27. $90 \text{ mph} = \frac{90 \text{ mi}}{1 \text{ hr}} \cdot \frac{1 \text{ hr}}{60 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} \cdot \frac{5280 \text{ ft}}{1 \text{ mi}} = 132 \text{ ft/sec}$; $m = \frac{0.3125 \text{ lb}}{32 \text{ ft/sec}^2} = \frac{0.3125}{32}$ slugs;

$W = \left(\frac{1}{2}\right)\left(\frac{0.3125 \text{ lb}}{32 \text{ ft/sec}^2}\right)(132 \text{ ft/sec})^2 \approx 85.1 \text{ ft} \cdot \text{lb}$

28. weight $= 1.6 \text{ oz} = 0.1 \text{ lb} \Rightarrow m = \frac{0.1 \text{ lb}}{32 \text{ ft/sec}^2} = \frac{1}{320}$ slugs; $W = \left(\frac{1}{2}\right)\left(\frac{1}{320} \text{ slugs}\right)(280 \text{ ft/sec})^2 = 122.5 \text{ ft} \cdot \text{lb}$

29. weight $= 2 \text{ oz} = \frac{1}{8} \text{ lb} \Rightarrow m = \frac{\frac{1}{8}}{32}$ slugs $= \frac{1}{256}$ slugs; $124 \text{ mph} = \frac{(124)(5280)}{(60)(60)} \approx 181.87 \text{ ft/sec}$;

$W = \left(\frac{1}{2}\right)\left(\frac{1}{256} \text{ slugs}\right)(181.87 \text{ ft/sec})^2 \approx 64.6 \text{ ft} \cdot \text{lb}$

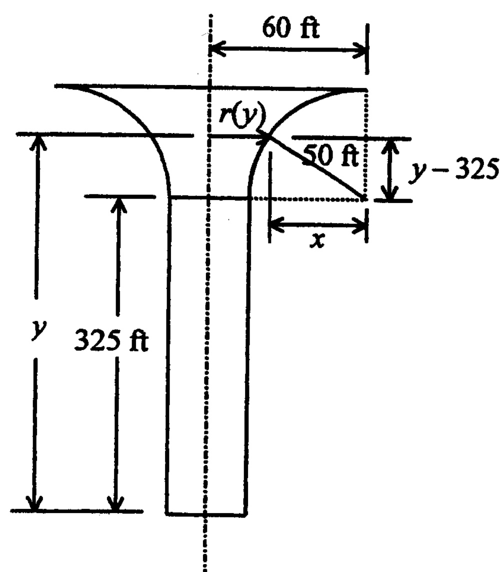
30. weight $= 14.5 \text{ oz} = \frac{14.5}{16} \text{ lb} \Rightarrow m = \frac{14.5}{(16)(32)}$ slugs; $W = \left(\frac{1}{2}\right)\left(\frac{14.5}{(16)(32)} \text{ slugs}\right)(88 \text{ ft/sec})^2 \approx 109.7 \text{ ft} \cdot \text{lb}$

31. weight $= 6.5 \text{ oz} = \frac{6.5}{16} \text{ lb} \Rightarrow m = \frac{6.5}{(16)(32)}$ slugs; $W = \left(\frac{1}{2}\right)\left(\frac{6.5}{(16)(32)} \text{ slugs}\right)(132 \text{ ft/sec})^2 \approx 110.6 \text{ ft} \cdot \text{lb}$

32. $F = (18 \text{ lb/ft})x \Rightarrow W = \int_0^{1/4} 18x\,dx = \left[9x^2\right]_0^{1/4} = \frac{9}{16} \text{ ft} \cdot \text{lb}$. Now $W = \frac{1}{2}mv^2 - \frac{1}{2}mv_1^2$, where $W = \frac{9}{16} \text{ ft} \cdot \text{lb}$,

$m = \frac{\frac{1}{8}}{32} = \frac{1}{256}$ slugs and $v_1 = 0$ ft/sec. Thus, $\frac{9}{16} \text{ ft} \cdot \text{lb.} = \left(\frac{1}{2}\right)\left(\frac{1}{256} \text{ slugs}\right)v^2 \Rightarrow v = 12\sqrt{2}$ ft/sec. With $v = 0$ at the top of the bearing's path and $v = 12\sqrt{2} - 32t \Rightarrow t = \frac{3\sqrt{2}}{8}$ sec when the bearing is at the top of its path. The height the bearing reaches is $s = 12\sqrt{2}\,t - 16t^2 \Rightarrow$ at $t = \frac{3\sqrt{2}}{8}$ the bearing reaches a height of

$(12\sqrt{2})\left(\frac{3\sqrt{2}}{8}\right) - (16)\left(\frac{3\sqrt{2}}{8}\right)^2 = 9 - \frac{16 \cdot 18}{64} = 4\frac{1}{2}$ ft



33. (a) From the diagram,

$r(y) = 60 - x = 60 - \sqrt{50^2 - (y - 325)^2}$

for $325 \le y \le 375$ ft.

(b) The volume of a horizontal slice of the funnel is $\Delta V \approx \pi\,[r(y)]^2\,dy$

$= \pi\left[60 - \sqrt{2500 - (y - 325)^2}\right]\Delta y.$

(c) The work required to lift the single slice of water is $\Delta W \approx 62.4\Delta V(375 - y)$

$= 62.4(375 - y)\pi\left[60 - \sqrt{2500 - (y - 325)^2}\right]\Delta y.$

The total work to pump out the funnel is

$$W = \int_{325}^{375} 62.4\pi(375 - y)\left[60 - \sqrt{2500 - (y - 325)^2}\right]dy = 6.3358 \cdot 10^7 \text{ ft} \cdot \text{lb}.$$

34. (a) From the result in Example 8, the work to pump out the throat is 1,353,869,354 ft · lb. Therefore, the total work required to pump out the throat and the funnel is 1,353,869,354 + 63,358,000 = 1,417,227,354 ft · lb.

(b) In horsepower-hours, the work required to pump out the glory hole is $\frac{1{,}417{,}227{,}354}{1.98 \cdot 10^6}$ = 7158. Therefore, it would take $\frac{715.8 \text{ hp} \cdot \text{h}}{1000 \text{ hp}} = 0.7158$ hours ≈ 43 minutes.

35. We imagine the milkshake divided into thin slabs by planes perpendicular to the y-axis at the points of a partition of the interval [0, 7]. The typical slab between the planes at y and y + Δy has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{y+17.5}{14}\right)^2 \Delta y \text{ in}^3$. The force F(y) required to lift this slab is equal to its weight: $F(y) = \frac{4}{9}\Delta V = \frac{4\pi}{9}\left(\frac{y+17.5}{14}\right)^2 \Delta y$ oz. The distance through which F(y) must act to lift this slab to the level of 1 inch above the top is about (8 − y) in. The work done lifting the slab is about $\Delta W = \left(\frac{4\pi}{9}\right)\frac{(y+17.5)^2}{14^2}(8-y)\,\Delta y$ in · oz. The work done lifting all the slabs from y = 0 to y = 7 is approximately $W = \sum_0^7 \frac{4\pi}{9 \cdot 14^2}(y+17.5)^2(8-y)\,\Delta y$ in · oz which is a Riemann sum. The work is the limit of these sums as the norm of the partition goes to zero: $W = \int_0^7 \frac{4\pi}{9 \cdot 14^2}(y+17.5)^2(8-y)\,dy$

$$= \frac{4\pi}{9 \cdot 14^2}\int_0^7 \left(2450 - 26.25y - 27y^2 - y^3\right) dy = \frac{4\pi}{9 \cdot 14^2}\left[-\frac{y^4}{4} - 9y^3 - \frac{26.25}{2}y^2 + 2450y\right]_0^7$$

$$= \frac{4\pi}{9 \cdot 14^2}\left[-\frac{7^4}{4} - 9 \cdot 7^3 - \frac{26.25}{2} \cdot 7^2 + 2450 \cdot 7\right] \approx 91.32 \text{ in} \cdot \text{oz} \approx 0.476 \text{ ft} \cdot \text{lb}$$

36. We fill the pipe and the tank.
To find the work required to fill the tank follow Example 6 with radius = 10 ft. Then $\Delta V = \pi \cdot 100\,\Delta y \text{ ft}^3$. The force required will be $F = 62.4 \cdot \Delta V = 62.4 \cdot 100\pi\,\Delta y = 6240\pi\,\Delta y$ lb. The distance through which F must act is y so the work done lifting the slab is about $\Delta W_1 = 6240\pi \cdot y \cdot \Delta y$ lb · ft. The work it takes to lift all the water into the tank is: $W_1 \approx \sum_{360}^{385} \Delta W_1 = \sum_{360}^{385} 6240\pi \cdot y \cdot \Delta y$ lb · ft. Taking the limit we end up with $W_1 = \int_{360}^{385} 6240\pi y\,dy = 6240\pi\left[\frac{y^2}{2}\right]_{360}^{385} = \frac{6240\pi}{2}\left[385^2 - 360^2\right] \approx 182{,}557{,}949$ ft · lb

To find the work required to fill the pipe, do as in part (a), but take the radius to be $\frac{4}{2}$ in $= \frac{1}{6}$ ft. Then $\Delta V = \pi \cdot \frac{1}{36}\Delta y \text{ ft}^3$ and $F = 62.4 \cdot \Delta V = \frac{62.4\pi}{36}\Delta y$. Also take different limits of summation and integration: $W_2 \approx \sum_0^{360} \Delta W_2 \Rightarrow W_2 = \int_0^{360} \frac{62.4}{36}\pi y\,dy = \frac{62.4\pi}{36}\left[\frac{y^2}{2}\right]_0^{360} = \left(\frac{62.4\pi}{36}\right)\left(\frac{360^2}{2}\right) \approx 352{,}864$ ft · lb.

The total work is $W = W_1 + W_2 \approx 182{,}557{,}949 + 352{,}864 \approx 182{,}910{,}813$ ft · lb. The time it takes to fill the tank and the pipe is Time $= \frac{W}{1650} \approx \frac{182{,}910{,}813}{1650} \approx 110{,}855$ sec ≈ 31 hr

37. Work $= \int_{6,370,000}^{35,780,000} \frac{1000\,MG}{r^2}\,dr = 1000\,MG \int_{6,370,000}^{35,780,000} \frac{dr}{r^2} = 1000\,MG\left[-\frac{1}{r}\right]_{6,370,000}^{35,780,000}$

$= (1000)(5.975 \cdot 10^{24})(6.672 \cdot 10^{-11})\left(\frac{1}{6{,}370{,}000} - \frac{1}{35{,}780{,}000}\right) \approx 5.144 \times 10^{10}$ J

38. (a) Let ρ be the x-coordinate of the second electron. Then $r^2 = (\rho-1)^2 \Rightarrow W = \int_{-1}^{0} F(\rho)\,d\rho$

$= \int_{-1}^{0} \frac{(23 \times 10^{-29})}{(\rho-1)^2}\,d\rho = -\left[\frac{23 \times 10^{-29}}{\rho-1}\right]_{-1}^{0} = (23 \times 10^{-29})\left(1 - \frac{1}{2}\right) = 11.5 \times 10^{-29}$

(b) $W = W_1 + W_2$ where W_1 is the work done against the field of the first electron and W_2 is the work done against the field of the second electron. Let ρ be the x-coordinate of the third electron. Then $r_1^2 = (\rho-1)^2$ and $r_2^2 = (\rho+1)^2 \Rightarrow W_1 = \int_3^5 \frac{23 \times 10^{-29}}{r_1^2}\,d\rho = \int_3^5 \frac{23 \times 10^{-29}}{(\rho-1)^2}\,d\rho = -23 \times 10^{-29}\left[\frac{1}{\rho-1}\right]_3^5$

$= (-23 \times 10^{-29})\left(\frac{1}{4} - \frac{1}{2}\right) = \frac{23}{4} \times 10^{-29}$, and $W_2 = \int_3^5 \frac{23 \times 10^{-29}}{r_2^2}\,d\rho = \int_3^5 \frac{23 \times 10^{-29}}{(\rho+1)^2}\,d\rho$

$= -23 \times 10^{-29}\left[\frac{1}{\rho+1}\right]_3^5 = (-23 \times 10^{-29})\left(\frac{1}{6} - \frac{1}{4}\right) = \frac{23 \times 10^{-29}}{12}(3-2) = \frac{23}{12} \times 10^{-29}$. Therefore

$W = W_1 + W_2 = \left(\frac{23}{4} \times 10^{-29}\right) + \left(\frac{23}{12} \times 10^{-29}\right) = \frac{23}{3} \times 10^{-29} \approx 7.67 \times 10^{-29}$ J

5.6 FLUID FORCES

1. To find the width of the plate at a typical depth y, we first find an equation for the line of the plate's right-hand edge: $y = x - 5$. If we let x denote the width of the right-hand half of the triangle at depth y, then $x = 5 + y$ and the total width is $L(y) = 2x = 2(5+y)$. The depth of the strip is $(-y)$. The force exerted by the water against one side of the plate is therefore $F = \int_{-5}^{-2} w(-y) \cdot L(y)\,dy = \int_{-5}^{-2} 62.4 \cdot (-y) \cdot 2(5+y)\,dy$

$= 124.8 \int_{-5}^{-2} (-5y - y^2)\,dy = 124.8\left[-\frac{5}{2}y^2 - \frac{1}{3}y^3\right]_{-5}^{-2} = 124.8\left[\left(-\frac{5}{2} \cdot 4 + \frac{1}{3} \cdot 8\right) - \left(-\frac{5}{2} \cdot 25 + \frac{1}{3} \cdot 125\right)\right]$

$= (124.8)\left(\frac{105}{2} - \frac{117}{3}\right) = (124.8)\left(\frac{315 - 234}{6}\right) = 1684.8$ lb

2. An equation for the line of the plate's right-hand edge is $y = x - 3 \Rightarrow x = y + 3$. Thus the total width is $L(y) = 2x = 2(y+3)$. The depth of the strip is $(2 - y)$. The force exerted by the water is

$$F = \int_{-3}^{0} w(2-y)L(y)\,dy = \int_{-3}^{0} 62.4 \cdot (2-y) \cdot 2(3+y)\,dy = 124.8 \int_{-3}^{0} \left(6 - y - y^2\right) dy = 124.8\left[6y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-3}^{0}$$
$$= (-124.8)\left(-18 - \tfrac{9}{2} + 9\right) = (-124.8)\left(-\tfrac{27}{2}\right) = 1684.8 \text{ lb}$$

3. Using the coordinate system of Exercise 2, we find the equation for the line of the plate's right-hand edge is $y = x - 3 \Rightarrow x = y + 3$. Thus the total width is $L(y) = 2x = 2(y+3)$. The depth of the strip changes to $(4 - y)$

$$\Rightarrow F = \int_{-3}^{0} w(4-y)L(y)\,dy = \int_{-3}^{0} 62.4 \cdot (4-y) \cdot 2(y+3)\,dy = 124.8 \int_{-3}^{0} \left(12 + y - y^2\right) dy$$
$$= 124.8\left[12y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-3}^{0} = (-124.8)\left(-36 + \tfrac{9}{2} + 9\right) = (-124.8)\left(-\tfrac{45}{2}\right) = 2808 \text{ lb}$$

4. Using the coordinate system of Exercise 2, we see that the equation for the line of the plate's right-hand edge remains the same: $y = x - 3 \Rightarrow x = 3 + y$ and $L(y) = 2x = 2(y+3)$. The depth of the strip changes to $(-y)$

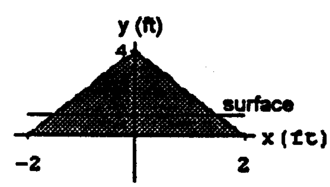
$$\Rightarrow F = \int_{-3}^{0} w(-y)L(y)\,dy = \int_{-3}^{0} 62.4 \cdot (-y) \cdot 2(y+3)\,dy = 124.8 \int_{-3}^{0} \left(-y^2 - 3y\right) dy = 124.8\left[-\frac{y^3}{3} - \frac{3}{2}y^2\right]_{-3}^{0}$$
$$= (-124.8)\left(\tfrac{27}{3} - \tfrac{27}{2}\right) = \frac{(-124.8)(27)(2-3)}{6} = 561.6 \text{ lb}$$

5. Using the coordinate system of Exercise 2, we find the equation for the line of the plate's right-hand edge to be $y = 2x - 4 \Rightarrow x = \frac{y+4}{2}$ and $L(y) = 2x = y + 4$. The depth of the strip is $(1 - y)$.

(a) $$F = \int_{-4}^{0} w(1-y)L(y)\,dy = \int_{-4}^{0} 62.4 \cdot (1-y)(y+4)\,dy = 62.4 \int_{-4}^{0} \left(4 - 3y - y^2\right) dy = 62.4\left[4y - \frac{3y^2}{2} - \frac{y^3}{3}\right]_{-4}^{0}$$
$$= (-62.4)\left[(-4)(4) - \frac{(3)(16)}{2} + \frac{64}{3}\right] = (-62.4)\left(-16 - 24 + \tfrac{64}{3}\right) = \frac{(-62.4)(-120+64)}{3} = 1164.8 \text{ lb}$$

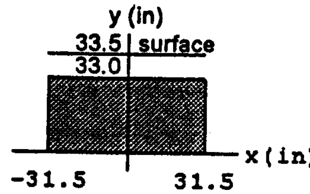
(b) $$F = (-64.0)\left[(-4)(4) - \frac{(3)(16)}{2} + \frac{64}{3}\right] = \frac{(-64.0)(-120+64)}{3} \approx 1194.7 \text{ lb}$$

6. Using the coordinate system given, we find an equation for the line of the plate's right-hand edge to be $y = -2x + 4$ $\Rightarrow x = \frac{4-y}{2}$ and $L(y) = 2x = 4 - y$. The depth of the strip is $(1 - y) \Rightarrow F = \int_{0}^{1} w(1-y)(4-y)\,dy$



$$= 62.4 \int_{0}^{1} \left(y^2 - 5y + 4\right) dy = 62.4\left[\frac{y^3}{3} - \frac{5y^2}{2} + 4y\right]_{0}^{1} = (62.4)\left(\tfrac{1}{3} - \tfrac{5}{2} + 4\right) = (62.4)\left(\tfrac{2 - 15 + 24}{6}\right)$$
$$= \frac{(62.4)(11)}{6} = 114.4 \text{ lb}$$

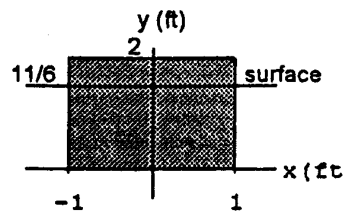
7. Using the coordinate system given in the accompanying figure, we see that the total width is L(y) = 63 and the depth of the strip is $(33.5 - y) \Rightarrow F = \int_0^{33} w(33.5 - y)L(y)\,dy$



$$= \int_0^{33} \frac{64}{12^3} \cdot (33.5 - y) \cdot 63\,dy = \left(\frac{64}{12^3}\right)(63) \int_0^{33} (33.5 - y)\,dy$$

$$= \left(\frac{64}{12^3}\right)(63)\left[33.5y - \frac{y^2}{2}\right]_0^{33} = \left(\frac{64 \cdot 63}{12^3}\right)\left[(33.5)(33) - \frac{33^2}{2}\right] = \frac{(64)(63)(33)(67-33)}{(2)(12^3)} = 1309 \text{ lb}$$

8. (a) Use the coordinate system given in the accompanying figure. The depth of the strip is $\left(\frac{11}{6} - y\right)$ ft



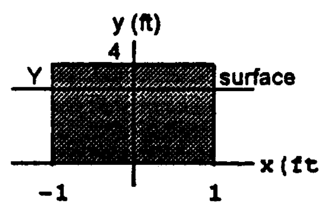
$$\Rightarrow F = \int_0^{11/6} w\left(\frac{11}{6} - y\right)(\text{width})\,dy$$

$$= (62.4)(\text{width}) \int_0^{11/6} \left(\frac{11}{6} - y\right) dy$$

$$= (62.4)(\text{width})\left[\frac{11}{6}y - \frac{y^2}{2}\right]_0^{11/6} = (62.4)(\text{width})\left[\left(\frac{11}{6}\right)^2 \cdot \frac{1}{2}\right] \Rightarrow F_{\text{end}} = (62.4)(2)\left(\frac{121}{36}\right)\left(\frac{1}{2}\right) \approx 209.73 \text{ lb and}$$

$F_{\text{side}} = (62.4)(4)\left(\frac{121}{36}\right)\left(\frac{1}{2}\right) \approx 419.47$ lb

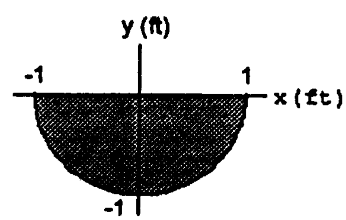
(b) Use the coordinate system given in the accompanying figure. Find Y from the condition that the entire volume of the water is conserved (no spilling): $\frac{11}{6} \cdot 2 \cdot 4 = 2 \cdot 2 \cdot Y$ $\Rightarrow Y = \frac{11}{3}$ ft. The depth of a typical strip is $\left(\frac{11}{3} - y\right)$ ft and the total width is L(y) = 2 ft. Thus,



$$F = \int_0^{11/3} w\left(\frac{11}{3} - y\right)L(y)\,dy = \int_0^{11/3} (62.4)\left(\frac{11}{3} - y\right) \cdot 2\,dy = (62.4)(2)\left[\frac{11}{3}y - \frac{y^2}{2}\right]_0^{11/3} = (62.4)(2)\left[\left(\frac{1}{2}\right)\left(\frac{11}{3}\right)^2\right]$$

$$= \frac{(62.4)(121)}{9} \approx 838.93 \text{ lb} \Rightarrow \text{the fluid force doubles.}$$

9. Using the coordinate system given in the accompanying figure, we see that the right-hand edge is $x = \sqrt{1 - y^2}$ for $-1 \le y \le 0$ so the total width is $L(y) = 2x = 2\sqrt{1 - y^2}$ and the depth of the strip is $(-y)$. The force exerted by the water is therefore



$$F = \int_{-1}^{0} w \cdot (-y) \cdot 2\sqrt{1 - y^2}\,dy$$

$$= 62.4 \int_{-1}^{0} \sqrt{1-y^2}\, d(1-y^2) = 62.4\left[\frac{2}{3}(1-y^2)^{3/2}\right]_{-1}^{0} = (62.4)\left(\frac{2}{3}\right)(1-0) = 41.6 \text{ lb}$$

10. Using the same coordinate system as in Exercise 9, the right-hand edge is $x = \sqrt{3^2 - y^2}$ and the total width is $L(y) = 2x = 2\sqrt{9-y^2}$. The depth of the strip is $(-y)$. The force exerted by the milk is therefore

$$F = \int_{-3}^{0} w\cdot(-y)\cdot 2\sqrt{9-y^2}\, dy = 64.5 \int_{-3}^{0} \sqrt{9-y^2}\, d(9-y^2) = 64.5\left[\frac{2}{3}(9-y^2)^{3/2}\right]_{-3}^{0} = (64.5)\left(\frac{2}{3}\right)(27-0)$$

$$= (64.5)(18) = 1161 \text{ lb}$$

11. The coordinate system is given in the text. The right-hand edge is $x = \sqrt{y}$ and the total width is $L(y) = 2x = 2\sqrt{y}$.

(a) The depth of the strip is $(2-y)$ so the force exerted by the liquid on the gate is $F = \int_0^1 w(2-y)L(y)\, dy$

$$= \int_0^1 50(2-y)\cdot 2\sqrt{y}\, dy = 100 \int_0^1 (2-y)\sqrt{y}\, dy = 100 \int_0^1 \left(2y^{1/2} - y^{3/2}\right) dy = 100\left[\frac{4}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^1$$

$$= 100\left(\frac{4}{3} - \frac{2}{5}\right) = \left(\frac{100}{15}\right)(20-6) = 93.33 \text{ lb}$$

(b) Suppose that H is the maximum height to which the container can be filled without exceeding its design limitation. The depth of a typical strip is $(H-y)$ and the force is $F = \int_0^1 w(H-y)L(y)\, dy = F_{max}$, where $F_{max} = 160$ lb. Therefore, $F_{max} = w \int_0^1 (H-y)\cdot 2\sqrt{y}\, dy = 100 \int_0^1 (H-y)\sqrt{y}\, dy$

$$= 100 \int_0^1 \left(Hy^{1/2} - y^{3/2}\right) dy = 100\left[\frac{2}{3}Hy^{3/2} - \frac{2}{5}y^{5/2}\right]_0^1 = 100\left(\frac{2H}{3} - \frac{2}{5}\right) = \left(\frac{100}{15}\right)(10H-6).$$ When

$F_{max} = 160$ lb we have $160 = \left(\frac{100}{15}\right)(10H-6) \Rightarrow 10H - 6 = 24 \Rightarrow H = 3$ ft

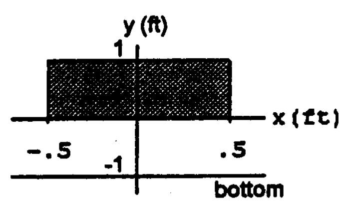
12. Use the coordinate system given in the accompanying figure. The total width is $L(y) = 1$.

(a) The depth of the strip is $(3-1) - y = (2-y)$ ft. The force exerted by the fluid in the window is

$$F = \int_0^1 w(2-y)L(y)\, dy = 62.4 \int_0^1 (2-y)\cdot 1\, dy = (62.4)\left[2y - \frac{y^2}{2}\right]_0^1 = (62.4)\left(2 - \frac{1}{2}\right) = \frac{(62.4)(3)}{2} = 93.6 \text{ lb}$$

(b) Suppose that H is the maximum height to which the tank can be filled without exceeding its design limitation. This means that the depth of a typical strip is $(H-1) - y$ and the force is

$$F = \int_0^1 w[(H-1) - y]L(y)\, dy = F_{max},$$ where

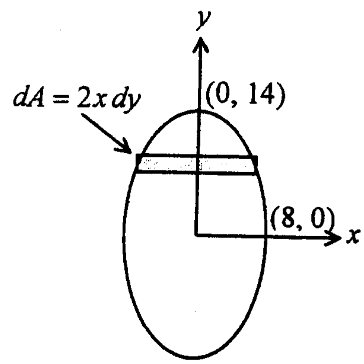


$F_{max} = 312$ lb. Thus, $F_{max} = w \int_0^1 [(H-1)-y]\cdot 1\, dy = (62.4)\left[(H-1)y - \frac{y^2}{2}\right]_0^1 = (62.4)\left(H - \frac{3}{2}\right)$

$= \left(\frac{62.4}{2}\right)(2H-3) = -93.6 + 62.4H$. Then $F_{max} = -93.6 + 62.4H \Rightarrow 312 = -93.6 + 62.4H \Rightarrow H = \frac{405.6}{62.4}$

$= 6.5$ ft

13. (a) The equation of the ellipse for the penstock gate is $\left(\frac{x}{8}\right)^2 + \left(\frac{y}{14}\right)^2 = 1$ or $49x^2 + 16y^2 = 3136 \Rightarrow x = \frac{\sqrt{3136 - 16y^2}}{7}$, where y is measured from the center of the ellipse.



(b) $L(y) = 2x = \frac{2}{7}\sqrt{3136 - 16y^2}$

(c) $\Delta F \approx 62.4[389 - (y + 115)](2x\, \Delta y) = 124.8(274 - y)\, \frac{\sqrt{3136 - 16y^2}}{7}\, \Delta y$. Therefore,

$F = \int_{-14}^{14} 17.829(274 - y)\sqrt{3136 - 16y^2}\, dy = 6.0159 \cdot 10^6$ lb $= 3008$ tons.

14. (a) After 9 hours of filling there are $V = 1000 \cdot 9 = 9000$ cubic feet of water in the pool. The level of the water is $h = \frac{V}{\text{Area}}$, where Area $= 50 \cdot 30 = 1500 \Rightarrow h = \frac{9000}{1500} = 6$ ft. The depth of the typical horizontal strip at level y is then $(6 - y)$ where y is measured up from the bottom of the pool. An equation for the drain plate's right-hand edge is $y = x \Rightarrow$ total width is $L(y) = 2x = 2y$. Thus the force against the drain plate is

$F = \int_0^1 w(6-y)L(y)\, dy = 62.4 \int_0^1 (6-y)\cdot 2y\, dy = (62.4)(2) \int_0^1 (6y - y^2) = (62.4)(2)\left[\frac{6y^2}{2} - \frac{y^3}{3}\right]_0^1$

$= (124.8)\left(3 - \frac{1}{3}\right) = (124.8)\left(\frac{8}{3}\right) = 332.8$ lb

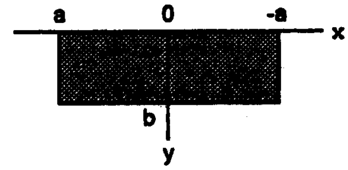
(b) Suppose that h is the maximum height. Then, the depth of a typical strip is $(h - y)$ and the force

$F = \int_0^1 w(h-y)L(y)\, dy = F_{max}$, where $F_{max} = 520$ lb. Hence, $F_{max} = (62.4) \int_0^1 (h-y)\cdot 2y\, dy$

$= 124.8 \int_0^1 (hy - y^2)\, dy = (124.8)\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^1 = (124.8)\left(\frac{h}{2} - \frac{1}{3}\right) = (20.8)(3h - 2) \Rightarrow \frac{520}{20.8} = 3h - 2$

$\Rightarrow h = \frac{27}{3} = 9$ ft

15. (a) The pressure at level y is $p(y) = w \cdot y \Rightarrow$ the average pressure is $\bar{p} = \frac{1}{b}\int_0^b p(y)\,dy = \frac{1}{b}\int_0^b w \cdot y\,dy = \frac{1}{b}w\left[\frac{y^2}{2}\right]_0^b$ $= \left(\frac{w}{b}\right)\left(\frac{b^2}{2}\right) = \frac{wb}{2}$. This is the pressure at level $\frac{b}{2}$, which is the pressure at the middle of the plate.



(b) The force exerted by the fluid is $F = \int_0^b w(\text{depth})(\text{length})\,dy = \int_0^b w \cdot y \cdot a\,dy$ $= (w \cdot a)\int_0^b y\,dy = (w \cdot a)\left[\frac{y^2}{2}\right]_0^b = w\left(\frac{ab^2}{2}\right) = \left(\frac{wb}{2}\right)(ab) = \bar{p} \cdot \text{Area}$, where $\bar{p}$ is the average value of the pressure (see part (a)).

16. When the water reaches the top of the tank the force on the movable side is $\int_{-2}^{0} (62.4)\left(2\sqrt{4-y^2}\right)(-y)\,dy$ $= (62.4)\int_{-2}^{0}\left(4-y^2\right)^{1/2}(-2y)\,dy = (62.4)\left[\frac{2}{3}\left(4-y^2\right)^{3/2}\right]_{-2}^{0} = (62.4)\left(\frac{2}{3}\right)\left(4^{3/2}\right) = 332.8$ ft·lb. The force compressing the spring is $F = 100x$, so when the tank is full we have $332.8 = 100x \Rightarrow x \approx 3.33$ ft. Therefore the movable end does not reach the required 5 ft to allow drainage $\Rightarrow$ the tank will overflow.

17. (a) An equation of the right-hand edge is $y = \frac{3}{2}x \Rightarrow x = \frac{2}{3}y$ and $L(y) = 2x = \frac{4y}{3}$. The depth of the strip is $(3-y) \Rightarrow F = \int_0^3 w(3-y)L(y)\,dy = \int_0^3 (62.4)(3-y)\left(\frac{4}{3}y\right)dy = (62.4)\cdot\left(\frac{4}{3}\right)\int_0^3 \left(3y-y^2\right)dy$ $= (62.4)\left(\frac{4}{3}\right)\left[\frac{3}{2}y^2 - \frac{y^3}{3}\right]_0^3 = (62.4)\left(\frac{4}{3}\right)\left[\frac{27}{2} - \frac{27}{3}\right] = (62.4)\left(\frac{4}{3}\right)\left(\frac{27}{6}\right) = 374.4$ lb

(b) We want to find a new water level Y such that $F_Y = \frac{1}{2}(374.4) = 187.2$ lb. The new depth of the strip is $(Y-y)$, and Y is the new upper limit of integration. Thus, $F_Y = \int_0^Y w(Y-y)L(y)\,dy$ $= 62.4\int_0^Y (Y-y)\left(\frac{4}{3}y\right)dy = (62.4)\left(\frac{4}{3}\right)\int_0^Y \left(Yy - y^2\right)dy = (62.4)\left(\frac{4}{3}\right)\left[Y\cdot\frac{y^2}{2} - \frac{y^3}{3}\right]_0^Y = (62.4)\left(\frac{4}{3}\right)\left(\frac{Y^3}{2} - \frac{Y^3}{3}\right)$ $= (62.4)\left(\frac{2}{9}\right)Y^3$. Therefore $Y^3 = \frac{9F_Y}{2\cdot(62.4)} = \frac{(9)(187.2)}{124.8} \Rightarrow Y = \sqrt[3]{\frac{(9)(187.2)}{124.8}} = \sqrt[3]{13.5} \approx 2.3811$ ft. So, $\Delta Y = 3 - Y \approx 3 - 2.3811 \approx 0.6189$ ft ≈ 7.5 in to the nearest half inch

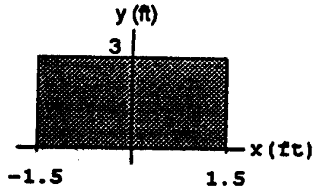
(c) No, it does not matter how long the trough is. The fluid pressure and the resulting force depend only on depth of the water.

18. (a) Using the given coordinate system we see that the total width is $L(y) = 3$ and the depth of the strip is $(3 - y)$.

Thus, $F = \int_0^3 w(3-y)L(y)\,dy = \int_0^3 (62.4)(3-y)\cdot 3\,dy$

$= (62.4)(3)\int_0^3 (3-y)\,dy = (62.4)(3)\left[3y - \frac{y^2}{2}\right]_0^3$

$= (62.4)(3)\left(9 - \frac{9}{2}\right) = (62.4)(3)\left(\frac{9}{2}\right) = 842.4$ lb



(b) Find a new water level Y such that $F_Y = (0.75)(842.4 \text{ lb}) = 631.8$ lb. The new depth of the strip is $(Y - y)$ and Y is the new upper limit of integration. Thus, $F_Y = \int_0^Y w(Y-y)L(y)\,dy$

$= 62.4\int_0^Y (Y-y)\cdot 3\,dy = (62.4)(3)\int_0^Y (Y-y)\,dy = (62.4)(3)\left[Yy - \frac{y^2}{2}\right]_0^Y = (62.4)(3)\left(Y^2 - \frac{Y^2}{2}\right)$

$= (62.4)(3)\left(\frac{Y^2}{2}\right)$. Therefore, $Y = \sqrt{\frac{2F_Y}{(62.4)(3)}} = \sqrt{\frac{1263.6}{187.2}} = \sqrt{6.75} \approx 2.598$ ft. So, $\Delta Y = 3 - Y$

$\approx 3 - 2.598 \approx 0.402$ ft ≈ 4.8 in

19. Use the same coordinate system as in Exercise 20 with $L(y) = 3.75$ and the depth of a typical strip being $(7.75 - y)$. Then $F = \int_0^{7.75} w(7.75-y)L(y)\,dy = \left(\frac{64.5}{12^3}\right)(3.75)\int_0^{7.75}(7.75-y)\,dy = \left(\frac{64.5}{12^3}\right)(3.75)\left[7.75y - \frac{y^2}{2}\right]_0^{7.75}$

$= \left(\frac{64.5}{12^3}\right)(3.75)\frac{(7.75)^2}{2} \approx 4.2$ lb

20. The force against the base is $F_{base} = pA = whA = w\cdot h\cdot(\text{length})(\text{width}) = \left(\frac{57}{12^3}\right)(10)(5.75)(3.5) \approx 6.64$ lb.

To find the fluid force against each side, use the coordinate system described in Exercise 10 with the depth of a typical strip being $(10 - y)$: $F = \int_0^{10} w(10-y)\binom{\text{width of}}{\text{the side}}\,dy = \left(\frac{57}{12^3}\right)\binom{\text{width of}}{\text{the side}}\left[10y - \frac{y^2}{2}\right]_0^{10}$

$= \left(\frac{57}{12^3}\right)\binom{\text{width of}}{\text{the side}}\left(\frac{100}{2}\right) \Rightarrow F_{end} = \left(\frac{57}{12^3}\right)(50)(3.5) \approx 5.773$ lb and $F_{side} = \left(\frac{57}{12^3}\right)(50)(5.75) \approx 9.484$ lb

21. Suppose that h is the maximum height. Using the coordinate system given in the text, we find an equation for the line of the end plate's right-hand edge is $y = \frac{5}{2}x \Rightarrow x = \frac{2}{5}y$. The total width is $L(y) = 2x = \frac{4}{5}y$ and the depth of the typical horizontal strip at level y is $(h - y)$. Then the force is $F = \int_0^h w(h-y)L(y)\,dy = F_{max}$,

where $F_{max} = 6667$ lb. Hence, $F_{max} = w \int_0^h (h-y)\cdot\frac{4}{5}y\,dy = (62.4)\left(\frac{4}{5}\right)\int_0^h (hy-y^2)\,dy$

$= (62.4)\left(\frac{4}{5}\right)\left[\frac{hy^2}{2}-\frac{y^3}{3}\right]_0^h = (62.4)\left(\frac{4}{5}\right)\left(\frac{h^3}{2}-\frac{h^3}{3}\right) = (62.4)\left(\frac{4}{5}\right)\left(\frac{1}{6}\right)h^3 = (10.4)\left(\frac{4}{5}\right)h^3 \Rightarrow h = \sqrt[3]{\left(\frac{5}{4}\right)\left(\frac{F_{max}}{10.4}\right)}$

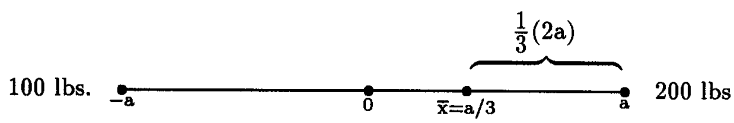
$= \sqrt[3]{\left(\frac{5}{4}\right)\left(\frac{6667}{10.4}\right)} \approx 9.288$ ft. The volume of water which the tank can hold is $V = \frac{1}{2}(\text{Base})(\text{Height})\cdot 30$, where

Height $= h$ and $\frac{1}{2}(\text{Base}) = \frac{2}{5}h \Rightarrow V = \left(\frac{2}{5}h^2\right)(30) = 12h^2 \approx 12(9.288)^2 \approx 1035\ \text{ft}^3$.

5.7 MOMENTS AND CENTERS OF MASS

1. Because the children are balanced, the moment of the system about the origin must be equal to zero: $5\cdot 80 = x\cdot 100 \Rightarrow x = 4$ ft, the distance of the 100-lb child from the fulcrum.

2. Suppose the log has length 2a. Align the log along the x-axis so the 100-lb end is placed at $x = -a$ and the 200-lb end at $x = a$. Then the center of mass $\bar{x}$ satisfies $\bar{x} = \frac{100(-a)+200(a)}{300} \Rightarrow \bar{x} = \frac{a}{3}$. That is, $\bar{x}$ is located at a distance $a - \frac{a}{3} = \frac{2a}{3} = \frac{1}{3}(2a)$ which is $\frac{1}{3}$ of the length of the log from the 200-lb (heavier) end (see figure) or $\frac{2}{3}$ of the way from the lighter end toward the heavier end.



3. The center of mass of each rod is in its center (see Example 1). The rod system is equivalent to two point masses located at the centers of the rods at coordinates $\left(\frac{L}{2},0\right)$ and $\left(0,\frac{L}{2}\right)$. Therefore $\bar{x} = \frac{m_y}{m}$ $= \frac{x_1m_1+x_2m_2}{m_1+m_2} = \frac{\frac{L}{2}\cdot m+0}{m+m} = \frac{L}{4}$ and $\bar{y} = \frac{m_x}{m} = \frac{y_1m_2+y_2m_2}{m_1+m_2} = \frac{0+\frac{L}{2}\cdot m}{m+m} = \frac{L}{4} \Rightarrow \left(\frac{L}{4},\frac{L}{4}\right)$ is the center of mass location.

4. Let the rods have lengths $x = L$ and $y = 2L$. The center of mass of each rod is in its center (see Example 1). The rod system is equivalent to two point masses located at the centers of the rods at coordinates $\left(\frac{L}{2},0\right)$ and $(0,L)$. Therefore $\bar{x} = \frac{\frac{L}{2}\cdot m+0\cdot 2m}{m+2m} = \frac{L}{6}$ and $\bar{y} = \frac{0\cdot m+L\cdot 2m}{m+2m} = \frac{2L}{3} \Rightarrow \left(\frac{L}{6},\frac{2L}{3}\right)$ is the center of mass location.

5. $M_0 = \int_0^2 x\cdot 4\,dx = \left[4\frac{x^2}{2}\right]_0^2 = 4\cdot\frac{4}{2} = 8;\ M = \int_0^2 4\,dx = [4x]_0^2 = 4\cdot 2 = 8 \Rightarrow \bar{x} = \frac{M_0}{M} = 1$

6. $M_0 = \int_1^3 x\cdot 4\,dx = \left[4\frac{x^2}{2}\right]_1^3 = \frac{4}{2}(9-1) = 16;\ M = \int_1^3 4\,dx = [4x]_1^3 = 12-4 = 8 \Rightarrow \bar{x} = \frac{M_0}{M} = \frac{16}{8} = 2$

7. $M_0 = \int_0^3 x\left(1+\frac{x}{3}\right) dx = \int_0^3 \left(x+\frac{x^2}{3}\right) dx = \left[\frac{x^2}{2}+\frac{x^3}{9}\right]_0^3 = \left(\frac{9}{2}+\frac{27}{9}\right) = \frac{15}{2}$; $M = \int_0^3 \left(1+\frac{x}{3}\right) dx = \left[x+\frac{x^2}{6}\right]_0^3$

$= 3+\frac{9}{6} = \frac{9}{2} \Rightarrow \bar{x} = \frac{M_0}{M} = \frac{\left(\frac{15}{2}\right)}{\left(\frac{9}{2}\right)} = \frac{15}{9} = \frac{5}{3}$

8. $M_0 = \int_0^4 x\left(2-\frac{x}{4}\right) dx = \int_0^4 \left(2x-\frac{x^2}{4}\right) dx = \left[x^2-\frac{x^3}{12}\right]_0^4 = \left(16-\frac{64}{12}\right) = 16-\frac{16}{3} = 16\cdot\frac{2}{3} = \frac{32}{3}$;

$M = \int_0^4 \left(2-\frac{x}{4}\right) dx = \left[2x-\frac{x^2}{8}\right]_0^4 = 8-\frac{16}{8} = 6 \Rightarrow \bar{x} = \frac{M_0}{M} = \frac{32}{3\cdot 6} = \frac{16}{9}$

9. $M_0 = \int_1^4 x\left(1+\frac{1}{\sqrt{x}}\right) dx = \int_1^4 (x+x^{1/2})\, dx = \left[\frac{x^2}{2}+\frac{2x^{3/2}}{3}\right]_1^4 = \left(8+\frac{16}{3}\right)-\left(\frac{1}{2}+\frac{2}{3}\right) = \frac{15}{2}+\frac{14}{3} = \frac{45+28}{6} = \frac{73}{6}$;

$M = \int_1^4 (1+x^{-1/2})\, dx = [x+2x^{1/2}]_1^4 = (4+4)-(1+2) = 5 \Rightarrow \bar{x} = \frac{M_0}{M} = \frac{\left(\frac{73}{6}\right)}{5} = \frac{73}{30}$

10. $M_0 = \int_{1/4}^1 x\cdot 3(x^{-3/2}+x^{-5/2})\, dx = 3\int_{1/4}^1 (x^{-1/2}+x^{-3/2})\, dx = 3\left[2x^{1/2}-\frac{2}{x^{1/2}}\right]_{1/4}^1 = 3\left[(2-2)-\left(2\cdot\frac{1}{2}-\frac{2}{\left(\frac{1}{2}\right)}\right)\right]$

$= 3(4-1) = 9$; $M = 3\int_{1/4}^1 (x^{-3/2}+x^{-5/2})\, dx = 3\left[\frac{-2}{x^{1/2}}-\frac{2}{3x^{3/2}}\right]_{1/4}^1 = 3\left[\left(-2-\frac{2}{3}\right)-\left(-4-\frac{16}{3}\right)\right] = 3\left(2+\frac{14}{3}\right)$

$= 6+14 = 20 \Rightarrow \bar{x} = \frac{M_0}{M} = \frac{9}{20}$

11. $M_0 = \int_0^1 x(2-x)\, dx + \int_1^2 x\cdot x\, dx = \int_0^1 (2x-x^2)\, dx + \int_1^2 x^2\, dx = \left[\frac{2x^2}{2}-\frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3}\right]_1^2 = \left(1-\frac{1}{3}\right)+\left(\frac{8}{3}-\frac{1}{3}\right)$

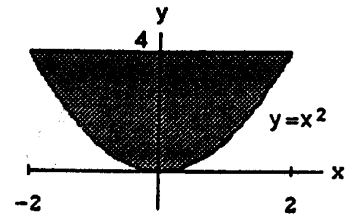
$= \frac{9}{3} = 3$; $M = \int_0^1 (2-x)\, dx + \int_1^2 x\, dx = \left[2x-\frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2}\right]_1^2 = \left(2-\frac{1}{2}\right)+\left(\frac{4}{2}-\frac{1}{2}\right) = 3 \Rightarrow \bar{x} = \frac{M_0}{M} = 1$

12. $M_0 = \int_0^1 x(x+1)\, dx + \int_1^2 2x\, dx = \int_0^1 (x^2+x)\, dx + \int_1^2 2x\, dx = \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1 + [x^2]_1^2 = \left(\frac{1}{3}+\frac{1}{2}\right)+(4-1)$

$= 3+\frac{5}{6} = \frac{23}{6}$; $M = \int_0^1 (x+1)\, dx + \int_1^2 2\, dx = \left[\frac{x^2}{2}+x\right]_0^1 + [2x]_1^2 = \left(\frac{1}{2}+1\right)+(4-2) = 2+\frac{3}{2} = \frac{7}{2}$

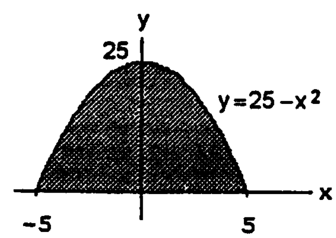
$\Rightarrow \bar{x} = \frac{M_0}{M} = \left(\frac{23}{6}\right)\left(\frac{2}{7}\right) = \frac{23}{21}$

13. Since the plate is symmetric about the y-axis and its density is constant, the distribution of mass is symmetric about the y-axis and the center of mass lies on the y-axis. This means that $\bar{x} = 0$. It remains to find $\bar{y} = \frac{M_x}{M}$. We model the distribution of mass with *vertical* strips. The typical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{x^2+4}{2}\right)$, length: $4 - x^2$, width: dx, area: $dA = (4 - x^2)\,dx$, mass: $dm = \delta\,dA = \delta(4 - x^2)\,dx$. The moment of the strip about the x-axis is $\tilde{y}\,dm = \left(\frac{x^2+4}{2}\right)\delta(4 - x^2)\,dx = \frac{\delta}{2}(16 - x^4)\,dx$. The moment of the plate about the x-axis is $M_x = \int \tilde{y}\,dm = \int_{-2}^{2} \frac{\delta}{2}(16 - x^4)\,dx = \frac{\delta}{2}\left[16x - \frac{x^5}{5}\right]_{-2}^{2} = \frac{\delta}{2}\left[\left(16\cdot 2 - \frac{2^5}{5}\right) - \left(-16\cdot 2 + \frac{2^5}{5}\right)\right]$ $= \frac{\delta\cdot 2}{2}\left(32 - \frac{32}{5}\right) = \frac{128\delta}{5}$. The mass of the plate is $M = \int \delta(4 - x^2)\,dx = \delta\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 2\delta\left(8 - \frac{8}{3}\right) = \frac{32\delta}{3}$.



Therefore $\bar{y} = \frac{M_x}{M} = \frac{\left(\frac{128\delta}{5}\right)}{\left(\frac{32\delta}{3}\right)} = \frac{12}{5}$. The plate's center of mass is the point $(\bar{x}, \bar{y}) = \left(0, \frac{12}{5}\right)$.

14. Applying the symmetry argument analogous to the one in Exercise 13, we find $\bar{x} = 0$. To find $\bar{y} = \frac{M_x}{M}$, we use the *vertical* strips technique. The typical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{25 - x^2}{2}\right)$, length: $25 - x^2$, width: dx, area: $dA = (25 - x^2)\,dx$, mass: $dm = \delta\,dA = \delta(25 - x^2)\,dx$.



The moment of the strip about the x-axis is $\tilde{y}\,dm = \left(\frac{25 - x^2}{2}\right)\delta(25 - x^2)\,dx = \frac{\delta}{2}(25 - x^2)^2\,dx$. The moment of the plate about the x-axis is

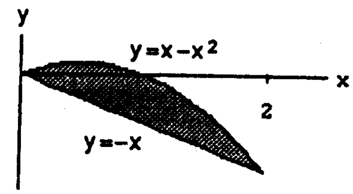
$$M_x = \int \tilde{y}\,dm = \int_{-5}^{5} \frac{\delta}{2}(25 - x^2)^2\,dx = \frac{\delta}{2}\int_{-5}^{5} (625 - 50x^2 + x^4)\,dx = \frac{\delta}{2}\left[625x - \frac{50}{3}x^3 + \frac{x^5}{5}\right]_{-5}^{5}$$

$$= 2\cdot\frac{\delta}{2}\left(625\cdot 5 - \frac{50}{3}\cdot 5^3 + \frac{5^5}{5}\right) = \delta\cdot 625\left(5 - \frac{10}{3} + 1\right) = \delta\cdot 625\cdot\left(\frac{8}{3}\right).$$ The mass of the plate is

$$M = \int dm = \int_{-5}^{5} \delta(25 - x^2)\,dx = \delta\left[25x - \frac{x^3}{3}\right]_{-5}^{5} = 2\delta\left(5^3 - \frac{5^3}{3}\right) = \frac{4}{3}\delta\cdot 5^3.$$ Therefore $\bar{y} = \frac{M_x}{M}$

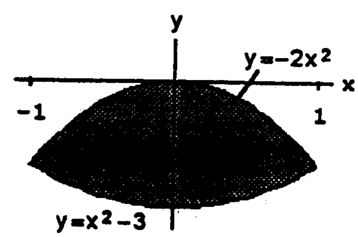
$$= \frac{\delta\cdot 5^4\cdot\left(\frac{8}{3}\right)}{\delta\cdot 5^3\cdot\left(\frac{4}{3}\right)} = 10.$$ The plate's center of mass is the point $(\bar{x}, \bar{y}) = (0, 10)$.

15. Intersection points: $x - x^2 = -x \Rightarrow 2x - x^2 = 0$ $\Rightarrow x(2-x) = 0 \Rightarrow x = 0$ or $x = 2$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{(x-x^2)+(-x)}{2}\right)$ $= \left(x, -\frac{x^2}{2}\right)$, length: $(x-x^2)-(-x) = 2x - x^2$, width: dx, area: $dA = (2x - x^2)\,dx$, mass: $dm = \delta\,dA = \delta(2x-x^2)\,dx$.



The moment of the strip about the x-axis is $\tilde{y}\,dm = \left(-\frac{x^2}{2}\right)\delta(2x-x^2)\,dx$; about the y-axis it is $\tilde{x}\,dm = x\cdot\delta(2x-x^2)\,dx$. Thus, $M_x = \int \tilde{y}\,dm = -\int_0^2 \left(\frac{\delta}{2}x^2\right)(2x-x^2)\,dx = -\frac{\delta}{2}\int_0^2 (2x^3 - x^4)\,dx$ $= -\frac{\delta}{2}\left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2 = -\frac{\delta}{2}\left(2^3 - \frac{2^5}{5}\right) = -\frac{\delta}{2}\cdot 2^3\left(1-\frac{4}{5}\right) = -\frac{4\delta}{5}$; $M_y = \int \tilde{x}\,dm = \int_0^2 x\cdot\delta(2x-x^2)\,dx$ $= \delta\int_0^2 (2x^2 - x^3) = \delta\left[\frac{2}{3}x^3 - \frac{x^4}{4}\right]_0^2 = \delta\left(2\cdot\frac{2^3}{3} - \frac{2^4}{4}\right) = \frac{\delta\cdot 2^4}{12} = \frac{4\delta}{3}$; $M = \int dm = \int_0^2 \delta(2x-x^2)\,dx$ $= \delta\int_0^2 (2x-x^2)\,dx = \delta\left[x^2 - \frac{x^3}{3}\right]_0^2 = \delta\left(4-\frac{8}{3}\right) = \frac{4\delta}{3}$. Therefore, $\bar{x} = \frac{M_y}{M} = \left(\frac{4\delta}{3}\right)\left(\frac{3}{4\delta}\right) = 1$ and $\bar{y} = \frac{M_x}{M}$ $= \left(-\frac{4\delta}{5}\right)\left(\frac{3}{4\delta}\right) = -\frac{3}{5} \Rightarrow (\bar{x},\bar{y}) = \left(1, -\frac{3}{5}\right)$ is the center of mass.

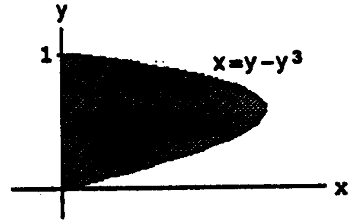
16. Intersection points: $x^2 - 3 = -2x^2 \Rightarrow 3x^2 - 3 = 0$ $\Rightarrow 3(x-1)(x+1) = 0 \Rightarrow x = -1$ or $x = 1$. Applying the symmetry argument analogous to the one in Exercise 13, we find $\bar{x} = 0$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{-2x^2 + (x^2-3)}{2}\right) = \left(x, \frac{-x^2-3}{2}\right)$, length: $-2x^2 - (x^2-3) = 3(1-x^2)$, width: dx,



area: $dA = 3(1-x^2)\,dx$, mass: $dm = \delta\,dA = 3\delta(1-x^2)\,dx$. The moment of the strip about the x-axis is $\tilde{y}\,dm = \frac{3}{2}\delta(-x^2-3)(1-x^2)\,dx = \frac{3}{2}\delta(x^4 + 3x^2 - x^2 - 3)\,dx = \frac{3}{2}\delta(x^4 + 2x^2 - 3)\,dx$; $M_x = \int \tilde{y}\,dm$ $= \frac{3}{2}\delta\int_{-1}^1 (x^4 + 2x^2 - 3)\,dx = \frac{3}{2}\delta\left[\frac{x^5}{5} + \frac{2x^3}{3} - 3x\right]_{-1}^1 = \frac{3}{2}\cdot\delta\cdot 2\left(\frac{1}{5} + \frac{2}{3} - 3\right) = 3\delta\left(\frac{3+10-45}{15}\right) = -\frac{32\delta}{5}$;

$M = \int dm = 3\delta\int_{-1}^1 (1-x^2)\,dx = 3\delta\left[x - \frac{x^3}{3}\right]_{-1}^1 = 3\delta\cdot 2\left(1-\frac{1}{3}\right) = 4\delta$. Therefore, $\bar{y} = \frac{M_x}{M} = -\frac{\delta\cdot 32}{5\cdot\delta\cdot 4} = -\frac{8}{5}$ $\Rightarrow (\bar{x},\bar{y}) = \left(0, -\frac{8}{5}\right)$ is the center of mass.

17. The typical *horizontal* strip has center of mass:



$(\tilde{x},\tilde{y}) = \left(\frac{y-y^3}{2}, y\right)$, length: $y-y^3$, width: dy,

area: $dA = (y-y^3)\,dy$, mass: $dm = \delta\,dA = \delta(y-y^3)\,dy$.

The moment of the strip about the y-axis is

$\tilde{x}\,dm = \delta\left(\frac{y-y^3}{2}\right)(y-y^3)\,dy = \frac{\delta}{2}(y-y^3)^2\,dy$

$= \frac{\delta}{2}(y^2-2y^4+y^6)\,dy$; the moment about the x-axis is $\tilde{y}\,dm = \delta y(y-y^3)\,dy = \delta(y^2-y^4)\,dy$. Thus,

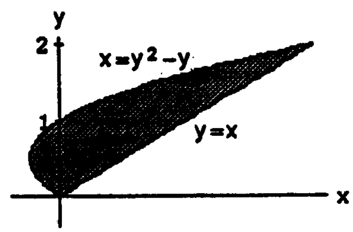
$M_x = \int \tilde{y}\,dm = \delta\int_0^1 (y^2-y^4)\,dy = \delta\left[\frac{y^3}{3}-\frac{y^5}{5}\right]_0^1 = \delta\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{2\delta}{15}$; $M_y = \int \tilde{x}\,dm = \frac{\delta}{2}\int_0^1 (y^2-2y^4+y^6)\,dy$

$= \frac{\delta}{2}\left[\frac{y^3}{3}-\frac{2y^5}{5}+\frac{y^7}{7}\right]_0^1 = \frac{\delta}{2}\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right) = \frac{\delta}{2}\left(\frac{35-42+15}{3\cdot5\cdot7}\right) = \frac{4\delta}{105}$; $M = \int dm = \delta\int_0^1 (y-y^3)\,dy$

$= \delta\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_0^1 = \delta\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\delta}{4}$. Therefore, $\bar{x} = \frac{M_y}{M} = \left(\frac{4\delta}{105}\right)\left(\frac{4}{\delta}\right) = \frac{16}{105}$ and $\bar{y} = \frac{M_x}{M} = \left(\frac{2\delta}{15}\right)\left(\frac{4}{\delta}\right) = \frac{8}{15}$

$\Rightarrow (\bar{x},\bar{y}) = \left(\frac{16}{105},\frac{8}{15}\right)$ is the center of mass.

18. Intersection points: $y = y^2 - y \Rightarrow y^2 - 2y = 0 \Rightarrow y(y-2)$ $= 0 \Rightarrow y = 0$ or $y = 2$. The typical *horizontal* strip has center of mass: $(\tilde{x},\tilde{y}) = \left(\frac{(y^2-y)+y}{2}, y\right) = \left(\frac{y^2}{2}, y\right)$,



length: $y-(y^2-y) = 2y-y^2$, width: dy,

area: $dA = (2y-y^2)\,dy$, mass: $dm = \delta\,dA = \delta(2y-y^2)\,dy$.

The moment about the y-axis is $\tilde{x}\,dm = \frac{\delta}{2}\cdot y^2(2y-y^2)\,dy$

$= \frac{\delta}{2}(2y^3-y^4)\,dy$; the moment about the x-axis is $\tilde{y}\,dm = \delta y(2y-y^2)\,dy = \delta(2y^2-y^3)\,dy$. Thus,

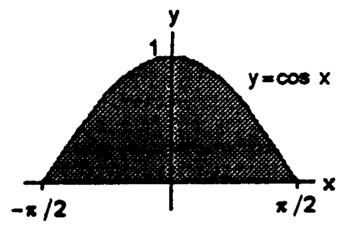
$M_x = \int \tilde{y}\,dm = \int_0^2 \delta(2y^2-y^3)\,dy = \delta\left[\frac{2y^3}{3}-\frac{y^4}{4}\right]_0^2 = \delta\left(\frac{16}{3}-\frac{16}{4}\right) = \frac{16\delta}{12}(4-3) = \frac{4\delta}{3}$; $M_y = \int \tilde{x}\,dm$

$= \int_0^2 \frac{\delta}{2}(2y^3-y^4)\,dy = \frac{\delta}{2}\left[\frac{y^4}{2}-\frac{y^5}{5}\right]_0^2 = \frac{\delta}{2}\left(8-\frac{32}{5}\right) = \frac{\delta}{2}\left(\frac{40-32}{5}\right) = \frac{4\delta}{5}$; $M = \int dm = \int_0^2 \delta(2y-y^2)\,dy$

$= \delta\left[y^2-\frac{y^3}{3}\right]_0^2 = \delta\left(4-\frac{8}{3}\right) = \frac{4\delta}{3}$. Therefore, $\bar{x} = \frac{M_y}{M} = \left(\frac{4\delta}{5}\right)\left(\frac{3}{4\delta}\right) = \frac{3}{5}$ and $\bar{y} = \frac{M_x}{M} = \left(\frac{4\delta}{3}\right)\left(\frac{3}{4\delta}\right) = 1$

$\Rightarrow (\bar{x},\bar{y}) = \left(\frac{3}{5},1\right)$ is the center of mass.

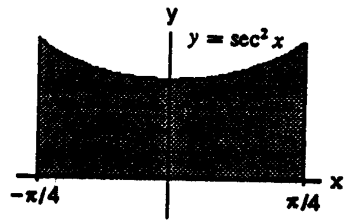
19. Applying the symmetry argument analogous to the one used in Exercise 13, we find $\bar{x} = 0$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\cos x}{2}\right)$, length: $\cos x$, width: dx, area: $dA = \cos x\, dx$, mass: $dm = \delta\, dA = \delta \cos x\, dx$. The moment of the strip about the x-axis is $\tilde{y}\, dm = \delta \cdot \frac{\cos x}{2} \cdot \cos x\, dx$ $= \frac{\delta}{2} \cos^2 x\, dx = \frac{\delta}{2}\left(\frac{1+\cos 2x}{2}\right) dx = \frac{\delta}{4}(1+\cos 2x)\, dx$; thus,



$M_x = \int \tilde{y}\, dm = \int_{-\pi/2}^{\pi/2} \frac{\delta}{4}(1+\cos 2x)\, dx = \frac{\delta}{4}\left[x + \frac{\sin 2x}{2}\right]_{-\pi/2}^{\pi/2} = \frac{\delta}{4}\left[\left(\frac{\pi}{2}+0\right)-\left(-\frac{\pi}{2}\right)\right] = \frac{\delta\pi}{4}$; $M = \int dm$

$= \delta \int_{-\pi/2}^{\pi/2} \cos x\, dx = \delta[\sin x]_{-\pi/2}^{\pi/2} = 2\delta$. Therefore, $\bar{y} = \frac{M_x}{M} = \frac{\delta\pi}{4 \cdot 2\delta} = \frac{\pi}{8} \Rightarrow (\bar{x}, \bar{y}) = \left(0, \frac{\pi}{8}\right)$ is the center of mass.

20. Applying the symmetry argument analogous to the one used in Exercise 13, we find $\bar{x} = 0$. The typical vertical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\sec^2 x}{2}\right)$, length: $\sec^2 x$, width: dx, area: $dA = \sec^2 x\, dx$, mass: $dm = \delta\, dA = \delta \sec^2 x\, dx$. The moment about the x-axis is $\tilde{y}\, dm = \left(\frac{\sec^2 x}{2}\right)(\delta \sec^2 x)\, dx$



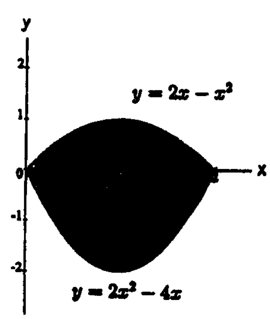
$= \frac{\delta}{2} \sec^4 x\, dx$. $M_x = \int_{-\pi/4}^{\pi/4} \tilde{y}\, dm = \frac{\delta}{2} \int_{-\pi/4}^{\pi/4} \sec^4 x\, dx = \frac{\delta}{2} \int_{-\pi/4}^{\pi/4} (\tan^2 x + 1)(\sec^2 x)\, dx$

$= \frac{\delta}{2} \int_{-\pi/4}^{\pi/4} (\tan x)^2 (\sec^2 x)\, dx + \frac{\delta}{2} \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx = \frac{\delta}{2}\left[\frac{(\tan x)^3}{3}\right]_{-\pi/4}^{\pi/4} + \frac{\delta}{2}[\tan x]_{-\pi/4}^{\pi/4}$

$= \frac{\delta}{2}\left[\frac{1}{3} - \left(-\frac{1}{3}\right)\right] + \frac{\delta}{2}[1-(-1)] = \frac{\delta}{3} + \delta = \frac{4\delta}{3}$; $M = \int dm = \delta \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx = \delta[\tan x]_{-\pi/4}^{\pi/4}$

$= \delta[1-(-1)] = 2\delta$. Therefore, $\bar{y} = \frac{M_x}{M} = \left(\frac{4\delta}{3}\right)\left(\frac{1}{2\delta}\right) = \frac{2}{3} \Rightarrow (\bar{x}, \bar{y}) = \left(0, \frac{2}{3}\right)$ is the center of mass.

21. Since the plate is symmetric about the line $x = 1$ and its density is constant, the distribution of mass is symmetric about this line and the center of mass lies on it. This means that $\bar{x} = 1$. The typical *vertical* strip has center of mass:
$(\tilde{x}, \tilde{y}) = \left(x, \frac{(2x-x^2)+(2x^2-4x)}{2}\right) = \left(x, \frac{x^2-2x}{2}\right)$,
length: $(2x-x^2)-(2x^2-4x) = -3x^2+6x = 3(2x-x^2)$,
width: dx, area: $dA = 3(2x-x^2)\, dx$, mass: $dm = \delta\, dA$
$= 3\delta(2x-x^2)\, dx$. The moment about the x-axis is



$\tilde{y}\, dm = \frac{3}{2}\delta(x^2-2x)(2x-x^2)\, dx = -\frac{3}{2}\delta(x^2-2x)^2\, dx = -\frac{3}{2}\delta(x^4-4x^3+4x^2)\, dx$. Thus, $M_x = \int \tilde{y}\, dm$

$$= -\int_0^2 \tfrac{3}{2}\delta(x^4 - 4x^3 + 4x^2)\,dx = -\tfrac{3}{2}\delta\left[\tfrac{x^5}{5} - x^4 + \tfrac{4}{3}x^3\right]_0^2 = -\tfrac{3}{2}\delta\left(\tfrac{2^5}{5} - 2^4 + \tfrac{4}{3}\cdot 2^3\right) = -\tfrac{3}{2}\delta\cdot 2^4\left(\tfrac{2}{5} - 1 + \tfrac{2}{3}\right)$$

$$= -\tfrac{3}{2}\delta\cdot 2^4\left(\tfrac{6-15+10}{15}\right) = -\tfrac{8\delta}{5};\ M = \int dm = \int_0^2 3\delta(2x - x^2)\,dx = 3\delta\left[x^2 - \tfrac{x^3}{3}\right]_0^2 = 3\delta\left(4 - \tfrac{8}{3}\right) = 4\delta.$$

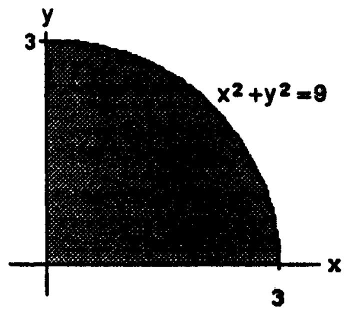
Therefore, $\bar{y} = \frac{M_x}{M} = \left(-\frac{8\delta}{5}\right)\left(\frac{1}{4\delta}\right) = -\frac{2}{5} \Rightarrow (\bar{x}, \bar{y}) = \left(1, -\frac{2}{5}\right)$ is the center of mass.

22. (a) Since the plate is symmetric about the line $x = y$ and its density is constant, the distribution of mass is symmetric about this line. This means that $\bar{x} = \bar{y}$. The typical *vertical* strip has center of mass:

$(\tilde{x}, \tilde{y}) = \left(x, \frac{\sqrt{9-x^2}}{2}\right)$, length: $\sqrt{9-x^2}$, width: dx,

area: $dA = \sqrt{9-x^2}\,dx$, mass: $dm = \delta\,dA = \delta\sqrt{9-x^2}\,dx$.

The moment about the x-axis is $\tilde{y}\,dm = \delta\left(\frac{\sqrt{9-x^2}}{2}\right)\sqrt{9-x^2}\,dx$



$= \frac{\delta}{2}(9 - x^2)\,dx$. Thus, $M_x = \int \tilde{y}\,dm = \int_0^3 \frac{\delta}{2}(9-x^2)\,dx = \frac{\delta}{2}\left[9x - \frac{x^3}{3}\right]_0^3 = \frac{\delta}{2}(27 - 9) = 9\delta$;

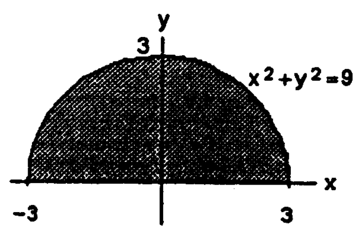
$M = \int dm = \int \delta\,dA = \delta\int dA = \delta(\text{Area of a quarter of a circle of radius } 3) = \delta\left(\frac{9\pi}{4}\right) = \frac{9\pi\delta}{4}$. Therefore,

$\bar{y} = \frac{M_x}{M} = (9\delta)\left(\frac{4}{9\pi\delta}\right) = \frac{4}{\pi} \Rightarrow (\bar{x}, \bar{y}) = \left(\frac{4}{\pi}, \frac{4}{\pi}\right)$ is the center of mass.

(b) Applying the symmetry argument analogous to the one used in Exercise 13, we find that $\bar{x} = 0$. The typical vertical strip has the same parameters as in part (a). Thus,

$M_x = \int \tilde{y}\,dm = \int_{-3}^{3} \frac{\delta}{2}(9-x^2)\,dx = 2\int_0^3 \frac{\delta}{2}(9-x^2)\,dx$

$= 2(9\delta) = 18\delta$; $M = \int dm = \int \delta\,dA = \delta\int dA$



$= \delta(\text{Area of a semi-circle of radius } 3) = \delta\left(\frac{9\pi}{2}\right) = \frac{9\pi\delta}{2}$. Therefore, $\bar{y} = \frac{M_x}{M} = (18\delta)\left(\frac{2}{9\pi\delta}\right) = \frac{4}{\pi}$, the same $\bar{y}$

as in part (a) $\Rightarrow (\bar{x}, \bar{y}) = \left(0, \frac{4}{\pi}\right)$ is the center of mass.

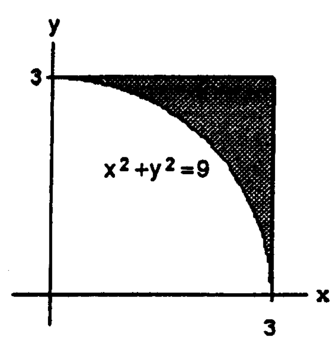
23. Since the plate is symmetric about the line $x = y$ and its density is constant, the distribution of mass is symmetric about this line. This means that $\bar{x} = \bar{y}$. The typical *vertical* strip has

center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{3 + \sqrt{9-x^2}}{2}\right)$,

length: $3 - \sqrt{9-x^2}$, width: dx, area: $dA = \left(3 - \sqrt{9-x^2}\right)dx$,

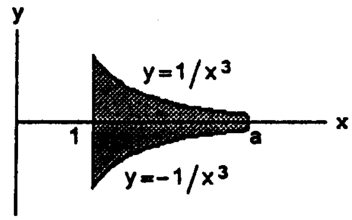
mass: $dm = \delta\,dA = \delta\left(3 - \sqrt{9-x^2}\right)dx$. The moment about the

x-axis is $\tilde{y}\,dm = \delta\frac{\left(3 + \sqrt{9-x^2}\right)\left(3 - \sqrt{9-x^2}\right)}{2}\,dx$



$= \frac{\delta}{2}[9-(9-x^2)]\,dx = \frac{\delta x^2}{2}\,dx$. Thus, $M_x = \int_0^3 \frac{\delta x^2}{2}\,dx = \frac{\delta}{6}[x^3]_0^3 = \frac{9\delta}{2}$. The area equals the area of a square with side length 3 minus one quarter the area of a disk with radius 3 $\Rightarrow A = 3^2 - \frac{\pi 9}{4} = \frac{9}{4}(4-\pi) \Rightarrow M = \delta A$ $= \frac{9\delta}{4}(4-\pi)$. Therefore, $\bar{y} = \frac{M_x}{M} = \left(\frac{9\delta}{2}\right)\left[\frac{4}{9\delta(4-\pi)}\right] = \frac{2}{4-\pi} \Rightarrow (\bar{x},\bar{y}) = \left(\frac{2}{4-\pi}, \frac{2}{4-\pi}\right)$ is the center of mass.

24. Applying the symmetry argument analogous to the one used in Exercise 13, we find that $\bar{y} = 0$. The typical *vertical* strip has



center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\frac{1}{x^3} - \frac{1}{x^3}}{2}\right) = (x, 0)$,

length: $\frac{1}{x^3} - \left(-\frac{1}{x^3}\right) = \frac{2}{x^3}$, width: dx, area: $dA = \frac{2}{x^3}\,dx$,

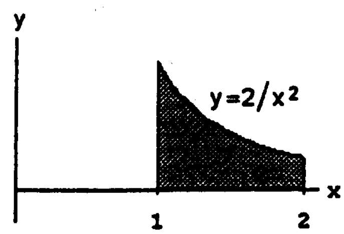
mass: $dm = \delta\,dA = \frac{2\delta}{x^3}\,dx$. The moment about the y-axis is

$\tilde{x}\,dm = x \cdot \frac{2\delta}{x^3}\,dx = \frac{2\delta}{x^2}\,dx$. Thus, $M_y = \int \tilde{x}\,dm = \int_1^a \frac{2\delta}{x^2}\,dx = 2\delta\left[-\frac{1}{x}\right]_1^a = 2\delta\left(-\frac{1}{a}+1\right) = \frac{2\delta(a-1)}{a}$;

$M = \int dm = \int_1^a \frac{2\delta}{x^3}\,dx = \delta\left[-\frac{1}{x^2}\right]_1^a = \delta\left(-\frac{1}{a^2}+1\right) = \frac{\delta(a^2-1)}{a^2}$. Therefore, $\bar{x} = \frac{M_y}{M} = \left[\frac{2\delta(a-1)}{a}\right]\left[\frac{a^2}{\delta(a^2-1)}\right]$

$= \frac{2a}{a+1} \Rightarrow (\bar{x},\bar{y}) = \left(\frac{2a}{a+1}, 0\right)$. Also, $\lim_{a\to\infty} \bar{x} = 2$.

25. $M_x = \int \tilde{y}\,dm = \int_1^2 \frac{\left(\frac{2}{x^2}\right)}{2} \cdot \delta \cdot \left(\frac{2}{x^2}\right) dx$



$= \int_1^2 \left(\frac{1}{x^2}\right)(x^2)\left(\frac{2}{x^2}\right) dx = \int_1^2 \frac{2}{x^2}\,dx = 2\int_1^2 x^{-2}\,dx$

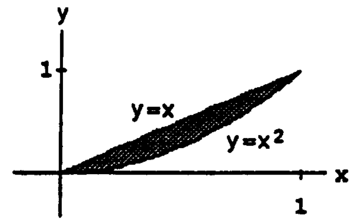
$= 2[-x^{-1}]_1^2 = 2\left[\left(-\frac{1}{2}\right) - (-1\right] = 2\left(\frac{1}{2}\right) = 1$;

$M_y = \int \tilde{x}\,dm = \int_1^2 x \cdot \delta \cdot \left(\frac{2}{x^2}\right) dx$

$= \int_1^2 x(x^2)\left(\frac{2}{x^2}\right) dx = 2\int_1^2 x\,dx = = 2\left[\frac{x^2}{2}\right]_1^2 = 2\left(2-\frac{1}{2}\right) = 4-1 = 3$; $M = \int dm = \int_1^2 \delta\left(\frac{2}{x^2}\right) dx$

$= \int_1^2 x^2\left(\frac{2}{x^2}\right) dx = 2\int_1^2 dx = 2[x]_1^2 = 2(2-1) = 2$. So $\bar{x} = \frac{M_y}{M} = \frac{3}{2}$ and $\bar{y} = \frac{M_x}{M} = \frac{1}{2} \Rightarrow (\bar{x},\bar{y}) = \left(\frac{3}{2}, \frac{1}{2}\right)$ is the center of mass.

26. We use the *vertical* strip approach:



$$M_x = \int \tilde{y}\, dm = \int_0^1 \frac{(x+x^2)}{2}(x-x^2)\cdot\delta\, dx = \frac{1}{2}\int_0^1 (x^2-x^4)\cdot 12x\, dx$$

$$= 6\int_0^1 (x^3-x^5)\, dx = 6\left[\frac{x^4}{4}-\frac{x^6}{6}\right]_0^1 = 6\left(\frac{1}{4}-\frac{1}{6}\right) = \frac{6}{4}-1 = \frac{1}{2};$$

$$M_y = \int \tilde{x}\, dm = \int_0^1 x(x-x^2)\cdot\delta\, dx = \int_0^1 (x^2-x^3)\cdot 12x\, dx = 12\int_0^1 (x^3-x^4)\, dx = 12\left[\frac{x^4}{4}-\frac{x^5}{5}\right]_0^1 = 12\left(\frac{1}{4}-\frac{1}{5}\right)$$

$$= \frac{12}{20} = \frac{3}{5};\ M = \int dm = \int_0^1 (x-x^2)\cdot\delta\, dx = 12\int_0^1 (x^2-x^3)\, dx = 12\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1 = 12\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{12}{12} = 1.\ \text{So}$$

$$\bar{x} = \frac{M_y}{M} = \frac{3}{5} \text{ and } \bar{y} = \frac{M_x}{M} = \frac{1}{2} \Rightarrow \left(\frac{3}{5},\frac{1}{2}\right) \text{ is the center of mass.}$$

27. (a) We use the shell method:

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_1^4 2\pi x\left[\frac{4}{\sqrt{x}}-\left(-\frac{4}{\sqrt{x}}\right)\right] dx$$

$$= 16\pi\int_1^4 \frac{x}{\sqrt{x}}\, dx = 16\pi\int_1^4 x^{1/2}\, dx = 16\pi\left[\frac{2}{3}x^{3/2}\right]_1^4$$

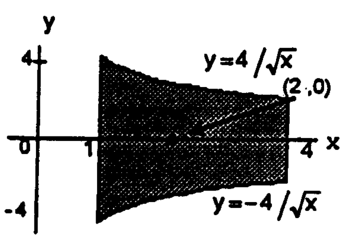
$$= 16\pi\left(\frac{2}{3}\cdot 8-\frac{2}{3}\right) = \frac{32\pi}{3}(8-1) = \frac{224\pi}{3}$$

(b) Since the plate is symmetric about the x-axis and its density $\delta(x) = \frac{1}{x}$ is a function of x alone, the distribution of its mass is symmetric about the x-axis. This means that $\bar{y} = 0$. We use the vertical strip approach to find $\bar{x}$: $M_y = \int \tilde{x}\, dm = \int_1^4 x\cdot\left[\frac{4}{\sqrt{x}}-\left(-\frac{4}{\sqrt{x}}\right)\right]\cdot\delta\, dx = \int_1^4 x\cdot\frac{8}{\sqrt{x}}\cdot\frac{1}{x}\, dx = 8\int_1^4 x^{-1/2}\, dx$

$$= 8\left[2x^{1/2}\right]_1^4 = 8(2\cdot 2-2) = 16;\ M = \int dm = \int_1^4 \left[\frac{4}{\sqrt{x}}-\left(\frac{-4}{\sqrt{x}}\right)\right]\cdot\delta\, dx = 8\int_1^4 \left(\frac{1}{\sqrt{x}}\right)\left(\frac{1}{x}\right) dx = 8\int_1^4 x^{-3/2}\, dx$$

$$= 8\left[-2x^{-1/2}\right]_1^4 = 8[-1-(-2)] = 8.\ \text{So } \bar{x} = \frac{M_y}{M} = \frac{16}{8} = 2 \Rightarrow (\bar{x},\bar{y}) = (2,0) \text{ is the center of mass.}$$

(c)



28. (a) We use the disk method:

$$V = \int_a^b \pi R^2(x)\,dx = \int_1^4 \pi\left(\frac{4}{x^2}\right)dx = 4\pi \int_1^4 x^{-2}\,dx$$

$$= 4\pi\left[-\frac{1}{x}\right]_1^4 = 4\pi\left[\frac{-1}{4} - (-1)\right] = \pi[-1+4] = 3\pi$$

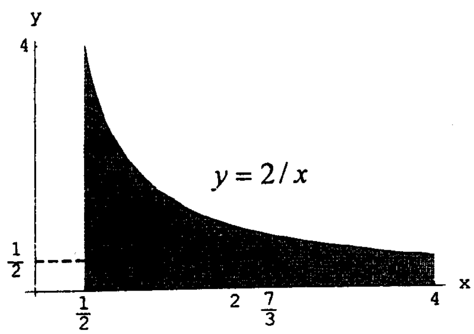
(b) We model the distribution of mass with vertical strips: $M_x = \int \tilde{y}\,dm = \int_1^4 \frac{\left(\frac{2}{x}\right)}{2}\cdot\left(\frac{2}{x}\right)\cdot\delta\,dx = \int_1^4 \frac{2}{x^2}\cdot\sqrt{x}\,dx$

$$= 2\int_1^4 x^{-3/2}\,dx = 2\left[\frac{-2}{\sqrt{x}}\right]_1^4 = 2[-1-(-2)] = 2;\ M_y = \int \tilde{x}\,dm = \int_1^4 x\cdot\frac{2}{x}\cdot\delta\,dx = 2\int_1^4 x^{1/2}\,dx$$

$$= 2\left[\frac{2x^{3/2}}{3}\right]_1^4 = 2\left[\frac{16}{3} - \frac{2}{3}\right] = \frac{28}{3};\ M = \int dm = \int_1^4 \frac{2}{x}\cdot\delta\,dx = 2\int_1^4 \frac{\sqrt{x}}{x}\,dx = 2\int_1^4 x^{-1/2}\,dx = 2\left[2x^{1/2}\right]_1^4$$

$= 2(4-2) = 4$. So $\overline{x} = \frac{M_y}{M} = \frac{\left(\frac{28}{3}\right)}{4} = \frac{7}{3}$ and $\overline{y} = \frac{M_x}{M} = \frac{2}{4} = \frac{1}{2} \Rightarrow (\overline{x}, \overline{y}) = \left(\frac{7}{3}, \frac{1}{2}\right)$ is the center of mass.

(c)



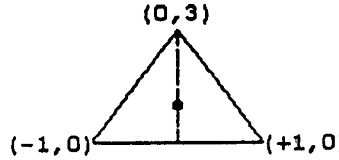
29. The mass of a horizontal strip is $dm = \delta\,dA = \delta L$, where L is the width of the triangle at a distance of y above its base on the x-axis as shown in the figure in the text. Also, by similar triangles we have $\frac{L}{b} = \frac{h-y}{h}$

$$\Rightarrow L = \frac{b}{h}(h-y).\ \text{Thus, } M_x = \int \tilde{y}\,dm = \int_0^h \delta y\left(\frac{b}{h}\right)(h-y)\,dy = \frac{\delta b}{h}\int_0^h (hy - y^2)\,dy = \frac{\delta b}{h}\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h$$

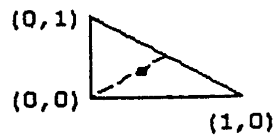
$$= \frac{\delta b}{h}\left(\frac{h^3}{2} - \frac{h^3}{3}\right) = \delta bh^2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\delta bh^2}{6};\ M = \int dm = \int_0^h \delta\left(\frac{b}{h}\right)(h-y)\,dy = \frac{\delta b}{h}\int_0^h (h-y)\,dy = \frac{\delta b}{h}\left[hy - \frac{y^2}{2}\right]_0^h$$

$= \frac{\delta b}{h}\left(h^2 - \frac{h^2}{2}\right) = \frac{\delta bh}{2}$. So $\overline{y} = \frac{M_x}{M} = \left(\frac{\delta bh^2}{6}\right)\left(\frac{2}{\delta bh}\right) = \frac{h}{3} \Rightarrow$ the center of mass lies above the base of the triangle one-third of the way toward the opposite vertex. Similarly the other two sides of the triangle can be placed on the x-axis and the same results will occur. Therefore the centroid does lie at the intersection of the medians, as claimed.

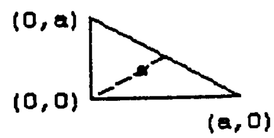
30. From the symmetry about the y-axis it follows that $\bar{x} = 0$. It also follows that the line through the points $(0,0)$ and $(0,3)$ is a median $\Rightarrow \bar{y} = \frac{1}{3}(3-0) = 1 \Rightarrow (\bar{x},\bar{y}) = (0,1)$.



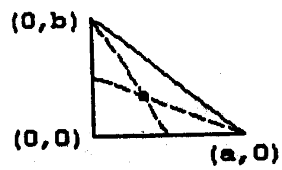
31. From the symmetry about the line $x = y$ it follows that $\bar{x} = \bar{y}$. It also follows that the line through the points $(0,0)$ and $\left(\frac{1}{2},\frac{1}{2}\right)$ is a median $\Rightarrow \bar{y} = \bar{x} = \frac{2}{3}\cdot\left(\frac{1}{2}-0\right) = \frac{1}{3}$ $\Rightarrow (\bar{x},\bar{y}) = \left(\frac{1}{3},\frac{1}{3}\right)$.



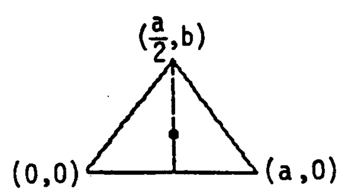
32. From the symmetry about the line $x = y$ it follows that $\bar{x} = \bar{y}$. It also follows that the line through the point $(0,0)$ and $\left(\frac{a}{2},\frac{a}{2}\right)$ is a median $\Rightarrow \bar{y} = \bar{x} = \frac{2}{3}\left(\frac{a}{2}-0\right) = \frac{1}{3}a$ $\Rightarrow (\bar{x},\bar{y}) = \left(\frac{a}{3},\frac{a}{3}\right)$.



33. The point of intersection of the median from the vertex $(0,b)$ to the opposite side has coordinates $\left(0,\frac{a}{2}\right) \Rightarrow \bar{y} = (b-0)\cdot\frac{1}{3}$ $= \frac{b}{3}$ and $\bar{x} = \left(\frac{a}{2}-0\right)\cdot\frac{2}{3} = \frac{a}{3} \Rightarrow (\bar{x},\bar{y}) = \left(\frac{a}{3},\frac{b}{3}\right)$.



34. From the symmetry about the line $x = \frac{a}{2}$ it follows that $\bar{x} = \frac{a}{2}$. It also follows that the line through the points $\left(\frac{a}{2},0\right)$ and $\left(\frac{a}{2},b\right)$ is a median $\Rightarrow \bar{y} = \frac{1}{3}(b-0) = \frac{b}{3}$ $\Rightarrow (\bar{x},\bar{y}) = \left(\frac{a}{2},\frac{b}{3}\right)$.



35. $y = x^{1/2} \Rightarrow dy = \frac{1}{2}x^{-1/2}\,dx \Rightarrow ds = \sqrt{(dx)^2+(dy)^2} = \sqrt{1+\frac{1}{4x}}\,dx$; $M_x = \delta\int_0^2 x\sqrt{1+\frac{1}{4x}}\,dx$

$$= \delta\int_0^2 x+\frac{1}{4}\,dx = \frac{2\delta}{3}\left[\left(x+\frac{1}{4}\right)^{3/2}\right]_0^2 = \frac{2\delta}{3}\left[\left(2+\frac{1}{4}\right)^{3/2}-\left(\frac{1}{4}\right)^{3/2}\right] = \frac{2\delta}{3}\left[\left(\frac{9}{4}\right)^{3/2}-\left(\frac{1}{4}\right)^{3/2}\right] = \frac{2\delta}{3}\left(\frac{27}{8}-\frac{1}{8}\right) = \frac{13\delta}{6}$$

36. $y = x^3 \Rightarrow dy = 3x^2\,dx \Rightarrow dx = \sqrt{(dx)^2+\left(3x^2\,dx\right)^2} = \sqrt{1+9x^4}\,dx$; $M_x = \delta\int_0^1 x^3\sqrt{1+9x^4}\,dx$;

$\left[u = 1+9x^4 \Rightarrow du = 36x^3\,dx \Rightarrow \frac{1}{36}\,du = x^3\,dx;\ x = 0 \Rightarrow u = 1,\ x = 1 \Rightarrow u = 10\right]$

$\rightarrow M_x = \delta \int_1^{10} \frac{1}{36} u^{1/2}\, du = \frac{\delta}{36}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{\delta}{54}\left(10^{3/2} - 1\right)$

37. From Example 6 we have $M_x = \int_0^{\pi} a(a \sin\theta)(k \sin\theta)\, d\theta = a^2k \int_0^{\pi} \sin^2\theta\, d\theta = \frac{a^2k}{2} \int_0^{\pi} (1 - \cos 2\theta)\, d\theta$

$= \frac{a^2k}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = \frac{a^2k\pi}{2}$; $M_y = \int_0^{\pi} a(a \cos\theta)(k \sin\theta)\, d\theta = a^2k \int_0^{\pi} \sin\theta \cos\theta\, d\theta = \frac{a^2k}{2}\left[\sin^2\theta\right]_0^{\pi} = 0$;

$M = \int_0^{\pi} ak \sin\theta\, d\theta = ak[-\cos\theta]_0^{\pi} = 2ak$. Therefore, $\overline{x} = \frac{M_y}{M} = 0$ and $\overline{y} = \frac{M_x}{M} = \left(\frac{a^2k\pi}{2}\right)\left(\frac{1}{2ak}\right) = \frac{a\pi}{4} \Rightarrow \left(0, \frac{a\pi}{4}\right)$

is the center of mass.

38. $M_x = \int \tilde{y}\, dm = \int_0^{\pi} (a \sin\theta) \cdot \delta \cdot a\, d\theta = \int_0^{\pi} \left(a^2 \sin\theta\right)\left(1 + k|\cos\theta|\right) d\theta$

$= a^2 \int_0^{\pi/2} (\sin\theta)(1 + k \cos\theta)\, d\theta + a^2 \int_{\pi/2}^{\pi} (\sin\theta)(1 - k \cos\theta)\, d\theta$

$= a^2 \int_0^{\pi/2} \sin\theta\, d\theta + a^2k \int_0^{\pi/2} \sin\theta \cos\theta\, d\theta + a^2 \int_{\pi/2}^{\pi} \sin\theta\, d\theta - a^2k \int_{\pi/2}^{\pi} \sin\theta \cos\theta\, d\theta$

$= a^2[-\cos\theta]_0^{\pi/2} + a^2k\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} + a^2[-\cos\theta]_{\pi/2}^{\pi} - a^2k\left[\frac{\sin^2\theta}{2}\right]_{\pi/2}^{\pi}$

$= a^2[0 - (-1)] + a^2k\left(\frac{1}{2} - 0\right)^{\pi/2} + a^2[-(-1) - 0] - a^2k\left(0 - \frac{1}{2}\right) = a^2 + \frac{a^2k}{2} + a^2 + \frac{a^2k}{2} = 2a^2 + a^2k = a^2(2 + k)$;

$M_y = \int \tilde{x}\, dm = \int_0^{\pi} (a \cos\theta) \cdot \delta \cdot a\, d\theta = \int_0^{\pi} \left(a^2 \cos\theta\right)\left(1 + k|\cos\theta|\right) d\theta$

$= a^2 \int_0^{\pi/2} (\cos\theta)(1 + k \cos\theta)\, d\theta + a^2 \int_{\pi/2}^{\pi} (\cos\theta)(1 - k \cos\theta)\, d\theta$

$= a^2 \int_0^{\pi/2} \cos\theta\, d\theta + a^2k \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta + a^2 \int_{\pi/2}^{\pi} \cos\theta\, d\theta - a^2k \int_{\pi/2}^{\pi} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$

$= a^2[\sin\theta]_0^{\pi/2} + \frac{a^2k}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} + a^2[\sin\theta]_{\pi/2}^{\pi} - \frac{a^2k}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/2}^{\pi}$

$= a^2(1 - 0) + \frac{a^2k}{2}\left[\left(\frac{\pi}{2} - 0\right) - (0 + 0)\right] + a^2(0 - 1) - \frac{a^2k}{2}\left[(\pi + 0) - \left(\frac{\pi}{2} + 0\right)\right] = a^2 + \frac{a^2k\pi}{4} - a^2 - \frac{a^2k\pi}{4} = 0$;

$M = \int_0^{\pi} \delta \cdot a\, d\theta = a \int_0^{\pi} (1 + k\,|\cos\theta|\, d\theta = a \int_0^{\pi/2} (1 + k \cos\theta)\, d\theta + a \int_{\pi/2}^{\pi} (1 - k \cos\theta)\, d\theta$

$= a[\theta + k\sin\theta]_0^{\pi/2} + a[\theta - k\sin\theta]_{\pi/2}^{\pi} = a\left[\left(\frac{\pi}{2}+k\right)-0\right] + a\left[(\pi+0)-\left(\frac{\pi}{2}-k\right)\right]$

$= \frac{a\pi}{2} + ak + a\left(\frac{\pi}{2}+k\right) = a\pi + 2ak = a(\pi+2k)$. So $\bar{x} = \frac{M_y}{M} = 0$ and $\bar{y} = \frac{M_x}{M} = \frac{a^2(2+k)}{a(\pi+2k)} = \frac{a(2+k)}{\pi+2k}$

$= \left(0, \frac{2a+ka}{\pi+2k}\right)$ is the center of mass.

39. Consider the curve as an infinite number of line segments joined together. From the derivation of arc length we have that the length of a particular segment is $ds = \sqrt{(dx)^2+(dy)^2}$. This implies that $M_x = \int \delta y\,ds$, $M_y = \int \delta x\,ds$ and $M = \int \delta\,ds$. If δ is constant, then $\bar{x} = \frac{M_y}{M} = \frac{\int x\,ds}{\int ds} = \frac{\int x\,ds}{\text{length}}$ and $\bar{y} = \frac{M_x}{M} = \frac{\int y\,ds}{\int ds} = \frac{\int y\,ds}{\text{length}}$.

40. Applying the symmetry argument analogous to the one used in Exercise 13, we find that $\bar{x} = 0$. The typical vertical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{a+\frac{x^2}{4p}}{2}\right)$, length: $a - \frac{x^2}{4p}$, width: dx, area: $dA = \left(a - \frac{x^2}{4p}\right)dx$, mass: $dm = \delta\,dA = \delta\left(a - \frac{x^2}{4p}\right)dx$. Thus, $M_x = \int \tilde{y}\,dm = \int_{-2\sqrt{pa}}^{2\sqrt{pa}} \frac{1}{2}\left(a+\frac{x^2}{4p}\right)\left(a-\frac{x^2}{4p}\right)\delta\,dx$

$= \frac{\delta}{2}\int_{-2\sqrt{pa}}^{2\sqrt{pa}} \left(a^2 - \frac{x^4}{16p^2}\right)dx = \frac{\delta}{2}\left[a^2x - \frac{x^5}{80p^2}\right]_{-2\sqrt{pa}}^{2\sqrt{pa}} = 2\cdot\frac{\delta}{2}\left[a^2x - \frac{x^5}{80p^2}\right]_0^{2\sqrt{pa}} = \delta\left(2a^2\sqrt{pa} - \frac{2^5p^2a^2\sqrt{pa}}{80p^2}\right)$

$= 2a^2\delta\sqrt{pa}\left(1 - \frac{16}{80}\right) = 2a^2\delta\sqrt{pa}\left(\frac{80-16}{80}\right) = 2a^2\delta\sqrt{pa}\left(\frac{64}{80}\right) = \frac{8a^2\delta\sqrt{pa}}{5}$; $M = \int dm = \delta\int_{-2\sqrt{pa}}^{2\sqrt{pa}}\left(a - \frac{x^2}{4p}\right)dx$

$= \delta\left[ax - \frac{x^3}{12p}\right]_{-2\sqrt{pa}}^{2\sqrt{pa}} = 2\cdot\delta\left[ax - \frac{x^3}{12p}\right]_0^{2\sqrt{pa}} = 2\delta\left(2a\sqrt{pa} - \frac{2^3pa\sqrt{pa}}{12p}\right) = 4a\delta\sqrt{pa}\left(1 - \frac{4}{12}\right) = 4a\delta\sqrt{pa}\left(\frac{12-4}{12}\right)$

$= \frac{8a\delta\sqrt{pa}}{3}$. So $\bar{y} = \frac{M_x}{M} = \left(\frac{8a^2\delta\sqrt{pa}}{5}\right)\left(\frac{3}{8a\delta\sqrt{pa}}\right) = \frac{3}{5}a$, as claimed.

41. A generalization of Example 6 yields $M_x = \int \tilde{y}\,dm = \int_{\pi/2-\alpha}^{\pi/2+\alpha} a^2\sin\theta\,d\theta = a^2[-\cos\theta]_{\pi/2-\alpha}^{\pi/2+\alpha}$

$= a^2\left[-\cos\left(\frac{\pi}{2}+\alpha\right) + \cos\left(\frac{\pi}{2}-\alpha\right)\right] = a^2(\sin\alpha + \sin\alpha) = 2a^2\sin\alpha$; $M = \int dm = \int_{\pi/2-\alpha}^{\pi/2+\alpha} a\,d\theta = a[\theta]_{\pi/2-\alpha}^{\pi/2+\alpha}$

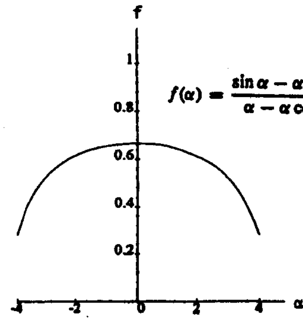
$= a\left[\left(\frac{\pi}{2}+\alpha\right)-\left(\frac{\pi}{2}-\alpha\right)\right] = 2a\alpha$. Thus, $\bar{y} = \frac{M_x}{M} = \frac{2a^2\sin\alpha}{2a\alpha} = \frac{a\sin\alpha}{\alpha}$. Now $s = a(2\alpha)$ and $a\sin\alpha = \frac{c}{2}$

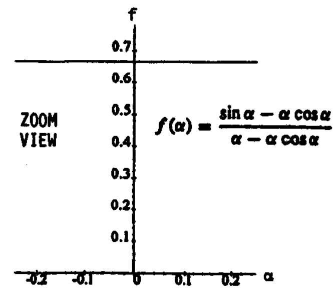
$\Rightarrow c = 2a \sin \alpha$. Then $\bar{y} = \frac{a(2a \sin \alpha)}{2a\alpha} = \frac{ac}{s}$, as claimed.

42. (a) First, we note that $\bar{y} = (\text{distance from origin to } \overline{AB}) + d \Rightarrow \frac{a \sin \alpha}{\alpha} = a \cos \alpha + d \Rightarrow d = \frac{a(\sin \alpha - \alpha \cos \alpha)}{\alpha}$.

Moreover, $h = a - a \cos \alpha \Rightarrow \frac{d}{h} = \frac{a(\sin \alpha - \alpha \cos \alpha)}{a(\alpha - \alpha \cos \alpha)} = \frac{\sin \alpha - \alpha \cos \alpha}{\alpha - \alpha \cos \alpha}$. The graphs below suggest that

$\lim_{\alpha \to 0^+} \frac{\sin \alpha - \alpha \cos \alpha}{\alpha - \alpha \cos \alpha} \approx \frac{2}{3}$.





(b) Equation (9): $\frac{d}{h} = \frac{\sin \alpha - \alpha \cos \alpha}{\alpha - \alpha \cos \alpha}$

α	0.2	0.4	0.6	0.8	1.0
$f(\alpha)$	0.666222	0.664879	0.662615	0.659389	0.655145

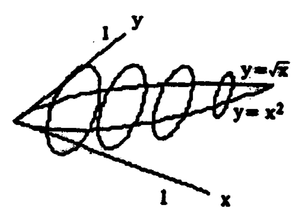
CHAPTER 5 PRACTICE EXERCISES

1. $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(\sqrt{x} - x^2\right)^2 = \frac{\pi}{4}\left(x - 2\sqrt{x} \cdot x^2 + x^4\right)$;

$a = 0, b = 1 \Rightarrow V = \int_a^b A(x)\,dx = \frac{\pi}{4}\int_0^1 \left(x - 2x^{5/2} + x^4\right) dx$

$= \frac{\pi}{4}\left[\frac{x^2}{2} - \frac{4}{7}x^{7/2} + \frac{x^5}{5}\right]_0^1 = \frac{\pi}{4}\left(\frac{1}{2} - \frac{4}{7} + \frac{1}{5}\right) = \frac{\pi}{4 \cdot 70}(35 - 40 + 14)$

$= \frac{9\pi}{280}$

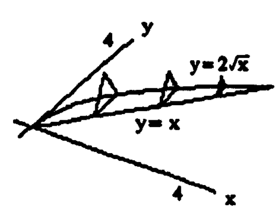


2. $A(x) = \frac{1}{2}(\text{side})^2\left(\sin \frac{\pi}{3}\right) = \frac{\sqrt{3}}{4}(2\sqrt{x} - x)^2 = \frac{\sqrt{3}}{4}\left(4x - 4x\sqrt{x} + x^2\right)$;

$a = 0, b = 4 \Rightarrow V = \int_a^b A(x)\,dx = \frac{\sqrt{3}}{4}\int_0^4 \left(4x - 4x^{3/2} + x^2\right) dx$

$= \frac{\sqrt{3}}{4}\left[2x^2 - \frac{8}{5}x^{5/2} + \frac{x^3}{3}\right]_0^4 = \frac{\sqrt{3}}{4}\left(32 - \frac{8 \cdot 32}{5} + \frac{64}{3}\right)$

$= \frac{32\sqrt{3}}{4}\left(1 - \frac{8}{5} + \frac{2}{3}\right) = \frac{8\sqrt{3}}{15}(15 - 24 + 10) = \frac{8\sqrt{3}}{15}$

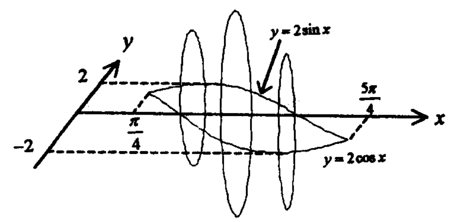


3. $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}(2 \sin x - 2 \cos x)^2$

$= \frac{\pi}{4} \cdot 4\left(\sin^2 x - 2 \sin x \cos x + \cos^2 x\right) = \pi(1 - \sin 2x);\ a = \frac{\pi}{4},$

$b = \frac{5\pi}{4} \Rightarrow V = \int_a^b A(x)\,dx = \pi \int_{\pi/4}^{5\pi/4} (1 - \sin 2x)\,dx$

$= \pi\left[x + \frac{\cos 2x}{2}\right]_{\pi/4}^{5\pi/4} = \pi\left[\left(\frac{5\pi}{4} + \frac{\cos\frac{5\pi}{2}}{2}\right) - \left(\frac{\pi}{4} - \frac{\cos\frac{\pi}{2}}{2}\right)\right] = \pi^2$



4. $A(x) = (\text{edge})^2 = \left(\left(\sqrt{6} - \sqrt{x}\right)^2 - 0\right)^2 = \left(\sqrt{6} - \sqrt{x}\right)^4 = 36 - 24\sqrt{6}\sqrt{x} + 36x - 4\sqrt{6}\,x^{3/2} + x^2;$

$a = 0,\ b = 6 \Rightarrow V = \int_a^b A(x)\,dx = \int_0^6 \left(36 - 24\sqrt{6}\sqrt{x} + 36x - 4\sqrt{6}\,x^{3/2} + x^2\right)dx$

$= \left[36x - 24\sqrt{6} \cdot \frac{2}{3}x^{3/2} + 18x^2 - 4\sqrt{6} \cdot \frac{2}{5}x^{5/2} + \frac{x^3}{3}\right]_0^6 = 216 - 16 \cdot \sqrt{6}\sqrt{6} \cdot 6 + 18 \cdot 6^2 - \frac{8}{5}\sqrt{6}\sqrt{6} \cdot 6^2 + \frac{6^3}{3}$

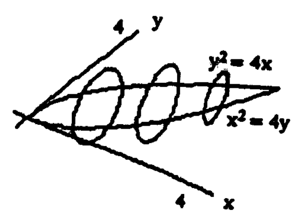
$= 216 - 576 + 648 - \frac{1728}{5} + 72 = 360 - \frac{1728}{5} = \frac{1800 - 1728}{5} = \frac{72}{5}$

5. $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(2\sqrt{x} - \frac{x^2}{4}\right)^2 = \frac{\pi}{4}\left(4x - x^{5/2} + \frac{x^4}{16}\right);$

$a = 0,\ b = 4 \Rightarrow V = \int_a^b A(x)\,dx = \frac{\pi}{4}\int_0^4 \left(4x - x^{5/2} + \frac{x^4}{16}\right)dx$

$= \frac{\pi}{4}\left[2x^2 - \frac{2}{7}x^{7/2} + \frac{x^5}{5 \cdot 16}\right]_0^4 = \frac{\pi}{4}\left(32 - 32 \cdot \frac{8}{7} + \frac{2}{5} \cdot 32\right)$

$= \frac{32\pi}{4}\left(1 - \frac{8}{7} + \frac{2}{5}\right) = \frac{8\pi}{35}(35 - 40 + 14) = \frac{72\pi}{35}$

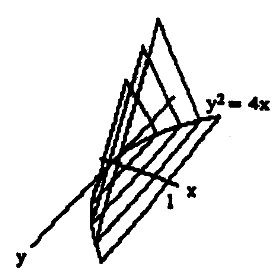


6. $A(x) = \frac{1}{2}(\text{edge})^2 \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{4}\left[2\sqrt{x} - \left(-2\sqrt{x}\right)\right]^2$

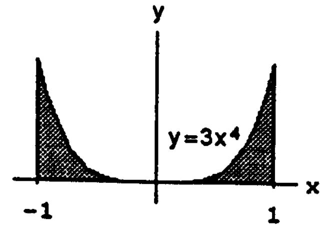
$= \frac{\sqrt{3}}{4}\left(4\sqrt{x}\right)^2 = 4\sqrt{3}\,x;\ a = 0,\ b = 1$

$\Rightarrow V = \int_a^b A(x)\,dx = \int_0^1 4\sqrt{3}\,x\,dx = \left[2\sqrt{3}\,x^2\right]_0^1$

$= 2\sqrt{3}$



7. (a) *disk method*:



$$V = \int_a^b \pi R^2(x)\,dx = \int_{-1}^{1} \pi\left(3x^4\right)^2 dx = \pi \int_{-1}^{1} 9x^8\,dx$$

$$= \pi\left[x^9\right]_{-1}^{1} = 2\pi$$

(b) *shell method*:

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^1 2\pi x\left(3x^4\right) dx = 2\pi\cdot 3 \int_0^1 x^5\,dx = 2\pi\cdot 3\left[\frac{x^6}{6}\right]_0^1 = \pi$$

Note: The lower limit of integration is 0 rather than -1.

(c) *shell method*:

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = 2\pi \int_{-1}^{1} (1-x)\left(3x^4\right) dx = 2\pi\left[\frac{3x^5}{5} - \frac{x^6}{2}\right]_{-1}^{1} = 2\pi\left[\left(\frac{3}{5}-\frac{1}{2}\right)-\left(-\frac{3}{5}-\frac{1}{2}\right)\right] = \frac{12\pi}{5}$$

(d) *washer method*:

$$R(x) = 3,\ r(x) = 3 - 3x^4 = 3\left(1-x^4\right) \Rightarrow V = \int_a^b \pi\left[R^2(x) - r^2(x)\right] dx = \int_{-1}^{1} \pi\left[9 - 9\left(1-x^4\right)^2\right] dx$$

$$= 9\pi \int_{-1}^{1} \left[1 - \left(1 - 2x^4 + x^8\right)\right] dx = 9\pi \int_{-1}^{1} \left(2x^4 - x^8\right) dx = 9\pi\left[\frac{2x^5}{5} - \frac{x^9}{9}\right]_{-1}^{1} = 18\pi\left[\frac{2}{5}-\frac{1}{9}\right] = \frac{2\pi\cdot 13}{5} = \frac{26\pi}{5}$$

8. (a) *washer method*:

$$R(x) = \frac{4}{x^3},\ r(x) = \frac{1}{2} \Rightarrow V = \int_a^b \left[R^2(x) - r^2(x)\right] dx = \int_1^2 \pi\left[\left(\frac{4}{x^3}\right)^2 - \left(\frac{1}{2}\right)^2\right] dx = \pi\left[-\frac{16}{5}x^{-5} - \frac{x}{4}\right]_1^2$$

$$= \pi\left[\left(\frac{-16}{5\cdot 32}-\frac{1}{2}\right)-\left(-\frac{16}{5}-\frac{1}{4}\right)\right] = \pi\left(-\frac{1}{10}-\frac{1}{2}+\frac{16}{5}+\frac{1}{4}\right) = \frac{\pi}{20}(-2-10+64+5) = \frac{57\pi}{20}$$

(b) *shell method*:

$$V = 2\pi \int_1^2 x\left(\frac{4}{x^3}-\frac{1}{2}\right) dx = 2\pi\left[-4x^{-1} - \frac{x^2}{4}\right]_1^2 = 2\pi\left[\left(-\frac{4}{2}-1\right)-\left(-4-\frac{1}{4}\right)\right] = 2\pi\left(\frac{5}{4}\right) = \frac{5\pi}{2}$$

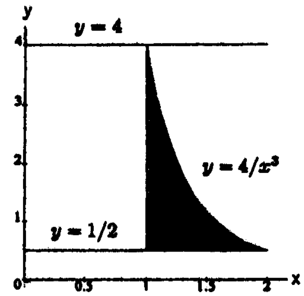
(c) *shell method*:

$$V = 2\pi \int_a^b \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = 2\pi \int_1^2 (2-x)\left(\frac{4}{x^3}-\frac{1}{2}\right) dx = 2\pi \int_1^2 \left(\frac{8}{x^3}-\frac{4}{x^2}-1+\frac{x}{2}\right) dx$$

$$= 2\pi\left[-\frac{4}{x^2}+\frac{4}{x}-x+\frac{x^2}{4}\right]_1^2 = 2\pi\left[(-1+2-2+1)-\left(-4+4-1+\frac{1}{4}\right)\right] = \frac{3\pi}{2}$$

(d) *disk method*:

$$V = \int_a^b \left[R^2(x) - r^2(x)\right] dx = \pi \int_1^2 \left[\left(\frac{7}{2}\right)^2 - \left(4-\frac{4}{x^3}\right)^2\right] dx = \frac{49\pi}{4} - 16\pi \int_1^2 \left(1 - 2x^{-3} + x^{-6}\right) dx$$



$$= \frac{49\pi}{4} - 16\pi\left[x + x^{-2} - \frac{x^{-5}}{5}\right]_1^2 = \frac{49\pi}{4} - 16\pi\left[\left(2+\frac{1}{4}-\frac{1}{5\cdot 32}\right)-\left(1+1-\frac{1}{5}\right)\right]$$

$$= \frac{49\pi}{4} - 16\pi\left(\frac{1}{4}-\frac{1}{160}+\frac{1}{5}\right) = \frac{49\pi}{4} - \frac{16\pi}{160}(40-1+32) = \frac{49\pi}{4} - \frac{71\pi}{10} = \frac{103\pi}{20}$$

9. (a) *disk method*:

$$V = \pi\int_1^5 \left(\sqrt{x-1}\right)^2 dx = \int_1^5 (x-1)\,dx = \pi\left[\frac{x^2}{2}-x\right]_1^5$$

$$= \pi\left[\left(\frac{25}{2}-5\right)-\left(\frac{1}{2}-1\right)\right] = \pi\left(\frac{24}{2}-4\right) = 8\pi$$

(b) *washer method*:

$$R(y) = 5,\ r(y) = y^2+1 \Rightarrow V = \int_c^d \pi\left[R^2(y)-r^2(y)\right]dy = \pi\int_{-2}^{2}\left[25-\left(y^2+1\right)^2\right]dy$$

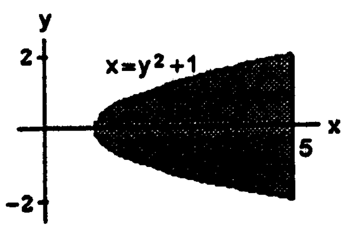
$$= \pi\int_{-2}^{2}\left(25-y^4-2y^2-1\right)dy = \pi\int_{-2}^{2}\left(24-y^4-2y^2\right)dy = \pi\left[24y-\frac{y^5}{5}-\frac{2}{3}y^3\right]_{-2}^{2} = 2\pi\left(24\cdot 2-\frac{32}{5}-\frac{2}{3}\cdot 8\right)$$

$$= 32\pi\left(3-\frac{2}{5}-\frac{1}{3}\right) = \frac{32\pi}{15}(45-6-5) = \frac{1088\pi}{15}$$

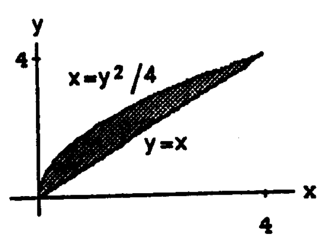
(c) *disk method*:

$$R(y) = 5-\left(y^2+1\right) = 4-y^2 \Rightarrow V = \int_c^d \pi R^2(y)\,dy = \int_{-2}^{2}\pi\left(4-y^2\right)^2 dy = \pi\int_{-2}^{2}\left(16-8y^2+y^4\right)dy$$

$$= \pi\left[16y-\frac{8y^3}{3}+\frac{y^5}{5}\right]_{-2}^{2} = 2\pi\left(32-\frac{64}{3}+\frac{32}{5}\right) = 64\pi\left(1-\frac{2}{3}+\frac{1}{5}\right) = \frac{64\pi}{15}(15-10+3) = \frac{512\pi}{15}$$



10. (a) *shell method*:



$$V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}dy = \int_0^4 2\pi y\left(y-\frac{y^2}{4}\right)dy$$

$$= 2\pi\int_0^4\left(y^2-\frac{y^3}{4}\right)dy = 2\pi\left[\frac{y^3}{3}-\frac{y^4}{16}\right]_0^4 = 2\pi\left(\frac{64}{3}-\frac{64}{4}\right)$$

$= \frac{2\pi}{12} \cdot 64 = \frac{32\pi}{3}$

(b) *shell method*:

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^4 2\pi x(2\sqrt{x} - x)\, dx = 2\pi \int_0^4 (2x^{3/2} - x^2)\, dx = 2\pi\left[\frac{4}{5}x^{5/2} - \frac{x^3}{3}\right]_0^4$$

$$= 2\pi\left(\frac{4}{5} \cdot 32 - \frac{64}{3}\right) = \frac{128\pi}{15}$$

(c) *shell method*:

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_0^4 2\pi(4-x)(2\sqrt{x} - x)\, dx = 2\pi \int_0^4 (8x^{1/2} - 4x - 2x^{3/2} + x^2)\, dx$$

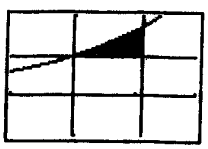
$$= 2\pi\left[\frac{16}{3}x^{3/2} - 2x^2 - \frac{4}{5}x^{5/2} + \frac{x^3}{3}\right]_0^4 = 2\pi\left(\frac{16}{3} \cdot 8 - 32 - \frac{4}{5} \cdot 32 + \frac{64}{3}\right) = 64\pi\left(\frac{4}{3} - 1 - \frac{4}{5} + \frac{2}{3}\right) = 64\pi\left(1 - \frac{4}{5}\right) = \frac{64\pi}{5}$$

(d) *shell method*:

$$V = \int_c^d 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^4 2\pi(4-y)\left(y - \frac{y^2}{4}\right) dy = 2\pi \int_0^4 \left(4y - y^2 - y^2 + \frac{y^3}{4}\right) dy$$

$$= 2\pi \int_0^4 \left(4y - 2y^2 + \frac{y^3}{4}\right) dy = 2\pi\left[2y^2 - \frac{2}{3}y^3 + \frac{y^4}{16}\right]_0^4 = 2\pi\left(32 - \frac{2}{3} \cdot 64 + 16\right) = 32\pi\left(2 - \frac{8}{3} + 1\right) = \frac{32\pi}{3}$$

11.



[−1, 2] by [−1, 2]

Use washer cross sections. A washer has inner radius $r = 1$, outer radius $R = e^{x/2}$, and area $\pi(R^2 - r^2)$ $= \pi(e^x - 1)$. The volume is $\int_0^{\ln 3} \pi(e^x - 1)\, dx = \pi[e^x - x]_0^{\ln 3} = \pi(3 - \ln 3 - 1) = \pi(2 - \ln 3)$.

12. *disk method*:

$$V = \pi \int_0^\pi (2 - \sin x)^2\, dx = \pi \int_0^\pi (4 - 4\sin x + \sin^2 x)\, dx = \pi \int_0^\pi \left(4 - 4\sin x + \frac{1 - \cos 2x}{2}\right) dx$$

$$= \pi\left[4x + 4\cos x + \frac{x}{2} - \frac{\sin 2x}{4}\right]_0^\pi = \pi\left[\left(4\pi - 4 + \frac{\pi}{2} - 0\right) - (0 + 4 + 0 - 0)\right] = \pi\left(\frac{9\pi}{2} - 8\right) = \frac{\pi}{2}(9\pi - 16)$$

13. (a) *disk method*:

$$V = \pi \int_0^2 (x^2 - 2x)^2\, dx = \pi \int_0^2 (x^4 - 4x^3 + 4x^2)\, dx = \pi\left[\frac{x^5}{5} - x^4 + \frac{4}{3}x^3\right]_0^2 = \pi\left(\frac{32}{5} - 16 + \frac{32}{3}\right)$$

$$= \frac{16\pi}{15}(6 - 15 + 10) = \frac{16\pi}{15}$$

(b) *disk method*:

$$V = 2\pi - \pi \int_0^2 \left[1 + (x^2 - 2x)\right]^2 dx = 2\pi - \pi \int_0^2 \left[1 + 2(x^2 - 2x) + (x^2 - 2x)^2\right] dx$$

$$= 2\pi - \pi \int_0^2 (1 + 2x^2 - 4x + x^4 - 4x^3 + 4x^2)\, dx = 2\pi - \pi \int_0^2 (x^4 - 4x^3 + 6x^2 - 4x + 1)\, dx$$

$$= 2\pi - \pi\left[\frac{x^5}{5} - x^4 + 2x^3 - 2x^2 + x\right]_0^2 = 2\pi - \pi\left(\frac{32}{5} - 16 + 16 - 8 + 2\right) = 2\pi - \frac{\pi}{5}(32 - 30) = 2\pi - \frac{2\pi}{5} = \frac{8\pi}{5}$$

(c) *shell method*:

$$V = \int_a^b 2\pi\left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dx = 2\pi \int_0^2 (2 - x)\left[-(x^2 - 2x)\right] dx = 2\pi \int_0^2 (2 - x)(2x - x^2)\, dx$$

$$= 2\pi \int_0^2 (4x - 2x^2 - 2x^2 + x^3)\, dx = 2\pi \int_0^2 (x^3 - 4x^2 + 4x)\, dx = 2\pi\left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2\right]_0^2 = 2\pi\left(4 - \frac{32}{3} + 8\right)$$

$$= \frac{2\pi}{3}(36 - 32) = \frac{8\pi}{3}$$

(d) *disk method*:

$$V = \pi \int_0^2 \left[2 - (x^2 - 2x)\right]^2 dx - \pi \int_0^2 2^2\, dx = \pi \int_0^2 \left[4 - 4(x^2 - 2x) + (x^2 - 2x)^2\right] dx - 8\pi$$

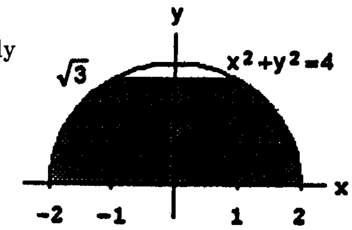
$$= \pi \int_0^2 (4 - 4x^2 + 8x + x^4 - 4x^3 + 4x^2)\, dx - 8\pi = \pi \int_0^2 (x^4 - 4x^3 + 8x + 4)\, dx - 8\pi$$

$$= \pi\left[\frac{x^5}{5} - x^4 + 4x^2 + 4x\right]_0^2 - 8\pi = \pi\left(\frac{32}{5} - 16 + 16 + 8\right) - 8\pi = \frac{\pi}{5}(32 + 40) - 8\pi = \frac{72\pi}{5} - \frac{40\pi}{5} = \frac{32\pi}{5}$$

14. *disk method*:

$$V = 2\pi \int_0^{\pi/4} 4\tan^2 x\, dx = 8\pi \int_0^{\pi/4} (\sec^2 x - 1)\, dx = 8\pi[\tan x - x]_0^{\pi/4} = 2\pi(4 - \pi)$$

15. The volume cut out is equivalent to the volume of the solid generated by revolving the region shown here about the x-axis. Using the *shell* method:



$$V = \int_c^d 2\pi\begin{pmatrix}\text{shell}\\\text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\\text{height}\end{pmatrix} dy = \int_0^{\sqrt{3}} 2\pi y\left[\sqrt{4-y^2} - \left(-\sqrt{4-y^2}\right)\right] dy$$

$$= 2\pi \int_0^{\sqrt{3}} 2y\sqrt{4-y^2}\, dy = -2\pi \int_0^{\sqrt{3}} \sqrt{4-y^2}\, d(4-y^2)$$

$$= (-2\pi)\left(\tfrac{2}{3}\right)\left[(4-y^2)^{3/2}\right]_0^{\sqrt{3}} = -\tfrac{4\pi}{3}(1-8) = \tfrac{28\pi}{3}$$

16. We rotate the region enclosed by the curve $y = \sqrt{12\left(1 - \frac{4x^2}{121}\right)}$ and the x-axis around the x-axis. To find the volume we use the *disk* method: $V = \int_a^b \pi R^2(x)\, dx = \int_{-11/2}^{11/2} \pi\left(\sqrt{12\left(1-\frac{4x^2}{121}\right)}\right)^2 dx = \pi \int_{-11/2}^{11/2} 12\left(1-\frac{4x^2}{121}\right) dx$

$$= 12\pi \int_{-11/2}^{11/2} \left(1 - \tfrac{4x^2}{121}\right) dx = 12\pi\left[x - \tfrac{4x^3}{363}\right]_{-11/2}^{11/2} = 24\pi\left[\tfrac{11}{2} - \left(\tfrac{4}{363}\right)\left(\tfrac{11}{2}\right)^3\right] = 132\pi\left[1 - \left(\tfrac{4}{363}\right)\left(\tfrac{11^2}{4}\right)\right]$$

$$= 132\pi\left(1 - \tfrac{1}{3}\right) = \tfrac{264\pi}{3} = 88\pi \approx 276 \text{ in}^3$$

17. $y = x^{1/2} - \frac{x^{3/2}}{3} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(\frac{1}{x} - 2 + x\right) \Rightarrow L = \int_1^4 \sqrt{1 + \frac{1}{4}\left(\frac{1}{x} - 2 + x\right)}\, dx$

$$\Rightarrow L = \int_1^4 \sqrt{\tfrac{1}{4}\left(\tfrac{1}{x} + 2 + x\right)}\, dx = \int_1^4 \sqrt{\tfrac{1}{4}\left(x^{-1/2} + x^{1/2}\right)^2}\, dx = \int_1^4 \tfrac{1}{2}\left(x^{-1/2} + x^{1/2}\right) dx = \tfrac{1}{2}\left[2x^{1/2} + \tfrac{2}{3}x^{3/2}\right]_1^4$$

$$= \tfrac{1}{2}\left[\left(4 + \tfrac{2}{3}\cdot 8\right) - \left(2 + \tfrac{2}{3}\right)\right] = \tfrac{1}{2}\left(2 + \tfrac{14}{3}\right) = \tfrac{10}{3}$$

18. $x = y^{2/3} \Rightarrow \frac{dx}{dy} = \frac{2}{3}x^{-1/3} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{4x^{-2/3}}{9} \Rightarrow L = \int_1^8 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}}\, dy$

$$= \int_1^8 \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\, dx = \frac{1}{3}\int_1^8 \sqrt{9x^{2/3}+4}\,\left(x^{-1/3}\right) dx;\ \left[u = 9x^{2/3} + 4 \Rightarrow du = 6y^{-1/3}\, dy;\ x = 1 \Rightarrow u = 13,\right.$$

$$\left.x = 8 \Rightarrow u = 40\right] \to L = \frac{1}{18}\int_{13}^{40} u^{1/2}\, du = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{13}^{40} = \frac{1}{27}\left[40^{3/2} - 13^{3/2}\right] \approx 7.634$$

19. $y = \frac{5}{12}x^{6/5} - \frac{5}{8}x^{4/5} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{1/5} - \frac{1}{2}x^{-1/5} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(x^{2/5} - 2 + x^{-2/5}\right)$

$\Rightarrow L = \int_1^{32} \sqrt{1 + \frac{1}{4}\left(x^{2/5} - 2 + x^{-2/5}\right)}\, dx \Rightarrow L = \int_1^{32} \sqrt{\frac{1}{4}\left(x^{2/5} + 2 + x^{-2/5}\right)}\, dx = \int_1^{32} \sqrt{\frac{1}{4}\left(x^{1/5} + x^{-1/5}\right)^2}\, dx$

$= \int_1^{32} \frac{1}{2}\left(x^{1/5} + x^{-1/5}\right) dx = \frac{1}{2}\left[\frac{5}{6}x^{6/5} + \frac{5}{4}x^{4/5}\right]_1^{32} = \frac{1}{2}\left[\left(\frac{5}{6}\cdot 2^6 + \frac{5}{4}\cdot 2^4\right) - \left(\frac{5}{6} + \frac{5}{4}\right)\right] = \frac{1}{2}\left(\frac{315}{6} + \frac{75}{4}\right)$

$= \frac{1}{48}(1260 + 450) = \frac{1710}{48} = \frac{285}{8}$

20. $x = \frac{1}{12}y^3 + \frac{1}{y} \Rightarrow \frac{dx}{dy} = \frac{1}{4}y^2 - \frac{1}{y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4} \Rightarrow L = \int_1^2 \sqrt{1 + \left(\frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4}\right)}\, dy$

$= \int_1^2 \sqrt{\frac{1}{16}y^4 + \frac{1}{2} + \frac{1}{y^4}}\, dy = \int_1^2 \sqrt{\left(\frac{1}{4}y^2 + \frac{1}{y^2}\right)^2}\, dy = \int_1^2 \left(\frac{1}{4}y^2 + \frac{1}{y^2}\right) dy = \left[\frac{1}{12}y^3 - \frac{1}{y}\right]_1^2$

$= \left(\frac{8}{12} - \frac{1}{2}\right) - \left(\frac{1}{12} - 1\right) = \frac{7}{12} + \frac{1}{2} = \frac{13}{12}$

21. $\frac{dx}{dt} = -5 \sin t + 5 \sin 5t$ and $\frac{dy}{dt} = 5 \cos t - 5 \cos 5t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{(5 \sin 5t - 5 \sin t)^2 + (5 \cos t - 5 \cos 5t)^2}$

$= 5\sqrt{\sin^2 5t - 2 \sin t \sin 5t + \sin^2 t + \cos^2 t - 2 \cos t \cos 5t + \cos^2 5t} = 5\sqrt{2 - 2(\sin t \sin 5t + \cos t \cos 5t)}$

$= 5\sqrt{2(1 - \cos 4t)} = 5\sqrt{4\left(\frac{1}{2}\right)(1 - \cos 4t)} = 10\sqrt{\sin^2 2t} = 10|\sin 2t| = 10 \sin 2t$ (since $0 \le t \le \pi/2$)

$\Rightarrow \text{Length} = \int_0^{\pi/2} 10 \sin 2t\, dt = (-5 \cos 2t)\Big|_{t=0}^{\pi/2} = (-5)(-1) - (-5)(1) = 10$

22. $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 2 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(2t)^2 + (2)^2} = 2\sqrt{t^2 + 1}$

$\Rightarrow \text{Length} = \int_0^1 2\sqrt{t^2 + 1}\, dt = \sqrt{2} + \ln\left(\sqrt{2} + 1\right) \approx 2.29559$ (Integral evaluated on TI-92 Plus calculator.)

23. Since $\frac{dL}{dx} = \frac{1}{x} + f'(x)$ must equal $\sqrt{1 + (f'(x))^2}$, $1 + (f'(x))^2 = \frac{1}{x^2} + \frac{2}{x}f'(x) + (f'(x))^2$, and $f'(x) = \frac{1}{2}x - \frac{1}{2x}$. Then $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x + C$, and the requirement to pass through $(1,1)$ means that $C = \frac{3}{4}$. The function is $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x + \frac{3}{4} = \frac{x^2 - 2\ln x + 3}{4}$.

24. $x = t^2$ and $y = \frac{t^3}{3} - t$, $-\sqrt{3} \le t \le \sqrt{3} \Rightarrow \frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = t^2 - 1 \Rightarrow$ Length $= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(2t)^2 + (t^2-1)^2}\,dt$

$= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{t^4 + 2t^2 + 1}\,dt = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(t^2+1)^2}\,dt = \int_{-\sqrt{3}}^{\sqrt{3}} (t^2+1)\,dt = \left[\frac{t^3}{3} + t\right]_{-\sqrt{3}}^{\sqrt{3}} = 4\sqrt{3}$

25. The equipment alone: the force required to lift the equipment is equal to its weight $\Rightarrow F_1(x) = 100$ N. The work done is $W_1 = \int_a^b F_1(x)\,dx = \int_0^{40} 100\,dx = [100x]_0^{40} = 4000$ J; the rope alone: the force required to lift the rope is equal to the weight of the rope paid out at elevation $x \Rightarrow F_2(x) = 0.8(40 - x)$. The work done is $W_2 = \int_a^b F_2(x)\,dx = \int_0^{40} 0.8(40-x)\,dx = 0.8\left[40x - \frac{x^2}{2}\right]_0^{40} = 0.8\left(40^2 - \frac{40^2}{2}\right) = \frac{(0.8)(1600)}{2} = 640$ J; the total work is $W = W_1 + W_2 = 4000 + 640 = 4640$ J

26. The force required to lift the water is equal to the water's weight, which varies steadily from $8 \cdot 800$ lb to $8 \cdot 400$ lb over the 4750 ft elevation. When the truck is x ft off the base of Mt. Washington, the water weight is $F(x) = 8 \cdot 800 \cdot \left(\frac{2 \cdot 4750 - x}{2 \cdot 4750}\right) = (6400)\left(1 - \frac{x}{9500}\right)$ lb. The work done is $W = \int_a^b F(x)\,dx$

$= \int_0^{4750} 6400\left(1 - \frac{x}{9500}\right)dx = 6400\left[x - \frac{x^2}{2 \cdot 9500}\right]_0^{4750} = 6400\left(4750 - \frac{4750^2}{4 \cdot 4750}\right) = \left(\frac{3}{4}\right)(6400)(4750)$

$= 22{,}800{,}000$ ft · lb

27. Force constant: $F = kx \Rightarrow 20 = k \cdot 1 \Rightarrow k = 20$ lb/ft; the work to stretch the spring 1 ft is

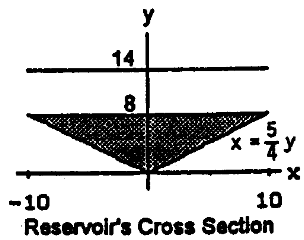
$W = \int_0^1 kx\,dx = k\int_0^1 x\,dx = \left[20\frac{x^2}{2}\right]_0^1 = 10$ ft · lb; the work to stretch the spring an additional foot is

$W = \int_1^2 kx\,dx = k\int_1^2 x\,dx = 20\left[\frac{x^2}{2}\right]_1^2 = 20\left(\frac{4}{2} - \frac{1}{2}\right) = 20\left(\frac{3}{2}\right) = 30$ ft · lb

28. Force constant: $F = kx \Rightarrow 200 = k(0.8) \Rightarrow k = 250$ N/m; the 300 N force stretches the spring $x = \frac{F}{k}$ $= \frac{300}{250} = 1.2$ m; the work required to stretch the spring that far is then $W = \int_0^{1.2} F(x)\,dx = \int_0^{1.2} 250x\,dx$

$= [125x^2]_0^{1.2} = 125(1.2)^2 = 180$ J

29. We imagine the water divided into thin slabs by planes perpendicular to the y-axis at the points of a partition of the interval $[0, 8]$. The typical slab between the planes at y and $y + \Delta y$ has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{5}{4}y\right)^2 = \frac{25\pi}{16}y^2\,\Delta y \text{ ft}^3$. The force F(y) required to lift this slab is equal to its weight: $F(y) = 62.4\,\Delta V = \frac{(62.4)(25)}{16}\pi y^2\,\Delta y$ lb. The distance through which F(y) must act to lift this slab to the level 6 ft above the top is about $(6 + 8 - y)$ ft, so the work done lifting the slab is about $\Delta W = \frac{(62.4)(25)}{16}\pi y^2(14 - y)\,\Delta y \text{ ft}\cdot\text{lb}$. The work done lifting all the slabs from $y = 0$ to $y = 8$ to the level 6 ft above the top is approximately $W \approx \sum_0^8 \frac{(62.4)(25)}{16}\pi y^2\,(14 - y)\,\Delta y$ ft·lb so the work to pump the water is the limit of these Riemann sums as the norm of the partition goes to zero:



$$W = \int_0^8 \frac{(62.4)(25)}{(16)}\pi y^2(14 - y)\,dy = \frac{(62.4)(25)\pi}{16}\int_0^8 (14y^2 - y^3)\,dy = (62.4)\left(\frac{25\pi}{16}\right)\left[\frac{14}{3}y^3 - \frac{y^4}{4}\right]_0^8$$

$$= (62.4)\left(\frac{25\pi}{16}\right)\left(\frac{14}{3}\cdot 8^3 - \frac{8^4}{4}\right) \approx 418{,}208.81 \text{ ft}\cdot\text{lb}$$

30. The same as in Exercise 29, but change the distance through which F(y) must act to $(8 - y)$ rather than $(6 + 8 - y)$. Also change the upper limit of integration from 8 to 5. The integral is:

$$W = \int_0^5 \frac{(62.4)(25)\pi}{16} y^2(8 - y)\,dy = (62.4)\left(\frac{25\pi}{16}\right)\int_0^5 (8y^2 - y^3)\,dy = (62.4)\left(\frac{25\pi}{16}\right)\left[\frac{8}{3}y^3 - \frac{y^4}{4}\right]_0^5$$

$$= (62.4)\left(\frac{25\pi}{16}\right)\left(\frac{8}{3}\cdot 5^3 - \frac{5^4}{4}\right) \approx 54{,}241.56 \text{ ft}\cdot\text{lb}$$

31. The tank's cross section looks like the figure in Exercise 29 with right edge given by $x = \frac{5}{10}y = \frac{y}{2}$. A typical horizontal slab has volume $\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{y}{2}\right)^2\Delta y = \frac{\pi}{4}y^2\,\Delta y$. The force required to lift this slab is its weight: $F(y) = 60\cdot\frac{\pi}{4}y^2\,\Delta y$. The distance through which F(y) must act is $(2 + 10 - y)$ ft, so the work to pump the liquid is $W = 60\int_0^{10} \pi(12 - y)\left(\frac{y^2}{4}\right)dy = 15\pi\left[\frac{12y^3}{3} - \frac{y^4}{4}\right]_0^{10} = 22{,}500\pi$ ft·lb; the time needed to empty the tank is $\frac{22{,}500 \text{ ft}\cdot\text{lb}}{275 \text{ ft}\cdot\text{lb/sec}} \approx 257$ sec

32. A typical horizontal slab has volume about $\Delta V = (20)(2x)\Delta y = (20)\left(2\sqrt{16 - y^2}\right)\Delta y$ and the force required to lift this slab is its weight $F(y) = (57)(20)\left(2\sqrt{16 - y^2}\right)\Delta y$. The distance through which F(y) must act is $(6 + 4 - y)$ ft, so the work to pump the olive oil from the half-full tank is

$$W = 57\int_{-4}^{0} (10 - y)(20)\left(2\sqrt{16 - y^2}\right)dy = 2880\int_{-4}^{0} 10\sqrt{16 - y^2}\,dy + 1140\int_{-4}^{0}\left(16 - y^2\right)^{1/2}(-2y)\,dy$$

$= 22{,}800 \cdot (\text{area of a quarter circle having radius 4}) + \frac{2}{3}(1140)\left[\left(16 - y^2\right)^{3/2}\right]_{-4}^{0} = (22{,}800)(4\pi) + 48{,}640$
$= 334{,}153.25 \text{ ft} \cdot \text{lb}$

33. $F = \int_a^b W \cdot \begin{pmatrix}\text{strip}\\ \text{depth}\end{pmatrix} \cdot L(y)\, dy \Rightarrow F = 2\int_0^2 (62.4)(2 - y)(2y)\, dy = 249.6 \int_0^2 \left(2y - y^2\right) dy = 249.6\left[y^2 - \frac{y^3}{3}\right]_0^2$
$= (249.6)\left(4 - \frac{8}{3}\right) = (249.6)\left(\frac{4}{3}\right) = 332.8 \text{ lb}$

34. $F = \int_a^b W \cdot \begin{pmatrix}\text{strip}\\ \text{depth}\end{pmatrix} \cdot L(y)\, dy \Rightarrow F = \int_0^{5/6} 75\left(\frac{5}{6} - y\right)(2y + 4)\, dy = 75 \int_0^{5/6} \left(\frac{5}{3}y + \frac{10}{3} - 2y^2 - 4y\right) dy$
$= 75 \int_0^{5/6} \left(\frac{10}{3} - \frac{7}{3}y - 2y^2\right) dy = 75\left[\frac{10}{3}y - \frac{7}{6}y^2 - \frac{2}{3}y^3\right]_0^{5/6} = (75)\left[\left(\frac{50}{18}\right) - \left(\frac{7}{6}\right)\left(\frac{25}{36}\right) - \left(\frac{2}{3}\right)\left(\frac{125}{216}\right)\right]$
$= (75)\left(\frac{25}{9} - \frac{175}{216} - \frac{250}{3 \cdot 216}\right) = \left(\frac{75}{9 \cdot 216}\right)(25 \cdot 216 - 175 \cdot 9 - 250 \cdot 3) = \frac{(75)(3075)}{9 \cdot 216} \approx 118.63 \text{ lb.}$

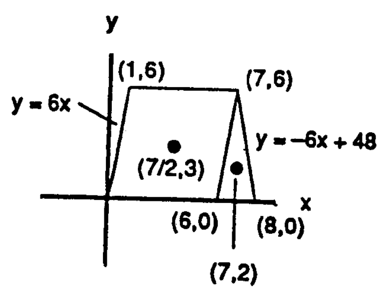
35. $F = \int_a^b W \cdot \begin{pmatrix}\text{strip}\\ \text{depth}\end{pmatrix} \cdot L(y)\, dy \Rightarrow F = 62.4 \int_0^4 (9 - y)\left(2 \cdot \frac{\sqrt{y}}{2}\right) dy = 62.4 \int_0^4 \left(9y^{1/2} - y^{3/2}\right) dy$
$= 62.4\left[6y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^4 = (62.4)\left(6 \cdot 8 - \frac{2}{5} \cdot 32\right) = \left(\frac{62.4}{5}\right)(48 \cdot 5 - 64) = \frac{(62.4)(176)}{5} = 2196.48 \text{ lb}$

36. $F = 62.4 \int_0^6 (10 - y)\left[\left(8 - \frac{y}{6}\right) - \left(\frac{y}{6}\right)\right] dy$

$= \frac{62.4}{3} \int_0^6 \left(240 - 34y + y^2\right) dy$

$= \frac{62.4}{3}\left[240y - 17y^2 + \frac{y^3}{3}\right]_0^6 = \frac{62.4}{3}(1440 - 612 + 72)$

$= 18{,}720 \text{ lb.}$



37. $\frac{dy}{dx} = x^2\sqrt{y} \Rightarrow dy = x^2 y^{1/2}\, dx \Rightarrow y^{-1/2}\, dy = x^2\, dx \Rightarrow \int y^{-1/2}\, dy = \int x^2\, dx \Rightarrow 2y^{1/2} = \frac{x^3}{3} + \overline{C}$
$\Rightarrow 2y^{1/2} - \frac{1}{3}x^3 = \overline{C} \Rightarrow y = \left(\frac{x^2}{6} + C\right)^2$ where $C = \frac{\overline{C}}{2}$.

38. $\frac{dy}{dx} = e^{2x+3y} \Rightarrow dy = e^{2x}e^{3y}\, dx \Rightarrow e^{-3y}\, dy = e^{2x}\, dx \Rightarrow \int e^{-3y}\, dy = \int e^{2x}\, dx$
$\Rightarrow -\frac{1}{3}e^{-3y} = \frac{1}{2}e^{2x} + C_1 \Rightarrow 3e^{2x} + 2e^{-3y} = C$, where $C = -6C_1$

39. $x\frac{dy}{dx} = y \ln x \Rightarrow \frac{dy}{y} = \frac{\ln x}{x}\,dx \Rightarrow \int \frac{dy}{y} = \int \frac{\ln x}{x}\,dx \Rightarrow \ln|y| = \int \frac{\ln x}{x}\,dx$ (Let $u = \ln x \Rightarrow du = \frac{dx}{x}$)

$\Rightarrow \ln|y| = \int u\,du = \frac{u^2}{2} + C_1 = \frac{1}{2}(\ln x)^2 + C_1 \Rightarrow |y| = e^{\frac{1}{2}(\ln x)^2 + C_1} = e^{C_1}e^{\frac{1}{2}(\ln x)^2} = C_2 e^{\frac{1}{2}(\ln x)^2}$

$\Rightarrow y = \pm C_2 e^{\frac{1}{2}(\ln x)^2} \Rightarrow y = Ce^{\frac{1}{2}(\ln x)^2}$ where $C_2 = e^{C_1}$ and $C = \pm C_2$

40. $\frac{1}{\sin x}\frac{dy}{dx} = e^{\cos x - y} = e^{\cos x}e^{-y} \Rightarrow e^y\,dy = e^{\cos x}\sin x\,dx \Rightarrow \int e^y\,dy = \int e^{\cos x}\sin x\,dx$

$\Rightarrow e^y = \int e^{\cos x}\sin x\,dx$ (Let $u = \cos x \Rightarrow -du = \sin x\,dx$) $\Rightarrow e^y = -\int e^u\,du \Rightarrow e^y = -e^u + C$

$\Rightarrow e^y + e^{\cos x} = C$

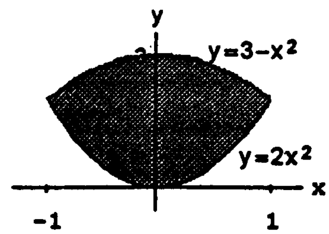
41. $\frac{\sec^2\sqrt{x}}{\sqrt{x}}\,dx = dt \Rightarrow \int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\,dx = \int dt \Rightarrow 2\int \sec^2 u\,du = \int dt \Rightarrow 2\tan u = t + C \Rightarrow 2\tan\sqrt{x} = t + C,$

where $u = \sqrt{x}$ and $du = \frac{1}{2}x^{-1/2}\,dx$

42. $(x\cos^2 t)\frac{dx}{dt} = \sin t \Rightarrow x\,dx = \frac{\sin t}{\cos^2 t}\,dt \Rightarrow \int x\,dx = \int \frac{\sin t}{\cos^2 t}\,dt \Rightarrow \frac{x^2}{2} = \frac{1}{\cos t} + C_1 \Rightarrow x^2 = 2\sec t + C,$

where $C = 2C_1$ and $C \geq -2$

43. Intersection points: $3 - x^2 = 2x^2 \Rightarrow 3x^2 - 3 = 0$
$\Rightarrow 3(x-1)(x+1) = 0 \Rightarrow x = -1$ or $x = 1$. Applying the symmetry argument analogous to the one used in Exercise 5.6.13, we find that $\bar{x} = 0$. The typical *vertical* strip has



center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{2x^2 + (3 - x^2)}{2}\right) = \left(x, \frac{x^2+3}{2}\right),$

length: $(3 - x^2) - 2x^2 = 3(1 - x^2)$, width: dx,

area: $dA = 3(1 - x^2)\,dx$, and mass: $dm = \delta \cdot dA = 3\delta(1 - x^2)\,dx$

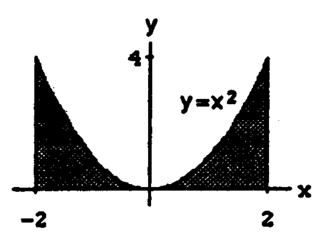
$\Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = \frac{3}{2}\delta(x^2+3)(1-x^2)\,dx = \frac{3}{2}\delta(-x^4 - 2x^2 + 3)\,dx$

$\Rightarrow M_x = \int \tilde{y}\,dm = \frac{3}{2}\delta\int_{-1}^{1}(-x^4 - 2x^2 + 3)\,dx = \frac{3}{2}\delta\left[-\frac{x^5}{5} - \frac{2x^3}{3} + 3x\right]_{-1}^{1} = 3\delta\left(-\frac{1}{5} - \frac{2}{3} + 3\right)$

$= \frac{3\delta}{15}(-3 - 10 + 45) = \frac{32\delta}{5}$; $M = \int dm = 3\delta\int_{-1}^{1}(1 - x^2)\,dx = 3\delta\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 6\delta\left(1 - \frac{1}{3}\right) = 4\delta$

$\Rightarrow \bar{y} = \frac{M_x}{M} = \frac{32\delta}{5 \cdot 4\delta} = \frac{8}{5}$. Therefore, the centroid is $(\bar{x}, \bar{y}) = \left(0, \frac{8}{5}\right)$.

44. Applying the symmetry argument analogous to the one used in Exercise 5.7.13, we find that $\bar{x} = 0$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{x^2}{2}\right)$, length: x^2, width: dx, area: $dA = x^2\,dx$, mass: $dm = \delta \cdot dA = \delta x^2\,dx$

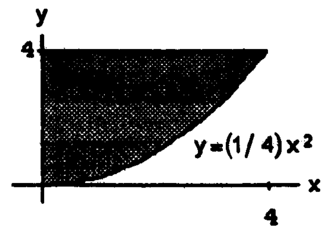


$\Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = \frac{\delta}{2}x^2 \cdot x^2\,dx = \frac{\delta}{2}x^4\,dx$

$\Rightarrow M_x = \int \tilde{y}\,dm = \frac{\delta}{2}\int_{-2}^{2} x^4\,dx = \frac{\delta}{10}\left[x^5\right]_{-2}^{2} = \frac{2\delta}{10}(2^5) = \frac{32\delta}{5}$;

$M = \int dm = \delta\int_{-2}^{2} x^2\,dx = \delta\left[\frac{x^3}{3}\right]_{-2}^{2} = \frac{2\delta}{3}(2^3) = \frac{16\delta}{3} \Rightarrow \bar{y} = \frac{M_x}{M} = \frac{32\cdot\delta\cdot 3}{5\cdot 16\cdot\delta} = \frac{6}{5}$. Therefore, the centroid is $(\bar{x},\bar{y}) = \left(0,\frac{6}{5}\right)$.

45. The typical *vertical* strip has: center of mass: $(\tilde{x},\tilde{y}) = \left(x, \frac{4+\frac{x^2}{4}}{2}\right)$, length: $4-\frac{x^2}{4}$, width: dx, area: $dA = \left(4-\frac{x^2}{4}\right)dx$, mass: $dm = \delta\cdot dA = \delta\left(4-\frac{x^2}{4}\right)dx \Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = \delta\cdot\frac{\left(4+\frac{x^2}{4}\right)}{2}\left(4-\frac{x^2}{4}\right)dx = \frac{\delta}{2}\left(16-\frac{x^4}{16}\right)dx$; the moment about the y-axis is $\tilde{x}\,dm = \delta\left(4-\frac{x^2}{4}\right)\cdot x\,dx = \delta\left(4x-\frac{x^3}{4}\right)dx$. Thus, $M_x = \int \tilde{y}\,dm = \frac{\delta}{2}\int_0^4\left(16-\frac{x^4}{16}\right)dx$

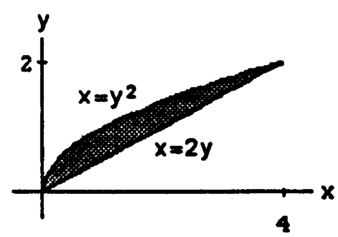


$= \frac{\delta}{2}\left[16x-\frac{x^5}{5\cdot 16}\right]_0^4 = \frac{\delta}{2}\left[64-\frac{64}{5}\right] = \frac{128\delta}{5}$; $M_y = \int \tilde{x}\,dm = \delta\int_0^4\left(4x-\frac{x^3}{4}\right)dx = \delta\left[2x^2-\frac{x^4}{16}\right]_0^4$

$= \delta(32-16) = 16\delta$; $M = \int dm = \delta\int_{-0}^{4}\left(4-\frac{x^2}{4}\right)dx = \delta\left[4x-\frac{x^3}{12}\right]_0^4 = \delta\left(16-\frac{64}{12}\right) = \frac{32\delta}{3}$

$\Rightarrow \bar{x} = \frac{M_y}{M} = \frac{16\cdot\delta\cdot 3}{32\cdot\delta} = \frac{3}{2}$ and $\bar{y} = \frac{M_x}{M} = \frac{128\cdot\delta\cdot 3}{5\cdot 32\cdot\delta} = \frac{12}{5}$. Therefore, the centroid is $(\bar{x},\bar{y}) = \left(\frac{3}{2},\frac{12}{5}\right)$.

46. A typical *horizontal* strip has: center of mass: $(\tilde{x},\tilde{y}) = \left(\frac{y^2+2y}{2}, y\right)$, length: $2y-y^2$, width: dy, area: $dA = (2y-y^2)\,dy$, mass: $dm = \delta\cdot dA = \delta(2y-y^2)\,dy$; the moment about the x-axis is $\tilde{y}\,dm = \delta\cdot y\cdot(2y-y^2)\,dy = \delta(2y^2-y^3)$; the moment about the y-axis is $\tilde{x}\,dm = \delta\cdot\frac{(y^2+2y)}{2}\cdot(2y-y^2)\,dy = \frac{\delta}{2}(4y^2-y^4)\,dy$

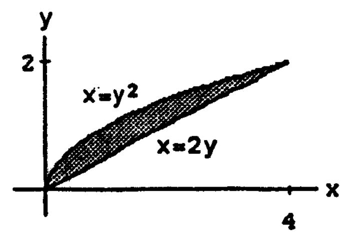


$\Rightarrow M_x = \int \tilde{y}\,dm = \delta\int_0^2(2y^2-y^3)\,dy = \delta\left[\frac{2}{3}y^3-\frac{y^4}{4}\right]_0^2 = \delta\left(\frac{2}{3}\cdot 8-\frac{16}{4}\right) = \delta\left(\frac{16}{3}-\frac{16}{4}\right) = \frac{\delta\cdot 16}{12} = \frac{4\delta}{3}$;

$M_y = \int \tilde{x}\,dm = \frac{\delta}{2}\int_0^2(4y^2-y^4)\,dy = \frac{\delta}{2}\left[\frac{4}{3}y^3-\frac{y^5}{5}\right]_0^2 = \frac{\delta}{2}\left(\frac{4\cdot 8}{3}-\frac{32}{5}\right) = \frac{32\delta}{15}$; $M = \int dm = \delta\int_0^2(2y-y^2)\,dy$

$= \delta\left[y^2-\frac{y^3}{3}\right]_0^2 = \delta\left(4-\frac{8}{3}\right) = \frac{4\delta}{3} \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{\delta\cdot 32\cdot 3}{15\cdot\delta\cdot 4} = \frac{8}{5}$ and $\bar{y} = \frac{M_x}{M} = \frac{4\cdot\delta\cdot 3}{3\cdot 4\cdot\delta} = 1$. Therefore, the centroid is $(\bar{x},\bar{y}) = \left(\frac{8}{5}, 1\right)$.

47. A typical horizontal strip has: center of mass: $(\tilde{x},\tilde{y}) = \left(\frac{y^2+2y}{2}, y\right)$, length: $2y - y^2$, width: dy, area: $dA = (2y - y^2)\,dy$, mass: $dm = \delta \cdot dA = (1+y)(2y-y^2)\,dy \Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = y(1+y)(2y-y^2)\,dy = (2y^2+2y^3-y^3-y^4)\,dy$ $= (2y^2+y^3-y^4)\,dy$; the moment about the y-axis is



$\tilde{x}\,dm = \left(\frac{y^2+2y}{2}\right)(1+y)(2y-y^2)\,dy = \frac{1}{2}(4y^2-y^4)(1+y)\,dy$

$= \frac{1}{2}(4y^2+4y^3-y^4-y^5)\,dy \Rightarrow M_x = \int \tilde{y}\,dm = \int_0^2 (2y^2+y^3-y^4)\,dy = \left[\frac{2}{3}y^3+\frac{y^4}{4}-\frac{y^5}{5}\right]_0^2$

$= \left(\frac{16}{3}+\frac{16}{4}-\frac{32}{5}\right) = 16\left(\frac{1}{3}+\frac{1}{4}-\frac{2}{5}\right) = \frac{16}{60}(20+15-24) = \frac{4}{15}(11) = \frac{44}{15}$; $M_y = \int \tilde{x}\,dm$

$= \int_0^2 \frac{1}{2}(4y^2+4y^3-y^4-y^5)\,dy = \frac{1}{2}\left[\frac{4}{3}y^3+y^4-\frac{y^5}{5}-\frac{y^6}{6}\right]_0^2 = \frac{1}{2}\left(\frac{4\cdot 2^3}{3}+2^4-\frac{2^5}{5}-\frac{2^6}{6}\right)$

$= 4\left(\frac{4}{3}+2-\frac{4}{5}-\frac{8}{6}\right) = 4\left(2-\frac{4}{5}\right) = \frac{24}{5}$; $M = \int dm = \int_0^2 (1+y)(2y-y^2)\,dy = \int_0^2 (2y+y^2-y^3)\,dy$

$= \left[y^2+\frac{y^3}{3}-\frac{y^4}{4}\right]_0^2 = \left(4+\frac{8}{3}-\frac{16}{4}\right) = \frac{8}{3} \Rightarrow \bar{x} = \frac{M_y}{M} = \left(\frac{24}{5}\right)\left(\frac{3}{8}\right) = \frac{9}{5}$ and $\bar{y} = \frac{M_x}{M} = \left(\frac{44}{15}\right)\left(\frac{3}{8}\right) = \frac{44}{40} = \frac{11}{10}$. Therefore, the center of mass is $(\bar{x},\bar{y}) = \left(\frac{9}{5}, \frac{11}{10}\right)$.

48. A typical vertical strip has: center of mass: $(\tilde{x},\tilde{y}) = \left(x, \frac{3}{2x^{3/2}}\right)$, length: $\frac{3}{x^{3/2}}$, width: dx, area: $dA = \frac{3}{x^{3/2}}\,dx$, mass: $dm = \delta \cdot dA = \delta \cdot \frac{3}{x^{3/2}}\,dx \Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = \frac{3}{2x^{3/2}} \cdot \frac{3}{x^{3/2}}\,dx = \frac{9}{2x^3}\,dx$; the moment about the y-axis is $\tilde{x}\,dm = x \cdot \frac{3}{x^{3/2}}\,dx = \frac{3}{x^{1/2}}\,dx$.

(a) $M_x = \delta \int_1^9 \frac{1}{2}\left(\frac{9}{x^3}\right)dx = \frac{9\delta}{2}\left[-\frac{x^{-2}}{2}\right]_1^9 = \frac{20\delta}{9}$; $M_y = \delta \int_1^9 x\left(\frac{3}{x^{3/2}}\right)dx = 3\delta\left[2x^{1/2}\right]_1^9 = 12\delta$;

$M = \delta \int_1^9 \frac{3}{x^{3/2}}\,dx = -6\delta\left[x^{-1/2}\right]_1^9 = 4\delta \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{12\delta}{4\delta} = 3$ and $\bar{y} = \frac{M_x}{M} = \frac{\left(\frac{20\delta}{9}\right)}{4\delta} = \frac{5}{9}$

(b) $M_x = \int_1^9 \frac{x}{2}\left(\frac{9}{x^3}\right)dx = \frac{9}{2}\left[-\frac{1}{x}\right]_1^9 = 4$; $M_y = \int_1^9 x^2\left(\frac{3}{x^{3/2}}\right)dx = \left[2x^{3/2}\right]_1^9 = 52$; $M = \int_1^9 x\left(\frac{3}{x^{3/2}}\right)dx$

$= 6\left[x^{1/2}\right]_1^9 = 12 \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{13}{3} = 3$ and $\bar{y} = \frac{M_x}{M} = \frac{1}{3}$

CHAPTER 5 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. $V = \pi \int_0^a [f(x)]^2\, dx = a^2 + a \Rightarrow \pi \int_0^x [f(t)]^2\, dt = x^2 + x$ for all $x > a \Rightarrow \pi[f(x)]^2 = 2x + 1 \Rightarrow f(x) = \sqrt{\frac{2x+1}{\pi}}$

2. By the shell method we have $2\pi b^3 = 2\pi \int_0^b xf(x)\, dx \Rightarrow x^3 = \int_0^x tf(t)\, dt$, where $x > 0$. By the Fundamental Theorem of Calculus we have $3x^2 = xf(x) \Rightarrow f(x) = 3x$.

3. $s(x) = Cx \Rightarrow \int_0^x \sqrt{1 + [f'(t)]^2}\, dt = Cx \Rightarrow \sqrt{1 + [f'(x)]^2} = C \Rightarrow f'(x) = \sqrt{C^2 - 1}$ for $C \ge 1$

   $\Rightarrow f(x) = \int_0^x \sqrt{C^2 - 1}\, dt + k$. Then $f(0) = a \Rightarrow a = 0 + k \Rightarrow f(x) = \int_0^x \sqrt{C^2 - 1}\, dt + a \Rightarrow f(x) = x\sqrt{C^2 - 1} + a$, where $C \ge 1$.

4. (a) The graph of $f(x) = \sin x$ traces out a path from $(0,0)$ to $(\alpha, \sin \alpha)$ whose length is $L = \int_0^\alpha \sqrt{1 + \cos^2\theta}\, d\theta$.

   The line segment from $(0,0)$ to $(\alpha, \sin \alpha)$ has length $\sqrt{(\alpha - 0)^2 + (\sin \alpha - 0)^2} = \sqrt{\alpha^2 + \sin^2\alpha}$. Since the shortest distance between two points is the length of the straight line segment joining them, we have immediately that $\int_0^\alpha \sqrt{1 + \cos^2\theta}\, d\theta > \sqrt{\alpha^2 + \sin^2\alpha}$ if $0 < \alpha \le \frac{\pi}{2}$.

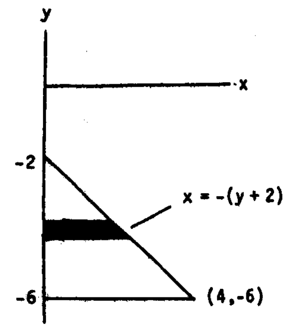
   (b) In general, if $y = f(x)$ is continuously differentiable and $f(0) = 0$, then $\int_0^\alpha \sqrt{1 + [f'(t)]^2}\, dt > \sqrt{\alpha^2 + f^2(\alpha)}$ for $\alpha > 0$.

5. Converting to pounds and feet, $2\ \text{lb/in} = \frac{2\ \text{lb}}{1\ \text{in}} \cdot \frac{12\ \text{in}}{1\ \text{ft}} = 24\ \text{lb/ft}$. Thus, $F = 24x \Rightarrow W = \int_0^{1/2} 24x\, dx$

   $= \left[12x^2\right]_0^{1/2} = 3\ \text{ft} \cdot \text{lb}$. Since $W = \frac{1}{2}mv_0^2 - \frac{1}{2}mv_1^2$, where $W = 3\ \text{ft} \cdot \text{lb}$, $m = \left(\frac{1}{10}\ \text{lb}\right)\left(\frac{1}{32\ \text{ft/sec}^2}\right)$

   $= \frac{1}{320}$ slugs, and $v_1 = 0$ ft/sec, we have $3 = \left(\frac{1}{2}\right)\left(\frac{1}{320} v_0^2\right) \Rightarrow v_0^2 = 3 \cdot 640$. For the projectile height, $s = -16t^2 + v_0 t$ (since $s = 0$ at $t = 0$) $\Rightarrow \frac{ds}{dt} = v = -32t + v_0$. At the top of the ball's path, $v = 0 \Rightarrow t = \frac{v_0}{32}$ and the height is $s = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = \frac{v_0^2}{64} = \frac{3 \cdot 640}{64} = 30$ ft.

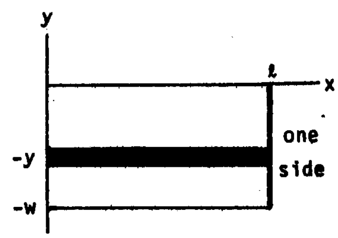
6. The submerged triangular plate is depicted in the figure at the right. The hypotenuse of the triangle has slope -1 $\Rightarrow y-(-2)=-(x-0) \Rightarrow x=-(y+2)$ is an equation of the hypotenuse. Using a typical horizontal strip, the fluid pressure is $F=\int (62.4)\cdot\begin{pmatrix}\text{strip}\\ \text{depth}\end{pmatrix}\cdot\begin{pmatrix}\text{strip}\\ \text{length}\end{pmatrix} dy$



$$= \int_{-6}^{-2} (62.4)(-y)[-(y+2)]\,dy = 62.4\int_{-6}^{-2}\left(y^2+2y\right)dy$$

$$= 62.4\left[\frac{y^3}{3}+y^2\right]_{-6}^{-2} = (62.4)\left[\left(-\frac{8}{3}+4\right)-\left(-\frac{216}{3}+36\right)\right] = (62.4)\left(\frac{208}{3}-32\right) = \frac{(62.4)(112)}{3} \approx 2329.6 \text{ lb}$$

7. Consider a rectangular plate of length ℓ and width w. The length is parallel with the surface of the fluid of weight density ω. The force on one side of the plate is $F=\omega\int_{-w}^{0}(-y)(\ell)\,dy = -\omega\ell\left[\frac{y^2}{2}\right]_{-w}^{0} = \frac{\omega\ell w^2}{2}$. The average force on one side of the plate is $F_{av} = \frac{\omega}{w}\int_{-w}^{0}(-y)\,dy$ $= \frac{\omega}{w}\left[-\frac{y^2}{2}\right]_{-w}^{0} = \frac{\omega w}{2}$. Therefore the force $\frac{\omega\ell w^2}{2} = \left(\frac{\omega w}{2}\right)(\ell w)$ = (the average pressure up and down)$\cdot$(the area of the plate).



8. For y measured downward from the fluid's surface the width of a horizontal strip is $2(y-2)$ when $2 \le y \le 8$ and it is $12-2(y-8)$ when $8 \le y \le 14$. Using the hint given in the Exercise, the fluid force on the plate is $F = 2\int_{2}^{8} w_1 y(y-2)\,dy + 2\int_{8}^{14}[8w_1 = w_2(y-8)](14-y)\,dy = (216w_1)+(288w_1+72w_2)$ $= 504w_1 + 72w_2$

9. $\frac{dA}{dt} = kA,\ A(0) = A_0 \Rightarrow \frac{dA}{A} = k\,dt \Rightarrow \ln A = kt + \ln C$ or $A = Ce^{kt}$

Apply the initial condition: $A_0 = Ce^0$ or $C = A_0$. Thus, $A = A_0e^{kt}$.

After 24 hours, $\frac{6}{7}$ remains so we can solve for k: $\frac{6}{7}A_0 = A_0e^{k(24)}$ implies $k = \frac{\ln\frac{6}{7}}{24} = -0.0064$.

The half-life is then obtained as follows: $\frac{1}{2}A_0 = A_0e^{(-0.0064)t} \Rightarrow t = \frac{\ln\frac{1}{2}}{-0.0064} = 251.4747$ hours.

To be reduced to $\frac{1}{5}$ of original: $\frac{1}{5}A_0 = A_0e^{(-0.0064)t}$ implies $t = \frac{\ln\frac{1}{2}}{-0.0064} = 108.3042$ hours.

10. $\frac{dP}{dt} = kP$, $P(0) = 2$, $P(2) = 3$. $\frac{dP}{P} = k\,dt$ implies $\ln|P| = kt + C$ or $P(t) = C_1e^{kt}$. $P(0) = 2 = C_1$ and $P(2) = 3 = 2e^{2k}$ implies $\frac{3}{2} = e^{2k}$ or, $k = \frac{1}{2}\ln\left(\frac{3}{2}\right) \approx 0.2027$. Thus, $P(t) = 2e^{0.2027t}$. The year 1985 implies $t = 6$, so $P(6) \approx 2e^{0.2027(6)} = 2(3.3743) = 6.7487$ million.

11. $\frac{dT}{dt} = k(T - 70)$, $T(0) = 200$, $T(1) = 100$. $\frac{dT}{T-70} = k\,dt$ implies $\ln|T - 70| = kt + C$, or $T = 70 + C_1e^{kt}$. $T(0) = 200 = 70 + C_1$ implies $C_1 = 130$. $T(1) = 190 = 70 + 130e^k$ implies $130e^k = 120$ or $k = \ln\left(\frac{12}{13}\right) \approx -0.08$. Thus, $T(t) = 60 + 130e^{-0.08t}$. The 3:30 p.m. implies $t = 3.5$ and $T(3.5) = 70 + 130e^{-0.08(3.5)}$ $\approx 70 + 130(0.7557) = 168.25$°F.

12. $\frac{dT}{dt} = k(T - 70)$, $T(0) = 94.6$ (Set $t = 0$ at 11:30 p.m.), $T(1) = 93.4$
(Linear or separable)

$$\frac{dT}{dt} - kT = -70k \Rightarrow \mu = e^{\int -k\,dt} = e^{-kt} \Rightarrow e^{-kt}T = \int -70ke^{-kt}\,dt = \frac{-70k}{-k}e^{-kt} + C, \text{ or } T = 70 + Ce^{kt}.$$

<u>$t = 0$, $T = 94.6$:</u>

$94.6 = 70 + Ce^0$, $C = 24.6 \Rightarrow T = 70 + 24.6e^{kt}$

<u>$t = 1$, $T = 93.4$:</u>

$$93.4 = 70 + 24.6e^k \Rightarrow \ln\frac{23.4}{24.6} = \ln e^k \Rightarrow k \approx -0.0500104206 \Rightarrow T = 70 + 24.6e^{-0.0500104206t}$$

<u>$T = 98.6$, find t:</u>

$$98.6 = 70 + 24.6e^{-0.0500104206t} \Rightarrow \ln\frac{28.6}{24.6} = \ln e^{-0.0500104206t}, \text{ or } t = \frac{\ln\frac{28.6}{24.6}}{-0.0500104206} \approx -3.01 \text{ hours.}$$

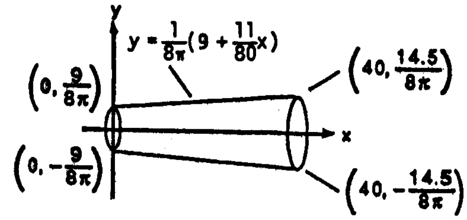
Time of death = 11:30 p.m. minus 3 hours $\approx$ 8:30 p.m.

13. From the symmetry of $y = 1 - x^n$, n even, about the y-axis for $-1 \le x \le 1$, we have $\bar{x} = 0$. To find $\bar{y} = \frac{M_x}{M}$, we use the vertical strips technique. The typical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{1-x^n}{2}\right)$, length: $1 - x^n$, width: dx, area: $dA = (1 - x^n)\,dx$, mass: $dm = 1 \cdot dA = (1 - x^n)\,dx$. The moment of the strip about the x-axis is $\tilde{y}\,dm = \frac{(1-x^n)^2}{2}\,dx \Rightarrow M_x = \int_{-1}^{1} \frac{(1-x^n)^2}{2}\,dx = 2\int_0^1 \frac{1}{2}(1 - 2x^n + x^{2n})\,dx = \left[x - \frac{2x^{n+1}}{n+1} + \frac{x^{2n+1}}{2n+1}\right]_0^1$

$$= 1 - \frac{2}{n+1} + \frac{1}{2n+1} = \frac{(n+1)(2n+1) - 2(2n+1) + (n+1)}{(n+1)(2n+1)} = \frac{2n^2 + 3n + 1 - 4n - 2 + n + 1}{(n+1)(2n+1)} = \frac{2n^2}{(n+1)(2n+1)}.$$

Also, $M = \int_{-1}^{1} dA = \int_{-1}^{1} (1 - x^n)\,dx = 2\int_0^1 (1 - x^n)\,dx = 2\left[x - \frac{x^{n+1}}{n+1}\right]_0^1 = 2\left(1 - \frac{1}{n+1}\right) = \frac{2n}{n+1}$. Therefore,

$\bar{y} = \frac{M_x}{M} = \frac{2n^2}{(n+1)(2n+1)} \cdot \frac{(n+1)}{2n} = \frac{n}{2n+1} \Rightarrow \left(0, \frac{n}{2n+1}\right)$ is the location of the centroid. As $n \to \infty$, $\bar{y} \to \frac{1}{2}$ so the limiting position of the centroid is $\left(0, \frac{1}{2}\right)$.



14. Align the telephone pole along the x-axis as shown in the accompanying figure. The slope of the top length of pole is $\frac{\left(\frac{14.5}{8\pi} - \frac{9}{8\pi}\right)}{40} = \frac{1}{8\pi}\cdot\frac{1}{40}\cdot(14.5 - 9) = \frac{5.5}{8\pi\cdot 40} = \frac{11}{8\pi\cdot 80}$. Thus, $y = \frac{9}{8\pi} + \frac{11}{8\pi\cdot 80}x = \frac{1}{8\pi}\left(9 + \frac{11}{80}x\right)$ is an equation of the line representing the top of the pole. Then, $M_y = \int_a^b x\cdot\pi y^2\,dx = \pi\int_0^{40} x\left[\frac{1}{8\pi}\left(9 + \frac{11}{80}x\right)\right]^2 dx$ $= \frac{1}{64\pi}\int_0^{40} x\left(9 + \frac{11}{80}x\right)^2 dx$; $M = \int_a^b \pi y^2\,dx = \pi\int_0^{40}\left[\frac{1}{8\pi}\left(9 + \frac{11}{80}x\right)\right]^2 dx = \frac{1}{64\pi}\int_0^{40}\left(9 + \frac{11}{80}x\right)^2 dx$. Thus, $\bar{x} = \frac{M_y}{M} \approx \frac{129{,}700}{5623.\bar{3}} \approx 23.06$ (using a calculator to compute the integrals). By symmetry about the x-axis, $\bar{y} = 0$ so the center of mass is about 23 ft from the top of the pole.

15. (a) Consider a single vertical strip with center of mass $(\tilde{x}, \tilde{y})$. If the plate lies to the right of the line, then the moment of this strip about the line x = b is $(\tilde{x} - b)\,dm = (\tilde{x} - b)\,\delta\,dA \Rightarrow$ the plate's first moment about x = b is the integral $\int (x-b)\delta\,dA = \int \delta x\,dA - \int \delta b\,dA = M_y - b\delta A$.

(b) If the plate lies to the left of the line, the moment of a vertical strip about the line x = b is $(b - \tilde{x})\,dm = (b - \tilde{x})\,\delta\,dA \Rightarrow$ the plate's first moment about x = b is $\int (b-x)\delta\,dA = \int b\delta\,dA - \int \delta x\,dA$ $= b\delta A - M_y$.

16. (a) By symmetry of the plate about the x-axis, $\bar{y} = 0$. A typical vertical strip has center of mass: $(\tilde{x}, \tilde{y}) = (x, 0)$, length: $4\sqrt{ax}$, width: dx, area: $4\sqrt{ax}\,dx$, mass: $dm = \delta\,dA = kx\cdot 4\sqrt{ax}\,dx$, for some proportionality constant k. The moment of the strip about the y-axis is $M_y = \int \tilde{x}\,dm = \int_0^a 4kx^2\sqrt{ax}\,dx$ $= 4k\sqrt{a}\int_0^a x^{5/2}\,dx = 4k\sqrt{a}\left[\frac{2}{7}x^{7/2}\right]_0^a = 4ka^{1/2}\cdot\frac{2}{7}a^{7/2} = \frac{8ka^4}{7}$. Also, $M = \int dm = \int_0^a 4kx\sqrt{ax}\,dx$ $= 4k\sqrt{a}\int_0^a x^{3/2}\,dx = 4k\sqrt{a}\left[\frac{2}{5}x^{5/2}\right]_0^a = 4ka^{1/2}\cdot\frac{2}{5}a^{5/2} = \frac{8ka^3}{5}$. Thus, $\bar{x} = \frac{M_y}{M} = \frac{8ka^4}{7}\cdot\frac{5}{8ka^3} = \frac{5}{7}a$ $\Rightarrow (\bar{x}, \bar{y}) = \left(\frac{5a}{7}, 0\right)$ is the center of mass.

(b) A typical horizontal strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(\frac{\frac{y^2}{4a} + a}{2}, 0\right) = \left(\frac{y^2 + 4a^2}{8a}, 0\right)$, length: $a - \frac{y^2}{4a}$, width: dy, area: $\left(a - \frac{y^2}{4a}\right)dy$, mass: $dm = \delta\,dA = |y|\left(a - \frac{y^2}{4a}\right)dy$. Thus, $M_x = \int \tilde{y}\,dm$

$= \int_{-2a}^{2a} y|y|\left(a - \frac{y^2}{4a}\right) dy = \int_{-2a}^{0} -y^2\left(a - \frac{y^2}{4a}\right) dy + \int_{0}^{2a} y^2\left(a - \frac{y^2}{4a}\right) dy$

$= \int_{-2a}^{0} \left(-ay^2 + \frac{y^4}{4a}\right) dy + \int_{0}^{2a} \left(ay^2 - \frac{y^4}{4a}\right) dy = \left[-\frac{a}{3}y^3 + \frac{y^5}{20a}\right]_{-2a}^{0} + \left[\frac{a}{3}y^3 - \frac{y^5}{20a}\right]_{0}^{2a}$

$= -\frac{8a^4}{3} + \frac{32a^5}{20a} + \frac{8a^4}{3} - \frac{32a^5}{20a} = 0$; $M_y = \int \tilde{x}\, dm = \int_{-2a}^{2a} \left(\frac{y^2 + 4a^2}{8a}\right)|y|\left(a - \frac{y^2}{4a}\right) dy$

$= \frac{1}{8a} \int_{-2a}^{2a} |y|(y^2 + 4a^2)\left(\frac{4a^2 - y^2}{4a}\right) dy = \frac{1}{32a^2} \int_{-2a}^{2a} |y|(16a^4 - y^4)\, dy$

$= \frac{1}{32a^2} \int_{-2a}^{0} (-16a^4y + y^5)\, dy + \frac{1}{32a^2} \int_{0}^{2a} (16a^4 - y^5)\, dy = \frac{1}{32a^2}\left[-8a^4y^2 + \frac{y^6}{6}\right]_{-2a}^{0} + \frac{1}{32a^2}\left[8a^4y^2 - \frac{y^6}{6}\right]_{0}^{2a}$

$= \frac{1}{32a^2}\left[8a^4 \cdot 4a^2 - \frac{64a^6}{6}\right] + \frac{1}{32a^2}\left[8a^4 \cdot 4a^2 - \frac{64a^6}{6}\right] = \frac{1}{16a^2}\left(32a^6 - \frac{32a^6}{3}\right) = \frac{1}{16a^2} \cdot \frac{2}{3}(32a^6) = \frac{4}{3}a^4$;

$M = \int dm = \int_{-2a}^{2a} |y|\left(\frac{4a^2 - y^2}{4a}\right) dy = \frac{1}{4a} \int_{-2a}^{2a} |y|(4a^2 - y^2)\, dy$

$= \frac{1}{4a} \int_{-2a}^{0} (-4a^2y + y^3)\, dy + \frac{1}{4a} \int_{0}^{2a} (4a^2y - y^3)\, dy = \frac{1}{4a}\left[-2a^2 + \frac{y^4}{4}\right]_{-2a}^{0} + \frac{1}{4a}\left[2a^2y^2 - \frac{y^4}{4}\right]_{0}^{2a}$

$= 2 \cdot \frac{1}{4a}\left(2a^2 \cdot 4a^2 - \frac{16a^4}{4}\right) = \frac{1}{2a}(8a^4 - 4a^4) = 2a^3$. Therefore, $\bar{x} = \frac{M_y}{M} = \left(\frac{4}{3}a^4\right)\left(\frac{1}{2a^3}\right) = \frac{2a}{3}$ and

$\bar{y} = \frac{M_x}{M} = 0$ is the center of mass.

17. (a) On $[0, a]$ a typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\sqrt{b^2 - x^2} + \sqrt{a^2 - x^2}}{2}\right)$,

length: $\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}$, width: dx, area: $dA = \left(\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}\right) dx$, mass: $dm = \delta\, dA$ $= \delta\left(\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}\right) dx$. On $[a, b]$ a typical *vertical* strip has center of mass:

$(\tilde{x}, \tilde{y}) = \left(x, \frac{\sqrt{b^2 - x^2}}{2}\right)$, length: $\sqrt{b^2 - x^2}$, width: dx, area: $dA = \sqrt{b^2 - x^2}\, dx$,

mass: $dm = \delta\, dA = \delta\sqrt{b^2 - x^2}\, dx$. Thus, $M_x = \int \tilde{y}\, dm$

$= \int_{0}^{a} \frac{1}{2}\left(\sqrt{b^2 - x^2} + \sqrt{a^2 - x^2}\right)\delta\left(\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}\right) dx + \int_{a}^{b} \frac{1}{2}\sqrt{b^2 - x^2}\, \delta\sqrt{b^2 - x^2}\, dx$

$= \frac{\delta}{2} \int_{0}^{a} [(b^2 - x^2) - (a^2 - x^2)]\, dx + \frac{\delta}{2} \int_{a}^{b} (b^2 - x^2)\, dx = \frac{\delta}{2} \int_{0}^{a} (b^2 - a^2)\, dx + \frac{\delta}{2} \int_{a}^{b} (b^2 - x^2)\, dx$

$= \frac{\delta}{2}[(b^2 - a^2)x]_0^a + \frac{\delta}{2}\left[b^2x - \frac{x^3}{3}\right]_a^b = \frac{\delta}{2}[(b^2 - a^2)a] + \frac{\delta}{2}\left[\left(b^3 - \frac{b^3}{3}\right) - \left(b^2a - \frac{a^3}{3}\right)\right]$

$= \frac{\delta}{2}(ab^2 - a^3) + \frac{\delta}{2}\left(\frac{2}{3}b^3 - ab^2 + \frac{a^3}{3}\right) = \frac{\delta b^3}{3} - \frac{\delta a^3}{3} = \delta\left(\frac{b^3 - a^3}{3}\right)$; $M_y = \int \tilde{x}\, dm$

$= \int_0^a x\delta\left(\sqrt{b^2 - x^2} - \sqrt{a^2 - x^2}\right) dx + \int_a^b x\delta\sqrt{b^2 - x^2}\, dx$

$= \delta \int_0^a x(b^2 - x^2)^{1/2}\, dx - \delta \int_0^a x(a^2 - x^2)^{1/2}\, dx + \delta \int_a^b x(b^2 - x^2)^{1/2}\, dx$

$= \frac{-\delta}{2}\left[\frac{2(b^2 - x^2)^{3/2}}{3}\right]_0^a + \frac{\delta}{2}\left[\frac{2(a^2 - x^2)^{3/2}}{3}\right]_0^a - \frac{\delta}{2}\left[\frac{2(b^2 - x^2)^{3/2}}{3}\right]_a^b$

$= -\frac{\delta}{3}\left[(b^2 - a^2)^{3/2} - (b^2)^{3/2}\right] + \frac{\delta}{3}\left[0 - (a^2)^{3/2}\right] - \frac{\delta}{3}\left[0 - (b^2 - a^2)^{3/2}\right] = \frac{\delta b^3}{3} - \frac{\delta a^3}{3} = \frac{\delta(b^3 - a^3)}{3} = M_x$;

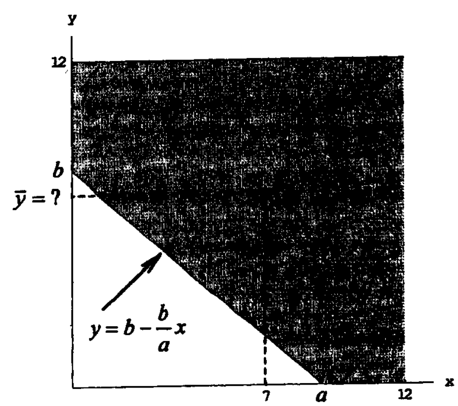
We calculate the mass geometrically: $M = \delta A = \delta\left(\frac{\pi b^2}{4}\right) - \delta\left(\frac{\pi a^2}{4}\right) = \frac{\delta\pi}{4}(b^2 - a^2)$. Thus, $\bar{x} = \frac{M_y}{M}$

$= \frac{\delta(b^3 - a^3)}{3} \cdot \frac{4}{\delta\pi(b^2 - a^2)} = \frac{4}{3\pi}\left(\frac{b^3 - a^3}{b^2 - a^2}\right) = \frac{4}{3\pi}\frac{(b-a)(a^2 + ab + b^2)}{(b-a)(b+a)} = \frac{4(a^2 + ab + b^2)}{3\pi(a+b)}$; likewise

$\bar{y} = \frac{M_x}{M} = \frac{4(a^2 + ab + b^2)}{3\pi(a+b)}$.

(b) $\lim_{a \to b} \frac{4}{3\pi}\left(\frac{a^2 + ab + b^2}{a+b}\right) = \left(\frac{4}{3\pi}\right)\left(\frac{b^2 + b^2 + b^2}{b+b}\right) = \left(\frac{4}{3\pi}\right)\left(\frac{3b^2}{2b}\right) = \frac{2b}{\pi} \Rightarrow (\bar{x}, \bar{y}) = \left(\frac{2b}{\pi}, \frac{2b}{\pi}\right)$ is the limiting position of the centroid as $a \to b$. This is the centroid of a circle of radius a (and we note the two circles coincide when $a = b$).

18. Assume that the x and y intercepts of the triangular corner are a and b, respectively. Then the equation of the sloped edge of the triangle is $y = b - (b/a)x$. The x-coordinate of the centroid must be greater than 6 in. because the triangular cutout will shift the centroid to the right of the center of the square. Therefore, we assume that $\bar{x} = 7$ in. Using vertical strips of area and noting that $\frac{1}{2}ab = 36$ in.^{2}, we calculate $\bar{x}$ as follows:



$\bar{x} = 7 \text{ in.} = \frac{\int_0^a x\left(12 - \left(b - \frac{b}{a}x\right)\right) dx + \int_a^{12} 12x\, dx}{144 - 36}$

$= \frac{6a^2 - \frac{ba^2}{2} + \frac{ba^3}{3a} + 12\left(72 - \frac{a^2}{2}\right)}{108} = \frac{6a^2 - \left(\frac{1}{2}ab\right)a + \frac{2}{3}\left(\frac{1}{2}ab\right)a + 864 - 6a^2}{108} = \frac{-36a + 24a + 864}{108}$

Solving for a and b gives $a = 9$ in. and $b = 8$ in. Next we calculate $\bar{y}$ using horizontal strips of area, but first we express the equation of the sloped edge in terms of y as $x = 9 - \left(\frac{9}{8}\right)y$.

$$\bar{y} = \frac{\int_0^8 y\left(12-\left(9-\frac{9}{8}y\right)\right)dy + \int_8^{12} 12y\,dy}{144-36} = \frac{6\left(8^2\right)-\frac{9\left(8^2\right)}{2}+\frac{9\left(8^3\right)}{3(8)}+12\left(72-\frac{8^2}{2}\right)}{108} = \frac{64}{9}\text{ in.} \approx 7.11\text{ in.}$$

Therefore, the centroid is $\frac{64}{9}$ in. from the bottom of the square.

NOTES:

TECHNOLOGY NOTES: