CHAPTER 6 TRANSCENDENTAL FUNCTIONS AND DIFFERENTIAL EQUATIONS

6.1 LOGARITHMS

1. $y = \ln 3x \Rightarrow y' = \left(\frac{1}{3x}\right)(3) = \frac{1}{x}$

2. $y = \ln(\theta + 1) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\theta + 1}\right)(1) = \frac{1}{\theta + 1}$

3. $y = \ln x^3 \Rightarrow \frac{dy}{dx} = \left(\frac{1}{x^3}\right)(3x^2) = \frac{3}{x}$

4. $y = (\ln x)^3 \Rightarrow \frac{dy}{dx} = 3(\ln x)^2 \cdot \frac{d}{dx}(\ln x) = \frac{3(\ln x)^2}{x}$

5. $y = t(\ln t)^2 \Rightarrow \frac{dy}{dt} = (\ln t)^2 + 2t(\ln t) \cdot \frac{d}{dt}(\ln t) = (\ln t)^2 + \frac{2t \ln t}{t} = (\ln t)^2 + 2 \ln t$

6. $y = t\sqrt{\ln t} = t(\ln t)^{1/2} \Rightarrow \frac{dy}{dt} = (\ln t)^{1/2} + \frac{1}{2}t(\ln t)^{-1/2} \cdot \frac{d}{dt}(\ln t) = (\ln t)^{1/2} + \frac{t(\ln t)^{-1/2}}{2t}$
$= (\ln t)^{1/2} + \frac{1}{2(\ln t)^{1/2}}$

7. $y = \frac{x^4}{4} \ln x - \frac{x^4}{16} \Rightarrow \frac{dy}{dx} = x^3 \ln x + \frac{x^4}{4} \cdot \frac{1}{x} - \frac{4x^3}{16} = x^3 \ln x$

8. $y = \frac{1 + \ln t}{t} \Rightarrow \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (1 + \ln t)(1)}{t^2} = \frac{1 - 1 - \ln t}{t^2} = -\frac{\ln t}{t^2}$

9. $y = \frac{\ln t}{t} \Rightarrow \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (\ln t)(1)}{t^2} = \frac{1 - \ln t}{t^2}$

10. $y = \frac{x \ln x}{1 + \ln x} \Rightarrow y' = \frac{(1 + \ln x)\left(\ln x + x \cdot \frac{1}{x}\right) - (x \ln x)\left(\frac{1}{x}\right)}{(1 + \ln x)^2} = \frac{(1 + \ln x)^2 - \ln x}{(1 + \ln x)^2} = 1 - \frac{\ln x}{(1 + \ln x)^2}$

11. $y = \frac{\ln x}{1 + \ln x} \Rightarrow y' = \frac{(1 + \ln x)\left(\frac{1}{x}\right) - (\ln x)\left(\frac{1}{x}\right)}{(1 + \ln x)^2} = \frac{\frac{1}{x} + \frac{\ln x}{x} - \frac{\ln x}{x}}{(1 + \ln x)^2} = \frac{1}{x(1 + \ln x)^2}$

12. $y = \ln(\ln(\ln x)) \Rightarrow y' = \frac{1}{\ln(\ln x)} \cdot \frac{d}{dx}(\ln(\ln x)) = \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{d}{dx}(\ln x) = \frac{1}{x(\ln x)\ln(\ln x)}$

13. $y = \theta[\sin(\ln \theta) + \cos(\ln \theta)] \Rightarrow \frac{dy}{d\theta} = [\sin(\ln \theta) + \cos(\ln \theta)] + \theta\left[\cos(\ln \theta) \cdot \frac{1}{\theta} - \sin(\ln \theta) \cdot \frac{1}{\theta}\right]$
$= \sin(\ln \theta) + \cos(\ln \theta) + \cos(\ln \theta) - \sin(\ln \theta) = 2\cos(\ln \theta)$

14. $y = \ln(\sec\theta + \tan\theta) \Rightarrow \frac{dy}{d\theta} = \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta} = \frac{\sec\theta(\tan\theta + \sec\theta)}{\tan\theta + \sec\theta} = \sec\theta$

15. $y = \ln\frac{1}{x\sqrt{x+1}} = -\ln x - \frac{1}{2}\ln(x+1) \Rightarrow y' = -\frac{1}{x} - \frac{1}{2}\left(\frac{1}{x+1}\right) = -\frac{2(x+1)+x}{2x(x+1)} = -\frac{3x+2}{2x(x+1)}$

16. $y = \sqrt{\ln\sqrt{t}} = \left(\ln t^{1/2}\right)^{1/2} \Rightarrow \frac{dy}{dt} = \frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2} \cdot \frac{d}{dt}\left(\ln t^{1/2}\right) = \frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2} \cdot \frac{1}{t^{1/2}} \cdot \frac{d}{dt}\left(t^{1/2}\right)$

$= \frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2} \cdot \frac{1}{t^{1/2}} \cdot \frac{1}{2}t^{-1/2} = \frac{1}{4t\sqrt{\ln\sqrt{t}}}$

17. $y = \frac{1+\ln t}{1-\ln t} \Rightarrow \frac{dy}{dt} = \frac{(1-\ln t)\left(\frac{1}{t}\right) - (1+\ln t)\left(\frac{-1}{t}\right)}{(1-\ln t)^2} = \frac{\frac{1}{t} - \frac{\ln t}{t} + \frac{1}{t} + \frac{\ln t}{t}}{(1-\ln t)^2} = \frac{2}{t(1-\ln t)^2}$

18. $y = \frac{1+(\ln t)^2}{1-(\ln t)^2} \Rightarrow \frac{dy}{dt} = \frac{\left(1-(\ln t)^2\right)\left(\frac{2\ln t}{2}\right) - \left(1+(\ln t)^2\right)\left(-\frac{2\ln t}{t}\right)}{\left(1-(\ln t)^2\right)^2} = \frac{4\ln t}{t\left(1-(\ln t)^2\right)^2}$

19. $y = \ln(\sec(\ln\theta)) \Rightarrow \frac{dy}{d\theta} = \frac{1}{\sec(\ln\theta)} \cdot \frac{d}{d\theta}(\sec(\ln\theta)) = \frac{\sec(\ln\theta)\tan(\ln\theta)}{\sec(\ln\theta)} \cdot \frac{d}{d\theta}(\ln\theta) = \frac{\tan(\ln\theta)}{\theta}$

20. $y = \ln\left(\frac{\left(x^2+1\right)^5}{\sqrt{1-x}}\right) = 5\ln\left(x^2+1\right) - \frac{1}{2}\ln(1-x) \Rightarrow y' = \frac{5\cdot 2x}{x^2+1} - \frac{1}{2}\left(\frac{1}{1-x}\right)(-1) = \frac{10x}{x^2+1} + \frac{1}{2(1-x)}$

21. $y = \log_2 5\theta = \frac{\ln 5\theta}{\ln 2} \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\ln 2}\right)\left(\frac{1}{5\theta}\right)(5) = \frac{1}{\theta\ln 2}$

22. $y = \log_4 x + \log_4 x^2 = \frac{\ln x}{\ln 4} + \frac{\ln x^2}{\ln 4} = \frac{\ln x}{\ln 4} + 2\,\frac{\ln x}{\ln 4} = 3\,\frac{\ln x}{\ln 4} \Rightarrow y' = \frac{3}{x\ln 4}$

23. $y = \log_2 r \cdot \log_4 r = \left(\frac{\ln r}{\ln 2}\right)\left(\frac{\ln r}{\ln 4}\right) = \frac{\ln^2 r}{(\ln 2)(\ln 4)} \Rightarrow \frac{dy}{dr} = \left[\frac{1}{(\ln 2)(\ln 4)}\right](2\ln r)\left(\frac{1}{r}\right) = \frac{2\ln r}{r(\ln 2)(\ln 4)}$

24. $y = \log_3\left(\left(\frac{x+1}{x-1}\right)^{\ln 3}\right) = \frac{\ln\left(\frac{x+1}{x-1}\right)^{\ln 3}}{\ln 3} = \frac{(\ln 3)\ln\left(\frac{x+1}{x-1}\right)}{\ln 3} = \ln\left(\frac{x+1}{x-1}\right) = \ln(x+1) - \ln(x-1)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{(x+1)(x-1)}$

25. $y = \theta\sin(\log_7\theta) = \theta\sin\left(\frac{\ln\theta}{\ln 7}\right) \Rightarrow \frac{dy}{d\theta} = \sin\left(\frac{\ln\theta}{\ln 7}\right) + \theta\left[\cos\left(\frac{\ln\theta}{\ln 7}\right)\right]\left(\frac{1}{\theta\ln 7}\right) = \sin(\log_7\theta) + \frac{1}{\ln 7}\cos(\log_7\theta)$

26. $y = 3\log_8(\log_2 t) = \frac{3\ln(\log_2 t)}{\ln 8} = \frac{3\ln\left(\frac{\ln t}{\ln 2}\right)}{\ln 8} \Rightarrow \frac{dy}{dt} = \left(\frac{3}{\ln 8}\right)\left[\frac{1}{(\ln t)/(\ln 2)}\right]\left(\frac{1}{t\ln 2}\right) = \frac{3}{t(\ln t)(\ln 8)}$

$= \frac{1}{t(\ln t)(\ln 2)}$

27. $y = \int_{x^2/2}^{x^2} \ln \sqrt{t}\, dt \Rightarrow \frac{dy}{dx} = \left(\ln \sqrt{x^2}\right) \cdot \frac{d}{dx}(x^2) - \left(\ln \sqrt{\frac{x^2}{2}}\right) \cdot \frac{d}{dx}\left(\frac{x^2}{2}\right) = 2x \ln |x| - x \ln \frac{|x|}{\sqrt{2}}$

28. $y = \int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t\, dt \Rightarrow \frac{dy}{dx} = \left(\ln \sqrt[3]{x}\right) \cdot \frac{d}{dx}\left(\sqrt[3]{x}\right) - \left(\ln \sqrt{x}\right) \cdot \frac{d}{dx}\left(\sqrt{x}\right) = \left(\ln \sqrt[3]{x}\right)\left(\frac{1}{3}x^{-2/3}\right) - \left(\ln \sqrt{x}\right)\left(\frac{1}{2}x^{-1/2}\right)$

$= \frac{\ln \sqrt[3]{x}}{3\sqrt[3]{x^2}} - \frac{\ln \sqrt{x}}{2\sqrt{x}}$

29. $\int_{-3}^{-2} \frac{1}{x}\, dx = \left[\ln |x|\right]_{-3}^{-2} = \ln 2 - \ln 3 = \ln \frac{2}{3}$

30. $\int_{-1}^{0} \frac{3}{3x-2}\, dx = \left[\ln |3x-2|\right]_{-1}^{0} = \ln 2 - \ln 5 = \ln \frac{2}{5}$

31. $\int \frac{2y}{y^2-25}\, dy = \ln \left|y^2 - 25\right| + C$

32. $\int \frac{8r}{4r^2-5}\, dr = \ln \left|4r^2 - 5\right| + C$

33. $\int_0^{\pi} \frac{\sin t}{2-\cos t}\, dt = \left[\ln |2-\cos t|\right]_0^{\pi} = \ln 3 - \ln 1 = \ln 3$; or let $u = 2 - \cos t \Rightarrow du = \sin t\, dt$ with $t = 0$

$\Rightarrow u = 1$ and $t = \pi \Rightarrow u = 3 \Rightarrow \int_0^{\pi} \frac{\sin t}{2-\cos t}\, dt = \int_1^3 \frac{1}{u}\, du = \left[\ln |u|\right]_1^3 = \ln 3 - \ln 1 = \ln 3$

34. $\int_0^{\pi/3} \frac{4 \sin \theta}{1 - 4\cos \theta}\, d\theta = \left[\ln |1 - 4\cos \theta|\right]_0^{\pi/3} = \ln |1-2| = -\ln 3 = \ln \frac{1}{3}$; or let $u = 1 - 4\cos \theta \Rightarrow du = 4 \sin \theta\, d\theta$

with $\theta = 0 \Rightarrow u = -3$ and $\theta = \frac{\pi}{3} \Rightarrow u = -1 \Rightarrow \int_0^{\pi/3} \frac{4 \sin \theta}{1 - 4\cos \theta}\, d\theta = \int_{-3}^{-1} \frac{1}{u}\, du = \left[\ln |u|\right]_{-3}^{-1} = -\ln 3 = \ln \frac{1}{3}$

35. Let $u = \ln x \Rightarrow du = \frac{1}{x}\, dx$; $x = 1 \Rightarrow u = 0$ and $x = 2 \Rightarrow u = \ln 2$;

$\int_1^2 \frac{2 \ln x}{x}\, dx = \int_0^{\ln 2} 2u\, du = \left[u^2\right]_0^{\ln 2} = (\ln 2)^2$

36. Let $u = \ln x \Rightarrow du = \frac{1}{x}\, dx$; $x = 2 \Rightarrow u = \ln 2$ and $x = 4 \Rightarrow u = \ln 4$;

$\int_2^4 \frac{dx}{x \ln x} = \int_{\ln 2}^{\ln 4} \frac{1}{u}\, du = [\ln u]_{\ln 2}^{\ln 4} = \ln (\ln 4) - \ln (\ln 2) = \ln\left(\frac{\ln 4}{\ln 2}\right) = \ln\left(\frac{\ln 2^2}{\ln 2}\right) = \ln\left(\frac{2 \ln 2}{\ln 2}\right) = \ln 2$

37. Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$; $x = 2 \Rightarrow u = \ln 2$ and $x = 4 \Rightarrow u = \ln 4$;

$$\int_2^4 \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\ln 4} u^{-2}\,du = \left[-\frac{1}{u}\right]_{\ln 2}^{\ln 4} = -\frac{1}{\ln 4} + \frac{1}{\ln 2} = -\frac{1}{\ln 2^2} + \frac{1}{\ln 2} = -\frac{1}{2\ln 2} + \frac{1}{\ln 2} = \frac{1}{2\ln 2} = \frac{1}{\ln 4}$$

38. Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$; $x = 2 \Rightarrow u = \ln 2$ and $x = 16 \Rightarrow u = \ln 16$;

$$\int_2^{16} \frac{dx}{2x\sqrt{\ln x}} = \frac{1}{2}\int_{\ln 2}^{\ln 16} u^{-1/2}\,du = \left[u^{1/2}\right]_{\ln 2}^{\ln 16} = \sqrt{\ln 16} - \sqrt{\ln 2} = \sqrt{4\ln 2} - \sqrt{\ln 2} = 2\sqrt{\ln 2} - \sqrt{\ln 2} = \sqrt{\ln 2}$$

39. Let $u = 6 + 3\tan t \Rightarrow du = 3\sec^2 t\,dt$;

$$\int \frac{3\sec^2 t}{6 + 3\tan t}\,dt = \int \frac{du}{u} = \ln|u| + C = \ln|6 + 3\tan t| + C$$

40. Let $u = 2 + \sec y \Rightarrow du = \sec y \tan y\,dy$;

$$\int \frac{\sec y \tan y}{2 + \sec y}\,dy = \int \frac{du}{u} = \ln|u| + C = \ln|2 + \sec y| + C$$

41. Let $u = \cos\frac{x}{2} \Rightarrow du = -\frac{1}{2}\sin\frac{x}{2}\,dx \Rightarrow -2\,du = \sin\frac{x}{2}\,dx$; $x = 0 \Rightarrow u = 1$ and $x = \frac{\pi}{2} \Rightarrow u = \frac{1}{\sqrt{2}}$;

$$\int_0^{\pi/2} \tan\frac{x}{2}\,dx = \int_0^{\pi/2} \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\,dx = -2\int_1^{1/\sqrt{2}} \frac{du}{u} = \left[-2\ln|u|\right]_1^{1/\sqrt{2}} = -2\ln\frac{1}{\sqrt{2}} = 2\ln\sqrt{2} = \ln 2$$

42. Let $u = \sin t \Rightarrow du = \cos t\,dt$; $t = \frac{\pi}{4} \Rightarrow u = \frac{1}{\sqrt{2}}$ and $t = \frac{\pi}{2} \Rightarrow u = 1$;

$$\int_{\pi/4}^{\pi/2} \cot t\,dt = \int_{\pi/4}^{\pi/2} \frac{\cos t}{\sin t}\,dt = \int_{1/\sqrt{2}}^{1} \frac{du}{u} = \left[\ln|u|\right]_{1/\sqrt{2}}^{1} = -\ln\frac{1}{\sqrt{2}} = \ln\sqrt{2}$$

43. Let $u = \sin\frac{\theta}{3} \Rightarrow du = \frac{1}{3}\cos\frac{\theta}{3}\,d\theta \Rightarrow 6\,du = 2\cos\frac{\theta}{3}\,d\theta$; $\theta = \frac{\pi}{2} \Rightarrow u = \frac{1}{2}$ and $\theta = \pi \Rightarrow u = \frac{\sqrt{3}}{2}$;

$$\int_{\pi/2}^{\pi} 2\cot\frac{\theta}{3}\,d\theta = \int_{\pi/2}^{\pi} \frac{2\cos\frac{\theta}{3}}{\sin\frac{\theta}{3}}\,d\theta = 6\int_{1/2}^{\sqrt{3}/2} \frac{du}{u} = 6\left[\ln|u|\right]_{1/2}^{\sqrt{3}/2} = 6\left(\ln\frac{\sqrt{3}}{2} - \ln\frac{1}{2}\right) = 6\ln\sqrt{3} = \ln 27$$

44. Let $u = \cos 3x \Rightarrow du = -3\sin 3x\,dx \Rightarrow -2\,du = 6\sin 3x\,dx$; $x = 0 \Rightarrow u = 1$ and $x = \frac{\pi}{12} \Rightarrow u = \frac{1}{\sqrt{2}}$;

$$\int_0^{\pi/12} 6\tan 3x\,dx = \int_0^{\pi/12} \frac{6\sin 3x}{\cos 3x}\,dx = -2\int_1^{1/\sqrt{2}} \frac{du}{u} = -2\left[\ln|u|\right]_1^{1/\sqrt{2}} = -2\ln\frac{1}{\sqrt{2}} - \ln 1 = 2\ln\sqrt{2} = \ln 2$$

45. $\int \frac{dx}{2\sqrt{x} + 2x} = \int \frac{dx}{2\sqrt{x}(1 + \sqrt{x})}$; let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$; $\int \frac{dx}{2\sqrt{x}(1 + \sqrt{x})} = \int \frac{du}{u} = \ln|u| + C$

$= \ln\left|1+\sqrt{x}\right| + C = \ln\left(1+\sqrt{x}\right) + C$

46. Let $u = \sec x + \tan x \Rightarrow du = \left(\sec x \tan x + \sec^2 x\right) dx = (\sec x)(\tan x + \sec x)\ dx \Rightarrow \sec x\ dx = \frac{du}{u}$;

$$\int \frac{\sec x\ dx}{\sqrt{\ln(\sec x + \tan x)}} = \int \frac{du}{u\sqrt{\ln u}} = \int (\ln u)^{-1/2} \cdot \frac{1}{u}\ du = 2(\ln u)^{1/2} + C = 2\sqrt{\ln(\sec x + \tan x)} + C$$

47. (a) $f(x) = \ln(\cos x) \Rightarrow f'(x) = -\frac{\sin x}{\cos x} = -\tan x = 0 \Rightarrow x = 0$; $f'(x) > 0$ for $-\frac{\pi}{4} \le x < 0$ and $f'(x) < 0$ for $0 < x \le \frac{\pi}{3} \Rightarrow$ there is a relative maximum at $x = 0$ with $f(0) = \ln(\cos 0) = \ln 1 = 0$; $f\left(-\frac{\pi}{4}\right) = \ln\left(\cos\left(-\frac{\pi}{4}\right)\right)$ $= \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln 2$ and $f\left(\frac{\pi}{3}\right) = \ln\left(\cos\left(\frac{\pi}{3}\right)\right) = \ln\frac{1}{2} = -\ln 2$. Therefore, the absolute minimum occurs at $x = \frac{\pi}{3}$ with $f\left(\frac{\pi}{3}\right) = -\ln 2$ and the absolute maximum occurs at $x = 0$ with $f(0) = 0$.

(b) $f(x) = \cos(\ln x) \Rightarrow f'(x) = \frac{-\sin(\ln x)}{x} = 0 \Rightarrow x = 1$; $f'(x) > 0$ for $\frac{1}{2} \le x < 1$ and $f'(x) < 0$ for $1 < x \le 2$ $\Rightarrow$ there is a relative maximum at $x = 1$ with $f(1) = \cos(\ln 1) = \cos 0 = 1$; $f\left(\frac{1}{2}\right) = \cos\left(\ln\left(\frac{1}{2}\right)\right)$ $= \cos(-\ln 2) = \cos(\ln 2)$ and $f(2) = \cos(\ln 2)$. Therefore, the absolute minimum occurs at $x = \frac{1}{2}$ and $x = 2$ with $f\left(\frac{1}{2}\right) = f(2) = \cos(\ln 2)$, and the absolute maximum occurs at $x = 1$ with $f(1) = 1$.

48. (a) $f(x) = x - \ln x \Rightarrow f'(x) = 1 - \frac{1}{x}$; if $x > 1$, then $f'(x) > 0$ which means that $f(x)$ is increasing

(b) $f(1) = 1 - \ln 1 = 1 \Rightarrow f(x) = x - \ln x > 0$, if $x > 1$ by part (a) $\Rightarrow \ln x < x$ if $x > 1$

49. $\int_1^5 (\ln 2x - \ln x)\ dx = \int_1^5 (-\ln x + \ln 2 + \ln x)\ dx = (\ln 2)\int_1^5 dx = (\ln 2)(5-1) = \ln 2^4 = \ln 16$

50. $A = \int_{-\pi/4}^{0} -\tan x\ dx + \int_0^{\pi/3} \tan x\ dx = \int_{-\pi/4}^{0} \frac{-\sin x}{\cos x}\ dx - \int_0^{\pi/3} \frac{-\sin x}{\cos x}\ dx = \left[\ln|\cos x|\right]_{-\pi/4}^{0} - \left[\ln|\cos x|\right]_0^{\pi/3}$

$= \left(\ln 1 - \ln\frac{1}{\sqrt{2}}\right) - \left(\ln\frac{1}{2} - \ln 1\right) = \ln\sqrt{2} + \ln 2 = \frac{3}{2}\ln 2$

51. $\frac{dy}{dx} = 1 + \frac{1}{x}$ at $(1,3) \Rightarrow y = x + \ln|x| + C$; $y = 3$ at $x = 1 \Rightarrow C = 2 \Rightarrow y = x + \ln|x| + 2$

52. $\frac{d^2y}{dx^2} = \sec^2 x \Rightarrow \frac{dy}{dx} = \tan x + C$ and $1 = \tan 0 + C \Rightarrow \frac{dy}{dx} = \tan x + 1 \Rightarrow y = \int (\tan x + 1)\ dx$

$= \ln|\sec x| + x + C_1$ and $0 = \ln|\sec 0| + 0 + C_1 \Rightarrow C_1 = 0 \Rightarrow y = \ln|\sec x| + x$

53. $\int \frac{\log_{10} x}{x}\ dx = \int \left(\frac{\ln x}{\ln 10}\right)\left(\frac{1}{x}\right) dx$; $\left[u = \ln x \Rightarrow du = \frac{1}{x}\ dx\right]$

$\Rightarrow \int \left(\frac{\ln x}{\ln 10}\right)\left(\frac{1}{x}\right) dx = \frac{1}{\ln 10}\int u\ du = \left(\frac{1}{\ln 10}\right)\left(\frac{1}{2}u^2\right) + C = \frac{(\ln x)^2}{2\ln 10} + C$

54. $\int_1^4 \frac{\ln 2 \log_2 x}{x}\,dx = \int_1^4 \left(\frac{\ln 2}{x}\right)\left(\frac{\ln x}{\ln 2}\right)dx = \int_1^4 \frac{\ln x}{x}\,dx = \left[\frac{1}{2}(\ln x)^2\right]_1^4 = \frac{1}{2}[(\ln 4)^2 - (\ln 1)^2] = \frac{1}{2}(\ln 4)^2$

$= \frac{1}{2}(2\ln 2)^2 = 2(\ln 2)^2$

55. $\int_0^2 \frac{\log_2(x+2)}{x+2}\,dx = \frac{1}{\ln 2}\int_0^2 [\ln(x+2)]\left(\frac{1}{x+2}\right)dx = \left(\frac{1}{\ln 2}\right)\left[\frac{(\ln(x+2))^2}{2}\right]_0^2 = \left(\frac{1}{\ln 2}\right)\left[\frac{(\ln 4)^2}{2} - \frac{(\ln 2)^2}{2}\right]$

$= \left(\frac{1}{\ln 2}\right)\left[\frac{4(\ln 2)^2}{2} - \frac{(\ln 2)^2}{2}\right] = \frac{3}{2}\ln 2$

56. $\int_0^9 \frac{2\log_{10}(x+1)}{x+1}\,dx = \frac{2}{\ln 10}\int_0^9 \ln(x+1)\left(\frac{1}{x+1}\right)dx = \left(\frac{2}{\ln 10}\right)\left[\frac{(\ln(x+1))^2}{2}\right]_0^9 = \left(\frac{2}{\ln 10}\right)\left[\frac{(\ln 10)^2}{2} - \frac{(\ln 1)^2}{2}\right]$

$= \ln 10$

57. $\int \frac{dx}{x\log_{10} x} = \int \left(\frac{\ln 10}{\ln x}\right)\left(\frac{1}{x}\right)dx = (\ln 10)\int \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)dx;\ \left[u = \ln x \Rightarrow du = \frac{1}{x}\,dx\right]$

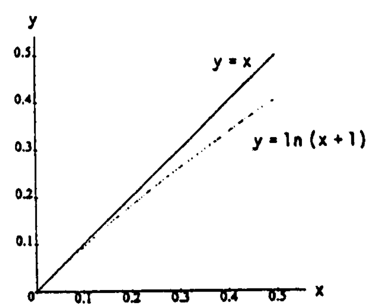
$\Rightarrow (\ln 10)\int \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)dx = (\ln 10)\int \frac{1}{u}\,du = (\ln 10)\ln|u| + C = (\ln 10)\ln|\ln x| + C$

58. $\int \frac{dx}{x(\log_8 x)^2} = \int \frac{dx}{x\left(\frac{\ln x}{\ln 8}\right)^2} = (\ln 8)^2\int \frac{(\ln x)^{-2}}{x}\,dx = (\ln 8)^2\,\frac{(\ln x)^{-1}}{-1} + C = -\frac{(\ln 8)^2}{\ln x} + C$

59. (a) $L(x) = f(0) + f'(0)\cdot x$, and $f(x) = \ln(1+x) \Rightarrow f'(x)\big|_{x=0} = \frac{1}{1+x}\Big|_{x=0} = 1 \Rightarrow L(x) = \ln 1 + 1\cdot x \Rightarrow L(x) = x$

(b) On $[0, 0.1]$, $f(x)$ and $L(x)$are both increasing because $f'(x) = \frac{1}{1+x} > 0$ and $L'(x) = 1 > 0$ for $0 \le x \le 0.1$. In addition $0 \le x \le 0.1 \Rightarrow 1 \le 1+x \le 1.1 \Rightarrow \frac{1}{1.1} \le \frac{1}{x+1} \le 1 \Rightarrow L'(x) \ge f'(x) \Rightarrow E(x) = f(x) - L(x)$ is non-increasing on $[0, 0.1]$ because $E'(x) = f'(x) - L' \le 0$ on the interval. Therefore, the largest error is $|f(0.1) - L(0.1)| = |\ln(1.1) - 1.1| \approx 0.00469$.

(c) The approximation $y = x$ for $\ln(1+x)$ is best for smaller positive values of x on the interval $[0, 0.1]$ as seen on the graph. As x increases so does the magnitude of the error $|\ln(x) - x|$. From the graph, an upper bound for the magnitude of the error is $|\ln(1.1) - 0.1| \approx 0.00469$ which is consistent with the analytical result obtained in part (b).



60. $\ln(1.2) = \ln(1+.2) \approx 0.2$, $\ln(.8) = \ln(1+(-0.2)) \approx -0.2$; with Simpson's rule for $n = 2$, $\ln(1.2) = \int_1^{1.2} \frac{1}{t}\,dt$

≈ 0.182323232 and $\ln(0.8) = \int_1^{0.8} \frac{1}{t}\,dt \approx -0.223148148$; alternatively, $\ln(1.2) = \ln(1+0.2) = \int_0^{0.2} \frac{1}{1+t}\,dt$

≈ 0.182323232 and $\ln(0.8) = \int_0^{-0.2} \frac{1}{1+t}\,dt \approx -0.223148148.$

6.2 EXPONENTIAL FUNCTIONS

1. $y = e^{-2x/3} \Rightarrow y' = -\frac{2}{3}e^{-2x/3}$

2. $y = e^{5-7x} \Rightarrow y' = e^{5-7x}\frac{d}{dx}(5-7x) \Rightarrow y' = -7e^{5-7x}$

3. $y = e^{\left(4\sqrt{x}+x^2\right)} \Rightarrow y' = e^{\left(4\sqrt{x}+x^2\right)}\frac{d}{dx}\left(4\sqrt{x}+x^2\right) \Rightarrow y' = \left(\frac{2}{\sqrt{x}}+2x\right)e^{\left(4\sqrt{x}+x^2\right)}$

4. $y = (1+3x)e^{-x} \Rightarrow y' = (1+3x)\left(-e^{-x}\right)3e^{-x} = (2-3x)e^{-x}$

5. $y = \left(x^2-2x+2\right)e^x \Rightarrow y' = (2x-2)e^x + \left(x^2-2x+2\right)e^x = x^2e^x$

6. $y = e^{\theta}(\sin\theta+\cos\theta) \Rightarrow y' = e^{\theta}(\sin\theta+\cos\theta) + e^{\theta}(\cos\theta-\sin\theta) = 2e^{\theta}\cos\theta$

7. $y = \ln\left(3\theta e^{-\theta}\right) = \ln 3 + \ln\theta + \ln e^{-\theta} = \ln 3 + \ln\theta - \theta \Rightarrow \frac{dy}{d\theta} = \frac{1}{\theta} - 1$

8. $y = \cos\left(e^{-\theta^2}\right) \Rightarrow \frac{dy}{d\theta} = -\sin\left(e^{-\theta^2}\right)\frac{d}{d\theta}\left(e^{-\theta^2}\right) = \left(-\sin\left(e^{-\theta^2}\right)\right)\left(e^{-\theta^2}\right)\frac{d}{d\theta}\left(-\theta^2\right) = 2\theta e^{-\theta^2}\sin\left(e^{-\theta^2}\right)$

9. $y = \ln\left(2e^{-t}\sin t\right) = \ln 2 + \ln e^{-t} + \ln\sin t = \ln 2 - t + \ln\sin t \Rightarrow \frac{dy}{dt} = -1 + \left(\frac{1}{\sin t}\right)\frac{d}{dt}(\sin t) = -1 + \frac{\cos t}{\sin t}$
$= \frac{\cos t - \sin t}{\sin t}$

10. $y = \ln\frac{e^{\theta}}{1+e^{\theta}} = \ln e^{\theta} - \ln\left(1+e^{\theta}\right) = \theta - \ln\left(1+e^{\theta}\right) \Rightarrow \frac{dy}{d\theta} = 1 - \left(\frac{1}{1+e^{\theta}}\right)\frac{d}{d\theta}\left(1+e^{\theta}\right) = 1 - \frac{e^{\theta}}{1+e^{\theta}} = \frac{1}{1+e^{\theta}}$

11. $y = \ln\frac{\sqrt{\theta}}{1+\sqrt{\theta}} = \ln\sqrt{\theta} - \ln\left(1+\sqrt{\theta}\right) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\sqrt{\theta}}\right)\frac{d}{d\theta}\left(\sqrt{\theta}\right) - \left(\frac{1}{1+\sqrt{\theta}}\right)\frac{d}{d\theta}\left(1+\sqrt{\theta}\right)$
$= \left(\frac{1}{\sqrt{\theta}}\right)\left(\frac{1}{2\sqrt{\theta}}\right) - \left(\frac{1}{1+\sqrt{\theta}}\right)\left(\frac{1}{2\sqrt{\theta}}\right) = \frac{\left(1+\sqrt{\theta}\right)-\sqrt{\theta}}{2\theta\left(1+\sqrt{\theta}\right)} = \frac{1}{2\theta\left(1+\sqrt{\theta}\right)} = \frac{1}{2\theta\left(1+\theta^{1/2}\right)}$

12. $y = e^{\sin t}\left(\ln t^2+1\right) \Rightarrow \frac{dy}{dt} = e^{\sin t}(\cos t)\left(\ln t^2+1\right) + \frac{2}{t}e^{\sin t} = e^{\sin t}\left[\left(\ln t^2+1\right)(\cos t) + \frac{2}{t}\right]$

13. $\int_0^{\ln x} \sin e^t\,dt \Rightarrow y' = \left(\sin e^{\ln x}\right)\cdot\frac{d}{dx}(\ln x) = \frac{\sin x}{x}$

14. $y = \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t \, dt \Rightarrow y' = \left(\ln e^{2x}\right) \cdot \frac{d}{dx}\left(e^{2x}\right) - \left(\ln e^{4\sqrt{x}}\right) \cdot \frac{d}{dx}\left(e^{4\sqrt{x}}\right) = (2x)\left(2e^{2x}\right) - \left(4\sqrt{x}\right)\left(e^{4\sqrt{x}}\right) \cdot \frac{d}{dx}\left(4\sqrt{x}\right)$

$= 4xe^{2x} - 4\sqrt{x}\, e^{4\sqrt{x}}\left(\frac{2}{\sqrt{x}}\right) = 4xe^{2x} - 8e^{4\sqrt{x}}$

15. $\ln y = e^y \sin x \Rightarrow \left(\frac{1}{y}\right)y' = \left(y'e^y\right)(\sin x) + e^y \cos x \Rightarrow y'\left(\frac{1}{y} - e^y \sin x\right) = e^y \cos x$

$\Rightarrow y'\left(\frac{1 - ye^y \sin x}{y}\right) = e^y \cos x \Rightarrow y' = \frac{ye^y \cos x}{1 - ye^y \sin x}$

16. $\ln xy = e^{x+y} \Rightarrow \ln x + \ln y = e^{x+y} \Rightarrow \frac{1}{x} + \left(\frac{1}{y}\right)y' = \left(1 + y'\right)e^{x+y} \Rightarrow y'\left(\frac{1}{y} - e^{x+y}\right) = e^{x+y} - \frac{1}{x}$

$\Rightarrow y'\left(\frac{1 - ye^{x+y}}{y}\right) = \frac{xe^{x+y} - 1}{x} \Rightarrow y' = \frac{y\left(xe^{x+y} - 1\right)}{x\left(1 - ye^{x+y}\right)}$

17. $e^{2x} = \sin(x + 3y) \Rightarrow 2e^{2x} = \left(1 + 3y'\right)\cos(x + 3y) \Rightarrow 1 + 3y' = \frac{2e^{2x}}{\cos(x + 3y)} \Rightarrow 3y' = \frac{2e^{2x}}{\cos(x + 3y)} - 1$

$\Rightarrow y' = \frac{2e^{2x} - \cos(x + 3y)}{3\cos(x + 3y)}$

18. $\tan y = e^x + \ln x \Rightarrow \left(\sec^2 y\right)y' = e^x + \frac{1}{x} \Rightarrow y' = \frac{\left(xe^x + 1\right)\cos^2 y}{x}$

19. $\int \left(e^{3x} + 5e^{-x}\right) dx = \frac{e^{3x}}{3} - 5e^{-x} + C$

20. $\int_{\ln 2}^{\ln 3} e^x \, dx = \left[e^x\right]_{\ln 2}^{\ln 3} = e^{\ln 3} - e^{\ln 2} = 3 - 2 = 1$

21. $\int 8e^{(x+1)} \, dx = 8e^{(x+1)} + C$

22. $\int_{\ln 4}^{\ln 9} e^{x/2} \, dx = \left[2e^{x/2}\right]_{\ln 4}^{\ln 9} = 2\left[e^{(\ln 9)/2} - e^{(\ln 4)/2}\right] = 2\left(e^{\ln 3} - e^{\ln 2}\right) = 2(3 - 2) = 2$

23. Let $u = -r^{1/2} \Rightarrow du = -\frac{1}{2}r^{-1/2} \, dr \Rightarrow -2 \, du = r^{-1/2} \, dr$;

$\int \frac{e^{-\sqrt{r}}}{\sqrt{r}} \, dr = \int e^{-r^{1/2}} \cdot r^{-1/2} \, dr = -2 \int e^u \, du = -2e^{-r^{1/2}} + C = -2e^{-\sqrt{r}} + C$

24. Let $u = -t^2 \Rightarrow du = -2t \, dt \Rightarrow -du = 2t \, dt$;

$\int 2te^{-t^2} \, dt = -\int e^u \, du = -e^u + C = -e^{-t^2} + C$

25. Let $u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2}\,dx \Rightarrow -du = \frac{1}{x^2}\,dx$;

$$\int \frac{e^{1/x}}{x^2}\,dx = \int -e^u\,du = -e^u + C = -e^{1/x} + C$$

26. Let $u = -x^{-2} \Rightarrow du = 2x^{-3}\,dx \Rightarrow \frac{1}{2}\,du = x^{-3}\,dx$;

$$\int \frac{e^{-1/x^2}}{x^3}\,dx = \int e^{-x^{-2}} \cdot x^{-3}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{-x^{-2}} + C = \frac{1}{2}e^{-1/x^2} + C$$

27. Let $u = \tan\theta \Rightarrow du = \sec^2\theta\,d\theta$; $\theta = 0 \Rightarrow u = 0$, $\theta = \frac{\pi}{4} \Rightarrow u = 1$;

$$\int_0^{\pi/4} \left(1 + e^{\tan\theta}\right)\sec^2\theta\,d\theta = \int_0^{\pi/4} \sec^2\theta\,d\theta + \int_0^1 e^u\,du = [\tan\theta]_0^{\pi/4} + \left[e^u\right]_0^1 = \left[\tan\left(\frac{\pi}{4}\right) - \tan(0)\right] + \left(e^1 - e^0\right)$$

$$= (1 - 0) + (e - 1) = e$$

28. Let $u = \cot\theta \Rightarrow du = -\csc^2\theta\,d\theta$; $\theta = \frac{\pi}{4} \Rightarrow u = 1$, $\theta = \frac{\pi}{2} \Rightarrow u = 0$;

$$\int_{\pi/4}^{\pi/2} \left(1 + e^{\cot\theta}\right)\csc^2\theta\,d\theta = \int_{\pi/4}^{\pi/2} \csc^2\theta\,d\theta - \int_1^0 e^u\,du = [-\cot\theta]_{\pi/4}^{\pi/2} - \left[e^u\right]_1^0 = \left[-\cot\left(\frac{\pi}{2}\right) + \cot\left(\frac{\pi}{4}\right)\right] - \left(e^0 - e^1\right)$$

$$= (0 + 1) - (1 - e) = e$$

29. Let $u = \sec \pi t \Rightarrow du = \frac{1}{\pi}\sec \pi t \tan \pi t\,dt \Rightarrow \pi\,du = \sec \pi t \tan \pi t\,dt$;

$$\int e^{\sec(\pi t)}\sec(\pi t)\tan(\pi t)\,dt = \frac{1}{\pi}\int e^u\,du = \frac{e^u}{\pi} + C = \frac{e^{\sec(\pi t)}}{\pi} + C$$

30. Let $u = e^{x^2} \Rightarrow du = 2xe^{x^2}\,dx$; $x = 0 \Rightarrow u = 1$, $x = \sqrt{\ln\pi} \Rightarrow u = e^{\ln\pi} = \pi$;

$$\int_0^{\sqrt{\ln\pi}} 2xe^{x^2}\cos\left(e^{x^2}\right)dx = \int_1^{\pi} \cos u\,du = [\sin u]_1^{\pi} = \sin(\pi) - \sin(1) = -\sin(1) \approx -0.84147$$

31. Let $u = 1 + e^r \Rightarrow du = e^r\,dr$;

$$\int \frac{e^r}{1 + e^r}\,dr = \int \frac{1}{u}\,du = \ln|u| + C = \ln\left(1 + e^r\right) + C$$

32. $\int \frac{1}{1 + e^x}\,dx = \int \frac{e^{-x}}{e^{-x} + 1}\,dx$;

let $u = e^{-x} + 1 \Rightarrow du = -e^{-x}\,dx \Rightarrow -du = e^{-x}\,dx$;

$$\int \frac{e^{-x}}{e^{-x} + 1}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln\left(e^{-x} + 1\right) + C$$

33. $y = 2^x \Rightarrow y' = 2^x \ln 2$

34. $y = 5^{\sqrt{s}} \Rightarrow \frac{dy}{ds} = 5^{\sqrt{s}} (\ln 5)\left(\frac{1}{2}s^{-1/2}\right) = \left(\frac{\ln 5}{2\sqrt{s}}\right)5^{\sqrt{s}}$

35. $y = x^{\pi} \Rightarrow y' = \pi x^{(\pi-1)}$

36. $y = (\cos\theta)^{\sqrt{2}} \Rightarrow \frac{dy}{d\theta} = -\sqrt{2}\,(\cos\theta)^{\left(\sqrt{2}-1\right)}(\sin\theta)$

37. $y = 7^{\sec\theta}\ln 7 \Rightarrow \frac{dy}{d\theta} = \left(7^{\sec\theta}\ln 7\right)(\ln 7)(\sec\theta\tan\theta) = 7^{\sec\theta}(\ln 7)^2(\sec\theta\tan\theta)$

38. $y = 2^{\sin 3t} \Rightarrow \frac{dy}{dt} = \left(2^{\sin 3t}\ln 2\right)(\cos 3t)(3) = (3\cos 3t)\left(2^{\sin 3t}\right)(\ln 2)$

39. $y = t^{1-e} \Rightarrow \frac{dy}{dt} = (1-e)\,t^{-e}$

40. $y = (\ln\theta)^{\pi} \Rightarrow \frac{dy}{d\theta} = \pi(\ln\theta)^{(\pi-1)}\left(\frac{1}{\theta}\right) = \frac{\pi(\ln\theta)^{(\pi-1)}}{\theta}$

41. $y = \log_3\left(\left(\frac{x+1}{x-1}\right)^{\ln 3}\right) = \frac{\ln\left(\frac{x+1}{x-1}\right)^{\ln 3}}{\ln 3} = \frac{(\ln 3)\ln\left(\frac{x+1}{x-1}\right)}{\ln 3} = \ln\left(\frac{x+1}{x-1}\right) = \ln(x+1) - \ln(x-1)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{(x+1)(x-1)}$

42. $y = \log_5\sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}} = \log_5\left(\frac{7x}{3x+2}\right)^{(\ln 5)/2} = \frac{\ln\left(\frac{7x}{3x+2}\right)^{(\ln 5)/2}}{\ln 5} = \left(\frac{\ln 5}{2}\right)\left[\frac{\ln\left(\frac{7x}{3x+2}\right)}{\ln 5}\right] = \frac{1}{2}\ln\left(\frac{7x}{3x+2}\right)$

$= \frac{1}{2}\ln 7x - \frac{1}{2}\ln(3x+2) \Rightarrow \frac{dy}{dx} = \frac{7}{2\cdot 7x} - \frac{3}{2\cdot(3x+2)} = \frac{(3x+2)-3x}{2x(3x+2)} = \frac{1}{x(3x+2)}$

43. $y = \log_7\left(\frac{\sin\theta\cos\theta}{e^{\theta}2^{\theta}}\right) = \frac{\ln(\sin\theta) + \ln(\cos\theta) - \ln e^{\theta} - \ln 2^{\theta}}{\ln 7} = \frac{\ln(\sin\theta) + \ln(\cos\theta) - \theta - \theta\ln 2}{\ln 7}$

$\Rightarrow \frac{dy}{d\theta} = \frac{\cos\theta}{(\sin\theta)(\ln 7)} - \frac{\sin\theta}{(\cos\theta)(\ln 7)} - \frac{1}{\ln 7} - \frac{\ln 2}{\ln 7} = \left(\frac{1}{\ln 7}\right)(\cot\theta - \tan\theta - 1 - \ln 2)$

44. $y = \log_2\left(\frac{x^2e^2}{2\sqrt{x+1}}\right) = \frac{\ln x^2 + \ln e^2 - \ln 2 - \ln\sqrt{x+1}}{\ln 2} = \frac{2\ln x + 2 - \ln 2 - \frac{1}{2}\ln(x+1)}{\ln 2}$

$\Rightarrow y' = \frac{2}{x\ln 2} - \frac{1}{2(\ln 2)(x+1)} = \frac{4(x+1)-x}{2x(x+1)(\ln 2)} = \frac{3x+4}{2x(x+1)\ln 2}$

45. $y = (x+1)^x \Rightarrow \ln y = \ln(x+1)^x = x\ln(x+1) \Rightarrow \frac{y'}{y} = \ln(x+1) + x\cdot\frac{1}{(x+1)} \Rightarrow y' = (x+1)^x\left[\frac{x}{x+1} + \ln(x+1)\right]$

46. $y = t^{\sqrt{t}} = t^{\left(t^{1/2}\right)} \Rightarrow \ln y = \ln t^{\left(t^{1/2}\right)} = \left(t^{1/2}\right)(\ln t) \Rightarrow \frac{1}{y}\frac{dy}{dt} = \left(\frac{1}{2}t^{-1/2}\right)(\ln t) + t^{1/2}\left(\frac{1}{t}\right) = \frac{\ln t + 2}{2\sqrt{t}}$

$\Rightarrow \frac{dy}{dt} = \left(\frac{\ln t + 2}{2\sqrt{t}}\right)t^{\sqrt{t}}$

47. $y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} = (\sin x)(\ln x) \Rightarrow \frac{y'}{y} = (\cos x)(\ln x) + (\sin x)\left(\frac{1}{x}\right) = \frac{\sin x + x(\ln x)(\cos x)}{x}$

$\Rightarrow y' = x^{\sin x}\left[\frac{\sin x + x(\ln x)(\cos x)}{x}\right]$

48. $y = (\ln x)^{\ln x} \Rightarrow \ln y = (\ln x)\ln(\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{x}\right)\ln(\ln x) + (\ln x)\left(\frac{1}{\ln x}\right)\frac{d}{dx}(\ln x) = \frac{\ln(\ln x)}{x} + \frac{1}{x}$

$\Rightarrow y' = \left(\frac{\ln(\ln x) + 1}{x}\right)(\ln x)^{\ln x}$

49. Let $u = x^2 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}\,du = x\,dx$; $x = 1 \Rightarrow u = 1$, $x = \sqrt{2} \Rightarrow u = 2$;

$$\int_1^{\sqrt{2}} x2^{\left(x^2\right)}\,dx = \int_1^2 \left(\frac{1}{2}\right)2^u\,du = \frac{1}{2}\left[\frac{2^u}{\ln 2}\right]_1^2 = \left(\frac{1}{2\ln 2}\right)\left(2^2 - 2^1\right) = \frac{1}{\ln 2}$$

50. Let $u = \cos t \Rightarrow du = -\sin t\,dt \Rightarrow -\,du = \sin t\,dt$; $t = 0 \Rightarrow u = 1$, $t = \frac{\pi}{2} \Rightarrow u = 0$;

$$\int_0^{\pi/2} 7^{\cos t}\sin t\,dt = -\int_1^0 7^u\,du = \left[-\frac{7^u}{\ln 7}\right]_1^0 = \left(\frac{-1}{\ln 7}\right)\left(7^0 - 7\right) = \frac{6}{\ln 7}$$

51. Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$; $x = 1 \Rightarrow u = 0$, $x = 2 \Rightarrow u = \ln 2$;

$$\int_1^2 \frac{2^{\ln x}}{x}\,dx = \int_0^{\ln 2} 2^u\,du = \left[\frac{2^u}{\ln 2}\right]_0^{\ln 2} = \left(\frac{1}{\ln 2}\right)\left(2^{\ln 2} - 2^0\right) = \frac{2^{\ln 2} - 1}{\ln 2}$$

52. $\int 3x^{\sqrt{3}}\,dx = \frac{3x^{\left(\sqrt{3}+1\right)}}{\sqrt{3}+1} + C$

53. $\int x^{\left(\sqrt{2}-1\right)}\,dx = \frac{x^{\sqrt{2}}}{\sqrt{2}} + C$

54. $\int_0^3 \left(\sqrt{2}+1\right)x^{\sqrt{2}}\,dx = \left[x^{\left(\sqrt{2}+1\right)}\right]_0^3 = 3^{\left(\sqrt{2}+1\right)}$

55. $\int_1^e x^{(\ln 2)-1}\,dx = \left[\frac{x^{\ln 2}}{\ln 2}\right]_1^e = \frac{e^{\ln 2} - 1^{\ln 2}}{\ln 2} = \frac{2-1}{\ln 2} = \frac{1}{\ln 2}$

56. Let $u = \ln t \Rightarrow du = \frac{dt}{t}$; $u = 0$ when $t = 1$, and $u = x$ when $t = e^x$.

$$\int_1^{e^x} \frac{3^{\ln t}}{t}\,dt = \int_0^x 3^u\,du = \frac{3^u}{\ln 3}\Big|_0^x = \frac{1}{\ln 3}\left(3^x - 1\right)$$

57. $\frac{dy}{dt} = e^t\sin\left(e^t - 2\right) \Rightarrow y = \int e^t\sin\left(e^t - 2\right)dt$;

let $u = e^t - 2 \Rightarrow du = e^t\,dt \Rightarrow y = \int \sin u\,du = -\cos u + C = -\cos\left(e^t - 2\right) + C$; $y(\ln 2) = 0$

$\Rightarrow -\cos\left(e^{\ln 2}-2\right)+C=0 \Rightarrow -\cos(2-2)+C=0 \Rightarrow C=\cos 0=1$; thus, $y=1-\cos\left(e^t-2\right)$

58. $\frac{dy}{dt}=e^{-t}\sec^2\left(\pi e^{-t}\right) \Rightarrow y=\int e^{-t}\sec^2\left(\pi e^{-t}\right)dt$;

let $u=\pi e^{-t} \Rightarrow du=-\pi e^{-t}\,dt \Rightarrow -\frac{1}{\pi}\,du=e^{-t}\,dt \Rightarrow y=-\frac{1}{\pi}\int \sec^2 u\,du=-\frac{1}{\pi}\tan u+C$

$=-\frac{1}{\pi}\tan\left(\pi e^{-t}\right)+C$; $y(\ln 4)=\frac{2}{\pi} \Rightarrow -\frac{1}{\pi}\tan\left(\pi e^{-\ln 4}\right)+C=\frac{2}{\pi} \Rightarrow -\frac{1}{\pi}\tan\left(\pi\cdot\frac{1}{4}\right)+C=\frac{2}{\pi}$

$\Rightarrow -\frac{1}{\pi}(1)+C=\frac{2}{\pi} \Rightarrow C=\frac{3}{\pi}$; thus, $y=\frac{3}{\pi}-\frac{1}{\pi}\tan\left(\pi e^{-t}\right)$

59. $\frac{d^2y}{dx^2}=2e^{-x} \Rightarrow \frac{dy}{dx}=-2e^{-x}+C$; $x=0$ and $\frac{dy}{dx}=0 \Rightarrow 0=-2e^0+C \Rightarrow C=2$; thus $\frac{dy}{dx}=-2e^{-x}+2$

$\Rightarrow y=2e^{-x}+2x+C_1$; $x=0$ and $y=1 \Rightarrow 1=2e^0+C_1 \Rightarrow C_1=-1 \Rightarrow y=2e^{-x}+2x-1=2\left(e^{-x}+x\right)-1$

60. $\frac{d^2y}{dt^2}=1-e^{2t} \Rightarrow \frac{dy}{dt}=t-\frac{1}{2}e^{2t}+C$; $t=1$ and $\frac{dy}{dt}=0 \Rightarrow 0=1-\frac{1}{2}e^2+C \Rightarrow C=\frac{1}{2}e^2-1$; thus

$\frac{dy}{dt}=t-\frac{1}{2}e^{2t}+\frac{1}{2}e^2-1 \Rightarrow y=\frac{1}{2}t^2-\frac{1}{4}e^{2t}+\left(\frac{1}{2}e^2-1\right)t+C_1$; $t=1$ and $y=-1 \Rightarrow -1=\frac{1}{2}-\frac{1}{4}e^2+\frac{1}{2}e^2-1+C_1$

$\Rightarrow C_1=-\frac{1}{2}-\frac{1}{4}e^2 \Rightarrow y=\frac{1}{2}t^2-\frac{1}{4}e^{2t}+\left(\frac{1}{2}e^2-1\right)t-\left(\frac{1}{2}+\frac{1}{4}e^2\right)$

61. $f(x)=e^x-2x \Rightarrow f'(x)=e^x-2$; $f'(x)=0 \Rightarrow e^x=2 \Rightarrow x=\ln 2$; $f(0)=1$, the absolute maximum; $f(\ln 2)=2-2\ln 2\approx 0.613706$, the absolute minimum; $f(1)=e-2\approx 0.71828$, a relative or local maximum since $f''(x)=e^x$ is always positive

62. The function $f(x)=2e^{\sin(x/2)}$ has a maximum whenever $\sin\frac{x}{2}=1$ and a minimum whenever $\sin\frac{x}{2}=-1$. Therefore the maximums occur at $x=\pi+2k(2\pi)$ and the minimums occur at $x=3\pi+2k(2\pi)$, where k is any integer. The maximum is $2e\approx 5.43656$ and the minimum is $\frac{2}{e}\approx 0.73576$

63. $f(x)=x^2\ln\frac{1}{x} \Rightarrow f'(x)=2x\ln\frac{1}{x}+x^2\left(\frac{1}{\frac{1}{x}}\right)\left(-x^{-2}\right)=2x\ln\frac{1}{x}-x=-x(2\ln x+1)$; $f'(x)=0 \Rightarrow x=0$ or $\ln x=-\frac{1}{2}$. Since $x=0$ is not in the domain of f, $x=e^{-1/2}=\frac{1}{\sqrt{e}}$. Also, $f'(x)>0$ for $0<x<\frac{1}{\sqrt{e}}$ and $f'(x)<0$ for $x>\frac{1}{\sqrt{e}}$. Therefore, $f\left(\frac{1}{\sqrt{e}}\right)=\frac{1}{e}\ln\sqrt{e}=\frac{1}{e}\ln e^{1/2}=\frac{1}{2e}\ln e=\frac{1}{2e}$ is the absolute maximum value of f assumed at $x=\frac{1}{\sqrt{e}}$.

64. $\int_0^{\ln 3}\left(e^{2x}-e^x\right)dx=\left[\frac{e^{2x}}{2}-e^x\right]_0^{\ln 3}=\left(\frac{e^{2\ln 3}}{2}-e^{\ln 3}\right)-\left(\frac{e^0}{2}-e^0\right)=\left(\frac{9}{2}-3\right)-\left(\frac{1}{2}-1\right)=\frac{8}{2}-2=2$

65. Let $x=\frac{r}{k} \Rightarrow k=\frac{r}{x}$ and as $k\to\infty$, $x\to 0 \Rightarrow \lim_{k\to\infty}\left(1+\frac{r}{k}\right)^k=\lim_{x\to 0}(1+x)^{r/x}=\lim_{x\to 0}\left((1+x)^{1/x}\right)^r$

$= \left(\lim_{x\to 0} (1+x)^{1/x}\right)^r$, since u^r is continuous. However, $\lim_{x\to 0} (1+x)^{1/x} = e$ (by Theorem 2), therefore, $\lim_{k\to\infty} \left(1+\frac{r}{k}\right)^k = e^r$.

66. $L = \int_0^1 \sqrt{1+\frac{e^x}{4}}\, dx \Rightarrow \frac{dy}{dx} = \pm\frac{e^{x/2}}{2} \Rightarrow y = \pm e^{x/2} + C$; $y(0) = 0 \Rightarrow 0 = \pm e^0 + C \Rightarrow C = \mp 1 \Rightarrow y = e^{x/2} - 1$
$y = -e^{x/2} + 1$

67. Note that $y = \ln x$ and $e^y = x$ are the same curve; $\int_1^a \ln x\, dx$ = area under the curve between 1 and a; $\int_0^{\ln a} e^y\, dy$ = area to the left of the curve. The sum of these areas is equal to the area of the rectangle $\Rightarrow \int_1^a \ln x\, dx + \int_0^{\ln a} e^y\, dy = a \ln a$.

68. (a) $y = e^x \Rightarrow y'' = e^x > 0$ for all $x \Rightarrow$ the graph of $y = e^x$ is always concave upward

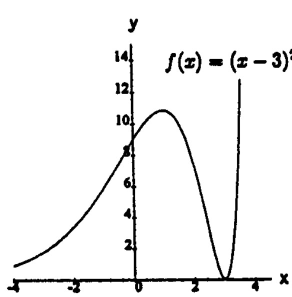
(b) area of the trapezoid ABCD $< \int_{\ln a}^{\ln b} e^x\, dx <$ area of the trapezoid AEFD $\Rightarrow \frac{1}{2}(AB + CD)(\ln b - \ln a)$ $< \int_{\ln a}^{\ln b} e^x\, dx < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a)$. Now $\frac{1}{2}(AB + CD)$ is the height of the midpoint $M = e^{(\ln a + \ln b)/2}$ since the curve containing the points B and C is linear $\Rightarrow e^{(\ln a + \ln b)/2}(\ln b - \ln a)$ $< \int_{\ln a}^{\ln b} e^x\, dx < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a)$

(c) $\int_{\ln a}^{\ln b} e^x\, dx = \left[e^x\right]_{\ln a}^{\ln b} = e^{\ln b} - e^{\ln a} = b - a$, so part (b) implies that

$e^{(\ln a + \ln b)/2}(\ln b - \ln a) < b - a < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a) \Rightarrow e^{(\ln a + \ln b)/2} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}$

$\Rightarrow e^{\ln(ab)/2} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2} \Rightarrow e^{\ln (ab)^{1/2}} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2} \Rightarrow \sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}$

69. $f(x) = (x-3)^2 e^x \Rightarrow f'(x) = 2(x-3)e^x + (x-3)^2 e^x$ $= (x-3)e^x(2 + x - 3) = (x-1)(x-3)e^x$; thus $f'(x) > 0$ for $x < 1$ or $x > 3$, and $f'(x) < 0$ for $1 < x < 3 \Rightarrow f(1) = 4e \approx 10.87$ is a local maximum and $f(3) = 0$ is a local minimum. Since $f(x) \geq 0$ for all x, $f(3) = 0$ is also an absolute minimum.



70. $e^{\ln x} = x$ for $x > 0$ and $\ln(e^x) = x$ for all x. My TI-92 Plus calculator gives correct results for $\ln(e^x)$, but for $x \le 0$ it gives $e^{\ln x} = x$, which is incorrect. Try graphing $e^{\ln x}$ for $-2 \le x \le 2$. To be correct the line $y = x$ should not show for $x \le 0$.

71. (a) $f(x) = e^x \Rightarrow f'(x) = e^x$; $L(x) = f(0) + f'(0)(x - 0) \Rightarrow L(x) = 1 + x$

(b) On $[0, 0.2]$, f(x) and L(x) are both increasing because $f'(x) = e^x > 0$ and $L'(x) = 1 > 0$ for $0 \le x \le 0.2$. Also, $f'(x) \ge L'(x)$ on the interval $[0, 0.2]$ since $e^x \ge 1$ on the interval $E(x) = f(x) - L(x)$ is non-decreasing on $[0, 0.2]$ since $E'(x) = f'(x) - L'(x) \ge 0$ on $[0, 0.2]$. Therefore, the largest error is $|E(0.2)| = |f(0.2) - L(0.2)| = |e^{0.2} - 1.2| \approx 0.02141$ on $[0, 0.2]$.

(c) Since f(x) is concave upward for all x, the tangent line lies below the curve $y = e^x$ for all x except at $x = 0$. Consequently, the linear approximation is never an overestimate.

72. Using Newton's Method: $f(x) = \ln(x) - 1 \Rightarrow f'(x) = \frac{1}{x} \Rightarrow x_{n+1} = x_n - \frac{\ln(x_n) - 1}{\left(\frac{1}{x_n}\right)} \Rightarrow x_{n+1} = x_n[2 - \ln(x_n)]$.

Then $x_1 = 2 \Rightarrow x_2 = 2.61370564$, $x_3 = 2.71624393$ and $x_5 = 2.71828183$. Many other methods may be used. For example, graph $y = \ln x - 1$ and determine the zero of y.

6.3 LINEAR FIRST ORDER DIFFERENTIAL EQUATIONS

1. $x\frac{dy}{dx} + y = e^x$, $x > 0$

Step 1: $\frac{dy}{dx} + \left(\frac{1}{x}\right)y = \frac{e^x}{x}$, $P(x) = \frac{1}{x}$, $Q(x) = \frac{e^x}{x}$

Step 2: $\int P(x)\,dx = \int \frac{1}{x}\,dx = \ln|x| = \ln x$, $x > 0$

Step 3: $v(x) = e^{\int P(x)\,dx} = e^{\ln x} = x$

Step 4: $y = \frac{1}{v(x)} \int v(x)\,Q(x)\,dx = \frac{1}{x} \int x\left(\frac{e^x}{x}\right)dx = \frac{1}{x}(e^x + C) = \frac{e^x + C}{x}$

2. $e^x\frac{dy}{dx} + 2e^x y = 1$

Step 1: $\frac{dy}{dx} + 2y = e^{-x}$, $P(x) = 2$, $Q(x) = e^{-x}$

Step 2: $\int P(x)\,dx = \int 2\,dx = 2x$

Step 3: $v(x) = e^{\int P(x)\,dx} = e^{2x}$

Step 4: $y = \frac{1}{e^{2x}} \int e^{2x} \cdot e^{-x}\,dx = \frac{1}{e^{2x}} \int e^x\,dx = \frac{1}{e^{2x}}(e^x + C) = e^{-x} + Ce^{-2x}$

3. $xy' + 3y = \frac{\sin x}{x^2}$, $x > 0$

Step 1: $\frac{dy}{dx} + \left(\frac{3}{x}\right)y = \frac{\sin x}{x^3}$, $P(x) = \frac{3}{x}$, $Q(x) = \frac{\sin x}{x^3}$

Step 2: $\int \frac{3}{x}\,dx = 3\ln|x| = \ln x^3$, $x > 0$

Step 3: $v(x) = e^{\ln x^3} = x^3$

Step 4: $y = \frac{1}{x^3}\int x^3\left(\frac{\sin x}{x^3}\right)dx = \frac{1}{x^3}\int \sin x\,dx = \frac{1}{x^3}(-\cos x + C) = \frac{C - \cos x}{x^3}$, $x > 0$

4. $y' + (\tan x)\,y = \cos^2 x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Step 1: $\frac{dy}{dx} + (\tan x)\,y = \cos^2 x$, $P(x) = \tan x$, $Q(x) = \cos^2 x$

Step 2: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| = \ln(\cos x)^{-1}$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Step 3: $v(x) = e^{\ln(\cos x)^{-1}} = (\cos x)^{-1}$

Step 4: $y = \frac{1}{(\cos x)^{-1}}\int (\cos x)^{-1}\cdot\cos^2 x\,dx = (\cos x)\int \cos x\,dx = (\cos x)(\sin x + C) = \sin x\cos x + C\cos x$

5. $x\frac{dy}{dx} + 2y = 1 - \frac{1}{x}$, $x > 0$

Step 1: $\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \frac{1}{x} - \frac{1}{x^2}$, $P(x) = \frac{2}{x}$, $Q(x) = \frac{1}{x} - \frac{1}{x^2}$

Step 2: $\int \frac{2}{x}\,dx = 2\ln|x| = \ln x^2$, $x > 0$

Step 3: $v(x) = e^{\ln x^2} = x^2$

Step 4: $y = \frac{1}{x^2}\int x^2\left(\frac{1}{x} - \frac{1}{x^2}\right)dx = \frac{1}{x^2}\int (x-1)\,dx = \frac{1}{x^2}\left(\frac{x^2}{2} - x + C\right) = \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2}$, $x > 0$

6. $(1+x)\,y' + y = \sqrt{x}$

Step 1: $\frac{dy}{dx} + \left(\frac{1}{1+x}\right)y = \frac{\sqrt{x}}{1+x}$, $P(x) = \frac{1}{1+x}$, $Q(x) = \frac{\sqrt{x}}{1+x}$

Step 2: $\int \frac{1}{1+x}\,dx = \ln(1+x)$, since $x > 0$

Step 3: $v(x) = e^{\ln(1+x)} = 1 + x$

Step 4: $y = \frac{1}{1+x}\int (1+x)\left(\frac{\sqrt{x}}{1+x}\right)dx = \frac{1}{1+x}\int \sqrt{x}\,dx = \left(\frac{1}{1+x}\right)\left(\frac{2}{3}x^{3/2} + C\right) = \frac{2x^{3/2}}{3(1+x)} + \frac{C}{1+x}$

7. $\frac{dy}{dx} - \frac{1}{2}y = \frac{1}{2}e^{x/2} \Rightarrow P(x) = -\frac{1}{2}$, $Q(x) = \frac{1}{2}e^{x/2} \Rightarrow \int P(x)\,dx = -\frac{1}{2}x \Rightarrow v(x) = e^{-x/2}$

$\Rightarrow y = \frac{1}{e^{-x/2}} \int e^{-x/2}\left(\frac{1}{2}e^{x/2}\right) dx = e^{x/2} \int \frac{1}{2}\, dx = e^{x/2}\left(\frac{1}{2}x + C\right) = \frac{1}{2}xe^{x/2} + Ce^{x/2}$

8. $\frac{dy}{dx} + 2y = 2xe^{-2x} \Rightarrow P(x) = 2,\ Q(x) = 2xe^{-2x} \Rightarrow \int P(x)\, dx = \int 2\, dx = 2x \Rightarrow v(x) = e^{2x}$

$\Rightarrow y = \frac{1}{e^{2x}} \int e^{2x}\left(2xe^{-2x}\right) dx = \frac{1}{e^{2x}} \int 2x\, dx = e^{-2x}\left(x^2 + C\right) = x^2e^{-2x} + Ce^{-2x}$

9. $\frac{dy}{dx} - \left(\frac{1}{x}\right)y = 2 \ln x \Rightarrow P(x) = -\frac{1}{x},\ Q(x) = 2 \ln x \Rightarrow \int P(x)\, dx = -\int \frac{1}{x}\, dx = -\ln x,\ x > 0$

$\Rightarrow v(x) = e^{-\ln x} = \frac{1}{x} \Rightarrow y = x \int \left(\frac{1}{x}\right)(2 \ln x)\, dx = x\left[(\ln x)^2 + C\right] = x(\ln x)^2 + Cx$

10. $\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \frac{\cos x}{x^2},\ x > 0 \Rightarrow P(x) = \frac{2}{x},\ Q(x) = \frac{\cos x}{x^2} \Rightarrow \int P(x)\, dx = \int \frac{2}{x}\, dx = 2 \ln|x| = \ln x^2,\ x > 0$

$\Rightarrow v(x) = e^{\ln x^2} = x^2 \Rightarrow y = \frac{1}{x^2} \int x^2\left(\frac{\cos x}{x^2}\right) dx = \frac{1}{x^2} \int \cos x\, dx = \frac{1}{x^2}(\sin x + C) = \frac{\sin x + C}{x^2}$

11. $\frac{ds}{dt} + \left(\frac{4}{t-1}\right)s = \frac{t+1}{(t-1)^3},\ t > 1 \Rightarrow P(t) = \frac{4}{t-1},\ Q(t) = \frac{t+1}{(t-1)^3} \Rightarrow \int P(t)\, dt = \int \frac{4}{t-1}\, dt$

$= 4 \ln|t-1| = \ln(t-1)^4 \Rightarrow v(t) = e^{\ln(t-1)^4} = (t-1)^4 \Rightarrow s = \frac{1}{(t-1)^4} \int (t-1)^4\left[\frac{t+1}{(t-1)^3}\right] dt$

$= \frac{1}{(t-1)^4} \int \left(t^2 - 1\right) dt = \frac{1}{(t-1)^4}\left(\frac{t^3}{3} - t + C\right) = \frac{t^3}{3(t-1)^4} - \frac{t}{(t-1)^4} + \frac{C}{(t-1)^4}$

12. $(t+1)\frac{ds}{dt} + 2s = 3(t+1) + \frac{1}{(t+1)^2},\ t > -1 \Rightarrow \frac{ds}{dt} + \left(\frac{2}{t+1}\right)s = 3 + \frac{1}{(t+1)^3} \Rightarrow P(t) = \frac{2}{t+1},\ Q(t) = 3 + (t+1)^{-3}$

$\Rightarrow \int P(t)\, dt = \int \frac{2}{t+1}\, dt = 2 \ln|t+1| = \ln(t+1)^2 \Rightarrow v(t) = e^{\ln(t+1)^2} = (t+1)^2$

$\Rightarrow s = \frac{1}{(t+1)^2} \int (t+1)^2\left[3 + (t+1)^{-3}\right] dt = \frac{1}{(t+1)^2} \int \left[3(t+1)^2 + (t+1)^{-1}\right] dt$

$= \frac{1}{(t+1)^2}\left[(t+1)^3 + \ln|t+1| + C\right] = (t+1) + (t+1)^{-2}\ln(t+1) + \frac{C}{(t+1)^2},\ t > -1$

13. $\frac{dr}{d\theta} + (\cot\theta)\, r = \sec\theta,\ 0 < \theta < \frac{\pi}{2} \Rightarrow P(\theta) = \cot\theta,\ Q(\theta) = \sec\theta \Rightarrow \int P(\theta)\, d\theta = \int \cot\theta\, d\theta = \ln|\sin\theta|$

$\Rightarrow v(\theta) = e^{\ln|\sin\theta|} = \sin\theta$ because $0 < \theta < \frac{\pi}{2} \Rightarrow r = \frac{1}{\sin\theta} \int (\sin\theta)(\sec\theta)\, d\theta$

$= \frac{1}{\sin\theta} \int \tan\theta\, d\theta = \frac{1}{\sin\theta}\left(\ln|\sec\theta| + C\right) = (\csc\theta)\left(\ln|\sec\theta| + C\right)$

14. $\tan\theta\, \frac{dr}{d\theta} + r = \sin^2\theta,\ 0 < \theta < \frac{\pi}{2} \Rightarrow \frac{dr}{d\theta} + \frac{r}{\tan\theta} = \frac{\sin^2\theta}{\tan\theta} \Rightarrow \frac{dr}{d\theta} + (\cot\theta)\, r = \sin\theta\cos\theta \Rightarrow P(\theta) = \cot\theta,\ Q(\theta)$

$= \sin\theta\cos\theta \Rightarrow \int P(\theta)\, d\theta = \int \cot\theta\, d\theta = \ln|\sin\theta| = \ln(\sin\theta)$ since $0 < \theta < \frac{\pi}{2} \Rightarrow v(\theta) = e^{\ln(\sin\theta)} = \sin\theta$

$\Rightarrow r = \frac{1}{\sin\theta}\int(\sin\theta)(\sin\theta\cos\theta)\,d\theta = \frac{1}{\sin\theta}\int\sin^2\theta\cos\theta\,d\theta = \left(\frac{1}{\sin\theta}\right)\left(\frac{\sin^3\theta}{3}+C\right) = \frac{\sin^2\theta}{3}+\frac{C}{\sin\theta}$

15. $\frac{dy}{dt}+2y=3 \Rightarrow P(t)=2,\ Q(t)=3 \Rightarrow \int P(t)\,dt = \int 2\,dt = 2t \Rightarrow v(t)=e^{2t} \Rightarrow y=\frac{1}{e^{2t}}\int 3e^{2t}\,dt$

$= \frac{1}{e^{2t}}\left(\frac{3}{2}e^{2t}+C\right);\ y(0)=1 \Rightarrow \frac{3}{2}+C=1 \Rightarrow C=-\frac{1}{2} \Rightarrow y=\frac{3}{2}-\frac{1}{2}e^{-2t}$

16. $\frac{dy}{dt}+\frac{2y}{t}=t^2,\ t>0 \Rightarrow P(t)=\frac{2}{t},\ Q(t)=t^2 \Rightarrow \int P(t)\,dt = 2\ln|t| \Rightarrow v(t)=e^{\ln t^2}=t^2 \Rightarrow y=\frac{1}{t^2}\int(t^2)(t^2)\,dt$

$= \frac{1}{t^2}\int t^4\,dt = \frac{1}{t^2}\left(\frac{t^5}{5}+C\right)=\frac{t^3}{5}+\frac{C}{t^2};\ y(2)=1 \Rightarrow \frac{8}{5}+\frac{C}{4}=1 \Rightarrow C=-\frac{12}{5} \Rightarrow y=\frac{t^3}{5}-\frac{12}{5t^2}$

17. $\frac{dy}{d\theta}+\left(\frac{1}{\theta}\right)y=\frac{\sin\theta}{\theta},\ \theta>0 \Rightarrow P(\theta)=\frac{1}{\theta},\ Q(\theta)=\frac{\sin\theta}{\theta} \Rightarrow \int P(\theta)\,d\theta=\ln|\theta| \Rightarrow v(\theta)=e^{\ln|\theta|}=|\theta|$

$\Rightarrow y=\frac{1}{|\theta|}\int|\theta|\left(\frac{\sin\theta}{\theta}\right)d\theta=\frac{1}{\theta}\int\theta\left(\frac{\sin\theta}{\theta}\right)d\theta$ for $\theta\neq 0 \Rightarrow y=\frac{1}{\theta}\int\sin\theta\,d\theta=\frac{1}{\theta}(-\cos\theta+C)$

$= -\frac{1}{\theta}\cos\theta+\frac{C}{\theta};\ y\left(\frac{\pi}{2}\right)=1 \Rightarrow C=\frac{\pi}{2} \Rightarrow y=-\frac{1}{\theta}\cos\theta+\frac{\pi}{2\theta}$

18. $\frac{dy}{d\theta}-\left(\frac{2}{\theta}\right)y=\theta^2\sec\theta\tan\theta,\ \theta>0 \Rightarrow P(\theta)=-\frac{2}{\theta},\ Q(\theta)=\theta^2\sec\theta\tan\theta \Rightarrow \int P(\theta)\,d\theta=-2\ln|\theta|$

$\Rightarrow v(\theta)=e^{-2\ln|\theta|}=\theta^{-2} \Rightarrow y=\frac{1}{\theta^{-2}}\int(\theta^{-2})(\theta^2\sec\theta\tan\theta)\,d\theta=\theta^2\int\sec\theta\tan\theta\,d\theta=\theta^2(\sec\theta+C)$

$= \theta^2\sec\theta+C\theta^2;\ y\left(\frac{\pi}{3}\right)=2 \Rightarrow 2=\left(\frac{\pi^2}{9}\right)(2)+C\left(\frac{\pi^2}{9}\right) \Rightarrow C=\frac{18}{\pi^2}-2 \Rightarrow y=\theta^2\sec\theta+\left(\frac{18}{\pi^2}-2\right)\theta^2$

19. $(x+1)\frac{dy}{dx}-2(x^2+x)y=\frac{e^{x^2}}{x+1},\ x>-1 \Rightarrow \frac{dy}{dx}-2\left[\frac{x(x+1)}{x+1}\right]y=\frac{e^{x^2}}{(x+1)^2} \Rightarrow \frac{dy}{dx}-2xy=\frac{e^{x^2}}{(x+1)^2} \Rightarrow P(x)=-2x,$

$Q(x)=\frac{e^{x^2}}{(x+1)^2} \Rightarrow \int P(x)\,dx=\int -2x\,dx=-x^2 \Rightarrow v(x)=e^{-x^2} \Rightarrow y=\frac{1}{e^{-x^2}}\int e^{-x^2}\left[\frac{e^{x^2}}{(x+1)^2}\right]dx$

$= e^{x^2}\int\frac{1}{(x+1)^2}\,dx=e^{x^2}\left[\frac{(x+1)^{-1}}{-1}+C\right]=-\frac{e^{x^2}}{x+1}+Ce^{x^2};\ y(0)=5 \Rightarrow -\frac{1}{0+1}+C=5 \Rightarrow -1+C=5$

$\Rightarrow C=6 \Rightarrow y=6e^{x^2}-\frac{e^{x^2}}{x+1}$

20. $\frac{dy}{dx}+xy=x \Rightarrow P(x)=x,\ Q(x)=x \Rightarrow \int P(x)\,dx=\int x\,dx=\frac{x^2}{2} \Rightarrow v(x)=e^{x^2/2} \Rightarrow y=\frac{1}{e^{x^2/2}}\int e^{x^2/2}\cdot x\,dx$

$= \frac{1}{e^{x^2/2}}\left(e^{x^2/2}+C\right)=1+\frac{C}{e^{x^2/2}};\ y(0)=-6 \Rightarrow 1+C=-6 \Rightarrow C=-7 \Rightarrow y=1-\frac{7}{e^{x^2/2}}$

21. $\frac{dy}{dt} - ky = 0 \Rightarrow P(t) = -k,\ Q(t) = 0 \Rightarrow \int P(t)\,dt = \int -k\,dt = -kt \Rightarrow v(t) = e^{-kt}$

$\Rightarrow y = \frac{1}{e^{-kt}} \int (e^{-kt})(0)\,dt = e^{kt}(0 + C) = Ce^{kt};\ y(0) = y_0 \Rightarrow C = y_0 \Rightarrow y = y_0 e^{kt}$

22. $\frac{dv}{dt} + \frac{k}{m}v = 0 \Rightarrow P(t) = \frac{k}{m},\ Q(t) = 0 \Rightarrow \int P(t)\,dt = \int \frac{k}{m}\,dt = \frac{k}{m}t = \frac{kt}{m} \Rightarrow v(t) = e^{kt/m}$

$\Rightarrow y = \frac{1}{e^{kt/m}} \int e^{kt/m} \cdot 0\,dt = \frac{C}{e^{kt/m}};\ v(0) = v_0 \Rightarrow \frac{C}{e^{k(0)/m}} = v_0 \Rightarrow C = v_0 \Rightarrow v = v_0 e^{-(k/m)t}$

23. $x \int \frac{1}{x}\,dx = x(\ln|x| + C) = x\ln|x| + Cx \Rightarrow$ (b) is correct

24. (a) $\frac{dx}{dt} = 1000 + 0.10x = 0.01(10{,}000 +) \Rightarrow dx = 0.1(10{,}000 + x)\,dt \Rightarrow \frac{dx}{x + 10{,}000} = 0.1\,dt$

$\Rightarrow \int \frac{dx}{x + 10{,}000} = \int 0.1\,dt \Rightarrow \ln|x + 10{,}000| = 0.1t + C_1 \Rightarrow e^{\ln|x+10{,}000|} = e^{0.1t + C_1}$

$\Rightarrow |x + 10{,}000| = e^{0.1t}e^{C_1} \Rightarrow x + 10{,}000 = \pm C_2 e^{0.1t} \Rightarrow x(t) = -10{,}000 + Ce^{0.1t}$, where $C_2 = e^{C_1}$ and $C = \pm C_2$. Apply the initial condition: $x(0) = 1000 = -10{,}000 + Ce^0 \Rightarrow C = 11{,}000$

$\Rightarrow x(t) = -10{,}000 + 11{,}000e^{0.1t}$

(b) $100{,}000 = -10{,}000 + 11{,}000e^{0.1t} \Rightarrow e^{0.1t} = 10 \Rightarrow t = 10\ln(10) \approx 23.03 \approx$ 23 years and 11 days.

25. Let $y(t)$ = the amount of salt in the container and $V(t)$ = the total volume of liquid in the tank at time t. Then, the departure rate is $\frac{y(t)}{V(t)}$ (the outflow rate).

(a) Rate entering $= \frac{2\text{ lb}}{\text{gal}} \cdot \frac{5\text{ gal}}{\text{min}} = 10$ lb/min

(b) Volume $= V(t) = 100\text{ gal} + (5t\text{ gal} - 4t\text{ gal}) = (100 + t)$ gal

(c) The volume at time t is $(100 + t)$ gal. The amount of salt in the tank at time t is y lbs. So the concentration at any time t is $\frac{y}{100 + t}$ lbs/gal. Then, Rate leaving $= \frac{y}{100 + t}$ (lbs/gal) $\cdot$ 4 (gal/min) $= \frac{4y}{100 + t}$ lbs/min

(d) $\frac{dy}{dt} = 10 - \frac{4y}{100 + t} \Rightarrow \frac{dy}{dt} + \left(\frac{4}{100 + t}\right)y = 10 \Rightarrow P(t) = \frac{4}{100 + t},\ Q(t) = 10 \Rightarrow \int P(t)\,dt = \int \frac{4}{100 + t}\,dt$

$= 4\ln(100 + t) \Rightarrow v(t) = e^{4\ln(100+t)} = (100 + t)^4 \Rightarrow y = \frac{1}{(100 + t)^4} \int (100 + t)^4(10\,dt)$

$= \frac{10}{(100 + t)^4}\left(\frac{(100 + t)^5}{5} + C\right) = 2(100 + t) + \frac{C}{(100 + t)^4};\ y(0) = 50 \Rightarrow 2(100 + 0) + \frac{C}{(100 + 0)^4} = 50$

$\Rightarrow C = -(150)(100)^4 \Rightarrow y = 2(100 + t) - \frac{(150)(100)^4}{(100 + t)^4} \Rightarrow y = 2(100 + t) - \frac{150}{\left(1 + \frac{t}{100}\right)^4}$

(e) $y(25) = 2(100+25) - \frac{(150)(100)^4}{(100+25)^4} \approx 188.56$ lbs $\Rightarrow$ concentration $= \frac{y(25)}{\text{volume}} \approx \frac{188.6}{125} \approx 1.5$ lb/gal

26. (a) $\frac{dV}{dt} = (5-3) = 2 \Rightarrow V = 100 + 2t$

The tank is full when $V = 200 = 100 + 2t \Rightarrow t = 50$ min.

(b) Let y(t) be the amount of concentrate in the tank at time t.

$\frac{dy}{dt} = \left(\frac{1}{2}\,\frac{\text{lb}}{\text{gal}}\right)\left(5\,\frac{\text{gal}}{\text{min}}\right) - \left(\frac{y}{100+2t}\,\frac{\text{lb}}{\text{gal}}\right)\left(3\,\frac{\text{gal}}{\text{min}}\right) \Rightarrow \frac{dy}{dt} = \frac{5}{2} - \frac{3}{2}\left(\frac{y}{50+t}\right) \Rightarrow \frac{dy}{dt} + \frac{3}{2(t+50)}y = \frac{5}{2}$

$Q(t) = \frac{5}{2}$; $P(t) = \frac{3}{2}\left(\frac{1}{t+50}\right) \Rightarrow \int P(t)\,dt = \frac{3}{2}\int \frac{1}{t+50}\,dt = \frac{3}{2}\ln(t+50)$ since $t + 50 > 0$

$v(t) = e^{\int P(t)\,dt} = e^{\frac{3}{2}\ln(t+50)} = (t+50)^{3/2}$

$y(t) = \frac{1}{(t+50)^{3/2}} \int \left(\frac{5}{2}\right)(t+50)^{3/2}\,dt = (t+50)^{-3/2}\left[(t+50)^{5/2} + C\right] \Rightarrow y(t) = t + 50 + \frac{C}{(t+50)^{3/2}}$

Apply the initial condition (i.e., distilled water in the tank at $t = 0$):

$y(0) = 0 = 50 + \frac{C}{50^{3/2}} \Rightarrow C = -50^{5/2} \Rightarrow y(t) = t + 50 - \frac{50^{5/2}}{(t+50)^{3/2}}$. When the tank is full at $t = 50$,

$y(50) = 100 - \frac{50^{3/2}}{100^{3/2}} \approx 83.22$ pounds of concentrate.

27. Let y be the amount of fertilizer in the tank at time t. Then rate entering $= 1\,\frac{\text{lb}}{\text{gal}} \cdot 1\,\frac{\text{gal}}{\text{min}} = 1\,\frac{\text{lb}}{\text{min}}$ and the volume in the tank at time t is $V(t) = 100$ (gal) $+ [1$ (gal/min) $- 3$ (gal/min)$]t$ min $= (100 - 2t)$ gal. Hence rate out $= \left(\frac{y}{100-2t}\right)3 = \frac{3y}{100-2t}$ lbs/min $\Rightarrow \frac{dy}{dt} = \left(1 - \frac{3y}{100-2t}\right)$ lbs/min $\Rightarrow \frac{dy}{dt} + \left(\frac{3}{100-2t}\right)y = 1$

$\Rightarrow P(t) = \frac{3}{100-2t}$, $Q(t) = 1 \Rightarrow \int P(t)\,dt = \int \frac{3}{100-2t}\,dt = \frac{3\ln(100-2t)}{-2} \Rightarrow v(t) = e^{(-3\ln(100-2t))/2}$

$= (100-2t)^{-3/2} \Rightarrow y = \frac{1}{(100-2t)^{-3/2}} \int (100-2t)^{-3/2}\,dt = (100-2t)^{3/2}\left[(100-2t)^{-1/2} + C\right]$

$= (100-2t) + C(100-2t)^{3/2}$; $y(0) = 0 \Rightarrow [100 - 2(0)] + C[100 - 2(0)]^{3/2} \Rightarrow C(100)^{3/2} = -100$

$\Rightarrow C = -(100)^{-1/2} = -\frac{1}{10} \Rightarrow y = (100-2t) - \frac{(100-2t)^{3/2}}{10}$. Let $\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -2 - \frac{\left(\frac{3}{2}\right)(100-2t)^{1/2}(-2)}{10}$

$= -2 + \frac{3\sqrt{100-2t}}{10} = 0 \Rightarrow 20 = 3\sqrt{100-2t} \Rightarrow 400 = 9(100-2t) \Rightarrow 400 = 900 - 18t \Rightarrow -500 = -18t$

$\Rightarrow t \approx 27.8$ min, the time to reach the maximum. The maximum amount is then

$y(27.8) = [100 - 2(27.8)] - \frac{[100 - 2(27.8)]^{3/2}}{10} \approx 14.8$ lb

28. Let $y = y(t)$ be the amount of carbon monoxide (CO) in the room at time t. The amount of CO entering the room is $\left(\frac{4}{100} \times \frac{3}{10}\right) = \frac{12}{1000}$ ft^3/min, and the amount of CO leaving the room is $\left(\frac{y}{4500}\right)\left(\frac{3}{10}\right) = \frac{y}{15{,}000}$ ft^3/min.

Thus, $\frac{dy}{dt} = \frac{12}{1000} - \frac{y}{15{,}000} \Rightarrow \frac{dy}{dt} + \frac{1}{15{,}000} y = \frac{12}{1000} \Rightarrow P(t) = \frac{1}{15{,}000}$, $Q(t) = \frac{12}{1000} \Rightarrow v(t) = e^{t/15{,}000}$

$\Rightarrow y = \frac{1}{e^{t/15{,}000}} \int \frac{12}{1000} e^{t/15{,}000}\, dt \Rightarrow y = e^{-t/15{,}000}\left(\frac{12 \cdot 15{,}000}{1000} e^{t/1500} + C\right) = e^{-t/15{,}000}\left(180e^{t/15{,}000} + C\right)$;

$y(0) = 0 \Rightarrow 0 = 1(180 + C) \Rightarrow C = -180 \Rightarrow y = 180 - 180e^{-t/15{,}000}$. When the concentration of CO is 0.01%

in the room, the amount of CO satisfies $\frac{y}{4500} = \frac{.01}{100} \Rightarrow y = 0.45\ \text{ft}^3$. When the room contains this amount we

have $0.45 = 180 - 180e^{-t/15{,}000} \Rightarrow \frac{179.55}{180} = e^{-t/15{,}000} \Rightarrow t = -15{,}000 \ln\left(\frac{179.55}{180}\right) \approx 37.55$ min.

29. Steady State $= \frac{V}{R}$ and we want $i = \frac{1}{2}\left(\frac{V}{R}\right) \Rightarrow \frac{1}{2}\left(\frac{V}{R}\right) = \frac{V}{R}\left(1 - e^{-Rt/L}\right) \Rightarrow \frac{1}{2} = 1 - e^{-Rt/L} \Rightarrow -\frac{1}{2} = -e^{-Rt/L}$

$\Rightarrow \ln \frac{1}{2} = -\frac{Rt}{L} \Rightarrow -\frac{L}{R} \ln \frac{1}{2} = t \Rightarrow t = \frac{L}{R} \ln 2$ sec

30. (a) $\frac{di}{dt} + \frac{R}{L} i = 0 \Rightarrow \frac{1}{i}\, di = -\frac{R}{L}\, dt \Rightarrow \ln i = -\frac{Rt}{L} + C_1 \Rightarrow i = e^{C_1} e^{-Rt/L} = Ce^{-Rt/L}$; $i(0) = I \Rightarrow I = C$

$\Rightarrow i = Ie^{-Rt/L}$ amp

(b) $\frac{1}{2} I = Ie^{-Rt/L} \Rightarrow e^{-Rt/L} = \frac{1}{2} \Rightarrow -\frac{Rt}{L} = \ln \frac{1}{2} = -\ln 2 \Rightarrow t = \frac{L}{R} \ln 2$ sec

(c) $t = \frac{L}{R} \Rightarrow i = Ie^{(-R/L)(L/R)} = Ie^{-1}$ amp

31. (a) $t = \frac{3L}{R} \Rightarrow i = \frac{V}{R}\left(1 - e^{(-R/L)(3L/R)}\right) = \frac{V}{R}\left(1 - e^{-3}\right) \approx 0.9502\, \frac{V}{R}$ amp, or about 95% of the steady state value

(b) $t = \frac{2L}{R} \Rightarrow i = \frac{V}{R}\left(1 - e^{(-R/L)(2L/R)}\right) = \frac{V}{R}\left(1 - e^{-2}\right) \approx 0.8647\, \frac{V}{R}$ amp, or about 86% of the steady state value

32. (a) $\frac{di}{dt} + \frac{R}{L} i = \frac{V}{L} \Rightarrow P(t) = \frac{R}{L}$, $Q(t) = \frac{V}{L} \Rightarrow \int P(t)\, dt = \int \frac{R}{L}\, dt = \frac{Rt}{L} \Rightarrow v(t) = e^{Rt/L}$

$\Rightarrow i = \frac{1}{e^{Rt/L}} \int e^{Rt/L}\left(\frac{V}{L}\right) dt = \frac{1}{e^{Rt/L}}\left[\frac{L}{R} e^{Rt/L}\left(\frac{V}{L}\right) + C\right] = \frac{V}{R} + Ce^{-(R/L)t}$

(b) $i(0) = 0 \Rightarrow \frac{V}{R} + C = 0 \Rightarrow C = -\frac{V}{R} \Rightarrow i = \frac{V}{R} - \frac{V}{R} e^{-Rt/L}$

(c) $i = \frac{V}{R} \Rightarrow \frac{di}{dt} = 0 \Rightarrow \frac{di}{dt} + \frac{R}{L} i = 0 + \left(\frac{R}{L}\right)\left(\frac{V}{R}\right) = \frac{V}{L} \Rightarrow i = \frac{V}{R}$ is a solution of Eq. (11); $i = Ce^{-(R/L)t}$

$\Rightarrow \frac{di}{dt} = -\frac{RC}{L} e^{-(R/L)t} \Rightarrow \frac{di}{dt} + \frac{R}{L} i = -\frac{RC}{L} e^{-(R/L)t} + \frac{R}{L}\left(Ce^{-(R/L)t}\right) = 0 \Rightarrow i = Ce^{-(R/L)t}$ satisfies

$\frac{di}{dt} + \frac{R}{L} i = 0$

6.4 EULER'S METHOD; POPULATION MODELS

1. $y_1 = y_0 + x_0(1 - y_0)\, dx = 0 + 1(1 - 0)(0.2) = 0.2$,

$y_2 = y_1 + x_1(1 - y_1)\, dx = 0.2 + 1.2(1 - 0.2)(0.2) = 0.392$,

$y_3 = y_2 + x_2(1 - y_2)\ dx = 0.392 + 1.4(1 - 0.392)(0.2) = 0.5622;$

$\frac{dy}{1-y} = x\ dx \Rightarrow -\ln|1-y| = \frac{x^2}{2} + C;\ x = 1,\ y = 0 \Rightarrow -\ln 1 = \frac{1}{2} + C \Rightarrow C = -\frac{1}{2} \Rightarrow \ln|1-y| = -\frac{x^2}{2} + \frac{1}{2}$

$\Rightarrow y = 1 - e^{(1-x^2)/2} \Rightarrow y(1.2) \approx 0.1975,\ y(1.4) \approx 0.3812,\ y(1.6) \approx 0.5416$

2. $y_1 = y_0 + \left(1 - \frac{y_0}{x_0}\right) dx = -1 + \left(1 - \frac{-1}{2}\right)(0.5) = -0.25,$

$y_2 = y_1 + \left(1 - \frac{y_1}{x_1}\right) dx = -0.25 + \left(1 - \frac{-0.25}{2.5}\right)(0.5) = 0.3,$

$y_3 = y_2 + \left(1 - \frac{y_2}{x_2}\right) dx = 0.3 + \left(1 - \frac{0.3}{3}\right)(0.5) = 0.75;$

$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = 1 \Rightarrow P(x) = \frac{1}{x},\ Q(x) = 1 \Rightarrow \int P(x)\ dx = \int \frac{1}{x}\ dx = \ln|x| = \ln x,\ x > 0 \Rightarrow v(x) = e^{\ln x} = x$

$\Rightarrow y = \frac{1}{x}\int x \cdot 1\ dx = \frac{1}{x}\left(\frac{x^2}{2} + C\right);\ x = 2,\ y = -1 \Rightarrow -1 = 1 + \frac{C}{2} \Rightarrow C = -4 \Rightarrow y = \frac{x}{2} - \frac{4}{x} \Rightarrow y(2.5) = \frac{2.5}{2} - \frac{4}{2.5}$

$= -0.35;\ y(3.0) = \frac{3}{2} - \frac{4}{3} \approx 0.1667,\ y(3.5) = \frac{3.5}{2} - \frac{4}{3.5} = \frac{4.25}{7} \approx 0.6071$

3. $y_1 = y_0 + (2x_0y_0 + 2y_0)\ dx = 3 + [2(0)(3) + 2(3)](0.2) = 4.2,$

$y_2 = y_1 + (2x_1y_1 + 2y_1)\ dx = 4.2 + [2(0.2)(4.2) + 2(4.2)](0.2) = 6.216,$

$y_3 = y_2 + (2x_2y_2 + 2y_2)\ dx = 6.216 + [2(0.4)(6.216) + 2(6.216)](0.2) = 9.6970;$

$\frac{dy}{dx} = 2y(x+1) \Rightarrow \frac{dy}{y} = 2(x+1)\ dx \Rightarrow \ln|y| = (x+1)^2 + C;\ x = 0,\ y = 3 \Rightarrow \ln 3 = 1 + C \Rightarrow C = \ln 3 - 1$

$\Rightarrow \ln y = (x+1)^2 + \ln 3 - 1 \Rightarrow y = e^{(x+1)^2 + \ln 3 - 1} = e^{\ln 3}e^{x^2+2x} = 3e^{x(x+2)} \Rightarrow y(0.2) \approx 4.6581,$

$y(0.4) \approx 7.8351,\ y(0.6) \approx 14.2765$

4. $y_1 = y_0 + y_0^2(1 + 2x_0)\ dx = 1 + 1^2[1 + 2(-1)](0.5) = 0.5,$

$y_2 = y_1 + y_1^2(1 + 2x_1)\ dx = 0.5 + (0.5)^2[1 + 2(-0.5)](0.5) = 0.5,$

$y_3 = y_2 + y_2^2(1 + 2x_2)\ dx = 0.5 + (0.5)^2[1 + 2(0)](0.5) = 0.625;$

$\frac{dy}{y^2} = (1 + 2x)\ dx \Rightarrow -\frac{1}{y} = x + x^2 + C;\ x = -1,\ y = 1 \Rightarrow -1 = -1 + (-1)^2 + C \Rightarrow C = -1 \Rightarrow \frac{1}{y} = 1 - x - x^2$

$\Rightarrow y = \frac{1}{1 - x - x^2} \Rightarrow y(-0.5) = 0.8,\ y(0) = 1,\ y(0.5) = 4$

5. $y_1 = 1 + 1(0.2) = 1.2,$

$y_2 = 1.2 + (1.2)(0.2) = 1.44,$

$y_3 = 1.44 + (1.44)(0.2) = 1.728,$

$y_4 = 1.728 + (1.728)(0.2) = 2.0736,$

$y_5 = 2.0736 + (2.0736)(0.2) = 2.48832;$

$\frac{dy}{y} = dx \Rightarrow \ln y = x + C_1 \Rightarrow y = Ce^x;\ y(0) = 1 \Rightarrow 1 = Ce^0 \Rightarrow C = 1 \Rightarrow y = e^x \Rightarrow y(1) = e \approx 2.7183$

6. $y_1 = 2 + \left(\frac{2}{1}\right)(0.2) = 2.4,$

$y_2 = 2.4 + \left(\frac{2.4}{1.2}\right)(0.2) = 2.8,$

$y_3 = 2.8 + \left(\frac{2.8}{1.4}\right)(0.2) = 3.2,$

$y_4 = 3.2 + \left(\frac{3.2}{1.6}\right)(0.2) = 3.6,$

$y_5 = 3.6 + \left(\frac{3.6}{1.8}\right)(0.2) = 4;$

$\frac{dy}{y} = \frac{dx}{x} \Rightarrow \ln y = \ln x + C \Rightarrow y = kx;\ y(1) = 2 \Rightarrow 2 = k \Rightarrow y = 2x \Rightarrow y(2) = 4$

7. Let $z_n = y_{n-1} + 2y_{n-1}(x_{n-1} + 1)\,dx$ and $y_n = y_{n-1} + (y_{n-1}(x_{n-1} + 1) + z_n(x_n + 1))\,dx$ with $x_0 = 0$, $y_0 = 3$, and $dx = 0.2$. The exact solution is $y = 3e^{x(x+2)}$. Using a programmable calculator or a spreadsheet (We used a spreadsheet) gives the values in the following table.

x	z	y-approx	y-exact	Error
0	---	3	3	0
0.2	4.2	4.608	4.658122	0.050122
0.4	6.81984	7.623475	7.835089	0.211614
0.6	11.89262	13.56369	14.27646	0.712777

8. Let $z_n = y_{n-1} + x_{n-1}(1 - y_{n-1})\,dx$ and $y_n = y_{n-1} + \left(\frac{x_{n-1}(1 - y_{n-1}) + x_n(1 - z_n)}{2}\right)dx$ with $x_0 = 1$, $y_0 = 0$, and $dx = 0.2$. The exact solution is $y = 1 - e^{(1-x^2)/2}$. Using a programmable calculator or a spreadsheet (We used a spreadsheet) gives the values in the following table.

x	z	y-approx	y-exact	Error
1	---	0	0	0
1.2	0.2	0.196	0.197481	0.001481
1.4	0.38896	0.378026	0.381217	0.003191
1.6	0.552178	0.536753	0.541594	0.004841

9. (a) $\frac{dP}{dt} = 0.0015P(150 - P) = \frac{0.225}{150}P(150 - P) = \frac{k}{M}P(M - P)$

Thus, $k = 0.225$ and $M = 150$, and $P = \frac{M}{1 + Ae^{-k/t}} = \frac{150}{1 + Ae^{-0.225t}}$

Initial condition: $P(0) = 6 \Rightarrow 6 = \frac{150}{1 + Ae^0} \Rightarrow 1 + A = 25 \Rightarrow A = 24$

Formula: $P = \frac{150}{1 + 24^{-0.225t}}$

(b) $100 = \frac{150}{1 + 24e^{-0.225t}} \Rightarrow 1 + 24e^{-0.2225t} = \frac{3}{2} \Rightarrow 24e^{-0.225t} = \frac{1}{2} \Rightarrow e^{-0.225t} = \frac{1}{48} \Rightarrow -0.225t = -\ln 48$

$\Rightarrow t = \frac{\ln 48}{0.225} \approx 17.21$ weeks

$125 = \frac{150}{1 + 24e^{-0.225t}} \Rightarrow 1 + 24e^{-0.2225t} = \frac{6}{5} \Rightarrow 24e^{-0.225t} = \frac{1}{5} \Rightarrow e^{-0.225t} = \frac{1}{120} \Rightarrow -0.225t = -\ln 120$

$\Rightarrow t = \frac{\ln 120}{0.225} \approx 21.28$

It will take about 17.21 weeks to reach 100 guppies, and about 21.28 weeks to reach 125 guppies.

10. (a) $\frac{dP}{dt} = 0.0004P(250 - P) = \frac{0.1}{250}P(250 - P) = \frac{k}{M}P(M - P)$

Thus, k = 0.1 and M = 250, and $P = \frac{M}{1 + Ae^{-kt}} = \frac{250}{1 + Ae^{-0.1t}}$

Initial condition: P(0) = 28, where t = 0 represents the year 1970.

$28 = \frac{250}{1 + Ae^0} \Rightarrow 28(1 + A) = 250 \Rightarrow A = \frac{250}{28} - 1 = \frac{111}{14} \approx 7.9286$

Formula: $P(t) = \frac{250}{1 + 111e^{-0.1t}/14}$, or approximately $P(t) = \frac{250}{1 + 7.9286e^{-0.1t}}$

(b) The population P(t) will round to 250 when $P(t) \geq 249.5 \Rightarrow 249.5 = \frac{250}{1 + 111e^{-0.1t}/14}$

$\Rightarrow 249.5\left(1 + \frac{111e^{-0.1t}}{14}\right) = 250 \Rightarrow \frac{(249.5)(111e^{-0.1t})}{14} = 0.5 \Rightarrow e^{-0.1t} = \frac{14}{55{,}389} \Rightarrow -0.1t = \ln \frac{14}{55{,}389}$

$\Rightarrow t = 10(\ln 55{,}389 - \ln 14) \approx 82.8$

It will take about 83 years.

11. (a) Using the general solution from Example 6, part (c),

$\frac{dy}{dt} = (0.08875 \times 10^{-7})(8 \times 10^7 - y)y \Rightarrow y(t) = \frac{M}{1 + Ae^{-rMt}} = \frac{8 \times 10^7}{1 + Ae^{-(0.08875)(8)t}} = \frac{8 \times 10^7}{1 + Ae^{-0.71t}}$

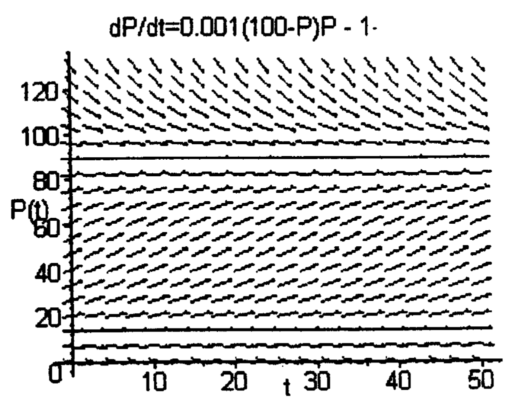
Apply the initial condition:

$y(0) = 1.6 \times 10^7 = \frac{8 \times 10^7}{1 + A} \Rightarrow A = \frac{8}{1.6} - 1 = 4 \Rightarrow y(1) = \frac{8 \times 10^7}{1 + 4e^{-0.71}} \approx 2.69671 \times 10^7$ kg.

(b) $y(t) = 4 \times 10^7 = \frac{8 \times 10^7}{1 + 4e^{-0.71t}} \Rightarrow 4e^{-0.71t} = 1 \Rightarrow t = -\frac{\ln(1/4)}{0.71} \approx 1.95253$ years.

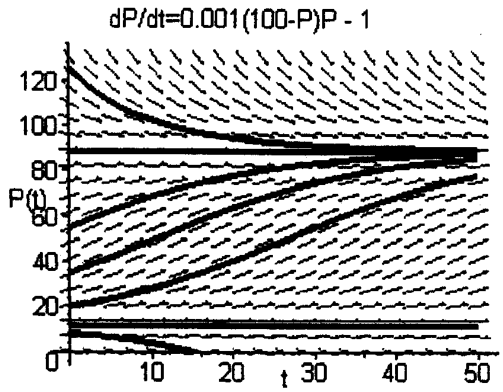
12. (a) If a part of the population leaves or is removed from the environment (e.g., a preserve or a region) each year, then c would represent the rate of reduction of the population due to this removal and/or migration. When grizzly bears become a nuisance (e.g., feeding on livestock) or threaten human safety, they are often relocated to other areas or even eliminated, but only after relocation efforts fail. In addition, bears are killed, sometimes accidentally and sometimes maliciously. For an environment that has a capacity of about 100 bears, a realistic value for c would probably be between 0 and 4.

(b)



Equilibrium solutions: $\frac{dP}{dt} = 0 = 0.001(100 - P)P - 1 \Rightarrow P^2 - 100P + 1000 = 0 \Rightarrow P_{eq} \approx 11.27$ (unstable) and $P_{eq} \approx 88.73$ (stable).

(c)



For $0 < P(0) \leq 11$, the bear population will eventually disappear, for $12 \leq P(0) \leq 88$, the population will grow to about 89, for $P(0) = 89$, the population will remain at about 89, and for $P(0) > 89$, the population will decrease to about 89 bears.

13. (a) $\frac{dy}{dt} = 1 + y \Rightarrow dy = (1 + y)\,dt \Rightarrow \frac{dy}{1+y} = dt \Rightarrow \ln|1 + y| = t + C_1 \Rightarrow e^{\ln|1+y|} = e^{t+C_1} \Rightarrow |1 + y| = e^t e^{C_1}$ $1 + y = \pm C_2 e^t \Rightarrow y = Ce^t - 1$, where $C_2 = e^{C_1}$ and $C = \pm C_2$. Apply the initial condition: $y(0) = 1 = Ce^0 - 1 \Rightarrow C = 2 \Rightarrow y = 2e^t - 1$.

(b) $\frac{dy}{dt} = 0.5(400 - y)y \Rightarrow dy = 0.5(400 - y)y\,dt \Rightarrow \frac{dy}{y(400 - y)} = 0.5\,dt$. Using the partial fraction decomposition in Example 6, part (c), we obtain $\frac{1}{400}\left(\frac{1}{y} + \frac{1}{400 - y}\right)dy = 0.5\,dt \Rightarrow \left(\frac{1}{y} + \frac{1}{400 - y}\right)dy$ $= 200\,dt \Rightarrow \int\left(\frac{1}{y} - \frac{1}{y - 400}\right)dy = \int 200\,dt \Rightarrow \ln|y| - \ln|y - 400| = 200t + C_1 \Rightarrow \ln\left|\frac{y}{y - 400}\right| = 200t + C_1$ $\Rightarrow e^{\ln\left|\frac{y}{y-400}\right|} = e^{200t+C_1} = e^{200t}e^{C_1} \Rightarrow \left|\frac{y}{y - 400}\right| = C_2 e^{200t}$ (where $C_2 = e^{C_1}$) $\Rightarrow \frac{y}{y - 400} = \pm C_2 e^{200t}$ $\Rightarrow \frac{y}{y - 400} = Ce^{200t}$ (where $C = \pm C_2$) $\Rightarrow y = Ce^{200t}y - 400Ce^{200t} \Rightarrow (1 - Ce^{200t})y = -400Ce^{200t}$ $\Rightarrow y = \frac{400Ce^{200t}}{Ce^{200t} - 1} \Rightarrow y = \frac{400}{1 - \frac{1}{C}e^{-200t}} = \frac{400}{1 + Ae^{-200t}}$, where $A = -\frac{1}{C}$. Apply the initial condition: $y(0) = 2 = \frac{400}{1 + Ae^0} \Rightarrow A = 199 \Rightarrow y(t) = \frac{400}{1 + 199e^{-200t}}$.

14. $\frac{dP}{dt} = r(M - P)P \Rightarrow dP = r(M - P)P\,dt \Rightarrow \frac{dP}{P(M - P)} = r\,dt$. Using the partial fraction decomposition in Example 6, part (c), we obtain $\frac{1}{M}\left(\frac{1}{P} + \frac{1}{M - P}\right)dP = r\,dt \Rightarrow \left(\frac{1}{P} + \frac{1}{M - P}\right)dP = rM\,dt \Rightarrow \int\left(\frac{1}{P} - \frac{1}{P - M}\right)dP$ $= \int rM\,dt \Rightarrow \ln|P| - \ln|P - M| = (rM)t + C_1 \Rightarrow \ln\left|\frac{P}{P - M}\right| = (rM)t + C_1 \Rightarrow e^{\ln\left|\frac{P}{P-M}\right|} = e^{(rM)t+C_1}$

$= e^{(rM)t}e^{C_1} \Rightarrow \left|\frac{P}{P-M}\right| = C_2e^{(rM)t}$ (where $C_2 = e^{C_1}$) $\Rightarrow \frac{P}{P-M} = \pm C_2e^{(rM)t} \Rightarrow \frac{P}{P-M} = Ce^{(rM)t}$ (where $C = \pm C_2$) $\Rightarrow P = Ce^{(rM)t}P - MCe^{(rM)t} \Rightarrow (1 - Ce^{(rM)t})P = -MCe^{(rM)t} \Rightarrow P = \frac{MCe^{(rM)t}}{Ce^{(rM)t} - 1}$

$\Rightarrow P = \frac{M}{1 - \frac{1}{C}e^{-(rM)t}} = \frac{M}{1 + Ae^{-(rM)t}}$, where $A = -\frac{1}{C}$.

15. (a) $\frac{dP}{dt} = kP^2 \Rightarrow \int P^{-2}\, dP = \int k\, dt \Rightarrow -P^{-1} = kt + C \Rightarrow P = \frac{-1}{kt + C}$

Initial condition: $P(0) = P_0 \Rightarrow P_0 = -\frac{1}{C} \Rightarrow C = -\frac{1}{P_0}$

Solution: $P = -\frac{1}{kt - (1/P_0)} = \frac{P_0}{1 - kP_0t}$

(b) There is a vertical asymptote at $t = \frac{1}{kP_0}$

16. (a) $\frac{dP}{dt} = r(M-P)(P-m) \Rightarrow \frac{dP}{dt} = r(1200-P)(P-100) \Rightarrow \frac{1}{(1200-P)(P-100)}\frac{dP}{dt} = r$

$\Rightarrow \frac{1100}{(1200-P)(P-100)}\frac{dP}{dt} = 1100r \Rightarrow \frac{(P-100)+(1200-P)}{(1200-P)(P-100)}\frac{dP}{dt} = 1100r$

$\Rightarrow \left(\frac{1}{1200-P} + \frac{1}{P-100}\right)\frac{dP}{dt} = 1100r$

(b) $\left(\frac{1}{1200-P} + \frac{1}{P-100}\right)dP = 1100r\, dt \Rightarrow \int \left(\frac{1}{1200-P} + \frac{1}{P-100}\right)dP = \int 1100r\, dt$

$\Rightarrow \ln(1200-P) + \ln(P-100) = 1100rt + C_1 \Rightarrow \left|\frac{P-100}{1200-P}\right| = 1100rt + C_1 \Rightarrow \frac{P-100}{1200-P}$

$= \pm e^{C_1}e^{1100rt} \Rightarrow \frac{P-100}{1200-P} = Ce^{1100rt}$ where $C = \pm e^{C_1} \Rightarrow P - 100 = 1200Ce^{1100rt} - CPe^{1100rt}$

$\Rightarrow P(1 + Ce^{1100rt}) = 1200Ce^{1100rt} + 100 \Rightarrow P = \frac{1200Ce^{1100rt} + 100}{Ce^{1100rt} + 1} \Rightarrow \frac{1200 + \frac{100}{C}e^{-1100rt}}{1 + \frac{1}{C}e^{-1100rt}}$

$\Rightarrow P = \frac{1200 + 100Ae^{-1100rt}}{1 + Ae^{-1100rt}}$ where $A = \frac{1}{C}$.

Apply the initial condition: $300 = \frac{1200 + 100A}{1 + A} = 300 + 300A = 1200 + 100A \Rightarrow A = \frac{9}{2}$

$\Rightarrow P = \frac{2400 + 900e^{-1100rt}}{2 + 9e^{-1100rt}}$. (Note that $P \to 1200$ as $t \to \infty$.)

(c) $\frac{dP}{dt} = r(M-P)(P-m) \Rightarrow \frac{dP}{dt} = r(M-P)(P-m) \Rightarrow \frac{1}{(M-P)(P-m)}\frac{dP}{dt} = r \Rightarrow \frac{M-m}{(M-P)(P-m)}\frac{dP}{dt}$

$= (M-m)r \Rightarrow \frac{(P-m)+(M-P)}{(M-P)(P-m)}\frac{dP}{dt} = (M-m)r \Rightarrow \left(\frac{1}{M-P} + \frac{1}{P-m}\right)\frac{dP}{dt} = (M-m)r$

$\Rightarrow \int \left(\frac{1}{M-P} + \frac{1}{P-m}\right)dP = \int (M-m)r\, dt \Rightarrow -\ln(M-P) + \ln(P-m) = (M-m)rt + C_1$

$\Rightarrow \ln\left|\frac{P-m}{M-P}\right| = (M-m)rt + C_1 \Rightarrow \frac{P-m}{M-P} = \pm e^{C_1} e^{(M-m)rt} \Rightarrow \frac{P-m}{M-P} = Ce^{(M-m)rt}$ where $C = \pm e^{C_1}$

$\Rightarrow P - m = MCe^{(M-m)rt} - CPe^{(M-m)rt} \Rightarrow P(1 + Ce^{(M-m)rt}) = MCe^{(M-m)rt} + m \Rightarrow P = \frac{MCe^{(M-m)rt} + m}{Ce^{(M-m)rt} + 1}$

$\Rightarrow P = \frac{M + \frac{m}{C}e^{-(M-m)rt}}{1 + \frac{1}{C}e^{-(M-m)rt}} \Rightarrow P = \frac{M + mAe^{-(M-m)rt}}{1 + Ae^{-(M-m)rt}}$ where $A = \frac{1}{C}$.

Apply the initial condition $P(0) = P_0$:

$P_0 = \frac{M + mA}{1 + A} \Rightarrow P_0 + P_0A = M + mA \Rightarrow A = \frac{M - P_0}{P_0 - m} \Rightarrow P = \frac{M(P_0 - m) + m(M - P_0)e^{-(M-m)rt}}{(P_0 - m) + (M - P_0)e^{-(M-m)rt}}$

(Note that $P \to M$ as $t \to \infty$ provided $P_0 > m$.)

17. $\frac{dy}{dx} = 2xe^{x^2}$, $y(0) = 2 \Rightarrow y_{n+1} = y_n + 2x_n e^{x_n^2}\, dx = y_n + 2x_n e^{x_n^2}(0.1) = y_n + 0.2x_n e^{x_n^2}$

On a TI-92 Plus calculator home screen, type the following commands:

2 STO> y: 0 STO> x:y (enter)
y+0.2*x*e^(x^2) STO> y: x+0.1 STO>x: y (enter, 10 times)

The last value displayed gives $y_{Euler}(1) \approx 3.45835$

The exact solution: $dy = 2xe^{x^2}\, dx \Rightarrow y = e^{x^2} + C$; $y(0) = 2 = e^0 + C \Rightarrow C = 1 \Rightarrow y = 1 + e^{x^2}$

$\Rightarrow y_{exact}(1) = 1 + e \approx 3.71828$

18. $\frac{dy}{dx} = y + e^x - 2$, $y(0) = 2 \Rightarrow y_{n+1} = y_n + (y_n + e^{x_n} - 2)\, dx = y_n + 0.5(y_n + e^{x_n} - 2)$

On a TI-92 Plus calculator home screen, type the following commands:

2 STO> y: 0 STO> x:y (enter)

y+0.5*$(y + e^x - 2)$ STO> y: x+0.5 STO>x: y (enter, 4 times)

The last value displayed gives $y_{Euler}(2) \approx 9.82187$

The exact solution: $\frac{dy}{dx} - y = e^x - 2 \Rightarrow P(x) = -1,\ Q(x) = e^x - 2 \Rightarrow \int P(x)\, dx = -x \Rightarrow v(x) = e^{-x}$

$\Rightarrow y = \frac{1}{e^{-x}} \int e^{-x}(e^x - 2)\, dx = e^x(x + 2e^{-x} + C)$; $y(0) = 2 \Rightarrow 2 = 2 + C \Rightarrow C = 0$

$\Rightarrow y = xe^x + 2 \Rightarrow y_{exact}(2) = 2e^2 + 2 \approx 16.7781$

19. $y_1 = -1 + \left[\frac{(-1)^2}{\sqrt{1}}\right](0.5) = -0.5,$

$y_2 = -0.5 + \left[\frac{(-0.5)^2}{\sqrt{1.5}}\right](0.5) = -0.39794,$

$y_3 = -0.39794 + \left[\frac{(-0.39794)^2}{\sqrt{2}}\right](0.5) = -0.34195,$

$y_4 = -0.34195 + \left[\frac{(-0.34195)^2}{\sqrt{2.5}}\right](0.5) = -0.30497,$

$y_5 = -0.27812,\ y_6 = -0.25745,\ y_7 = -0.24088,\ y_8 = -0.2272;$

$\frac{dy}{y^2} = \frac{dx}{\sqrt{x}} \Rightarrow -\frac{1}{y} = 2\sqrt{x} + C;\ y(1) = -1 \Rightarrow 1 = 2 + C \Rightarrow C = -1 \Rightarrow y = \frac{1}{1-2\sqrt{x}} \Rightarrow y(5) = \frac{1}{1-2\sqrt{5}} \approx -0.2880$

20. $y_1 = 1 + (1 - e^0)\left(\frac{1}{3}\right) = 1,$

$y_2 = 1 + (1 - e^{2/3})\left(\frac{1}{3}\right) = 0.68409,$

$y_3 = 0.68409 + (0.68409 - e^{4/3})\left(\frac{1}{3}\right) = -0.35244,$

$y_4 = -0.35244 + (-0.35244 - e^{6/3})\left(\frac{1}{3}\right) = -2.93294,$

$y_5 = -2.93294 + (-2.93294 - e^{8/3})\left(\frac{1}{3}\right) = -8.70789,$

$y_6 = -8.70789 + (-8.70789 - e^{10/3})\left(\frac{1}{3}\right) = -20.95439;$

$y' - y = -e^{2x} \Rightarrow P(x) = -1,\ Q(x) = -e^{2x} \Rightarrow \int P(x)\,dx = -x \Rightarrow v(x) = e^{-x} \Rightarrow y = \frac{1}{e^{-x}} \int e^{-x}(-e^{2x})\,dx$

$= e^x(-e^x + C);\ y(0) = 1 \Rightarrow 1 = -1 + C \Rightarrow C = 2 \Rightarrow y = -e^{2x} + 2e^x \Rightarrow y(2) = -e^4 + 2e^2 \approx -39.8200$

21. (a) $\frac{dy}{dx} = 2y^2(x-1) \Rightarrow \frac{dy}{y^2} = 2(x-1)\,dx \Rightarrow \int y^{-2}\,dy = \int (2x-2)\,dx \Rightarrow -y^{-1} = x^2 - 2x + C$

Initial value: $y(2) = -\frac{1}{2} \Rightarrow 2 = 2^2 - 2(2) + C \Rightarrow C = 2$

Solution: $-y^{-1} = x^2 - 2x + 2$ or $y = -\frac{1}{x^2 - 2x + 2}$

$y(3) = -\frac{1}{3^2 - 2(3) + 2} = -\frac{1}{5} = -0.2$

(b) To find the approximation, set $y_1 = 2y^2(x-1)$ and use EULERT with initial values $x = 2$ and $y = -\frac{1}{2}$ and step size 0.2 for 5 points. This gives $y(3) \approx -0.1851$; error ≈ 0.0149.

(c) Use step size 0.1 for 10 points. This gives $y(3) \approx -0.1929$; error ≈ 0.0071.

(d) Use step size 0.05 for 20 points. This gives $y(3) \approx -0.1965$; error ≈ 0.0035.

22. (a) $\frac{dy}{dx} = y - 1 \Rightarrow \int \frac{dy}{y-1} = \int dx \Rightarrow \ln|y-1| = x + C \Rightarrow |y-1| = e^{x+C} \Rightarrow y - 1 = \pm e^C e^x$

$\Rightarrow y = Ae^x + 1$

Initial condition: $y(0) = 3 \Rightarrow 3 = Ae^0 + 1 \Rightarrow A = 2$

Solution: $y = 2e^x + 1$

$y(1) = 2e + 1 \approx 6.4366$

(b) To find the approximation, set $y_1 = y - 1$ and use a graphing calculator or CAS with initial values $x = 0$ and $y = 3$ and step size 0.2 for 5 points. This gives $y(1) \approx 5.9766$; error ≈ 0.4599.

(c) Use step size 0.1 for 10 points. This gives $y(1) \approx 6.1875$; error ≈ 0.2491.

(d) Use step size 0.05 for 20 points. This gives $y(1) \approx 6.3066$; error ≈ 0.1300.

23. The exact solution is $y = \frac{-1}{x^2 - 2x + 2}$, so $y(3) = -0.2$. To find the approximation, let $z_n = y_{n-1} + 2y_{n-1}^2(x_n - 1)\,dx$ and $y_n = y_{n-1} + (y_{n-1}^2(x_{n-1} - 1) + z_n^2(x_n^2 - 1))\,dx$ with initial values $x_0 = 2$ and $y_0 = -\frac{1}{2}$. Use a spreadsheet, graphing calculator, or CAS as indicated in parts (a) through (d).
   (a) Use $dx = 0.2$ with 5 steps to obtain $y(3) \approx -0.2024 \Rightarrow$ error ≈ 0.0024.
   (b) Use $dx = 0.1$ with 10 steps to obtain $y(3) \approx -0.2005 \Rightarrow$ error ≈ 0.0005.
   (c) Use $dx = 0.05$ with 20 steps to obtain $y(3) \approx -0.2001 \Rightarrow$ error ≈ 0.0001.
   (d) Each time the step size is cut in half, the error is reduced to approximately one-fourth of what it was for the larger step size.

24. The exact solution is $y = 2e^x + 1$, so $y(1) = 2e^1 + 1 \approx 6.4366$. To find the approximate solution let $z_n = y_{n-1} + (y_{n-1} - 1)\,dx$ and $y_n = y_{n-1} + \left(\frac{y_{n-1} + z_n - 2}{2}\right) dx$ with initial value $y_n = 3$. Use a spreadsheet, graphing calculator, or CAS as indicated in parts (a) through (d).
   (a) Use $dx = 0.2$ with 5 steps to obtain $y(1) \approx 6.4054 \Rightarrow$ error ≈ 0.0311.
   (b) Use $dx = 0.1$ with 10 steps to obtain $y(1) \approx 6.4282 \Rightarrow$ error ≈ 0.0084.
   (c) Use $dx = 0.05$ with 20 steps to obtain $y(1) \approx 6.4344 \Rightarrow$ error ≈ 0.0022.
   (d) Each time the step size is cut in half, the error is reduced to approximately one-fourth of what it was for the larger step size.

25-30. Example CAS commands:

Maple:
```
with(plots): with(DEtools):
a:=-4; b:=4;
eq:= D(y)(x) = x + y;
plot1:= dfieldplot(eq,[x,y], x=a..b, y=-4..4, scaling=CONSTRAINED);
display({plot1});
gen_sol:= dsolve({eq},y(x));
tograph:= {seq(subs(_C1=i, gen_sol), i = {-1,0,1,3,9})};
plot2:=implicitplot(tograph, x=a..b, y=-4..4, scaling=CONSTRAINED):
display({plot1,plot2}, title = `Direction Field and Solution Curves`);
eulerapprox:= proc(f,x0,y0,n) local i,j,h;
  x(0):= evalf(x0);
  y(0):= evalf(y0);
     h:= (b-a)/n;
for i from 1 to n do
  y(i):= evalf(y(i-1) + h*f(x(i-1),y(i-1))):  x(i):= x(i-1) + h  od;
[[x(j),y(j)] $j=0..n];
end;
rhs(eq);
f:= unapply(%,(x,y));
eulerapprox(f,0,-7/10,4);
plot3:= plot(%, style=LINE,scaling=CONSTRAINED, title=`Euler Approximation`):
display({plot3});
y(0):= 'y(0)';
partsol:= dsolve({eq, y(0)=-7/10}, y(x));
plot4: implicitplot(partsol,x=-1..8,y=-3..40,scaling=CONSTRAINED):
display({plot3, plot4}, title=`Actual Solution & Euler Approximation`);
```

Mathematica:

```
Need package for plotting vector fields:
<< "Graphics`PlotField`"
Also load package to improve solving of ODE's:
<< Calculus`DSolve`
SetOptions[PlotVectorField, PlotPoints -> 6];
Clear[x,y,yp,h]
Note: here we define "eulerstep" to find the next Euler point, given the
current one, assuming that the variables "a" (initial point), "b" (final
point), and "n" (# of steps) have been defined, along with the function
"yp[x,y]" (which specifies the derivative). Then the whole Euler solution
is given as a list of points {x,y} by:
  NestList[ eulerstep, N[{a,ya}], n ]
where "ya" is the initial value.
eulerstep[{x_,y_}]  :=  {x+h, y+h*N[yp[x,y]]  }
h  :=  N[(b-a)/n]

yp[x_,y_] := x+y
{a,b} = {0,1}; ya = 1;
{xmin,xmax} = {-4,4}; {ymin,ymax} = {-4,4};
p1 = PlotVectorField[{1,yp[x,y]},{x,xmin,xmax},{y,ymin,ymax},
 ScaleFunction -> (1&) ]
ode = y'[x] == yp[x,y[x]]
DSolve[ ode, y[x], x ]
gensol = y[x] /. First[%]
sols = Map[ (gensol /. C[1] -> #)&, {-2,-1,0,1,2} ]
p2 = Plot[ Evaluate[sols], {x,xmin,xmax} ]
Show[ {p1, p2}, PlotRange -> {Automatic,{ymin,ymax}} ]
DSolve[ {ode, y[a] == ya}, y[x], x ]
partsol = y[x] /. First[%]
p3 = Plot[ partsol, {x,a,b} ]
n = 10;
approx1 = NestList[ eulerstep, N[{a,ya}], n ];
p4 = ListPlot[ approx1, PlotJoined -> True ]
Show[{p4,p3)}]

Here's an alternate approach to plotting the two solutions (simpler but less
obvious):

Show[p3, Epilog -> {Line[approx]} ]

n = 25;
approx2 = NestList[ eulerstep, N[{a,ya}], n ];
p4 = ListPlot[ approx2, PlotJoined -> True ]
Show[{p4,p3}]
n = 50;
approx3 = NestList[ eulerstep, N[{a,ya}], n ];
p4 = ListPlot[ approx3, PlotJoined -> True ]
Show[{p4,p3}]
n = 100;
approx4 = NestList[ eulerstep, N[{a,ya}], n ];
p4 = ListPlot[ approx4, PlotJoined -> True ]
Show[{p4,p3}]
yb = partsol /. x -> b // N
err1 = Last[approx1][[2]] - yb
percent1 = err1/yb * 100
```

```
err2 = Last[approx2][[2]] - yb
percent2 = err2/yb * 100
err3 = Last[approx3][[2]] - yb
percent3 = err3/yb * 100
err4 = Last[approx4][[2]] - yb
percent4 = err4/yb * 100
```

31. Example CAS commands:

Maple:

```
with(plots): with(DEtools):
eq:= D(y)(x) = f - y;
eq1:= subs(f = 2*x, eq);
eq2:= subs(f=sin(2*x), eq);
eq3:= subs(f=3*exp(x/2), eq);
eq4:= subs(f=2*exp(-x/2)*cos(2*x), eq);
partsol1:= dsolve({eq1,y(0)=0}, y(x));
plot1:= implicitplot(partsol1, x=-2..6, y=-1..10, scaling=CONSTRAINED):
display(plot1);
partsol2:= dsolve({eq2, y(0)=0}, y(x));
plot2:= implicitplot(partsol2, x=-2..6, y=-1..4, scaling=CONSTRAINED):
display(plot2);
partsol3:= dsolve({eq3, y(0)=0}, y(x));
plot3:= implicitplot(partsol3, x=-2..6, y=-2..10, scaling=CONSTRAINED):
display(plot3);
partsol4:= dsolve({eq4, y(0)=0}, y(x));
plot4:= implicitplot(partsol4, x=-2..6, y=-3..2, scaling=CONSTRAINED):
display(plot4);
display({plot1,plot2,plot3,plot4});
```

Mathematica:

```
Clear[x,y,f]
ode = y'[x] + y[x] == f[x]
a = 0; ya = 0;
{xmin,xmax} = {-2,6};
f[x_] = 2x
DSolve[ {ode, y[a] == ya}, y[x], x ]
sol1 = y[x] /. First[%]
Plot[ sol1, {x,xmin,xmax} ]
f[x_] = Sin[2x]
DSolve[ {ode, y[a] == ya}, y[x], x ]
sol2 = y[x] /. First[%]
Plot[ sol2, {x,xmin,xmax} ]
f[x_] = 3 Exp[x/2]
DSolve[ {ode, y[a] == ya}, y[x], x ]
sol3 = y[x] /. First[%]
Plot[ sol3, {x,xmin,xmax} ]
f[x_] = 2 Exp[-x/2] Cos[2x]
DSolve[ {ode, y[a] == ya}, y[x], x ]
sol4 = y[x] /. First[%]
Plot[ sol4, {x,xmin,xmax} ]
Plot[ {sol1, sol2, sol3, sol4}, {x,xmin,xmax} ]
```

32. Example CAS commands:

Maple:

```
with(plots): with(DEtools):
a:=-3; b:=3;
eq:=D(y)(x)=(3*x^2+4*x+2)/(2*(y-1));
plot1:=dfieldplot(eq,[x,y],x=a..b,y=a..b,scaling=CONSTRAINED):
display({plot1});
right:=int(numer(rhs(eq)),x);
left:=int(denom(rhs(eq)),y);
sol:=left = right + C;
tograph:={seq(subs(C=i,sol),i={-6,-4,-2,0,2,4,6})};
plot2:=implicitplot(tograph,x=a..b,y=a..b, scaling=CONSTRAINED):
display(plot2);
DEplot(eq,y(x),x=a..b,{[0,-1]},y=a..b);
```

Mathematica:

```
yp[x_,y_] := (3x^2 + 4x + 2) / (2(y-1))
a = 0; ya = -1;
{xmin,xmax} = {-3,3}; {ymin,ymax} = {-3,3};
p1 = PlotVectorField[{1,yp[x,y]}, {x,xmin,xmax}, {y,ymin,ymax},
  ScaleFunction -> (1&) ]
impeqn =
  Integrate[Denominator[yp[x,y]],y] ==
  Integrate[Numerator[yp[x,y]],x] + C[1]
<< Graphics`ImplicitPlot`
eqns = Map[ (impeqn /. C[1] -> #)&,
  {-6,-4,-2,0,2,4,6} ];
p2 = ImplicitPlot[ Evaluate[eqns], {x,xmin,xmax} ]
Show[ {p1, p2} ]
impeqn /. {x -> 0, y -> -1}
Solve[%,C[1]]
parteqn = impeqn /. First[%]
ImplicitPlot[ Evaluate[parteqn], {x,xmin,xmax} ]
```

6.5 HYPERBOLIC FUNCTIONS

1. $\sinh x = -\frac{3}{4} \Rightarrow \cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\left(-\frac{3}{4}\right)^2} = \sqrt{1+\frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(-\frac{3}{4}\right)}{\left(\frac{5}{4}\right)} = -\frac{3}{5}$, $\coth x = \frac{1}{\tanh x} = -\frac{5}{3}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{4}{5}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = -\frac{4}{3}$

2. $\sinh x = \frac{4}{3} \Rightarrow \cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{4}{3}\right)}{\left(\frac{5}{3}\right)} = \frac{4}{5}$, $\coth x = \frac{1}{\tanh x} = \frac{5}{4}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{3}{5}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{3}{4}$

3. $\cosh x = \frac{17}{15}$, $x > 0 \Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\left(\frac{17}{15}\right)^2 - 1} = \sqrt{\frac{289}{225} - 1} = \sqrt{\frac{64}{225}} = \frac{8}{15}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{8}{15}\right)}{\left(\frac{17}{15}\right)} = \frac{8}{17}$, $\coth x = \frac{1}{\tanh x} = \frac{17}{8}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{15}{17}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{15}{8}$

4. $\cosh x = \frac{13}{5},\ x > 0 \Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\frac{169}{25} - 1} = \sqrt{\frac{144}{25}} = \frac{12}{5}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{12}{5}\right)}{\left(\frac{13}{5}\right)} = \frac{12}{13}$, $\coth x = \frac{1}{\tanh x} = \frac{13}{12}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{5}{13}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{5}{12}$

5. $2 \cosh(\ln x) = 2\left(\frac{e^{\ln x} + e^{-\ln x}}{2}\right) = e^{\ln x} + \frac{1}{e^{\ln x}} = x + \frac{1}{x}$

6. $\sinh(2 \ln x) = \frac{e^{2\ln x} - e^{-2\ln x}}{2} = \frac{e^{\ln x^2} - e^{\ln x^{-2}}}{2} = \frac{\left(x^2 - \frac{1}{x^2}\right)}{2} = \frac{x^4 - 1}{2x^2}$

7. $\cosh 5x + \sinh 5x = \frac{e^{5x} + e^{-5x}}{2} + \frac{e^{5x} - e^{-5x}}{2} = e^{5x}$

8. $\cosh 3x - \sinh 3x = \frac{e^{3x} + e^{-3x}}{2} - \frac{e^{3x} - e^{-3x}}{2} = e^{-3x}$

9. $(\sinh x + \cosh x)^4 = \left(\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}\right)^4 = (e^x)^4 = e^{4x}$

10. $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x) = \ln(\cosh^2 x - \sinh^2 x) = \ln 1 = 0$

11. (a) $\sinh 2x = \sinh(x + x) = \sinh x \cosh x + \cosh x \sinh x = 2 \sinh x \cosh x$

(b) $\cosh 2x = \cosh(x + x) = \cosh x \cosh x + \sinh x \sin x = \cosh^2 x + \sinh^2 x$

12. $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}\left[(e^x + e^{-x}) + (e^x - e^{-x})\right]\left[(e^x + e^{-x}) - (e^x - e^{-x})\right]$
$= \frac{1}{4}(2e^x)(2e^{-x}) = \frac{1}{4}(4e^0) = \frac{1}{4}(4) = 1$

13. $y = 6 \sinh \frac{x}{3} \Rightarrow \frac{dy}{dx} = 6\left(\cosh \frac{x}{3}\right)\left(\frac{1}{3}\right) = 2 \cosh \frac{x}{3}$

14. $y = \frac{1}{2} \sinh(2x + 1) \Rightarrow \frac{dy}{dx} = \frac{1}{2}[\cosh(2x + 1)](2) = \cosh(2x + 1)$

15. $y = 2\sqrt{t} \tanh \sqrt{t} = 2t^{1/2} \tanh t^{1/2} \Rightarrow \frac{dy}{dt} = \left[\operatorname{sech}^2(t^{1/2})\right]\left(\frac{1}{2}t^{-1/2}\right)(2t^{1/2}) + (\tanh t^{1/2})(t^{-1/2})$
$= \operatorname{sech}^2 \sqrt{t} + \frac{\tanh \sqrt{t}}{\sqrt{t}}$

16. $y = t^2 \tanh \frac{1}{t} = t^2 \tanh t^{-1} \Rightarrow \frac{dy}{dt} = \left[\operatorname{sech}^2(t^{-1})\right](-t^{-2})(t^2) + (2t)(\tanh t^{-1}) = -\operatorname{sech}^2 \frac{1}{t} + 2t \tanh \frac{1}{t}$

17. $y = \ln(\sinh z) \Rightarrow \frac{dy}{dz} = \frac{\cosh z}{\sinh z} = \coth z$

18. $y = \ln(\cosh z) \Rightarrow \frac{dy}{dz} = \frac{\sinh z}{\cosh z} = \tanh z$

19. $y = (\operatorname{sech} \theta)(1 - \ln \operatorname{sech} \theta) \Rightarrow \frac{dy}{d\theta} = \left(-\frac{-\operatorname{sech} \theta \tanh \theta}{\operatorname{sech} \theta}\right)(\operatorname{sech} \theta) + (-\operatorname{sech} \theta \tanh \theta)(1 - \ln \operatorname{sech} \theta)$
$= \operatorname{sech} \theta \tanh \theta - (\operatorname{sech} \theta \tanh \theta)(1 - \ln \operatorname{sech} \theta) = (\operatorname{sech} \theta \tanh \theta)[1 - (1 - \ln \operatorname{sech} \theta)]$
$= (\operatorname{sech} \theta \tanh \theta)(\ln \operatorname{sech} \theta)$

20. $y = (\text{csch}\ \theta)(1 - \ln \text{csch}\ \theta) \Rightarrow \frac{dy}{d\theta} = (\text{csch}\ \theta)\left(-\frac{-\text{csch}\ \theta \coth\theta}{\text{csch}\ \theta}\right) + (1 - \ln \text{csch}\ \theta)(-\text{csch}\ \theta \coth\theta)$

$= \text{csch}\ \theta \coth\theta - (1 - \ln \text{csch}\ \theta)(\text{csch}\ \theta \coth\theta) = (\text{csch}\ \theta \coth\theta)(1 - 1 + \ln \text{csch}\ \theta) = (\text{csch}\ \theta \coth\theta)(\ln \text{csch}\ \theta)$

21. $y = \ln \cosh v - \frac{1}{2}\tanh^2 v \Rightarrow \frac{dy}{dv} = \frac{\sinh v}{\cosh v} - \left(\frac{1}{2}\right)(2\tanh v)(\text{sech}^2 v) = \tanh v - (\tanh v)(\text{sech}^2 v)$

$= (\tanh v)(1 - \text{sech}^2 v) = (\tanh v)(\tanh^2 v) = \tanh^3 v$

22. $y = \ln \sinh v - \frac{1}{2}\coth^2 v \Rightarrow \frac{dy}{dv} = \frac{\cosh v}{\sinh v} - \left(\frac{1}{2}\right)(2\coth v)(-\text{csch}^2 v) = \coth v + (\coth v)(\text{csch}^2 v)$

$= (\coth v)(1 + \text{csch}^2 v) = (\coth v)(\coth^2 v) = \coth^3 v$

23. $y = (x^2+1)\,\text{sech}(\ln x) = (x^2+1)\left(\frac{2}{e^{\ln x} + e^{-\ln x}}\right) = (x^2+1)\left(\frac{2}{x + x^{-1}}\right) = (x^2+1)\left(\frac{2x}{x^2+1}\right) = 2x \Rightarrow \frac{dy}{dx} = 2$

24. $y = (4x^2-1)\,\text{csch}(\ln 2x) = (4x^2-1)\left(\frac{2}{e^{\ln 2x} + e^{-\ln 2x}}\right) = (4x^2-1)\left(\frac{2}{2x - (2x)^{-1}}\right) = (4x^2-1)\left(\frac{4x}{4x^2-1}\right)$

$= 4x \Rightarrow \frac{dy}{dx} = 4$

25. $y = \sinh^{-1}\sqrt{x} = \sinh^{-1}(x^{1/2}) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{2}\right)x^{-1/2}}{\sqrt{1 + (x^{1/2})^2}} = \frac{1}{2\sqrt{x}\sqrt{1+x}} = \frac{1}{2\sqrt{x(1+x)}}$

26. $y = \cosh^{-1} 2\sqrt{x+1} = \cosh^{-1}\left(2(x+1)^{1/2}\right) \Rightarrow \frac{dy}{dx} = \frac{(2)\left(\frac{1}{2}\right)(x+1)^{-1/2}}{\sqrt{\left[2(x+1)^{1/2}\right]^2 - 1}} = \frac{1}{\sqrt{x+1}\sqrt{4x+3}} = \frac{1}{\sqrt{4x^2+7x+3}}$

27. $y = (1-\theta)\tanh^{-1}\theta \Rightarrow \frac{dy}{d\theta} = (1-\theta)\left(\frac{1}{1-\theta^2}\right) + (-1)\tanh^{-1}\theta = \frac{1}{1+\theta} - \tanh^{-1}\theta$

28. $y = (\theta^2 + 2\theta)\tanh^{-1}(\theta+1) \Rightarrow \frac{dy}{d\theta} = (\theta^2+2\theta)\left[\frac{1}{1-(\theta+1)^2}\right] + (2\theta+2)\tanh^{-1}(\theta+1)$

$= \frac{\theta^2+2\theta}{-\theta^2-2\theta} + (2\theta+2)\tanh^{-1}(\theta+1) = (2\theta+2)\tanh^{-1}(\theta+1) - 1$

29. $y = (1-t)\coth^{-1}\sqrt{t} = (1-t)\coth^{-1}(t^{1/2}) \Rightarrow \frac{dy}{dt} = (1-t)\left[\frac{\left(\frac{1}{2}\right)t^{-1/2}}{1-(t^{1/2})^2}\right] + (-1)\coth^{-1}(t^{1/2}) = \frac{1}{2\sqrt{t}} - \coth^{-1}\sqrt{t}$

30. $y = (1-t^2)\coth^{-1} t \Rightarrow \frac{dy}{dt} = (1-t^2)\left(\frac{1}{1-t^2}\right) + (-2t)\coth^{-1} t = 1 - 2t\coth^{-1} t$

31. $y = \cos^{-1} x - x\,\text{sech}^{-1} x \Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} - \left[x\left(\frac{-1}{x\sqrt{1-x^2}}\right) + (1)\,\text{sech}^{-1} x\right] = \frac{-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} - \text{sech}^{-1} x$

$= -\text{sech}^{-1} x$

32. $y = \ln x + \sqrt{1-x^2}\,\text{sech}^{-1} x = \ln x + (1-x^2)^{1/2}\,\text{sech}^{-1} x \Rightarrow \frac{dy}{dx}$

$= \frac{1}{x} + (1-x^2)^{1/2}\left(\frac{-1}{x\sqrt{1-x^2}}\right) + \left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x)\,\text{sech}^{-1} x = \frac{1}{x} - \frac{1}{x} - \frac{x}{\sqrt{1-x^2}}\,\text{sech}^{-1} x = \frac{-x}{\sqrt{1-x^2}}\,\text{sech}^{-1} x$

33. $y = \text{csch}^{-1}\left(\frac{1}{2}\right)^\theta \Rightarrow \frac{dy}{d\theta} = -\frac{\left[\ln\left(\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)^\theta}{\left(\frac{1}{2}\right)^\theta\sqrt{1+\left[\left(\frac{1}{2}\right)^\theta\right]^2}} = -\frac{\ln(1)-\ln(2)}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}} = \frac{\ln 2}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}}$

34. $y = \text{csch}^{-1} 2^\theta \Rightarrow \frac{dy}{d\theta} = -\frac{(\ln 2)\, 2^\theta}{2^\theta\sqrt{1+(2^\theta)^2}} = \frac{-\ln 2}{\sqrt{1+2^{2\theta}}}$

35. $y = \sinh^{-1}(\tan x) \Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{\sqrt{1+(\tan x)^2}} = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \frac{\sec^2 x}{|\sec x|} = \frac{|\sec x|\,|\sec x|}{|\sec x|} = |\sec x|$

36. $y = \cosh^{-1}(\sec x) \Rightarrow \frac{dy}{dx} = \frac{(\sec x)(\tan x)}{\sqrt{\sec^2 x - 1}} = \frac{(\sec x)(\tan x)}{\sqrt{\tan^2 x}} = \frac{(\sec x)(\tan x)}{|\tan x|} = \sec x,\ 0 < x < \frac{\pi}{2}$

37. (a) If $y = \tan^{-1}(\sinh x) + C$, then $\frac{dy}{dx} = \frac{\cosh x}{1+\sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \text{sech}\, x$, which verifies the formula

(b) If $y = \sin^{-1}(\tanh x) + C$, then $\frac{dy}{dx} = \frac{\text{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\text{sech}^2 x}{\text{sech}\, x} = \text{sech}\, x$, which verifies the formula

38. If $y = \frac{x^2}{2}\,\text{sech}^{-1} x - \frac{1}{2}\sqrt{1-x^2} + C$, then $\frac{dy}{dx} = x\,\text{sech}^{-1} x + \frac{x^2}{2}\left(\frac{-1}{x\sqrt{1-x^2}}\right) + \frac{2x}{4\sqrt{1-x^2}} = x\,\text{sech}^{-1} x$, which verifies the formula

39. If $y = \frac{x^2-1}{2}\coth^{-1} x + \frac{x}{2} + C$, then $\frac{dy}{dx} = x\coth^{-1} x + \left(\frac{x^2-1}{2}\right)\left(\frac{1}{1-x^2}\right) + \frac{1}{2} = x\coth^{-1} x$, which verifies the formula

40. If $y = x\tanh^{-1} x + \frac{1}{2}\ln(1-x^2) + C$, then $\frac{dy}{dx} = \tanh^{-1} x + x\left(\frac{1}{1-x^2}\right) + \frac{1}{2}\left(\frac{-2x}{1-x^2}\right) = \tanh^{-1} x$, which verifies the formula

41. $\int \sinh 2x\, dx = \frac{1}{2}\int \sinh u\, du$, where $u = 2x$ and $du = 2\, dx$

$= \frac{\cosh u}{2} + C = \frac{\cosh 2x}{2} + C$

42. $\int \sinh \frac{x}{5}\, dx = 5\int \sinh u\, du$, where $u = \frac{x}{5}$ and $du = \frac{1}{5}\, dx$

$= 5\cosh u + C = 5\cosh \frac{x}{5} + C$

43. $\int 6\cosh\left(\frac{x}{2} - \ln 3\right) dx = 12 \int \cosh u\, du$, where $u = \frac{x}{2} - \ln 3$ and $du = \frac{1}{2}\, dx$

$= 12 \sinh u + C = 12 \sinh\left(\frac{x}{2} - \ln 3\right) + C$

44. $\int 4\cosh(3x - \ln 2)\, dx = \frac{4}{3} \int \cosh u\, du$, where $u = 3x - \ln 2$ and $du = 3\, dx$

$= \frac{4}{3} \sinh u + C = \frac{4}{3} \sinh(3x - \ln 2) + C$

45. $\int \tanh \frac{x}{7}\, dx = 7 \int \frac{\sinh u}{\cosh u}\, du$, where $u = \frac{x}{7}$ and $du = \frac{1}{7}\, dx$

$= 7 \ln|\cosh u| + C_1 = 7 \ln\left|\cosh \frac{x}{7}\right| + C_1 = 7 \ln\left|\frac{e^{x/7} + e^{-x/7}}{2}\right| + C_1 = 7 \ln\left|e^{x/7} + e^{-x/7}\right| - 7 \ln 2 + C_1$

$= 7 \ln\left(e^{x/7} + e^{-x/7}\right) + C$, since $e^{x/7} + e^{-x/7} > 0$ for all x.

46. $\int \coth \frac{\theta}{\sqrt{3}}\, d\theta = \sqrt{3} \int \frac{\cosh u}{\sinh u}\, du$, where $u = \frac{\theta}{\sqrt{3}}$ and $du = \frac{d\theta}{\sqrt{3}}$

$= \sqrt{3} \ln|\sinh u| + C_1 = \sqrt{3} \ln\left|\sinh \frac{\theta}{\sqrt{3}}\right| + C_1 = \sqrt{3} \ln\left|\frac{e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}}{2}\right| + C_1$

$= \sqrt{3} \ln\left|e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}\right| - \sqrt{3} \ln 2 + C_1 = \sqrt{3} \ln\left|e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}\right| + C$

47. $\int \operatorname{sech}^2\left(x - \frac{1}{2}\right) dx = \int \operatorname{sech}^2 u\, du$, where $u = \left(x - \frac{1}{2}\right)$ and $du = dx$

$= \tanh u + C = \tanh\left(x - \frac{1}{2}\right) + C$

48. $\int \operatorname{csch}^2(5 - x)\, dx = -\int \operatorname{csch}^2 u\, du$, where $u = (5 - x)$ and $du = -dx$

$= -(-\coth u) + C = \coth u + C = \coth(5 - x) + C$

49. $\int \frac{\operatorname{sech}\sqrt{t}\tanh\sqrt{t}}{\sqrt{t}}\, dt = 2 \int \operatorname{sech} u \tanh u\, du$, where $u = \sqrt{t} = t^{1/2}$ and $du = \frac{dt}{2\sqrt{t}}$

$= 2(-\operatorname{sech} u) + C = -2 \operatorname{sech}\sqrt{t} + C$

50. $\int \frac{\operatorname{csch}(\ln t)\coth(\ln t)}{t}\, dt = \int \operatorname{csch} u \coth u\, du$, where $u = \ln t$ and $du = \frac{dt}{t}$

$= -\operatorname{csch} u + C = -\operatorname{csch}(\ln t) + C$

51. $\int_{\ln 2}^{\ln 4} \coth x\, dx = \int_{\ln 2}^{\ln 4} \frac{\cosh x}{\sinh x}\, dx = \int_{3/4}^{15/8} \frac{1}{u}\, du = [\ln|u|]_{3/4}^{15/8} = \ln\left|\frac{15}{8}\right| - \ln\left|\frac{3}{4}\right| = \ln\left|\frac{15}{8} \cdot \frac{4}{3}\right| = \ln \frac{5}{2}$,

where $u = \sinh x$, $du = \cosh x\, dx$, the lower limit is $\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - \left(\frac{1}{2}\right)}{2} = \frac{3}{4}$ and the upper limit is $\sinh(\ln 4) = \frac{e^{\ln 4} - e^{-\ln 4}}{2} = \frac{4 - \left(\frac{1}{4}\right)}{2} = \frac{15}{8}$

52. $\int_0^{\ln 2} \tanh 2x \, dx = \int_0^{\ln 2} \frac{\sinh 2x}{\cosh 2x} \, dx = \frac{1}{2} \int_1^{17/8} \frac{1}{u} \, du = \frac{1}{2}\left[\ln|u|\right]_1^{17/8} = \frac{1}{2}\left[\ln\left(\frac{17}{8}\right) - \ln 1\right] = \frac{1}{2} \ln \frac{17}{8}$, where $u = \cosh 2x$, $du = 2\sinh(2x)\,dx$, the lower limit is $\cosh 0 = 1$ and the upper limit is $\cosh(2\ln 2) = \cosh(\ln 4)$ $= \frac{e^{\ln 4} + e^{-\ln 4}}{2} = \frac{4 + \left(\frac{1}{4}\right)}{2} = \frac{17}{8}$

53. $\int_{-\ln 4}^{-\ln 2} 2e^{\theta} \cosh\theta \, d\theta = \int_{-\ln 4}^{-\ln 2} 2e^{\theta}\left(\frac{e^{\theta} + e^{-\theta}}{2}\right) d\theta = \int_{-\ln 4}^{-\ln 2} \left(e^{2\theta} + 1\right) d\theta = \left[\frac{e^{2\theta}}{2} + \theta\right]_{-\ln 4}^{-\ln 2}$

$= \left(\frac{e^{-2\ln 2}}{2} - \ln 2\right) - \left(\frac{e^{-2\ln 4}}{2} - \ln 4\right) = \left(\frac{1}{8} - \ln 2\right) - \left(\frac{1}{32} - \ln 4\right) = \frac{3}{32} - \ln 2 + 2\ln 2 = \frac{3}{32} + \ln 2$

54. $\int_0^{\ln 2} 4e^{-\theta} \sinh\theta \, d\theta = \int_0^{\ln 2} 4e^{-\theta}\left(\frac{e^{\theta} - e^{-\theta}}{2}\right) d\theta = 2\int_0^{\ln 2} \left(1 - e^{-2\theta}\right) d\theta = 2\left[\theta + \frac{e^{-2\theta}}{2}\right]_0^{\ln 2}$

$= 2\left[\left(\ln 2 + \frac{e^{-2\ln 2}}{2}\right) - \left(0 + \frac{e^{-0}}{2}\right)\right] = 2\left(\ln 2 + \frac{1}{8} - \frac{1}{2}\right) = 2\ln 2 + \frac{1}{4} - 1 = \ln 4 - \frac{3}{4}$

55. $\int_{-\pi/4}^{\pi/4} \cosh(\tan\theta)\sec^2\theta \, d\theta = \int_{-1}^{1} \cosh u \, du = [\sinh u]_{-1}^{1} = \sinh(1) - \sinh(-1) = \left(\frac{e^1 - e^{-1}}{2}\right) - \left(\frac{e^{-1} - e^1}{2}\right)$

$= \frac{e - e^{-1} - e^{-1} + e}{2} = e - e^{-1}$, where $u = \tan\theta$, $du = \sec^2\theta\,d\theta$, the lower limit is $\tan\left(-\frac{\pi}{4}\right) = -1$ and the upper limit is $\tan\left(\frac{\pi}{4}\right) = 1$

56. $\int_0^{\pi/2} 2\sinh(\sin\theta)\cos\theta \, d\theta = 2\int_0^1 \sinh u \, du = 2[\cosh u]_0^1 = 2(\cosh 1 - \cosh 0) = 2\left(\frac{e + e^{-1}}{2} - 1\right)$

$= e + e^{-1} - 2$, where $u = \sin\theta$, $du = \cos\theta\,d\theta$, the lower limit is $\sin 0 = 0$ and the upper limit is $\sin\left(\frac{\pi}{2}\right) = 1$

57. $\int_1^2 \frac{\cosh(\ln t)}{t} \, dt = \int_0^{\ln 2} \cosh u \, du = [\sinh u]_0^{\ln 2} = \sinh(\ln 2) - \sinh(0) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} - 0 = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$, where $u = \ln t$, $du = \frac{1}{t}\,dt$, the lower limit is $\ln 1 = 0$ and the upper limit is $\ln 2$

58. $\int_1^4 \frac{8\cosh\sqrt{x}}{\sqrt{x}} \, dx = 16\int_1^2 \cosh u \, du = 16[\sinh u]_1^2 = 16(\sinh 2 - \sinh 1) = 16\left[\left(\frac{e^2 - e^{-2}}{2}\right) - \left(\frac{e - e^{-1}}{2}\right)\right]$

$= 8\left(e^2 - e^{-2} - e + e^{-1}\right)$, where $u = \sqrt{x} = x^{1/2}$, $du = \frac{1}{2}x^{-1/2} = \frac{dx}{2\sqrt{x}}$, the lower limit is $\sqrt{1} = 1$ and the upper limit is $\sqrt{4} = 2$

59. $\int_{-\ln 2}^{0} \cosh^2\left(\frac{x}{2}\right) dx = \int_{-\ln 2}^{0} \frac{\cosh x + 1}{2}\, dx = \frac{1}{2} \int_{-\ln 2}^{0} (\cosh x + 1)\, dx = \frac{1}{2}[\sinh x + x]_{-\ln 2}^{0}$

$= \frac{1}{2}[(\sinh 0 + 0) - (\sinh(-\ln 2) - \ln 2)] = \frac{1}{2}\left[(0+0) - \left(\frac{e^{-\ln 2} - e^{\ln 2}}{2} - \ln 2\right)\right] = \frac{1}{2}\left[-\frac{\left(\frac{1}{2}\right) - 2}{2} + \ln 2\right]$

$= \frac{1}{2}\left(1 - \frac{1}{4} + \ln 2\right) = \frac{3}{8} + \frac{1}{2}\ln 2 = \frac{3}{8} + \ln\sqrt{2}$

60. $\int_{0}^{\ln 10} 4\sinh^2\left(\frac{x}{2}\right) dx = \int_{0}^{\ln 10} 4\left(\frac{\cosh x - 1}{2}\right) dx = 2\int_{0}^{\ln 10} (\cosh x - 1)\, dx = 2[\sinh x - x]_0^{\ln 10}$

$= 2[(\sinh(\ln 10) - \ln 10) - (\sinh 0 - 0)] = e^{\ln 10} - e^{-\ln 10} - 2\ln 10 = 10 - \frac{1}{10} - 2\ln 10 = 9.9 - 2\ln 10$

61. $\sinh^{-1}\left(\frac{-5}{12}\right) = \ln\left(-\frac{5}{12} + \sqrt{\frac{25}{144} + 1}\right) = \ln\left(\frac{2}{3}\right)$

62. $\cosh^{-1}\left(\frac{5}{3}\right) = \ln\left(\frac{5}{3} + \sqrt{\frac{25}{9} - 1}\right) = \ln 3$

63. $\tanh^{-1}\left(-\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1-(1/2)}{1+(1/2)}\right) = -\frac{\ln 3}{2}$

64. $\coth^{-1}\left(\frac{5}{4}\right) = \frac{1}{2}\ln\left(\frac{(9/4)}{(1/4)}\right) = \frac{1}{2}\ln 9 = \ln 3$

65. $\operatorname{sech}^{-1}\left(\frac{3}{5}\right) = \ln\left(\frac{1 + \sqrt{1 - (9/25)}}{(3/5)}\right) = \ln 3$

66. $\operatorname{csch}^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \ln\left(-\sqrt{3} + \frac{\sqrt{4/3}}{(1/\sqrt{3})}\right) = \ln(-\sqrt{3} + 2)$

67. (a) $\int_{0}^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}} = \left[\sinh^{-1}\frac{x}{2}\right]_0^{2\sqrt{3}} = \sinh^{-1}\sqrt{3} - \sinh 0 = \sinh^{-1}\sqrt{3}$

(b) $\sinh^{-1}\sqrt{3} = \ln\left(\sqrt{3} + \sqrt{3+1}\right) = \ln(\sqrt{3} + 2)$

68. (a) $\int_{0}^{1/3} \frac{6\, dx}{\sqrt{1+9x^2}} = 2\int_{0}^{1} \frac{dx}{\sqrt{a^2+u^2}}$, where $u = 3x$, $du = 3\, dx$, $a = 1$

$= [2\sinh^{-1} u]_0^1 = 2(\sinh^{-1} 1 - \sinh^{-1} 0) = 2\sin^{-1} 1$

(b) $2\sinh^{-1} 1 = 2\ln(1 + \sqrt{1^2+1}) = 2\ln(1+\sqrt{2})$

69. (a) $\int_{5/4}^{2} \frac{1}{1-x^2}\, dx = [\coth^{-1} x]_{5/4}^{2} = \coth^{-1} 2 - \coth^{-1}\frac{5}{4}$

(b) $\coth^{-1} 2 - \coth^{-1}\frac{5}{4} = \frac{1}{2}\left[\ln 3 - \ln\left(\frac{9/4}{1/4}\right)\right] = \frac{1}{2}\ln\frac{1}{3}$

70. (a) $\int_0^{1/2} \frac{1}{1-x^2}\,dx = \left[\tanh^{-1} x\right]_0^{1/2} = \tanh^{-1}\frac{1}{2} - \tanh^{-1} 0 = \tanh^{-1}\frac{1}{2}$

(b) $\tanh^{-1}\frac{1}{2} = \frac{1}{2}\ln\left(\frac{1+(1/2)}{1-(1/2)}\right) = \frac{1}{2}\ln 3$

71. (a) $\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}} = \int_{4/5}^{12/13} \frac{du}{u\sqrt{a^2-u^2}}$, where $u = 4x$, $du = 4\,dx$, $a = 1$

$= \left[-\operatorname{sech}^{-1} u\right]_{4/5}^{12/13} = -\operatorname{sech}^{-1}\frac{12}{13} + \operatorname{sech}^{-1}\frac{4}{5}$

(b) $-\operatorname{sech}^{-1}\frac{12}{13} + \operatorname{sech}^{-1}\frac{4}{5} = -\ln\left(\frac{1+\sqrt{1-(12/13)^2}}{(12/13)}\right) + \ln\left(\frac{1+\sqrt{1-(4/5)^2}}{(4/5)}\right)$

$= -\ln\left(\frac{13+\sqrt{169-144}}{12}\right) + \ln\left(\frac{5+\sqrt{25-16}}{4}\right) = \ln\left(\frac{5+3}{4}\right) - \ln\left(\frac{13+5}{12}\right) = \ln 2 - \ln\frac{3}{2}$

$= \ln\left(2\cdot\frac{2}{3}\right) = \ln\frac{4}{3}$

72. (a) $\int_1^2 \frac{dx}{x\sqrt{4+x^2}} = \left[-\frac{1}{2}\operatorname{csch}^{-1}\left|\frac{x}{2}\right|\right]_1^2 = -\frac{1}{2}\left(\operatorname{csch}^{-1} 1 - \operatorname{csch}^{-1}\frac{1}{2}\right) = \frac{1}{2}\left(\operatorname{csch}^{-1}\frac{1}{2} - \operatorname{csch}^{-1} 1\right)$

(b) $\frac{1}{2}\left(\operatorname{csch}^{-1}\frac{1}{2} - \operatorname{csch}^{-1} 1\right) = \frac{1}{2}\left[\ln\left(2+\frac{\sqrt{5/4}}{(1/2)}\right) - \ln\left(1+\sqrt{2}\right)\right] = \frac{1}{2}\ln\left(\frac{2+\sqrt{5}}{1+\sqrt{2}}\right)$

73. (a) $\int_0^{\pi} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx = \int_0^0 \frac{1}{\sqrt{1+u^2}}\,du = \left[\sinh^{-1} u\right]_0^0 = \sinh^{-1} 0 - \sinh^{-1} 0 = 0$, where $u = \sin x$, $du = \cos x\,dx$

(b) $\sinh^{-1} 0 - \sinh^{-1} 0 = \ln\left(0+\sqrt{0+1}\right) - \ln\left(0+\sqrt{0+1}\right) = 0$

74. (a) $\int_1^{e} \frac{dx}{x\sqrt{1+(\ln x)^2}} = \int_0^1 \frac{du}{\sqrt{a^2+u^2}}$, where $u = \ln x$, $du = \frac{1}{x}\,dx$, $a = 1$

$= \left[\sinh^{-1} u\right]_0^1 = \sinh^{-1} 1 - \sinh^{-1} 0 = \sinh^{-1} 1$

(b) $\sinh^{-1} 1 - \sinh^{-1} 0 = \ln\left(1+\sqrt{1^2+1}\right) - \ln\left(0+\sqrt{0^2+1}\right) = \ln\left(1+\sqrt{2}\right)$

75. (a) Let $E(x) = \frac{f(x)+f(-x)}{2}$ and $O(x) = \frac{f(x)-f(-x)}{2}$. Then $E(x) + O(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$ $= \frac{2f(x)}{2} = f(x)$. Also, $E(-x) = \frac{f(-x)+f(-(-x))}{2} = \frac{f(x)+f(-x)}{2} = E(x) \Rightarrow E(x)$ is even, and $O(-x) = \frac{f(-x)-f(-(-x))}{2} = -\frac{f(x)-f(-x)}{2} = -O(x) \Rightarrow O(x)$ is odd. Consequently, $f(x)$ can be written as a sum of an even and an odd function.

(b) $f(x) = \frac{f(x)+f(-x)}{2}$ because $\frac{f(x)-f(-x)}{2} = 0$ and $f(x) = \frac{f(x)-f(-x)}{2}$ because $\frac{f(x)+f(-x)}{2} = 0$; thus $f(x) = \frac{2f(x)}{2} + 0$ and $f(x) = 0 + \frac{2f(x)}{2}$

76. $y = \sinh^{-1} x \Rightarrow x = \sinh y \Rightarrow x = \frac{e^y - e^{-y}}{2} \Rightarrow 2x = e^y - \frac{1}{e^y} \Rightarrow 2xe^y = e^{2y} - 1 \Rightarrow e^{2y} - 2xe^y - 1 = 0$ $\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} \Rightarrow e^y = x + \sqrt{x^2+1} \Rightarrow \sinh^{-1} x = y = \ln\left(x + \sqrt{x^2+1}\right)$. Since $e^y > 0$, we cannot choose $e^y = x - \sqrt{x^2+1}$ because $x - \sqrt{x^2+1} < 0$.

77. (a) $m\frac{dv}{dt} = mg - kv^2 \Rightarrow \frac{m\frac{dv}{dt}}{mg - kv^2} = 1 \Rightarrow \frac{\frac{1}{g}\frac{dv}{dt}}{1 - \frac{kv^2}{mg}} = 1 \Rightarrow \frac{\sqrt{\frac{k}{mg}}\,dv}{1 - \sqrt{v\left(\frac{k}{mg}\right)}^2} = \sqrt{\frac{kg}{m}}\,dt \Rightarrow \int \frac{\sqrt{\frac{k}{mg}}\,dv}{1 - \sqrt{v\left(\frac{k}{mg}\right)}^2}\,dv$

$= \int \sqrt{\frac{kg}{m}}\,dt \Rightarrow \tanh^{-1}\left(\sqrt{\frac{k}{mg}}\,v\right) = \sqrt{\frac{kg}{m}}\,t + C \Rightarrow v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}\,t + C\right)$; $v(0) = 0 \Rightarrow C = 0$

$\Rightarrow v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}\,t\right)$

(b) $\lim\limits_{t\to\infty} v = \lim\limits_{t\to\infty} \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}\,t\right) = \sqrt{\frac{mg}{k}} \lim\limits_{t\to\infty} \tanh\left(\sqrt{\frac{kg}{m}}\,t\right) = \sqrt{\frac{mg}{k}}(1) = \sqrt{\frac{mg}{k}}$

(c) $\sqrt{\frac{160}{0.005}} = \sqrt{\frac{1{,}600{,}000}{5}} = \frac{400}{\sqrt{5}} = 80\sqrt{5} \approx 178.89$ ft/sec

78. (a) $s(t) = a\cos kt + b\sin kt \Rightarrow \frac{ds}{dt} = -ak\sin kt + bk\cos kt \Rightarrow \frac{d^2s}{dt^2} = -ak^2\cos kt - bk^2\sin kt$ $= -k^2(a\cos kt + \sin kt) = -k^2 s(t) \Rightarrow$ acceleration is proportional to s. The negative constant $-k^2$ implies that the acceleration is directed toward the origin.

(b) $s(t) = a\cosh kt + b\sinh kt \Rightarrow \frac{ds}{dt} = ak\sinh kt + bk\cosh kt \Rightarrow \frac{d^2s}{dt^2} = ak^2\cosh kt + bk^2\sinh kt$ $= k^2(a\cosh kt + \sinh kt) = k^2 s(t) \Rightarrow$ acceleration is proportional to s. The positive constant k^2 implies that the acceleration is directed away from the origin.

79. $\frac{dy}{dx} = \frac{-1}{x\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} \Rightarrow y = \int \frac{-1}{x\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx \Rightarrow y = \operatorname{sech}^{-1}(x) - \sqrt{1-x^2} + C$; $x = 1$ and $y = 0 \Rightarrow C = 0 \Rightarrow y = \operatorname{sech}^{-1}(x) - \sqrt{1-x^2}$

80. To find the length of the curve: $y = \frac{1}{a}\cosh ax \Rightarrow y' = \sinh ax \Rightarrow s = \int_0^b \sqrt{1 + (\sinh ax)^2}\,dx$ $\Rightarrow s = \int_0^b \cosh ax\,dx = \left[\frac{1}{a}\sinh ax\right]_0^b = \frac{1}{a}\sinh ab$. Then the area under the curve is $A = \int_0^b \frac{1}{a}\cosh ax\,dx$ $= \left[\frac{1}{a^2}\sinh ax\right]_0^b = \frac{1}{a^2}\sinh ab = \left(\frac{1}{a}\right)\left(\frac{1}{a}\sinh ab\right)$ which is the area of the rectangle of height $\frac{1}{a}$ and length s as claimed.

81. $V = \pi \int_0^2 \left(\cosh^2 x - \sinh^2 x\right) dx = \pi \int_0^2 1\, dx = 2\pi$

82. $V = 2\pi \int_0^{\ln \sqrt{3}} \operatorname{sech}^2 x\, dx = 2\pi[\tanh x]_0^{\ln \sqrt{3}} = 2\pi\left[\frac{\sqrt{3}-(1/\sqrt{3})}{\sqrt{3}+(1/\sqrt{3})}\right] = \pi$

83. $y = \frac{1}{2}\cosh 2x \Rightarrow y' = \sinh 2x \Rightarrow L = \int_0^{\ln \sqrt{5}} \sqrt{1+(\sinh 2x)^2}\, dx = \int_0^{\ln \sqrt{5}} \cosh 2x\, dx = \left[\frac{1}{2}\sinh 2x\right]_0^{\ln \sqrt{5}}$

$= \left[\frac{1}{2}\left(\frac{e^{2x}-e^{-2x}}{2}\right)\right]_0^{\ln \sqrt{5}} = \frac{1}{4}\left(5-\frac{1}{5}\right) = \frac{6}{5}$

84. (a) Let the point located at $(\cosh x, 0)$ be called T. Then $A(u)$ = area of the triangle ΔOTP minus the area under the curve $y = \sqrt{x^2-1}$ from A to T $\Rightarrow A(u) = \frac{1}{2}\cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2-1}\, dx.$

(b) $A(u) = \frac{1}{2}\cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2-1}\, dx \Rightarrow A'(u) = \frac{1}{2}\left(\cosh^2 u + \sinh^2 u\right) - \left(\sqrt{\cosh^2 u - 1}\right)(\sinh u)$

$= \frac{1}{2}\cosh^2 u + \frac{1}{2}\sinh^2 u - \sinh^2 u = \frac{1}{2}\left(\cosh^2 u - \sinh^2 u\right) = \left(\frac{1}{2}\right)(1) = \frac{1}{2}$

(c) $A'(u) = \frac{1}{2} \Rightarrow A(u) = \frac{u}{2} + C$, and from part (a) we have $A(0) = 0 \Rightarrow C = 0 \Rightarrow A(u) = \frac{u}{2} \Rightarrow u = 2A$

85. (a) $y = \frac{H}{w}\cosh\left(\frac{w}{H}x\right) \Rightarrow \tan\phi = \frac{dy}{dx} = \left(\frac{H}{w}\right)\left[\frac{w}{H}\sinh\left(\frac{w}{H}x\right)\right] = \sinh\left(\frac{w}{H}x\right)$

(b) The tension at P is given by $T\cos\phi = H \Rightarrow T = H\sec\phi = H\sqrt{1+\tan^2\phi} = H\sqrt{1+\left(\sinh\frac{w}{H}x\right)^2}$

$= H\cosh\left(\frac{w}{H}x\right) = w\left(\frac{H}{w}\right)\cosh\left(\frac{w}{H}x\right) = wy$

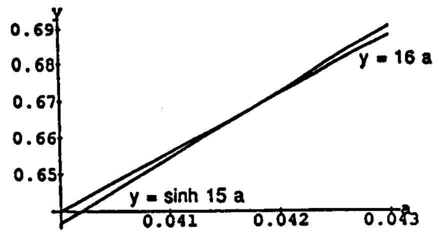
86. $s = \frac{1}{a}\sinh ax \Rightarrow \sinh ax = as \Rightarrow ax = \sinh^{-1} as \Rightarrow x = \frac{1}{a}\sinh^{-1} as$; $y = \frac{1}{a}\cosh ax = \frac{1}{a}\sqrt{\cosh^2 ax}$

$= \frac{1}{a}\sqrt{\sinh^2 ax + 1} = \frac{1}{a}\sqrt{a^2s^2+1} = \sqrt{s^2+\frac{1}{a^2}}$

87. (a) Since the cable is 32 ft long, $s = 16$ and $x = 15$. From Exercise 88, $x = \frac{1}{a}\sinh^{-1} as \Rightarrow 15a = \sinh^{-1} 16a$

$\Rightarrow \sinh 15a = 16a.$

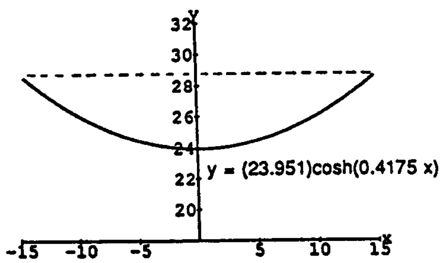
(b) The intersection is near $(0.042, 0.672)$.



(c) Newton's method indicates that at $a \approx 0.0417525$ the curves $y = 16a$ and $y = \sinh 15a$ intersect.

(d) $T = wy \approx (2\text{ lb})\left(\frac{1}{0.0417525}\right) \approx 47.90\text{ lb}$

(e) The sag is about 4.8 ft.



CHAPTER 6 PRACTICE EXERCISES

1. $\int e^x \sin(e^x)\,dx = \int \sin u\,du$, where $u = e^x$ and $du = e^x\,dx$,

   $= -\cos u + C = -\cos(e^x) + C$

2. $\int e^t \cos(3e^t - 2)\,dt = \frac{1}{3}\int \cos u\,du$, where $u = 3e^t - 2$ and $du = 3e^t\,dt$,

   $= \frac{1}{3}\sin u + C = \frac{1}{3}\sin(3e^t - 2) + C$

3. $\int e^x \sec^2(e^x - 7)\,dx = \int \sec^2 u\,du$, where $u = e^x - 7$ and $du = e^x\,dx$,

   $= \tan u + C = \tan(e^x - 7) + C$

4. $\int e^y \csc(e^y + 1)\cot(e^y + 1)\,dy = \int \csc u \cot u\,du$, where $u = e^y + 1$ and $du = e^y\,dy$,

   $= -\csc u + C = -\csc(e^y + 1) + C$

5. $\int (\sec^2 x)e^{\tan x}\,dx = \int e^u\,du$, where $u = \tan x$ and $du = \sec^2 x\,dx$,

$= e^u + C = e^{\tan x} + C$

6. $\int (\csc^2 x) e^{\cot x}\, dx = -\int e^u\, du$, where $u = \cot x$ and $du = -\csc^2 x\, dx$,

$= -e^u + C = -e^{\cot x} + C$

7. $\int_{-1}^{1} \frac{1}{3x-4}\, dx = \frac{1}{3}\int_{-7}^{-1} \frac{1}{u}\, du$, where $u = 3x - 4$, $du = 3\, dx$; $x = -1 \Rightarrow u = -7$, $x = 1 \Rightarrow u = -1$,

$= \frac{1}{3}[\ln|u|]_{-7}^{-1} = \frac{1}{3}[\ln|-1| - \ln|-7|] = \frac{1}{3}[0 - \ln 7] = -\frac{\ln 7}{3}$

8. $\int_{1}^{e} \frac{\sqrt{\ln x}}{x}\, dx = \int_{0}^{1} u^{1/2}\, du$, where $u = \ln x$, $du = \frac{1}{x}\, dx$; $x = 1 \Rightarrow u = 0$, $x = e \Rightarrow u = 1$,

$= \left[\frac{2}{3}u^{3/2}\right]_0^1 = \left[\frac{2}{3}1^{3/2} - \frac{2}{3}0^{3/2}\right] = \frac{2}{3}$

9. $\int_{0}^{\pi} \tan\left(\frac{x}{3}\right) dx = \int_{0}^{\pi} \frac{\sin\left(\frac{x}{3}\right)}{\cos\left(\frac{x}{3}\right)}\, dx = -3\int_{1}^{1/2} \frac{1}{u}\, du$,

where $u = \cos\left(\frac{x}{3}\right)$, $du = -\frac{1}{3}\sin\left(\frac{x}{3}\right) dx$; $x = 0 \Rightarrow u = 1$, $x = \pi \Rightarrow u = \frac{1}{2}$,

$= -3[\ln|u|]_1^{1/2} = -3\left[\ln\left|\frac{1}{2}\right| - \ln|1|\right] = -3\ln\frac{1}{2} = \ln 2^3 = \ln 8$

10. $\int_{1/6}^{1/4} 2\cot \pi x\, dx = 2\int_{1/6}^{1/4} \frac{\cos \pi x}{\sin \pi x}\, dx = \frac{2}{\pi}\int_{1/2}^{1/\sqrt{2}} \frac{1}{u}\, du$,

where $u = \sin \pi x$, $du = \pi \cos \pi x\, dx$; $x = \frac{1}{6} \Rightarrow u = \frac{1}{2}$, $x = \frac{1}{4} \Rightarrow u = \frac{1}{\sqrt{2}}$,

$= \frac{2}{\pi}[\ln|u|]_{1/2}^{1/\sqrt{2}} = \frac{2}{\pi}\left[\ln\left|\frac{1}{\sqrt{2}}\right| - \ln\left|\frac{1}{2}\right|\right] = \frac{2}{\pi}\left[\ln 1 - \frac{1}{2}\ln 2 - \ln 1 + \ln 2\right] = \frac{2}{\pi}\left[\frac{1}{2}\ln 2\right] = \frac{\ln 2}{\pi}$

11. $\int_{0}^{4} \frac{2t}{t^2 - 25}\, dt = \int_{-25}^{-9} \frac{1}{u}\, du$, where $u = t^2 - 25$, $du = 2t\, dt$; $t = 0 \Rightarrow u = -25$, $t = 4 \Rightarrow u = -9$,

$= [\ln|u|]_{-25}^{-9} = \ln|-9| - \ln|-25| = \ln 9 - \ln 25 = \ln\frac{9}{25}$

12. $\int_{-\pi/2}^{\pi/6} \frac{\cos t}{1 - \sin t}\, dt = -\int_{2}^{1/2} \frac{1}{u}\, du$, where $u = 1 - \sin t$, $du = -\cos t\, dt$; $t = -\frac{\pi}{2} \Rightarrow u = 2$, $t = \frac{\pi}{6} \Rightarrow u = \frac{1}{2}$,

$= -[\ln|u|]_2^{1/2} = -\left[\ln\left|\frac{1}{2}\right| - \ln|2|\right] = -\ln 1 + \ln 2 + \ln 2 = 2\ln 2 = \ln 4$

13. $\int \frac{\tan(\ln v)}{v}\,dv = \int \tan u\,du = \int \frac{\sin u}{\cos u}\,du$, where $u = \ln v$ and $du = \frac{1}{v}\,dv$,

$= -\ln|\cos u| + C = -\ln|\cos(\ln v)| + C$

14. $\int \frac{1}{v \ln v}\,dv = \int \frac{1}{u}\,du$, where $u = \ln v$ and $du = \frac{1}{v}\,dv$,

$= \ln|u| + C = \ln|\ln v| + C$

15. $\int \frac{(\ln x)^{-3}}{x}\,dx = \int u^{-3}\,du$, where $u = \ln x$ and $du = \frac{1}{x}\,dx$,

$= \frac{u^{-2}}{-2} + C = -\frac{1}{2}(\ln x)^{-2} + C$

16. $\int \frac{\ln(x-5)}{x-5}\,dx = \int u\,du$, where $u = \ln(x-5)$ and $du = \frac{1}{x-5}\,dx$,

$= \frac{u^2}{2} + C = \frac{[\ln(x-5)]^2}{2} + C$

17. $\int \frac{1}{r}\csc^2(1 + \ln r)\,dr = \int \csc^2 u\,du$, where $u = 1 + \ln r$ and $du = \frac{1}{r}\,dr$,

$= -\cot u + C = -\cot(1 + \ln r) + C$

18. $\int \frac{\cos(1 - \ln v)}{v}\,dv = -\int \cos u\,du$, where $u = 1 - \ln v$ and $du = -\frac{1}{v}\,dv$,

$= -\sin u + C = -\sin(1 - \ln v) + C$

19. $\int x3^{x^2}\,dx = \frac{1}{2}\int 3^u\,du$, where $u = x^2$ and $du = 2x\,dx$,

$= \frac{1}{2\ln 3}(3^u) + C = \frac{1}{2\ln 3}\left(3^{x^2}\right) + C$

20. $\int 2^{\tan x}\sec^2 x\,dx = \int 2^u\,du$, where $u = \tan x$ and $du = \sec^2 x\,dx$,

$= \frac{1}{\ln 2}(2^u) + C = \frac{2^{\tan x}}{\ln 2} + C$

21. $\int_1^7 \frac{3}{x}\,dx = 3\int_1^7 \frac{1}{x}\,dx = 3[\ln|x|]_1^7 = 3(\ln 7 - \ln 1) = 3\ln 7$

22. $\int_1^{32} \frac{1}{5x}\,dx = \frac{1}{5}\int_1^{32} \frac{1}{x}\,dx = \frac{1}{5}[\ln|x|]_1^{32} = \frac{1}{5}(\ln 32 - \ln 1) = \frac{1}{5}\ln 32 = \ln(\sqrt[5]{32}) = \ln 2$

23. $\int_{-2}^{-1} e^{-(x+1)}\,dx = -\int_{1}^{0} e^u\,du$, where $u = -(x+1)$, $du = -dx$; $x = -2 \Rightarrow u = 1$, $x = -1 \Rightarrow u = 0$

$= -\left[e^u\right]_1^0 = -\left(e^0 - e^1\right) = e - 1$

24. $\int_{-\ln 2}^{0} e^{2w}\,dw = \frac{1}{2}\int_{\ln(1/4)}^{0} e^u\,du$, where $u = 2w$, $du = 2\,dw$; $w = -\ln 2 \Rightarrow u = \ln\frac{1}{4}$, $w = 0 \Rightarrow u = 0$

$= \frac{1}{2}\left[e^u\right]_{\ln(1/4)}^0 = \frac{1}{2}\left[e^0 - e^{\ln(1/4)}\right] = \frac{1}{2}\left(1 - \frac{1}{4}\right) = \frac{3}{8}$

25. $\int_1^3 \frac{[\ln(v+1)]^2}{v+1}\,dv = \int_1^3 [\ln(v+1)]^2 \frac{1}{v+1}\,dv = \int_{\ln 2}^{\ln 4} u^2\,du$,

where $u = \ln(v+1)$, $du = \frac{1}{v+1}dv$; $v = 1 \Rightarrow u = \ln 2$, $v = 3 \Rightarrow u = \ln 4$;

$= \frac{1}{3}\left[u^3\right]_{\ln 2}^{\ln 4} = \frac{1}{3}\left[(\ln 4)^3 - (\ln 2)^3\right] = \frac{1}{3}\left[(2\ln 2)^3 - (\ln 2)^3\right] = \frac{(\ln 2)^3}{3}(8-1) = \frac{7}{3}(\ln 2)^3$

26. $\int_2^4 (1+\ln t)(t\ln t)\,dt = \int_2^4 (t\ln t)(1+\ln t)\,dt = \int_{2\ln 2}^{4\ln 4} u\,du$,

where $u = t\ln t$, $du = \left((t)\left(\frac{1}{t}\right) + (\ln t)(1)\right)dt = (1+\ln t)\,dt$; $t = 2 \Rightarrow u = 2\ln 2$, $t = 4 \Rightarrow u = 4\ln 4$,

$= \frac{1}{2}\left[u^2\right]_{2\ln 2}^{4\ln 4} = \frac{1}{2}\left[(4\ln 4)^2 - (2\ln 2)^2\right] = \frac{1}{2}\left[(8\ln 2)^2 - (2\ln 2)^2\right] = \frac{(2\ln 2)^2}{2}(16-1) = 30(\ln 2)^2$

27. $\int_1^8 \frac{\log_4\theta}{\theta}\,d\theta = \frac{1}{\ln 4}\int_1^8 (\ln\theta)\left(\frac{1}{\theta}\right)d\theta = \frac{1}{\ln 4}\int_0^{\ln 8} u\,du$, where $u = \ln\theta$, $du = \frac{1}{\theta}\,d\theta$, $\theta = 1 \Rightarrow u = 0$, $\theta = 8 \Rightarrow u = \ln 8$

$= \frac{1}{2\ln 4}\left[u^2\right]_0^{\ln 8} = \frac{1}{\ln 16}\left[(\ln 8)^2 - 0^2\right] = \frac{(3\ln 2)^2}{4\ln 2} = \frac{9\ln 2}{4}$

28. $\int_1^e \frac{8(\ln 3)(\log_3\theta)}{\theta}\,d\theta = \int_1^e \frac{8(\ln 3)(\ln\theta)}{\theta(\ln 3)}\,d\theta = 8\int_1^e (\ln\theta)\left(\frac{1}{\theta}\right)d\theta = 8\int_0^1 u\,du$,

where $u = \ln\theta$, $du = \frac{1}{\theta}\,d\theta$; $\theta = 1 \Rightarrow u = 0$, $\theta = e \Rightarrow u = 1$,

$= 4\left[u^2\right]_0^1 = 4\left(1^2 - 0^2\right) = 4$

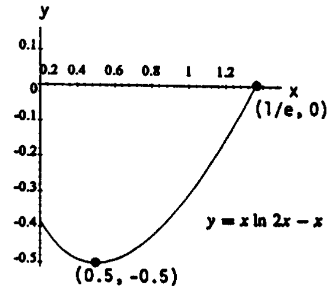
29. $\int \frac{2\,dy}{\sqrt{1+25y^2}} = \frac{2}{5}\int \frac{du}{\sqrt{1+u^2}}$, where $u = 5y$ and $du = 5\,dy$,

$= \frac{2}{5}\sin^{-1}u + C = \frac{2}{5}\sin^{-1}(5y) + C$

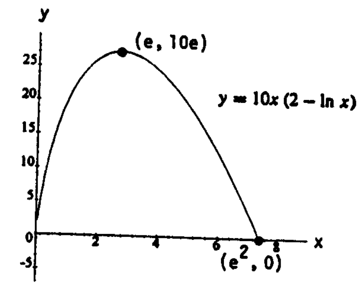
30. $\int \frac{3\,dy}{4-49y^2} = \frac{3}{4}\int \frac{dy}{1-\frac{49y^2}{4}} = \frac{3}{4}\int \frac{2}{7}\left(\frac{du}{1-u^2}\right) = \frac{3}{14}\int \frac{du}{1-u^2}$, where $u = \frac{7}{2}y$ and $du = \frac{2}{7}\,dy$,

$= \frac{3}{14}\tanh^{-1}u + C = \frac{3}{14}\tanh^{-1}\left(\frac{7}{2}y\right) + C$ for $\left|\frac{7}{2}y\right| < 1 \Rightarrow -\frac{2}{7} < y < \frac{2}{7}$

31. $y = x\ln 2x - x \Rightarrow y' = x\left(\frac{2}{2x}\right) + \ln(2x) - 1 = \ln 2x$; solving $y' = 0 \Rightarrow x = \frac{1}{2}$; $y' > 0$ for $x > \frac{1}{2}$ and $y' < 0$ for $x < \frac{1}{2} \Rightarrow$ relative minimum of $-\frac{1}{2}$ at $x = \frac{1}{2}$; $f\left(\frac{1}{2e}\right) = -\frac{1}{e}$ and $f\left(\frac{e}{2}\right) = 0 \Rightarrow$ absolute minimum is $-\frac{1}{2}$ at $x = \frac{1}{2}$ and the absolute maximum is 0 at $x = \frac{e}{2}$



32. $y = 10x(2 - \ln x) \Rightarrow y' = 10(2 - \ln x) - 10x\left(\frac{1}{x}\right)$
$= 20 - 10\ln x - 10 = 10(1 - \ln x)$; solving $y' = 0$
$\Rightarrow x = e$; $y' < 0$ for $x > e$ and $y' > 0$ for $x < e$
$\Rightarrow$ relative maximum at $x = e$ of $10e$;
$y(e^2) = 10e^2(2 - 2\ln e) = 0 \Rightarrow$ absolute minimum is 0 at $x = e^2$ and the absolute maximum is $10e$ at $x = e$



33. $A = \int_1^e \frac{2\ln x}{x}\,dx = \int_0^1 2u\,du = \left[u^2\right]_0^1 = 1$, where $u = \ln x$ and $du = \frac{1}{x}\,dx$; $x = 1 \Rightarrow u = 0$, $x = e \Rightarrow u = 1$

34. (a) $A_1 = \int_{10}^{20} \frac{1}{x}\,dx = \left[\ln|x|\right]_{10}^{20} = \ln 20 - \ln 10 = \ln\frac{20}{10} = \ln 2$, and $A_2 = \int_1^2 \frac{1}{x}\,dx = \left[\ln|x|\right]_1^2 = \ln 2 - \ln 1 = \ln 2$

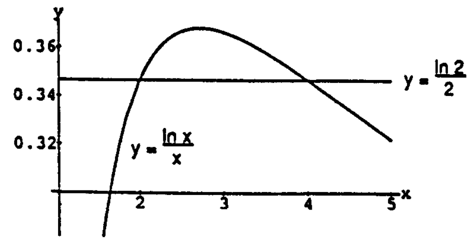
(b) $A_1 = \int_{ka}^{kb} \frac{1}{x}\,dx = \left[\ln|x|\right]_{ka}^{kb} = \ln kb - \ln ka = \ln\frac{kb}{ka} = \ln\frac{b}{a} = \ln b - \ln a$, and $A_2 = \int_a^b \frac{1}{x}\,dx = \left[\ln|x|\right]_a^b$ $= \ln b - \ln a$

35. $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$; $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \left(\frac{1}{x}\right)\sqrt{x} = \frac{1}{\sqrt{x}} \Rightarrow \left.\frac{dy}{dt}\right|_{e^2} = \frac{1}{e}$ m/sec

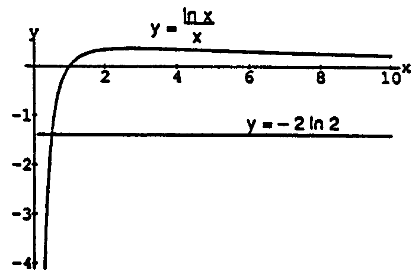
36. $V = \pi\int_{1/4}^{4} \left(\frac{1}{2\sqrt{x}}\right)^2 dx = \frac{\pi}{4}\int_{1/4}^{4} \frac{1}{x}\,dx = \frac{\pi}{4}\left[\ln|x|\right]_{1/4}^{4} = \frac{\pi}{4}\left(\ln 4 - \ln\frac{1}{4}\right) = \frac{\pi}{4}\ln 16 = \frac{\pi}{4}\ln\left(2^4\right) = \pi\ln 2$

37. The two functions differ by $\ln\frac{5}{3}$ because $K = \ln(5x) - \ln(3x) = \ln 5 + \ln x - \ln 3 - \ln x = \ln 5 - \ln 3 = \ln\frac{5}{3}$.

38. (a) No, there are two intersections: one at $x = 2$ and the other at $x = 4$



(b) Yes, because there is only one intersection point.



39. Force = Mass times Acceleration (Newton's Second Law) or $F = ma$. Let $a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v\,\frac{dv}{ds}$. Then $ma = -mgR^2s^{-2} \Rightarrow a = -gR^2s^{-2} \Rightarrow v\,\frac{dv}{ds} = -gR^2s^{-2} \Rightarrow v\,dv = -gR^2s^{-2}\,ds \Rightarrow \int v\,dv = \int -gR^2s^{-2}\,ds$ $\Rightarrow \frac{v^2}{2} = \frac{gR^2}{s} + C \Rightarrow v^2 = \frac{2gR^2}{s} + 2C_1 = \frac{2gR^2}{s} + C$. When $t = 0$, $v = v_0$ and $s = R \Rightarrow v_0^2 = \frac{2gR^2}{R} + C$ $\Rightarrow C = v_0^2 - 2gR \Rightarrow v^2 = \frac{2gR^2}{s} + v_0^2 - 2gR$

40. If $v_0 = \sqrt{2gR}$, then $v^2 = \frac{2gR^2}{s} \Rightarrow v = \sqrt{\frac{2gR^2}{s}}$, since $v \ge 0$ if $v_0 \ge \sqrt{2gR}$. Then $\frac{ds}{dt} = \frac{\sqrt{2gR^2}}{\sqrt{s}}$

$\Rightarrow \sqrt{s}\,ds = \sqrt{2gR^2}\,dt \Rightarrow \int s^{1/2}\,ds = \int \sqrt{2gR^2}\,dt \Rightarrow \frac{2}{3}s^{3/2} = \left(\sqrt{2gR^2}\right)t + C_1 \Rightarrow s^{3/2} = \left(\frac{3}{2}\sqrt{2gR^2}\right)t + C$;

$t = 0$ and $s = R \Rightarrow R^{3/2} = \left(\frac{3}{2}\sqrt{2gR^2}\right)(0) + C \Rightarrow C = R^{3/2} \Rightarrow s^{3/2} = \left(\frac{3}{2}\sqrt{2gR^2}\right)t + R^{3/2}$

$= \left(\frac{3}{2}R\sqrt{2g}\right)t + R^{3/2} = R^{3/2}\left[\left(\frac{3}{2}R^{-1/2}\sqrt{2g}\right)t + 1\right] = R^{3/2}\left[\left(\frac{3\sqrt{2gR}}{2R}\right)t + 1\right]$

$= R^{3/2}\left[\left(\frac{3v_0}{2R}\right)t + 1\right] \Rightarrow s = R\left[1 + \left(\frac{3v_0}{2R}\right)t\right]^{2/3}$

41. $y' \cos x - y \sin x = \sin 2x$; Noting that $\frac{d}{dx}(y \cos x) = y' \cos x - y \sin x$, we can rewrite the differential equation as $\frac{d}{dx}(y \cos x) = \sin 2x \Rightarrow y \cos x = -\frac{1}{2}\cos 2x + C \Rightarrow y = \frac{-\cos 2x}{2 \cos x} + \frac{C}{\cos x}$; $y(0) = 1 \Rightarrow 1 = -\frac{1}{2} + C$ $\Rightarrow C = \frac{3}{2} \Rightarrow y = \frac{3 - \cos 2x}{2 \cos x}$

42. $\frac{dy}{dx} = -\frac{y \ln y}{1+x^2} \Rightarrow \frac{dy}{y \ln y} = -\frac{dx}{1+x^2} \Rightarrow \int \frac{\left(\frac{1}{y}\right)}{\ln y}\, dy = -\int \frac{1}{1+x^2}\, dx \Rightarrow \ln(\ln y) = -\tan^{-1} x + C$; $x = 0$ and $y = e^2 \Rightarrow \ln(\ln e^2) = -\tan^{-1} 0 + C \Rightarrow \ln 2 = C \Rightarrow \ln(\ln y) = -\tan^{-1} x + \ln 2$

$\Rightarrow e^{\ln(\ln y)} = e^{(-\tan^{-1} x + \ln 2)} \Rightarrow \ln y = e^{(-\tan^{-1} x + \ln 2)} \Rightarrow e^{\ln y} = e^{e^{(-\tan^{-1} x + \ln 2)}} \Rightarrow y = e^{e^{(-\tan^{-1} x + \ln 2)}}$ or $y = \exp(\exp(-\tan^{-1} x + \ln 2))$

43. $\frac{dy}{dx} + \left(\frac{2}{x+1}\right)y = \frac{x}{x+1} \Rightarrow P(x) = \frac{2}{x+1},\ Q(x) = \frac{x}{x+1} \Rightarrow \int P(x)\, dx = \int \left(\frac{2}{x+1}\right) dx = 2 \ln|x+1| = \ln(x+1)^2$

$\Rightarrow v(x) = e^{\ln(x+1)^2} = (x+1)^2 \Rightarrow y = \frac{1}{(x+1)^2} \int (x+1)^2 \left(\frac{x}{x+1}\right) dx = \frac{1}{(x+1)^2} \int (x^2 + x)\, dx$

$= \left(\frac{1}{x+1}\right)^2 \left(\frac{x^3}{3} + \frac{x^2}{2} + C\right)$; $x = 0$ and $y = 1 \Rightarrow 1 = 0 + 0 + C \Rightarrow C = 1 \Rightarrow y = \frac{1}{(x+1)^2} \cdot \left(\frac{x^3}{3} + \frac{x^2}{2} + 1\right)$

44. $\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \frac{x^2+1}{x} \Rightarrow P(x) = \frac{2}{x},\ Q(x) = \frac{x^2+1}{x} \Rightarrow \int P(x)\, dx = \int \frac{2}{x}\, dx = 2 \ln|x| = \ln x^2 \Rightarrow v(x) = e^{\ln x^2}$

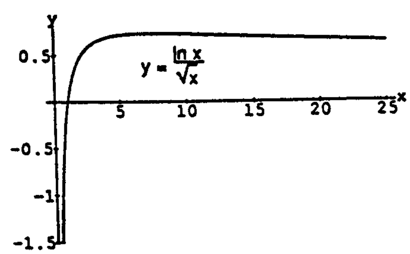
$= x^2 \Rightarrow y = \frac{1}{x^2} \int x^2 \left(\frac{x^2+1}{x}\right) dx = \frac{1}{x^2} \int (x^3 + x)\, dx = \frac{1}{x^2}\left(\frac{x^4}{4} + \frac{x^2}{2} + C\right) = \frac{x^2}{4} + \frac{1}{2} + \frac{C}{x^2}$; $x = 1$ and $y = 1$

$\Rightarrow 1 = \frac{1}{4} + \frac{1}{2} + C \Rightarrow C = \frac{1}{4} \Rightarrow y = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}$

45. (a) $y = \frac{\ln x}{\sqrt{x}} \Rightarrow y' = \frac{1}{x\sqrt{x}} - \frac{\ln x}{2x^{3/2}} = \frac{2 - \ln x}{2x\sqrt{x}}$

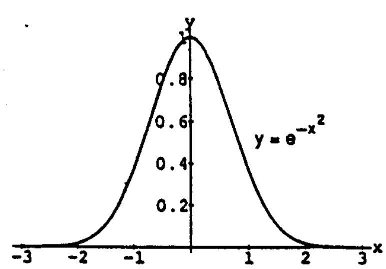
$\Rightarrow y'' = -\frac{3}{4}x^{-5/2}(2 - \ln x) - \frac{1}{2}x^{-5/2} = x^{-5/2}\left(\frac{3}{4}\ln x - 2\right)$;

solving $y' = 0 \Rightarrow \ln x = 2 \Rightarrow x = e^2$; $y' < 0$ for $x > e^2$ and and $y' > 0$ for $x < e^2 \Rightarrow$ a maximum of $\frac{2}{e}$; $y'' = 0$ $\Rightarrow \ln x = \frac{8}{3} \Rightarrow x = e^{8/3}$; the curve is concave down on $\left(0, e^{8/3}\right)$ and concave up on $\left(e^{8/3}, \infty\right)$

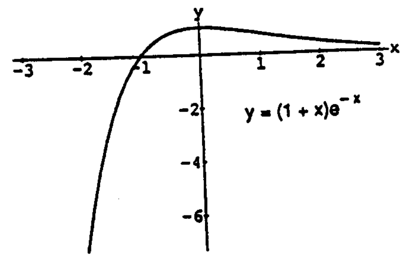


(b) $y = e^{-x^2} \Rightarrow y' = -2xe^{-x^2} \Rightarrow y'' = -2e^{-x^2} + 4x^2e^{-x^2}$;

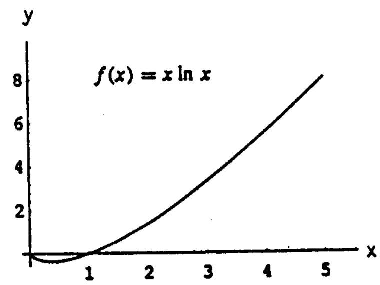
solving $y' = 0 \Rightarrow x = 0$; $y' < 0$ for $x > 0$ and $y' > 0$ for $x < 0 \Rightarrow$ a maximum at $x = 0$ of $e^0 = 1$; there are points of inflection at $x = \pm\frac{1}{\sqrt{2}}$; the curve is concave down for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and concave up otherwise



(c) $y = (1+x)e^{-x} \Rightarrow y' = e^{-x} - (1+x)e^{-x} = -xe^{-x}$ $\Rightarrow y'' = -e^{-x} + xe^{-x} = (x-1)e^{-x}$; solving $y' = 0$ $\Rightarrow -xe^{-x} = 0 \Rightarrow x = 0$; $y' < 0$ for $x > 0$ and $y' > 0$ for $x < 0 \Rightarrow$ a maximum at $x = 0$ of $(1+0)e^0 = 1$; there is a point of inflection at $x = 1$ and the curve is concave up for $x > 1$ and concave down for $x < 1$



46. $y = x \ln x \Rightarrow y' = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$; solving $y' = 0$ $\Rightarrow \ln x + 1 = 0 \Rightarrow \ln x = -1 \Rightarrow x = e^{-1}$; $y' > 0$ for $x > e^{-1}$ and $y' < 0$ for $x < e^{-1} \Rightarrow$ a minimum of $e^{-1} \ln e^{-1}$ $= -\frac{1}{e}$ at $x = e^{-1}$



47. Since the half life is 5700 years and $A(t) = A_0e^{kt}$ we have $\frac{A_0}{2} = A_0e^{5700k} \Rightarrow \frac{1}{2} = e^{5700k} \Rightarrow \ln(0.5) = 5700k$ $\Rightarrow k = \frac{\ln(0.5)}{5700}$. With 10% of the original carbon-14 remaining we have $0.1A_0 = A_0e^{\frac{\ln(0.5)}{5700}t} \Rightarrow 0.1 = e^{\frac{\ln(0.5)}{5700}t}$ $\Rightarrow \ln(0.1) = \frac{\ln(0.5)}{5700}t \Rightarrow t = \frac{(5700)\ln(0.1)}{\ln(0.5)} \approx 18{,}935$ years (rounded to the nearest year).

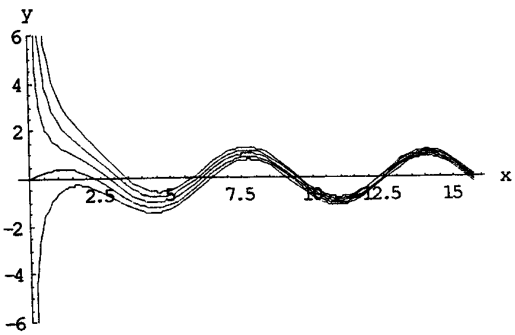
48. $T - T_s = (T_o - T_s)e^{-kt} \Rightarrow 180 - 40 = (220 - 40)e^{-k/4}$, time in hours, $\Rightarrow k = -4\ln\left(\frac{7}{9}\right) = 4\ln\left(\frac{9}{7}\right) \Rightarrow 70 - 40$ $= (220 - 40)e^{-4\ln(9/7)t} \Rightarrow t = \frac{\ln 6}{4\ln\left(\frac{9}{7}\right)} \approx 1.78$ hr ≈ 107 min, the total time $\Rightarrow$ the time it took to cool from 180° F to 70° F was $107 - 15 = 92$ min

49. (a) $xy' + y = x\cos x \Rightarrow y' + \frac{1}{x}y = \cos x \Rightarrow P(x) = \frac{1}{x} \Rightarrow \int P(x)\,dx = \ln x$ since $x > 0$

$v(x) = e^{\int P(x)\,dx} = x \Rightarrow y = \frac{1}{v(x)}\int P(x)\cos x\,dx = \frac{1}{x}\int x\cos x\,dx$. Integration by parts: let $u = x$ and $dv = \cos x\,dx$ and $v = \sin x \Rightarrow \int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$;

Therefore, $y = \frac{x\sin x + \cos x + C}{x} = \sin x + \frac{C + \cos x}{x}$.

(b) The graphs, from the bottom curve to the top, are for $C = -2, -1, 0, 1, 2$, respectively.



As $x \to \infty$, all of the solution curves get closer to sin x regardless of the value for C. As $x \to 0^+$, the solution curves get close to $\frac{C+1}{x}$ if $C \neq -1$, and if $C = -1$, then $\lim_{x \to 0^+} y(x) = 0$.

50. Use the Fundamental Theorem of Calculus.

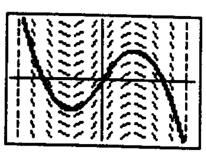
$$y' = \frac{d}{dx}\left(\int_0^x \sin t^2 \, dt\right) + \frac{d}{dx}(x^3 + x + 2) = (\sin x^2) = (3x^2 + 1)$$

$$y'' = \frac{d}{dx}(\sin x^2 + 3x^2 + 1) = (\cos x^2)(2x) + 6x = 2x \cos(x^2) + 6x$$

Thus, the differential equation is satisfied. Verify the initial conditions: $y'(0) = (\sin 0^2) + 3(0)^2 + 1 = 1$,

$$y(0) = \int_0^0 \sin(t^2)\, dt + 0^3 + 0 + 2 = 2$$

51.



[−10, 10] by [−10, 10]

52. To find the approximate values let $y_n = y_{n-1} + (y_{n-1} + \cos x_{n-1})(0.1)$ with $x_0 = 0$, $y_0 = 0$, and 20 steps. Use a spreadsheet, graphing calculator, or CAS to obtain the values in the following table.

x	y
0	0
0.1	0.1000
0.2	0.2095
0.3	0.3285
0.4	0.4568
0.5	0.5946
0.6	0.7418
0.7	0.8986
0.8	1.0649
0.9	1.2411
1.0	1.4273

x	y
1.1	1.6241
1.2	1.8319
1.3	2.0513
1.4	2.2832
1.5	2.5285
1.6	2.7884
1.7	3.0643
1.8	3.3579
1.9	3.6709
2.0	4.0057

53. To find the approximate solution let $z_n = y_{n-1} + ((2 - y_{n-1})(2x_{n-1} + 3))(0.1)$ and

$$y_n = y_{n-1} + \left(\frac{(2 - y_{n-1})(2x_{n-1} + 3) + (2 - z_n)(2x_n + 3)}{2}\right)(0.1)$$ with initial values $x_0 = -3$, $y_0 = 1$,

and 20 steps. Use a spreadsheet, graphing calculator, or CAS to obtain the values in the following table.

x	y	x	y
−3	1	−1.9	−5.9686
−2.9	0.6680	−1.8	−6.5456
−2.8	0.2599	−1.7	−6.9831
−2.7	−0.2294	−1.6	−7.2562
−2.6	−0.8011	−1.5	−7.3488
−2.5	−1.4509	−1.4	−7.2553
−2.4	−2.1687	−1.3	−6.9813
−2.3	−2.9374	−1.2	−6.5430
−2.2	−3.7333	−1.1	−5.9655
−2.1	−4.5268	−1.0	−5.2805
−2.0	−5.2840		

54. To estimate y(3), let $z_n = y_{n-1} + \left(\frac{x_{n-1} - 2y_{n-1}}{x_{n-1} + 1}\right)(0.05)$ and $y_n = y_{n-1} + \frac{1}{2}\left(\frac{x_{n-1} - 2y_{n-1}}{x_{n-1} + 1} + \frac{x_n - 2z_n}{x_n + 1}\right)(0.05)$ with initial values $x_0 = 0$, $y_0 = 1$, and 60 steps. Use a spreadsheet, programmable calculator, or CAS to obtain $y(3) \approx 0.9063$.

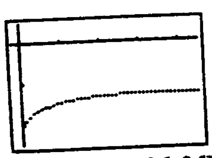
55. To estimate y(4), let $y_n = y_{n-1} + \left(\frac{x_{n-1}^2 - 2y_{n-1} + 1}{x_{n-1}}\right)(0.05)$ with initial values $x_0 = 1$, $y_0 = 1$, and 60 steps.

Use a spreadsheet, programmable calculator, or CAS to obtain $y(4) \approx 4.4974$.

56. Let $y_n = y_{n-1} + \left(\frac{1}{e^{x_{n-1} + y_{n-1} + 2}}\right)(dx)$ with starting values $x_0 = 0$ and $y_0 = -2$, and steps 0f 0.1 and −0.1.

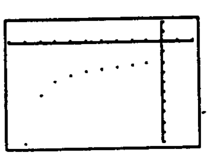
Use a spreadsheet, programmable calculator, or CAS to generate the following graphs.

(a)



[−0.2, 4.5] by [−2.5, 0.5]

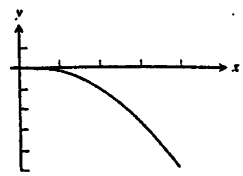
(b) Note that we choose a small interval of x-values because the y-values decrease very rapidly and our calculator cannot handle the calculations for $x \le -1$. (This occurs because the analytic solution is $y = -2 + \ln(2 - e^{-x})$, which has an asymptote at $x = -\ln 2 \approx -0.69$. Obviously, the Euler approximations are misleading for $x \le -0.7$.)



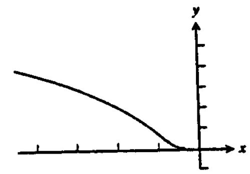
[−1, 0.2] by [−10, 2]

57. Let $z_n = y_{n-1} - \left(\frac{x_{n-1}^2 + y_{n-1}}{e^{y_{n-1}} + x_{n-1}}\right)(dx)$ and $y_n = y_{n-1} + \frac{1}{2}\left(\frac{x_{n-1}^2 + y_{n-1}}{e^{y_{n-1}} + x_{n-1}} + \frac{x_n^2 + z_n}{e^{z_n} + x_n}\right)(dx)$ with starting values $x_0 = 0$, $y_0 = 0$, and steps of 0.1 and −0.1. Use a spreadsheet, programmable calculator, or CAS to generate the following graphs.

(a)



(b)



58. (a) $\frac{dP}{dt} = 0.002P\left(1 - \frac{P}{800}\right) \Rightarrow \frac{dP}{dt} = 0.002P\left(\frac{800-P}{800}\right) \Rightarrow \frac{800}{P(800-P)}\,dP = 0.002\,dt \Rightarrow \frac{(800-P)+P}{P(800-P)} = 0.002\,dt$

$\Rightarrow \int\left(\frac{1}{P} + \frac{1}{800-P}\right)dP = 0.002\,dt \Rightarrow \ln|P| - \ln|800-P| = 0.002t + C \Rightarrow \ln\left|\frac{P}{800-P}\right| = 0.002t + C$

$\Rightarrow \ln\left|\frac{800-P}{P}\right| = -0.002t - C \Rightarrow \left|\frac{800-P}{P}\right| = e^{-0.002t-C} \Rightarrow \frac{800-P}{P} = \pm e^{-C}e^{-0.002t}$

$\Rightarrow \frac{800}{P} - 1 = Ae^{-0.002t} \Rightarrow P = \frac{800}{1 + Ae^{-0.002t}}$

Initial condition: $P(0) = 50 \Rightarrow 50 = \frac{800}{1 + Ae^0} \Rightarrow 1 + A = 16 \Rightarrow A = 15$

Solution: $P = \frac{800}{1 + 15e^{-.002t}}$

(b) $\frac{dP}{dt} = 0.002P\left(1 - \frac{P}{800}\right)$, $P(0) = 50 \Rightarrow P_{n+1} = P_n + 0.002P_n\left(1 - \frac{P_n}{800}\right)dt = P_n + 0.001P_n\left(1 - \frac{P_n}{800}\right)$

On a TI-02 Plus calculator home screen, type the following commands:

50 STO> p:0 STO> t:p (enter)
p+0.001*p*(1 − p/800) STO> p:t+0.5 STO> t:p (enter, 40 times)

The last value displayed gives $P_{Euler}(20) \approx 51.9073$

From part (a), $P_{exact}(20) = \frac{800}{1 + 15e^{-0.002(20)}} \approx 51.9081 \Rightarrow \left|\frac{P_{Euler}(20) - P_{exact}(20)}{P_{exact}(20}\right| \times 100\%$

$= \left|\frac{51.9073 - 51.9081}{51.9081}\right| \times 100\% \approx 0.154\%$

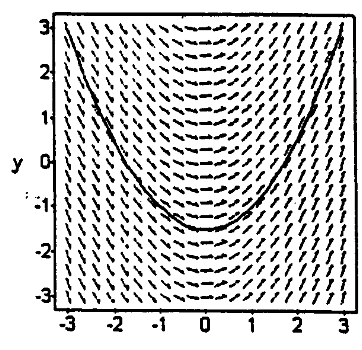
59.

x	1	1.2	1.4	1.6	1.8	2.0
y	−1	−0.8	−0.56	−0.28	0.04	0.4

$\frac{dy}{dx} = x \Rightarrow dy = x\,dx \Rightarrow y = \frac{x^2}{2} + C$; $x = 1$ and $y = -1$

$\Rightarrow -1 = \frac{1}{2} + C \Rightarrow C = -\frac{3}{2} \Rightarrow y\text{ (exact)} = \frac{x^2}{2} - \frac{3}{2}$

$\Rightarrow y(2) = \frac{2^2}{2} - \frac{3}{2} = \frac{1}{2}$ is the exact value



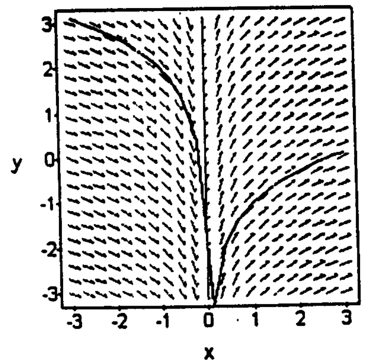
60.

x	1	1.2	1.4	1.6	1.8	2.0
y	−1	−0.8	−0.6333	−0.4904	−0.3654	−0.2544

$\frac{dy}{dx} = \frac{1}{x} \Rightarrow dy = \frac{1}{x}\,dx \Rightarrow y = \ln|x| + C$; $x = 1$ and $y = -1$

$\Rightarrow -1 = \ln 1 + C \Rightarrow C = -1 \Rightarrow y \text{ (exact)} = \ln|x| - 1$

$\Rightarrow y(2) = \ln 2 - 1 \approx -0.3069$ is the exact value



61.

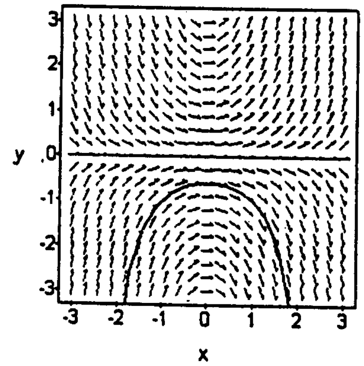
x	1	1.2	1.4	1.6	1.8	2.0
y	−1	−1.2	−1.488	−1.9046	−2.5141	−3.4192

$\frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = x\,dx \Rightarrow \ln|y| = \frac{x^2}{2} + C \Rightarrow y = e^{\frac{x^2}{2}+C} = e^{x^2/2} \cdot e^C$

$= C_1 e^{x^2/2}$; $x = 1$ and $y = -1 \Rightarrow -1 = C_1 e^{1/2} \Rightarrow C_1 = -e^{-1/2}$

$\Rightarrow y \text{ (exact)} = -e^{-1/2} \cdot e^{x^2/2} = -e^{(x^2-1)/2} \Rightarrow y(2) = -e^{3/2}$

≈ -4.4817 is the exact value



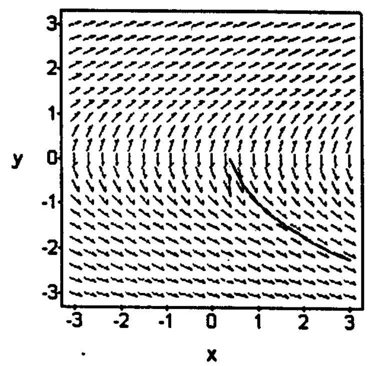
62.

x	1	1.2	1.4	1.6	1.8	2.0
y	−1	−1.2	−1.3667	−1.5130	−1.6452	−1.7668

$\frac{dy}{dx} = \frac{1}{y} \Rightarrow y\,dy = dx \Rightarrow \frac{y^2}{2} = x + C$; $x = 1$ and $y = -1$

$\Rightarrow \frac{1}{2} = 1 + C \Rightarrow C = -\frac{1}{2} \Rightarrow y^2 = 2x - 1 \Rightarrow y \text{ (exact)} = -\sqrt{2x-1}$

$\Rightarrow y(2) = -\sqrt{3} \approx -1.7321$ is the exact value



CHAPTER 6 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. $A_1 = \int_1^e \frac{2\log_2 x}{x}\,dx = \frac{2}{\ln 2}\int_1^e \frac{\ln x}{x}\,dx = \left[\frac{(\ln x)^2}{\ln 2}\right]_1^e = \frac{1}{\ln 2}$; $A_2 = \int_1^e \frac{2\log_4 x}{4}\,dx = \frac{2}{\ln 4}\int_1^e \frac{\ln x}{x}\,dx$

$= \left[\frac{(\ln x)^2}{2\ln 2}\right]_1^e = \frac{1}{2\ln 2} \Rightarrow A_1 : A_2 = 2:1$

2. $f(x) = e^{g(x)} \Rightarrow f'(x) = e^{g(x)}\,g'(x)$, where $g'(x) = \frac{x}{1+x^4} \Rightarrow f'(2) = e^0\left(\frac{2}{1+16}\right) = \frac{2}{17}$

3. (a) $\frac{df}{dx} = \frac{2 \ln e^x}{e^x} \cdot e^x = 2x$

(b) $f(0) = \int_1^1 \frac{2 \ln t}{t}\, dt = 0$

(c) $\frac{df}{dx} = 2x \Rightarrow f(x) = x^2 + C$; $f(0) = 0 \Rightarrow C = 0 \Rightarrow f(x) = x^2 \Rightarrow$ the graph of f(x) is a parabola

4. (a) The figure shows that $\frac{\ln e}{e} > \frac{\ln \pi}{\pi} \Rightarrow \pi \ln e > e \ln \pi \Rightarrow \ln e^\pi > \ln \pi^e \Rightarrow e^\pi > \pi^e$

(b) $y = \frac{\ln x}{x} \Rightarrow y' = \left(\frac{1}{x}\right)\left(\frac{1}{x}\right) - \frac{\ln x}{x^2} \Rightarrow \frac{1 - \ln x}{x^2}$; solving $y' = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$; $y' < 0$ for $x > e$ and $y' > 0$ for $0 < x < e \Rightarrow$ an absolute maximum occurs at $x = e$

5. The area of the shaded region is $\int_0^1 \sin^{-1} x\, dx = \int_0^1 \sin^{-1} y\, dy$, which is the same as the area of the region to the left of the curve $y = \sin x$ (and part of the rectangle formed by the coordinate axes and dashed lines $y = 1$, $x = \frac{\pi}{2}$). The area of the rectangle is $\frac{\pi}{2} = \int_0^1 \sin^{-1} y\, dy + \int_0^{\pi/2} \sin x\, dx$, so we have

$$\frac{\pi}{2} = \int_0^1 \sin^{-1} x\, dx + \int_0^{\pi/2} \sin x\, dx \Rightarrow \int_0^{\pi/2} \sin x\, dx = \frac{\pi}{2} - \int_0^1 \sin^{-1} x\, dx.$$

6. (a) slope of $L_3 <$ slope of $L_2 <$ slope of $L_1 \Rightarrow \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$

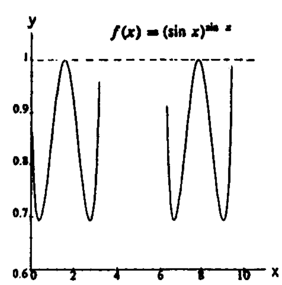
(b) area of small (shaded) rectangle < area under curve < area of large rectangle

$$\Rightarrow \frac{1}{b}(b - a) < \int_a^b \frac{1}{x}\, dx < \frac{1}{a}(b - a) \Rightarrow \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$$

7. Method 1: Use a CAS or a numerical integral function on a calculator or spreadsheet to define $y_1 = x^2 \ln x$ and $y_2 = \frac{d}{dx}\left(\frac{x^3 \ln x}{3} - \frac{x^3}{9} + C\right)$, then compare the graph of y_1 with that of y_2. The graphs should be the same.

Method 2: Use a CAS or a numerical integration function on a calculator or spreadsheet to define $y_1 = \int_a^x t^2 \ln t\, dt$ and $y_2 = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C$ (you pick an $a > 0$ and any value for C), then compare the graph of y_1 with that of y_2. The graphs should be the same except for a vertical translation.

8. In the interval $\pi < x < 2\pi$ the function $\sin x < 0$ $\Rightarrow (\sin x)^{\sin x}$ is not defined for all values in that interval or its translation by 2π.



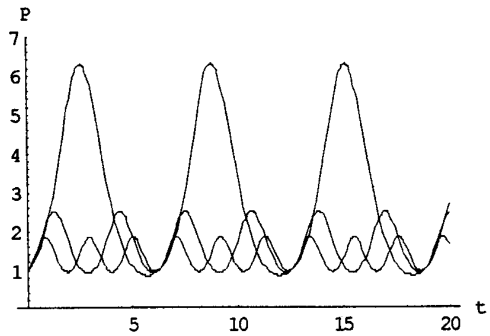
9. (a) $\frac{dP}{dt} = kP\cos(at-b) \Rightarrow \frac{dP}{P} = k\cos(at-b)\,dt \Rightarrow \ln|P| = \frac{k}{a}\sin(at-b) + C$

$\Rightarrow |P| = e^{\frac{k}{a}\sin(at-b)+C_1} = e^{C_1}e^{\frac{k}{a}\sin(at-b)} = C_2e^{\frac{k}{a}\sin(at-b)} \Rightarrow P = \pm C_2e^{\frac{k}{a}\sin(at-b)} = Ce^{\frac{k}{a}\sin(at-b)}$ where

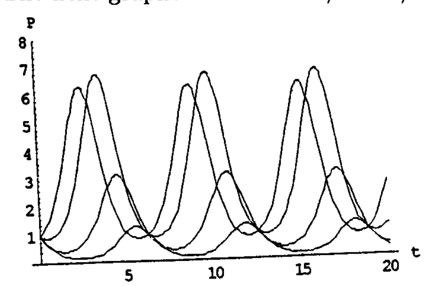
$C_2 = e^{C_1}$ and $C = \pm C_2$; $P(0) = P_0 \Rightarrow Ce^{-\frac{k}{a}\sin b} \Rightarrow C = P_0e^{\frac{k}{a}\sin b} \Rightarrow P = P_0e^{\frac{k}{a}\sin b}e^{\frac{k}{a}\sin(at-b)}$

$= P_0e^{\frac{k}{a}(\sin b+\sin(at-b))}$

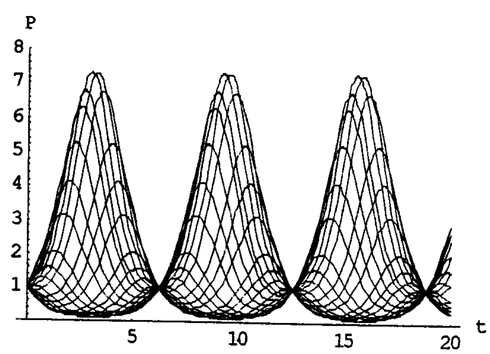
(b) The following graphs are for a = 1, 2, 3, b = 1, k = 1



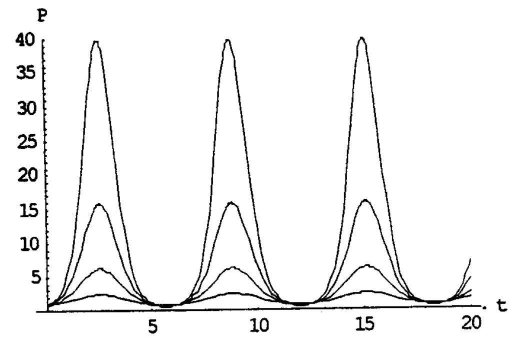
The graphs show that as a increases, the frequency of the oscillations increases and the amplitude decreases. The next graphs are for a = 1, b = 1, 2, 3, 4, k = 1



The graphs show that as b increases the solution curve shifts to the right and the amplitude oscillates. An interesting picture is obtained by graphing the solution curves for a = 1, b = 1, 2, ..., 20, k = 1 and is shown in the next graph.



The next graphs are for a = 1, b = 1, k = 0.5, 1.0, 1.5, 2.0



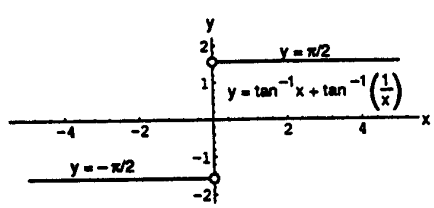
The graphs show that as k increases, the maximum population increases and the minimum population decreases, getting closer to zero. In the long term, all the population appears to oscillate for all values of the parameters.

10. (a) $\frac{dy}{dt} = k\,\frac{A}{V}(c - y) \Rightarrow dy = -k\,\frac{A}{V}(y - c)\,dt \Rightarrow \frac{dy}{y - c} = -k\,\frac{A}{V}\,dt \Rightarrow \int \frac{dy}{y - c} = -\int k\,\frac{A}{V}\,dt \Rightarrow \ln|y - c|$
$= -k\,\frac{A}{V}t + C_1 \Rightarrow y - c = \pm e^{C_1}e^{-k\frac{A}{V}t}$. Apply the initial condition, $y(0) = y_0 \Rightarrow y_0 = c + C \Rightarrow C = y_0 - c$
$\Rightarrow y = c + (y_0 - c)e^{-k\frac{A}{V}t}$

(b) Steady state solution: $y_\infty = \lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left[c + (y_0 - c)e^{-k\frac{A}{V}t}\right] = c + (y_0 - c)(0) = c$

11. $y = \tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right) \Rightarrow y' = \frac{1}{1 + x^2} + \frac{\left(-\frac{1}{x^2}\right)}{\left(1 + \frac{1}{x^2}\right)}$

$= \frac{1}{1 + x^2} - \frac{1}{1 + x^2} = 0 \Rightarrow \tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right)$ is a constant
and the constant is $\frac{\pi}{2}$ for $x > 0$; it is $-\frac{\pi}{2}$ for $x < 0$ since
$\tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right)$ is odd. Next the $\lim_{x\to 0^+}\left[\tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right)\right]$
$= 0 + \frac{\pi}{2} = \frac{\pi}{2}$ and $\lim_{x\to 0^-}\left(\tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right)\right) = 0 + \left(-\frac{\pi}{2}\right) = -\frac{\pi}{2}$



NOTES: