CHAPTER 4 INTEGRATION

4.1 INDEFINITE INTEGRALS

1. (a) $3x^2$ (b) $\frac{x^8}{8}$ (c) $\frac{x^8}{8} - 3x^2 + 8x$

2. (a) x^{-3} (b) $-\frac{x^{-3}}{3}$ (c) $-\frac{x^{-3}}{3} + x^2 + 3x$

3. (a) $\frac{1}{x^2}$ (b) $\frac{-1}{4x^2}$ (c) $\frac{x^4}{4} + \frac{1}{2x^2}$

4. (a) $\sqrt{x^3}$ (b) $\sqrt{x}$ (c) $\frac{2}{3}\sqrt{x^3} + 2\sqrt{x}$

5. (a) $x^{2/3}$ (b) $x^{1/3}$ (c) $x^{-1/3}$

6. (a) $\cos(\pi x)$ (b) $-3\cos x$ (c) $\frac{-\cos(\pi x)}{\pi} + \cos(3x)$

7. (a) $\tan x$ (b) $2\tan\left(\frac{x}{3}\right)$ (c) $-\frac{2}{3}\tan\left(\frac{3x}{2}\right)$

8. (a) $\sec x$ (b) $\frac{4}{3}\sec(3x)$ (c) $\frac{2}{\pi}\sec\left(\frac{\pi x}{2}\right)$

9. $\int (x+1)\, dx = \frac{x^2}{2} + x + C$

10. $\int \left(3t^2 + \frac{t}{2}\right) dt = t^3 + \frac{t^2}{4} + C$

11. $\int \left(\frac{1}{x} - \frac{5}{x^2+1}\right) dx = \ln|x| - 5\tan^{-1} x + C$

12. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx = \int \left(x^{-2} - x^2 - \frac{1}{3}\right) dx = \frac{x^{-1}}{-1} - \frac{x^3}{3} - \frac{1}{3}x + C = -\frac{1}{x} - \frac{x^3}{3} - \frac{x}{3} + C$

13. $\int (e^{-x} + 4^x)\, dx = -e^{-x} + \frac{4^x}{\ln 4} + C$

14. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx = \int \left(x^{1/2} + x^{1/3}\right) dx = \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{4/3}}{\frac{4}{3}} + C = \frac{2}{3}x^{3/2} + \frac{3}{4}x^{4/3} + C$

15. $\int \left(\frac{2}{\sqrt{1-y^2}} - \frac{1}{y^{1/4}}\right) dy = 2\sin^{-1} y - \frac{4}{3}y^{3/4} + C$

$$\int \frac{dx}{2x-1} = \int \frac{du}{2u} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln|2x-1|+C = \ln\sqrt{|2x-1|}+C$$

30. Let $u = x^2+4 \Rightarrow du = 2x\,dx \Rightarrow \frac{du}{2} = x\,dx$

$$\int \frac{x\,dx}{x^2+4} = \int \frac{du}{2u} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln|x^2+4|+C = \frac{1}{2}\ln(x^2+4)+C = \ln\sqrt{x^2+4}+C$$

31. Let $u = 2-\cos t \Rightarrow du = \sin t\,dt$

$$\int \frac{\sin t}{2-\cos t}\,dt = \int \frac{du}{u} = \ln|u|+C = \ln|2-\cos t|+C$$

32. $\int \frac{1}{\sin t\cos t}\,dt = \int \sec t\csc t\,dt = \sec t + C$

33. $\int \frac{1}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\int \frac{2}{\sqrt{1-(2x)^2}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$, where $u = 2x$ and $du = 2\,dx$

$$= \frac{1}{2}\sin^{-1}u + C = \frac{1}{2}\sin^{-1}(2x)+C$$

34. $\int \frac{dx}{9+3x^2} = \frac{1}{9}\int \frac{dx}{1+3x^2/9}$; Let $u = \frac{\sqrt{3}x}{3} \Rightarrow du = \frac{\sqrt{3}}{3}dx \Rightarrow \sqrt{3}\,du = dx$;

$$\frac{1}{9}\int \frac{dx}{1+3x^2/9} = \left(\frac{\sqrt{3}}{9}\right)\int \frac{du}{1+u^2} = \frac{\sqrt{3}}{9}\tan^{-1}u + C = \frac{\sqrt{3}}{9}\tan^{-1}\frac{\sqrt{3}x}{3}+C = \frac{1}{3\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+C$$

35. $\int \frac{dx}{x\sqrt{25x^2-2}} = \int \frac{du}{u\sqrt{u^2-2}}$, where $u = 5x$ and $du = 5\,dx$

$$= \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{u}{\sqrt{2}}\right|+C = \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{5x}{\sqrt{2}}\right|+C$$

36. $\int \frac{3\,dr}{\sqrt{1-4(r-1)^2}} = \frac{3}{2}\int \frac{du}{\sqrt{1-u^2}}$, where $u = 2(r-1)$ and $du = 2\,dr$

$$= \frac{3}{2}\sin^{-1}u + C = \frac{3}{2}\sin^{-1}2(r-1)+C$$

37. $\int \frac{dx}{1+(3x+1)^2} = \frac{1}{3}\int \frac{du}{1+u^2}$, where $u = 3x+1$ and $du = 3\,dx$

$$= \frac{1}{3}\tan^{-1}u + C = \frac{1}{3}\tan^{-1}(3x+1)+C$$

38. $\int \frac{y\,dy}{\sqrt{1-y^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$, where $u = y^2$ and $du = 2y\,dy$

$= \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1} y^2 + C$

39. Let $u = e^x \Rightarrow du = e^x\,dx$; $\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{e^x}{1+(e^x)^2}\,dx = \int \frac{du}{1+u^2} = \tan^{-1} u + C = \tan^{-1}(e^x) + C$

40. Let $u = \ln t \Rightarrow du = \frac{1}{t}\,dt$; $\int \frac{4\,dt}{t(1+\ln^2 t)} = 4\int \frac{du}{1+u^2} = 4\tan^{-1} u + C = 4\tan^{-1}(\ln t) + C$

41. (a) Let $u = \tan x \Rightarrow du = \sec^2 x\,dx$; $v = u^3 \Rightarrow dv = 3u^2\,du \Rightarrow 6\,dv = 18u^2\,du$; $w = 2 + v \Rightarrow dw = dv$

$$\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx = \int \frac{18u^2}{(2+u^3)^2}\,du = \int \frac{6\,dv}{(2+v)^2} = \int \frac{6\,dw}{w^2} = 6\int w^{-2}\,dw = -6w^{-1} + C = -\frac{6}{2+v} + C$$

$$= -\frac{6}{2+u^3} + C = -\frac{6}{2+\tan^3 x} + C$$

(b) Let $u = \tan^3 x \Rightarrow du = 3\tan^2 x\sec^2 x\,dx \Rightarrow 6\,du = 18\tan^2 x\sec^2 x\,dx$; $v = 2 + u \Rightarrow dv = du$

$$\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx = \int \frac{6\,du}{(2+u)^2} = \int \frac{6\,dv}{v^2} = -\frac{6}{v} + C = -\frac{6}{2+u} + C = -\frac{6}{2+\tan^3 x} + C$$

(c) Let $u = 2 + \tan^3 x \Rightarrow du = 3\tan^2 x\sec^2 x\,dx \Rightarrow 6\,du = 18\tan^2 x\sec^2 x\,dx$

$$\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx = \int \frac{6\,du}{u^2} = -\frac{6}{u} + C = -\frac{6}{2+\tan^3 x} + C$$

42. (a) Let $u = x - 1 \Rightarrow du = dx$; $v = \sin u \Rightarrow dv = \cos u\,du$; $w = 1 + v^2 \Rightarrow dw = 2v\,dv \Rightarrow \frac{1}{2}\,dw = v\,dv$

$$\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx = \int \sqrt{1+\sin^2 u}\,\sin u\cos u\,du = \int v\sqrt{1+v^2}\,dv$$

$$= \int \frac{1}{2}\sqrt{w}\,dw = \frac{1}{3}w^{3/2} + C = \frac{1}{3}(1+v^2)^{3/2} + C = \frac{1}{3}(1+\sin^2 u)^{3/2} + C = \frac{1}{3}(2+\sin^2(x-1))^{3/2} + C$$

(b) Let $u = \sin(x-1) \Rightarrow du = \cos(x-1)\,dx$; $v = 1 + u^2 \Rightarrow dv = 2u\,du \Rightarrow \frac{1}{2}\,dv = u\,du$

$$\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx = \int u\sqrt{1+u^2}\,du = \int \frac{1}{2}\sqrt{v}\,dv = \int \frac{1}{2}v^{1/2}\,dv$$

$$= \left(\frac{1}{2}\left(\frac{2}{3}\right)v^{3/2}\right) + C = \frac{1}{3}v^{3/2} + C = \frac{1}{3}(1+u^2)^{3/2} + C = \frac{1}{3}(1+\sin^2(x-1))^{3/2} + C$$

(c) Let $u = 1 + \sin^2(x-1) \Rightarrow du = 2\sin(x-1)\cos(x-1)\,dx \Rightarrow \frac{1}{2}\,du = \sin(x-1)\cos(x-1)\,dx$

$$\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx = \int \frac{1}{2}\sqrt{u}\,du = \int \frac{1}{2}u^{1/2}\,du = \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{1}{3}(1+\sin^2(x-1))^{3/2} + C$$

43. Let $u = 3(2r-1)^2 + 6 \Rightarrow du = 6(2r-1)(2)\,dr \Rightarrow \frac{1}{12}\,du = (2r-1)\,dr$; $v = \sqrt{u} \Rightarrow dv = \frac{1}{2\sqrt{u}}\,du \Rightarrow \frac{1}{6}\,dv$
$= \frac{1}{12\sqrt{u}}\,du$

$$\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr = \int \left(\frac{\cos\sqrt{u}}{\sqrt{u}}\right)\left(\frac{1}{12}\,du\right) = \int (\cos v)\left(\frac{1}{6}\,dv\right) = \frac{1}{6}\sin v + C = \frac{1}{6}\sin\sqrt{u} + C$$
$$= \frac{1}{6}\sin\sqrt{3(2r-1)^2+6} + C$$

44. Let $u = \cos\sqrt{\theta} \Rightarrow du = \left(-\sin\sqrt{\theta}\right)\left(\frac{1}{2\sqrt{\theta}}\right)d\theta \Rightarrow -2\,du = \frac{\sin\sqrt{\theta}}{\sqrt{\theta}}\,d\theta$

$$\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta = \int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\sqrt{\cos^3\sqrt{\theta}}}\,d\theta = \int \frac{-2\,du}{u^{3/2}} = -2\int u^{-3/2}\,du = -2\left(-2u^{-1/2}\right) + C = \frac{4}{\sqrt{u}} + C$$
$$= \frac{4}{\sqrt{\cos\sqrt{\theta}}} + C$$

45. Let $u = 3t^2 - 1 \Rightarrow du = 6t\,dt \Rightarrow 2\,du = 12t\,dt$

$s = \int 12t\left(3t^2-1\right)^3 dt = \int u^3\,(2\,du) = 2\left(\frac{1}{4}u^4\right) + C = \frac{1}{2}u^4 + C = \frac{1}{2}\left(3t^2-1\right)^4 + C$;

$s = 3$ when $t = 1 \Rightarrow 3 = \frac{1}{2}(3-1)^4 + C \Rightarrow 3 = 8 + C \Rightarrow C = -5 \Rightarrow s = \frac{1}{2}\left(3t^2-1\right)^4 - 5$

46. Let $u = x^2 + 8 \Rightarrow du = 2x\,dx \Rightarrow 2\,du = 4x\,dx$

$y = \int 4x\left(x^2+8\right)^{-1/3} dx = \int u^{-1/3}\,(2\,du) = 2\left(\frac{3}{2}u^{2/3}\right) + C = 3u^{2/3} + C = 3\left(x^2+8\right)^{2/3} + C$;

$y = 0$ when $x = 0 \Rightarrow 0 = 3(8)^{2/3} + C \Rightarrow C = -12 \Rightarrow y = 3\left(x^2+8\right)^{2/3} - 12$

47. Let $u = t + \frac{\pi}{12} \Rightarrow du = dt$

$s = \int 8\sin^2\left(t + \frac{\pi}{12}\right) dt = \int 8\sin^2 u\,du = 8\left(\frac{u}{2} - \frac{1}{4}\sin 2u\right) + C = 4\left(t + \frac{\pi}{12}\right) - 2\sin\left(2t + \frac{\pi}{6}\right) + C$;

$s = 8$ when $t = 0 \Rightarrow 8 = 4\left(\frac{\pi}{12}\right) - 2\sin\left(\frac{\pi}{6}\right) + C \Rightarrow C = 8 - \frac{\pi}{3} + 1 = 9 - \frac{\pi}{3} \Rightarrow s = 4t - 2\sin\left(2t + \frac{\pi}{6}\right) + 9$

48. $\frac{dy}{dx} = 1 + \frac{1}{x}$ at $(1,3) \Rightarrow y = x + \ln|x| + C$; $y = 3$ at $x = 1 \Rightarrow C = 2 \Rightarrow y = x + \ln|x| + 2$

49. $\frac{dy}{dt} = e^t \sin\left(e^t - 2\right) \Rightarrow y = \int e^t \sin\left(e^t - 2\right) dt$;

let $u = e^t - 2 \Rightarrow du = e^t\,dt \Rightarrow y = \int \sin u\,du = -\cos u + C = -\cos\left(e^t - 2\right) + C$; $y(\ln 2) = 0$

$\Rightarrow -\cos\left(e^{\ln 2} - 2\right) + C = 0 \Rightarrow -\cos(2-2) + C = 0 \Rightarrow C = \cos 0 = 1$; thus, $y = 1 - \cos\left(e^t - 2\right)$

50. $\frac{dy}{dt} = e^{-t}\sec^2\left(\pi e^{-t}\right) \Rightarrow y = \int e^{-t}\sec^2\left(\pi e^{-t}\right) dt$;

let $u = \pi e^{-t} \Rightarrow du = -\pi e^{-t}\,dt \Rightarrow -\frac{1}{\pi}\,du = e^{-t}\,dt \Rightarrow y = -\frac{1}{\pi}\int \sec^2 u\,du = -\frac{1}{\pi}\tan u + C$

$= -\frac{1}{\pi}\tan\left(\pi e^{-t}\right) + C;\ y(\ln 4) = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi}\tan\left(\pi e^{-\ln 4}\right) + C = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi}\tan\left(\pi\cdot\frac{1}{4}\right) + C = \frac{2}{\pi}$

$\Rightarrow -\frac{1}{\pi}(1) + C = \frac{2}{\pi} \Rightarrow C = \frac{3}{\pi}$; thus, $y = \frac{3}{\pi} - \frac{1}{\pi}\tan\left(\pi e^{-t}\right)$

51. Let $u = 2t - \frac{\pi}{2} \Rightarrow du = 2\,dt \Rightarrow -2\,du = -4\,dt$

$\frac{ds}{dt} = \int -4\sin\left(2t - \frac{\pi}{2}\right)dt = \int (\sin u)(-2\,du) = 2\cos u + C_1 = 2\cos\left(2t - \frac{\pi}{2}\right) + C_1;$

at $t = 0$ and $\frac{ds}{dt} = 100$ we have $100 = 2\cos\left(-\frac{\pi}{2}\right) + C_1 \Rightarrow C_1 = 100 \Rightarrow \frac{ds}{dt} = 2\cos\left(2t - \frac{\pi}{2}\right) + 100$

$\Rightarrow s = \int \left(2\cos\left(2t - \frac{\pi}{2}\right) + 100\right)dt = \int (\cos u + 50)\,du = \sin u + 50u + C_2 = \sin\left(2t - \frac{\pi}{2}\right) + 50\left(2t - \frac{\pi}{2}\right) + C_2;$

at $t = 0$ and $s = 0$ we have $0 = \sin\left(-\frac{\pi}{2}\right) + 50\left(-\frac{\pi}{2}\right) + C_2 \Rightarrow C_2 = 1 + 25\pi$

$\Rightarrow s = \sin\left(2t - \frac{\pi}{2}\right) + 100t - 25\pi + (1 + 25\pi) \Rightarrow s = \sin\left(2t - \frac{\pi}{2}\right) + 100t + 1$

52. $\frac{d^2y}{dx^2} = \sec^2 x \Rightarrow \frac{dy}{dx} = \tan x + C$ and $1 = \tan 0 + C \Rightarrow \frac{dy}{dx} = \tan x + 1 \Rightarrow y = \int (\tan x + 1)\,dx$

$= \ln|\sec x| + x + C_1$ and $0 = \ln|\sec 0| + 0 + C_1 \Rightarrow C_1 = 0 \Rightarrow y = \ln|\sec x| + x$

53. $\frac{d^2y}{dx^2} = 2e^{-x} \Rightarrow \frac{dy}{dx} = -2e^{-x} + C$; $x = 0$ and $\frac{dy}{dx} = 0 \Rightarrow 0 = -2e^0 + C \Rightarrow C = 2$; thus $\frac{dy}{dx} = -2e^{-x} + 2$

$\Rightarrow y = 2e^{-x} + 2x + C_1$; $x = 0$ and $y = 1 \Rightarrow 1 = 2e^0 + C_1 \Rightarrow C_1 = -1 \Rightarrow y = 2e^{-x} + 2x - 1 = 2\left(e^{-x} + x\right) - 1$

54. $\frac{d^2y}{dt^2} = 1 - e^{2t} \Rightarrow \frac{dy}{dt} = t - \frac{1}{2}e^{2t} + C$; $t = 1$ and $\frac{dy}{dt} = 0 \Rightarrow 0 = 1 - \frac{1}{2}e^2 + C \Rightarrow C = \frac{1}{2}e^2 - 1$; thus

$\frac{dy}{dt} = t - \frac{1}{2}e^{2t} + \frac{1}{2}e^2 - 1 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t + C_1$; $t = 1$ and $y = -1 \Rightarrow -1 = \frac{1}{2} - \frac{1}{4}e^2 + \frac{1}{2}e^2 - 1 + C_1$

$\Rightarrow C_1 = -\frac{1}{2} - \frac{1}{4}e^2 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t - \left(\frac{1}{2} + \frac{1}{4}e^2\right)$

55. $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} \Rightarrow dy = \frac{dx}{\sqrt{1 - x^2}} \Rightarrow y = \sin^{-1} x + C$; $x = 0$ and $y = 0 \Rightarrow 0 = \sin^{-1} 0 + C \Rightarrow C = 0 \Rightarrow y = \sin^{-1} x$

56. $\frac{dy}{dx} = \frac{1}{x^2 + 1} - 1 \Rightarrow dy = \left(\frac{1}{1 + x^2} - 1\right)dx \Rightarrow y = \tan^{-1}(x) - x + C$; $x = 0$ and $y = 1 \Rightarrow 1 = \tan^{-1} 0 - 0 + C$

$\Rightarrow C = 1 \Rightarrow y = \tan^{-1}(x) - x + 1$

57. $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}} \Rightarrow dy = \frac{dx}{x\sqrt{x^2 - 1}} \Rightarrow y = \sec^{-1}|x| + C$; $x = 2$ and $y = \pi \Rightarrow \pi = \sec^{-1} 2 + C \Rightarrow C = \pi - \sec^{-1} 2$

$= \pi - \frac{\pi}{3} = \frac{2\pi}{3} \Rightarrow y = \sec^{-1}(x) + \frac{2\pi}{3},\ x > 1$

58. $\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}} \Rightarrow dy = \left(\frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}}\right) dx \Rightarrow y = \tan^{-1} x - 2 \sin^{-1} x + C$; $x = 0$ and $y = 2$

$\Rightarrow 2 = \tan^{-1} 0 - 2 \sin^{-1} 0 + C \Rightarrow C = 2 \Rightarrow y = \tan^{-1} x - 2 \sin^{-1} x + 2$

59. Let $u = 2t \Rightarrow du = 2\ dt \Rightarrow 3\ du = 6\ dt$

$s = \int 6 \sin 2t\ dt = \int (\sin u)(3\ du) = -3 \cos u + C = -3 \cos 2t + C;$

at $t = 0$ and $s = 0$ we have $0 = -3 \cos 0 + C \Rightarrow C = 3 \Rightarrow s = 3 - 3 \cos 2t \Rightarrow s\left(\frac{\pi}{2}\right) = 3 - 3 \cos(\pi) = 6$ m

60. Let $u = \pi t \Rightarrow du = \pi\ dt \Rightarrow \pi\ du = \pi^2\ dt$

$v = \int \pi^2 \cos \pi t\ dt = \int (\cos u)(\pi\ du) = \pi \sin u + C_1 = \pi \sin(\pi t) + C_1;$

at $t = 0$ and $v = 8$ we have $8 = \pi(0) + C_1 \Rightarrow C_1 = 8 \Rightarrow v = \frac{ds}{dt} = \pi \sin(\pi t) + 8 \Rightarrow s = \int (\pi \sin(\pi t) + 8)\ dt$

$= \int \sin u\ du + 8t + C_2 = -\cos(\pi t) + 8t + C_2$; at $t = 0$ and $s = 0$ we have $0 = -1 + C_2 \Rightarrow C_2 = 1$

$\Rightarrow s = 8t - \cos(\pi t) + 1 \Rightarrow s(1) = 8 - \cos \pi + 1 = 10$ m

61. $\frac{d}{dx}\left(\sin^{-1} \frac{x}{a} + C\right) = \frac{d}{dx} \sin^{-1} \frac{x}{a} + \frac{dC}{dx} = \frac{1}{\sqrt{1-(x/a)^2}} \cdot \frac{1}{a} = \frac{1}{\sqrt{a^2 - x^2}}$, which verifies formula 10.

62. $\frac{d}{dx}\left(\frac{1}{a} \tan^{-1} \frac{x}{a} + C\right) = \frac{1}{a} \frac{d}{dx} \tan^{-1} \frac{x}{a} + \frac{dC}{dx} = \frac{1}{a\left(1+(x/a)^2\right)} \cdot \frac{1}{a} + 0 = \frac{1}{a^2 + x^2}$, which verifies formula 11.

63. (a) $\int \frac{1}{\sqrt{3^2 - x^2}}\ dx = \sin^{-1} \frac{x}{3} + C$ (b) $\int \frac{1}{\left(\sqrt{3}\right)^2 + x^2}\ dx = \frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + C$

64. (a) $\int \frac{1}{\sqrt{16 - 25x^2}}\ dx = \frac{1}{5} \int \frac{1}{\sqrt{(4/5)^2 - x^2}}\ dx = \frac{1}{5} \sin^{-1} \frac{5x}{4} + C$

(b) $\int \frac{8}{1 + 4x^2}\ dx = 2 \int \frac{1}{x^2 + (1/2)^2}\ dx = 2\left[2 \tan^{-1} 2x + \overline{C}\right] = 4 \tan^{-1} 2x + C$, where $C = 2\overline{C}$

65. $\int \sec x\ dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\ dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\ dx$

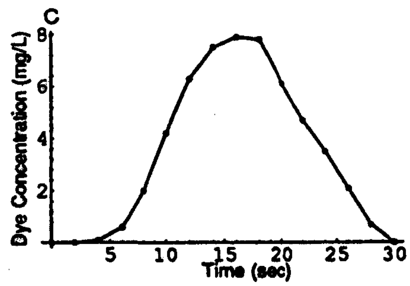
$= \begin{bmatrix} \text{Let } u = \sec x + \tan x \text{ and} \\ du = (\sec^2 x + \sec x \tan x)\ dx \end{bmatrix} = \int \frac{du}{u} = \ln|\sec x + \tan x| + C$

66. All three integrations are correct. In each case, the derivative of the function on the right is the integrand on the left, and each formula has an arbitrary constant for generating the remaining antiderivatives. Moreover, $\sin^2 x + C_1 = 1 - \cos^2 x + C_1 \Rightarrow C_2 = 1 + C_1$; also $-\cos^2 x + C_2 = -\frac{\cos 2x}{2} - \frac{1}{2} + C_2 \Rightarrow C_3 = C_2 - \frac{1}{2} = C_1 + \frac{1}{2}$.

4.3 ESTIMATING WITH FINITE SUMS

1. Using values of the function taken from the graph at the midpoints of the intervals,
Area $\approx (0.25)(2) + (1.0)(2) + (2.0)(2) + (3.25)(2) + (4.0)(2) + (4.0)(2)$
$+ (3.35)(2) + (2.25)(2) + (1.3)(2) + (0.75)(2) + (0.25)(2) = 44.8$ mg·sec/L. Cardiac output
$= \frac{\text{amount of dye}}{\text{area under curve}} \times 60 \approx \frac{5 \text{ mg}}{44.5 \text{ mg}\cdot\text{sec/L}} \times 60 \frac{\text{sec}}{\text{min}} \approx 6.7$ L/min.

2. Using values of the function taken from the graph at the midpoints of the intervals,
Area $\approx 0(2) + (0.1)(2) + (0.4)(2) + (1.2)(2) + (3.2)(2)$
$+ (5.3)(2) + (6.8)(2) + (7.6)(2) + (7.7)(2) + (6.9)(2)$
$+ (5.6)(2) + (4.0)(2) + (2.8)(2) + (1.6)(2) + (0.2)(2)$
$= 107.0$ mg· sec/L.
Cardiac output $= \frac{\text{dye concentration}}{\text{area estimate}} \times 60$
$= \frac{10 \text{ mg}}{107.0 \text{ mg}\cdot\text{sec/L}} \times 60 \frac{\text{sec}}{\text{min}} = 5.61$ L/min.



3. (a) $D \approx (0)(1) + (12)(1) + (22)(1) + (10)(1) + (5)(1) + (13)(1) + (11)(1) + (6)(1) + (2)(1) + (6)(1) = 87$ inches
(b) $D \approx (12)(1) + (22)(1) + (10)(1) + (5)(1) + (13)(1) + (11)(1) + (6)(1) + (2)(1) + (6)(1) + (0)(1) = 87$ inches

4. (a) $D \approx (1)(300) + (1.2)(300) + (1.7)(300) + (2.0)(300) + (1.8)(300) + (1.6)(300) + (1.4)(300) + (1.2)(300)$
$+ (1.0)(300) + (1.8)(300) + (1.5)(300) + (1.2)(300) = 5200$ meters (NOTE: 5 minutes = 300 seconds)
(b) $D \approx (1.2)(300) + (1.7)(300) + (2.0)(300) + (1.8)(300) + (1.6)(300) + (1.4)(300) + (1.2)(300) + (1.0)(300)$
$+ (1.8)(300) + (1.5)(300) + (1.2)(300) + (0)(300) = 4920$ meters (NOTE: 5 minutes = 300 seconds)

5. (a) $D \approx (0)(10) + (44)(10) + (15)(10) + (35)(10) + (30)(10) + (44)(10) + (35)(10) + (15)(10) + (22)(10)$
$+ (35)(10) + (44)(10) + (30)(10) = 3490$ feet ≈ 0.66 miles
(b) $D \approx (44)(10) + (15)(10) + (35)(10) + (30)(10) + (44)(10) + (35)(10) + (15)(10) + (22)(10) + (35)(10)$
$+ (44)(10) + (30)(10) + (35)(10) = 3840$ feet ≈ 0.73 miles

6. (a) The distance traveled will be the area under the curve. We will use the approximate velocities at the midpoints of each time interval to approximate this area using rectangles. Thus,
$D \approx (20)(0.001) + (51)(0.001) + (72)(0.001) + (89)(0.001) + (102)(0.001) + (112)(0.001) + (120.5)(0.001)$
$+ (128.5)(0.001) + (134.5)(0.001) + (139.5)(0.001) \approx 0.969$ miles
(b) Roughly, after 0.0063 hours, the car would have gone 0.485 miles, where 0.0060 hours = 22.7 sec. At 22.7 sec, the velocity was approximately 120 mi/hr.

7. (a) $S_4 = \pi\left[\sqrt{16-(-2)^2}\right]^2(2) + \pi\left[\sqrt{16-0^2}\right]^2(2) + \pi\left[\sqrt{16-(2)^2}\right]^2(2) = \pi[(16-4) + (16-0) + (16-4)](2)$
$= 80\pi$
(b) $\frac{|V - S_4|}{V} = \frac{\left|\left(\frac{256}{3}\right)\pi - 80\pi\right|}{\left(\frac{256}{3}\right)\pi} = \frac{16}{256} \approx 6\%$

8. (a) $S_5 = \pi[(25-(-3)^2)+(25-(-1)^2)+(25-(1)^2)+(25-(3)^2)](2) = \pi(16+24+24+16)(2) = 160\pi$

(b) $V = \frac{4}{3}\pi r^3 = \frac{500\pi}{3} \Rightarrow \frac{|V-S_5|}{V} = \frac{\left|\left(\frac{500}{3}\right)\pi - 160\pi\right|}{\left(\frac{500}{3}\right)\pi} = \frac{20}{500} = 4\%$

9. (a) $S_8 = \pi\left[(16-0^2)+\left(16-\left(\frac{1}{2}\right)^2\right)+(16-(1)^2)+\left(16-\left(\frac{3}{2}\right)^2\right)+(16-(2)^2)+\left(16-\left(\frac{5}{2}\right)^2\right)\right.$

$\left.+(16-(3)^2)+\left(16-\left(\frac{7}{2}\right)\right)^2\right]\left(\frac{1}{2}\right) = \frac{\pi}{2}\left[128-\frac{1}{4}-1-\frac{9}{4}-4-\frac{25}{4}-9-\frac{49}{4}\right] = \frac{372\pi}{8} = \frac{93\pi}{2}$, overestimates

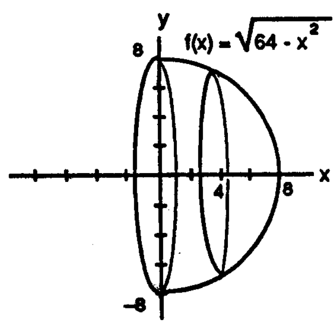
(b) $V = \frac{2}{3}\pi r^3 = \frac{128\pi}{3} \Rightarrow \frac{|V-S_8|}{V} = \frac{\left|\left(\frac{128}{3}\right)\pi - \left(\frac{93}{2}\right)\pi\right|}{\left(\frac{128}{3}\right)\pi} = \frac{23}{256} \approx 9\%$

10. (a) $S_8 = \pi\left[\left(16-\left(\frac{1}{2}\right)^2\right)+(16-(1)^2)+\left(16-\left(\frac{3}{2}\right)^2\right)+(16-(2)^2)+\left(16-\left(\frac{5}{2}\right)^2\right)+(16-(3)^2)\right.$

$\left.+\left(16-\left(\frac{7}{2}\right)^2\right)\right]\left(\frac{1}{2}\right) = \frac{\pi}{2}\left[112-\frac{1}{4}-1-\frac{9}{4}-4-\frac{25}{4}-9-\frac{49}{4}\right] = \frac{308\pi}{8} = \frac{77\pi}{2}$, underestimates

(b) $\frac{|V-S_8|}{V} = \frac{\left|\left(\frac{128}{3}\right)\pi - \left(\frac{77}{2}\right)\pi\right|}{\left(\frac{128}{3}\right)\pi} = \frac{25}{256} \approx 10\%$

11. (a) To have the same orientation as the hemisphere in Exercise 10, tip the bowl sideways (assume the water is ice). The water covers the interval [4, 8]. The function which will give us the values of the radii of the approximating cylinders is the equation of the upper semicircle formed by intersecting the hemisphere with the xy-plane, $f(x) = \sqrt{64-x^2}$. Using $\Delta x = \frac{1}{2}$ and left-endpoints for each interval



$\Rightarrow S_8 = \pi\left[(64-(4)^2)+\left(64-\left(\frac{9}{2}\right)^2\right)\right.$

$+(64-(5)^2)+\left(64-\left(\frac{11}{2}\right)^2\right)+(64-(6)^2)$

$\left.+\left(64-\left(\frac{13}{2}\right)^2\right)+(64-(7)^2)+\left(64-\left(\frac{15}{2}\right)^2\right)\right]\left(\frac{1}{2}\right) = \frac{\pi}{2}\left(512-16-\frac{81}{4}-25-\frac{121}{4}-36-\frac{169}{4}-49-\frac{225}{4}\right)$

$= \frac{\pi}{2}\left(386-\frac{596}{4}\right) = \frac{\pi}{8}(1544-596) = \frac{948}{8}\pi = 118.5\pi;$

(b) $\frac{|V-S_8|}{V} = \frac{\left|\left(\frac{320}{3}\right)\pi - \left(\frac{948}{8}\right)\pi\right|}{\left(\frac{320}{3}\right)\pi} = \frac{2844-2560}{2560} \approx 11\%$

12. We are using boxes (rectangular parallelepipeds) that are 30 feet wide, 5 feet long, and h(x) feet deep to approximate the volume of water in the pool.

(a) Using left-hand endpoints in the table: $S = (30)(5)(6.0) + (30)(5)(8.2) + (30)(5)(9.1) + (30)(5)(9.9)$
$+ (30)(5)(10.5) + (30)(5)(11.0) + (30)(500(11.5) + (30)(5)(11.9) + (30)(5)(12.3) + (30)(5)(12.7) = 15{,}465\ \text{ft}^3$.

(b) Using right-hand endpoints in the table: $S = (30)(5)(8.2) + (30)(5)(9.1) + (30)(5)(9.9) + (30)(5)(10.5)$
$+ (30)(5)(11.0) + (30)(5)(11.5) + (30)(5)(11.9) + (30)(5)(12.3) + (30)(5)(12.7) + (30)(5)(13) = 16{,}515\ \text{ft}^3$.

13. (a) $S_5 = \pi\left[\left(\sqrt{0}\right)^2 + \left(\sqrt{1}\right)^2 + \left(\sqrt{2}\right)^2 + \left(\sqrt{3}\right)^2 + \left(\sqrt{4}\right)^2\right](1) = 10\pi$, underestimates the volume.

(b) $\frac{|V - S_5|}{V} = \frac{\left(\frac{25}{2}\right)\pi - 10\pi}{\left(\frac{25}{2}\right)\pi} = \frac{5}{25} = 20\%$

14. (a) $S_5 = \pi\left[\left(\sqrt{1}\right)^2 + \left(\sqrt{2}\right)^2 + \left(\sqrt{3}\right)^2 + \left(\sqrt{4}\right)^2 + \left(\sqrt{5}\right)^2\right](1) = 15\pi$, overestimates the volume.

(b) $\frac{|V - S_5|}{V} = \frac{15\pi - \left(\frac{25}{2}\right)\pi}{\left(\frac{25}{2}\right)\pi} = \frac{5}{25} = 20\%$

15. (a) Because the acceleration is decreasing, an upper estimate is obtained using left end-points in summing acceleration $\cdot \Delta t$. Thus, $\Delta t = 1$ and speed $\approx [32.00 + 19.41 + 11.77 + 7.14 + 4.33](1) = 74.65$ ft/sec

(b) Using right end-points we obtain a lower estimate: speed $\approx [19.41 + 11.77 + 7.14 + 4.33 + 2.63](1)$ $= 45.28$ ft/sec

(c) Upper estimates for the speed at each second are:

t	0	1	2	3	4	5
v	0	32.00	51.41	63.18	70.32	74.65

Thus, the distance fallen when $t = 3$ seconds is $s \approx [32.00 + 51.41 + 63.18](1) = 146.59$ ft.

16. (a) The speed is a decreasing function of time $\Rightarrow$ left end-points give an upper estimate for the height (distance) attained. Also

t	0	1	2	3	4	5
v	400	368	336	304	272	240

gives the time-velocity table by subtracting the constant $g = 32$ from the speed at each time increment $\Delta t = 1$ sec. Thus, the speed ≈ 240 ft/sec after 5 seconds.

(b) A lower estimate for height attained is $h \approx [368 + 336 + 304 + 272 + 240](1) = 1520$ ft.

17. Partition $[0, 2]$ into the four subintervals $\left[0, \frac{1}{2}\right]$, $\left[\frac{1}{2}, 1\right]$, $\left[1, \frac{3}{2}\right]$, and $\left[\frac{3}{2}, 2\right]$. The midpoints of these subintervals are $m_1 = \frac{1}{4}$, $m_2 = \frac{3}{4}$, $m_3 = \frac{5}{4}$, and $m_4 = \frac{7}{4}$. The heights of the four approximating

rectangles are $f(m_1) = \left(\frac{1}{4}\right)^3 = \frac{1}{64}$, $f(m_2) = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$, $f(m_3) = \left(\frac{5}{4}\right)^3 = \frac{125}{64}$, and $f(m_4) = \left(\frac{7}{4}\right)^3 = \frac{343}{64}$

$\Rightarrow$ Average value $\approx \dfrac{\frac{1}{64}+\frac{27}{64}+\frac{125}{64}+\frac{343}{64}}{4} = \dfrac{1+27+125+343}{4\cdot 64} = \dfrac{496}{256} = \dfrac{31}{16}$. Notice that the average value is

approximated by $\dfrac{\left(\frac{1}{4}\right)^3+\left(\frac{3}{4}\right)^3+\left(\frac{5}{4}\right)^3+\left(\frac{7}{4}\right)^3}{4} = \frac{1}{2}\left[\left(\frac{1}{4}\right)^3\left(\frac{1}{2}\right)+\left(\frac{3}{4}\right)^3\left(\frac{1}{2}\right)+\left(\frac{5}{4}\right)^3\left(\frac{1}{2}\right)+\left(\frac{7}{4}\right)^3\left(\frac{1}{2}\right)\right]$

$= \dfrac{1}{\text{length of }[0,2]}\cdot\left[\begin{matrix}\text{approximate area under}\\ \text{curve } f(x) = x^3\end{matrix}\right]$. We use this observation in solving the next several exercises.

18. Partition [1,9] into the four subintervals [1,3], [3,5], [5,7], and [7,9]. The midpoints of these subintervals are $m_1 = 2$, $m_2 = 4$, $m_3 = 6$, and $m_4 = 8$. The heights of the four approximating rectangles are $f(m_1) = \frac{1}{2}$, $f(m_2) = \frac{1}{4}$, $f(m_3) = \frac{1}{6}$, and $f(m_4) = \frac{1}{8}$. The width of each rectangle is $\Delta x = 2$. Thus, Area $\approx 2\left(\frac{1}{2}\right)+2\left(\frac{1}{4}\right)+2\left(\frac{1}{6}\right)+2\left(\frac{1}{8}\right) = \frac{25}{12} \Rightarrow$ average value $\approx \dfrac{\text{area}}{\text{length of }[1,9]} = \dfrac{\left(\frac{25}{12}\right)}{8} = \dfrac{25}{96}$.

19. Partition [0,2] into the four subintervals [0,0.5], [0.5,1], [1,1.5], and [1.5,2]. The midpoints of the subintervals are $m_1 = 0.25$, $m_2 = 0.75$, $m_3 = 1.25$, and $m_4 = 1.75$. The heights of the four approximating rectangles are $f(m_1) = \frac{1}{2}+\sin^2\frac{\pi}{4} = \frac{1}{2}+\frac{1}{2} = 1$, $f(m_2) = \frac{1}{2}+\sin^2\frac{3\pi}{4} = \frac{1}{2}+\frac{1}{2} = 1$, $f(m_3) = \frac{1}{2}+\sin^2\frac{5\pi}{4} = \frac{1}{2}+\left(-\frac{1}{\sqrt{2}}\right)^2$ $= \frac{1}{2}+\frac{1}{2} = 1$, and $f(m_4) = \frac{1}{2}+\sin^2\frac{7\pi}{4} = \frac{1}{2}+\left(-\frac{1}{\sqrt{2}}\right)^2 = 1$. The width of each rectangle is $\Delta x = \frac{1}{2}$. Thus, Area $\approx (1+1+1+1)\left(\frac{1}{2}\right) = 2 \Rightarrow$ average value $\approx \dfrac{\text{area}}{\text{length of }[0,2]} = \dfrac{2}{2} = 1$.

20. Partition [0,4] into the four subintervals [0,1], [1,2,], [2,3], and [3,4]. The midpoints of the subintervals are $m_1 = \frac{1}{2}$, $m_2 = \frac{3}{2}$, $m_3 = \frac{5}{2}$, and $m_4 = \frac{7}{2}$. The heights of the four approximating rectangles are

$f(m_1) = 1-\left(\cos\left(\dfrac{\pi\left(\frac{1}{2}\right)}{4}\right)\right)^4 = 1-\left(\cos\left(\frac{\pi}{8}\right)\right)^4 = 0.27145$ (to 5 decimal places),

$f(m_2) = 1-\left(\cos\left(\dfrac{\pi\left(\frac{3}{2}\right)}{4}\right)\right)^4 = 1-\left(\cos\left(\frac{3\pi}{8}\right)\right)^4 = 0.97855$, $f(m_3) = 1-\left(\cos\left(\dfrac{\pi\left(\frac{5}{2}\right)}{4}\right)\right)^4 = 1-\left(\cos\left(\frac{5\pi}{8}\right)\right)^4$

$= 0.97855$, and $f(m_4) = 1-\left(\cos\left(\dfrac{\pi\left(\frac{7}{2}\right)}{4}\right)\right)^4 = 1-\left(\cos\left(\frac{7\pi}{8}\right)\right)^4 = 0.27145$. The width of each rectangle is $\Delta x = 1$. Thus, Area $\approx (0.27145)(1)+(0.97855)(1)+(0.97855)(1)+(0.27145)(1) = 2.5 \Rightarrow$ average value $\approx \dfrac{\text{area}}{\text{length of }[0,4]} = \dfrac{2.5}{4} = \dfrac{5}{8}$.

21. Since the leakage is increasing, an upper estimate uses right end-points and a lower estimate uses left end-points:

(a) upper estimate $= (70)(1) + (97)(1) + (136)(1) + (190)(1) + (265)(1) = 758$ gal,

lower estimate $= (50)(1) + (70)(1) + (97)(1) + (136)(1) + (190)(1) = 543$ gal.

(b) upper estimate $= (70 + 97 + 136 + 190 + 265 + 369 + 516 + 720) = 2363$ gal,

lower estimate $= (50 + 70 + 97 + 136 + 190 + 265 + 369 + 516) = 1693$ gal.

(c) worst case: $2363 + 720t = 25{,}000 \Rightarrow t \approx 31.4$ hrs;

best case: $1693 + 720t = 25{,}000 \Rightarrow t \approx 32.4$ hrs

22. Since the pollutant release increases over time, an upper estimate uses right end-points and a lower estimate uses left end-points:

(a) upper estimate $= (0.2)(30) + (0.25)(30) + (0.27)(30) + (0.34)(30) + (0.45)(30) + (0.52)(30) = 60.9$ tons

lower estimate $= (0.05)(30) + (0.2)(30) + (0.25)(30) + (0.27)(30) + (0.34)(30) + (0.45)(30) = 46.8$ tons

(b) Using the lower (best case) estimate: $46.8 + (0.52)(30) + (0.63)(30) + (0.70)(30) + (0.81)(30) = 126.6$ tons, so near the end of September 125 tons of pollutants will have been released.

23. (a) The diagonal of the square has length 2, so the side length is $\sqrt{2}$. Area $= \left(\sqrt{2}\right)^2 = 2$

(b) Think of the octagon as a collection of 16 right triangles with a hypotenuse of length 1 and an acute angle measuring $\frac{2\pi}{16} = \frac{\pi}{8}$.

Area $= 16\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right) = 4\sin\frac{\pi}{4} = 2\sqrt{2} \approx 2.828$

(c) Think of the 16-gon as a collection of 32 right triangles with a hypotenuse of length 1 and an acute angle measuring $\frac{2\pi}{32} = \frac{\pi}{16}$.

Area $= 32\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{16}\right)\left(\cos\frac{\pi}{16}\right) = 8\sin\frac{\pi}{8} \approx 3.061$

(d) Each area is less than the area of the circle, π. As n increases, the area approaches π.

24. (a) Each of the isosceles triangles is made up of two right triangles having hypotenuse 1 and an acute angle measuring $\frac{2\pi}{2n} = \frac{\pi}{n}$. The area of each isosceles triangle is $A_T = 2\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{n}\right)\left(\cos\frac{\pi}{n}\right) = \frac{1}{2}\sin\frac{2\pi}{n}$.

(b) The area of the polygon is $A_P = nA_T = \frac{n}{2}\sin\frac{2\pi}{n}$, so $\lim\limits_{n\to\infty} A_P = \lim\limits_{n\to\infty} \frac{n}{2}\sin\frac{2\pi}{n} = \lim\limits_{n\to\infty} \pi \cdot \frac{\sin\frac{2\pi}{n}}{\left(\frac{2\pi}{n}\right)} = \pi$

(c) Multiply each area by r^2.

$A_T = \frac{1}{2}r^2 \sin\frac{2\pi}{n}$

$A_P = \frac{n}{2}r^2 \sin\frac{2\pi}{n}$

$\lim\limits_{n\to\infty} A_P = \pi r^2$

25-28. Example CAS commands:

Maple:

```
with(student):
f:=x -> sin(x); a:= 0; b:= Pi;
plot(f(x),x=a..b);
n:= 1000;
```

```
middlebox(f(x),x=a..b,n);
middlesum(f(x),x=a..b,n);
average:= evalf(%)/(b−a);
fsolve(f(x)=average,x);
```

Mathematica:

```
Clear[x]
f[x_] = Sin[x]
{a,b} = {0,Pi};
Plot[ f[x], {x,a,b} ]
n = 100; dx = (b−a)/n;
Table[ N[f[x]], {x,a+dx/2,b,dx} ];
fave = (Plus @@ %) /n
n = 200; dx = (b−a)/n;
Table[ N[f[x]], {x,a+dx/2,b,dx} ];
fave = (Plus @@ %)/n
n = 1000; dx = (b−a)/n;
Table[ N[f[x]], {x,a+dx/2,b,dx} ];
fave = (Plus @@ %)/n
FindRoot[ f[x] − fave, {x,a} ]
```

4.4 RIEMANN SUMS AND DEFINITE INTEGRALS

1. $\sum_{k=1}^{2} \frac{6k}{k+1} = \frac{6(1)}{1+1} + \frac{6(2)}{2+1} = \frac{6}{2} + \frac{12}{3} = 7$

2. $\sum_{k=1}^{3} \frac{k-1}{k} = \frac{1-1}{1} + \frac{2-1}{2} + \frac{3-1}{3} = 0 + \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$

3. $\sum_{k=1}^{4} \cos k\pi = \cos(1\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) = -1 + 1 - 1 + 1 = 0$

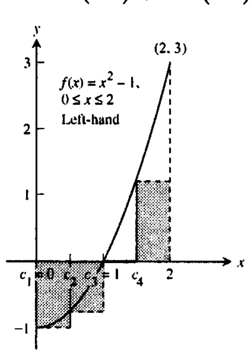
4. $\sum_{k=1}^{5} \sin k\pi = \sin(1\pi) + \sin(2\pi) + \sin(3\pi) + \sin(4\pi) + \sin(5\pi) = 0 + 0 + 0 + 0 + 0 = 0$

5. $\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k} = (-1)^{1+1} \sin \frac{\pi}{1} + (-1)^{2+1} \sin \frac{\pi}{2} + (-1)^{3+1} \sin \frac{\pi}{3} = 0 - 1 + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}-2}{2}$

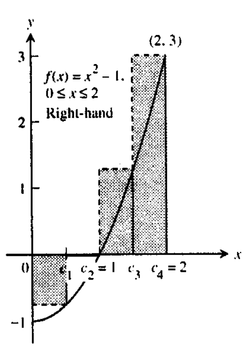
6. $\sum_{k=1}^{4} (-1)^{k} \cos k\pi = (-1)^{1} \cos(1\pi) + (-1)^{2} \cos(2\pi) + (-1)^{3} \cos(3\pi) + (-1)^{4} \cos(4\pi)$

$= -(-1) + 1 - (-1) + 1 = 4$

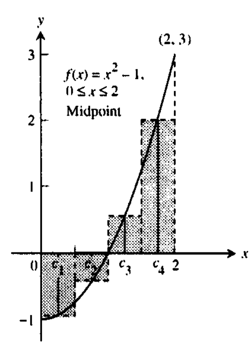
7. (a)



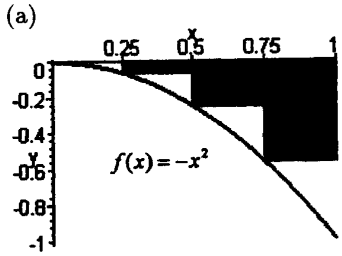
(b)



(c)



8. (a)



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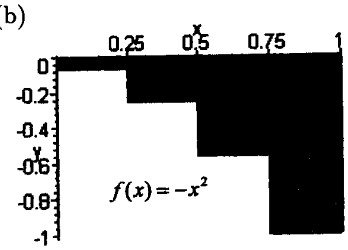
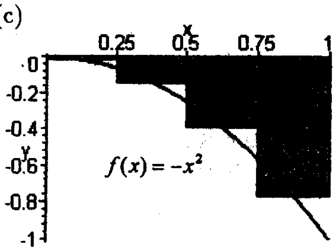
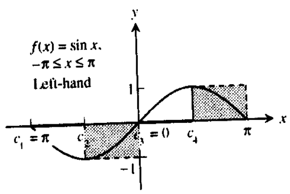


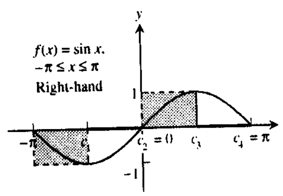
(c)



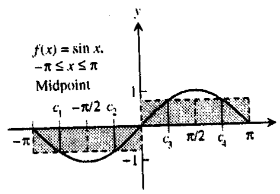
9. (a)



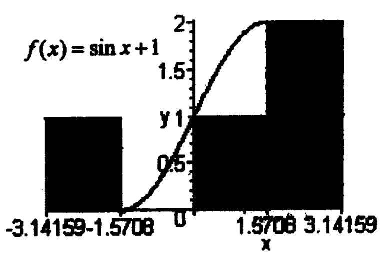
(b)



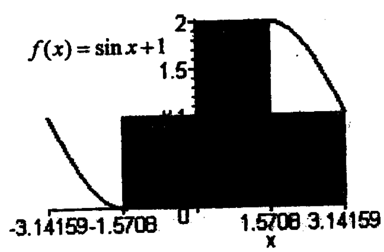
(c)



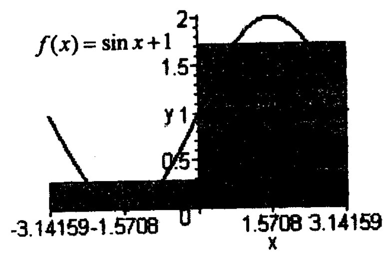
10. (a)



(b)



(c)



11. $\int_0^2 x^2\, dx$

12. $\int_{-1}^0 2x^3\, dx$

13. $\int_{-7}^5 \left(x^2 - 3x\right) dx$

14. $\int_2^3 \frac{1}{1-x}\, dx$

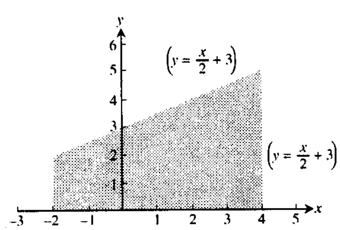
15. $\int_0^1 \sqrt{4-x^2}\, dx$

16. $\int_{-\pi/4}^0 (\sec x)\, dx$

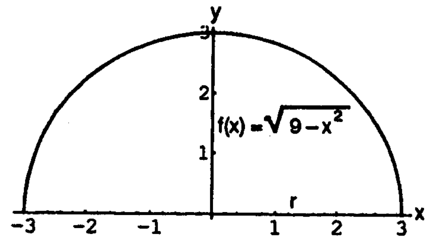
17. The area of the trapezoid is $A = \frac{1}{2}(B+b)h$

$= \frac{1}{2}(5+2)(6) = 21 \Rightarrow \int_{-2}^4 \left(\frac{x}{2}+3\right) dx$

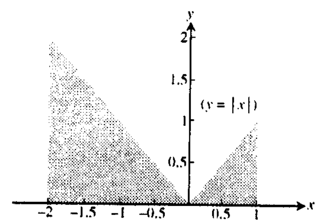
$= 21$ square units



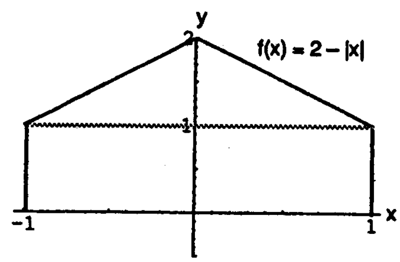
18. The area of the semicircle is $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2$
$= \frac{9}{2}\pi \Rightarrow \int_{-3}^{3} \sqrt{9 - x^2}\,dx = \frac{9}{2}\pi$ square units



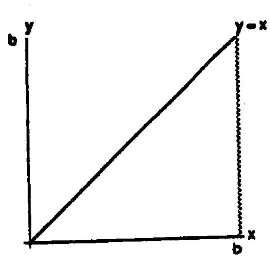
19. The area of the triangle on the left is $A = \frac{1}{2}bh = \frac{1}{2}(2)(2)$
$= 2$. The area of the triangle on the right is $A = \frac{1}{2}bh$
$= \frac{1}{2}(1)(1) = \frac{1}{2}$. Then, the total area is $2.5 \Rightarrow \int_{-2}^{1} |x|\,dx$
$= 2.5$ square units



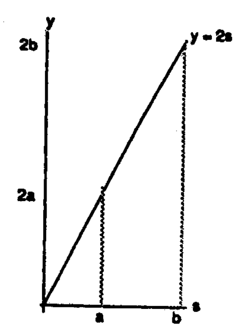
20. The area of the triangular peak is $A = \frac{1}{2}bh = \frac{1}{2}(2)(1) = 1$.
The area of the rectangular base is $S = \ell w = (2)(1) = 2$.
Then the total area is $3 \Rightarrow \int_{-1}^{1} (2 - |x|)\,dx = 3$ square units



21. $\int_{0}^{b} x\,dx = \frac{1}{2}(b)(b) = \frac{b^2}{2}$ square units



22. $\int_{a}^{b} 2s\,ds = \frac{1}{2}b(2b) - \frac{1}{2}a(2a) = b^2 - a^2$ square units



23. The graph of $f(x) = 1 - x$ on the interval $[0, 1]$ forms a right isosceles triangle in the first quadrant with its two legs, each of length one, lying on the coordinate axes. The area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$, which is also the value of the integral $\int_0^1 (1-x)\,dx = \frac{1}{2}$, therefore, $av(f) = \frac{1}{1-0}\int_0^1 (1-x)\,dx = (1)\left(\frac{1}{2}\right) = \frac{1}{2}$.

24. The graph of $f(x) = |x|$ on the interval $[-1, 1]$ forms two congruent isosceles right triangles one in the first and the other in the second quadrant. The total area of these two triangles is $A = 2\left(\frac{1}{2}bh\right) = 2\left(\frac{1}{2}\cdot 1\cdot 1\right) = 1$, which is also the value of the integral $\int_{-1}^1 |x|\,dx$, therefore, $av(f) = \frac{1}{1-(-1)}\int_{-1}^1 |x|\,dx = \frac{1}{2}(1) = \frac{1}{2}$.

25. The function $f(x) = \sqrt{1-x^2}$ on the interval $[0, 1]$ forms a quarter-circular area of radius 1 lying in the first quadrant with its center on the origin. The area of this quarter-circle is $A = \frac{\pi}{4}r^2 = \frac{\pi}{4}$, which is also the value of the integral $\int_0^1 \sqrt{1-x^2}\,dx$, therefore, $av(f) = \frac{1}{1-0}\int_0^1 \sqrt{1-x^2}\,dx = (1)\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$.

26. The function $f(x) = \sqrt{1-(x-2)^2}$ on the interval $[1, 2]$ forms a quarter-circular area of radius 1, lying in the first quadrant with the center of the circle on the point $(2, 0)$. The area of this quarter circle is $A = \frac{\pi}{4}r^2 = \frac{\pi}{4}$, which is also the value of the integral $\int_1^2 \sqrt{1-(x-2)^2}\,dx$, therefore,

$$av(f) = \frac{1}{2-1}\int_1^2 \sqrt{1-(x-2)^2} = (1)\left(\frac{\pi}{4}\right) = \frac{\pi}{4}.$$

27. (a) $\int_2^2 g(x)\,dx = 0$

(b) $\int_5^1 g(x)\,dx = -\int_1^5 g(x)\,dx = -8$

(c) $\int_1^2 3f(x)\,dx = 3\int_1^2 f(x)\,dx = 3(-4) = -12$

(d) $\int_2^5 f(x)\,dx = \int_1^5 f(x)\,dx - \int_1^2 f(x)\,dx = 6 - (-4) = 10$

(e) $\int_1^5 [f(x) - g(x)]\,dx = \int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx = 6 - 8 = -2$

(f) $\int_1^5 [4f(x) - g(x)]\,dx = 4\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx = 4(6) - 8 = 16$

28. (a) $\int_1^9 -2f(x)\,dx = -2\int_1^9 f(x)\,dx = -2(-1) = 2$

(b) $\int_7^9 [f(x)+h(x)]\,dx = \int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx = 5+4 = 9$

(c) $\int_7^9 [2f(x)-3h(x)]\,dx = 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx = 2(5)-3(4) = -2$

(d) $\int_9^1 f(x)\,dx = -\int_1^9 f(x)\,dx = -(-1) = 1$

(e) $\int_1^7 f(x)\,dx = \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx = -1-5 = -6$

(f) $\int_9^7 [h(x)-f(x)]\,dx = \int_7^9 [f(x)-h(x)]\,dx = \int_7^9 f(x)\,dx - \int_7^9 h(x)\,dx = 5-4 = 1$

29. (a) $\int_1^2 f(u)\,du = \int_1^2 f(x)\,dx = 5$

(b) $\int_1^2 \sqrt{3}f(z)\,dz = \sqrt{3}\int_1^2 f(z)\,dz = 5\sqrt{3}$

(c) $\int_2^1 f(t)\,dt = -\int_1^2 f(t)\,dt = -5$

(d) $\int_1^2 [-f(x)]\,dx = -\int_1^2 f(x)\,dx = -5$

30. (a) $\int_0^{-3} g(t)\,dt = -\int_{-3}^0 g(t)\,dt = -\sqrt{2}$

(b) $\int_{-3}^0 g(u)\,du = \int_{-3}^0 g(t)\,dt = \sqrt{2}$

(c) $\int_{-3}^0 [-g(x)]\,dx = -\int_{-3}^0 g(x)\,dx = -\sqrt{2}$

(d) $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr = \frac{1}{\sqrt{2}}\int_{-3}^0 g(t)\,dt = \left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{2}\right) = 1$

31. (a) $\int_3^4 f(z)\,dz = \int_0^4 f(z)\,dz - \int_0^3 f(z)\,dz = 7-3 = 4$

(b) $\int_4^3 f(t)\,dt = -\int_3^4 f(t)\,dt = -4$

32. (a) $\int_1^3 h(r)\,dr = \int_{-1}^3 h(r)\,dr - \int_{-1}^1 h(r)\,dr = 6-0 = 6$

(b) $-\int_3^1 h(u)\,du = -\left(-\int_1^3 h(u)\,du\right) = \int_1^3 h(u)\,du = 6$

33. To find where $x-x^2 \ge 0$, let $x-x^2 = 0 \Rightarrow x(1-x) = 0 \Rightarrow x = 0$ or $x = 1$. If $0 < x < 1$, then $x^2 < x$ $\Rightarrow 0 < x-x^2 \Rightarrow a = 0$ and $b = 1$ maximize the integral.

34. To find where $x^4 - 2x^2 \le 0$, let $x^4 - 2x^2 = 0 \Rightarrow x^2(x^2-2) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{2}$. By the sign graph, $+ + + + + + \underset{-\sqrt{2}}{0} - - \underset{0}{0} - - \underset{\sqrt{2}}{0} + + + + + + + +$, we can see that $x^4 - 2x^2 \le 0$ on $[-\sqrt{2}, \sqrt{2}] \Rightarrow a = -\sqrt{2}$ and $b = \sqrt{2}$ minimize the integral.

35. By the constant multiple rule, $\int_a^b k\, dx = k \int_a^b 1\, dx$. The Riemann sums definition of the definite integral gives

$$\int_a^b 1\, dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \Delta x_k, \text{ and if } \Delta x_k = \frac{b-a}{n}, \text{ then } \lim_{\|P\| \to 0} \sum_{k=1}^{n} \Delta x_k = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \frac{b-a}{n}$$

$$= \lim_{\|P\| \to 0} \left(\frac{b-a}{n} \sum_{k=1}^{n} 1 \right) = \lim_{\|P\| \to 0} \left(\frac{b-a}{n} \cdot n \right) = \lim_{\|P\| \to 0} (b-a) = b - a. \text{ Therefore, } \int_a^b k\, dx = k(b-a),$$

for any k.

36. If $f(x) \ge 0$ on $[a,b]$, then $\min f \ge 0$ and $\max f \ge 0$ on $[a,b]$. Now, $(b-a)\min f \le \int_a^b f(x)\, dx \le (b-a)\max f$.

Then $b \ge a \Rightarrow b - a \ge 0 \Rightarrow (b-a)\min f \ge 0 \Rightarrow \int_a^b f(x)\, dx \ge 0$.

37. $f(x) = \frac{1}{1+x^2}$ is decreasing on $[0,1] \Rightarrow$ maximum value of f occurs at $0 \Rightarrow \max f = f(0) = 1$; minimum value of f occurs at $1 \Rightarrow \min f = f(1) = \frac{1}{1+1^2} = \frac{1}{2}$. Therefore, $(1-0)\min f \le \int_0^1 \frac{1}{1+x^2}\, dx \le (1-0)\max f$

$\Rightarrow \frac{1}{2} \le \int_0^1 \frac{1}{1+x^2}\, dx \le 1$. That is, an upper bound $= 1$ and a lower bound $= \frac{1}{2}$.

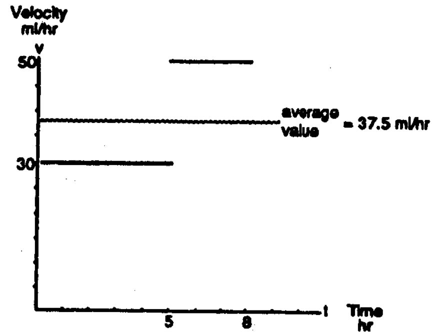
38. See Exercise 37 above. On $[0, 0.5]$, $\max f = \frac{1}{1+0^2} = 1$, $\min f = \frac{1}{1+(0.5)^2} = 0.8$. Therefore

$(0.5-0)\min f \le \int_0^{0.5} f(x)\, dx \le (0.5-0)\max f \Rightarrow 0.4 \le \int_0^{0.5} \frac{1}{1+x^2}\, dx \le 0.5$. On $[0.5, 1]$, $\max f = \frac{1}{1+(0.5)^2} = 0.8$

and $\min f = \frac{1}{1+1^2} = 0.5$. Thus $(1-0.5)\min f \le \int_{0.5}^{1} \frac{1}{1+x^2}\, dx \le (1-0.5)\max f \Rightarrow 0.25 \le \int_{0.5}^{1} \frac{1}{1+x^2}\, dx \le 0.4$.

Then $0.25 + 0.4 \le \int_0^{0.5} \frac{1}{1+x^2}\, dx + \int_{0.5}^{1} \frac{1}{1+x^2}\, dx \le 0.5 + 0.4 \Rightarrow 0.65 \le \int_0^1 \frac{1}{1+x^2}\, dx \le 0.9$.

39. The car drove the first 150 miles in 5 hours and the second 150 miles in 3 hours, which means it drove 300 miles in 8 hours, for an average of $\frac{300}{8}$ mi/hr $= 37.5$ mi/hr. In terms of average values of functions, the function whose average value we seek is

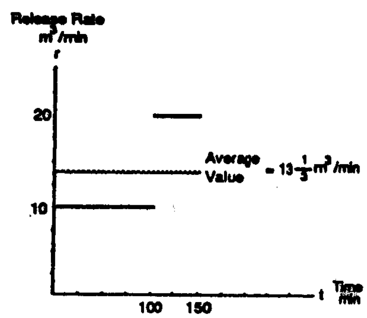


$$v(t) = \begin{cases} 30, & 0 \le t \le 5 \\ 50, & 5 < t \le 8 \end{cases}$$, and the average value is

$\frac{(30)(5)+(50)(3)}{8} = 37.5$ mph. It does not help to consider

$$v(s) = \begin{cases} 30, & 0 \le s \le 150 \\ 50, & 150 < s \le 300 \end{cases}$$ whose average value is $\frac{(30)(150)+(50)(150)}{300} = 40$ (mph)/mi because we want the average speed with respect to time, not distance.

40. The dam released 1000 m^3 of water in 100 min and then released another 1000 m^3 of water in 50 min, for a total of 2000 m^3 in 150 min, which averages to $\frac{2000}{150}$ $= \frac{40}{3}$ m^3/min. In terms of average values of functions, the function whose average value we seek is



$$r(t) = \begin{cases} 10, & 0 \le t \le 100 \\ 20, & 100 < 1 \le 150 \end{cases}$$, and the average value is

$\frac{(10)(100)+(20)(50)}{150} = \frac{40}{3}$ m^3/min.

41-46. Example CAS commands:

Maple:

```
with(student):
f:= x -> x ∧ 2 + 1; a:= 0; b:= 1;
n:=20;
leftbox(f(x),x=a..b,n);
leftsum(f(x),x=a..b,n);
evalf(%);
rightbox(f(x),x=a..b,n);
rightsum(f(x),x=a..b,n);
evalf(%);
middlebox(f(x),x=a..b,n);
middlesum(f(x),x=a..b,n);
evalf(%);
```

Mathematica:

This CAS does not have the leftbox, leftsum, etc. commands. Here are definitions of 3 functions that plot the boxes and also return the Riemann sum, using either left endpoints, right endpoints, or midpoints of each subinterval for the values of the function. The arguments to each are:

```
f: a pure function of one variable
{a,b}: the interval
n: the (positive integer) number of subintervals
plotopts: (optional) options for the plot

LeftSum[f_, {a_,b_},n_,plotopts___] := Module[
{x, dx = (b-a)/n, xvals, yvals, boxes},
xvals = Table[ N[x], {x,a,b-dx,dx} ];
yvals = Map[ f, xvals ] // N;
boxes = MapThread[
  Line[{(#1,0},{#1,#3},{#2,#3},{#2,0}}]&,
  {xvals,xvals+dx,yvals} ];
Plot[ f[x], {x,a,b}, Epilog -> boxes, plotopts ];
(Plus @@ yvals)*dx // N
]
RightSum[f_,{a_,b_},n_,plotopts___] := Module[
{x, dx = (b-a)/n, xvals, yvals, boxes},
xvals = Table[ N[x], {x,a+dx,b,dx} ];
yvals = Map[ f, xvals ] // N;
boxes = MapThread[
  Line[{{#1,0},{#1,#3},{#2,#3},{#2,0}}]&,
  {xvals-dx,xvals,yvals} ];
Plot[ f[x], {x,a,b}, Epilog -> boxes, plotopts ];
(Plus @@ yvals)*dx // N
]
MiddleSum[f_,{a_,b_},n_,plotopts___] := Module[
{x, dx = (b-a)/n, xvals, yvals, boxes},
xvals = Table[ N[x], {x,a+dx/2,b,dx} ];
yvals = Map[ f, xvals ] // N;
boxes = MapThread[
  Line[{{#1,0},{#1,#3},{#2,#3},{#2,0}}]&,
  {xvals-dx/2,xvals+dx/2,yvals} ];
Plot[ f[x], {x,a,b}, Epilog -> boxes, plotopts ];
(Plus @@ yvals)*dx // N
]
Clear[x]
f[x_] = x ∧ 2 + 1
{a,b} = {0,1};
n = 20;
LeftSum[ f, {a,b}, n ]
RightSum[ f, {a,b}, n ]
MiddleSum[ f, {a,b}, n ]
```

4.5 THE MEAN VALUE AND FUNDAMENTAL THEOREMS

1. $\int_{-2}^{0} (2x+5)\,dx = \left[x^2+5x\right]_{-2}^{0} = \left(0^2+5(0)\right)-\left((-2)^2+5(-2)\right) = 6$

2. $\int_{0}^{4} \left(3x-\frac{x^3}{4}\right) dx = \left[\frac{3x^2}{2}-\frac{x^4}{16}\right]_{0}^{4} = \left(\frac{3(4)^2}{2}-\frac{4^4}{16}\right)-\left(\frac{3(0)^2}{2}-\frac{(0)^4}{16}\right) = 8$

3. $\int_0^1 \left(x^2+\sqrt{x}\right) dx = \left[\frac{x^3}{3}+\frac{2}{3}x^{3/2}\right]_0^1 = \left(\frac{1}{3}+\frac{2}{3}\right)-0 = 1$

4. $\int_{-2}^{-1} \frac{2}{x^2}\, dx = \int_{-2}^{-1} 2x^{-2}\, dx = \left[-2x^{-1}\right]_{-2}^{-1} = \left(\frac{-2}{-1}\right)-\left(\frac{-2}{-2}\right) = 1$

5. $\int_0^{\pi} (1+\cos x)\, dx = [x+\sin x]_0^{\pi} = (\pi+\sin \pi)-(0+\sin 0) = \pi$

6. $\int_0^{\pi/3} 2\sec^2 x\, dx = [2\tan x]_0^{\pi/3} = \left(2\tan\left(\frac{\pi}{3}\right)\right)-(2\tan 0) = 2\sqrt{3}-0 = 2\sqrt{3}$

7. $\int_{\pi/4}^{3\pi/4} \csc\theta\cot\theta\, d\theta = [-\csc\theta]_{\pi/4}^{3\pi/4} = \left(-\csc\left(\frac{3\pi}{4}\right)\right)-\left(-\csc\left(\frac{\pi}{4}\right)\right) = -\sqrt{2}-\left(-\sqrt{2}\right) = 0$

8. $\int_0^{\pi/2} \frac{1+\cos 2t}{2}\, dt = \frac{1}{2}\left(t+\frac{\sin 2t}{2}\right)\Big|_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2}+\frac{\sin 2(\pi/2)}{2}\right)-\left(0-\frac{\sin 2(0)}{2}\right)\right] = \frac{\pi}{4}$

9. $\int_{-\pi/2}^{\pi/2} (8y^2+\sin y)\, dy = \left[\frac{8y^3}{3}-\cos y\right]_{-\pi/2}^{\pi/2} = \left(\frac{8\left(\frac{\pi}{2}\right)^3}{3}-\cos\frac{\pi}{2}\right)-\left(\frac{8\left(-\frac{\pi}{2}\right)^3}{3}-\cos\left(-\frac{\pi}{2}\right)\right) = \frac{2\pi^3}{3}$

10. $\int_{-1}^{1} (r+1)^2\, dr = \int_{-1}^{1} (r^2+2r+1)\, dr = \left[\frac{r^3}{3}+r^2+r\right]_{-1}^{1} = \left(\frac{1^3}{3}+1^2+1\right)-\left(\frac{(-1)^3}{3}+(-1)^2+(-1)\right) = \frac{8}{3}$

11. $\int_1^{\sqrt{2}} \left(\frac{u^2}{2}-\frac{1}{u^5}\right) du = \int_1^{\sqrt{2}} \left(\frac{u^2}{2}-u^{-5}\right) du = \left(\frac{u^3}{6}+\frac{u^{-4}}{4}\right)\Big|_1^{\sqrt{2}} = \left[\left(\frac{\left(\sqrt{2}\right)^3}{6}+\frac{1}{4\left(\sqrt{2}\right)^4}\right)-\left(\frac{(1)^3}{6}+\frac{1}{4(1)^4}\right)\right]$

$= \frac{\sqrt{2}}{3}+\frac{1}{16}-\frac{1}{6}-\frac{1}{4} = \frac{16\sqrt{2}+3-8-12}{48} = \frac{16\sqrt{2}-17}{48}$

12. $\int_4^9 \frac{1-\sqrt{u}}{\sqrt{u}}\, du = \int_4^9 \left(u^{-1/2}-1\right) du = \left(2u^{1/2}-u\right)\Big|_4^9 = \left[\left(2\sqrt{9}-9\right)-\left(2\sqrt{4}-4\right)\right] = -3-(0) = -3$

13. Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow \frac{du}{2} = x\, dx \Rightarrow \int xe^{x^2}\, dx = \int \frac{1}{2}e^u\, du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$

$\Rightarrow \int_0^1 xe^{x^2}\, dx = \frac{1}{2}e^{x^2}\Big|_0^1 = \frac{e^1}{2}-\frac{e^0}{2} = \frac{e-1}{2}$

14. $\int_0^{\ln 2} e^{3x}\,dx = \frac{e^{3x}}{3}\Big|_0^{\ln 2} = \frac{e^{3\ln 2}}{3} - \frac{e^{3(0)}}{3} = \frac{e^{\ln(2^3)} - 1}{3} = \frac{2^3 - 1}{3} = \frac{7}{3} = 2\frac{1}{3}$

15. (a) $\int_0^{\sqrt{x}} \cos t\,dt = [\sin t]_0^{\sqrt{x}} = \sin\sqrt{x} - \sin 0 = \sin\sqrt{x} \Rightarrow \frac{d}{dx}\left(\int_0^{\sqrt{x}} \cos t\,dt\right) = \frac{d}{dx}(\sin\sqrt{x}) = \cos\sqrt{x}\left(\frac{1}{2}x^{-1/2}\right)$
$= \frac{\cos\sqrt{x}}{2\sqrt{x}}$

(b) $\frac{d}{dx}\left(\int_0^{\sqrt{x}} \cos t\,dt\right) = (\cos\sqrt{x})\left(\frac{d}{dx}(\sqrt{x})\right) = (\cos\sqrt{x})\left(\frac{1}{2}x^{-1/2}\right) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$

16. (a) $\int_1^{\sin x} 3t^2\,dt = [t^3]_1^{\sin x} = \sin^3 x - 1 \Rightarrow \frac{d}{dx}\left(\int_1^{\sin x} 3t^2\,dt\right) = \frac{d}{dx}(\sin^3 x - 1) = 3\sin^2 x\cos x$

(b) $\frac{d}{dx}\left(\int_1^{\sin x} 3t^2\,dt\right) = (3\sin^2 x)\left(\frac{d}{dx}(\sin x)\right) = 3\sin^2 x\cos x$

17. (a) $\int_0^{t^4} \sqrt{u}\,du = \int_0^{t^4} u^{1/2}\,du = \left[\frac{2}{3}u^{3/2}\right]_0^{t^4} = \frac{2}{3}(t^4)^{3/2} - 0 = \frac{2}{3}t^6 \Rightarrow \frac{d}{dt}\left(\int_0^{t^4} \sqrt{u}\,du\right) = \frac{d}{dt}\left(\frac{2}{3}t^6\right) = 4t^5$

(b) $\frac{d}{dt}\left(\int_0^{t^4} \sqrt{u}\,du\right) = \sqrt{t^4}\left(\frac{d}{dt}(t^4)\right) = t^2(4t^3) = 4t^5$

18. (a) $\int_0^{\tan\theta} \sec^2 y\,dy = [\tan y]_0^{\tan\theta} = \tan(\tan\theta) - 0 = \tan(\tan\theta) \Rightarrow \frac{d}{d\theta}\left(\int_0^{\tan\theta} \sec^2 y\,dy\right) = \frac{d}{d\theta}(\tan(\tan\theta))$
$= (\sec^2(\tan\theta))\sec^2\theta$

(b) $\frac{d}{d\theta}\left(\int_0^{\tan\theta} \sec^2 y\,dy\right) = (\sec^2(\tan\theta))\left(\frac{d}{d\theta}(\tan\theta)\right) = (\sec^2(\tan\theta))\sec^2\theta$

19. $y = \int_0^x \sqrt{1+t^2}\,dt \Rightarrow \frac{dy}{dx} = \sqrt{1+x^2}$

20. $y = \int_1^x \frac{1}{t}\,dt \Rightarrow \frac{dy}{dx} = \frac{1}{x},\ x > 0$

21. $y = \int_{\sqrt{x}}^0 \sin t^2\,dt \Rightarrow y = -\int_0^{\sqrt{x}} \sin t^2\,dt \Rightarrow \frac{dy}{dx} = -\left(\sin\left(\sqrt{x}\right)^2\right)\left(\frac{d}{dx}(\sqrt{x})\right) = -(\sin x)\left(\frac{1}{2}x^{-1/2}\right) = -\frac{\sin x}{2\sqrt{x}}$

22. $y = \int_0^{x^2} \cos\sqrt{t}\, dt \Rightarrow \frac{dy}{dx} = \left(\cos\sqrt{x^2}\right)\left(\frac{d}{dx}(x^2)\right) = 2x\cos x$

23. $y = \int_1^{x^{1/3}} e^{\left(t^3+1\right)}\, dt$; let $u = x^{1/3} \Rightarrow \frac{du}{dx} = \frac{1}{3}x^{-2/3}$ and $y = \int_1^u e^{\left(t^3+1\right)}\, dt$

From the Fundamental Theorem, Part 1, $\frac{dy}{du} = e^{\left(u^3+1\right)}$ so that the Chain Rule gives

$$\frac{d}{dx}\int_1^{x^{1/3}} e^{\left(t^3+1\right)}\, dt = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(e^{\left(u^3+1\right)}\right)\left(\frac{1}{3}x^{-2/3}\right) = \frac{e^{(x+1)}}{3x^{2/3}}.$$

24. $y = \int_{e^x}^{e} \ln t\, dt$, $x > 1$; let $u = e^x \Rightarrow \frac{du}{dx} = e^x$ and $y = \int_u^e \ln t\, dt = -\int_e^u \ln t\, dt.$

From the Fundamental Theorem, Part 1, $\frac{dy}{du} = -\ln u$ so that the Chain Rule gives

$$\frac{d}{dx}\int_{e^x}^{e} \ln t\, dt = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (-\ln u)\, e^x = e^x(-\ln e^x) = e^x \ln e^{-x} = -xe^x.$$

25. Let $u = 1 - 2x \Rightarrow du = -2\, dx$

$$\int (1-2x)^3\, dx = \int -\frac{1}{2}u^3\, du = -\frac{1}{8}u^4 + C \Rightarrow \int_0^1 (1-2x)^3\, dx = \left[-\frac{1}{8}(1-2x)^4\right]_0^1 = -\frac{1}{8}(-1)^4 - \left(-\frac{1}{8}\right)(1)^4 = 0$$

26. Let $u = t^2 + 1 \Rightarrow du = 2t\, dt$

$$\int t\sqrt{t^2+1}\, dt = \int \frac{1}{2}u^{1/2}\, du = \frac{1}{3}u^{3/2} + C \Rightarrow \int_0^1 t\sqrt{t^2+1}\, dt = \left[\frac{1}{3}\left(t^2+1\right)^{3/2}\right]_0^1 = \frac{1}{3}(2)^{3/2} - \frac{1}{3}(1)^{3/2}$$

$$= \frac{1}{3}\left(2\sqrt{2} - 1\right)$$

27. Let $u = 1 + \frac{\theta}{2} \Rightarrow du = \frac{1}{2}\, d\theta$

$$\int \sin^2\left(1+\frac{\theta}{2}\right) d\theta = \int 2\sin^2 u\, du = 2\left(\frac{u}{2} - \frac{1}{4}\sin 2u\right) + C \Rightarrow \int_0^{\pi} \sin^2\left(1+\frac{\theta}{2}\right) d\theta = \left[\left(1+\frac{\theta}{2}\right) - \frac{1}{2}\sin(2+\theta)\right]_0^{\pi}$$

$$= \left[\left(1+\frac{\pi}{2}\right) - \frac{1}{2}\sin(2+\pi)\right] - \left(1 - \frac{1}{2}\sin 2\right) = \frac{\pi}{2} + \sin 2$$

28. Let $u = \sin\frac{x}{4} \Rightarrow du = \frac{1}{4}\cos\frac{x}{4}\, dx$

$$\int \sin^2\frac{x}{4}\cos\frac{x}{4}\, dx = \int 4u^2\, du = \frac{4}{3}u^3 + C \Rightarrow \int_0^{\pi} \sin^2\frac{x}{4}\cos\frac{x}{4}\, dx = \left[\frac{4}{3}\sin^3\frac{x}{4}\right]_0^{\pi} = \frac{4}{3}\sin^3\frac{\pi}{4} - \frac{4}{3}\sin^3 0$$

$$= \frac{4}{3}\left(\frac{\sqrt{2}}{2}\right)^3 = \frac{2\sqrt{2}}{6} = \frac{\sqrt{2}}{3}$$

29. Using the method in Example 6, $y = \int_2^x \sec t \, dt + 3$. Evaluating the integral gives $y = \ln|\sec t + \tan t|\Big|_2^x + 3$ (see Exercise 65 in Section 4.2) $\Rightarrow y = \ln\left|\frac{\sec x + \tan x}{\sec 2 + \tan 2}\right| + 3$.

30. Using the method in Example 6, $y = \int_1^x t\sqrt{1+t^2}\, dt - 2$. Evaluating the integral gives $y = \frac{1}{3}\left(1+t^2\right)^{3/2}\Big|_0^x - 2$

$= \frac{1}{3}\left(1+x^2\right)^{3/2} - \frac{2\sqrt{2}}{3} - 2 = \frac{1}{3}\left[\left(1+x^2\right)^{1/2} - 2\sqrt{2} - 6\right].$

31. $\frac{dy}{dt} = e^t \sin\left(e^t - 2\right) \Rightarrow y = \int e^t \sin\left(e^t - 2\right) dt;$

let $u = e^t - 2 \Rightarrow du = e^t\, dt \Rightarrow y = \int \sin u\, du = -\cos u + C = -\cos\left(e^t - 2\right) + C$; $y(\ln 2) = 0$

$\Rightarrow -\cos\left(e^{\ln 2} - 2\right) + C = 0 \Rightarrow -\cos(2-2) + C = 0 \Rightarrow C = \cos 0 = 1$; thus, $y = 1 - \cos\left(e^t - 2\right)$

32. $\frac{d^2y}{dt^2} = 1 - e^{2t} \Rightarrow \frac{dy}{dt} = t - \frac{1}{2}e^{2t} + C$; $t = 1$ and $\frac{dy}{dt} = 0 \Rightarrow 0 = 1 - \frac{1}{2}e^2 + C \Rightarrow C = \frac{1}{2}e^2 - 1$; thus

$\frac{dy}{dt} = t - \frac{1}{2}e^{2t} + \frac{1}{2}e^2 - 1 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t + C_1$; $t = 1$ and $y = -1 \Rightarrow -1 = \frac{1}{2} - \frac{1}{4}e^2 + \frac{1}{2}e^2 - 1 + C_1$

$\Rightarrow C_1 = -\frac{1}{2} - \frac{1}{4}e^2 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t - \left(\frac{1}{2} + \frac{1}{4}e^2\right)$

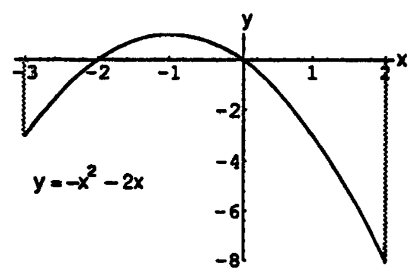
33. $-x^2 - 2x = 0 \Rightarrow -x(x+2) = 0 \Rightarrow x = 0$ or $x = -2$; Area

$= -\int_{-3}^{-2} \left(-x^2 - 2x\right) dx + \int_{-2}^{0} \left(-x^2 - 2x\right) dx - \int_{0}^{2} \left(-x^2 - 2x\right) dx$

$= -\left[-\frac{x^3}{3} - x^2\right]_{-3}^{-2} + \left[-\frac{x^3}{3} - x^2\right]_{-2}^{0} - \left[-\frac{x^3}{3} - x^2\right]_{0}^{2}$

$= -\left[\left(-\frac{(-2)^3}{3} - (-2)^2\right) - \left(-\frac{(-3)^3}{3} - (-3)^2\right)\right]$

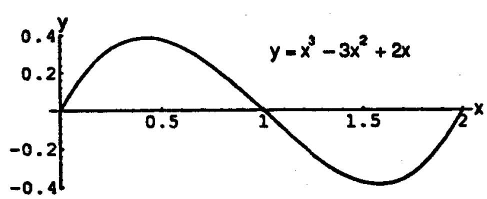
$+\left[\left(-\frac{0^3}{3} - 0^2\right) - \left(-\frac{(-2)^3}{3} - (-2)^2\right)\right] - \left[\left(-\frac{2^3}{3} - 2^2\right) - \left(-\frac{0^3}{3} - 0^2\right)\right] = \frac{28}{3}$



34. $x^3 - 3x^2 + 2x = 0 \Rightarrow x\left(x^2 - 3x + 2\right) = 0$

$\Rightarrow x(x-2)(x-1) = 0$, $x = 0, 1$, or 2;

Area $= \int_0^1 \left(x^3 - 3x^2 + 2x\right) dx - \int_1^2 \left(x^3 - 3x^2 + 2x\right) dx$



$= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 - \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2$

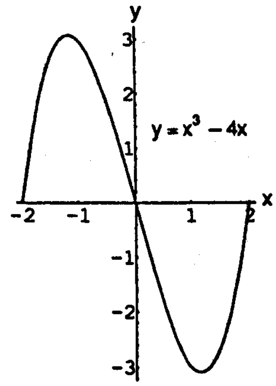
$= \left(\frac{1^4}{4} - 1^3 + 1^2\right) - \left(\frac{0^4}{4} - 0^3 + 0^2\right) - \left[\left(\frac{2^4}{4} - 2^3 + 2^2\right) - \left(\frac{1^4}{4} - 1^3 + 1^2\right)\right] = \frac{1}{2}$

35. $x^3 - 4x = 0 \Rightarrow x(x^2 - 4) = 0 \Rightarrow x(x-2)(x+2) = 0$

$\Rightarrow x = 0,\ 2,\ \text{or } -2;\ \text{Area} = \int_{-2}^{0} (x^3 - 4x)\,dx - \int_{0}^{2} (x^3 - 4x)\,dx$

$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^0 - \left[\frac{x^4}{4} - 2x^2\right]_0^2 = \left(\frac{0^4}{4} - 2(0)^2\right)$

$- \left(\frac{(-2)^4}{4} - 2(-2)^2\right) - \left[\left(\frac{2^4}{4} - 2(2)^2\right) - \left(\frac{0^4}{4} - 2(0)^2\right)\right] = 8$



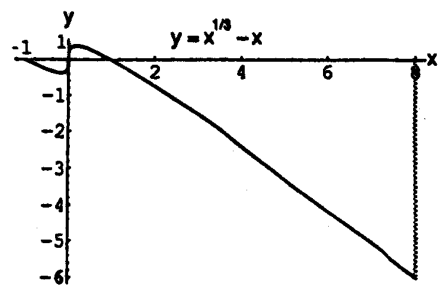
36. $x^{1/3} - x = 0 \Rightarrow x^{1/3}(1 - x^{2/3}) = 0 \Rightarrow x^{1/3} = 0$ or
$1 - x^{2/3} = 0 \Rightarrow x = 0$ or $1 = x^{2/3} \Rightarrow x = 0$ or
$1 = x^2 \Rightarrow x = 0$ or ± 1;

$\text{Area} = -\int_{-1}^{0} (x^{1/3} - x)\,dx + \int_{0}^{1} (x^{1/3} - x)\,dx - \int_{1}^{8} (x^{1/3} - x)\,dx$

$= -\left[\frac{3}{4}x^{4/3} - \frac{x^2}{2}\right]_{-1}^0 + \left[\frac{3}{4}x^{4/3} - \frac{x^2}{2}\right]_0^1 - \left[\frac{3}{4}x^{4/3} - \frac{x^2}{2}\right]_1^8$

$= -\left[\left(\frac{3}{4}(0)^{4/3} - \frac{0^2}{2}\right) - \left(\frac{3}{4}(-1)^{4/3} - \frac{(-1)^2}{2}\right)\right] + \left[\left(\frac{3}{4}(1)^{4/3} - \frac{1^2}{2}\right) - \left(\frac{3}{4}(0)^{4/3} - \frac{0^2}{2}\right)\right]$

$- \left[\left(\frac{3}{4}(8)^{4/3} - \frac{8^2}{2}\right) - \left(\frac{3}{4}(1)^{4/3} - \frac{1^2}{2}\right)\right] = \frac{1}{4} + \frac{1}{4} - \left(12 - \frac{64}{2}\right) + \frac{1}{4} = \frac{83}{4}$



37. The area of the rectangle bounded by the lines $y = 2$, $y = 0$, $x = \pi$, and $x = 0$ is 2π. The area under the curve $y = 1 + \cos x$ on $[0, \pi]$ is $\int_0^{\pi} (1 + \cos x)\,dx = [x + \sin x]_0^{\pi} = (\pi + \sin \pi) - (0 + \sin 0) = \pi$. Therefore the area of the shaded region is $2\pi - \pi = \pi$.

38. The area of the rectangle bounded by the lines $y = 2$, $y = 0$, $t = -\frac{\pi}{4}$, and $t = 1$ is $2\left(1 - \left(-\frac{\pi}{4}\right)\right) = 2 + \frac{\pi}{2}$. The area under the curve $y = \sec^2 t$ on $\left[-\frac{\pi}{4}, 0\right]$ is $\int_{-\pi/4}^{0} \sec^2 t\,dt = [\tan t]_{-\pi/4}^{0} = \tan 0 - \tan\left(-\frac{\pi}{4}\right) = 1$. The area under the curve $y = 1 - t^2$ on $[0, 1]$ is $\int_0^1 (1 - t^2)\,dt = \left[t - \frac{t^3}{3}\right]_0^1 = \left(1 - \frac{1^3}{3}\right) - \left(0 - \frac{0^3}{3}\right) = \frac{2}{3}$. Thus, the total area under the curves on $\left[-\frac{\pi}{4}, 1\right]$ is $1 + \frac{2}{3} = \frac{5}{3}$. Therefore the area of the shaded region is $\left(2 + \frac{\pi}{2}\right) - \frac{5}{3} = \frac{1}{3} + \frac{\pi}{2}$.

39. $\frac{dc}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2} \Rightarrow c = \int_0^x \frac{1}{2}t^{-1/2}dt = \left[t^{1/2}\right]_0^x = \sqrt{x}$

(a) $c(100) - c(1) = \sqrt{100} - \sqrt{1} = \$9.00$ (b) $c(400) - c(100) = \sqrt{400} - \sqrt{100} = \$10.00$

40. $r = \int_0^3 \left(2 - \frac{2}{(x+1)^2}\right) dx = 2\int_0^3 \left(1 - \frac{1}{(x+1)^2}\right) dx = 2\left[x - \left(\frac{-1}{x+1}\right)\right]_0^3 = 2\left[\left(3 + \frac{1}{(3+1)}\right) - \left(0 + \frac{1}{(0+1)}\right)\right]$

$= 2\left[3\frac{1}{4} - 1\right] = 2\left(2\frac{1}{4}\right) = 4.5$ or \$4500

41. (a) $v = \frac{ds}{dt} = \frac{d}{dt}\int_0^t f(x)\,dx = f(t) \Rightarrow v(5) = f(5) = 2$ m/sec

(b) $a = \frac{df}{dt}$ is negative since the slope of the tangent line at $t = 5$ is negative.

(c) $s = \int_0^3 f(x)\,dx = \frac{1}{2}(3)(3) = \frac{9}{2}$ m since the integral is the area of the triangle formed by $y = f(x)$, the x-axis, and $x = 3$.

(d) $t = 6$ since after $t = 6$ to $t = 9$, the region lies below the x-axis.

(e) At $t = 4$ and $t = 7$, since there are horizontal tangents there.

(f) Toward the origin between $t = 6$ and $t = 9$ since the velocity is negative on this interval. Away from the origin between $t = 0$ and $t = 6$ since the velocity is positive there.

(g) Right or positive side, because the integral of f from 0 to 9 is positive, there being more area above the x-axis than below it.

42. (a) $v = \frac{ds}{dt} = \frac{d}{dt}\int_0^t f(x)\,dx = f(t) \Rightarrow v(3) = f(3) = 0$ m/sec.

(b) $a = \frac{df}{dt}$ is positive, since the slope of the tangent line at $t = 3$ is positive.

(c) At $t = 3$, the particle's position is $\int_0^3 f(x)\,dx = \frac{1}{2}(3)(-6) = -9$.

(d) The particle passes through the origin at $t = 6$ because $s(6) = \int_0^6 f(x)\,dx = 0$.

(e) At $t = 7$, since there is a horizontal tangent there.

(f) The particle starts at the origin and moves away to the left for $0 < t < 3$. It moves back toward the origin for $3 < t < 6$, passes through the origin at $t = 6$, and moves away to the right for $t > 6$.

(g) Right side, since its position at $t = 9$ is positive, there being more area above the x-axis than below it.

43. $\int_4^8 \pi(64-x^2)\,dx = \pi\left[64x - \frac{x^3}{3}\right]_4^8 = \pi\left[\left(512 - \frac{512}{3}\right) - \left(256 - \frac{64}{3}\right)\right] = \pi\left(256 - \frac{448}{3}\right) = \frac{320\pi}{3}$

44. $\int_0^5 \pi\left(\sqrt{x}\right)^2 dx = \pi \int_0^5 x\,dx = \pi\left[\frac{x^2}{2}\right]_0^5 = \pi\left(\frac{25}{2} - \frac{0}{2}\right) = \frac{25\pi}{2}$

45. $\int_1^x f(t)\,dt = x^2 - 2x + 1 \Rightarrow f(x) = \frac{d}{dx}\int_1^x f(t)\,dt = \frac{d}{dx}\left(x^2 - 2x + 1\right) = 2x - 2$

46. $\int_0^x f(t)\,dt = x\cos\pi x \Rightarrow f(x) = \frac{d}{dx}\int_1^x f(t)\,dt = \cos\pi x - \pi x\sin\pi x \Rightarrow f(4) = \cos\pi(4) - \pi(4)\sin\pi(4) = 1$

47. $f(x) = 2 - \int_2^{x+1} \frac{9}{1+t}\,dt \Rightarrow f'(x) = -\frac{9}{1+(x+1)} = \frac{-9}{x+2} \Rightarrow f'(1) = -3;\ f(1) = 2 - \int_2^{1+1} \frac{9}{1+t}\,dt = 2 - 0 = 2;$

$L(x) = f'(1)(x-1) + f(1) = -3(x-1) + 2 = -3x + 5$

48. $g(x) = 3 + \int_1^{x^2} \sec(t-1)\,dt \Rightarrow g'(x) = \left(\sec\left(x^2-1\right)\right)(2x) = 2x\sec\left(x^2-1\right) \Rightarrow g'(-1) = 2(-1)\sec\left((-1)^2-1\right)$

$= -2;\ g(-1) = 3 + \int_1^{(-1)^2} \sec(t-1)\,dt = 3 + \int_1^1 \sec(t-1)\,dt = 3 + 0 = 3;\ L(x) = g'(-1)(x-(-1)) + g(-1)$

$= -2(x+1) + 3 = -2x + 1$

49. (a) True: since f is continuous, g is differentiable by Part 1 of the Fundamental Theorem of Calculus.
   (b) True: g is continuous because it is differentiable.
   (c) True, since $g'(1) = f(1) = 0$.
   (d) False, since $g''(1) = f'(1) > 0$.
   (e) True, since $g'(1) = 0$ and $g''(1) = f'(1) > 0$.
   (f) False: $g''(x) = f'(x) > 0$, so g'' never changes sign.
   (g) True, since $g'(1) = f(1) = 0$ and $g'(x) = f(x)$ is an increasing function of x (because $f'(x) > 0$).

50. (a) True: by Part 1 of the Fundamental Theorem of Calculus, $h'(x) = f(x)$. Since f is differentiable for all x, h has a second derivative for all x.
   (b) True: they are continuous because they are differentiable.
   (c) True, since $h'(1) = f(1) = 0$.
   (d) True, since $h'(1) = 0$ and $h''(1) = f'(1) < 0$.
   (e) False, since $h''(1) = f'(1) < 0$.
   (f) False, since $h''(x) = f'(x) < 0$ never changes sign.

(g) True, since $h'(1) = f(1) = 0$ and $h'(x) = f(x)$ is a decreasing function of x (because $f'(x) < 0$).

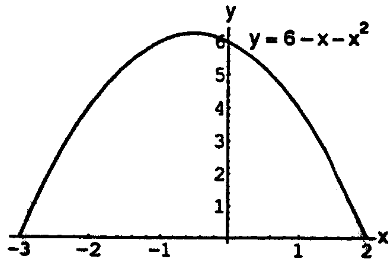
51. (a) $6 - x - x^2 = 0 \Rightarrow x^2 + x - 6 = 0$

$\Rightarrow (x+3)(x-2) = 0 \Rightarrow x = -3$ or $x = 2$;

$$\text{Area} = \int_{-3}^{2} (6 - x - x^2)\,dx = \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$$

$$= \left(6(2) - \frac{2^2}{2} - \frac{2^3}{3}\right) - \left(6(-3) - \frac{(-3)^2}{2} - \frac{(-3)^3}{3}\right)$$

$$= \frac{125}{6}$$



(b) $y' = -1 - 2x = 0 \Rightarrow x = -\frac{1}{2}$; $y' > 0$ for $x < -\frac{1}{2}$ and $y' < 0$ for $x > -\frac{1}{2} \Rightarrow x = -\frac{1}{2}$ yields a local maximum

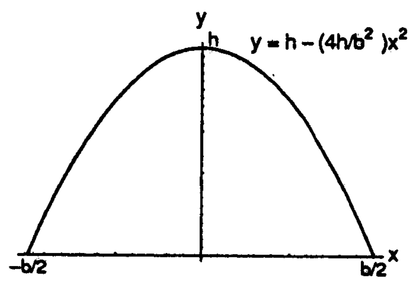
$\Rightarrow \text{height} = y\left(-\frac{1}{2}\right) = \frac{25}{4}$

(c) Base $= 2 - (-3) = 5$, height $= y\left(-\frac{1}{2}\right) = \frac{25}{4} \Rightarrow \text{Area} = \frac{2}{3}(\text{Base})(\text{Height}) = \frac{2}{3}(5)\left(\frac{25}{4}\right) = \frac{125}{6}$

(d) $$\text{Area} = \int_{-b/2}^{b/2} \left(h - \left(\frac{4h}{b^2}\right)x^2\right) dx = \left[hx - \frac{4hx^3}{3b^2}\right]_{-b/2}^{b/2}$$

$$= \left(h\left(\frac{b}{2}\right) - \frac{4h\left(\frac{b}{2}\right)^3}{3b^2}\right) - \left(h\left(-\frac{b}{2}\right) - \frac{4h\left(-\frac{b}{2}\right)^3}{3b^2}\right)$$

$$= \left(\frac{bh}{2} - \frac{bh}{6}\right) - \left(-\frac{bh}{2} + \frac{bh}{6}\right) = bh - \frac{bh}{3} = \frac{2}{3}bh$$



52. (a) $$\left(\frac{1}{\frac{1}{60} - 0}\right) \int_{0}^{1/60} V_{max} \sin 120\pi t\,dt = 60\left[-V_{max}\left(\frac{1}{120\pi}\right)\cos(120\pi t)\right]_0^{1/60} = -\frac{V_{max}}{2\pi}[\cos 2\pi - \cos 0]$$

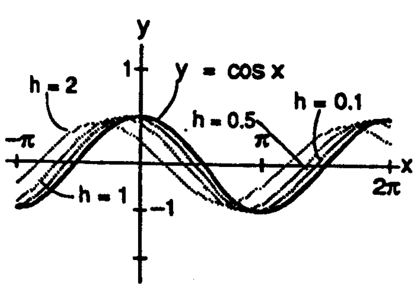
$$= -\frac{V_{max}}{2\pi}[1 - 1] = 0$$

(b) $V_{max} = \sqrt{2}\,V_{rms} = \sqrt{2}\,(240) \approx 339$ volts

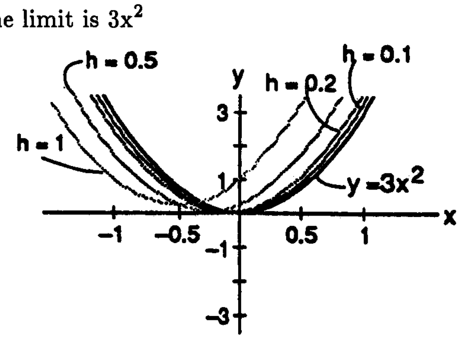
(c) $$\int_0^{1/60} (V_{max})^2 \sin^2 120\pi t\,dt = (V_{max})^2 \int_0^{1/60} \left(\frac{1 - \cos 240\pi t}{2}\right) dt = \frac{(V_{max})^2}{2} \int_0^{1/60} (1 - \cos 240\pi t)\,dt$$

$$= \frac{(V_{max})^2}{2}\left[t - \left(\frac{1}{240\pi}\right)\sin 240\pi t\right]_0^{1/60} = \frac{(V_{max})^2}{2}\left[\left(\frac{1}{60} - \left(\frac{1}{240\pi}\right)\sin(2\pi)\right) - \left(0 - \left(\frac{1}{240\pi}\right)\sin(0)\right)\right] = \frac{(V_{max})^2}{120}$$

53.



54. The limit is $3x^2$



55-58. Example CAS commands:

Maple:

```
f:=x -> x ∧ 3 − 4*x ∧ 2 + 3*x;
F:=x -> int(f(t),t=0..x);
plot({F(x),f(x)},x=0..3.75);
solve(diff(F(x),x),x);
plot({diff(f(x),x),F(x)}, x=0..3);
map(evalf,[solve(diff(f(x),x)=0)]);
```

Mathematica:

```
Clear[x]
{a,b} = {0,2Pi}; f[x_] = Sin[2x] Cos[x/3]
F[x_] = Integrate[ f[t], {t,a,x} ]
Plot[ {f[x],F[x]}, {x,a,b} ]
x /. Map[
   FindRoot[ F′ [x] == 0, {x,#} ]&,
   {2,3,5,6} ]
x /. Map[
   FindRoot[ f′[x] == 0, {x,#} ]&,
   {1,2,4,5,6} ]
```

59-62. Example CAS commands:

Maple:

```
f:=x -> sqrt(1 −x ∧ 2);
u:=x -> x ∧ 2;
F:=x -> int(f(t),t=1..u(x));
dFx:=diff(F(x),x);
simplify(%);
solve(dFx=0,x);
dFxx:=diff(F(x),x$2);
simplify(%);
solve(dFxx=0,x);
evalf(%);
plot (F(x),x=−1..1);
```

Mathematica:

```
a = 1; u[x_] = x ∧ 2; f[x_] = Sqrt[ 1 − x ∧ 2 ]
F[x_] = Integrate[ f[t], {t,a,u[x]} ]
F′[x]
x /. NSolve[ F′[x] == 0, x ]
F″[x]
x /. NSolve [ F″[x] == 0, x]
Plot[ F[x], {x,−1,1} ]
```

63. In Maple type diff(int(f(x),x=a..u(x)),x); or, in Mathematica type $\partial_{x} \int_{a}^{u[x]} f[t]\, dt$

64. In Maple type diff(int(f(x),x=a..u(x)),x,x); or in Mathematica type $\partial_{x,x} \left(\int_{a}^{u[x]} f[t]\, dt \right)$

4.6 SUBSTITUTION IN DEFINITE INTEGRALS

1. (a) Let $u = y + 1 \Rightarrow du = dy$; $y = 0 \Rightarrow u = 1$, $y = 3 \Rightarrow u = 4$

$$\int_0^3 \sqrt{y+1}\,dy = \int_1^4 u^{1/2}\,du = \left[\frac{2}{3}u^{3/2}\right]_1^4 = \left(\frac{2}{3}\right)(4)^{3/2} - \left(\frac{2}{3}\right)(1)^{3/2} = \left(\frac{2}{3}\right)(8) - \left(\frac{2}{3}\right)(1) = \frac{14}{3}$$

(b) Use the same substitution for u as in part (a); $y = -1 \Rightarrow u = 0$, $y = 0 \Rightarrow u = 1$

$$\int_{-1}^0 \sqrt{y+1}\,dy = \int_0^1 u^{1/2}\,du = \left[\frac{2}{3}u^{3/2}\right]_0^1 = \left(\frac{2}{3}\right)(1)^{3/2} - 0 = \frac{2}{3}$$

2. (a) Let $u = \tan x \Rightarrow du = \sec^2 x\,dx$; $x = 0 \Rightarrow u = 0$, $x = \frac{\pi}{4} \Rightarrow u = 1$

$$\int_0^{\pi/4} \tan x \sec^2 x\,dx = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1^2}{2} - 0 = \frac{1}{2}$$

(b) Use the same substitution as in part (a); $x = -\frac{\pi}{4} \Rightarrow u = -1$, $x = 0 \Rightarrow u = 0$

$$\int_{-\pi/4}^0 \tan x \sec^2 x\,dx = \int_{-1}^0 u\,du = \left[\frac{u^2}{2}\right]_{-1}^0 = 0 - \frac{1}{2} = -\frac{1}{2}$$

3. (a) Let $u = \cos x \Rightarrow du = -\sin x\,dx \Rightarrow -du = \sin x\,dx$; $x = 0 \Rightarrow u = 1$, $x = \pi \Rightarrow u = -1$

$$\int_0^{\pi} 3\cos^2 x \sin x\,dx = \int_1^{-1} -3u^2\,du = \left[-u^3\right]_1^{-1} = -(-1)^3 - \left(-(1)^3\right) = 2$$

(b) Use the same substitution as in part (a); $x = 2\pi \Rightarrow u = 1$, $x = 3\pi \Rightarrow u = -1$

$$\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\,dx = \int_1^{-1} -3u^2\,du = 2$$

4. (a) Let $u = t^2 + 1 \Rightarrow du = 2t\,dt \Rightarrow \frac{1}{2}\,du = t\,dt$; $t = 0 \Rightarrow u = 1$, $t = \sqrt{7} \Rightarrow u = 8$

$$\int_0^{\sqrt{7}} t\left(t^2+1\right)^{1/3}\,dt = \int_1^8 \frac{1}{2}u^{1/3}\,du = \left[\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)u^{4/3}\right]_1^8 = \left(\frac{3}{8}\right)(8)^{4/3} - \left(\frac{3}{8}\right)(1)^{4/3} = \frac{45}{8}$$

(b) Use the same substitution as in part (a); $t = -\sqrt{7} \Rightarrow u = 8$, $t = 0 \Rightarrow u = 1$

$$\int_{-\sqrt{7}}^0 t\left(t^2+1\right)^{1/3}\,dt = \int_8^1 \frac{1}{2}u^{1/3}\,du = -\int_1^8 \frac{1}{2}u^{1/3}\,du = -\frac{45}{8}$$

5. (a) Let $u = 4 + r^2 \Rightarrow du = 2r\,dr \Rightarrow \frac{1}{2}\,du = r\,dr$; $r = -1 \Rightarrow u = 5$, $r = 1 \Rightarrow u = 5$

$$\int_{-1}^1 \frac{5r}{\left(4+r^2\right)^2}\,dr = 5\int_5^5 \frac{1}{2}u^{-2}\,du = 0$$

(b) Use the same substitution as in part (a); $r = 0 \Rightarrow u = 4$, $r = 1 \Rightarrow u = 5$

$$\int_0^1 \frac{5r}{(4+r^2)^2}\,dr = 5\int_4^5 \tfrac{1}{2}u^{-2}\,du = 5\left[-\tfrac{1}{2}u^{-1}\right]_4^5 = 5\left(-\tfrac{1}{2}(5)^{-1}\right) - 5\left(-\tfrac{1}{2}(4)^{-1}\right) = \tfrac{1}{8}$$

6. (a) Let $u = x^2 + 1 \Rightarrow du = 2x\,dx \Rightarrow 2\,du = 4x\,dx$; $x = 0 \Rightarrow u = 1$, $x = \sqrt{3} \Rightarrow u = 4$

$$\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx = \int_1^4 \frac{2}{\sqrt{u}}\,du = \int_1^4 2u^{-1/2}\,du = \left[4u^{1/2}\right]_1^4 = 4(4)^{1/2} - 4(1)^{1/2} = 4$$

(b) Use the same substitution as in part (a); $x = -\sqrt{3} \Rightarrow u = 4$, $x = \sqrt{3} \Rightarrow u = 4$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx = \int_4^4 \frac{2}{\sqrt{u}}\,du = 0$$

7. (a) Let $u = 4 + 3\sin z \Rightarrow du = 3\cos z\,dz \Rightarrow \frac{1}{3}\,du = \cos z\,dz$; $z = 0 \Rightarrow u = 4$, $z = 2\pi \Rightarrow u = 4$

$$\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz = \int_4^4 \frac{1}{\sqrt{u}}\left(\tfrac{1}{3}\,du\right) = 0$$

(b) Use the same substitution as in part (a); $z = -\pi \Rightarrow u = 4 + 3\sin(-\pi) = 4$, $z = \pi \Rightarrow u = 4$

$$\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz = \int_4^4 \frac{1}{\sqrt{u}}\left(\tfrac{1}{3}\,du\right) = 0$$

8. Let $u = t^5 + 2t \Rightarrow du = (5t^4 + 2)\,dt$; $t = 0 \Rightarrow u = 0$, $t = 1 \Rightarrow u = 3$

$$\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\,dt = \int_0^3 u^{1/2}\,du = \left[\tfrac{2}{3}u^{3/2}\right]_0^3 = \tfrac{2}{3}(3)^{3/2} - \tfrac{2}{3}(0)^{3/2} = 2\sqrt{3}$$

9. Let $u = 1 + \sqrt{y} \Rightarrow du = \frac{dy}{2\sqrt{y}}$; $y = 1 \Rightarrow u = 2$, $y = 4 \Rightarrow u = 3$

$$\int_1^4 \frac{dy}{2\sqrt{y}\,(1+\sqrt{y})^2} = \int_2^3 \frac{1}{u^2}\,du = \int_2^3 u^{-2}\,du = \left[-u^{-1}\right]_2^3 = \left(-\tfrac{1}{3}\right) - \left(-\tfrac{1}{2}\right) = \tfrac{1}{6}$$

10. Let $u = \cos 2\theta \Rightarrow du = -2\sin 2\theta\,d\theta \Rightarrow -\frac{1}{2}\,du = \sin 2\theta\,d\theta$; $\theta = 0 \Rightarrow u = 1$, $\theta = \frac{\pi}{6} \Rightarrow u = \cos 2\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$$\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\,d\theta = \int_1^{1/2} u^{-3}\left(-\tfrac{1}{2}\,du\right) = -\tfrac{1}{2}\int_1^{1/2} u^{-3}\,du = \left[-\tfrac{1}{2}\left(\frac{u^{-2}}{-2}\right)\right]_1^{1/2} = \frac{1}{4\left(\frac{1}{2}\right)^2} - \frac{1}{4(1)^2} = \frac{3}{4}$$

11. Let $u = 1 - \sin 2t \Rightarrow du = -2\cos 2t\,dt \Rightarrow -\frac{1}{2}\,du = \cos 2t\,dt$; $t = 0 \Rightarrow u = 1$, $t = \frac{\pi}{4} \Rightarrow u = 0$

$$\int_0^{\pi/4} (1-\sin 2t)^{3/2}\cos 2t\,dt = \int_1^0 -\tfrac{1}{2}u^{3/2}\,du = \left[-\tfrac{1}{2}\left(\tfrac{2}{5}u^{5/2}\right)\right]_1^0 = \left(-\tfrac{1}{5}(0)^{5/2}\right)-\left(-\tfrac{1}{5}(1)^{5/2}\right)=\tfrac{1}{5}$$

12. Let $u = 4y - y^2 + 4y^3 + 1 \Rightarrow du = (4 - 2y + 12y^2)\,dy$; $y = 0 \Rightarrow u = 1$, $y = 1 \Rightarrow u = 4(1) - (1)^2 + 4(1)^3 + 1 = 8$

$$\int_0^1 \left(4y - y^2 + 4y^3 + 1\right)^{-2/3}\left(12y^2 - 2y + 4\right)dy = \int_1^8 u^{-2/3}\,du = \left[3u^{1/3}\right]_1^8 = 3(8)^{1/3} - 3(1)^{1/3} = 3$$

13. Let $u = e^{\sin x} \Rightarrow du = e^{\sin x}\cos x\,dx$; $x = 0 \Rightarrow u = 1$, $x = \pi/2 \Rightarrow u = e$

$$\int_0^{\pi/2} e^{\sin x}\cos x\,dx = \int_1^e du = u\big|_1^e = e - 1$$ (Note: Letting $u = \sin x$ also works.)

14. Let $u = \tan\theta \Rightarrow du = \sec^2\theta\,d\theta$; $\theta = 0 \Rightarrow u = 0$, $\theta = \frac{\pi}{4} \Rightarrow u = 1$;

$$\int_0^{\pi/4} \left(1 + e^{\tan\theta}\right)\sec^2\theta\,d\theta = \int_0^{\pi/4}\sec^2\theta\,d\theta + \int_0^1 e^u\,du = [\tan\theta]_0^{\pi/4} + \left[e^u\right]_0^1 = \left[\tan\left(\tfrac{\pi}{4}\right) - \tan(0)\right] + \left(e^1 - e^0\right)$$
$$= (1-0) + (e-1) = e$$

15. Let $u = e^v \Rightarrow du = e^v\,dv \Rightarrow 2\,du = 2e^v\,dv$; $v = \ln\frac{\pi}{6} \Rightarrow u = \frac{\pi}{6}$, $v = \ln\frac{\pi}{2} \Rightarrow u = \frac{\pi}{2}$;

$$\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^v\cos e^v\,dv = 2\int_{\pi/6}^{\pi/2}\cos u\,du = [2\sin u]_{\pi/6}^{\pi/2} = 2\left[\sin\left(\tfrac{\pi}{2}\right) - \sin\left(\tfrac{\pi}{6}\right)\right] = 2\left(1 - \tfrac{1}{2}\right) = 1$$

16. Let $u = e^{x^2} \Rightarrow du = 2xe^{x^2}\,dx$; $x = 0 \Rightarrow u = 1$, $x = \sqrt{\ln\pi} \Rightarrow u = e^{\ln\pi} = \pi$;

$$\int_0^{\sqrt{\ln\pi}} 2xe^{x^2}\cos\left(e^{x^2}\right)dx = \int_1^{\pi}\cos u\,du = [\sin u]_1^{\pi} = \sin(\pi) - \sin(1) = -\sin(1) \approx -0.84147$$

17. $\frac{dy}{dt} = e^{-t}\sec^2\left(\pi e^{-t}\right) \Rightarrow y = \int e^{-t}\sec^2\left(\pi e^{-t}\right)dt$;

let $u = \pi e^{-t} \Rightarrow du = -\pi e^{-t}\,dt \Rightarrow -\frac{1}{\pi}\,du = e^{-t}\,dt \Rightarrow y = -\frac{1}{\pi}\int\sec^2 u\,du = -\frac{1}{\pi}\tan u + C$

$= -\frac{1}{\pi}\tan\left(\pi e^{-t}\right) + C$; $y(\ln 4) = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi}\tan\left(\pi e^{-\ln 4}\right) + C = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi}\tan\left(\pi\cdot\frac{1}{4}\right) + C = \frac{2}{\pi}$

$\Rightarrow -\frac{1}{\pi}(1) + C = \frac{2}{\pi} \Rightarrow C = \frac{3}{\pi}$; thus, $y = \frac{3}{\pi} - \frac{1}{\pi}\tan\left(\pi e^{-t}\right)$

18. $\frac{d^2y}{dx^2} = 2e^{-x} \Rightarrow \frac{dy}{dx} = -2e^{-x} + C$; $x = 0$ and $\frac{dy}{dx} = 0 \Rightarrow 0 = -2e^0 + C \Rightarrow C = 2$; thus $\frac{dy}{dx} = -2e^{-x} + 2$

$\Rightarrow y = 2e^{-x} + 2x + C_1$; $x = 0$ and $y = 1 \Rightarrow 1 = 2e^0 + C_1 \Rightarrow C_1 = -1 \Rightarrow y = 2e^{-x} + 2x - 1 = 2\left(e^{-x} + x\right) - 1$

19. For the sketch given, $a = 0$, $b = \pi$; $f(x) - g(x) = 1 - \cos^2 x = \sin^2 x = \frac{1 - \cos 2x}{2}$;

$$A = \int_0^{\pi} \frac{(1 - \cos 2x)}{2}\,dx = \frac{1}{2}\int_0^{\pi} (1 - \cos 2x)\,dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}[(\pi - 0) - (0 - 0)] = \frac{\pi}{2}$$

20. For the sketch given, $a = -\frac{\pi}{3}$, $b = \frac{\pi}{3}$; $f(t) - g(t) = \frac{1}{2}\sec^2 t - \left(-4\sin^2 t\right) = \frac{1}{2}\sec^2 t + 4\sin^2 t$;

$$A = \int_{-\pi/3}^{\pi/3} \left(\frac{1}{2}\sec^2 t + 4\sin^2 t\right) dt = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \sec^2 t\,dt + 4\int_{-\pi/3}^{\pi/3} \sin^2 t\,dt = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \sec^2 t\,dt + 4\int_{-\pi/3}^{\pi/3} \frac{(1 - \cos 2t)}{2}\,dt$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \sec^2 t\,dt + 2\int_{-\pi/3}^{\pi/3} (1 - \cos 2t)\,dt = \frac{1}{2}[\tan t]_{-\pi/3}^{\pi/3} + 2\left[t - \frac{1}{2}\sin 2t\right]_{-\pi/3}^{\pi/3} = \sqrt{3} + 4\cdot\frac{\pi}{3} - \sqrt{3} = \frac{4\pi}{3}$$

21. For the sketch given, $a = -2$, $b = 2$; $f(x) - g(x) = 2x^2 - \left(x^4 - 2x^2\right) = 4x^2 - x^4$;

$$A = \int_{-2}^{2} \left(4x^2 - x^4\right) dx = \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_{-2}^{2} = \left(\frac{32}{3} - \frac{32}{5}\right) - \left[-\frac{32}{3} - \left(-\frac{32}{5}\right)\right] = \frac{64}{3} - \frac{64}{5} = \frac{320 - 192}{15} = \frac{128}{15}$$

22. For the sketch given, $a = -1$, $b = 1$; $f(x) - g(x) = x^2 - \left(-2x^4\right) = x^2 + 2x^4$;

$$A = \int_{-1}^{1} \left(x^2 + 2x^4\right) dx = \left[\frac{x^3}{3} + \frac{2x^5}{5}\right]_{-1}^{1} = \left(\frac{1}{3} + \frac{2}{5}\right) - \left[-\frac{1}{3} + \left(-\frac{2}{5}\right)\right] = \frac{2}{3} + \frac{4}{5} = \frac{10 + 12}{15} = \frac{22}{15}$$

23. AREA = A1 + A2

A1: For the sketch given, $a = -3$ and we find b by solving the equations $y = x^2 - 4$ and $y = -x^2 - 2x$ simultaneously for x: $x^2 - 4 = -x^2 - 2x \Rightarrow 2x^2 + 2x - 4 = 0 \Rightarrow 2(x + 2)(x - 1) \Rightarrow x = -2$ or $x = 1$ so

$b = -2$: $f(x) - g(x) = \left(x^2 - 4\right) - \left(-x^2 - 2x\right) = 2x^2 + 2x - 4 \Rightarrow A1 = \int_{-3}^{-2} \left(2x^2 + 2x - 4\right) dx$

$$= \left[\frac{2x^3}{3} + \frac{2x^2}{2} - 4x\right]_{-3}^{-2} = \left(-\frac{16}{3} + 4 + 8\right) - (-18 + 9 + 12) = 9 - \frac{16}{3} = \frac{11}{3};$$

A2: For the sketch given, $a = -2$ and $b = 1$: $f(x) - g(x) = \left(-x^2 - 2x\right) - \left(x^2 - 4\right) = -2x^2 - 2x + 4$

$$\Rightarrow A2 = -\int_{-2}^{1} \left(2x^2 + 2x - 4\right) dx = -\left[\frac{2x^3}{3} + x^2 - 4x\right]_{-2}^{1} = -\left(\frac{2}{3} + 1 - 4\right) + \left(-\frac{16}{3} + 4 + 8\right)$$

$$= -\frac{2}{3} - 1 + 4 - \frac{16}{3} + 4 + 8 = 9;$$

Therefore, AREA $= A1 + A2 = \frac{11}{3} + 9 = \frac{38}{3}$

24. AREA = A1 + A2 + A3

A1: For the sketch given, $a = -2$ and $b = -1$: $f(x) - g(x) = (-x + 2) - \left(4 - x^2\right) = x^2 - x - 2$

$$\Rightarrow A1 = \int_{-2}^{-1} \left(x^2 - x - 2\right) dx = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-2}^{-1} = \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(-\frac{8}{3} - \frac{4}{2} + 4\right) = \frac{7}{3} - \frac{1}{2} = \frac{14 - 3}{6} = \frac{11}{6};$$

A2: For the sketch given, a = −1 and b = 2: $f(x) - g(x) = (4 - x^2) - (-x+2) = -(x^2 - x - 2)$

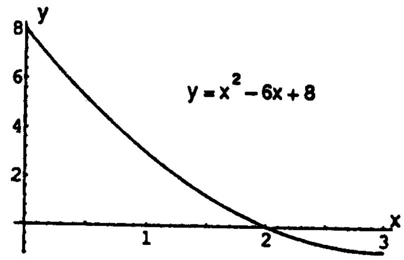
$$\Rightarrow A2 = -\int_{-1}^{2} (x^2 - x - 2)\,dx = -\left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{2} = -\left(\frac{8}{3} - \frac{4}{2} - 4\right) + \left(-\frac{1}{3} - \frac{1}{2} + 2\right) = -3 + 8 - \frac{1}{2} = \frac{9}{2};$$

A3: For the sketch given, a = 2 and b = 3: $f(x) - g(x) = (-x+2) - (4 - x^2) = x^2 - x - 2$

$$\Rightarrow A3 = \int_{2}^{3} (x^2 - x - 2)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{2}^{3} = \left(\frac{27}{3} - \frac{9}{2} - 6\right) - \left(\frac{8}{3} - \frac{4}{2} - 4\right) = 9 - \frac{9}{2} - \frac{8}{3};$$

Therefore, AREA = A1 + A2 + A3 $= \frac{11}{6} + \frac{9}{2} + \left(9 - \frac{9}{2} - \frac{8}{3}\right) = 9 - \frac{5}{6} = \frac{49}{6}$

25. $x^2 - 6x + 8 = 0 \Rightarrow (x-4)(x-2) = 0 \Rightarrow x = 4$ or $x = 2$, the x-intercepts.



(a) $\int_0^3 (x^2 - 6x + 8)\,dx = \int_0^3 x^2\,dx - 6\int_0^3 x\,dx + \int_0^3 8\,dx$

$$= \frac{3^3}{3} - 6\left(\frac{3^2}{2} - \frac{0^2}{2}\right) + 8(3-0) = 6$$

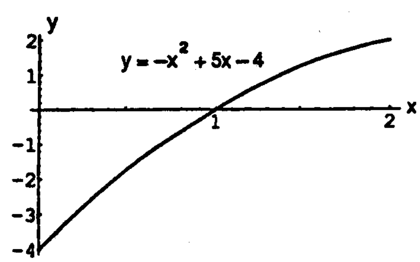
(b) Area $= \int_0^2 (x^2 - 6x + 8)\,dx + \left(-\int_2^3 (x^2 - 6x + 8)\,dx\right)$

$$= \left(\int_0^2 x^2\,dx - 6\int_0^2 x\,dx + \int_0^2 8\,dx\right) - \left(\int_2^3 x^2\,dx - 6\int_2^3 x\,dx + \int_2^3 8\,dx\right)$$

$$= \left[\frac{2^3}{3} - 6\left(\frac{2^2}{2} - \frac{0^2}{2}\right) + 8(2-0)\right] - \left(\int_0^3 x^2\,dx - \int_0^2 x^2\,dx - 6\left(\frac{3^2}{2} - \frac{2^2}{2}\right) + 8(3-2)\right)$$

$$= \left(\frac{8}{3} - 12 + 16\right) - \left(\frac{3^3}{3} - \frac{2^3}{3} - 15 + 8\right) = \frac{22}{3} = 7\tfrac{1}{3}$$

26. $-x^2 + 5x - 4 = 0 \Rightarrow x^2 - 5x + 4 = 0 \Rightarrow (x-4)(x-1) = 0$ $\Rightarrow x = 4$ or $x = 1$, the x-intercepts.



(a) $\int_0^2 (-x^2 + 5x - 4)\,dx = -\int_0^2 x^2\,dx + 5\int_0^2 x\,dx - \int_0^2 4\,dx$

$$= -\frac{2^3}{3} + 5\left(\frac{2^2}{2} - \frac{0^2}{2}\right) - 4(2-0) = -\frac{2}{3}$$

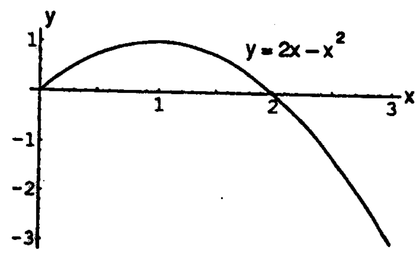
(b) Area $= -\int_0^1 (-x^2 + 5x - 4)\,dx + \int_1^2 (-x^2 + 5x - 4)\,dx$

$$= \int_0^1 x^2\,dx - 5\int_0^1 x\,dx + \int_0^1 4\,dx + \int_1^2 -x^2\,dx + 5\int_1^2 x\,dx - \int_1^2 4\,dx$$

$$= \frac{1^3}{3} - 5\left(\frac{1^2}{2} - \frac{0^2}{2}\right) + 4(1-0) + \left(\int_0^2 -x^2\,dx - \int_0^1 -x^2\,dx\right) + 5\left(\frac{2^2}{2} - \frac{1^2}{2}\right) - 4(2-1)$$

$$= \frac{1}{3} - \frac{5}{2} + 4 - \frac{2^3}{3} + \frac{1^3}{3} + \frac{15}{2} - 4 = 3$$

27. $2x - x^2 = 0 \Rightarrow x(2-x) = 0 \Rightarrow x = 0$ or $x = 2$, the x-intercepts.



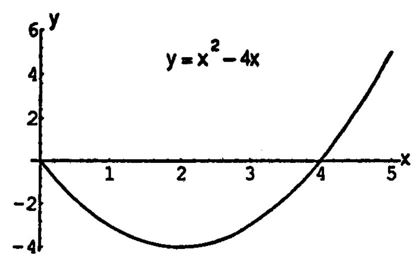
(a) $\int_0^3 (2x - x^2)\,dx = 2\int_0^3 x\,dx - \int_0^3 x^2\,dx$

$$= 2\left(\frac{3^2}{2} - \frac{0^2}{2}\right) - \frac{3^3}{3} = 0$$

(b) Area $= \int_0^2 (2x - x^2)\,dx - \int_2^3 (2x - x^2)\,dx = 2\int_0^2 x\,dx - \int_0^2 x^2\,dx - \left(2\int_2^3 x\,dx - \int_2^3 x^2\,dx\right)$

$$= 2\left(\frac{2^2}{2} - \frac{0^2}{2}\right) - \frac{2^3}{3} - 2\left(\frac{3^2}{2} - \frac{2^2}{2}\right) + \left(\int_0^3 x^2\,dx - \int_0^2 x^2\,dx\right) = 4 - \frac{8}{3} - 5 + \frac{3^3}{3} - \frac{2^3}{3} = \frac{8}{3}$$

28. $x^2 - 4x = 0 \Rightarrow x(x-4) = 0 \Rightarrow x = 0$ or $x = 4$, the x-intercepts.



(a) $\int_0^5 (x^2 - 4x)\,dx = \int_0^5 x^2\,dx - 4\int_0^5 x\,dx$

$$= \frac{5^3}{3} - 4\left(\frac{5^2}{2} - \frac{0^2}{2}\right) = -\frac{25}{3}$$

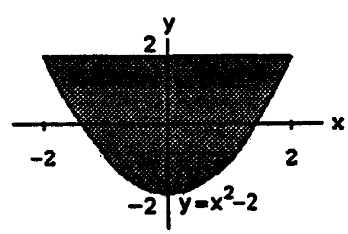
(b) Area $= -\int_0^4 (x^2 - 4x)\,dx + \int_4^5 (x^2 - 4x)\,dx$

$$= -\int_0^4 x^2\,dx + 4\int_0^4 x\,dx + \int_4^5 x^2\,dx - 4\int_4^5 x\,dx = -\frac{4^3}{3} + 4\left(\frac{4^2}{2} - \frac{0^2}{2}\right) + \left(\int_0^5 x^2\,dx - \int_0^4 x^2\,dx\right) - 4\left(\frac{5^2}{2} - \frac{4^2}{2}\right)$$

$$= -\frac{64}{3} + 32 + \frac{5^3}{3} - \frac{4^3}{3} - 18 = 13$$

29. $a = -2,\ b = 2$;

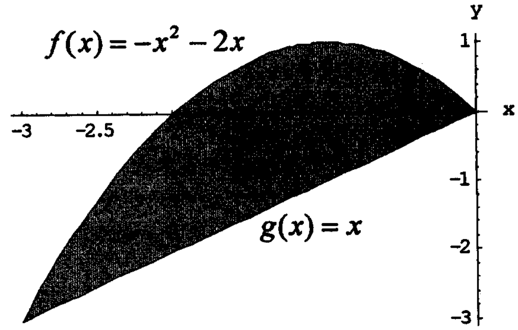
$f(x) - g(x) = 2 - (x^2 - 2) = 4 - x^2$



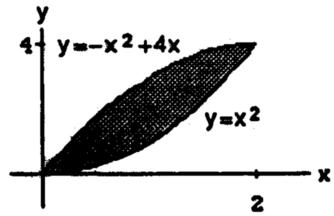
$$\Rightarrow A = \int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$

$$= 2\cdot\left(\frac{24}{3} - \frac{8}{3}\right) = \frac{32}{3}$$

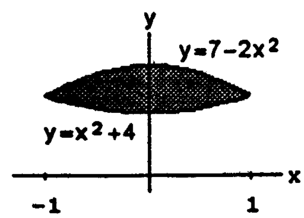
30. Limits of integration: $-x^2 - 2x = x \Rightarrow x^2 = -3x$
$\Rightarrow x(x+3) = 0 \Rightarrow a = -3$ and $b = 0$
$f(x) - g(x) = (-x^2 - 2x) - x = -x^2 - 3x$
$\Rightarrow A = \int_{-3}^{0} (-x^2 - 3x)\,dx = \left[\frac{-3x^2}{2} - \frac{x^3}{3}\right]_{-3}^{0}$
$= 0 - \left(-\frac{27}{2} + \frac{27}{3}\right) = \frac{9}{2}$



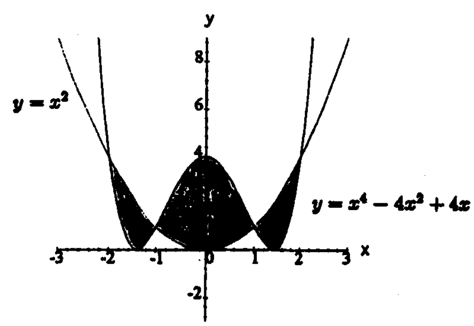
31. Limits of integration: $x^2 = -x^2 + 4x \Rightarrow 2x^2 - 4x = 0$
$\Rightarrow 2x(x-2) = 0 \Rightarrow a = 0$ and $b = 2$;
$f(x) - g(x) = (-x^2 + 4x) - x^2 = -2x^2 + 4x$
$\Rightarrow A = \int_{0}^{2} (-2x^2 + 4x)\,dx = \left[\frac{-2x^3}{3} + \frac{4x^2}{2}\right]_{0}^{2}$
$= -\frac{16}{3} + \frac{16}{2} = \frac{-32 + 48}{6} = \frac{8}{3}$



32. Limits of integration: $7 - 2x^2 = x^2 + 4 \Rightarrow 3x^2 - 3 = 0$
$\Rightarrow 3(x-1)(x+1) = 0 \Rightarrow a = -1$ and $b = 1$;
$f(x) - g(x) = (7 - 2x^2) - (x^2 + 4) = 3 - 3x^2$
$\Rightarrow A = \int_{-1}^{1} (3 - 3x^2)\,dx = 3\left[x - \frac{x^3}{3}\right]_{-1}^{1}$
$= 3\left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right] = 6\left(\frac{2}{3}\right) = 4$



33. Limits of integration: $x^4 - 4x^2 + 4 = x^2$
$\Rightarrow x^4 - 5x^2 + 4 = 0 \Rightarrow (x^2 - 4)(x^2 - 1) = 0$
$\Rightarrow (x+2)(x-2)(x+1)(x-1) = 0 \Rightarrow x = -2, -1, 1, 2$;
$f(x) - g(x) = (x^4 - 4x^2 + 4) - x^2 = x^4 - 5x^2 + 4$ and
$g(x) - f(x) = x^2 - (x^4 - 4x^2 + 4) = -x^4 + 5x^2 - 4$



$\Rightarrow A = \int_{-2}^{-1} (-x^4 + 5x^2 - 4)\,dx + \int_{-1}^{1} (x^4 - 5x^2 + 4)\,dx + \int_{1}^{2} (-x^4 + 5x^2 - 4)\,dx$

$= \left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\right]_{-2}^{-1} + \left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right]_{-1}^{1} + \left[\frac{-x^5}{5} + \frac{5x^3}{3} - 4x\right]_{1}^{2} = \left(\frac{1}{5} - \frac{5}{3} + 4\right) - \left(\frac{32}{5} - \frac{40}{3} + 8\right) + \left(\frac{1}{5} - \frac{5}{3} + 4\right)$

$-\left(-\frac{1}{5} + \frac{5}{3} - 4\right) + \left(-\frac{32}{5} + \frac{40}{3} - 8\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right) = -\frac{60}{5} + \frac{60}{3} = \frac{300 - 180}{15} = 8$

34. Limits of integration: $y = |x^2 - 4| = \begin{cases} x^2 - 4, & x \le -2 \text{ or } x \ge 2 \\ 4 - x^2, & -2 \le x \le 2 \end{cases}$

for $x \le -2$ and $x \ge 2$: $x^2 - 4 = \frac{x^2}{2} + 4$

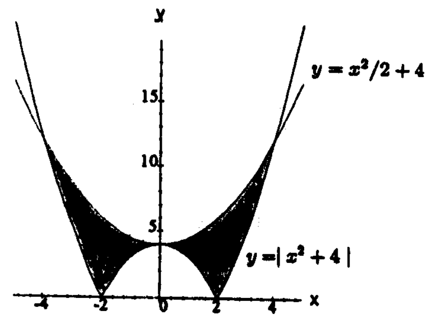
$\Rightarrow 2x^2 - 8 = x^2 + 8 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4;$

for $-2 \le x \le 2$: $4 - x^2 = \frac{x^2}{2} + 4 \Rightarrow 8 - 2x^2 = x^2 + 8$

$\Rightarrow x^2 = 0 \Rightarrow x = 0$; by symmetry of the graph,

$$A = 2\int_0^2 \left[\left(\frac{x^2}{2}+4\right)-\left(4-x^2\right)\right]dx + 2\int_2^4 \left[\left(\frac{x^2}{2}+4\right)-\left(x^2-4\right)\right]dx$$

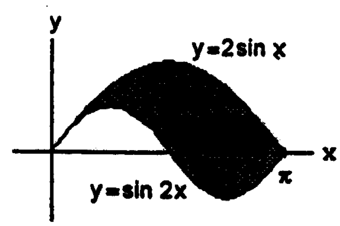
$$= 2\left[\frac{x^3}{2}\right]_0^2 + 2\left[8x - \frac{x^3}{6}\right]_2^4 = 2\left(\frac{8}{2}-0\right) + 2\left(32 - \frac{64}{6} - 16 + \frac{8}{6}\right) = 40 - \frac{56}{3} = \frac{64}{3}$$



35. $a = 0$, $b = \pi$; $f(x) - g(x) = 2 \sin x - \sin 2x$

$$\Rightarrow A = \int_0^\pi (2 \sin x - \sin 2x)\,dx = \left[-2 \cos x + \frac{\cos 2x}{2}\right]_0^\pi$$

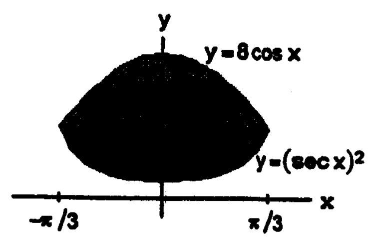
$$= \left[-2(-1) + \frac{1}{2}\right] - \left(-2 \cdot 1 + \frac{1}{2}\right) = 4$$



36. $a = -\frac{\pi}{3}$, $b = \frac{\pi}{3}$; $f(x) - g(x) = 8 \cos x - \sec^2 x$

$$\Rightarrow A = \int_{-\pi/3}^{\pi/3} \left(8 \cos x - \sec^2 x\right) dx = [8 \sin x - \tan x]_{-\pi/3}^{\pi/3}$$

$$= \left(8 \cdot \frac{\sqrt{3}}{2} - \sqrt{3}\right) - \left(-8 \cdot \frac{\sqrt{3}}{2} + \sqrt{3}\right) = 6\sqrt{3}$$



37. $A = A1 + A2$

$a_1 = -1$, $b_1 = 0$ and $a_2 = 0$, $b_2 = 1$;

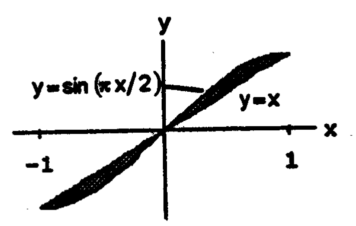
$f_1(x) - g_1(x) = x - \sin\left(\frac{\pi x}{2}\right)$ and $f_2(x) - g_2(x) = \sin\left(\frac{\pi x}{2}\right) - x$

$\Rightarrow$ by symmetry about the origin,

$$A_1 + A_2 = 2A_1 \Rightarrow A = 2\int_0^1 \left[\sin\left(\frac{\pi x}{2}\right) - x\right]dx$$

$$= 2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{x^2}{2}\right]_0^1 = 2\left[\left(-\frac{2}{\pi}\cdot 0 - \frac{1}{2}\right) - \left(-\frac{2}{\pi}\cdot 1 - 0\right)\right]$$

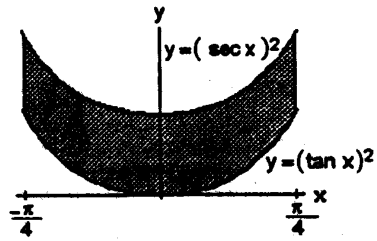
$$= 2\left(\frac{2}{\pi} - \frac{1}{2}\right) = 2\left(\frac{4-\pi}{2\pi}\right) = \frac{4-\pi}{\pi}$$



38. $a = -\frac{\pi}{4},\ b = \frac{\pi}{4};\ f(x) - g(x) = \sec^2 x - \tan^2 x$

$$\Rightarrow A = \int_{-\pi/4}^{\pi/4} \left(\sec^2 x - \tan^2 x\right) dx = \int_{-\pi/4}^{\pi/4} \left[\sec^2 x - \left(\sec^2 x - 1\right)\right] dx$$

$$= \int_{-\pi/4}^{\pi/4} 1 \cdot dx = [x]_{-\pi/4}^{\pi/4} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$



39. $A = A_1 + A_2$

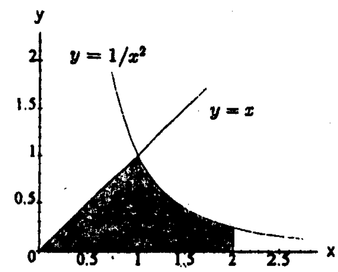
Limits of integration: $y = x$ and $y = \frac{1}{x^2} \Rightarrow x = \frac{1}{x^2},\ x \neq 0$

$$\Rightarrow x^3 = 1 \Rightarrow x = 1\ ,\ f_1(x) - g_1(x) = x - 0 = x$$

$$\Rightarrow A_1 = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2};\ f_2(x) - g_2(x) = \frac{1}{x^2} - 0$$

$$= x^{-2} \Rightarrow A_2 = \int_1^2 x^{-2}\,dx = \left[\frac{-1}{x}\right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2};$$

$$A = A_1 + A_2 = \frac{1}{2} + \frac{1}{2} = 1$$

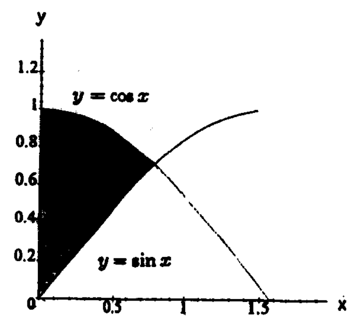


40. Limits of integration: $\sin x = \cos x \Rightarrow x = \frac{\pi}{4} \Rightarrow a = 0$

and $b = \frac{\pi}{4};\ f(x) - g(x) = \cos x - \sin x$

$$\Rightarrow A = \int_0^{\pi/4} (\cos x - \sin x)\,dx = [\sin x + \cos x]_0^{\pi/4}$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) = \sqrt{2} - 1$$



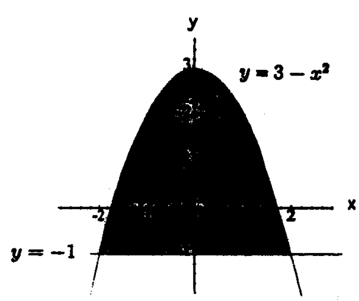
41. Limits of integration: $y = 3 - x^2$ and $y = -1$

$$\Rightarrow 3 - x^2 = -1 \Rightarrow x^2 = 4 \Rightarrow a = -2 \text{ and } b = 2;$$

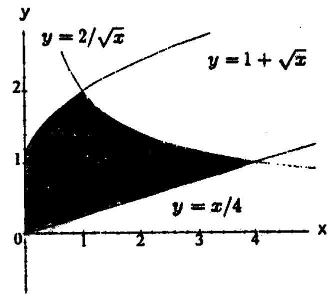
$$f(x) - g(x) = \left(3 - x^2\right) - (-1) = 4 - x^2$$

$$\Rightarrow A = \int_{-2}^{2} \left(4 - x^2\right) dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = 16 - \frac{16}{3} = \frac{32}{3}$$



42. Limits of integration: $y = 1 + \sqrt{x}$ and $y = \frac{2}{\sqrt{x}}$



$\Rightarrow 1 + \sqrt{x} = \frac{2}{\sqrt{x}},\ x \neq 0 \Rightarrow \sqrt{x} + x = 2 \Rightarrow x = (2-x)^2$

$\Rightarrow x = 4 - 4x + x^2 \Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x-4)(x-1) = 0 \Rightarrow x = 1, 4$ (but $x = 4$ does not satisfy the equation); $y = \frac{2}{\sqrt{x}}$ and $y = \frac{x}{4} \Rightarrow \frac{2}{\sqrt{x}} = \frac{x}{4}$

$\Rightarrow 8 = x\sqrt{x} \Rightarrow 64 = x^3 \Rightarrow x = 4$ since $x > 0$;

Therefore, AREA $= A_1 + A_2$: $f_1(x) - g_1(x) = (1 + x^{1/2}) - \frac{x}{4}$

$$\Rightarrow A_1 = \int_0^1 \left(1 + x^{1/2} - \frac{x}{4}\right) dx = \left[x + \frac{2}{3}x^{3/2} - \frac{x^2}{8}\right]_0^1 = \left(1 + \frac{2}{3} - \frac{1}{8}\right) - 0 = \frac{37}{24};\ f_2(x) - g_2(x) = 2x^{-1/2} - \frac{x}{4}$$

$$\Rightarrow A_2 = \int_1^4 \left(2x^{-1/2} - \frac{x}{4}\right) dx = \left[4x^{1/2} - \frac{x^2}{8}\right]_1^4 = \left(4 \cdot 2 - \frac{16}{8}\right) - \left(4 - \frac{1}{8}\right) = 4 - \frac{15}{8} = \frac{17}{8};$$

Therefore, AREA $= A_1 + A_2 = \frac{37}{24} + \frac{17}{8} = \frac{37+51}{24} = \frac{88}{24} = \frac{11}{3}$

43-46. Example CAS commands:

Maple:

```
p:=x^2*cos(x);
q:=x^3 - x;
plot({p,q}, x=-2..2,-2..2);
intpt1:=fsolve(p=q,x=-2..0);
intpt2:=fsolve(p=q,x=0..2);
intone:=Int(q-p,x=intpt1..0);
inttwo:=Int(p-q,x=0..intpt2);
evalf(intone+inttwo);
```

Mathematica:

```
Clear[x]
f[x_] = x^2 Cos[x]
g[x_] = x^3 - x
Plot[ {f[x],g[x]}, {x,-4,4} ]

Here, need to use FindRoot for each crossing; can do all together using Map
over initial guesses.

pts = x /. Map[
    FindRoot[ f[x] == g[x], {x,#} ]&,
    {-1,0,1} ]
i1 = NIntegrate[ f[x] - g[x], {x,pts[[1]],pts[[2]]}]
i2 = NIntegrate[ g[x] - f[x], {x,pts[[2]],pts[[3]]}]
i1 + i2
```

4.7 NUMERICAL INTEGRATION

1. $\int_1^2 x\,dx$

I. (a) For n = 4, $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{h}{2} = \frac{1}{8}$;

$\sum mf(x_i) = 12 \Rightarrow T = \frac{1}{8}(12) = \frac{3}{2}$;

$f(x) = x \Rightarrow f'(x) = 1 \Rightarrow f'' = 0 \Rightarrow M = 0$
$\Rightarrow |E_T| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	5/4	5/4	2	5/2
x_2	3/2	3/2	2	3
x_3	7/4	7/4	2	7/2
x_4	2	2	1	2

(b) $\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2} \Rightarrow |E_T| = \int_1^2 x\,dx - T = 0$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

II. (a) For n = 4, $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{h}{3} = \frac{1}{12}$;

$\sum mf(x_i) = 18 \Rightarrow S = \frac{1}{12}(18) = \frac{3}{2}$;

$f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	5/4	5/4	4	5
x_2	3/2	3/2	2	3
x_3	7/4	7/4	4	7
x_4	2	2	1	2

(b) $\int_1^2 x\,dx = \frac{3}{2} \Rightarrow |E_S| = \int_1^2 x\,dx - S = \frac{3}{2} - \frac{3}{2} = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

2. $\int_1^3 (2x-1)\,dx$

I. (a) For n = 4, $h = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{h}{2} = \frac{1}{4}$;

$\sum mf(x_i) = 24 \Rightarrow T = \frac{1}{4}(24) = 6$;

$f(x) = 2x - 1 \Rightarrow f'(x) = 2 \Rightarrow f'' = 0 \Rightarrow M = 0$
$\Rightarrow |E_T| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	3/2	2	2	4
x_2	2	3	2	6
x_3	5/2	4	2	8
x_4	3	5	1	5

(b) $\int_1^3 (2x-1)\,dx = \left[x^2 - x\right]_1^3 = (9-3) - (1-1) = 6 \Rightarrow |E_T| = \int_1^3 (2x-1)\,dx - T = 6 - 6 = 0$