CHAPTER 5 APPLICATIONS OF INTEGRALS

5.1 VOLUMES BY SLICING AND ROTATION ABOUT AN AXIS

1. (a) $A = \pi(\text{radius})^2$ and radius $= \sqrt{1-x^2} \Rightarrow A(x) = \pi(1-x^2)$

   (b) $A = \text{width} \cdot \text{height}$, width $=$ height $= 2\sqrt{1-x^2} \Rightarrow A(x) = 4(1-x^2)$

   (c) $A = (\text{side})^2$ and diagonal $= \sqrt{2}(\text{side}) \Rightarrow A = \frac{(\text{diagonal})^2}{2}$; diagonal $= 2\sqrt{1-x^2} \Rightarrow A(x) = 2(1-x^2)$

   (d) $A = \frac{\sqrt{3}}{4}(\text{side})^2$ and side $= 2\sqrt{1-x^2} \Rightarrow A(x) = \sqrt{3}(1-x^2)$

2. (a) $A = \pi(\text{radius})^2$ and radius $= \sqrt{x} \Rightarrow A(x) = \pi x$

   (b) $A = \text{width} \cdot \text{height}$, width $=$ height $= 2\sqrt{x} \Rightarrow A(x) = 4x$

   (c) $A = (\text{side})^2$ and diagonal $= \sqrt{2}(\text{side}) \Rightarrow A = \frac{(\text{diagonal})^2}{2}$; diagonal $= 2\sqrt{x} \Rightarrow A(x) = 2x$

   (d) $A = \frac{\sqrt{3}}{4}(\text{side})^2$ and side $= 2\sqrt{x} \Rightarrow A(x) = \sqrt{3}x$

3. $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{\left(\sqrt{x}-(-\sqrt{x})\right)^2}{2} = 2x$ (see Exercise 1c); $a = 0$, $b = 4$;

   $V = \int_a^b A(x)\,dx = \int_0^4 2x\,dx = \left[x^2\right]_0^4 = 16$

4. $A(x) = \frac{\pi(\text{diameter})^2}{4} = \frac{\pi\left[(2-x^2)-x^2\right]^2}{4} = \frac{\pi\left[2(1-x^2)\right]^2}{4} = \pi(1-2x^2+x^4)$; $a = -1$, $b = 1$;

   $V = \int_a^b A(x)\,dx = \int_{-1}^1 \pi(1-2x^2+x^4)\,dx = \pi\left[x - \frac{2}{3}x^3 + \frac{x^5}{5}\right]_{-1}^1 = 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16\pi}{15}$

5. (a) $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left[\frac{1}{\sqrt{1+x^2}} - \left(-\frac{1}{\sqrt{1+x^2}}\right)\right]^2 = \frac{\pi}{1+x^2} \Rightarrow V = \int_a^b A(x)\,dx = \int_{-1}^1 \frac{\pi\,dx}{1+x^2}$

   $= \pi\left[\tan^{-1} x\right]_{-1}^1 = (\pi)(2)\left(\frac{\pi}{4}\right) = \frac{\pi^2}{2}$

   (b) $A(x) = (\text{edge})^2 = \left[\frac{1}{\sqrt{1+x^2}} - \left(-\frac{1}{\sqrt{1+x^2}}\right)\right]^2 = \frac{4}{1+x^2} \Rightarrow V = \int_a^b A(x)\,dx = \int_{-1}^1 \frac{4\,dx}{1+x^2}$

   $= 4\left[\tan^{-1} x\right]_{-1}^1 = 4\left[\tan^{-1}(1) - \tan^{-1}(-1)\right] = 4\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 2\pi$

6. (a) $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(\frac{2}{\sqrt[4]{1-x^2}} - 0\right)^2 = \frac{\pi}{4}\left(\frac{4}{\sqrt{1-x^2}}\right) = \frac{\pi}{\sqrt{1-x^2}} \Rightarrow V = \int_a^b A(x)\,dx$

$$= \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{\pi}{\sqrt{1-x^2}}\,dx = \pi\left[\sin^{-1} x\right]_{-\sqrt{2}/2}^{\sqrt{2}/2} = \pi\left[\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) - \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right] = \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{2}$$

(b) $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{1}{2}\left(\frac{2}{\sqrt[4]{1-x^2}} - 0\right)^2 = \frac{2}{\sqrt{1-x^2}} \Rightarrow V = \int_a^b A(x)\,dx = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{2}{\sqrt{1-x^2}}\,dx$

$$= 2\left[\sin^{-1} x\right]_{-\sqrt{2}/2}^{\sqrt{2}/2} = 2\left(\frac{\pi}{4} \cdot 2\right) = \pi$$

7. (a) STEP 1) $A(x) = \frac{1}{2}(\text{side}) \cdot (\text{side}) \cdot \left(\sin \frac{\pi}{3}\right) = \frac{1}{2} \cdot \left(2\sqrt{\sin x}\right) \cdot \left(2\sqrt{\sin x}\right)\left(\sin \frac{\pi}{3}\right) = \sqrt{3}\sin x$

STEP 2) $a = 0,\ b = \pi$

STEP 3) $V = \int_a^b A(x)\,dx = \sqrt{3}\int_0^{\pi} \sin x\,dx = \left[-\sqrt{3}\cos x\right]_0^{\pi} = \sqrt{3}(1+1) = 2\sqrt{3}$

(b) STEP 1) $A(x) = (\text{side})^2 = \left(2\sqrt{\sin x}\right)\left(2\sqrt{\sin x}\right) = 4\sin x$

STEP 2) $a = 0,\ b = \pi$

STEP 3) $V = \int_a^b A(x)\,dx = \int_0^{\pi} 4\sin x\,dx = \left[-4\cos x\right]_0^{\pi} = 8$

8. (a) STEP 1) $A(x) = \frac{\pi(\text{diameter})^2}{4} = \frac{\pi}{4}(\sec x - \tan x)^2 = \frac{\pi}{4}\left(\sec^2 x + \tan^2 x - 2\sec x \tan x\right)$

$$= \frac{\pi}{4}\left[\sec^2 x + \left(\sec^2 x - 1\right) - 2\,\frac{\sin x}{\cos^2 x}\right]$$

STEP 2) $a = -\frac{\pi}{3},\ b = \frac{\pi}{3}$

STEP 3) $V = \int_a^b A(x)\,dx = \int_{-\pi/3}^{\pi/3} \frac{\pi}{4}\left(2\sec^2 x - 1 - \frac{2\sin x}{\cos^2 x}\right)dx = \frac{\pi}{4}\left[2\tan x - x + 2\left(-\frac{1}{\cos x}\right)\right]_{-\pi/3}^{\pi/3}$

$$= \frac{\pi}{4}\left[2\sqrt{3} - \frac{\pi}{3} + 2\left(-\frac{1}{\left(\frac{1}{2}\right)}\right) - \left(-2\sqrt{3} + \frac{\pi}{3} + 2\left(-\frac{1}{\left(\frac{1}{2}\right)}\right)\right)\right] = \frac{\pi}{4}\left(4\sqrt{3} - \frac{2\pi}{3}\right)$$

(b) STEP 1) $A(x) = (\text{edge})^2 = (\sec x - \tan x)^2 = \left(2\sec^2 x - 1 - 2\,\frac{\sin x}{\cos^2 x}\right)$

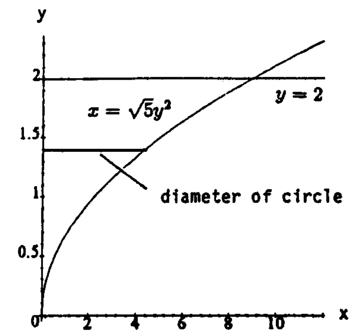
STEP 2) $a = -\frac{\pi}{3},\ b = \frac{\pi}{3}$

STEP 3) $V = \int_a^b A(x)\,dx = \int_{-\pi/3}^{\pi/3} \left(2\sec^2 x - 1 - \frac{2\sin x}{\cos^2 x}\right)dx = 2\left(2\sqrt{3} - \frac{\pi}{3}\right) = 4\sqrt{3} - \frac{2\pi}{3}$

9. $A(y) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(\sqrt{5}y^2 - 0\right)^2 = \frac{5\pi}{4}y^4$;

$c = 0,\ d = 2;\ V = \int_c^d A(y)\,dy = \int_0^2 \frac{5\pi}{4}y^4\,dy$

$= \left[\left(\frac{5\pi}{4}\right)\left(\frac{y^5}{5}\right)\right]_0^2 = \frac{\pi}{4}(2^5 - 0) = 8\pi$



10. $A(y) = \frac{1}{2}(\text{leg})(\text{leg}) = \frac{1}{2}\left[\sqrt{1-y^2} - \left(-\sqrt{1-y^2}\right)\right]^2 = \frac{1}{2}\left(2\sqrt{1-y^2}\right)^2 = 2\left(1-y^2\right)$; $c = -1,\ d = 1$;

$V = \int_c^d A(y)\,dy = \int_{-1}^{1} 2\left(1-y^2\right)dy = 2\left[y - \frac{y^3}{3}\right]_{-1}^{1} = 4\left(1 - \frac{1}{3}\right) = \frac{8}{3}$

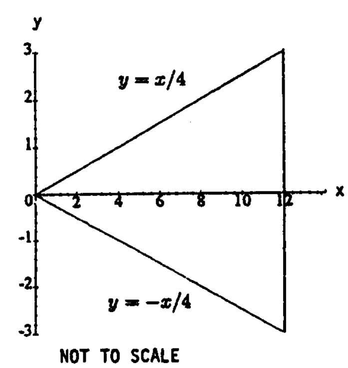
11. (a) It follows from Cavalieri's Theorem that the volume of a column is the same as the volume of a right prism with a square base of side length s and altitude h. Thus, STEP 1) $A(x) = (\text{side length})^2 = s^2$; STEP 2) $a = 0,\ b = h$; STEP 3) $V = \int_a^b A(x)\,dx = \int_0^h s^2\,dx = s^2h$

(b) From Cavalieri's Theorem we conclude that the volume of the column is the same as the volume of the prism described above, regardless of the number of turns $\Rightarrow V = s^2h$

12. 1) The solid and the cone have the same altitude of 12.

2) The cross sections of the solid are disks of diameter $x - \left(\frac{x}{2}\right) = \frac{x}{2}$. If we place the vertex of the cone at the origin of the coordinate system and make its axis of symmetry coincide with the x-axis then the cone's cross sections will be circular disks of diameter $\frac{x}{4} - \left(-\frac{x}{4}\right) = \frac{x}{2}$ (see accompanying figure).

3) The solid and the cone have equal altitudes and identical parallel cross sections. From Cavalieri's Theorem we conclude that the solid and the cone have the same volume.



13. $R(x) = y = 1 - \frac{x}{2} \Rightarrow V = \int_0^2 \pi[R(x)]^2\,dx = \pi\int_0^2 \left(1 - \frac{x}{2}\right)^2 dx = \pi\int_0^2 \left(1 - x + \frac{x^2}{4}\right)dx = \pi\left[x - \frac{x^2}{2} + \frac{x^3}{12}\right]_0^2$

$= \pi\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \frac{2\pi}{3}$

14. $R(y) = x = \frac{3y}{2} \Rightarrow V = \int_0^2 \pi[R(y)]^2\, dy = \pi \int_0^2 \left(\frac{3y}{2}\right)^2 dy = \pi \int_0^2 \frac{9}{4}y^2\, dy = \pi\left[\frac{3}{4}y^3\right]_0^2 = \pi \cdot \frac{3}{4} \cdot 8 = 6\pi$

15. $R(x) = \tan\left(\frac{\pi}{4}y\right)$; $u = \frac{\pi}{4}y \Rightarrow du = \frac{\pi}{4}\,dy \Rightarrow 4\,du = \pi\,dy$; $y = 0 \Rightarrow u = 0$, $y = 1 \Rightarrow u = \frac{\pi}{4}$;

$V = \int_0^1 \pi[R(y)]^2\, dy = \pi \int_0^1 \left[\tan\left(\frac{\pi}{4}y\right)\right]^2 dy = 4\int_0^{\pi/4} \tan^2 u\, du = 4\int_0^{\pi/4} \left(-1 + \sec^2 u\right) du = 4[-u + \tan u]_0^{\pi/4}$

$= 4\left(-\frac{\pi}{4} + 1 - 0\right) = 4 - \pi$

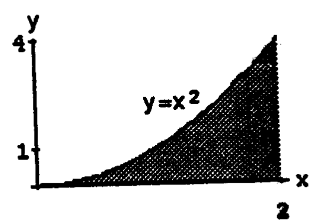
16. $R(x) = \sin x \cos x$; $R(x) = 0 \Rightarrow a = 0$ and $b = \frac{\pi}{2}$ are the limits of integration; $V = \int_0^{\pi/2} \pi[R(x)]^2\, dx$

$= \pi \int_0^{\pi/2} (\sin x \cos x)^2\, dx = \pi \int_0^{\pi/2} \frac{(\sin 2x)^2}{4}\, dx$; $\left[u = 2x \Rightarrow du = 2\,dx \Rightarrow \frac{du}{8} = \frac{dx}{4}; x = 0 \Rightarrow u = 0,\right.$

$\left. x = \frac{\pi}{2} \Rightarrow u = \pi\right] \to V = \pi \int_0^{\pi} \frac{1}{8}\sin^2 u\, du = \frac{\pi}{8}\left[\frac{u}{2} - \frac{1}{4}\sin 2u\right]_0^{\pi} = \frac{\pi}{8}\left[\left(\frac{\pi}{2} - 0\right) - 0\right] = \frac{\pi^2}{16}$

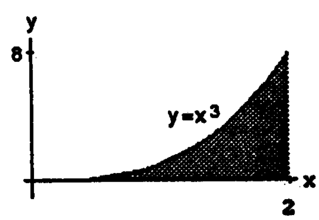
17. $R(x) = x^2 \Rightarrow V = \int_0^2 \pi[R(x)]^2\, dx = \pi \int_0^2 \left(x^2\right)^2 dx$

$= \pi \int_0^2 x^4\, dx = \pi\left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}$



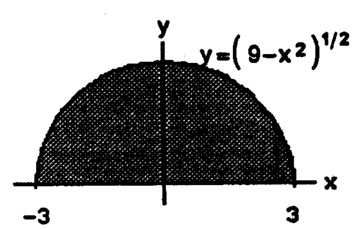
18. $R(x) = x^3 \Rightarrow V = \int_0^2 \pi[R(x)]^2\, dx = \pi \int_0^2 \left(x^3\right)^2 dx$

$= \pi \int_0^2 x^6\, dx = \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$



19. $R(x) = \sqrt{9 - x^2} \Rightarrow V = \int_{-3}^{3} \pi[R(x)]^2\, dx = \pi \int_{-3}^{3} \left(9 - x^2\right) dx$

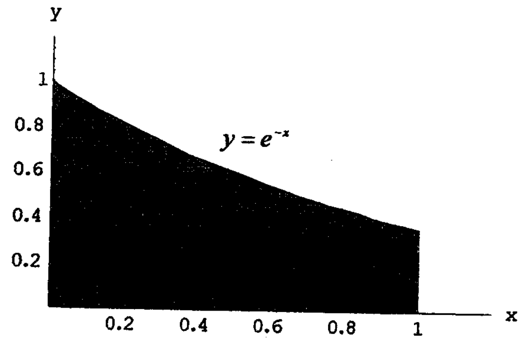
$= \pi\left[9x - \frac{x^3}{3}\right]_{-3}^{3} = 2\pi\left[9(3) - \frac{27}{3}\right] = 2 \cdot \pi \cdot 18 = 36\pi$



20. $R(x) = e^{-x}$; $V = \int_0^1 \pi[R(x)]^2\,dx = \int_0^1 \pi\left(e^{-x}\right)^2 dx$

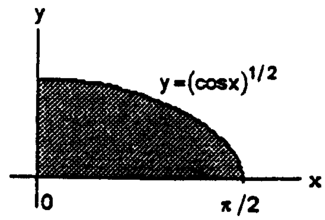
$= \pi \int_0^1 e^{-2x}\,dx = -\frac{\pi}{2}e^{-2x}\Big|_0^1 = -\frac{\pi}{2}\left(e^{-2} - 1\right)$

$= \frac{\pi}{2}\left(1 - \frac{1}{e^2}\right) = \frac{\pi\left(e^2 - 1\right)}{2e^2}$



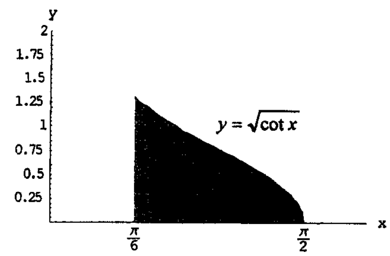
21. $R(x) = \sqrt{\cos x} \Rightarrow V = \int_0^{\pi/2} \pi[R(x)]^2\,dx = \pi \int_0^{\pi/2} \cos x\,dx$

$= \pi[\sin x]_0^{\pi/2} = \pi(1 - 0) = \pi$



22. $R(x) = \sqrt{\cot x} \Rightarrow V = \pi \int_{\pi/6}^{\pi/2} \cot x\,dx = \pi \int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x}\,dx$

$= \pi\left[\ln(\sin x)\right]_{\pi/6}^{\pi/2} = \pi\left(\ln 1 - \ln \frac{1}{2}\right) = \pi \ln 2$



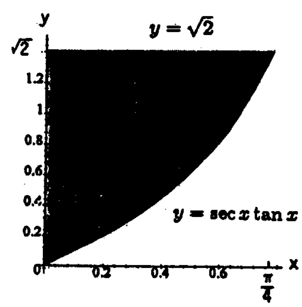
23. $R(x) = \sqrt{2} - \sec x \tan x \Rightarrow V = \int_0^{\pi/4} \pi[R(x)]^2\,dx$

$= \pi \int_0^{\pi/4} \left(\sqrt{2} - \sec x \tan x\right)^2 dx$

$= \pi \int_0^{\pi/4} \left(2 - 2\sqrt{2}\sec x \tan x + \sec^2 x \tan^2 x\right) dx$

$= \pi\left(\int_0^{\pi/4} 2\,dx - 2\sqrt{2}\int_0^{\pi/4} \sec x \tan x\,dx + \int_0^{\pi/4} (\tan x)^2 \sec^2 x\,dx\right)$

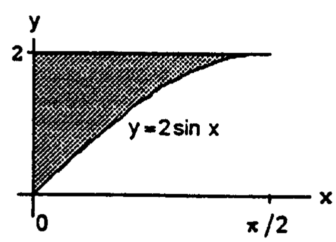
$= \pi\left([2x]_0^{\pi/4} - 2\sqrt{2}\,[\sec x]_0^{\pi/4} + \left[\frac{\tan^3 x}{3}\right]_0^{\pi/4}\right) = \pi\left[\left(\frac{\pi}{2} - 0\right) - 2\sqrt{2}\left(\sqrt{2} - 1\right) + \frac{1}{3}\left(1^3 - 0\right)\right] = \pi\left(\frac{\pi}{2} + 2\sqrt{2} - \frac{11}{3}\right)$



24. $R(x) = 2 - 2\sin x = 2(1 - \sin x) \Rightarrow V = \int_0^{\pi/2} \pi[R(x)]^2\,dx$

$= \pi \int_0^{\pi/2} 4(1 - \sin x)^2\,dx = 4\pi \int_0^{\pi/2} \left(1 + \sin^2 x - 2\sin x\right) dx$

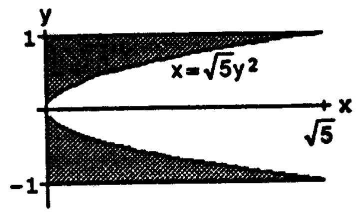
$= 4\pi \int_0^{\pi/2} \left[1 + \frac{1}{2}(1 - \cos 2x) - 2\sin x\right] dx$



$$= 4\pi \int_0^{\pi/2} \left(\frac{3}{2} - \frac{\cos 2x}{2} - 2 \sin x\right) = 4\pi\left[\frac{3}{2}x - \frac{\sin 2x}{4} + 2 \cos x\right]_0^{\pi/2} = 4\pi\left[\left(\frac{3\pi}{4} - 0 + 0\right) - (0 - 0 + 2)\right] = \pi(3\pi - 8)$$

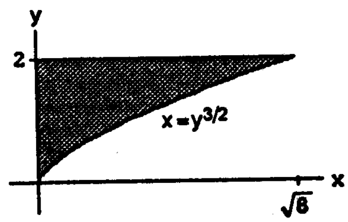
25. $R(y) = \sqrt{5} \cdot y^2 \Rightarrow V = \int_{-1}^{1} \pi[R(y)]^2 \, dy = \pi \int_{-1}^{1} 5y^4 \, dy$

$= \pi\left[y^5\right]_{-1}^{1} = \pi[1 - (-1)] = 2\pi$



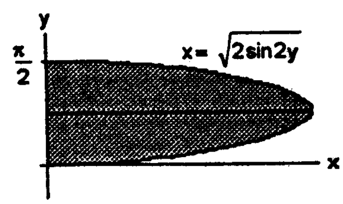
26. $R(y) = y^{3/2} \Rightarrow V = \int_0^2 \pi[R(y)]^2 \, dy = \pi \int_0^2 y^3 \, dy$

$= \pi\left[\frac{y^4}{4}\right]_0^2 = 4\pi$



27. $R(y) = \sqrt{2 \sin 2y} \Rightarrow V = \int_0^{\pi/2} \pi[R(y)]^2 \, dy = \pi \int_0^{\pi/2} 2 \sin 2y \, dy$

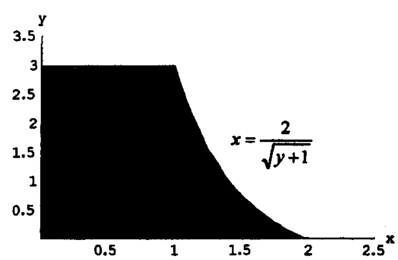
$= \pi\left[-\cos 2y\right]_0^{\pi/2} = \pi[1 - (-1)] = 2\pi$



28. $R(y) = \frac{2}{\sqrt{y+1}} \Rightarrow V = \pi \int_0^3 \left(\frac{2}{\sqrt{y+1}}\right)^2 dy$

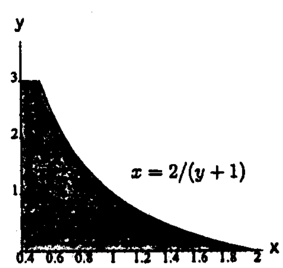
$= 4\pi \int_0^3 \frac{1}{y+1} \, dy = 4\pi\left[\ln|y+1|\right]_0^3$

$= 4\pi(\ln 4 - \ln 1) = 4\pi \ln 4$



29. $R(y) = \frac{2}{y+1} \Rightarrow V = \int_0^3 \pi[R(y)]^2 \, dy = 4\pi \int_0^3 \frac{1}{(y+1)^2} \, dy$

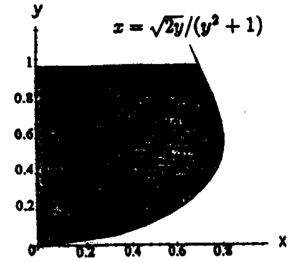
$= 4\pi\left[\frac{-1}{y+1}\right]_0^3 = 4\pi\left[-\frac{1}{4} - (-1)\right] = 3\pi$



30. $R(y) = \frac{\sqrt{2y}}{y^2+1} \Rightarrow V = \int_0^1 \pi[R(y)]^2\,dy = \pi \int_0^1 2y\left(y^2+1\right)^{-2}\,dy;$

$\left[u = y^2+1 \Rightarrow du = 2y\,dy;\ y = 0 \Rightarrow u = 1,\ y = 1 \Rightarrow u = 2\right]$

$\rightarrow V = \pi \int_1^2 u^{-2}\,du = \pi\left[-\frac{1}{u}\right]_1^2 = \pi\left[-\frac{1}{2}-(-1)\right] = \frac{\pi}{2}$



31. For the sketch given, $a = -\frac{\pi}{2}$, $b = \frac{\pi}{2}$; $R(x) = 1$, $r(x) = \sqrt{\cos x}$; $V = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right)dx$

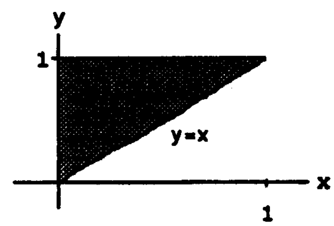
$= \int_{-\pi/2}^{\pi/2} \pi(1-\cos x)\,dx = 2\pi \int_0^{\pi/2} (1-\cos x)\,dx = 2\pi[x - \sin x]_0^{\pi/2} = 2\pi\left(\frac{\pi}{2}-1\right) = \pi^2 - 2\pi$

32. For the sketch given, $c = 0$, $d = \frac{\pi}{4}$; $R(y) = 1$, $r(y) = \tan y$; $V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right)dy$

$= \pi \int_0^{\pi/4} \left(1 - \tan^2 y\right)dy = \pi \int_0^{\pi/4} \left(2 - \sec^2 y\right)dy = \pi[2y - \tan y]_0^{\pi/4} = \pi\left(\frac{\pi}{2}-1\right) = \frac{\pi^2}{2} - \pi$

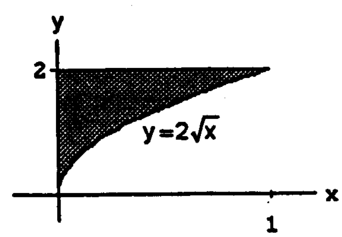
33. $r(x) = x$ and $R(x) = 1 \Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right)dx$

$= \int_0^1 \pi\left(1-x^2\right)dx = \pi\left[x - \frac{x^3}{3}\right]_0^1 = \pi\left[\left(1-\frac{1}{3}\right)-0\right] = \frac{2\pi}{3}$



34. $r(x) = 2\sqrt{x}$ and $R(x) = 2 \Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right)dx$

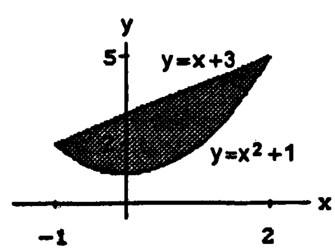
$= \pi \int_0^1 (4-4x)\,dx = 4\pi\left[x - \frac{x^2}{2}\right]_0^1 = 4\pi\left(1-\frac{1}{2}\right) = 2\pi$



35. $r(x) = x^2+1$ and $R(x) = x+3 \Rightarrow V = \int_{-1}^2 \pi\left([R(x)]^2 - [r(x)]^2\right)dx$

$= \pi \int_{-1}^2 \left[(x+3)^2 - \left(x^2+1\right)^2\right]dx$

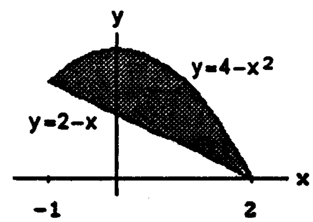
$= \pi \int_{-1}^2 \left[\left(x^2+6x+9\right) - \left(x^4+2x^2+1\right)\right]dx$



$= \pi \int_{-1}^{2} (-x^4 - x^2 + 6x + 8)\,dx = \pi\left[-\frac{x^5}{5} - \frac{x^3}{3} + \frac{6x^2}{2} + 8x\right]_{-1}^{2} = \pi\left[\left(-\frac{32}{5} - \frac{8}{3} + \frac{24}{2} + 16\right) - \left(\frac{1}{5} + \frac{1}{3} + \frac{6}{2} - 8\right)\right]$

$= \pi\left(-\frac{33}{5} - 3 + 28 - 3 + 8\right) = \pi\left(\frac{5 \cdot 30 - 33}{5}\right) = \frac{117\pi}{5}$

36. $r(x) = 2 - x$ and $R(x) = 4 - x^2 \Rightarrow V = \int_{-1}^{2} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$



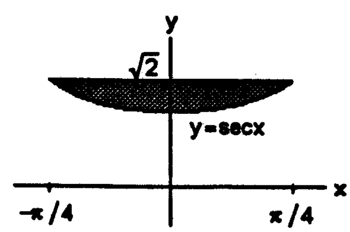
$= \pi \int_{-1}^{2} \left[\left(4 - x^2\right)^2 - (2 - x)^2\right] dx$

$= \pi \int_{-1}^{2} \left[\left(16 - 8x^2 + x^4\right) - \left(4 - 4x + x^2\right)\right] dx$

$= \pi \int_{-1}^{2} \left(12 + 4x - 9x^2 + x^4\right) dx = \pi\left[12x + 2x^2 - 3x^3 + \frac{x^5}{5}\right]_{-1}^{2} = \pi\left[\left(24 + 8 - 24 + \frac{32}{5}\right) - \left(-12 + 2 + 3 - \frac{1}{5}\right)\right]$

$= \pi\left(15 + \frac{33}{5}\right) = \frac{108\pi}{5}$

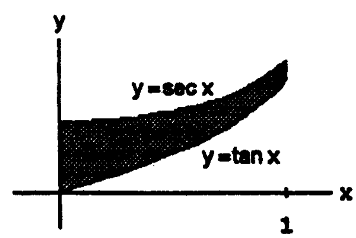
37. $r(x) = \sec x$ and $R(x) = \sqrt{2} \Rightarrow V = \int_{-\pi/4}^{\pi/4} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$



$= \pi \int_{-\pi/4}^{\pi/4} \left(2 - \sec^2 x\right) dx = \pi[2x - \tan x]_{-\pi/4}^{\pi/4}$

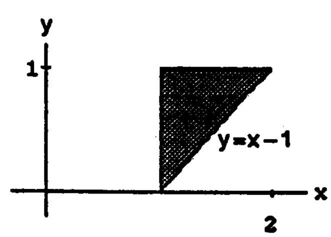
$= \pi\left[\left(\frac{\pi}{2} - 1\right) - \left(-\frac{\pi}{2} + 1\right)\right] = \pi(\pi - 2)$

38. $R(x) = \sec x$ and $r(x) = \tan x \Rightarrow V = \int_{0}^{1} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$



$= \pi \int_{0}^{1} \left(\sec^2 x - \tan^2 x\right) dx = \pi \int_{0}^{1} 1\,dx = \pi[x]_0^1 = \pi$

39. $r(y) = 1$ and $R(y) = 1 + y \Rightarrow V = \int_{0}^{1} \pi\left([R(y)]^2 - [r(y)]^2\right) dy$



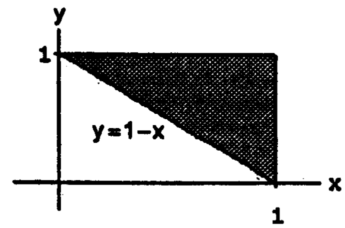
$= \pi \int_{0}^{1} \left[(1 + y)^2 - 1\right] dy = \pi \int_{0}^{1} \left(1 + 2y + y^2 - 1\right) dy$

$= \pi \int_{0}^{1} \left(2y + y^2\right) dy = \pi\left[y^2 + \frac{y^3}{3}\right]_{0}^{1} = \pi\left(1 + \frac{1}{3}\right) = \frac{4\pi}{3}$

40. $R(y) = 1$ and $r(y) = 1 - y \Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

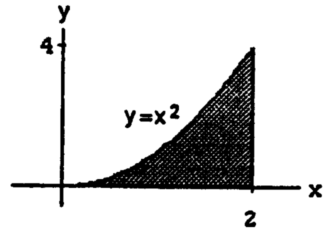
$= \pi \int_0^1 \left[1 - (1-y)^2\right] dy = \pi \int_0^1 \left[1 - \left(1 - 2y + y^2\right)\right] dy$

$= \pi \int_0^1 \left(2y - y^2\right) dy = \pi\left[y^2 - \frac{y^3}{3}\right]_0^1 = \pi\left(1 - \frac{1}{3}\right) = \frac{2\pi}{3}$



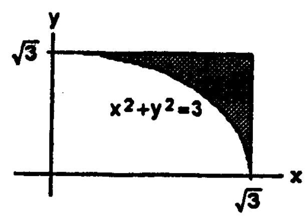
41. $R(y) = 2$ and $r(y) = \sqrt{y} \Rightarrow V = \int_0^4 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^4 (4 - y)\, dy = \pi\left[4y - \frac{y^2}{2}\right]_0^4 = \pi(16 - 8) = 8\pi$



42. $R(y) = \sqrt{3}$ and $r(y) = \sqrt{3 - y^2} \Rightarrow V = \int_0^{\sqrt{3}} \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^{\sqrt{3}} \left[3 - \left(3 - y^2\right)\right] dy = \pi \int_0^{\sqrt{3}} y^2\, dy = \pi\left[\frac{y^3}{3}\right]_0^{\sqrt{3}} = \pi\sqrt{3}$

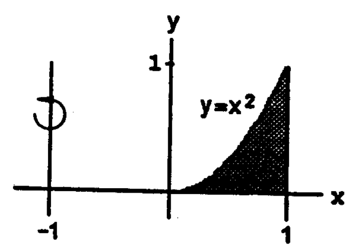


43. $R(y) = 2$ and $r(y) = 1 + \sqrt{y} \Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^1 \left[4 - \left(1 + \sqrt{y}\right)^2\right] dy = \pi \int_0^1 \left(4 - 1 - 2\sqrt{y} - y\right) dy$

$= \pi \int_0^1 \left(3 - 2\sqrt{y} - y\right) dy = \pi\left[3y - \frac{4}{3}y^{3/2} - \frac{y^2}{2}\right]_0^1$

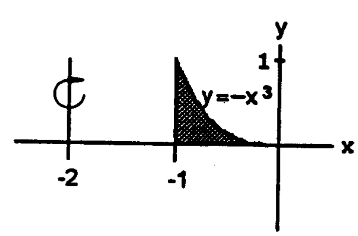
$= \pi\left(3 - \frac{4}{3} - \frac{1}{2}\right) = \pi\left(\frac{18 - 8 - 3}{6}\right) = \frac{7\pi}{6}$



44. $R(y) = 2 - y^{1/3}$ and $r(y) = 1 \Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^1 \left[\left(2 - y^{1/3}\right)^2 - 1\right] dy = \pi \int_0^1 \left(4 - 4y^{1/3} + y^{2/3} - 1\right) dy$

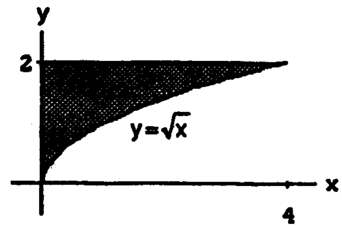
$= \pi \int_0^1 \left(3 - 4y^{1/3} + y^{2/3}\right) dy = \pi\left[3y - 3y^{4/3} + \frac{3y^{5/3}}{5}\right]_0^1$



$= \pi\left(3 - 3 + \frac{3}{5}\right) = \frac{3\pi}{5}$

45. (a) $r(x) = \sqrt{x}$ and $R(x) = 2 \Rightarrow V = \int_0^4 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_0^4 (4 - x)\, dx = \pi\left[4x - \frac{x^2}{2}\right]_0^4 = \pi(16 - 8) = 8\pi$



(b) $r(y) = 0$ and $R(y) = y^2 \Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^2 y^4\, dy = \pi\left[\frac{y^5}{5}\right]_0^2 = \frac{32\pi}{5}$

(c) $r(x) = 0$ and $R(x) = 2 - \sqrt{x} \Rightarrow V = \int_0^4 \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_0^4 \left(2 - \sqrt{x}\right)^2 dx$

$= \pi \int_0^4 \left(4 - 4\sqrt{x} + x\right) dx = \pi\left[4x - \frac{8x^{3/2}}{3} + \frac{x^2}{2}\right]_0^4 = \pi\left(16 - \frac{64}{3} + \frac{16}{2}\right) = \frac{8\pi}{3}$

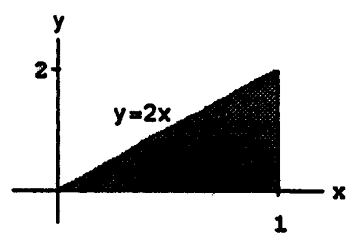
(d) $r(y) = 4 - y^2$ and $R(y) = 4 \Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \pi \int_0^2 \left[16 - \left(4 - y^2\right)^2\right] dy$

$= \pi \int_0^2 \left(16 - 16 + 8y^2 - y^4\right) dy = \pi \int_0^2 \left(8y^2 - y^4\right) dy = \pi\left[\frac{8}{3}y^3 - \frac{y^5}{5}\right]_0^2 = \pi\left(\frac{64}{3} - \frac{32}{5}\right) = \frac{224\pi}{15}$

46. (a) $r(y) = 0$ and $R(y) = 1 - \frac{y}{2} \Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^2 \left(1 - \frac{y}{2}\right)^2 dy = \pi \int_0^2 \left(1 - y + \frac{y^2}{4}\right) dy$

$= \pi\left[y - \frac{y^2}{2} + \frac{y^3}{12}\right]_0^2 = \pi\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \frac{2\pi}{3}$

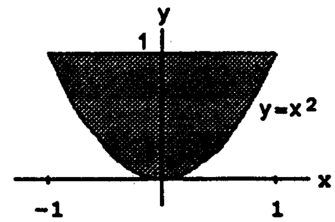


(b) $r(y) = 1$ and $R(y) = 2 - \frac{y}{2} \Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^2 \left[\left(2 - \frac{y}{2}\right)^2 - 1\right] dy = \pi \int_0^2 \left(4 - 2y + \frac{y^2}{4} - 1\right) dy = \pi \int_0^2 \left(3 - 2y + \frac{y^2}{4}\right) dy = \pi\left[3y - y^2 + \frac{y^3}{12}\right]_0^2$

$= \pi\left(6 - 4 + \frac{8}{12}\right) = \pi\left(2 + \frac{2}{3}\right) = \frac{8\pi}{3}$

47. (a) $r(x) = 0$ and $R(x) = 1 - x^2 \Rightarrow V = \int_{-1}^{1} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$



$= \pi \int_{-1}^{1} \left(1 - x^2\right)^2 dx = \pi \int_{-1}^{1} \left(1 - 2x^2 + x^4\right) dx$

$= \pi\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1} = 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 2\pi\left(\frac{15 - 10 + 3}{15}\right)$

$= \frac{16\pi}{15}$

(b) $r(x) = 1$ and $R(x) = 2 - x^2 \Rightarrow V = \int_{-1}^{1} \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_{-1}^{1} \left[\left(2 - x^2\right)^2 - 1\right] dx$

$= \pi \int_{-1}^{1} \left(4 - 4x^2 + x^4 - 1\right) dx = \pi \int_{-1}^{1} \left(3 - 4x^2 + x^4\right) dx = \pi\left[3x - \frac{4}{3}x^3 + \frac{x^5}{5}\right]_{-1}^{1} = 2\pi\left(3 - \frac{4}{3} + \frac{1}{5}\right)$

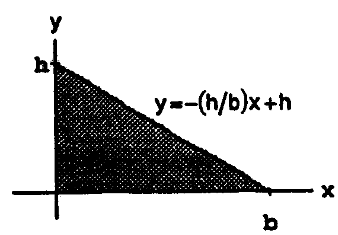
$= \frac{2\pi}{15}(45 - 20 + 3) = \frac{56\pi}{15}$

(c) $r(x) = 1 + x^2$ and $R(x) = 2 \Rightarrow V = \int_{-1}^{1} \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_{-1}^{1} \left[4 - \left(1 + x^2\right)^2\right] dx$

$= \pi \int_{-1}^{1} \left(4 - 1 - 2x^2 - x^4\right) dx = \pi \int_{-1}^{1} \left(3 - 2x^2 - x^4\right) dx = \pi\left[3x - \frac{2}{3}x^3 - \frac{x^5}{5}\right]_{-1}^{1} = 2\pi\left(3 - \frac{2}{3} - \frac{1}{5}\right)$

$= \frac{2\pi}{15}(45 - 10 - 3) = \frac{64\pi}{15}$

48. (a) $r(x) = 0$ and $R(x) = -\frac{h}{b}x + h \Rightarrow V = \int_{0}^{b} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$



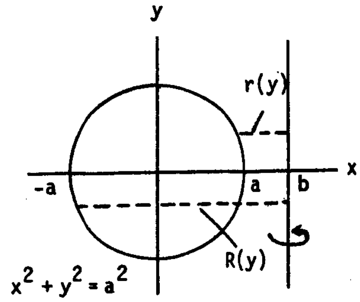
$= \pi \int_{0}^{b} \left(-\frac{h}{b}x + h\right)^2 dx = \pi \int_{0}^{b} \left(\frac{h^2}{b^2}x^2 - \frac{2h^2}{b}x + h^2\right) dx$

$= \pi h^2\left[\frac{x^3}{3b^2} - \frac{x^2}{b} + x\right]_{0}^{b} = \pi h^2\left(\frac{b}{3} - b + b\right) = \frac{\pi h^2 b}{3}$

(b) $r(y) = 0$ and $R(y) = b\left(1 - \frac{y}{h}\right) \Rightarrow V = \int_{0}^{h} \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \pi b^2 \int_{0}^{h} \left(1 - \frac{y}{h}\right)^2 dy$

$= \pi b^2 \int_{0}^{h} \left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right) dy = \pi b^2\left[y - \frac{y^2}{h} + \frac{y^3}{3h^2}\right]_{0}^{h} = \pi b^2\left(h - h + \frac{h}{3}\right) = \frac{\pi b^2 h}{3}$

49. $R(y) = b + \sqrt{a^2 - y^2}$ and $r(y) = b - \sqrt{a^2 - y^2}$



$\Rightarrow V = \int_{-a}^{a} \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_{-a}^{a} \left[\left(b + \sqrt{a^2 - y^2}\right)^2 - \left(b - \sqrt{a^2 - y^2}\right)^2\right] dy$

$= \pi \int_{-a}^{a} 4b\sqrt{a^2 - y^2}\, dy = 4b\pi \int_{-a}^{a} \sqrt{a^2 - y^2}\, dy$

$= 4b\pi \cdot$ area of semicircle of radius $a = 4b\pi \cdot \frac{\pi a^2}{2} = 2a^2b\pi^2$

50. (a) A cross section has radius $r = \sqrt{2y}$ and area $\pi r^2 = 2\pi y$. The volume is $\int_0^5 2\pi y\, dy = \pi\left[y^2\right]_0^5 = 25\pi$.

(b) $V(h) = \int A(h)\, dh$, so $\frac{dV}{dh} = A(h)$. Therefore $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = A(h) \cdot \frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{1}{A(h)} \cdot \frac{dV}{dt}$

For $h = 4$, the area is $2\pi(4) = 8\pi$, so $\frac{dh}{dt} = \frac{1}{8\pi} \cdot 3\, \frac{\text{units}^3}{\text{sec}} = \frac{3}{8\pi} \cdot \frac{\text{units}^3}{\text{sec}}$.

51. (a) $R(y) = \sqrt{a^2 - y^2} \Rightarrow V = \pi \int_{-a}^{h-a} \left(a^2 - y^2\right) dy = \pi\left[a^2y - \frac{y^3}{3}\right]_{-a}^{h-a} = \pi\left[a^2h - a^3 - \frac{(h-a)^3}{3} - \left(-a^3 + \frac{a^3}{3}\right)\right]$

$= \pi\left[a^2h - \frac{1}{3}\left(h^3 - 3h^2a + 3ha^2 - a^3\right) - \frac{a^3}{3}\right] = \pi\left(a^2h - \frac{h^3}{3} + h^2a - ha^2\right) = \frac{\pi h^2(3a - h)}{3}$

(b) Given $\frac{dV}{dt} = 0.2\ \text{m}^3/\text{sec}$ and $a = 5$ m, find $\left.\frac{dh}{dt}\right|_{h=4}$. From part (a), $V(h) = \frac{\pi h^2(15 - h)}{3} = 5\pi h^2 - \frac{\pi h^3}{3}$

$\Rightarrow \frac{dV}{dh} = 10\pi h - \pi h^2 \Rightarrow \frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = \pi h(10 - h)\frac{dh}{dt} \Rightarrow \left.\frac{dh}{dt}\right|_{h=4} = \frac{0.2}{4\pi(10 - 4)} = \frac{1}{(20\pi)(6)} = \frac{1}{120\pi}$ m/sec.

52. Partition the appropriate interval in the axis of revolution and measure the radius $r(x)$ of the shadow region at these points. Then use an approximation such as the trapezoidal rule to estimate the integral $\int_a^b \pi r^2(x)\, dx$.

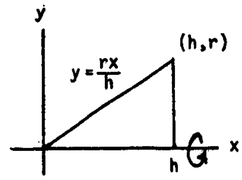
53. The cross section of a solid right circular cylinder with a cone removed is a disk with radius R from which a disk of radius h has been removed (figure provided). Thus its area is $A_1 = \pi R^2 - \pi h^2 = \pi\left(R^2 - h^2\right)$. The cross section of the hemisphere is a disk of radius $\sqrt{R^2 - h^2}$ (figure provided). Therefore its area is $A_2 = \pi\left(\sqrt{R^2 - h^2}\right)^2 = \pi\left(R^2 - h^2\right)$. We can see that $A_1 = A_2$. The altitudes of both solids are R. Applying Cavalieri's Theorem we find Volume of Hemisphere = (Volume of Cylinder) – (Volume of Cone) $= \left(\pi R^2\right)R - \frac{1}{3}\pi\left(R^2\right)R = \frac{2}{3}\pi R^3$.

54. (a) $R(x) = \sqrt{a^2 - x^2} \Rightarrow V = \int_{-a}^{a} \pi[R(x)]^2\, dx = \pi \int_{-a}^{a} \left(a^2 - x^2\right) dx = \pi\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a}$

$= \pi\left[\left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right] = 2\pi\left(\frac{2a^3}{3}\right) = \frac{4}{3}\pi a^3$, the volume of a sphere of radius a

(b) $R(x) = \frac{rx}{h} \Rightarrow V = \int_0^h \pi[R(x)]^2\,dx = \pi \int_0^h \frac{r^2x^2}{h^2}\,dx$

$= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \left(\frac{\pi r^2}{h^2}\right)\left(\frac{h^3}{3}\right) = \frac{1}{3}\pi r^2 h$, the volume of a cone of radius r and height h



55. $R(y) = \sqrt{256 - y^2} \Rightarrow V = \int_{-16}^{-7} \pi[R(y)]^2\,dy = \pi \int_{-16}^{-7} (256 - y^2)\,dy = \pi\left[256y - \frac{y^3}{3}\right]_{-16}^{-7}$

$= \pi\left[(256)(-7) + \frac{7^3}{3} - \left((256)(-16) + \frac{16^3}{3}\right)\right] = \pi\left(\frac{7^3}{3} + 256(16-7) - \frac{16^3}{3}\right) = 1053\pi\ \text{cm}^3 \approx 3308\ \text{cm}^3$

56. $R(x) = \frac{x}{12}\sqrt{36 - x^2} \Rightarrow V = \int_0^6 \pi[R(x)]^2\,dx = \pi \int_0^6 \frac{x^2}{144}(36 - x^2)\,dx = \frac{\pi}{144}\int_0^6 (36x^2 - x^4)\,dx$

$= \frac{\pi}{144}\left[12x^3 - \frac{x^5}{5}\right]_0^6 = \frac{\pi}{144}\left(12 \cdot 6^3 - \frac{6^5}{5}\right) = \frac{\pi \cdot 6^3}{144}\left(12 - \frac{36}{5}\right) = \left(\frac{216\pi}{144}\right)\left(\frac{60-36}{5}\right) = \frac{36\pi}{5}\ \text{cm}^3$. The plumb bob will weigh about $W = (8.5)\left(\frac{36\pi}{5}\right) \approx 192$ gm, to the nearest gram

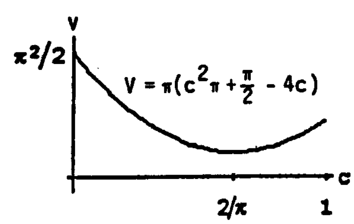
57. (a) $R(x) = |c - \sin x|$, so $V = \pi \int_0^\pi [R(x)]^2\,dx = \pi \int_0^\pi (c - \sin x)^2\,dx = \pi \int_0^\pi (c^2 - 2c\sin x + \sin^2 x)\,dx$

$= \pi \int_0^\pi \left(c^2 - 2c\sin x + \frac{1 - \cos 2x}{2}\right)dx = \pi \int_0^\pi \left(c^2 + \frac{1}{2} - 2c\sin x - \frac{\cos 2x}{2}\right)dx$

$= \pi\left[\left(c^2 + \frac{1}{2}\right)x + 2c\cos x - \frac{\sin 2x}{4}\right]_0^\pi = \pi\left[\left(c^2\pi + \frac{\pi}{2} - 2c - 0\right) - (0 + 2c - 0)\right] = \pi\left(c^2\pi + \frac{\pi}{2} - 4c\right)$. Let $V(c) = \pi\left(c^2\pi + \frac{\pi}{2} - 4c\right)$. We find the extreme values of V(c): $\frac{dV}{dc} = \pi(2c\pi - 4) = 0 \Rightarrow c = \frac{2}{\pi}$ is a critical point, and $V\left(\frac{2}{\pi}\right) = \pi\left(\frac{4}{\pi} + \frac{\pi}{2} - \frac{8}{\pi}\right) = \pi\left(\frac{\pi}{2} - \frac{4}{\pi}\right) = \frac{\pi^2}{2} - 4$; Evaluate V at the endpoints: $V(0) = \frac{\pi^2}{2}$ and $V(1) = \pi\left(\frac{3}{2}\pi - 4\right) = \frac{\pi^2}{2} - (4 - \pi)\pi$. Now we see that the function's absolute minimum value is $\frac{\pi^2}{2} - 4$, taken on at the critical point $c = \frac{2}{\pi}$. (See also the accompanying graph.)

(b) From the discussion in part (a) we conclude that the function's absolute maximum value is $\frac{\pi^2}{2}$, taken on at the endpoint $c = 0$.

(c) The graph of the solid's volume as a function of c for $0 \le c \le 1$ is given at the right. As c moves away from [0, 1] the volume of the solid increases without bound. If we approximate the solid as a set of solid disks, we



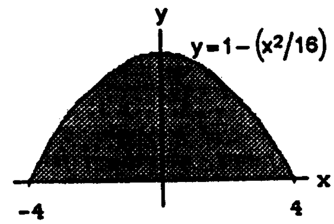
can see that the radius of a typical disk increases without bounds as c moves away from [0, 1].

58. (a) $R(x) = 1 - \frac{x^2}{16} \Rightarrow V = \int_{-4}^{4} \pi[R(x)]^2\,dx = \pi \int_{-4}^{4} \left(1 - \frac{x^2}{16}\right)^2 dx$

$= \pi \int_{-4}^{4} \left(1 - \frac{x^2}{8} + \frac{x^4}{16^2}\right) dx = \pi\left[x - \frac{x^3}{24} + \frac{x^5}{5 \cdot 16^2}\right]_{-4}^{4}$

$= 2\pi\left(4 - \frac{4^3}{24} + \frac{4^5}{5 \cdot 16^2}\right) = 2\pi\left(4 - \frac{8}{3} + \frac{4}{5}\right)$

$= \frac{2\pi}{15}(60 - 40 + 12) = \frac{64\pi}{15}\text{ ft}^3$



(b) The helicopter will be able to fly $\left(\frac{64\pi}{15}\right)(7.481)(2) \approx 201$ additional miles.

59. (a) Using $d = \frac{C}{\pi}$, and $A = \pi\left(\frac{d}{2}\right)^2 = \frac{C^2}{4\pi}$ yields the following areas (in square inches, rounded to the nearest tenth: 2.3, 1.6, 1.5, 2.1, 3.2, 4.8, 7.0, 9.3, 10.7, 10.7, 9.3, 6.4, 3.2.

(b) If C(y) is the circumference as a function of y, then the area of a cross section is

$A(y) = \pi\left(\frac{C(y)/\pi}{2}\right)^2 = \frac{C^2(y)}{4\pi}$, and the volume is $\frac{1}{4\pi}\int_0^6 C^2(y)\,dy$.

(c) $\int_0^6 A(y)\,dy = \frac{1}{4\pi}\int_0^6 C^2(y)\,dy \approx \frac{1}{4\pi}\left(\frac{6-0}{24}\right)[5.4^2 + 2(4.5^2 + 4.4^2 + 5.1^2 + 6.3^2 + 7.8^2 + 9.4^2 + 10.8^2$
$+ 11.6^2 + 11.6^2 + 10.8^2 + 9.0^2) + 6.3^2] \approx 34.7\text{ in.}^3$

(d) $V = \frac{1}{4\pi}\int_0^6 C^2(y)\,dy \approx \frac{1}{4\pi}\left(\frac{6-0}{36}\right)\left[5.4^2 + 4(4.5^2) + 2(4.4^2) + 4(5.1^2) + 2(6.3^2) + 4(7.8^2) + 2(9.4^2)\right.$
$\left. + 4(10.8^2) + 2(11.6^2) + 4(11.6^2) + 2(10.8^2) + 4(9.0^2) + 6.3^2\right] = 34.792\text{ in.}^3$

by Simpson's rule. The Simpson's rule estimate should be more accurate than the trapezoid estimate. The error in the Simpson's estimate is proportional to $h^4 = 0.0625$ whereas the error in the trapezoid estimate is proportional to $h^2 = 0.25$, a larger number when $h = 0.5$ in.

60. (a) Displacement Volume $V \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n)$, $x_0 = 0$, $x_n = 10 - h$,

$h = 2.54,\ n = 10 \Rightarrow V = \int_{x_0}^{x_n} A(x)\,dx \approx \frac{2.54}{3}[0 + 4(1.07) + 2(3.84) + 4(7.82) + 2(12.20) + 4(15.18)$

$+ 2(16.14) + 4(14.00) + 2(9.21) + 4(3.24) + 0] = \frac{2.54}{3}(4.28 + 7.68 + 31.28 + 24.4 + 60.72 + 32.28$

$+ 56 + 18.42 + 12.96) = \frac{2.54}{3}(248.02) = 209.99 \approx 210\text{ ft}^3$

(b) The weight of water displaced is approximately $64 \cdot 210 = 13{,}440$ lb

(c) The volume of a prism $= (25.4) \cdot (16.14) = 409.96 \approx 410\text{ ft}^3$. Thus, the prismatic coefficient is $\frac{210\text{ ft}^3}{410\text{ ft}^3} \approx 0.51$

5.2 MODELING VOLUME USING CYLINDRICAL SHELLS

1. For the sketch given, a = 0, b = 2;

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx = \int_0^2 2\pi x\left(1+\frac{x^2}{4}\right)dx = 2\pi\int_0^2\left(x+\frac{x^3}{4}\right)dx = 2\pi\left[\frac{x^2}{2}+\frac{x^4}{16}\right]_0^2 = 2\pi\left(\frac{4}{2}+\frac{16}{16}\right)$$

$= 2\pi \cdot 3 = 6\pi$

2. For the sketch given, a = 0, b = 2;

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx = \int_0^2 2\pi x\left(2-\frac{x^2}{4}\right)dx = 2\pi\int_0^2\left(2x-\frac{x^3}{4}\right)dx = 2\pi\left[x^2-\frac{x^4}{16}\right]_0^2 = 2\pi(4-1) = 6\pi$$

3. For the sketch given, c = 0, d = $\sqrt{2}$;

$$V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^{\sqrt{2}} 2\pi y\cdot(y^2)dy = 2\pi\int_0^{\sqrt{2}} y^3\,dy = 2\pi\left[\frac{y^4}{4}\right]_0^{\sqrt{2}} = 2\pi$$

4. For the sketch given, c = 0, d = $\sqrt{3}$;

$$V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^{\sqrt{3}} 2\pi y\cdot[3-(3-y^2)]\,dy = 2\pi\int_0^{\sqrt{3}} y^3\,dy = 2\pi\left[\frac{y^4}{4}\right]_0^{\sqrt{3}} = \frac{9\pi}{2}$$

5. For the sketch given, a = 0, b = $\sqrt{3}$;

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx = \int_0^{\sqrt{3}} 2\pi x\cdot\left(\sqrt{x^2+1}\right)dx;$$

$[u = x^2+1 \Rightarrow du = 2x\,dx;\ x = 0 \Rightarrow u = 1,\ x = \sqrt{3} \Rightarrow u = 4]$

$$\Rightarrow V = \pi\int_1^4 u^{1/2}\,du = \pi\left[\frac{2}{3}u^{3/2}\right]_1^4 = \frac{2\pi}{3}\left(4^{3/2}-1\right) = \left(\frac{2\pi}{3}\right)(8-1) = \frac{14\pi}{3}$$

6. For the sketch given, a = 0, b = 3;

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx = \int_0^3 2\pi x\left(\frac{9x}{\sqrt{x^3+9}}\right)dx;$$

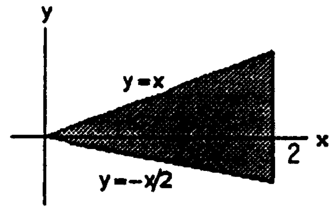
$[u = x^3+9 \Rightarrow du = 3x^2\,dx \Rightarrow 3\,du = 9x^2\,dx;\ x = 0 \Rightarrow u = 9,\ x = 3 \Rightarrow u = 36]$

$$\Rightarrow V = 2\pi\int_9^{36} 3u^{-1/2}\,du = 6\pi\left[2u^{1/2}\right]_9^{36} = 12\pi\left(\sqrt{36}-\sqrt{9}\right) = 36\pi$$

7. $a = 0,\ b = 2;$

$$V = \int_a^b 2\pi\left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dx = \int_0^2 2\pi x\left[x - \left(-\frac{x}{2}\right)\right] dx$$

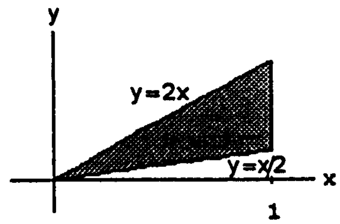
$$= \int_0^2 2\pi x^2 \cdot \frac{3}{2}\, dx = \pi \int_0^2 3x^2\, dx = \pi\left[x^3\right]_0^2 = 8\pi$$



8. $a = 0,\ b = 1;$

$$V = \int_a^b 2\pi\left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dx = \int_0^1 2\pi x\left(2x - \frac{x}{2}\right) dx$$

$$= \pi \int_0^1 2\left(\frac{3x^2}{2}\right) dx = \pi \int_0^1 3x^2\, dx = \pi\left[x^3\right]_0^1 = \pi$$

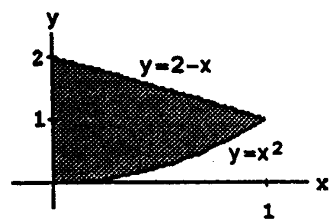


9. $a = 0,\ b = 1;$

$$V = \int_a^b 2\pi\left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dx = \int_0^1 2\pi x\left[(2 - x) - x^2\right] dx$$

$$= 2\pi \int_0^1 \left(2x - x^2 - x^3\right) dx = 2\pi\left[x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= 2\pi\left(1 - \frac{1}{3} - \frac{1}{4}\right) = 2\pi\left(\frac{12 - 4 - 3}{12}\right) = \frac{10\pi}{12} = \frac{5\pi}{6}$$

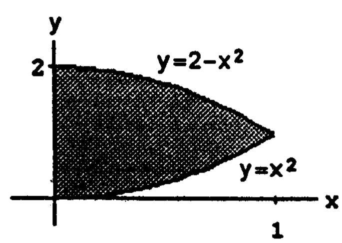


10. $a = 0,\ b = 1;$

$$V = \int_a^b 2\pi\left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dx = \int_0^1 2\pi x\left[\left(2 - x^2\right) - x^2\right] dx$$

$$= 2\pi \int_0^1 x\left(2 - 2x^2\right) dx = 4\pi \int_0^1 \left(x - x^3\right) dx$$

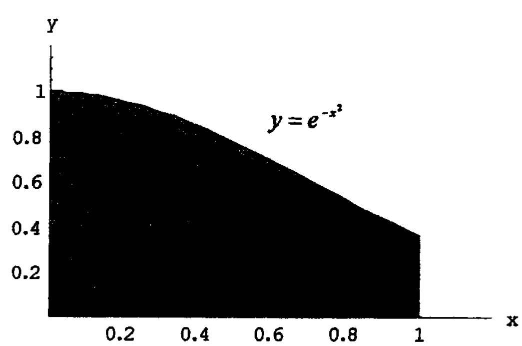
$$= 4\pi\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 4\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \pi$$



11. $a = 0,\ b = 1;\ V = \int_a^b 2\pi\left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dy$

$$= \int_0^1 2\pi x e^{-x^2}\, dx; \text{ Let } u = -x^2 \Rightarrow -du = 2x\, dx$$

$$\Rightarrow V = -\int_{u=0}^{-1} \pi e^u\, du = -\pi e^u\Big[_{u=0}^{-1} = \pi\left(1 - \frac{1}{e}\right) = \frac{\pi(e - 1)}{e}$$

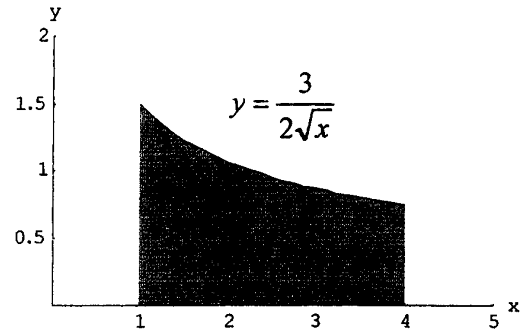


12. a = 1, b = 4;

$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_1^4 2\pi x\left(\frac{3}{2}x^{-1/2}\right) dx$

$= 3\pi \int_1^4 x^{1/2}\, dx = 3\pi\left[\frac{2}{3}x^{3/2}\right]_1^4 = 2\pi\left(4^{3/2} - 1\right)$

$= 2\pi(8-1) = 14\pi$



13. (a) $xf(x) = \begin{cases} x \cdot \frac{\sin x}{x}, & 0 < x \le \pi \\ x, & x = 0 \end{cases} \Rightarrow xf(x) = \begin{cases} \sin x, & 0 < x \le \pi \\ 0, & x = 0 \end{cases}$; since sin 0 = 0 we have

$xf(x) = \begin{cases} \sin x, & 0 < x \le \pi \\ \sin x, & x = 0 \end{cases} \Rightarrow xf(x) = \sin x,\ 0 \le x \le \pi$

(b) $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^\pi 2\pi x \cdot f(x)\, dx$ and $x \cdot f(x) = \sin x,\ 0 \le x \le \pi$ by part (a)

$\Rightarrow V = 2\pi \int_0^\pi \sin x\, dx = 2\pi[-\cos x]_0^\pi = 2\pi(-\cos \pi + \cos 0) = 4\pi$

14. (a) $xg(x) = \begin{cases} x \cdot \frac{\tan^2 x}{x}, & 0 < x \le \frac{\pi}{4} \\ x \cdot 0, & x = 0 \end{cases} \Rightarrow xg(x) = \begin{cases} \tan^2 x, & 0 < x \le \pi/4 \\ 0, & x = 0 \end{cases}$; since tan 0 = 0 we have

$xg(x) = \begin{cases} \tan^2 x, & 0 < x \le \pi/4 \\ \tan^2 x, & x = 0 \end{cases} \Rightarrow xg(x) = \tan^2 x,\ 0 \le x \le \pi/4$

(b) $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^{\pi/4} 2\pi x \cdot g(x)\, dx$ and $x \cdot g(x) = \tan^2 x,\ 0 \le x \le \pi/4$ by part (a)

$\Rightarrow V = 2\pi \int_0^{\pi/4} \tan^2 x\, dx = 2\pi \int_0^{\pi/4} \left(\sec^2 x - 1\right) dx = 2\pi[\tan x - x]_0^{\pi/4} = 2\pi\left(1 - \frac{\pi}{4}\right) = \frac{4\pi - \pi^2}{2}$

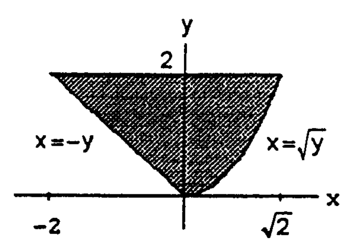
15. c = 0, d = 2;

$V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi y\left[\sqrt{y} - (-y)\right] dy$

$= 2\pi \int_0^2 \left(y^{3/2} + y^2\right) dy = 2\pi\left[\frac{2y^{5/2}}{5} + \frac{y^3}{3}\right]_0^2$

$= 2\pi\left[\frac{2}{5}\left(\sqrt{2}\right)^5 + \frac{2^3}{3}\right] = 2\pi\left(\frac{8\sqrt{2}}{5} + \frac{8}{3}\right) = 16\pi\left(\frac{\sqrt{2}}{5} + \frac{1}{3}\right)$

$= \frac{16\pi}{15}\left(3\sqrt{2} + 5\right)$

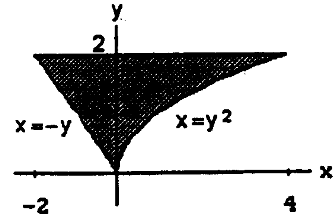


16. $c = 0$, $d = 2$;

$$V = \int_c^d 2\pi \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dy = \int_0^2 2\pi y\left[y^2 - (-y)\right] dy$$

$$= 2\pi \int_0^2 \left(y^3 + y^2\right) dy = 2\pi\left[\frac{y^4}{4} + \frac{y^3}{3}\right]_0^2 = 16\pi\left(\frac{2}{4} + \frac{1}{3}\right)$$

$$= 16\pi\left(\frac{5}{6}\right) = \frac{40\pi}{3}$$

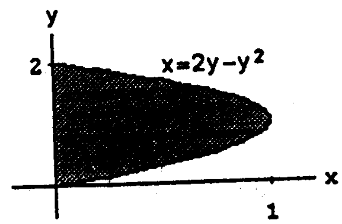


17. $c = 0$, $d = 2$;

$$V = \int_c^d 2\pi \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dy = \int_0^2 2\pi y\left(2y - y^2\right) dy$$

$$= 2\pi \int_0^2 \left(2y^2 - y^3\right) dy = 2\pi\left[\frac{2y^3}{3} - \frac{y^4}{4}\right]_0^2 = 2\pi\left(\frac{16}{3} - \frac{16}{4}\right)$$

$$= 32\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{32\pi}{12} = \frac{8\pi}{3}$$

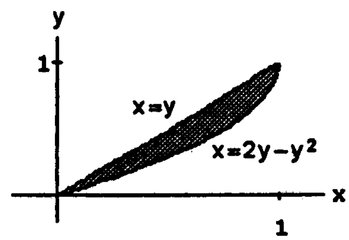


18. $c = 0$, $d = 1$;

$$V = \int_c^d 2\pi \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dy = \int_0^1 2\pi y\left(2y - y^2 - y\right) dy$$

$$= 2\pi \int_0^1 y\left(y - y^2\right) dy = 2\pi \int_0^1 \left(y^2 - y^3\right) dy$$

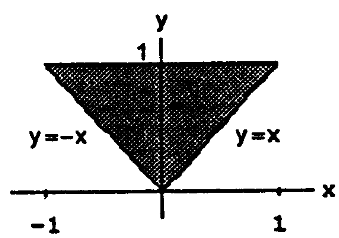
$$= 2\pi\left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$$



19. $c = 0$, $d = 1$;

$$V = \int_c^d 2\pi \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dy = 2\pi \int_0^1 y[y - (-y)]\, dy$$

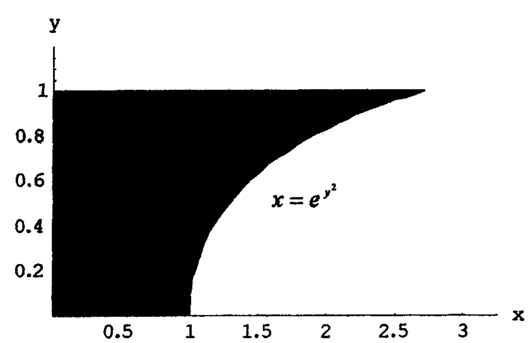
$$= 2\pi \int_0^1 2y^2\, dy = \frac{4\pi}{3}\left[y^3\right]_0^1 = \frac{4\pi}{3}$$



20. $c = 0$, $d = 1$; $V = \int_c^d 2\pi \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dy$

$$= \int_0^1 2\pi y e^{y^2}\, dy;\ \text{Let } u = y^2 \Rightarrow du = 2y\, dy$$

$$\Rightarrow V = \int_{u=0}^{1} \pi e^u\, du = \pi e^u \Big|_{u=0}^{1} = \pi(e - 1)$$

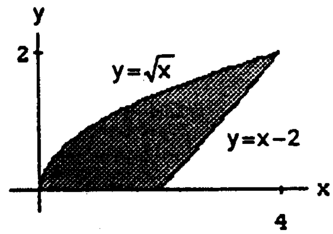


21. $c = 0,\ d = 2$;

$$V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi y[(2+y) - y^2]\, dy$$

$$= 2\pi \int_0^2 (2y + y^2 - y^3)\, dy = 2\pi \left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_0^2$$

$$= 2\pi\left(4 + \frac{8}{3} - \frac{16}{4}\right) = \frac{\pi}{6}(48 + 32 - 48) = \frac{16\pi}{3}$$

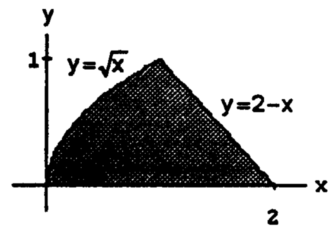


22. $c = 0,\ d = 1$;

$$V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi y[(2-y) - y^2]\, dy$$

$$= 2\pi \int_0^1 (2y - y^2 - y^3)\, dy = 2\pi \left[y^2 - \frac{y^3}{3} - \frac{y^4}{4}\right]_0^1$$

$$= 2\pi\left(1 - \frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}(12 - 4 - 3) = \frac{5\pi}{6}$$



23. (a) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi y \cdot 12(y^2 - y^3)\, dy = 24\pi \int_0^1 (y^3 - y^4)\, dy = 24\pi \left[\frac{y^4}{4} - \frac{y^5}{5}\right]_0^1$

$$= 24\pi\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{24\pi}{20} = \frac{6\pi}{5}$$

(b) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi(1-y)[12(y^2 - y^3)]\, dy = 24\pi \int_0^1 (1-y)(y^2 - y^3)\, dy$

$$= 24\pi \int_0^1 (y^2 - 2y^3 + y^4)\, dy = 24\pi \left[\frac{y^3}{3} - \frac{y^4}{2} + \frac{y^5}{5}\right]_0^1 = 24\pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = 24\pi\left(\frac{1}{30}\right) = \frac{4\pi}{5}$$

(c) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi\left(\frac{8}{5} - y\right)[12(y^2 - y^3)]\, dy = 24\pi \int_0^1 \left(\frac{8}{5} - y\right)(y^2 - y^3)\, dy$

$$= 24\pi \int_0^1 \left(\frac{8}{5}y^2 - \frac{13}{5}y^3 + y^4\right) dy = 24\pi \left[\frac{8}{15}y^3 - \frac{13}{20}y^4 + \frac{y^5}{5}\right]_0^1 = 24\pi\left(\frac{8}{15} - \frac{13}{20} + \frac{1}{5}\right) = \frac{24\pi}{60}(32 - 39 + 12)$$

$$= \frac{24\pi}{12} = 2\pi$$

(d) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi\left(y + \frac{2}{5}\right)[12(y^2 - y^3)]\, dy = 24\pi \int_0^1 \left(y + \frac{2}{5}\right)(y^2 - y^3)\, dy$

$$= 24\pi \int_0^1 \left(y^3 - y^4 + \frac{2}{5}y^2 - \frac{2}{5}y^3\right) dy = 24\pi \int_0^1 \left(\frac{2}{5}y^2 + \frac{3}{5}y^3 - y^4\right) dy = 24\pi \left[\frac{2}{15}y^3 + \frac{3}{20}y^4 - \frac{y^5}{5}\right]_0^1$$

$$= 24\pi\left(\frac{2}{15} + \frac{3}{20} - \frac{1}{5}\right) = \frac{24\pi}{60}(8 + 9 - 12) = \frac{24\pi}{12} = 2\pi$$

24. (a) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^2 2\pi y\left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\right]dy = \int_0^2 2\pi y\left(y^2 - \frac{y^4}{4}\right)dy = 2\pi\int_0^2 \left(y^3 - \frac{y^5}{4}\right)dy$

$= 2\pi\left[\frac{y^4}{4} - \frac{y^6}{24}\right]_0^2 = 2\pi\left(\frac{2^4}{4} - \frac{2^6}{24}\right) = 32\pi\left(\frac{1}{4} - \frac{4}{24}\right) = 32\pi\left(\frac{1}{4} - \frac{1}{6}\right) = 32\pi\left(\frac{2}{24}\right) = \frac{8\pi}{3}$

(b) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^2 2\pi(2-y)\left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\right]dy = \int_0^2 2\pi(2-y)\left(y^2 - \frac{y^4}{4}\right)dy$

$= 2\pi\int_0^2 \left(2y^2 - \frac{y^4}{2} - y^3 + \frac{y^5}{4}\right)dy = 2\pi\left[\frac{2y^3}{3} - \frac{y^5}{10} - \frac{y^4}{4} + \frac{y^6}{24}\right]_0^2 = 2\pi\left(\frac{16}{3} - \frac{32}{10} - \frac{16}{4} + \frac{64}{24}\right) = \frac{8\pi}{5}$

(c) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^2 2\pi(5-y)\left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\right]dy = \int_0^2 2\pi(5-y)\left(y^2 - \frac{y^4}{4}\right)dy$

$= 2\pi\int_0^2 \left(5y^2 - \frac{5}{4}y^4 - y^3 + \frac{y^5}{4}\right)dy = 2\pi\left[\frac{5y^3}{3} - \frac{5y^5}{20} - \frac{y^4}{4} + \frac{y^6}{24}\right]_0^2 = 2\pi\left(\frac{40}{3} - \frac{160}{20} - \frac{16}{4} + \frac{64}{24}\right) = 8\pi$

(d) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^2 2\pi\left(y + \frac{5}{8}\right)\left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\right]dy = \int_0^2 2\pi\left(y + \frac{5}{8}\right)\left(y^2 - \frac{y^4}{4}\right)dy$

$= 2\pi\int_0^2 \left(y^3 - \frac{y^5}{4} + \frac{5}{8}y^2 - \frac{5}{32}y^4\right)dy = 2\pi\left[\frac{y^4}{4} - \frac{y^6}{24} + \frac{5y^3}{24} - \frac{5y^5}{160}\right]_0^2 = 2\pi\left(\frac{16}{4} - \frac{64}{24} + \frac{40}{24} - \frac{160}{160}\right) = 4\pi$

25. (a) About the x-axis: $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^1 2\pi y\left(\sqrt{y} - y\right)dy = 2\pi\int_0^1 \left(y^{3/2} - y^2\right)dy$

$= 2\pi\left(\frac{2}{5}y^{5/2} - \frac{1}{3}y^3\right)\Big|_{y=0}^{1} = 2\pi\left(\frac{2}{5} - \frac{1}{3}\right) = \frac{2\pi}{15}$

About the y-axis: $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx = \int_0^1 2\pi x\left(x - x^2\right)dx = 2\pi\int_0^1 \left(x^2 - x^3\right)dx$

$= 2\pi\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_{x=0}^{1} = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$

(b) About the x-axis: $R(x) = x$ and $r(x) = x^2 \Rightarrow V = \int_a^b \pi\left[R(x)^2 - r(x)^2\right]dx = \int_0^1 \pi\left[x^2 - x^4\right]dx$

$= \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_{x=0}^{1} = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$

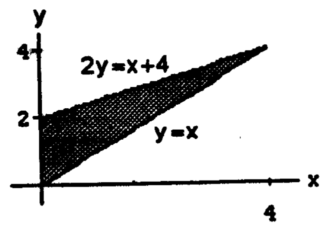
About the y-axis: $R(y) = \sqrt{y}$ and $r(y) = y \Rightarrow V = \int_c^d \pi\left[R(y)^2 - r(y)^2\right]dy = \int_0^1 \pi\left[y - y^2\right]dy$

$= \pi\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_{y=0}^{1} = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}$

26. (a) $V = \int_a^b \pi[R^2(x) - r^2(x)]\,dx = \pi \int_0^4 \left[\left(\frac{x}{2}+2\right)^2 - x^2\right] dx$

$= \pi \int_0^4 \left(-\frac{3}{4}x^2 + 2x + 4\right) dx = \pi\left[-\frac{x^3}{4} + x^2 + 4x\right]_0^4$

$= \pi(-16 + 16 + 16) = 16\pi$



(b) $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^4 2\pi x\left(\frac{x}{2}+2-x\right) dx = \int_0^4 2\pi x\left(2-\frac{x}{2}\right) dx = 2\pi \int_0^4 \left(2x - \frac{x^2}{2}\right) dx$

$= 2\pi\left[x^2 - \frac{x^3}{6}\right]_0^4 = 2\pi\left(16 - \frac{64}{6}\right) = \frac{32\pi}{3}$

(c) $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^4 2\pi(4-x)\left(\frac{x}{2}+2-x\right) dx = 2\pi \int_0^4 (4-x)\left(2-\frac{x}{2}\right) dx$

$= 2\pi \int_0^4 \left(8 - 4x + \frac{x^2}{2}\right) dx = 2\pi\left[8x - 2x^2 + \frac{x^3}{6}\right]_0^4 = 2\pi\left(32 - 32 + \frac{64}{6}\right) = \frac{64\pi}{3}$

(d) $V = \int_a^b \pi[R^2(x) - r^2(x)]\,dx = \int_0^4 \pi\left[(8-x)^2 - \left(6-\frac{x}{2}\right)^2\right] dx$

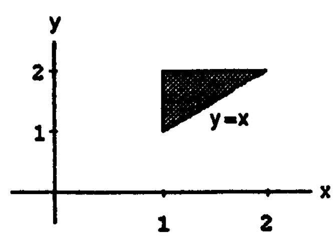
$= \pi \int_0^4 \left[(64 - 16x + x^2) - \left(36 - 6x + \frac{x^2}{4}\right)\right] dx = \pi \int_0^4 \left(\frac{3}{4}x^2 - 10x + 28\right) dx = \pi\left[\frac{x^3}{4} - 5x^2 + 28x\right]_0^4$

$= \pi[16 - (5)(16) + (7)(16)] = \pi(3)(16) = 48\pi$

27. (a) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_1^2 2\pi y(y-1)\,dy$

$= 2\pi \int_1^2 (y^2 - y)\,dy = 2\pi\left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^2 = 2\pi\left[\left(\frac{8}{3}-\frac{4}{2}\right) - \left(\frac{1}{3}-\frac{1}{2}\right)\right]$

$= 2\pi\left(\frac{7}{3} - 2 + \frac{1}{2}\right) = \frac{\pi}{3}(14 - 12 + 3) = \frac{5\pi}{3}$



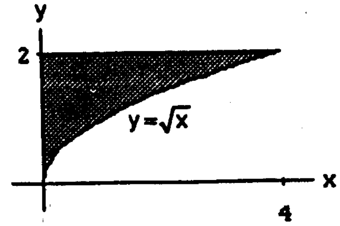
(b) $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_1^2 2\pi x(2-x)\,dx = 2\pi \int_1^2 (2x - x^2)\,dx = 2\pi\left[x^2 - \frac{x^3}{3}\right]_1^2$

$= 2\pi\left[\left(4-\frac{8}{3}\right) - \left(1-\frac{1}{3}\right)\right] = 2\pi\left[\left(\frac{12-8}{3}\right) - \left(\frac{3-1}{3}\right)\right] = 2\pi\left(\frac{4}{3}-\frac{2}{3}\right) = \frac{4\pi}{3}$

(c) $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_1^2 2\pi\left(\frac{10}{3} - x\right)(2-x)\,dx = 2\pi \int_1^2 \left(\frac{20}{3} - \frac{16}{3}x + x^2\right) dx$

$= 2\pi\left[\frac{20}{3}x - \frac{8}{3}x^2 + \frac{1}{3}x^3\right]_1^2 = 2\pi\left[\left(\frac{40}{3} - \frac{32}{3} + \frac{8}{3}\right) - \left(\frac{20}{3} - \frac{8}{3} + \frac{1}{3}\right)\right] = 2\pi\left(\frac{3}{3}\right) = 2\pi$

(d) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_1^2 2\pi(y-1)(y-1)\,dy = 2\pi \int_1^2 (y-1)^2 = 2\pi\left[\frac{(y-1)^3}{3}\right]_1^2 = \frac{2\pi}{3}$



28. (a) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi y(y^2 - 0)\, dy$

$= 2\pi \int_0^2 y^3\, dy = 2\pi\left[\frac{y^4}{4}\right]_0^2 = 2\pi\left(\frac{2^4}{4}\right) = 8\pi$

(b) $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^4 2\pi x(2 - \sqrt{x})\, dx = 2\pi \int_0^4 (2x - x^{3/2})\, dx = 2\pi\left[x^2 - \frac{2}{5}x^{5/2}\right]_0^4$

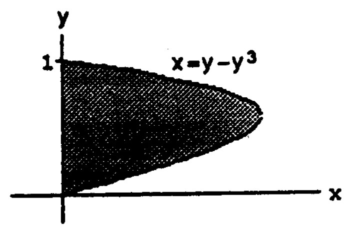
$= 2\pi\left(16 - \frac{2\cdot 2^5}{5}\right) = 2\pi\left(16 - \frac{64}{5}\right) = \frac{2\pi}{5}(80 - 64) = \frac{32\pi}{5}$

(c) $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^4 2\pi(4 - x)(2 - \sqrt{x})\, dx = 2\pi \int_0^4 (8 - 4x^{1/2} - 2x + x^{3/2})\, dx$

$= 2\pi\left[8x - \frac{8}{3}x^{3/2} - x^2 + \frac{2}{5}x^{5/2}\right]_0^4 = 2\pi\left(32 - \frac{64}{3} - 16 + \frac{64}{5}\right) = \frac{2\pi}{15}(240 - 320 + 192) = \frac{2\pi}{15}(112) = \frac{224\pi}{15}$

(d) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi(2 - y)(y^2)\, dy = 2\pi \int_0^2 (2y^2 - y^3)\, dy = 2\pi\left[\frac{2}{3}y^3 - \frac{y^4}{4}\right]_0^2$

$= 2\pi\left(\frac{16}{3} - \frac{16}{4}\right) = \frac{32\pi}{12}(4 - 3) = \frac{8\pi}{3}$



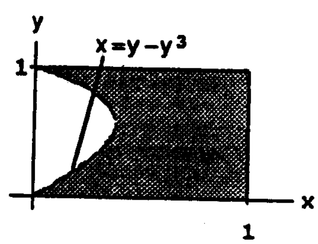
29. (a) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi y(y - y^3)\, dy$

$= \int_0^1 2\pi(y^2 - y^4)\, dy = 2\pi\left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{5}\right)$

$= \frac{4\pi}{15}$

(b) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi(1 - y)(y - y^3)\, dy = 2\pi \int_0^1 (y - y^2 - y^3 + y^4)\, dy$

$= 2\pi\left[\frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5}\right]_0^1 = 2\pi\left(\frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5}\right) = \frac{2\pi}{60}(30 - 20 - 15 + 12) = \frac{7\pi}{30}$



30. (a) $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi y[1 - (y - y^3)]\, dy$

$= 2\pi \int_0^1 (y - y^2 + y^4)\, dy = 2\pi\left[\frac{y^2}{2} - \frac{y^3}{3} + \frac{y^5}{5}\right]_0^1$

$= 2\pi\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{5}\right) = \frac{2\pi}{30}(15 - 10 + 6)$

$= \frac{11\pi}{15}$

(b) Use the washer method:

$$V = \int_c^d \pi[R^2(y) - r^2(y)]\,dy = \int_0^1 \pi\left[1^2 - (y - y^3)^2\right] dy = \pi \int_0^1 (1 - y^2 - y^6 + 2y^4)\,dy = \pi\left[y - \frac{y^3}{3} - \frac{y^7}{7} + \frac{2y^5}{5}\right]_0^1$$

$$= \pi\left(1 - \frac{1}{3} - \frac{1}{7} + \frac{2}{5}\right) = \frac{\pi}{105}(105 - 35 - 15 + 42) = \frac{97\pi}{105}$$

(c) Use the washer method:

$$V = \int_c^d \pi[R^2(y) - r^2(y)]\,dy = \int_0^1 \pi\left[[1 - (y - y^3)]^2 - 0\right] dy = \pi \int_0^1 \left[1 - 2(y - y^3) + (y - y^3)^2\right] dy$$

$$= \pi \int_0^1 (1 + y^2 + y^6 - 2y + 2y^3 - 2y^4)\,dy = \pi\left[y + \frac{y^3}{3} + \frac{y^7}{7} - y^2 + \frac{y^4}{2} - \frac{2y^5}{5}\right]_0^1 = \pi\left(1 + \frac{1}{3} + \frac{1}{7} - 1 + \frac{1}{2} - \frac{2}{5}\right)$$

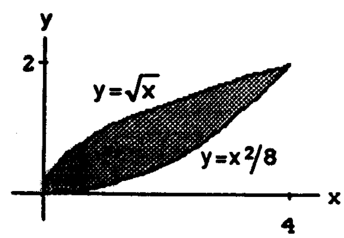
$$= \frac{\pi}{210}(70 + 30 + 105 - 2 \cdot 42) = \frac{121\pi}{210}$$

(d) $$V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi(1 - y)[1 - (y - y^3)]\,dy = 2\pi \int_0^1 (1 - y)(1 - y + y^3)\,dy$$

$$= 2\pi \int_0^1 (1 - y + y^3 - y + y^2 - y^4)\,dy = 2\pi \int_0^1 (1 - 2y + y^2 + y^3 - y^4)\,dy = 2\pi\left[y - y^2 + \frac{y^3}{3} + \frac{y^4}{4} - \frac{y^5}{5}\right]_0^1$$

$$= 2\pi\left(1 - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{5}\right) = \frac{2\pi}{60}(20 + 15 - 12) = \frac{23\pi}{30}$$

31. (a) $$V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi y\left(\sqrt{8y} - y^2\right) dy$$



$$= 2\pi \int_0^2 \left(2\sqrt{2}\,y^{3/2} - y^3\right) dy = 2\pi\left[\frac{4\sqrt{2}}{5}y^{5/2} - \frac{y^4}{4}\right]_0^2$$

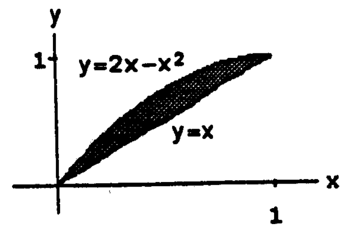
$$= 2\pi\left(\frac{4\sqrt{2}\cdot(\sqrt{2})^5}{5} - \frac{2^4}{4}\right) = 2\pi\left(\frac{4 \cdot 2^3}{5} - \frac{4 \cdot 4}{4}\right)$$

$$= 2\pi \cdot 4\left(\frac{8}{5} - 1\right) = \frac{8\pi}{5}(8 - 5) = \frac{24\pi}{5}$$

(b) $$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^4 2\pi x\left(\sqrt{x} - \frac{x^2}{8}\right) dx = 2\pi \int_0^4 \left(x^{3/2} - \frac{x^3}{8}\right) dx = 2\pi\left[\frac{2}{5}x^{5/2} - \frac{x^4}{32}\right]_0^4$$

$$= 2\pi\left(\frac{2 \cdot 2^5}{5} - \frac{4^4}{32}\right) = 2\pi\left(\frac{2^6}{5} - \frac{2^8}{32}\right) = \frac{\pi \cdot 2^7}{160}(32 - 20) = \frac{\pi \cdot 2^9 \cdot 3}{160} = \frac{\pi \cdot 2^4 \cdot 3}{5} = \frac{48\pi}{5}$$

32. (a) $V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_0^1 2\pi x\left[\left(2x-x^2\right)-x\right]dx$



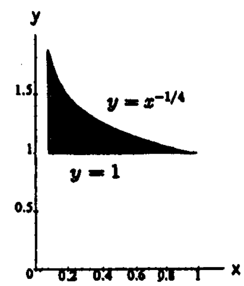
$=2\pi\int_0^1 x\left(x-x^2\right)dx=2\pi\int_0^1\left(x^2-x^3\right)dx$

$=2\pi\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=2\pi\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{\pi}{6}$

(b) $V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_0^1 2\pi(1-x)\left[\left(2x-x^2\right)-x\right]dx=2\pi\int_0^1(1-x)\left(x-x^2\right)dx$

$=2\pi\int_0^1\left(x-2x^2+x^3\right)dx=2\pi\left[\frac{x^2}{2}-\frac{2}{3}x^3+\frac{x^4}{4}\right]_0^1=2\pi\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)=\frac{2\pi}{12}(6-8+3)=\frac{\pi}{6}$

33. (a) $V=\int_a^b\pi\left[R^2(x)-r^2(x)\right]dx=\pi\int_{1/16}^1\left(x^{-1/2}-1\right)dx$



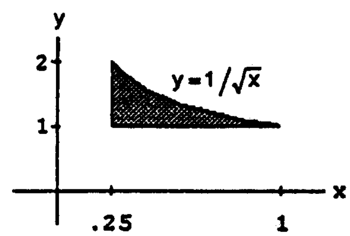
$=\pi\left[2x^{1/2}-x\right]_{1/16}^1=\pi\left[(2-1)-\left(2\cdot\frac{1}{4}-\frac{1}{16}\right)\right]$

$=\pi\left(1-\frac{7}{16}\right)=\frac{9\pi}{16}$

(b) $V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy=\int_1^2 2\pi y\left(\frac{1}{y^4}-\frac{1}{16}\right)dy=2\pi\int_1^2\left(y^{-3}-\frac{y}{16}\right)dy=2\pi\left[-\frac{1}{2}y^{-2}-\frac{y^2}{32}\right]_1^2$

$=2\pi\left[\left(-\frac{1}{8}-\frac{1}{8}\right)-\left(-\frac{1}{2}-\frac{1}{32}\right)\right]=2\pi\left(\frac{1}{4}+\frac{1}{32}\right)=\frac{2\pi}{32}(8+1)=\frac{9\pi}{16}$

34. (a) $V=\int_c^d\pi\left[R^2(y)-r^2(y)\right]dy=\int_{1/16}^1\pi\left(\frac{1}{y^4}-\frac{1}{16}\right)dy$



$=\pi\left[-\frac{1}{3}y^{-3}-\frac{y}{16}\right]_1^2=\pi\left[\left(-\frac{1}{24}-\frac{1}{8}\right)-\left(-\frac{1}{3}-\frac{1}{16}\right)\right]$

$=\frac{\pi}{48}(-2-6+16+3)=\frac{11\pi}{48}$

(b) $V=\int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dx=\int_{1/4}^1 2\pi x\left(\frac{1}{\sqrt{x}}-1\right)dx=2\pi\int_{1/4}^1\left(x^{1/2}-x\right)dx=2\pi\left[\frac{2}{3}x^{3/2}-\frac{x^2}{2}\right]_{1/4}^1$

$=2\pi\left[\left(\frac{2}{3}-\frac{1}{2}\right)-\left(\frac{2}{3}\cdot\frac{1}{8}-\frac{1}{32}\right)\right]=\pi\left(\frac{4}{3}-1-\frac{1}{6}+\frac{1}{16}\right)=\frac{\pi}{48}(4\cdot 16-48-8+3)=\frac{11\pi}{48}$

35. *Disc*: $V=V_1-V_2$

$V_1=\int_{a_1}^{b_1}\pi\left[R_1(x)\right]^2dx$ and $V_2=\int_{a_2}^{b_2}\pi\left[R_2(x)\right]^2$ with $R_1(x)=\sqrt{\frac{x+2}{3}}$ and $R_2(x)=\sqrt{x}$,

$a_1=-2,\ b_1=1;\ a_2=0,\ b_2=1\Rightarrow$ two integrals are required

Washer: $V = V_1 - V_2$

$$V_1 = \int_{a_1}^{b_1} \pi\left([R_1(x)]^2 - [r_1(x)]^2\right) dx \text{ with } R_1(x) = \sqrt{\frac{x+2}{3}} \text{ and } r_1(x) = 0;\ a_1 = -2 \text{ and } b_1 = 0;$$

$$V_2 = \int_{a_2}^{b_2} \pi\left([R_2(x)]^2 - [r_2(x)]^2\right) dx \text{ with } R_2(x) = \sqrt{\frac{x+2}{3}} \text{ and } r_2(x) = \sqrt{x};\ a_2 = 0 \text{ and } b_2 = 1$$

$\Rightarrow$ two integrals are required

Shell: $V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_c^d 2\pi y \binom{\text{shell}}{\text{height}} dy$ where shell height $= y^2 - (3y^2 - 2) = 2 - 2y^2$;

$c = 0$ and $d = 1$. Only *one* integral is required. It is, therefore preferable to use the *shell* method. However, whichever method you use, you will get $V = \pi$.

36. *Disc*: $V = V_1 - V_2 - V_3$

$$V_i = \int_{c_i}^{d_i} \pi[R_i(y)]^2\, dy,\ i = 1, 2, 3 \text{ with } R_1(y) = 1 \text{ and } c_1 = -1,\ d_1 = 1;\ R_2(y) = \sqrt{y} \text{ and } c_2 = 0 \text{ and } d_2 = 1;$$

$R_3(y) = (-y)^{1/4}$ and $c_3 = -1$, $d_3 = 0 \Rightarrow$ three integrals are required

Washer: $V = V_1 + V_2$

$$V_i = \int_{c_i}^{d_i} \pi\left([R_i(y)]^2 - [r_i(y)]^2\right) dy,\ i = 1, 2 \text{ with } R_1(y) = 1,\ r_1(y) = \sqrt{y},\ c_1 = 0 \text{ and } d_1 = 1;$$

$R_2(y) = 1$, $r_2(y) = (-y)^{1/4}$, $c_2 = -1$ and $d_2 = 0 \Rightarrow$ two integrals are required

Shell: $V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx$ where shell radius $= x$, shell height $= x^2 - (-x^4) = x^2 + x^4$,

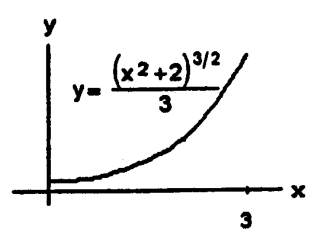
$a = 0$ and $b = 1 \Rightarrow$ only one integral is required. It is, therefore preferable to use the *shell* method. However, whichever method you use, you will get $V = \frac{5\pi}{6}$.

5.3 LENGTHS OF PLANE CURVES

1. $\frac{dy}{dx} = \frac{1}{3}\cdot\frac{3}{2}(x^2+2)^{1/2}\cdot 2x = \sqrt{(x^2+2)}\cdot x$

$$\Rightarrow L = \int_0^3 \sqrt{1+(x^2+2)x^2}\, dx = \int_0^3 \sqrt{1+2x^2+x^4}\, dx$$

$$= \int_0^3 \sqrt{(1+x^2)^2}\, dx = \int_0^3 (1+x^2)\, dx = \left[x + \frac{x^3}{3}\right]_0^3$$



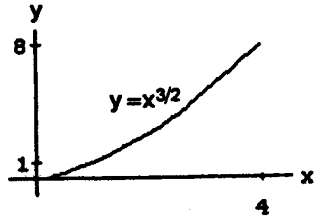
$= 3 + \frac{27}{3} = 12$

2. $\frac{dy}{dx} = \frac{3}{2}\sqrt{x} \Rightarrow L = \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx;\ \left[u = 1 + \frac{9}{4}x\right.$

$\Rightarrow du = \frac{9}{4}\, dx \Rightarrow \frac{4}{9}\, du = dx;\ x = 0 \Rightarrow u = 1;\ x = 4$

$\left.\Rightarrow u = 10\right] \to L = \int_1^{10} u^{1/2}\left(\frac{4}{9}\, du\right) = \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10}$

$= \frac{8}{27}\left(10\sqrt{10} - 1\right)$

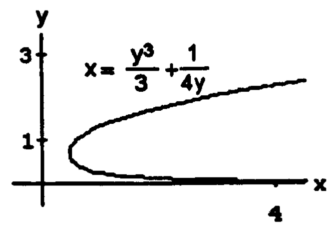


3. $\frac{dx}{dy} = y^2 - \frac{1}{4y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^4 - \frac{1}{2} + \frac{1}{16y^4}$

$\Rightarrow L = \int_1^3 \sqrt{1 + y^4 - \frac{1}{2} + \frac{1}{16y^4}}\, dy = \int_1^3 \sqrt{y^4 + \frac{1}{2} + \frac{1}{16y^4}}\, dy$

$= \int_1^3 \sqrt{\left(y^2 + \frac{1}{4y^2}\right)^2}\, dy = \int_1^3 \left(y^2 + \frac{1}{4y^2}\right) dy = \left[\frac{y^3}{3} - \frac{y^{-1}}{4}\right]_1^3$

$= \left(\frac{27}{3} - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = 9 - \frac{1}{12} - \frac{1}{3} + \frac{1}{4} = 9 + \frac{(-1-4+3)}{12} = 9 + \frac{(-2)}{12} = \frac{53}{6}$

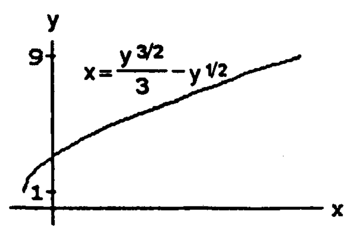


4. $\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y - 2 + \frac{1}{y}\right)$

$\Rightarrow L = \int_1^9 \sqrt{1 + \frac{1}{4}\left(y - 2 + \frac{1}{y}\right)}\, dy = \int_1^9 \sqrt{\frac{1}{4}\left(y + 2 + \frac{1}{y}\right)}\, dy$

$= \int_1^9 \frac{1}{2}\sqrt{\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2}\, dy = \frac{1}{2}\int_1^9 \left(y^{1/2} + y^{-1/2}\right) dy$

$= \frac{1}{2}\left[\frac{2}{3}y^{3/2} + 2y^{1/2}\right]_1^9 = \left[\frac{y^{3/2}}{3} + y^{1/2}\right]_1^9 = \left(\frac{3^3}{3} + 3\right) - \left(\frac{1}{3} + 1\right) = 11 - \frac{1}{3} = \frac{32}{3}$

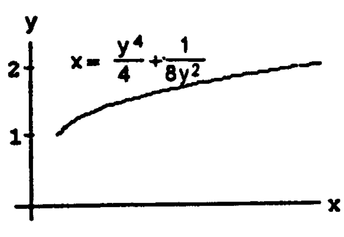


5. $\frac{dx}{dy} = y^3 - \frac{1}{4y^3} \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^6 - \frac{1}{2} + \frac{1}{16y^6}$

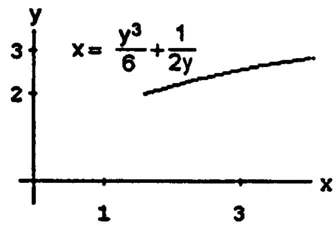
$\Rightarrow L = \int_1^2 \sqrt{1 + y^6 - \frac{1}{2} + \frac{1}{16y^6}}\, dy = \int_1^2 \sqrt{y^6 + \frac{1}{2} + \frac{1}{16y^6}}\, dy$

$= \int_1^2 \sqrt{\left(y^3 + \frac{y^{-3}}{4}\right)^2}\, dy = \int_1^2 \left(y^3 + \frac{y^{-3}}{4}\right) dy$

$= \left[\frac{y^4}{4} - \frac{y^{-2}}{8}\right]_1^2 = \left(\frac{16}{4} - \frac{1}{(16)(2)}\right) - \left(\frac{1}{4} - \frac{1}{8}\right) = 4 - \frac{1}{32} - \frac{1}{4} + \frac{1}{8} = \frac{128 - 1 - 8 + 4}{32} = \frac{123}{32}$



6. $\frac{dx}{dy}=\frac{y^2}{2}-\frac{1}{2y^2}\Rightarrow\left(\frac{dx}{dy}\right)^2=\frac{1}{4}\left(y^4-2+y^{-4}\right)$

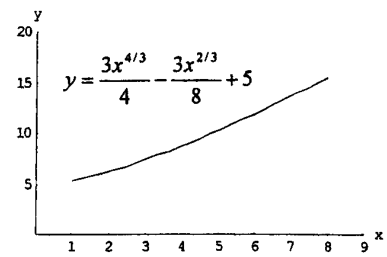


$$\Rightarrow L=\int_2^3\sqrt{1+\tfrac{1}{4}\left(y^4-2+y^{-4}\right)}\,dy=\int_2^3\sqrt{\tfrac{1}{4}\left(y^4+2+y^{-4}\right)}\,dy$$

$$=\frac{1}{2}\int_2^3\sqrt{\left(y^2+y^{-2}\right)^2}\,dy=\frac{1}{2}\int_2^3\left(y^2+y^{-2}\right)dy=\frac{1}{2}\left[\frac{y^3}{3}-y^{-1}\right]_2^3$$

$$=\frac{1}{2}\left[\left(\frac{27}{3}-\frac{1}{3}\right)-\left(\frac{8}{3}-\frac{1}{2}\right)\right]=\frac{1}{2}\left(\frac{26}{3}-\frac{8}{3}+\frac{1}{2}\right)=\frac{1}{2}\left(6+\frac{1}{2}\right)=\frac{13}{4}$$

7. $\frac{dy}{dx}=x^{1/3}-\frac{1}{4}x^{-1/3}\Rightarrow\left(\frac{dy}{dx}\right)^2=x^{2/3}-\frac{1}{2}+\frac{x^{-2/3}}{16}$

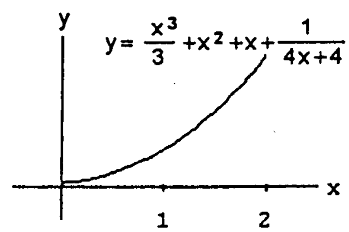


$$\Rightarrow L=\int_1^8\sqrt{1+x^{2/3}-\frac{1}{2}+\frac{x^{-2/3}}{16}}\,dx=\int_1^8\sqrt{x^{2/3}+\frac{1}{2}+\frac{x^{-2/3}}{16}}\,dx$$

$$=\int_1^8\sqrt{\left(x^{1/3}+\tfrac{1}{4}x^{-1/3}\right)^2}\,dx=\int_1^8\left(x^{1/3}+\tfrac{1}{4}x^{-1/3}\right)dx$$

$$=\left[\frac{3}{4}x^{4/3}+\frac{3}{8}x^{2/3}\right]_1^8=\frac{3}{8}\left[2x^{4/3}+x^{2/3}\right]_1^8=\frac{3}{8}\left[\left(2\cdot2^4+2^2\right)-(2+1)\right]=\frac{3}{8}(32+4-3)=\frac{99}{8}$$

8. $\frac{dy}{dx}=x^2+2x+1-\frac{4}{(4x+4)^2}=x^2+2x+1-\frac{1}{4}\frac{1}{(1+x)^2}$



$$=(1+x)^2-\frac{1}{4}\frac{1}{(1+x)^2}\Rightarrow\left(\frac{dy}{dx}\right)^2=(1+x)^4-\frac{1}{2}+\frac{1}{16(1+x)^4}$$

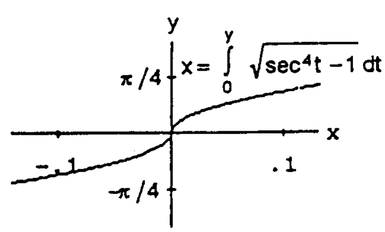
$$\Rightarrow L=\int_0^2\sqrt{1+(1+x)^4-\frac{1}{2}+\frac{(1+x)^{-4}}{16}}\,dx$$

$$=\int_0^2\sqrt{(1+x)^4+\frac{1}{2}+\frac{(1+x)^{-4}}{16}}\,dx=\int_0^2\sqrt{\left[(1+x)^2+\frac{(1+x)^{-2}}{4}\right]^2}\,dx=\int_0^2\left[(1+x)^2+\frac{(1+x)^{-2}}{4}\right]dx;$$

$[u=1+x\Rightarrow du=dx;\ x=0\Rightarrow u=1,\ x=2\Rightarrow u=3]$

$$\rightarrow L=\int_1^3\left(u^2+\tfrac{1}{4}u^{-2}\right)du=\left[\frac{u^3}{3}-\frac{1}{4}u^{-1}\right]_1^3=\left(9-\frac{1}{12}\right)-\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{108-1-4+3}{12}=\frac{106}{12}=\frac{53}{6}$$

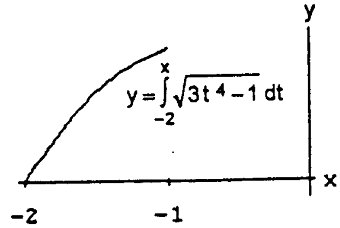
9. $\frac{dx}{dy}=\sqrt{\sec^4 y-1}\Rightarrow\left(\frac{dx}{dy}\right)^2=\sec^4 y-1$



$$\Rightarrow L=\int_{-\pi/4}^{\pi/4}\sqrt{1+\left(\sec^4 y-1\right)}\,dy=\int_{-\pi/4}^{\pi/4}\sec^2 y\,dy$$

$= [\tan y]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$

10. $\frac{dy}{dx} = \sqrt{3x^4 - 1} \Rightarrow \left(\frac{dy}{dx}\right)^2 = 3x^4 - 1$



$\Rightarrow L = \int_{-2}^{-1} \sqrt{1 + (3x^4 - 1)}\, dx = \int_{-2}^{-1} \sqrt{3}\, x^2\, dx$

$= \sqrt{3}\left[\frac{x^3}{3}\right]_{-2}^{-1} = \frac{\sqrt{3}}{3}[-1 - (-2)^3] = \frac{\sqrt{3}}{3}(-1 + 8) = \frac{7\sqrt{3}}{3}$

11. $\frac{dx}{dt} = -a \sin t$ and $\frac{dy}{dt} = a \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-a \sin t)^2 + (a \cos t)^2} = \sqrt{a^2(\sin^2 t + \cos^2 t)} = |a|$

$\Rightarrow \text{Length} = \int_0^{2\pi} |a|\, dt = |a| \int_0^{2\pi} dt = 2\pi|a|$, the circumference of a circle when $a > 0$.

12. $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = 1 + \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin t)^2 + (1 + \cos t)^2} = \sqrt{2 + 2 \cos t}$

$\Rightarrow \text{Length} = \int_0^{\pi} \sqrt{2 + 2 \cos t}\, dt = \sqrt{2} \int_0^{\pi} \sqrt{\left(\frac{1 - \cos t}{1 - \cos t}\right)(1 + \cos t)}\, dt = \sqrt{2} \int_0^{\pi} \sqrt{\frac{\sin^2 t}{1 - \cos t}}\, dt$

$= \sqrt{2} \int_0^{\pi} \frac{\sin t}{\sqrt{1 - \cos t}}\, dt$ (since $\sin t \ge 0$ on $[0, \pi]$); $[u = 1 - \cos t \Rightarrow du = \sin t\, dt;\ t = 0 \Rightarrow u = 0,$

$t = \pi \Rightarrow u = 2] \rightarrow \sqrt{2} \int_0^{2} u^{-1/2}\, du = \sqrt{2}\left[2u^{1/2}\right]_0^2 = 4$

13. $\frac{dx}{dt} = e^t - 1$ and $\frac{dy}{dt} = 2e^{t/2} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(e^t - 1)^2 + (2e^{t/2})^2} = \sqrt{e^{2t} + 2e^t + 1} = \sqrt{(e^t + 1)^2}$

$|e^t + 1| = e^t + 1$ (since $e^t + 1 > 0$ for all t) $\Rightarrow L = \int_0^3 (e^t + 1)\, dt = (e^t + t)\big|_0^3 = e^3 + 2$

14. $\frac{dx}{dt} = t$ and $\frac{dy}{dt} = (2t + 1)^{1/2} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t^2 + (2t + 1)} = \sqrt{(t + 1)^2} = |t + 1| = t + 1$ since $0 \le t \le 4$

$\Rightarrow \text{Length} = \int_0^4 (t + 1)\, dt = \left[\frac{t^2}{2} + t\right]_0^4 = (8 + 4) = 12$

15. $\frac{dx}{dt} = e^t(\cos t - \sin t)$ and $\frac{dy}{dt} = e^t(\sin t + \cos t) \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{e^{2t}((\cos t - \sin t)^2 + (\sin t + \cos t)^2)} = \sqrt{2e^{2t}(\sin^2 t + \cos^2 t)} = \sqrt{2}|e^t| = \sqrt{2}\,e^t$ (since $e^t > 0$ for all t)

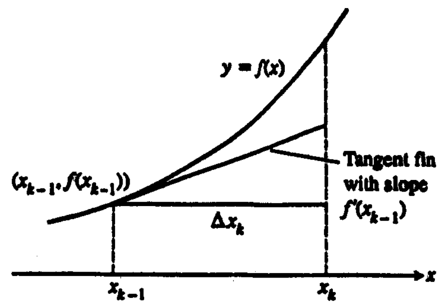
$\Rightarrow L = \int_0^{\pi} \sqrt{2}\,e^t\,dt = \sqrt{2}\,e^t\Big|_0^{\pi} = \sqrt{2}(e^{\pi} - 1)$

16. $\frac{dx}{dt} = 8t\cos t$ and $\frac{dy}{dt} = 8t\sin t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(8t\cos t)^2 + (8t\sin t)^2} = \sqrt{64t^2\cos^2 t + 64t^2\sin^2 t}$

$= |8t| = 8t$ since $0 \le t \le \frac{\pi}{2} \Rightarrow$ Length $= \int_0^{\pi/2} 8t\,dt = \left[4t^2\right]_0^{\pi/2} = \pi^2$

17. $\sqrt{2}\,a = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx,\ a \ge 0 \Rightarrow \sqrt{2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \Rightarrow \frac{dy}{dx} = \pm 1 \Rightarrow y = f(x) = \pm x + C$ where C is any real number.

18. (a) From the accompanying figure and definition of the differential (change along the tangent line) we see that $dy = f'(x_{k-1})\,\Delta x_k \Rightarrow$ length of kth tangent fin is $\sqrt{(\Delta x_k)^2 + (dy)^2} = \sqrt{(\Delta x_k)^2 + [f'(x_{k-1})\,\Delta x_k]^2}$.



(b) Length of curve $= \lim_{n\to\infty} \sum_{k=1}^{n}$ (length of kth tangent fin) $= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f'(x_{k-1})\,\Delta x_k]^2}$

$= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + [f'(x_{k-1})]^2}\,\Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$

19. (a) $\left(\frac{dy}{dx}\right)^2$ corresponds to $\frac{1}{4x}$ here, so take $\frac{dy}{dx}$ as $\frac{1}{2\sqrt{x}}$. Then $y = \sqrt{x} + C$, and since $(1,1)$ lies on the curve, $C = 0$. So $y = \sqrt{x}$ from $(1,1)$ to $(4,2)$.

(b) Only one. We know the derivative of the function and the value of the function at one value of x.

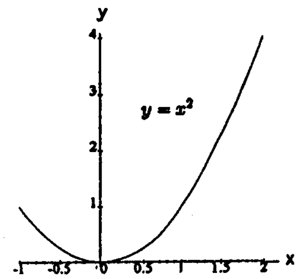
20. (a) $\left(\frac{dx}{dy}\right)^2$ corresponds to $\frac{1}{y^4}$ here, so take $\frac{dx}{dy}$ as $\frac{1}{y^2}$. Then $x = -\frac{1}{y} + C$ and, since $(0,1)$ lies on the curve, $C = 1$. So $y = \frac{1}{1-x}$.

(b) Only one. We know the derivative of the function and the value of the function at one value of x.

21. (a) $\frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)^2 = 4x^2 \Rightarrow L = \int_{-1}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_{-1}^{2} \sqrt{1 + 4x^2}\, dx$

(b)

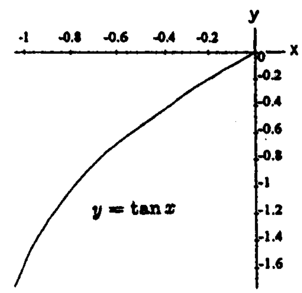


(c) $L \approx 6.13$

22. (a) $\frac{dy}{dx} = \sec^2 x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \sec^4 x$

$\Rightarrow L = \int_{-\pi/3}^{0} \sqrt{1 + \sec^4 x}\, dx$

(b)

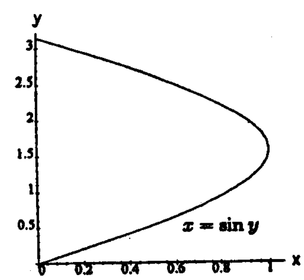


(c) $L \approx 2.06$

23. (a) $\frac{dx}{dy} = \cos y \Rightarrow \left(\frac{dx}{dy}\right)^2 = \cos^2 y$

$\Rightarrow L = \int_{0}^{\pi} \sqrt{1 + \cos^2 y}\, dy$

(b)



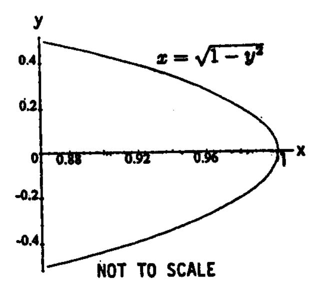
(c) $L \approx 3.82$

24. (a) $\frac{dx}{dy} = -\frac{y}{\sqrt{1-y^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{y^2}{1-y^2}$

$\Rightarrow L = \int_{-1/2}^{1/2} \sqrt{1 + \frac{y^2}{(1-y^2)}}\, dy = \int_{-1/2}^{1/2} \sqrt{\frac{1}{1-y^2}}\, dy$

$= \int_{-1/2}^{1/2} (1-y^2)^{-1/2}\, dy$

(b)

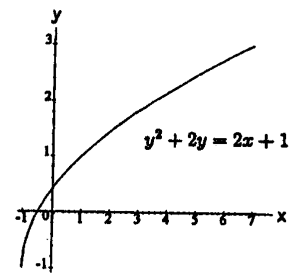


(c) $L \approx 1.05$

25. (a) $2y + 2 = 2\,\frac{dx}{dy} \Rightarrow \left(\frac{dx}{dy}\right)^2 = (y+1)^2$

$$\Rightarrow L = \int_{-1}^{3} \sqrt{1 + (y+1)^2}\,dy$$

(b)

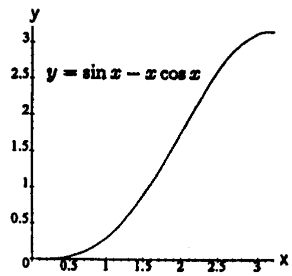


(c) $L \approx 9.29$

26. (a) $\frac{dy}{dx} = \cos x - \cos x + x \sin x \Rightarrow \left(\frac{dy}{dx}\right)^2 = x^2 \sin^2 x$

$$\Rightarrow L = \int_{0}^{\pi} \sqrt{1 + x^2 \sin^2 x}\,dx$$

(b)



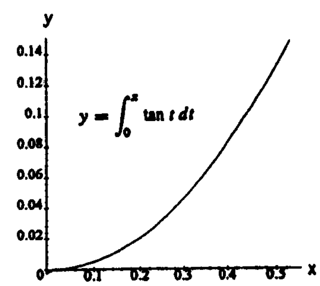
(c) $L \approx 4.70$

27. (a) $\frac{dy}{dx} = \tan x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x$

$$\Rightarrow L = \int_{0}^{\pi/6} \sqrt{1 + \tan^2 x}\,dx = \int_{0}^{\pi/6} \sqrt{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}}\,dx$$

$$= \int_{0}^{\pi/6} \frac{dx}{\cos x} = \int_{0}^{\pi/6} \sec x\,dx$$

(b)



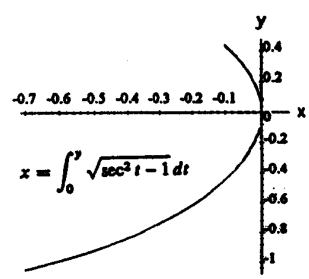
(c) $L \approx 0.55$

28. (a) $\frac{dx}{dy} = \sqrt{\sec^2 y - 1} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \sec^2 y - 1$

$$\Rightarrow L = \int_{-\pi/3}^{\pi/4} \sqrt{1 + \left(\sec^2 y - 1\right)}\,dy$$

$$= \int_{-\pi/3}^{\pi/4} |\sec y|\,dy = \int_{-\pi/3}^{\pi/4} \sec y\,dy$$

(b)



(c) $L \approx 2.20$

29. The length of the curve $y = \sin\left(\frac{3\pi}{20}x\right)$ from 0 to 20 is: $L = \int_0^{20} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$; $\frac{dy}{dx} = \frac{3\pi}{20}\cos\left(\frac{3\pi}{20}x\right) \Rightarrow \left(\frac{dy}{dx}\right)^2$ $= \frac{9\pi^2}{400}\cos^2\left(\frac{3\pi}{20}x\right) \Rightarrow L = \int_0^{20} \sqrt{1 + \frac{9\pi^2}{400}\cos^2\left(\frac{3\pi}{20}x\right)}\, dx$. Using numerical integration we find $L \approx 21.07$ in

30. First, we'll find the length of the cosine curve: $L = \int_{-25}^{25} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$; $\frac{dy}{dx} = -\frac{25\pi}{50}\sin\left(\frac{\pi x}{50}\right)$ $\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{\pi^2}{4}\sin^2\left(\frac{\pi x}{50}\right) \Rightarrow L = \int_{-25}^{25} \sqrt{1 + \frac{\pi^2}{4}\sin^2\left(\frac{\pi x}{50}\right)}\, dx$. Using a numerical integrator we find

$L \approx 73.1848$ ft. Surface area is: $A = \text{length} \cdot \text{width} \approx (73.1848)(300) = 21{,}955.44$ ft.
Cost $= 1.75A = (1.75)(21{,}955.44) = \$38{,}422.02$. Answers may vary slightly, depending on the numerical integration used.

31-36. Example CAS commands:

Maple:
```
xi:=(i,n) -> (a + (b−a)*i/n);
digits := 6:
f:= x -> sqrt(1−x^2); a:= −1: b:= 1:
n:=8;
segs := [seq([xi(i,n),f(xi(i,n))], i = 0..n)];i:= 'i':
plot({f(x),segs}, x=a..b);
approx:= sum(sqrt((xi(j,n) − xi(j−1,n))^2 + (f(xi(j,n))−f(xi(j−1,n)))^2),j=1..n);
evalf(approx);
int(sqrt(1+D(f)(x)^2), x = a..b);
evalf(%);
```

Mathematica:
```
Clear[x]
{a,b} = {−1,1}; f[x_] = Sqrt[ 1 − x^2 ]
p1 = Plot[ f[x], {x,a,b} ]
n = 8;
pts = Table[ {xn,f[xn]}, {xn,a,b,(b−a)/n} ] // N
Show[{ p1, Graphics[{Line[pts]}] }]
Sum[ Sqrt[
    (pts[[i+1,1]] − pts[[i,1]])^2 +
    (pts[[i+1,2]] − pts[[i,2]])^2 ] ,
    {i,1,n} ]
NIntegrate[ Sqrt[1+f' [x]^2], {x,a,b}]
```

5.4 FIRST ORDER SEPARABLE DIFFERENTIAL EQUATIONS

1. (a) $y = e^{-x} \Rightarrow y' = -e^{-x} \Rightarrow 2y' + 3y = 2\left(-e^{-x}\right) + 3e^{-x} = e^{-x}$

   (b) $y = e^{-x} + e^{-3x/2} \Rightarrow y' = -e^{-x} - \frac{3}{2}e^{-3x/2} \Rightarrow 2y' + 3y = 2\left(-e^{-x} - \frac{3}{2}e^{-3x/2}\right) + 3\left(e^{-x} + e^{-3x/2}\right) = e^{-x}$

(c) $y = e^{-x} + Ce^{-3x/2} \Rightarrow y' = -e^{-x} - \frac{3}{2}Ce^{-3x/2} \Rightarrow 2y' + 3y = 2\left(-e^{-x} - \frac{3}{2}Ce^{-3x/2}\right) + 3\left(e^{-x} + Ce^{-3x/2}\right) = e^{-x}$

2. (a) $y = -\frac{1}{x} \Rightarrow y' = \frac{1}{x^2} = \left(-\frac{1}{x}\right)^2 = y^2$

(b) $y = -\frac{1}{x+3} \Rightarrow y' = \frac{1}{(x+3)^2} = \left[-\frac{1}{(x+3)}\right]^2 = y^2$

(c) $y = \frac{1}{x+C} \Rightarrow y' = \frac{1}{(x+C)^2} = \left[-\frac{1}{x+C}\right]^2 = y^2$

3. $y = (x-2)\,e^{-x^2} \Rightarrow y' = e^{-x^2} + \left(-2xe^{-x^2}\right)(x-2) \Rightarrow y' = e^{-x^2} - 2xy;\ y(2) = (2-2)\,e^{-2^2} = 0$

4. $y = \frac{\cos x}{x} \Rightarrow y' = \frac{-x \sin x - \cos x}{x^2} \Rightarrow y' = -\frac{\sin x}{x} - \frac{1}{x}\left(\frac{\cos x}{x}\right) \Rightarrow y' = -\frac{\sin x}{x} - \frac{y}{x} \Rightarrow xy' = -\sin x - y$

$\Rightarrow xy' + y = -\sin x;\ y\left(\frac{\pi}{2}\right) = \frac{\cos(\pi/2)}{(\pi/2)} = 0$

5. $2\sqrt{xy}\,\frac{dy}{dx} = 1 \Rightarrow 2x^{1/2}y^{1/2}\,dy = dx \Rightarrow 2y^{1/2}\,dy = x^{-1/2}\,dx \Rightarrow \int 2y^{1/2}\,dy = \int x^{-1/2}\,dx \Rightarrow 2\left(\frac{2}{3}y^{3/2}\right)$

$= 2x^{1/2} + C_1 \Rightarrow \frac{2}{3}y^{3/2} - x^{1/2} = C$, where $C = \frac{1}{2}C_1$

6. $\frac{dy}{dx} = x^2\sqrt{y} \Rightarrow dy = x^2y^{1/2}\,dx \Rightarrow y^{-1/2}\,dy = x^2\,dx \Rightarrow \int y^{-1/2}\,dy = \int x^2\,dx \Rightarrow 2y^{1/2} = \frac{x^3}{3} + C$

$\Rightarrow 2y^{1/2} - \frac{1}{3}x^3 = C$

7. $\frac{dy}{dx} = e^{x-y} \Rightarrow dy = e^xe^{-y}\,dx \Rightarrow e^y\,dy = e^x\,dx \Rightarrow \int e^y\,dy = \int e^x\,dx \Rightarrow e^y = e^x + C \Rightarrow e^y - e^x = C$

8. $\frac{dy}{dx} = 3x^2e^{-y} \Rightarrow dy = 3x^2e^{-y}\,dx \Rightarrow e^y\,dy = 3x^2\,dx \Rightarrow \int e^y\,dy = \int 3x^2\,dx \Rightarrow e^y = x^3 + C \Rightarrow e^y - x^3 = C$

9. $\frac{dy}{dx} = \sqrt{y}\cos^2\sqrt{y} \Rightarrow dy = \left(\sqrt{y}\cos^2\sqrt{y}\right)dx \Rightarrow \frac{\sec^2\sqrt{y}}{\sqrt{y}}\,dy = dx \Rightarrow \int \frac{\sec^2\sqrt{y}}{\sqrt{y}}\,dy = \int dx$. In the integral on the left-hand side, substitute $u = \sqrt{y} \Rightarrow du = \frac{1}{2\sqrt{y}}\,dy \Rightarrow 2\,du = \frac{1}{\sqrt{y}}\,dy$, and we have

$2\int \sec^2 u\,du = \int dx \Rightarrow 2\tan u = x + C \Rightarrow -x + 2\tan\sqrt{y} = C$

10. $\sqrt{2xy}\,\frac{dy}{dx} = 1 \Rightarrow \sqrt{2}\sqrt{y}\,dy = \frac{1}{\sqrt{x}}\,dx \Rightarrow \sqrt{2}\,y^{1/2}\,dy = x^{-1/2}\,dx$

$\Rightarrow \sqrt{2}\int y^{1/2}\,dy = \int x^{-1/2}\,dx \Rightarrow \sqrt{2}\left(\frac{y^{3/2}}{\frac{3}{2}}\right) = \left(\frac{x^{1/2}}{\frac{1}{2}}\right) + C_1 \Rightarrow \sqrt{2}\,y^{3/2} = 3\sqrt{x} + \frac{3}{2}C_1$

$\Rightarrow \sqrt{2}\left(\sqrt{y}\right)^3 - 3\sqrt{x} = C$, where $C = \frac{3}{2}C_1$.

11. $\sqrt{x}\,\frac{dy}{dx} = e^{y+\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{e^y e^{\sqrt{x}}}{\sqrt{x}} \Rightarrow dy = \frac{e^y e^{\sqrt{x}}}{\sqrt{x}}\,dx \Rightarrow e^{-y}\,dy = \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx \Rightarrow \int e^{-y}\,dy = \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$

In the integral on the right-hand side, substitute $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2\,du = \frac{1}{\sqrt{x}}\,dx.$ and we have

$\int e^{-y}\,dy = 2\int e^u\,du \Rightarrow -e^{-y} = 2e^u + C_1 \Rightarrow e^{-y} + 2e^{\sqrt{x}} = C$, where $C = -C_1$.

12. $(\sec x)\,\frac{dy}{dx} = e^{y+\sin x} \Rightarrow \frac{dy}{dx} = e^{y+\sin x}\cos x \Rightarrow dy = \left(e^y e^{\sin x}\cos x\right)dx \Rightarrow e^{-y}\,dy = \left(e^{\sin x}\cos x\right)dx$

$\Rightarrow \int e^{-y}\,dy = \int \left(e^{\sin x}\cos x\right)dx \Rightarrow -e^{-y} = e^{\sin x} + C_1 \Rightarrow e^{-y} + e^{\sin x} = C$, where $C = -C_1$

13. $\frac{dy}{dx} = 2x\sqrt{1-y^2} \Rightarrow \frac{dy}{\sqrt{1-y^2}} = 2x\,dx \Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = \int 2x\,dx \Rightarrow \sin^{-1} y = x^2 + C \Rightarrow y = \sin(x^2 + C)$

14. $\frac{dy}{dx} = \frac{e^{2x-y}}{e^{x+y}} = e^{(2x-y)-(x+y)} = e^{x-2y} = \frac{e^x}{e^{2y}} \Rightarrow e^{2y}\,dy = e^x\,dx \Rightarrow \int e^{2y}\,dy = \int e^x\,dx \Rightarrow \frac{e^{2y}}{2} = e^x + C_1$

$\Rightarrow e^{2y} - 2e^x = C$, where $C = 2C_1$

15. (a) $\frac{dp}{dh} = kp \Rightarrow p = p_0e^{kh}$ where $p_0 = 1013$; $90 = 1013e^{20k} \Rightarrow k = \frac{\ln(90) - \ln(1013)}{20} \approx -0.121$

(b) $p = 1013e^{-6.05} \approx 2.389$ millibars

(c) $900 = 1013e^{(-0.121)h} \Rightarrow -0.121h = \ln\left(\frac{900}{1013}\right) \Rightarrow h = \frac{\ln(1013) - \ln(900)}{0.121} \approx 0.977$ km

16. $\frac{dy}{dt} = -0.6y \Rightarrow y = y_0e^{-0.6t}$; $y_0 = 100 \Rightarrow y = 100e^{-0.6t} \Rightarrow y = 100e^{-0.6} \approx 54.88$ grams when $t = 1$ hr

17. $A = A_0e^{kt} \Rightarrow 800 = 1000e^{10k} \Rightarrow k = \frac{\ln(0.8)}{10} \Rightarrow A = 1000e^{((\ln(0.8))/10)t}$, where A represents the amount of sugar that remains after time t. Thus after another 14 hrs, $A = 1000e^{((\ln(0.8)/10)24} \approx 585.35$ kg

18. $L(x) = L_0e^{-kx} \Rightarrow \frac{L_0}{2} = L_0e^{-18k} \Rightarrow \ln\frac{1}{2} = -18k \Rightarrow k = \frac{\ln 2}{18} \approx 0.0385 \Rightarrow L(x) = L_0e^{-0.0385x}$; when the intensity is one-tenth of the surface value, $\frac{L_0}{10} = L_0e^{-0.0385x} \Rightarrow \ln 10 = 0.0385x \Rightarrow x \approx 59.8$ ft

19. $V(t) = V_0e^{-t/40} \Rightarrow 0.1V_0 = V_0e^{-t/40}$ when the voltage is 10% of its original value $\Rightarrow t = -40\ln(0.1)$ ≈ 92.1 sec

20. $0.9P_0 = P_0e^k \Rightarrow k = \ln 0.9$; when the well's output falls to one-fifth of its present value $P = 0.2P_0$

$\Rightarrow 0.2P_0 = P_0e^{(\ln 0.9)t} \Rightarrow 0.2 = e^{(\ln 0.9)t} \Rightarrow \ln(0.2) = (\ln 0.9)t \Rightarrow t = \frac{\ln 0.2}{\ln 0.9} \approx 15.28$ yr

21. (a) $\frac{dQ}{dt} = r - kQ$, where k is a positive constant

(b) $\frac{dQ}{dt} = -k\left(Q - \frac{r}{k}\right) \Rightarrow \frac{dQ}{Q - \frac{r}{k}} = -k\ dt \Rightarrow \ln\left|Q - \frac{r}{k}\right| = -kt + C_1 \Rightarrow \left|Q - \frac{r}{k}\right| = e^{-kt+C_1} \Rightarrow \left|Q - \frac{r}{k}\right|$

$= e^{C_1}e^{-kt} \Rightarrow Q - \frac{r}{k} = \pm C_2 e^{-kt} \Rightarrow Q - \frac{r}{k} = Ce^{-kt}$, where $C_2 = e^{C_1}$ and $C = \pm C_2$.

$Q(0) = Q_0 \Rightarrow Q_0 = \frac{r}{k} + C \Rightarrow C = Q_0 - \frac{r}{k} \Rightarrow Q = \frac{r}{k} + \left(Q_0 - \frac{r}{k}\right)e^{-kt}$

(c) Since $k > 0$, $\lim_{t\to\infty} Q(t) = \lim_{t\to\infty}\left[\frac{r}{k} - \left(Q_0 - \frac{r}{k}\right)e^{-kt}\right] = \frac{r}{k}$.

22. $T - T_s = (T_0 - T_s)\,e^{-kt}$, $T_0 = 90°C$, $T_s = 20°C$, $T = 60°C \Rightarrow 60 - 20 = 70e^{-10k} \Rightarrow \frac{4}{7} = e^{-10k}$

$\Rightarrow k = \frac{\ln\left(\frac{7}{4}\right)}{10} \approx 0.05596$

(a) $35 - 20 = 70e^{-0.05596t} \Rightarrow t \approx 27.5$ min is the total time $\Rightarrow 27.5 - 10 = 17.5$ min longer to reach 35°C

(b) $T - T_s = (T_0 - T_s)\,e^{-kt}$, $T_0 = 90°C$, $T_s = -15°C \Rightarrow 35 + 15 = 105e^{-0.05596t} \Rightarrow t \approx 13.26$ min

23. $T - 65° = (T_0 - 65°)\,e^{-kt} \Rightarrow 35° - 65° = (T_0 - 65°)\,e^{-10k}$ and $50° - 65° = (T_0 - 65°)\,e^{-20k}$. Solving $-30° = (T_0 - 65°)\,e^{-10k}$ and $-15° = (T_0 - 65°)\,e^{-20k}$ simultaneously $\Rightarrow (T_0 - 65°)\,e^{-10k} = 2(T_0 - 65°)\,e^{-20k}$

$\Rightarrow e^{10k} = 2 \Rightarrow k = \frac{\ln 2}{10}$ and $-30° = \frac{T_0 - 65°}{e^{10k}} \Rightarrow -30°\left[e^{10\left(\frac{\ln 2}{10}\right)}\right] = T_0 - 65° \Rightarrow T_0 = 65° - 30°\left(e^{\ln 2}\right) = 65° - 60° = 5°$

24. $T - T_s = (T_0 - T_s)\,e^{-kt} \Rightarrow 39 - T_s = (46 - T_s)\,e^{-10k}$ and $33 - T_s = (46 - T_s)\,e^{-20k} \Rightarrow \frac{39 - T_s}{46 - T_s} = e^{-10k}$ and

$\frac{33 - T_s}{46 - T_s} = e^{-20k} = \left(e^{-10k}\right)^2 \Rightarrow \frac{33 - T_s}{46 - T_s} = \left(\frac{39 - T_s}{46 - T_s}\right)^2 \Rightarrow (33 - T_s)(46 - T_s) = (39 - T_s)^2 \Rightarrow 1518 - 79T_s + T_s^2$

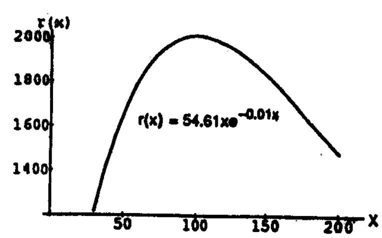
$= 1521 - 78T_s + T_s^2 \Rightarrow -T_s = 3 \Rightarrow T_s = -3°C$

25. (a) $\frac{dp}{dx} = -\frac{1}{100}p \Rightarrow \frac{dp}{p} = -\frac{1}{100}\,dx \Rightarrow \ln p = -\frac{1}{100}x + C \Rightarrow p = e^{(-0.01x+C)} = e^C e^{-0.01x} = C_1 e^{-0.01x}$;

$p(100) = 20.09 \Rightarrow 20.09 = C_1 e^{(-0.01)(100)} \Rightarrow C_1 = 20.09e \approx 54.61 \Rightarrow p(x) = 54.61e^{-0.01x}$ (in dollars)

(b) $p(10) = 54.61e^{(-0.01)(10)} = \$49.41$, and $p(90) = 54.61e^{(-0.01)(90)} = \$22.20$

(c) $r(x) = xp(x) \Rightarrow r'(x) = p(x) + xp'(x)$; $p'(x) = -.5461e^{-0.01x} \Rightarrow r'(x) = (54.61 - .5461x)e^{-0.01x}$. Thus, $r'(x) = 0 \Rightarrow 54.61 = .5461x \Rightarrow x = 100$. Since $r' > 0$ for any $x < 100$ and $r' < 0$ for $x > 100$, then $r(x)$ must be a maximum at $x = 100$.



26. (a) $\frac{dP}{dt} = kP \Rightarrow P = P_0e^{kt}$; Take the year 1750 to be $t = 0 \Rightarrow P = 728e^{kt}$; $P(50) = 906 \Rightarrow 906 = 728e^{50k}$

$\Rightarrow k = \frac{\ln\left(\frac{906}{728}\right)}{50} = 0.004375 \Rightarrow P = 728e^{0.004375t}$. In 1900, $t = 150$ and the model predicts

$P = 728e^{0.004375(150)} = 1403$ million or 1.403 billion people, which is less than the actual number.

In 1950, $t = 200$ and the model predicts $P = 728e^{0.004375(200)} = 1746$ million or 1.746 billion people which is also less than the actual value.

(b) Take the year 1850 to be $t = 0 \Rightarrow P = 1171e^{kt}$; $P(50) = 1608 \Rightarrow 1171e^{50k} \Rightarrow k = \frac{\ln\left(\frac{1608}{1171}\right)}{50} = 0.006343$

$\Rightarrow P = 1171e^{0.006343t}$. In 1950, $t = 100$ and the model predicts $P(100) = 1171e^{0.006343(100)} = 2208$ million or 2.208 billion people, which is less than the actual number.

(c) Take the year 1900 to be $t = 0 \Rightarrow P = 1608e^{kt}$; $P(50) = 2517 \Rightarrow 2517 = 1608e^{50k} \Rightarrow k = \frac{\ln\left(\frac{2517}{1608}\right)}{50}$

$= 0.008962 \Rightarrow P = 1608e^{0.008962t}$. In 1999, $t = 99$ and the model predicts $P(99) = 1608e^{0.008962(99)}$

$= 3905$ million or 3.905 billion people. This is about $\frac{2}{3}$ of the actual number of people in 1999. The values of k found in parts (a)–(c) of this problem suggests that the relative growth rate of the world's population; that is $\frac{dP/dt}{P} = k$, is increasing with time rather than remaining constant, as assumed for the model. The increase in the relative growth rate might be attributed to factors such as longer life spans due to improvements in health care, living conditions, etc.

27. $\frac{dy}{dt} = -ky$ (k is a positive constant and y is nonnegative) $\Rightarrow \frac{dy}{y} = -k\,dt \Rightarrow \ln y = -kt + C_1 \Rightarrow y = e^{-kt+C_1}$

$= e^{C_1}e^{-kt} = Ce^{-kt}$; $y(0) = y_0 \Rightarrow y_0 = Ce^0 \Rightarrow C = y_0 \Rightarrow y = y_0e^{-kt}$

28. (a) From Exercise 27, $y = y_0e^{-kt}$. Let $y = \frac{y_0}{2}$ and solve for t: $\frac{y_0}{2} = y_0e^{-kt} \Rightarrow e^{-kt} = \frac{1}{2} \Rightarrow -kt = \ln\left(\frac{1}{2}\right)$

$= \ln 1 - \ln 2 \Rightarrow t = \frac{0 - \ln 2}{-k} = \frac{\ln 2}{k}$. Therefore, the half-life is $\frac{\ln 2}{k}$.

(b) Half-life of polonium-210 $= \frac{\ln 2}{5 \times 10^{-3}} \approx 139$ days.

29. From Exercise 27, $y = y_0e^{-kt}$ and from Exercise 28(a), half-life $= \frac{\ln 2}{k} \Rightarrow k = \frac{\ln 2}{\text{half-life}} = \frac{\ln 2}{5700} \approx 0.0001216$.

Since 10% of the nuclei have decayed, 90% remain. Let $y = 0.9y_0$ and solve for t: $0.9y_0 = y_0e^{-0.0001216t}$

$\Rightarrow t = -\frac{\ln 0.9}{0.0001216} \approx 864$. Therefore, the sample is approximately 864 years old.

30. From Exercise 29, the half-life of carbon-14 is 5700 yr $\Rightarrow \frac{1}{2}c_0 = c_0e^{-k(5700)} \Rightarrow k = \frac{\ln 2}{5700} \approx 0.0001216$

$\Rightarrow c = c_0e^{-0.0001216t} \Rightarrow (0.445)c_0 = c_0e^{-0.0001216t} \Rightarrow t = \frac{\ln(0.445)}{-0.0001216} \approx 6659$ years

31. From Exercise 30, $k \approx 0.0001216$ for carbon-14.

(a) $c = c_0e^{-0.0001216t} \Rightarrow (0.17)c_0 = c_0e^{-0.0001216t} \Rightarrow t \approx 14{,}572.01$ years $\Rightarrow$ 12,572 BC

(b) $(0.18)c_0 = c_0e^{-0.0001216t} \Rightarrow t \approx 14{,}101.96$ years $\Rightarrow$ 12,102 BC

(c) $(0.16)c_0 = c_0e^{-0.0001216t} \Rightarrow t \approx 15{,}070.57$ years $\Rightarrow$ 13,071 BC

32. From Exercise 29, $k \approx 0.0001216$ for carbon-14. Thus, $c = c_0e^{-0.0001216t} \Rightarrow (0.995)c_0 = c_0e^{-0.0001216t}$ $\Rightarrow t = \frac{\ln(0.995)}{-0.0001216} \approx 41$ years old

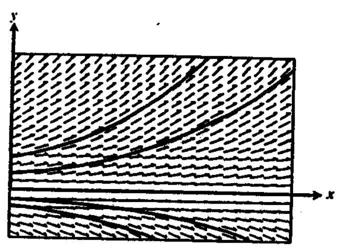
33. (a) distance coasted $= \frac{v_0m}{k} = \frac{(22)(5)}{\frac{1}{5}} = 550$ ft

(b) $v = v_0e^{-(k/m)t} \Rightarrow 1 = 22e^{-(1/25)t} \Rightarrow \ln 1 = \ln 22 - \frac{t}{25} \Rightarrow t = 25 \ln 22 \approx 77.28$ sec

34. (a) distance coasted $= \frac{v_0m}{k} = \frac{(22)(1{,}750{,}000)}{3000} = \frac{38{,}500}{3}$ ft ≈ 2.43 miles

(b) $v = v_0e^{-(k/m)t} \Rightarrow 1 = 22e^{(-3000/1{,}750{,}000)t} = 22e^{-3t/1750} \Rightarrow \ln 1 = \ln 22 - \frac{3t}{1750}$ $\Rightarrow t = \frac{1750 \ln 22}{3} \Rightarrow t \approx 1803.1$ sec ≈ 30 min

35.



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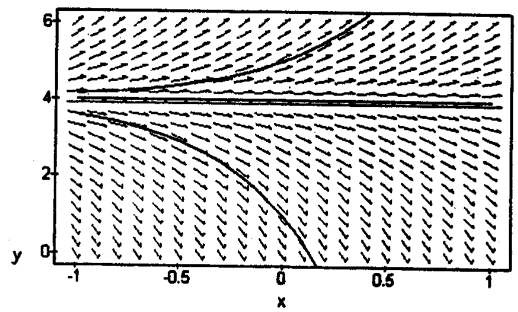
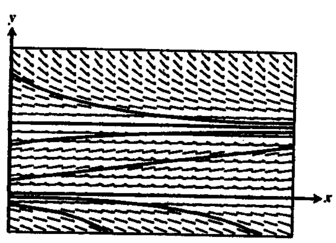
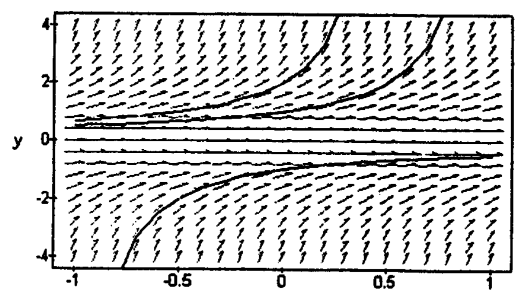


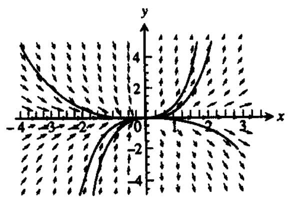
37.



38.



39.



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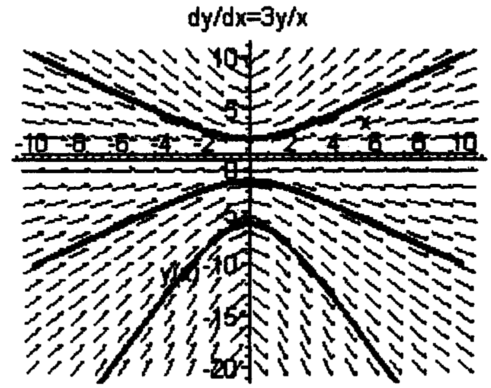


5.5 SPRINGS, PUMPING AND LIFTING

1. The force required to lift the water is equal to the water's weight, which varies steadily from 40 lb to 0 lb over the 20-ft lift. When the bucket is x ft off the ground, the water weighs: $F(x) = 40\left(\frac{20-x}{20}\right) = 40\left(1-\frac{x}{20}\right)$

$= 40 - 2x$ lb. The work done is: $W = \int_a^b F(x)\,dx = \int_0^{20} (40-2x)\,dx = \left[40x - x^2\right]_0^{20} = (40)(20) - 20^2$

$= 800 - 400 = 400$ ft · lb

2. The water's weight varies steadily from 16 lb to 8 lb over the 20-ft lift. When the bucket is x ft off the ground, the water weighs: $F(x) = 16\left(\frac{40-x}{40}\right) = 16\left(1-\frac{x}{40}\right) = 16 - \frac{2x}{5}$ lb. The work done is: $W = \int_a^b F(x)\,dx$

$= \int_0^{20} \left(16 - \frac{2x}{5}\right) dx = \left[16x - \frac{x^2}{5}\right]_0^{20} = (16)(20) - \frac{20^2}{5} = 320 - \frac{400}{5} = 320 - 80 = 240$ ft · lb

3. The force required to haul up the rope is equal to the rope's weight, which varies steadily and is proportional to x, the length of the rope still hanging: $F(x) = 0.624x$. The work done is: $W = \int_0^{50} F(x)\,dx = \int_0^{50} 0.624x\,dx$

$= 0.624\left[\frac{x^2}{2}\right]_0^{50} = 780$ J

4. The weight of sand varies steadily from 144 lb to 72 lb over the 18 ft length. When the bag is x ft off the ground, the sand weighs: $F(x) = 144\left(\frac{36-x}{36}\right) = 144\left(1-\frac{x}{36}\right)$. The work done is: $W = \int_a^b F(x)\,dx$

$= \int_0^{18} 144\left(1-\frac{x}{36}\right) dx = 144\left[x - \frac{x^2}{72}\right]_0^{18} = 144\left(18 - \frac{18^2}{18\cdot 4}\right) = 144\left(18 - \frac{18}{4}\right) = \frac{144\cdot 18\cdot 3}{4} = 36\cdot 18\cdot 3 = 1944$ ft · lb

5. The force required to lift the cable is equal to the weight of the cable paid out: $F(x) = (4.5)(180 - x)$ where x is the position of the car off the first floor. The work done is: $W = \int_0^{180} F(x)\,dx = 4.5\int_0^{180} (180 - x)\,dx$ $= 4.5\left[180x - \frac{x^2}{2}\right]_0^{180} = 4.5\left(180^2 - \frac{180^2}{2}\right) = \frac{4.5 \cdot 180^2}{2} = 72{,}900$ ft·lb

6. Since the force is acting toward the origin, it acts opposite to the positive x-direction. Thus $F(x) = -\frac{k}{x^2}$. The work done is $W = \int_a^b -\frac{k}{x^2}\,dx = k\int_a^b -\frac{1}{x^2}\,dx = k\left[\frac{1}{x}\right]_a^b = k\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{k(a-b)}{ab}$

7. The force against the piston is $F = pA$. If $V = Ax$, where x is the height of the cylinder, then $dV = A\,dx$ $\Rightarrow$ Work $= \int F\,dx = \int pA\,dx = \int_{(p_1, V_1)}^{(p_2, V_2)} p\,dV.$

8. $pV^{1.4} = c$, a constant $\Rightarrow p = cV^{-1.4}$. If $V_1 = 243$ in^3 and $p_1 = 50$ lb/in^3, then $c = (50)(243)^{1.4} = 109{,}350$ lb. Thus $W = \int_{243}^{32} 109{,}350V^{-1.4}\,dV = \left[-\frac{109{,}350}{0.4V^{0.4}}\right]_{243}^{32} = -\frac{109{,}350}{0.4}\left(\frac{1}{32^{0.4}} - \frac{1}{243^{0.4}}\right) = -\frac{109{,}350}{0.4}\left(\frac{1}{4} - \frac{1}{9}\right)$ $= -\frac{(109{,}350)(5)}{(0.4)(36)} = -37{,}968.75$ in·lb. Note that when a system is compressed, the work done is negative.

9. The force required to stretch the spring from its natural length of 2 m to a length of 5 m is $F(x) = kx$. The work done by F is $W = \int_0^3 F(x)\,dx = k\int_0^3 x\,dx = \frac{k}{2}\left[x^2\right]_0^3 = \frac{9k}{2}$. This work is equal to 1800 J $\Rightarrow \frac{9}{2}k = 1800$ $\Rightarrow k = 400$ N/m

10. (a) We find the force constant from Hooke's Law: $F = kx \Rightarrow k = \frac{F}{x} \Rightarrow k = \frac{800}{4} = 200$ lb/in

(b) The work done to stretch the spring 2 inches beyond its natural length is $W = \int_0^2 kx\,dx$ $= 200\int_0^2 x\,dx = 200\left[\frac{x^2}{2}\right]_0^2 = 200(2 - 0) = 400$ in·lb $= 33.3$ ft·lb

(c) We substitute $F = 1600$ into the equation $F = 200x$ to find $1600 = 200x \Rightarrow x = 8$ in

11. We find the force constant from Hooke's law: $F = kx$. A force of 2 N stretches the spring to 0.02 m $\Rightarrow 2 = k \cdot (0.02) \Rightarrow k = 100\,\frac{N}{m}$. The force of 4 N will stretch the rubber band y m, where $F = ky \Rightarrow y = \frac{F}{k}$ $\Rightarrow y = \frac{4N}{100\,\frac{N}{m}} \Rightarrow y = 0.04$ m $= 4$ cm. The work done to stretch the rubber band 0.04 m is $W = \int_0^{0.04} kx\,dx$

$$= 100 \int_0^{0.04} x\,dx = 100\left[\frac{x^2}{2}\right]_0^{0.04} = \frac{(100)(0.04)^2}{2} = 0.08 \text{ J}$$

12. We find the force constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} \Rightarrow k = \frac{90}{1} \Rightarrow k = 90 \frac{N}{m}$. The work done to stretch the spring 5 m beyond its natural length is $W = \int_0^5 kx\,dx = 90 \int_0^5 x\,dx = 90\left[\frac{x^2}{2}\right]_0^5 = (90)\left(\frac{25}{2}\right) = 1125$ J

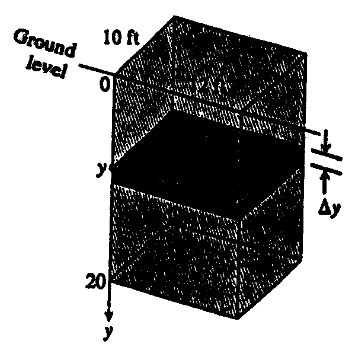
13. (a) We find the spring's constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} = \frac{21{,}714}{8-5} = \frac{21{,}714}{3} \Rightarrow k = 7238 \frac{\text{lb}}{\text{in}}$

(b) The work done to compress the assembly the first half inch is $W = \int_0^{0.5} kx\,dx = 7238 \int_0^{0.5} x\,dx$ $= 7238\left[\frac{x^2}{2}\right]_0^{0.5} = (7238)\frac{(0.5)^2}{2} = \frac{(7238)(0.25)}{2} \approx 905$ in · lb. The work done to compress the assembly the second half inch is: $W = \int_{0.5}^{1.0} kx\,dx = 7238 \int_{0.5}^{1.0} x\,dx = 7238\left[\frac{x^2}{2}\right]_{0.5}^{1.0} = \frac{7238}{2}[1-(0.5)^2] = \frac{(7238)(0.75)}{2}$ ≈ 2714 in · lb

14. First, we find the force constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} = \frac{150}{\left(\frac{1}{16}\right)} = 16 \cdot 150 = 2{,}400 \frac{\text{lb}}{\text{in}}$. If someone compresses the scale $x = \frac{1}{8}$ in, he/she must weigh $F = kx = 2{,}400\left(\frac{1}{8}\right) = 300$ lb. The work done to compress the scale this far is $W = \int_0^{1/8} kx\,dx = 2400\left[\frac{x^2}{2}\right]_0^{1/8} = \frac{2400}{2 \cdot 64} = 18.75$ lb · in. $= \frac{25}{16}$ ft · lb

15. We will use the coordinate system given.

(a) The typical slab between the planes at y and $y + \Delta y$ has a volume of $\Delta V = (10)(12)\,\Delta y = 120\,\Delta y$ ft^3. The force F required to lift the slab is equal to its weight: $F = 62.4\,\Delta V = 62.4 \cdot 120\,\Delta y$ lb. The distance through which F must act is about y ft, so the work done lifting the slab is about $\Delta W = \text{force} \times \text{distance}$ $= 62.4 \cdot 120 \cdot y \cdot \Delta y$ ft · lb. The work it takes to lift all the water is approximately $W \approx \sum_0^{20} \Delta W$ $= \sum_0^{20} 62.4 \cdot 120y \cdot \Delta y$ ft · lb. This is a Riemann sum for the function $62.4 \cdot 120y$ over the interval $0 \le y \le 20$.



The work of pumping the tank empty is the limit of these sums: $W = \int_0^{20} 62.4 \cdot 120y\,dy$

$$= (62.4)(120)\left[\frac{y^2}{2}\right]_0^{20} = (62.4)(120)\left(\frac{400}{2}\right) = (62.4)(120)(200) = 1{,}497{,}600 \text{ ft} \cdot \text{lb}$$

(b) The time t it takes to empty the full tank with $\left(\frac{5}{11}\right)$-hp motor is $t = \dfrac{W}{250\ \frac{ft \cdot lb}{sec}} = \dfrac{1{,}497{,}600\ ft \cdot lb}{250\ \frac{ft \cdot lb}{sec}}$ $= 5990.4$ sec $= 1.664$ hr $\Rightarrow t \approx 1$ hr and 40 min

(c) Following all the steps of part (a), we find that the work it takes to lower the water level 10 ft is $W = \int_0^{10} 62.4 \cdot 120y\, dy = (62.4)(120)\left[\frac{y^2}{2}\right]_0^{10} = (62.4)(120)\left(\frac{100}{2}\right) = 374{,}400$ ft $\cdot$ lb and the time is

$t = \dfrac{W}{250\ \frac{ft \cdot lb}{sec}} = 1497.6$ sec $= 0.416$ hr ≈ 25 min

(d) In a location where water weighs $62.26\ \frac{lb}{ft^3}$:

a) $W = (62.26)(24{,}000) = 1{,}494{,}240$ ft $\cdot$ lb.

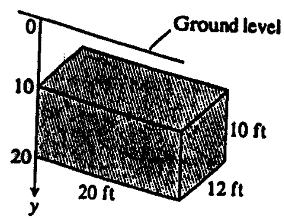
b) $t = \frac{1{,}494{,}240}{250} = 5976.96$ sec ≈ 1.660 hr $\Rightarrow t \approx 1$ hr and 40 min

In a location where water weighs $62.59\ \frac{lb}{ft^3}$

a) $W = (62.59)(24{,}000) = 1{,}502{,}160$ ft $\cdot$ lb

b) $t = \frac{1{,}502{,}160}{250} = 6008.64$ sec ≈ 1.669 hr $\Rightarrow t \approx 1$ hr and 40.1 min

16. We will use the coordinate system given.



(a) The typical slab between the planes at y and $y + \Delta y$ has a volume of $\Delta V = (20)(12)\,\Delta y = 240\,\Delta y$ ft^3. The force F required to lift the slab is equal to its weight: $F = 62.4\,\Delta V = 62.4 \cdot 240\,\Delta y$ lb. The distance through which F must act is about y ft, so the work done lifting the slab is about ΔW = force $\times$ distance $= 62.4 \cdot 240 \cdot y \cdot \Delta y$ ft $\cdot$ lb. The work it takes to lift all the water is approximately $W \approx \sum_{10}^{20} \Delta W$ $= \sum_{10}^{20} 62.4 \cdot 240y \cdot \Delta y$ ft $\cdot$ lb. This is a Riemann sum for the function $62.4 \cdot 240y$ over the interval $10 \le y \le 20$. The work it takes to empty the cistern is the limit of these sums: $W = \int_{10}^{20} 62.4 \cdot 240y\, dy$

$$= (62.4)(240)\left[\frac{y^2}{2}\right]_{10}^{20} = (62.4)(240)(200 - 50) = (62.4)(240)(150) = 2{,}246{,}400 \text{ ft} \cdot \text{lb}$$

(b) $t = \dfrac{W}{275\ \frac{ft \cdot lb}{sec}} = \dfrac{2{,}246{,}400\ ft \cdot lb}{275} \approx 8168.73$ sec ≈ 2.27 hours ≈ 2 hr and 16.1 min

(c) Following all the steps of part (a), we find that the work it takes to empty the tank halfway is

$$W = \int_{10}^{15} 62.4 \cdot 240y\, dy = (62.4)(240)\left[\frac{y^2}{2}\right]_{10}^{15} = (62.4)(240)\left(\frac{225}{2} - \frac{100}{2}\right) = (62.4)(240)\left(\frac{125}{2}\right) = 936{,}000 \text{ ft.}$$

Then the time is $t = \dfrac{W}{275\ \frac{ft \cdot lb}{sec}} = \dfrac{936{,}000}{275} \approx 3403.64$ sec ≈ 56.7 min

(d) In a location where water weighs 62.26 $\frac{\text{lb}}{\text{ft}^3}$:

a) $W = (62.26)(240)(150) = 2{,}241{,}360$ ft · lb.

b) $t = \frac{2{,}241{,}360}{275} = 8150.40$ sec $= 2.264$ hours ≈ 2 hr and 15.8 min

c) $W = (62.26)(240)\left(\frac{125}{2}\right) = 933{,}900$ ft · lb; $t = \frac{933{,}900}{275} = 3396$ sec ≈ 0.94 hours ≈ 56.6 min

In a location where water weighs 62.59 $\frac{\text{lb}}{\text{ft}^3}$

a) $W = (62.59)(240)(150) = 2{,}253{,}240$ ft · lb.

b) $t = \frac{2{,}253{,}240}{275} = 8193.60$ sec $= 2.276$ hours ≈ 2 hr and 16.61 min

c) $W = (62.59)(240)\left(\frac{125}{2}\right) = 938{,}850$ ft · lb; $t = \frac{938{,}850}{275} \approx 3414$ sec ≈ 0.95 hours ≈ 56.9 min

17. Using exactly the same procedure as done in Example 6 we change only the distance through which F must act: distance $\approx (10-y)$ m. Then $\Delta W = 245{,}000\pi(10-y)\,\Delta y$ J $\Rightarrow W \approx \sum_0^{10} \Delta W = \sum_0^{10} 245{,}000\pi(10-y)\,\Delta y$

$$\Rightarrow W = \int_0^{10} 245{,}000\pi(10-y)\,dy = 245{,}000\pi \int_0^{10} (10-y)\,dy = 245{,}000\pi\left[10y - \frac{y^2}{2}\right]_0^{10} = 245{,}000\pi\left(100 - \frac{100}{2}\right)$$

$\approx (245{,}000\pi)(50) \approx 38{,}484{,}510$ J

18. Exactly as done in Example 6 with the change in the upper limit of the sums and the integral: $W \approx \sum_0^5 \Delta W$

$$= \sum_0^5 245{,}000\pi(14-y)\,\Delta y \text{ J} \Rightarrow W = \int_0^5 245{,}000\pi(14-y)\,dy = 245{,}000\pi\left[14y - \frac{y^2}{2}\right]_0^5 = 245{,}000\pi\left(70 - \frac{25}{2}\right)$$

$= (245{,}000\pi)\left(\frac{115}{2}\right) \approx 44{,}257{,}186.5$ J

19. The typical slab between the planes at y and and $y + \Delta y$ has a volume of $\Delta V = \pi(\text{radius})^2(\text{thickness})$ $= \pi\left(\frac{20}{2}\right)^2 \Delta y = \pi \cdot 100\,\Delta y$ ft^3. The force F required to lift the slab is equal to its weight: $F = 51.2\,\Delta V = 51.2 \cdot 100\pi\,\Delta y$ lb $\Rightarrow F = 5120\pi\,\Delta y$ lb. The distance through which F must act is about $(30-y)$ ft. The work it takes to lift all the kerosene is approximately $W \approx \sum_0^{30} \Delta W$ $= \sum_0^{30} 5120\pi(30-y)\,\Delta y$ ft · lb which is a Riemann sum. The work to pump the tank dry is the limit of these sums: $W = \int_0^{30} 5120\pi(30-y)\,dy = 5120\pi\left[30y - \frac{y^2}{2}\right]_0^{30} = 5120\pi\left(\frac{900}{2}\right) = (5120)(450\pi)$

$\approx 7{,}238{,}229.47$ ft · lb

20. For both ways of filling the tank, the typical slab between the planes at y and $y + \Delta y$ has a volume of $\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi(2)^2\Delta y$. The force F required to lift this slab is equal to its weight: $F = 62.4\,\Delta V = \pi(4)(62.4)\,\Delta y$. The distance through which F must act *does* depend on the way of filling.

(a) If we pump the water through a hose attached to a valve in the bottom, the distance is $(15+y)$ so the work done lifting the slab is about $\Delta W_1 = (62.4)(4\pi)(15+y)\,\Delta y$. The work done lifting all the slabs is

$W_1 \approx \sum_0^6 (62.4)(4\pi)(15+y)\,\Delta y$ and taking the limit we get $W_1 = \int_0^6 (62.4)(4\pi)(15+y)\,dy$

$= (62.4)(4\pi)\left[15y + \frac{y^2}{2}\right]_0^6 = (62.4)(4\pi)\left(15\cdot 6 + \frac{36}{2}\right) = (62.4)(4\pi)(90+18) = (62.4)(4\pi)(108)$

$\approx 84{,}687.3$ ft · lb

(b) If we attach the hose to the rim of the tank and let the water pour in, the distance is $(15+6)$, so the work done by the *pump* on one slab is $\Delta W_2 = (62.4)(4\pi)(15+6)\,\Delta y$. The work done lifting all the slabs is:

$W_2 \approx \sum_0^6 (62.4)(4\pi)(15+6)\,\Delta y$ and taking the limit we get $W_2 = \int_0^6 (62.4)(4\pi)(15+6)\,dy$

$= (62.4)(4\pi)(21)\int_0^6 dy = (62.4)(4\pi)(126) \approx 98{,}801.8$ ft · lb. We see that $W_2 > W_1$ and if we assume that the pump produces a constant amount of work per hour then it takes more time to do work W_2.

21. (a) Follow all the steps of Example 7 but make the substitution of $64.5\,\frac{\text{lb}}{\text{ft}^3}$ for $57\,\frac{\text{lb}}{\text{ft}^3}$. Then,

$W = \int_0^8 \frac{64.5\pi}{4}(10-y)y^2\,dy = \frac{64.5\pi}{4}\left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^8 = \frac{64.5\pi}{4}\left(\frac{10\cdot 8^3}{3} - \frac{8^4}{4}\right) = \left(\frac{64.5\pi}{4}\right)(8^3)\left(\frac{10}{3} - 2\right)$

$= \frac{64.5\pi\cdot 8^3}{3} = 21.5\pi\cdot 8^3 \approx 34{,}583$ ft · lb

(b) Exactly as done in Example 7 but change the distance through which F acts to distance $\approx (13-y)$ ft.

Then $W = \int_0^8 \frac{57\pi}{4}(13-y)y^2\,dy = \frac{57\pi}{4}\left[\frac{13y^3}{3} - \frac{y^4}{4}\right]_0^8 = \frac{57\pi}{4}\left(\frac{13\cdot 8^3}{3} - \frac{8^4}{4}\right) = \left(\frac{57\pi}{4}\right)(8^3)\left(\frac{13}{3} - 2\right) = \frac{57\pi\cdot 8^3\cdot 7}{3\cdot 4}$

$= (19\pi)(8^2)(7)(2) \approx 53.482$ ft · lb

22. The typical slab between the planes of y and y+Δy has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness})$

$= \pi\left(\sqrt{y}\right)^2\Delta y = \pi y\,\Delta y$ m³. The force F(y) is equal to the slab's weight: $F(y) = 10{,}000\,\frac{N}{m^3}\cdot \Delta V$

$= \pi 10{,}000 y\,\Delta y$ N. The height of the tank is $4^2 = 16$ m. The distance through which F(y) must act to lift the slab to the level of the top of the tank is about $(16-y)$ m, so the work done lifting the slab is about

$\Delta W = 10{,}000\pi y(16-y)\,\Delta y$ N · m. The work done lifting all the slabs from $y = 0$ to $y = 16$ to the top is approximately $W \approx \sum_0^{16} 10{,}000\pi y(16-y)\Delta y$. Taking the limit of these Riemann sums, we get

$W = \int_0^{16} 10{,}000\pi y(16-y)\,dy = 10{,}000\pi\int_0^{16}(16y - y^2)\,dy = 10{,}000\pi\left[\frac{16y^2}{2} - \frac{y^3}{3}\right]_0^{16} = 10{,}000\pi\left(\frac{16^3}{2} - \frac{16^3}{3}\right)$

$= \frac{10{,}000 \cdot \pi \cdot 16^3}{6} \approx 21{,}446{,}605.9$ J

23. The typical slab between the planes at y and y+Δy has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness})$ $= \pi\left(\sqrt{25-y^2}\right)^2 \Delta y \text{ m}^3$. The force F(y) required to lift this slab is equal to its weight: $F(y) = 9800 \cdot \Delta V$ $= 9800\pi\left(\sqrt{25-y^2}\right)^2 \Delta y = 9800\pi(25-y^2)\Delta y$ N. The distance through which F(y) must act to lift the slab to the level of 4 m above the top of the reservoir is about $(4-y)$ m, so the work done is approximately $\Delta W \approx 9800\pi(25-y^2)(4-y)\,\Delta y$ N·m. The work done lifting all the slabs from y = −5 m to y = 0 m is approximately $W \approx \sum_{-5}^{0} 9800\pi(25-y^2)(4-y)\,\Delta y$ N·m. Taking the limit of these Riemann sums, we get

$$W = \int_{-5}^{0} 9800\pi(25-y^2)(4-y)\,dy = 9800\pi \int_{-5}^{0} (100 - 25y - 4y^2 + y^3)\,dy = 9800\pi\left[100y - \frac{25}{2}y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_{-5}^{0}$$

$$= -9800\pi\left(-500 - \frac{25 \cdot 25}{2} + \frac{4}{3} \cdot 125 + \frac{625}{4}\right) \approx 15{,}073{,}100 \text{ J}$$

24. The typical slab between the planes at y and y+Δy has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness})$ $= \pi\left(\sqrt{100-y^2}\right)^2 \Delta y = \pi(100-y^2)\Delta y \text{ ft}^3$. The force is $F(y) = \frac{56 \text{ lb}}{\text{ft}^3} \cdot \Delta V = 56\pi(100-y^2)\Delta y$ lb. The distance through which F(y) must act to lift the slab to the level of 2 ft above the top of the tank is about $(12-y)$ ft, so the work done is $\Delta W \approx 56\pi(100-y^2)(12-y)\,\Delta y$ lb·ft. The work done lifting all the slabs from y = 0 ft to y = 10 ft is approximately $W \approx \sum_{0}^{10} 56\pi(100-y^2)(12-y)\,\Delta y$ lb·ft. Taking the limit of these Riemann sums, we get $W = \int_0^{10} 56\pi(100-y^2)(12-y)\,dy = 56\pi \int_0^{10} (100-y^2)(12-y)\,dy$

$$= 56\pi \int_0^{10} (1200 - 100y - 12y^2 + y^3)\,dy = 56\pi\left[1200y - \frac{100y^2}{2} - \frac{12y^3}{3} + \frac{y^4}{4}\right]_0^{10}$$

$$= 56\pi\left(12{,}000 - \frac{10{,}000}{2} - 4 \cdot 1000 + \frac{10{,}000}{4}\right) = (56\pi)\left(12 - 5 - 4 + \frac{5}{2}\right)(1000) \approx 967{,}611 \text{ ft} \cdot \text{lb}.$$

It would cost (0.5)(967.611) = 483,805¢ = \$4838.05. Yes, we can afford to hire the firm.

25. $F = m\frac{dv}{dt} = m\frac{dv}{dx} \cdot \frac{dx}{dt} = mv\frac{dv}{dx}$ by the chain rule $\Rightarrow W = \int_{x_1}^{x_2} mv\frac{dv}{dx}\,dx = m\int_{x_1}^{x^2}\left(v\frac{dv}{dx}\right)dx = m\left[\frac{1}{2}v^2(x)\right]_{x_1}^{x_2}$ $= \frac{1}{2}m\left[v^2(x_2) - v^2(x_1)\right] = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$, as claimed.

26. weight = 2 oz $= \frac{2}{16}$ lb; mass $= \frac{\text{weight}}{32} = \frac{\frac{1}{8}}{32} = \frac{1}{256}$ slugs; $W = \left(\frac{1}{2}\right)\left(\frac{1}{256}\text{ slugs}\right)(160 \text{ ft/sec})^2 \approx 50$ ft·lb