave up	Concave down

Graphical support:



[−2, 4] by [−20, 20]

(a) $(-\infty, -0.53]$ and $[0.65, 2.88]$

(b) $[-0.53, 0.65]$ and $[2.88, \infty)$

(c) $(0, 2)$

(d) $(-\infty, 0)$ and $(2, \infty)$

(e) Local maxima: $(-0.53, 2.45)$ and $(2.88, 16.23)$; local minimum: $(0.65, -0.68)$
Note that the local maximum at $x \approx 2.88$ is also an absolute maximum.

(f) $(0, 1)$ and $(2, 9)$

25. $y' = \frac{2}{5}x^{-4/5}$

Intervals	$x < 0$	$0 < x$
Sign of y'	+	+
Behavior of y	Increasing	Increasing

$y'' = -\frac{8}{25}x^{-9/5}$

Intervals	$x < 0$	$0 < x$
Sign of y''	+	−
Behavior of y	Concave up	Concave down