CHAPTER 7 TECHNIQUES OF INTEGRATION, L'HÔPITAL'S RULE, AND IMPROPER INTEGRALS

7.1 BASIC INTEGRATION FORMULAS

1. $\int \frac{16x\,dx}{\sqrt{8x^2+1}}$; $\begin{bmatrix} u = 8x^2+1 \\ du = 16x\,dx \end{bmatrix} \to \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{8x^2+1} + C$

2. $\int \frac{3\cos x\,dx}{\sqrt{1+3\sin x}}$; $\begin{bmatrix} u = 1+3\sin x \\ du = 3\cos x\,dx \end{bmatrix} \to \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{1+3\sin x} + C$

3. $\int 3\sqrt{\sin v}\cos v\,dv$; $\begin{bmatrix} u = \sin v \\ du = \cos v\,dv \end{bmatrix} \to \int 3\sqrt{u}\,du = 3\cdot\frac{2}{3}u^{3/2} + C = 2(\sin v)^{3/2} + C$

4. $\int \cot^3 y\csc^2 y\,dy$; $\begin{bmatrix} u = \cot y \\ du = -\csc^2 y\,dy \end{bmatrix} \to \int u^3(-du) = -\frac{u^4}{4} + C = \frac{-\cot^4 y}{4} + C$

5. $\int_0^1 \frac{16x\,dx}{8x^2+2}$; $\begin{bmatrix} u = 8x^2+2 \\ du = 16x\,dx \\ x = 0 \Rightarrow u = 2,\quad x = 1 \Rightarrow u = 10 \end{bmatrix} \to \int_2^{10} \frac{du}{u} = [\ln|u|]_2^{10} = \ln 10 - \ln 2 = \ln 5$

6. $\int_{\pi/4}^{\pi/3} \frac{\sec^2 z\,dz}{\tan z}$; $\begin{bmatrix} u = \tan z \\ du = \sec^2 z\,dz \\ z = \frac{\pi}{4} \Rightarrow u = 1,\ z = \frac{\pi}{3} \Rightarrow u = \sqrt{3} \end{bmatrix} \to \int_1^{\sqrt{3}} \frac{1}{u}\,du = [\ln|u|]_1^{\sqrt{3}} = \ln\sqrt{3} - \ln 1 = \ln\sqrt{3}$

7. $\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$; $\begin{bmatrix} u = \sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\,dx \\ 2\,du = \frac{dx}{\sqrt{x}} \end{bmatrix} \to \int \frac{2\,du}{u} = 2\ln|u| + C = 2\ln(\sqrt{x}+1) + C$

8. $\int \frac{dx}{x-\sqrt{x}} = \int \frac{dx}{\sqrt{x}(\sqrt{x}-1)}$; $\begin{bmatrix} u = \sqrt{x}-1 \\ du = \frac{1}{2\sqrt{x}}\,dx \\ 2\,du = \frac{dx}{\sqrt{x}} \end{bmatrix} \to \int \frac{2\,du}{u} = 2\ln|u| + C = 2\ln\left|\sqrt{x}-1\right| + C$

9. $\int \cot(3-7x)\,dx;\ \begin{bmatrix} u = 3-7x \\ du = -7\,dx \end{bmatrix} \to -\frac{1}{7}\int \cot u\,du = -\frac{1}{7}\ln|\sin u| + C = -\frac{1}{7}\ln|\sin(3-7x)| + C$

10. $\int \csc(\pi x - 1)\,dx;\ \begin{bmatrix} u = \pi x - 1 \\ du = \pi\,dx \end{bmatrix} \to \int \csc u \cdot \frac{du}{\pi} = \frac{-1}{\pi}\ln|\csc u + \cot u| + C$

$= -\frac{1}{\pi}\ln|\csc(\pi x - 1) + \cot(\pi x - 1)| + C$

11. $\int e^{\theta}\csc(e^{\theta}+1)\,d\theta;\ \begin{bmatrix} u = e^{\theta}+1 \\ du = e^{\theta}\,d\theta \end{bmatrix} \to \int \csc u\,du = -\ln|\csc u + \cot u| + C = -\ln|\csc(e^{\theta}+1) + \cot(e^{\theta}+1)| + C$

12. $\int \frac{\cot(3+\ln x)}{x}\,dx;\ \begin{bmatrix} u = 3+\ln x \\ du = \frac{dx}{x} \end{bmatrix} \to \int \cot u\,du = \ln|\sin u| + C = \ln|\sin(3+\ln x)| + C$

13. $\int \sec\frac{t}{3}\,dt;\ \begin{bmatrix} u = \frac{t}{3} \\ du = \frac{dt}{3} \end{bmatrix} \to \int 3\sec u\,du = 3\ln|\sec u + \tan u| + C = 3\ln\left|\sec\frac{t}{3} + \tan\frac{t}{3}\right| + C$

14. $\int x\sec(x^2-5)\,dx;\ \begin{bmatrix} u = x^2-5 \\ du = 2x\,dx \end{bmatrix} \to \int \frac{1}{2}\sec u\,du = \frac{1}{2}\ln|\sec u + \tan u| + C$

$= \frac{1}{2}\ln|\sec(x^2-5) + \tan(x^2-5)| + C$

15. $\int \csc(s-\pi)\,ds;\ \begin{bmatrix} u = s-\pi \\ du = ds \end{bmatrix} \to \int \csc u\,du = -\ln|\csc u + \cot u| + C = -\ln|\csc(s-\pi) + \cot(s-\pi)| + C$

16. $\int \frac{1}{\theta^2}\csc\frac{1}{\theta}\,d\theta;\ \begin{bmatrix} u = \frac{1}{\theta} \\ du = \frac{-d\theta}{\theta^2} \end{bmatrix} \to \int -\csc u\,du = \ln|\csc u + \cot u| + C = \ln\left|\csc\frac{1}{\theta} + \cot\frac{1}{\theta}\right| + C$

17. $\int_0^{\sqrt{\ln 2}} 2xe^{x^2}\,dx;\ \begin{bmatrix} u = x^2 \\ du = 2x\,dx \\ x = 0 \Rightarrow u = 0,\ x = \sqrt{\ln 2} \Rightarrow u = \ln 2 \end{bmatrix} \to \int_0^{\ln 2} e^u\,du = \left[e^u\right]_0^{\ln 2} = e^{\ln 2} - e^0 = 2 - 1 = 1$

18. $\int_{\pi/2}^{\pi} \sin(y)\,e^{\cos y}\,dy;\ \begin{bmatrix} u = \cos y \\ du = -\sin y\,dy \\ x = \frac{\pi}{2} \Rightarrow u = 0,\ x = \pi \Rightarrow u = -1 \end{bmatrix} \to \int_0^{-1} -e^u\,du = \int_{-1}^{0} e^u\,du = \left[e^u\right]_{-1}^{0} = 1 - e^{-1} = \frac{e-1}{e}$

19. $\int e^{\tan v}\sec^2 v\,dv;\ \begin{bmatrix} u = \tan v \\ du = \sec^2 v\,dv \end{bmatrix} \to \int e^u\,du = e^u + C = e^{\tan v} + C$

20. $\int \frac{e^{\sqrt{t}}\,dt}{\sqrt{t}}$; $\begin{bmatrix} u = \sqrt{t} \\ du = \frac{dt}{2\sqrt{t}} \end{bmatrix} \to \int 2e^u\,du = 2e^u + C = 2e^{\sqrt{t}} + C$

21. $\int 3^{x+1}\,dx$; $\begin{bmatrix} u = x+1 \\ du = dx \end{bmatrix} \to \int 3^u\,du = \left(\frac{1}{\ln 3}\right)3^u + C = \frac{3^{(x+1)}}{\ln 3} + C$

22. $\int \frac{2^{\ln x}}{x}\,dx$; $\begin{bmatrix} u = \ln x \\ du = \frac{dx}{x} \end{bmatrix} \to \int 2^u\,du = \frac{2^u}{\ln 2} + C = \frac{2^{\ln x}}{\ln 2} + C$

23. $\int \frac{2^{\sqrt{w}}\,dw}{2\sqrt{w}}$; $\begin{bmatrix} u = \sqrt{w} \\ du = \frac{dw}{2\sqrt{w}} \end{bmatrix} \to \int 2^u\,du = \frac{2^u}{\ln 2} + C = \frac{2^{\sqrt{w}}}{\ln 2} + C$

24. $\int 10^{2\theta}\,d\theta$; $\begin{bmatrix} u = 2\theta \\ du = 2\,d\theta \end{bmatrix} \to \int \frac{1}{2}10^u\,du = \frac{10^u}{2\ln 10} + C = \frac{1}{2}\left(\frac{10^{2\theta}}{\ln 10}\right) + C$

25. $\int \frac{9\,du}{1+9u^2}$; $\begin{bmatrix} x = 3u \\ dx = 3\,du \end{bmatrix} \to \int \frac{3\,dx}{1+x^2} = 3\tan^{-1} x + C = 3\tan^{-1} 3u + C$

26. $\int \frac{4\,dx}{1+(2x+1)^2}$; $\begin{bmatrix} u = 2x+1 \\ du = 2\,dx \end{bmatrix} \to \int \frac{2\,du}{1+u^2} = 2\tan^{-1} u + C = 2\tan^{-1}(2x+1) + C$

27. $\int_0^{1/6} \frac{dx}{\sqrt{1-9x^2}}$; $\begin{bmatrix} u = 3x \\ du = 3\,dx \\ x = 0 \Rightarrow u = 0,\ x = \frac{1}{6} \Rightarrow u = \frac{1}{2} \end{bmatrix} \to \int_0^{1/2} \frac{1}{3}\frac{du}{\sqrt{1-u^2}} = \left[\frac{1}{3}\sin^{-1} u\right]_0^{1/2} = \frac{1}{3}\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{18}$

28. $\int_0^1 \frac{dt}{\sqrt{4-t^2}} = \left[\sin^{-1}\frac{t}{2}\right]_0^1 = \sin^{-1}\left(\frac{1}{2}\right) - 0 = \frac{\pi}{6}$

29. $\int \frac{2s\,ds}{\sqrt{1-s^4}}$; $\begin{bmatrix} u = s^2 \\ du = 2s\,ds \end{bmatrix} \to \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1} s^2 + C$

30. $\int \frac{2\,dx}{x\sqrt{1-4\ln^2 x}}$; $\begin{bmatrix} u = 2\ln x \\ du = \frac{2\,dx}{x} \end{bmatrix} \to \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(2\ln x) + C$

31. $\int \frac{6\,dx}{x\sqrt{25x^2-1}} = \int \frac{6\,dx}{5x\sqrt{x^2-\frac{1}{25}}} = \frac{6}{5}\cdot 5\sec^{-1}|5x| + C = 6\sec^{-1}|5x| + C$

32. $\int \frac{dr}{r\sqrt{r^2-9}} = \frac{1}{3}\sec^{-1}\left|\frac{r}{3}\right| + C$

33. $\int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x\,dx}{e^{2x}+1};\ \begin{bmatrix} u = e^x \\ du = e^x\,dx \end{bmatrix} \to \int \frac{du}{u^2+1} = \tan^{-1}u + C = \tan^{-1}e^x + C$

34. $\int \frac{dy}{\sqrt{e^{2y}-1}} = \int \frac{e^y\,dy}{e^y\sqrt{(e^y)^2-1}};\ \begin{bmatrix} u = e^y \\ du = e^y\,dy \end{bmatrix} \to \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C = \sec^{-1}e^y + C$

35. $\int_1^{e^{\pi/3}} \frac{dx}{x\cos(\ln x)};\ \begin{bmatrix} u = \ln x \\ du = \frac{dx}{x} \\ x = 1 \Rightarrow u = 0,\ x = e^{\pi/3} \Rightarrow u = \frac{\pi}{3} \end{bmatrix} \to \int_0^{\pi/3} \frac{du}{\cos u} = \int_0^{\pi/3} \sec u\,du = \left[\ln|\sec u + \tan u|\right]_0^{\pi/3}$

$= \ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \ln|\sec 0 + \tan 0| = \ln\left(2+\sqrt{3}\right) - \ln(1) = \ln\left(2+\sqrt{3}\right)$

36. $\int \frac{\ln x\,dx}{x+4x\ln^2 x} = \int \frac{\ln x\,dx}{x(1+4\ln^2 x)};\ \begin{bmatrix} u = \ln^2 x \\ du = \frac{2}{x}\ln x\,dx \end{bmatrix} \to \int \frac{1}{2}\frac{du}{1+4u} = \frac{1}{8}\ln|1+4u| + C = \frac{1}{8}\ln\left(1+4\ln^2 x\right) + C$

37. $\int_1^2 \frac{8\,dx}{x^2-2x+2} = 8\int_1^2 \frac{dx}{1+(x-1)^2};\ \begin{bmatrix} u = x-1 \\ du = dx \\ x = 1 \Rightarrow u = 0,\ x = 2 \Rightarrow u = 1 \end{bmatrix} \to 8\int_0^1 \frac{du}{1+u^2} = 8\left[\tan^{-1}u\right]_0^1$

$= 8\left(\tan^{-1}1 - \tan^{-1}0\right) = 8\left(\frac{\pi}{4} - 0\right) = 2\pi$

38. $\int_2^4 \frac{2\,dx}{x^2-6x+10} = 2\int_2^4 \frac{dx}{(x-3)^2+1};\ \begin{bmatrix} u = x-3 \\ du = dx \\ x = 2 \Rightarrow u = -1,\ x = 4 \Rightarrow u = 1 \end{bmatrix} \to 2\int_{-1}^1 \frac{du}{u^2+1} = 2\left[\tan^{-1}u\right]_{-1}^1$

$= 2\left[\tan^{-1}1 - \tan^{-1}(-1)\right] = 2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi$

39. $\int \frac{dt}{\sqrt{-t^2+4t-3}} = \int \frac{dt}{\sqrt{1-(t-2)^2}};\ \begin{bmatrix} u = t-2 \\ du = dt \end{bmatrix} \to \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}(t-2) + C$

40. $\int \frac{d\theta}{\sqrt{2\theta-\theta^2}} = \int \frac{d\theta}{\sqrt{1-(\theta-1)^2}};\ \begin{bmatrix} u = \theta-1 \\ du = d\theta \end{bmatrix} \to \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}(\theta-1) + C$

41. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}} = \int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}}; \begin{bmatrix} u = x+1 \\ du = dx \end{bmatrix} \to \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C = \sec^{-1}|x+1| + C,$
$|u| = |x+1| > 1$

42. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} = \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}}; \begin{bmatrix} u = x-2 \\ du = dx \end{bmatrix} \to \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C$
$= \sec^{-1}|x+2| + C, |u| = |x+2| > 1$

43. $\int (\sec x + \cot x)^2\, dx = \int (\sec^2 x + 2\sec x \cot x + \cot^2 x)\, dx = \int \sec^2 x\, dx + \int 2\csc x\, dx + \int (\csc^2 x - 1)\, dx$
$= \tan x - 2\ln|\csc x + \cot x| - \cot x - x + C$

44. $\int (\csc x - \tan x)^2\, dx = \int (\csc^2 x - 2\csc x \tan x + \tan^2 x)\, dx = \int \csc^2 x\, dx - \int 2\sec x\, dx + \int (\sec^2 x - 1)\, dx$
$= -\cot x - 2\ln|\sec x + \tan x| + \tan x - x + C$

45. $\int \csc x \sin 3x\, dx = \int (\csc x)(\sin 2x \cos x + \sin x \cos 2x)\, dx = \int (\csc x)(2\sin x \cos^2 x + \sin x \cos 2x)\, dx$
$= \int (2\cos^2 x + \cos 2x)\, dx = \int [(1 + \cos 2x) + \cos 2x]\, dx = \int (1 + 2\cos 2x)\, dx = x + \sin 2x + C$

46. $\int (\sin 3x \cos 2x - \cos 3x \sin 2x)\, dx = \int \sin(3x - 2x)\, dx = \int \sin x\, dx = -\cos x + C$

47. $\int \frac{x}{x+1}\, dx = \int \left(1 - \frac{1}{x+1}\right) dx = x - \ln|x+1| + C$

48. $\int \frac{x^2}{x^2+1}\, dx = \int \left(1 - \frac{1}{x^2+1}\right) dx = x - \tan^{-1} x + C$

49. $\int_{\sqrt{2}}^{3} \frac{2x^3}{x^2-1}\, dx = \int_{\sqrt{2}}^{3} \left(2x + \frac{2x}{x^2-1}\right) dx = \left[x^2 + \ln|x^2-1|\right]_{\sqrt{2}}^{3} = (9 + \ln 8) - (2 + \ln 1) = 7 + \ln 8$

50. $\int_{-1}^{3} \frac{4x^2-7}{2x+3}\, dx = \int_{-1}^{3} \left[(2x-3) + \frac{2}{2x+3}\right] dx = \left[x^2 - 3x + \ln|2x+3|\right]_{-1}^{3} = (9 - 9 + \ln 9) - (1 + 3 + \ln 1) = \ln 9 - 4$

51. $\int \frac{4t^3 - t^2 + 16t}{t^2+4}\, dt = \int \left[(4t-1) + \frac{4}{t^2+4}\right] dt = 2t^2 - t + 2\tan^{-1}\left(\frac{t}{2}\right) + C$

52. $\int \frac{2\theta^3 - 7\theta^2 + 7\theta}{2\theta - 5}\, d\theta = \int \left[(\theta^2 - \theta + 1) + \frac{5}{2\theta - 5}\right] d\theta = \frac{\theta^3}{3} - \frac{\theta^2}{2} + \theta + \frac{5}{2}\ln|2\theta - 5| + C$

53. $\int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1} x + \sqrt{1-x^2} + C$

54. $\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx = \int \frac{dx}{2\sqrt{x-1}} + \int \frac{dx}{x} = (x-1)^{1/2} + \ln|x| + C$

55. $\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\,dx = \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\,dx = [\tan x + \sec x]_0^{\pi/4} = (1+\sqrt{2}) - (0+1) = \sqrt{2}$

56. $\int_0^{1/2} \frac{2-8x}{1+4x^2}\,dx = \int_0^{1/2} \left(\frac{2}{1+4x^2} - \frac{8x}{1+4x^2}\right) dx = \left[\tan^{-1}(2x) - \ln\left|1+4x^2\right|\right]_0^{1/2}$
$= (\tan^{-1} 1 - \ln 2) - (\tan^{-1} 0 - \ln 1) = \frac{\pi}{4} - \ln 2$

57. $\int \frac{dx}{1+\sin x} = \int \frac{(1-\sin x)}{(1-\sin^2 x)}\,dx = \int \frac{(1-\sin x)}{\cos^2 x}\,dx = \int (\sec^2 x - \sec x \tan x)\,dx = \tan x - \sec x + C$

58. $1 + \cos x = 1 + \cos\left(2 \cdot \frac{x}{2}\right) = 2\cos^2 \frac{x}{2} \Rightarrow \int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2\left(\frac{x}{2}\right)} = \frac{1}{2}\int \sec^2\left(\frac{x}{2}\right) dx = \tan \frac{x}{2} + C$

59. $\int \frac{1}{\sec\theta + \tan\theta}\,d\theta = \int \frac{\cos\theta}{1+\sin\theta}\,d\theta; \begin{bmatrix} u = 1+\sin\theta \\ du = \cos\theta\,d\theta \end{bmatrix} \to \int \frac{du}{u} = \ln|u| + C = \ln|1+\sin\theta| + C$

60. $\int \frac{1}{\csc\theta + \cot\theta}\,d\theta = \int \frac{\sin\theta}{1+\cos\theta}\,d\theta; \begin{bmatrix} u = 1+\cos\theta \\ du = -\sin\theta\,d\theta \end{bmatrix} \to \int \frac{-du}{u} = -\ln|u| + C = -\ln|1+\cos\theta| + C$

61. $\int \frac{1}{1-\sec x}\,dx = \int \frac{\cos x}{\cos x - 1}\,dx = \int \left(1 + \frac{1}{\cos x - 1}\right) dx = \int \left(1 - \frac{1+\cos x}{\sin^2 x}\right) dx = \int \left(1 - \csc^2 x - \frac{\cos x}{\sin^2 x}\right) dx$
$= \int (1 - \csc^2 x - \csc x \cot x)\,dx = x + \cot x + \csc x + C$

62. $\int \frac{1}{1-\csc x}\,dx = \int \frac{\sin x}{\sin x - 1}\,dx = \int \left(1 + \frac{1}{\sin x - 1}\right) dx = \int \left(1 + \frac{\sin x + 1}{(\sin x - 1)(\sin x + 1)}\right) dx$
$= \int \left(1 - \frac{1+\sin x}{\cos^2 x}\right) dx = \int \left(1 - \sec^2 x - \frac{\sin x}{\cos^2 x}\right) dx = \int (1 - \sec^2 x - \sec x \tan x)\,dx = x - \tan x - \sec x + C$

63. $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\,dx = \int_0^{2\pi} \left|\sin \frac{x}{2}\right| dx; \begin{bmatrix} \sin \frac{x}{2} \ge 0 \\ \text{for } 0 \le \frac{x}{2} \le \pi \end{bmatrix} \to \int_0^{2\pi} \sin\left(\frac{x}{2}\right) dx = \left[-2\cos \frac{x}{2}\right]_0^{2\pi} = -2(\cos\pi - \cos 0)$
$= (-2)(-2) = 4$

64. $\int_0^{\pi} \sqrt{1-\cos 2x}\, dx = \int_0^{\pi} \sqrt{2}\,|\sin x|\, dx; \left[\begin{matrix}\sin x \ge 0 \\ \text{for } 0 \le x \le \pi\end{matrix}\right] \to \sqrt{2}\int_0^{\pi} \sin x\, dx = \left[-\sqrt{2}\cos x\right]_0^{\pi}$
$= -\sqrt{2}(\cos \pi - \cos 0) = 2\sqrt{2}$

65. $\int_{\pi/2}^{\pi} \sqrt{1+\cos 2t}\, dt = \int_{\pi/2}^{\pi} \sqrt{2}\,|\cos t|\, dt; \left[\begin{matrix}\cos t \le 0 \\ \text{for } \frac{\pi}{2} \le t \le \pi\end{matrix}\right] \to \int_{\pi/2}^{\pi} -\sqrt{2}\cos t\, dt = \left[-\sqrt{2}\sin t\right]_{\pi/2}^{\pi}$
$= -\sqrt{2}\left(\sin \pi - \sin \frac{\pi}{2}\right) = \sqrt{2}$

66. $\int_{-\pi}^{0} \sqrt{1+\cos t}\, dt = \int_{-\pi}^{0} \sqrt{2}\left|\cos \frac{t}{2}\right| dt; \left[\begin{matrix}\cot \frac{t}{2} \ge 0 \\ \text{for } -\pi \le t \le 0\end{matrix}\right] \to \int_{-\pi}^{0} \sqrt{2}\cos \frac{t}{2}\, dt = \left[2\sqrt{2}\sin \frac{t}{2}\right]_{-\pi}^{0}$
$= 2\sqrt{2}\left[\sin 0 - \sin\left(-\frac{\pi}{2}\right)\right] = 2\sqrt{2}$

67. $\int_{-\pi}^{0} \sqrt{1-\cos^2\theta}\, d\theta = \int_{-\pi}^{0} |\sin \theta|\, d\theta; \left[\begin{matrix}\sin \theta \le 0 \\ \text{for } -\pi \le \theta \le 0\end{matrix}\right] \to \int_{-\pi}^{0} -\sin \theta\, d\theta = [\cos \theta]_{-\pi}^{0} = \cos 0 - \cos(-\pi)$
$= 1 - (-1) = 2$

68. $\int_{\pi/2}^{\pi} \sqrt{1-\sin^2\theta}\, d\theta = \int_{\pi/2}^{\pi} |\cos \theta|\, d\theta; \left[\begin{matrix}\cos \theta \le 0 \\ \text{for } \frac{\pi}{2} \le \theta \le \pi\end{matrix}\right] \to \int_{\pi/2}^{\pi} -\cos \theta\, d\theta = [-\sin \theta]_{\pi/2}^{\pi} = -\sin \pi + \sin \frac{\pi}{2} = 1$

69. $\int_{-\pi/4}^{\pi/4} \sqrt{\tan^2 y + 1}\, dy = \int_{-\pi/4}^{\pi/4} |\sec y|\, dy; \left[\begin{matrix}\sec y \ge 0 \\ \text{for } -\frac{\pi}{4} \le y \le \frac{\pi}{4}\end{matrix}\right] \to \int_{-\pi/4}^{\pi/4} \sec y\, dy = \left[\ln|\sec y + \tan y|\right]_{-\pi/4}^{\pi/4}$
$= \ln\left|\sqrt{2}+1\right| - \ln\left|\sqrt{2}-1\right|$

70. $\int_{-\pi/4}^{0} \sqrt{\sec^2 y - 1}\, dy = \int_{-\pi/4}^{0} |\tan y|\, dy; \left[\begin{matrix}\tan y \le 0 \\ \text{for } -\frac{\pi}{4} \le y \le 0\end{matrix}\right] \to \int_{-\pi/4}^{0} -\tan y\, dy = \left[\ln|\cos y|\right]_{-\pi/4}^{0} = -\ln\left(\frac{1}{\sqrt{2}}\right)$
$= \ln \sqrt{2}$

71. $\int_{\pi/4}^{3\pi/4} (\csc x - \cot x)^2\, dx = \int_{\pi/4}^{3\pi/4} \left(\csc^2 x - 2\csc x \cot x + \cot^2 x\right) dx = \int_{\pi/4}^{3\pi/4} \left(2\csc^2 x - 1 - 2\csc x \cot x\right) dx$
$= \left[-2\cot x - x + 2\csc x\right]_{\pi/4}^{3\pi/4} = \left(-2\cot \frac{3\pi}{4} - \frac{3\pi}{4} + 2\csc \frac{3\pi}{4}\right) - \left(-2\cot \frac{\pi}{4} - \frac{\pi}{4} + 2\csc \frac{\pi}{4}\right)$
$= \left[-2(-1) - \frac{3\pi}{4} + 2\left(\sqrt{2}\right)\right] - \left[-2(1) - \frac{\pi}{4} + 2\left(\sqrt{2}\right)\right] = 4 - \frac{\pi}{2}$

72. $\int_0^{\pi/4} (\sec x + 4\cos x)^2\, dx = \int_0^{\pi/4} \left[\sec^2 x + 8 + 16\left(\frac{1+\sin 2x}{2}\right)\right] dx = \left[\tan x + 16x - 4\cos 2x\right]_0^{\pi/4}$
$= \left(\tan \frac{\pi}{4} + 4\pi - 4\cos \frac{\pi}{2}\right) - (\tan 0 + 0 - 4\cos 0) = 5 + 4\pi$

73. $\int \cos\theta \csc(\sin\theta)\, d\theta; \begin{bmatrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \to \int \csc u\, du = -\ln|\csc u + \cot u| + C$
$= -\ln|\csc(\sin\theta) + \cot(\sin\theta)| + C$

74. $\int \left(1 + \frac{1}{x}\right)\cot(x + \ln x)\, dx; \begin{bmatrix} u = x + \ln x \\ du = \left(1 + \frac{1}{x}\right) dx \end{bmatrix} \to \int \cot u\, du = \ln|\sin u| + C = \ln|\sin(x + \ln x)| + C$

75. $\int (\csc x - \sec x)(\sin x + \cos x)\, dx = \int (1 + \cot x - \tan x - 1)\, dx = \int \cot x\, dx - \int \tan x\, dx$
$= \ln|\sin x| + \ln|\cos x| + C$

76. $\int 3\sinh\left(\frac{x}{2} + \ln 5\right) dx = \begin{bmatrix} u = \frac{x}{2} + \ln 5 \\ 2\, du = dx \end{bmatrix} = 6\int \sinh u\, du = 6\cosh u + C = 6\cosh\left(\frac{x}{2} + \ln 5\right) + C$

77. $\int \frac{6\, dy}{\sqrt{y}(1+y)}; \begin{bmatrix} u = \sqrt{y} \\ du = \frac{1}{2\sqrt{y}}\, dy \end{bmatrix} \to \int \frac{12\, du}{1+u^2} = 12\tan^{-1} u + C = 12\tan^{-1}\sqrt{y} + C$

78. $\int \frac{dx}{x\sqrt{4x^2-1}} = \int \frac{2\, dx}{2x\sqrt{(2x)^2-1}}; \begin{bmatrix} u = 2x \\ du = 2\, dx \end{bmatrix} \to \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C = \sec^{-1}|2x| + C$

79. $\int \frac{7\, dx}{(x-1)\sqrt{x^2-2x-48}} = \int \frac{7\, dx}{(x-1)\sqrt{(x-1)^2-49}}; \begin{bmatrix} u = x-1 \\ du = dx \end{bmatrix} \to \int \frac{7\, du}{u\sqrt{u^2-49}} = 7\cdot\frac{1}{7}\sec^{-1}\left|\frac{u}{7}\right| + C$
$= \sec^{-1}\left|\frac{x-1}{7}\right| + C$

80. $\int \frac{dx}{(2x+1)\sqrt{4x^2+4x}} = \int \frac{dx}{(2x+1)\sqrt{(2x+1)^2-1}}; \begin{bmatrix} u = 2x+1 \\ du = 2\, dx \end{bmatrix} \to \int \frac{du}{2u\sqrt{u^2-1}} = \frac{1}{2}\sec^{-1}|u| + C$
$= \frac{1}{2}\sec^{-1}|2x+1| + C$

81. $\int \sec^2 t \tan(\tan t)\, dt; \begin{bmatrix} u = \tan t \\ du = \sec^2 t\, dt \end{bmatrix} \to \int \tan u\, du = -\ln|\cos u| + C = \ln|\sec u| + C = \ln|\sec(\tan t)| + C$

82. $\int \frac{dx}{x\sqrt{3+x^2}} = -\frac{1}{\sqrt{3}}\operatorname{csch}^{-1}\left|\frac{x}{\sqrt{3}}\right| + C$ (from Table 6.15)

83. (a) $\int \cos^3\theta\, d\theta = \int (\cos\theta)(1-\sin^2\theta)\, d\theta; \begin{bmatrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \to \int (1-u^2)\, du = u - \frac{u^3}{3} + C = \sin\theta - \frac{1}{3}\sin^3\theta + C$

(b) $\int \cos^5\theta\, d\theta = \int (\cos\theta)(1-\sin^2\theta)^2\, d\theta = \int (1-u^2)^2\, du = \int (1-2u^2+u^4)\, du = u - \frac{2}{3}u^3 + \frac{u^5}{5} + C$
$= \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + C$

(c) $\int \cos^9\theta\, d\theta = \int (\cos^8\theta)(\cos\theta)\, d\theta = \int (1-\sin^2\theta)^4(\cos\theta)\, d\theta$

84. (a) $\int \sin^3\theta\, d\theta = \int (1-\cos^2\theta)(\sin\theta)\, d\theta; \begin{bmatrix} u = \cos\theta \\ du = -\sin\theta\, d\theta \end{bmatrix} \to \int (1-u^2)(-du) = \frac{u^3}{3} - u + C$
$= -\cos\theta + \frac{1}{3}\cos^3\theta + C$

(b) $\int \sin^5\theta\, d\theta = \int (1-\cos^2\theta)^2(\sin\theta)\, d\theta = \int (1-u^2)^2(-du) = \int (-1+2u^2-u^4)\, du$
$= -\cos\theta + \frac{2}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta + C$

(c) $\int \sin^7\theta\, d\theta = \int (1-u^2)^3(-du) = \int (-1+3u^2-3u^4+u^6)\, du = -\cos\theta + \cos^3\theta - \frac{3}{5}\cos^5\theta + \frac{\cos^7\theta}{7} + C$

(d) $\int \sin^{13}\theta\, d\theta = \int (\sin^{12}\theta)(\sin\theta)\, d\theta = \int (1-\cos^2\theta)^6(\sin\theta)\, d\theta$

85. (a) $\int \tan^3\theta\, d\theta = \int (\sec^2\theta - 1)(\tan\theta)\, d\theta = \int \sec^2\theta\tan\theta\, d\theta - \int \tan\theta\, d\theta = \frac{1}{2}\tan^2\theta - \int \tan\theta\, d\theta$
$= \frac{1}{2}\tan^2\theta + \ln|\cos\theta| + C$

(b) $\int \tan^5\theta\, d\theta = \int (\sec^2\theta - 1)(\tan^3\theta)\, d\theta = \int \tan^3\theta\sec^2\theta\, d\theta - \int \tan^3\theta\, d\theta = \frac{1}{4}\tan^4\theta - \int \tan^3\theta\, d\theta$

(c) $\int \tan^7\theta\, d\theta = \int (\sec^2\theta - 1)(\tan^5\theta)\, d\theta = \int \tan^5\theta\sec^2\theta\, d\theta - \int \tan^5\theta\, d\theta = \frac{1}{6}\tan^6\theta - \int \tan^5\theta\, d\theta$

(d) $\int \tan^{2k+1}\theta\, d\theta = \int (\sec^2\theta - 1)(\tan^{2k-1}\theta)\, d\theta = \int \tan^{2k-1}\theta\sec^2\theta\, d\theta - \int \tan^{2k-1}\theta\, d\theta;$

$\begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{bmatrix} \to \int u^{2k-1}\, du - \int \tan^{2k-1}\theta\, d\theta = \frac{1}{2k}u^{2k} - \int \tan^{2k-1}\theta\, d\theta = \frac{1}{2k}\tan^{2k}\theta - \int \tan^{2k-1}\theta\, d\theta$

86. (a) $\int \cot^3\theta\, d\theta = \int (\csc^2\theta - 1)(\cot\theta)\, d\theta = \int \cot\theta\csc^2\theta\, d\theta - \int \cot\theta\, d\theta = -\frac{1}{2}\cot^2\theta - \int \cot\theta\, d\theta$
$= -\frac{1}{2}\cot^2\theta - \ln|\sin\theta| + C$

(b) $\int \cot^5\theta\, d\theta = \int (\csc^2\theta - 1)(\cot^3\theta)\, d\theta = \int \cot^3\theta\csc^2\theta\, d\theta - \int \cot^3\theta\, d\theta = -\frac{1}{4}\cot^4\theta - \int \cot^3\theta\, d\theta$

(c) $\int \cot^7\theta\, d\theta = \int (\csc^2\theta - 1)(\cot^5\theta)\, d\theta = \int \cot^5\theta\csc^2\theta\, d\theta - \int \cot^5\theta\, d\theta = -\frac{1}{6}\cot^6\theta - \int \cot^5\theta\, d\theta$

(d) $\int \cot^{2k+1}\theta\, d\theta = \int (\csc^2\theta - 1)(\cot^{2k-1}\theta)\, d\theta = \int \cot^{2k-1}\theta \csc^2\theta\, d\theta - \int \cot^{2k-1}\theta\, d\theta;$

$\begin{bmatrix} u = \cot\theta \\ du = -\csc^2\theta\, d\theta \end{bmatrix} \to -\int u^{2k-1}\, du - \int \cot^{2k-1}\theta\, d\theta = -\frac{1}{2k}u^{2k} - \int \cot^{2k-1}\theta\, d\theta$

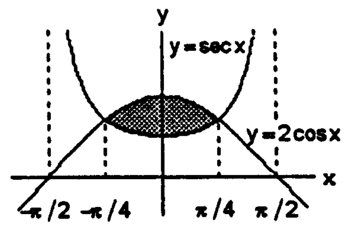
$= -\frac{1}{2k}\cot^{2k}\theta - \int \cot^{2k-1}\theta\, d\theta$

87. $A = \int_{-\pi/4}^{\pi/4} (2\cos x - \sec x)\, dx = [2\sin x - \ln|\sec x + \tan x|]_{-\pi/4}^{\pi/4}$

$= [\sqrt{2} - \ln(\sqrt{2}+1)] - [-\sqrt{2} - \ln(\sqrt{2}-1)]$

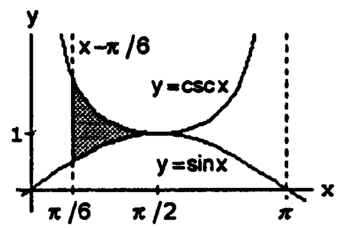
$= 2\sqrt{2} - \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 2\sqrt{2} - \ln\left(\frac{(\sqrt{2}+1)^2}{2-1}\right)$

$= 2\sqrt{2} - \ln(3+2\sqrt{2})$



88. $A = \int_{\pi/6}^{\pi/2} (\csc x - \sin x)\, dx = [-\ln|\csc x + \cot x| + \cos x]_{\pi/6}^{\pi/2}$

$= -\ln|1+0| + \ln|2+\sqrt{3}| - \frac{\sqrt{3}}{2} = \ln(2+\sqrt{3}) - \frac{\sqrt{3}}{2}$



89. $V = \int_{-\pi/4}^{\pi/4} \pi(2\cos x)^2\, dx - \int_{-\pi/4}^{\pi/4} \pi\sec^2 x\, dx = 4\pi \int_{-\pi/4}^{\pi/4} \cos^2 x\, dx - \pi \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx$

$= 2\pi \int_{-\pi/4}^{\pi/4} (1+\cos 2x)\, dx - \pi[\tan x]_{-\pi/4}^{\pi/4} = 2\pi\left[x + \frac{1}{2}\sin 2x\right]_{-\pi/4}^{\pi/4} - \pi[1-(-1)]$

$= 2\pi\left[\left(\frac{\pi}{4}+\frac{1}{2}\right) - \left(-\frac{\pi}{4}-\frac{1}{2}\right)\right] - 2\pi = 2\pi\left(\frac{\pi}{2}+1\right) - 2\pi = \pi^2$

90. $V = \int_{\pi/6}^{\pi/2} \pi\csc^2 x\, dx - \int_{\pi/6}^{\pi/2} \pi\sin^2 x\, dx = \pi \int_{\pi/6}^{\pi/2} \csc^2 x\, dx - \frac{\pi}{2} \int_{\pi/6}^{\pi/2} (1-\cos 2x)\, dx$

$= \pi[-\cot x]_{\pi/6}^{\pi/2} - \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\pi/6}^{\pi/2} = \pi[0-(-\sqrt{3})] - \frac{\pi}{2}\left[\left(\frac{\pi}{2}-0\right) - \left(\frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)\right]$

$= \pi\sqrt{3} - \frac{\pi}{2}\left(\frac{2\pi}{6} + \frac{\sqrt{3}}{4}\right) = \pi\left(\frac{7\sqrt{3}}{8} - \frac{\pi}{6}\right)$

91. $y = \ln(\cos x) \Rightarrow \frac{dy}{dx} = -\frac{\sin x}{\cos x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x = \sec^2 x - 1;\ L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_0^{\pi/3} \sqrt{1 + (\sec^2 x - 1)}\, dx = \int_0^{\pi/3} \sec x\, dx = [\ln|\sec x + \tan x|]_0^{\pi/3} = \ln|2 + \sqrt{3}| - \ln|1 + 0| = \ln(2 + \sqrt{3})$

92. $y = \ln(\sec x) \Rightarrow \frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x = \sec^2 x - 1;\ L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_0^{\pi/4} \sec x\, dx = [\ln|\sec x + \tan x|]_0^{\pi/4} = \ln|\sqrt{2} + 1| - \ln|1 + 0| = \ln(\sqrt{2} + 1)$

93. $\int \csc x\, dx = \int (\csc x)(1)\, dx = \int (\csc x)\left(\frac{\csc x + \cot x}{\csc x + \cot x}\right) dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\, dx;$

$\left[\begin{array}{c} u = \csc x + \cot x \\ du = (-\csc x \cot x - \csc^2 x)\, dx \end{array}\right] \to \int \frac{-du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C$

94. $\left[(x^2 - 1)(x + 1)\right]^{-2/3} = [(x-1)(x+1)^2]^{-2/3} = (x-1)^{-2/3}(x+1)^{-4/3} = (x+1)^{-2}\left[(x-1)^{-2/3}(x+1)^{2/3}\right]$

$= (x+1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} = (x+1)^{-2}\left(1 - \frac{2}{x+1}\right)^{-2/3}$

(a) $\int \left[(x^2 - 1)(x + 1)\right]^{-2/3} dx = \int (x+1)^{-2}\left(1 - \frac{2}{x+1}\right)^{-2/3} dx;\ \left[\begin{array}{c} u = \frac{1}{x+1} \\ du = -\frac{1}{(x+1)^2}\, dx \end{array}\right]$

$\to \int -(1 - 2u)^{-2/3}\, du = \frac{3}{2}(1 - 2u)^{1/3} + C = \frac{3}{2}\left(1 - \frac{2}{x+1}\right)^{1/3} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(b) $\int \left[(x^2 - 1)(x + 1)\right]^{-2/3} dx = \int (x+1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} dx;\ u = \left(\frac{x-1}{x+1}\right)^k$

$\Rightarrow du = k\left(\frac{x-1}{x+1}\right)^{k-1} \frac{[(x+1) - (x-1)]}{(x+1)^2}\, dx = 2k\frac{(x-1)^{k-1}}{(x+1)^{k+1}}\, dx;\ dx = \frac{(x+1)^2}{2k}\left(\frac{x+1}{x-1}\right)^{k-1} du$

$= \frac{(x+1)^2}{2k}\left(\frac{x-1}{x+1}\right)^{1-k} du$; then, $\int \left(\frac{x-1}{x+1}\right)^{-2/3} \frac{1}{2k}\left(\frac{x-1}{x+1}\right)^{1-k} du = \frac{1}{2k}\int \left(\frac{x-1}{x+1}\right)^{(1/3-k)} du$

$= \frac{1}{2k}\int \left(\frac{x-1}{x+1}\right)^{k(1/3k-1)} du = \frac{1}{2k}\int u^{(1/3k-1)}\, du = \frac{1}{2k}(3k)\, u^{1/3k} + C = \frac{3}{2}u^{1/3k} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(c) $\int \left[(x^2 - 1)(x + 1)\right]^{-2/3} dx = \int (x+1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} dx;$

$\left[\begin{array}{c} u = \tan^{-1} x \\ x = \tan u \\ dx = \frac{du}{\cos^2 u} \end{array}\right] \to \int \frac{1}{(\tan u + 1)^2}\left(\frac{\tan u - 1}{\tan u + 1}\right)^{-2/3}\left(\frac{du}{\cos^2 u}\right) = \int \frac{1}{(\sin u + \cos u)^2}\left(\frac{\sin u - \cos u}{\sin u + \cos u}\right)^{-2/3} du;$

$$\left[\begin{array}{l} \sin u + \cos u = \sin u + \sin\left(\frac{\pi}{2} - u\right) = 2 \sin \frac{\pi}{4} \cos\left(u - \frac{\pi}{4}\right) \\ \sin u - \cos u = \sin u - \sin\left(\frac{\pi}{2} - u\right) = 2 \cos \frac{\pi}{4} \sin\left(u - \frac{\pi}{4}\right) \end{array}\right] \to \int \frac{1}{2\cos^2\left(u - \frac{\pi}{4}\right)} \left[\frac{\sin\left(u - \frac{\pi}{4}\right)}{\cos\left(u - \frac{\pi}{4}\right)}\right]^{-2/3} du$$

$$= \frac{1}{2} \int \tan^{-2/3}\left(u - \frac{\pi}{4}\right) \sec^2\left(u - \frac{\pi}{4}\right) du = \frac{3}{2} \tan^{1/3}\left(u - \frac{\pi}{4}\right) + C = \frac{3}{2}\left[\frac{\tan u - \tan \frac{\pi}{4}}{1 + \tan u \tan \frac{\pi}{4}}\right]^{1/3} + C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$$

(d) $u = \tan^{-1} \sqrt{x} \Rightarrow \tan u = \sqrt{x} \Rightarrow \tan^2 u = x \Rightarrow dx = 2 \tan u\left(\frac{1}{\cos^2 u}\right) du = \frac{2 \sin u}{\cos^3 u} du = -\frac{2d(\cos u)}{\cos^3 u}$;

$$x - 1 = \tan^2 u - 1 = \frac{\sin^2 u - \cos^2 u}{\cos^2 u} = \frac{1 - 2\cos^2 u}{\cos^2 u};\ x + 1 = \tan^2 u + 1 = \frac{\cos^2 u + \sin^2 u}{\cos^2 u} = \frac{1}{\cos^2 u};$$

$$\int (x-1)^{-2/3}(x+1)^{-4/3}\, dx = \int \frac{\left(1 - 2\cos^2 u\right)^{-2/3}}{\left(\cos^2 u\right)^{-2/3}} \cdot \frac{1}{\left(\cos^2 u\right)^{-4/3}} \cdot \frac{-2d(\cos u)}{\cos^3 u}$$

$$= \int \left(1 - 2\cos^2 u\right)^{-2/3} \cdot (-2) \cdot \cos u \cdot d(\cos u) = \frac{1}{2} \int \left(1 - 2\cos^2 u\right)^{-2/3} \cdot d\left(1 - 2\cos^2 u\right)$$

$$= \frac{3}{2}\left(1 - 2\cos^2 u\right)^{1/3} + C = \frac{3}{2}\left[\frac{\left(\frac{1 - 2\cos^2 u}{\cos^2 u}\right)}{\left(\frac{1}{\cos^2 u}\right)}\right]^{1/3} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$$

(e) $u = \tan^{-1}\left(\frac{x-1}{2}\right) \Rightarrow \frac{x-1}{2} = \tan u \Rightarrow x + 1 = 2(\tan u + 1) \Rightarrow dx = \frac{2\, du}{\cos^2 u} = 2d(\tan u)$;

$$\int (x-1)^{-2/3}(x+1)^{-4/3}\, dx = \int (\tan u)^{-2/3}(\tan u + 1)^{-4/3} \cdot 2^{-2} \cdot 2 \cdot d(\tan u)$$

$$= \frac{1}{2} \int \left(1 - \frac{1}{\tan u + 1}\right)^{-2/3} d\left(1 - \frac{1}{\tan u + 1}\right) = \frac{3}{2}\left(1 - \frac{1}{\tan u + 1}\right)^{1/3} + C = \frac{3}{2}\left(1 - \frac{2}{x+1}\right)^{1/3} + C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$$

(f) $\left[\begin{array}{c} u = \cos^{-1} x \\ x = \cos u \\ dx = -\sin u\, du \end{array}\right] \to -\int \frac{\sin u\, du}{\sqrt[3]{\left(\cos^2 u - 1\right)^2 (\cos u + 1)^2}} = -\int \frac{\sin u\, du}{\left(\sin^{4/3} u\right)\left(4 \cos \frac{u}{2}\right)^{4/3}}$

$$= -\int \frac{du}{(\sin u)^{1/3}\left(4 \cos \frac{u}{2}\right)^{4/3}} = -\int \frac{du}{2\left(\sin \frac{u}{2}\right)^{4/3}\left(\cos \frac{u}{2}\right)^{5/3}} = -\frac{1}{2} \int \left(\frac{\cos \frac{u}{2}}{\sin \frac{u}{2}}\right)^{1/3} \frac{du}{\left(\cos^2 \frac{u}{2}\right)}$$

$$= -\int \tan^{-1/3}\left(\frac{u}{2}\right) d\left(\tan \frac{u}{2}\right) = -\frac{3}{2} \tan^{2/3} \frac{u}{2} + C = \frac{3}{2}\left(-\tan^2 \frac{u}{2}\right)^{1/3} + C = \frac{3}{2}\left(\frac{\cos u - 1}{\cos u + 1}\right)^{1/3} + C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$$

(g) $\int \left[(x^2-1)(x+1)\right]^{-2/3} dx;\ \begin{bmatrix} u = \cosh^{-1} x \\ x = \cosh u \\ dx = \sinh u \end{bmatrix} \to \int \frac{\sinh u\, du}{\sqrt[3]{(\cosh^2 u - 1)^2 (\cosh u + 1)^2}}$

$= \int \frac{\sinh u\, du}{\sqrt[3]{(\sinh^4 u)(\cosh u + 1)^2}} = \int \frac{du}{\sqrt[3]{(\sinh u)\left(4\cosh^4 \frac{u}{2}\right)}} = \frac{1}{2}\int \frac{du}{\sqrt[3]{\sinh\left(\frac{u}{2}\right)\cosh^5\left(\frac{u}{2}\right)}}$

$= \int \left(\tanh \frac{u}{2}\right)^{-1/3} d\left(\tanh \frac{u}{2}\right) = \frac{3}{2}\left(\tanh \frac{u}{2}\right)^{2/3} + C = \frac{3}{2}\left(\frac{\cosh x - 1}{\cosh x + 1}\right)^{1/3} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

7.2 INTEGRATION BY PARTS

1. $u = x,\ du = dx;\ dv = \sin \frac{x}{2}\, dx,\ v = -2\cos \frac{x}{2};$

$\int x \sin \frac{x}{2}\, dx = -2x \cos \frac{x}{2} - \int \left(-2\cos \frac{x}{2}\right) dx = -2x\cos\left(\frac{x}{2}\right) + 4\sin\left(\frac{x}{2}\right) + C$

2. $u = \theta,\ du = d\theta;\ dv = \cos \pi\theta\, d\theta,\ v = \frac{1}{\pi}\sin \pi\theta;$

$\int \theta \cos \pi\theta\, d\theta = \frac{\theta}{\pi}\sin \pi\theta - \int \frac{1}{\pi}\sin \pi\theta\, d\theta = \frac{\theta}{\pi}\sin \pi\theta + \frac{1}{\pi^2}\cos \pi\theta + C$

3.
$$\begin{array}{lcl} & & \cos t \\ t^2 & \xrightarrow{(+)} & \sin t \\ 2t & \xrightarrow{(-)} & -\cos t \\ 2 & \xrightarrow{(+)} & -\sin t \\ 0 & & \end{array}$$

$\int t^2 \cos t\, dt = t^2 \sin t + 2t\cos t - 2\sin t + C$

4.
$$\begin{array}{lcl} & & \sin x \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$

$\int x^2 \sin x\, dx = -x^2\cos x + 2x\sin x + 2\cos x + C$

5. $u = \ln x,\ du = \frac{dx}{x};\ dv = x\, dx,\ v = \frac{x^2}{2};$

$\int_1^2 x \ln x\, dx = \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2 \frac{x^2}{2}\frac{dx}{x} = 2\ln 2 - \left[\frac{x^2}{4}\right]_1^2 = 2\ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$

6. $u = \ln x,\ du = \frac{dx}{x};\ dv = x^3\,dx,\ v = \frac{x^4}{4};$

$$\int_1^e x^3 \ln x\,dx = \left[\frac{x^4}{4}\ln x\right]_1^e - \int_1^e \frac{x^4}{4}\frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16}\right]_1^e = \frac{3e^4+1}{16}$$

7. $u = \tan^{-1} y,\ du = \frac{dy}{1+y^2};\ dv = dy,\ v = y;$

$$\int \tan^{-1} y\,dy = y\tan^{-1} y - \int \frac{y\,dy}{(1+y^2)} = y\tan^{-1} y - \frac{1}{2}\ln\left(1+y^2\right) + C = y\tan^{-1} y - \ln\sqrt{1+y^2} + C$$

8. $u = \sin^{-1} y,\ du = \frac{dy}{\sqrt{1-y^2}};\ dv = dy,\ v = y;$

$$\int \sin^{-1} y\,dy = y\sin^{-1} y - \int \frac{y\,dy}{\sqrt{1-y^2}} = y\sin^{-1} y + \sqrt{1-y^2} + C$$

9. $u = x,\ du = dx;\ dv = \sec^2 x\,dx,\ v = \tan x;$

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x + \ln|\cos x| + C$$

10. $\int 4x\sec^2 2x\,dx;\ [y = 2x] \to \int y\sec^2 y\,dy = y\tan y - \int \tan y\,dy = y\tan y - \ln|\sec y| + C$

$= 2x\tan 2x - \ln|\sec 2x| + C$

11.

	e^x
x^3 $\xrightarrow{(+)}$	e^x
$3x^2$ $\xrightarrow{(-)}$	e^x
$6x$ $\xrightarrow{(+)}$	e^x
6 $\xrightarrow{(-)}$	e^x
0	

$$\int x^3e^x\,dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = \left(x^3 - 3x^2 + 6x - 6\right)e^x + C$$

12.

	e^{-p}
p^4 $\xrightarrow{(+)}$	$-e^{-p}$
$4p^3$ $\xrightarrow{(-)}$	e^{-p}
$12p^2$ $\xrightarrow{(+)}$	$-e^{-p}$
$24p$ $\xrightarrow{(-)}$	e^{-p}
24 $\xrightarrow{(+)}$	$-e^{-p}$
0	

$$\int p^4e^{-p}\,dp = -p^4e^{-p} - 4p^3e^{-p} - 12p^2e^{-p} - 24pe^{-p} - 24e^{-p} + C$$
$$= \left(-p^4 - 4p^3 - 12p^2 - 24p - 24\right)e^{-p} + C$$

13. e^x

$x^2 - 5x \xrightarrow{(+)} e^x$

$2x - 5 \xrightarrow{(-)} e^x$

$2 \xrightarrow{(+)} e^x$

0

$$\int (x^2 - 5x)e^x\,dx = (x^2 - 5x)e^x - (2x - 5)e^x + 2e^x + C = x^2e^x - 7xe^x + 7e^x + C$$
$$= (x^2 - 7x + 7)e^x + C$$

14. e^r

$r^2 + r + 1 \xrightarrow{(+)} e^r$

$2r + 1 \xrightarrow{(-)} e^r$

$2 \xrightarrow{(+)} e^r$

0

$$\int (r^2 + r + 1)e^r\,dr = (r^2 + r + 1)e^r - (2r + 1)\,e^r + 2e^r + C$$
$$= \left[(r^2 + r + 1) - (2r + 1) + 2\right]e^r + C = (r^2 - r + 2)e^r + C$$

15. e^x

$x^5 \xrightarrow{(+)} e^x$

$5x^4 \xrightarrow{(-)} e^x$

$20x^3 \xrightarrow{(+)} e^x$

$60x^2 \xrightarrow{(-)} e^x$

$120x \xrightarrow{(+)} e^x$

$120 \xrightarrow{(-)} e^x$

0

$$\int x^5e^x\,dx = x^5e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120xe^x - 120e^x + C$$
$$= (x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120)e^x + C$$

16. e^{4t}

$t^2 \xrightarrow{(+)} \frac{1}{4}e^{4t}$

$2t \xrightarrow{(-)} \frac{1}{16}e^{4t}$

$2 \xrightarrow{(+)} \frac{1}{64}e^{4t}$

0

$$\int t^2e^{4t}\,dt = \frac{t^2}{4}e^{4t} - \frac{2t}{16}e^{4t} + \frac{2}{64}e^{4t} + C = \frac{t^2}{4}e^{4t} - \frac{t}{8}e^{4t} + \frac{1}{32}e^{4t} + C$$
$$\left(\frac{t^2}{4} - \frac{t}{8} + \frac{1}{32}\right)e^{4t} + C$$

17. $\sin 2\theta$

$\theta^2 \xrightarrow{(+)} -\frac{1}{2}\cos 2\theta$

$2\theta \xrightarrow{(-)} -\frac{1}{4}\sin 2\theta$

$2 \xrightarrow{(+)} \frac{1}{8}\cos 2\theta$

0

$$\int_0^{\pi/2} \theta^2 \sin 2\theta \, d\theta = \left[-\frac{\theta^2}{2}\cos 2\theta + \frac{\theta}{2}\sin 2\theta + \frac{1}{4}\cos 2\theta\right]_0^{\pi/2}$$

$$= \left[-\frac{\pi^2}{8}\cdot(-1) + \frac{\pi}{4}\cdot 0 + \frac{1}{4}\cdot(-1)\right] - \left[0 + 0 + \frac{1}{4}\cdot 1\right] = \frac{\pi^2}{8} - \frac{1}{2} = \frac{\pi^2 - 4}{8}$$

18. $\cos 2x$

$x^3 \xrightarrow{(+)} \frac{1}{2}\sin 2x$

$3x^2 \xrightarrow{(-)} -\frac{1}{4}\cos 2x$

$6x \xrightarrow{(+)} -\frac{1}{8}\sin 2x$

$6 \xrightarrow{(-)} \frac{1}{16}\cos 2x$

0

$$\int_0^{\pi/2} x^3 \cos 2x \, dx = \left[\frac{x^3}{2}\sin 2x + \frac{3x^2}{4}\cos 2x - \frac{3x}{4}\sin 2x - \frac{3}{8}\cos 2x\right]_0^{\pi/2}$$

$$= \left[\frac{\pi^3}{16}\cdot 0 + \frac{3\pi^2}{16}\cdot(-1) - \frac{3\pi}{8}\cdot 0 - \frac{3}{8}\cdot(-1)\right] - \left[0 + 0 - 0 - \frac{3}{8}\cdot 1\right] = -\frac{3\pi^2}{16} + \frac{3}{4} = \frac{3(4-\pi^2)}{16}$$

19. $u = \sec^{-1} t$, $du = \frac{dt}{t\sqrt{t^2-1}}$; $dv = t\,dt$, $v = \frac{t^2}{2}$;

$$\int_{2/\sqrt{3}}^{2} t \sec^{-1} t \, dt = \left[\frac{t^2}{2}\sec^{-1} t\right]_{2/\sqrt{3}}^{2} - \int_{2/\sqrt{3}}^{2} \left(\frac{t^2}{2}\right)\frac{dt}{t\sqrt{t^2-1}} = \left(2\cdot\frac{\pi}{3} - \frac{2}{3}\cdot\frac{\pi}{6}\right) - \int_{2/\sqrt{3}}^{2} \frac{t\,dt}{2\sqrt{t^2-1}}$$

$$= \frac{5\pi}{9} - \left[\frac{1}{2}\sqrt{t^2-1}\right]_{2/\sqrt{3}}^{2} = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \sqrt{\frac{4}{3}-1}\right) = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right) = \frac{5\pi}{9} - \frac{\sqrt{3}}{3} = \frac{5\pi - 3\sqrt{3}}{9}$$

20. $u = \sin^{-1}(x^2)$, $du = \frac{2x\,dx}{\sqrt{1-x^4}}$; $dv = 2x\,dx$, $v = x^2$;

$$\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2)\, dx = \left[x^2 \sin^{-1}(x^2)\right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2 \cdot \frac{2x\,dx}{\sqrt{1-x^4}} = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) + \int_0^{1/\sqrt{2}} \frac{d(1-x^4)}{2\sqrt{1-x^4}}$$

$$= \frac{\pi}{12} + \left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}} = \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12}$$

21. $I = \int e^\theta \sin\theta \, d\theta$; $[u = \sin\theta,\ du = \cos\theta\, d\theta;\ dv = e^\theta d\theta,\ v = e^\theta] \Rightarrow I = e^\theta \sin\theta - \int e^\theta \cos\theta \, d\theta$;

$[u = \cos\theta,\ du = -\sin\theta\, d\theta;\ dv = e^\theta d\theta,\ v = e^\theta] \Rightarrow I = e^\theta \sin\theta - \left(e^\theta\cos\theta + \int e^\theta \sin\theta\, d\theta\right)$

$= e^\theta \sin\theta - e^\theta\cos\theta - I + C' \Rightarrow 2I = (e^\theta\sin\theta - e^\theta\cos\theta) + C' \Rightarrow I = \frac{1}{2}(e^\theta\sin\theta - e^\theta\cos\theta) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

22. $I=\int e^{-y}\cos y\,dy;\ \left[u=\cos y,\ du=-\sin y\,dy;\ dv=e^{-y}\,dy,\ v=-e^{-y}\right]$

$\Rightarrow I=-e^{-y}\cos y-\int\left(-e^{-y}\right)(-\sin y)\ dy=-e^{-y}\cos y-\int e^{-y}\sin y\,dy;\ [u=\sin y,\ du=\cos y\,dy;$

$dv=e^{-y}\,dy,\ v=-e^{-y}]\Rightarrow I=-e^{-y}\cos y-\left(-e^{-y}\sin y-\int\left(-e^{y}\right)\cos y\,dy\right)=-e^{-y}\cos y+e^{-y}\sin y-I+C'$

$\Rightarrow 2I=e^{-y}(\sin y-\cos y)+C'\Rightarrow I=\frac{1}{2}\left(e^{-y}\sin y-e^{-y}\cos y\right)+C$, where $C=\frac{C'}{2}$ is another arbitrary constant

23. $I=\int e^{2x}\cos 3x\,dx;\ \left[u=\cos 3x;\ du=-3\sin 3x\,dx,\ dv=e^{2x}\,dx;\ v=\frac{1}{2}e^{2x}\right]$

$\Rightarrow I=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{2}\int e^{2x}\sin 3x\,dx;\ \left[u=\sin 3x,\ du=3\cos 3x,\ dv=e^{2x}\,dx;\ v=\frac{1}{2}e^{2x}\right]$

$\Rightarrow I=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{2}\left(\frac{1}{2}e^{2x}\sin 3x-\frac{3}{2}\int e^{2x}\cos 3x\,dx\right)=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}e^{2x}\sin 3x-\frac{9}{4}I+C'$

$\Rightarrow \frac{13}{4}I=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}e^{2x}\sin 3x+C'\Rightarrow \frac{e^{2x}}{13}(3\sin 3x+2\cos 3x)+C$, where $C=\frac{4}{13}C'$

24. $\int e^{-2x}\sin 2x\,dx;\ [y=2x]\rightarrow\frac{1}{2}\int e^{-y}\sin y\,dy=I;\ [u=\sin y,\ du=\cos y\,dy;\ dv=e^{-y}\,dy,\ v=-e^{-y}]$

$\Rightarrow I=\frac{1}{2}\left(-e^{-y}\sin y+\int e^{-y}\cos y\,dy\right)[u=\cos u,\ du=-\sin y;\ dv=e^{-y}\,dy,\ v=-e^{-y}]$

$\Rightarrow I=-\frac{1}{2}e^{-y}\sin y+\frac{1}{2}\left(-e^{-y}\cos y-\int\left(-e^{-y}\right)(-\sin y)\,dy\right)=-\frac{1}{2}e^{-y}(\sin y+\cos y)-I+C'$

$\Rightarrow 2I=-\frac{1}{2}e^{-y}(\sin y+\cos y)+C'\Rightarrow I=-\frac{1}{4}e^{-y}(\sin y+\cos y)+C=-\frac{e^{-2x}}{4}(\sin 2x+\cos 2x)+C$, where $C=\frac{C'}{2}$

25. $\int e^{\sqrt{3s+9}}\,ds;\ \begin{bmatrix}3s+9=x^2\\ ds=\frac{2}{3}x\,dx\end{bmatrix}\rightarrow\int e^{x}\cdot\frac{2}{3}x\,dx=\frac{2}{3}\int xe^{x}\,dx;\ [u=x,\ du=dx;\ dv=e^{x}\,dx,\ v=e^{x}];$

$\frac{2}{3}\int xe^{x}\,dx=\frac{2}{3}\left(xe^{x}-\int e^{x}\,dx\right)=\frac{2}{3}\left(xe^{x}-e^{x}\right)+C=\frac{2}{3}\left(\sqrt{3s+9}\,e^{\sqrt{3s+9}}-e^{\sqrt{3s+9}}\right)+C$

26. $u=x,\ du=dx;\ dv=\sqrt{1-x}\,dx,\ v=-\frac{2}{3}\sqrt{(1-x)^3};$

$\int_0^1 x\sqrt{1-x}\,dx=\left[-\frac{2}{3}\sqrt{(1-x)^3}\,x\right]_0^1+\frac{2}{3}\int_0^1\sqrt{(1-x)^3}\,dx=\frac{2}{3}\left[-\frac{2}{5}(1-x)^{5/2}\right]_0^1=\frac{4}{15}$

27. $u=x,\ du=dx;\ dv=\tan^2 x\,dx,\ v=\int\tan^2 x\,dx=\int\frac{\sin^2 x}{\cos^2 x}\,dx=\int\frac{1-\cos^2 x}{\cos^2 x}\,dx=\int\frac{dx}{\cos^2 x}-\int dx$

$=\tan x-x;\ \int_0^{\pi/3}x\tan^2 x\,dx=[x(\tan x-x)]_0^{\pi/3}-\int_0^{\pi/3}(\tan x-x)\,dx=\frac{\pi}{3}\left(\sqrt{3}-\frac{\pi}{3}\right)+\left[\ln|\cos x|+\frac{x^2}{2}\right]_0^{\pi/3}$

$= \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \ln \frac{1}{2} + \frac{\pi^2}{18} = \frac{\pi\sqrt{3}}{3} - \ln 2 - \frac{\pi^2}{18}$

28. $u = \ln(x + x^2)$, $du = \frac{(2x+1)\,dx}{x + x^2}$; $dv = dx$, $v = x$; $\int \ln(x + x^2)\,dx = x \ln(x + x^2) - \int \frac{2x+1}{x(x+1)} \cdot x\,dx$

$= x \ln(x + x^2) - \int \frac{(2x+1)\,dx}{x+1} = x \ln(x + x^2) - \int \frac{2(x+1) - 1}{x+1}\,dx = x \ln(x + x^2) - 2x + \ln|x+1| + C$

29. $\int \sin(\ln x)\,dx; \begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \\ dx = e^u\,du \end{bmatrix} \rightarrow \int (\sin u)\,e^u\,du$. From Exercise 21, $\int (\sin u)\,e^u\,du = e^u\left(\frac{\sin u - \cos u}{2}\right) + C$

$= \frac{1}{2}[-x \cos(\ln x) + x \sin(\ln x)] + C$

30. $\int z(\ln z)^2\,dz; \begin{bmatrix} u = \ln z \\ du = \frac{1}{z}\,dz \\ dz = e^u\,du \end{bmatrix} \rightarrow \int e^u \cdot u^2 \cdot e^u\,du = \int e^{2u} \cdot u^2\,du;$

e^{2u}

$u^2 \xrightarrow{(+)} \frac{1}{2}e^{2u}$

$2u \xrightarrow{(-)} \frac{1}{4}e^{2u}$

$2 \xrightarrow{(+)} \frac{1}{8}e^{2u}$

0

$\int u^2 e^{2u}\,du = \frac{u^2}{2}e^{2u} - \frac{u}{2}e^{2u} + \frac{1}{4}e^{2u} + C = \frac{e^{2u}}{4}[2u^2 - 2u + 1] + C$

$= \frac{z^2}{4}[2(\ln z)^2 - 2\ln z + 1] + C$

31. $y = \int x^2 e^{4x}\,dx$

Let $u = x^2$ $\qquad dv = e^{4x}\,dx$

$du = 2x\,dx$ $\qquad v = \frac{1}{4}e^{4x}$

$y = (x^2)\left(\frac{1}{4}e^{4x}\right) - \int \left(\frac{1}{4}e^{4x}\right)(2x\,dx)$

$= \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\int xe^{4x}\,dx$

Let $u = x$ $\qquad dv = e^{4x}\,dx$

$du = dx$ $\qquad v = \frac{1}{4}e^{4x}$

$y = \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\left[(x)\left(\frac{1}{4}e^{4x}\right) - \int \left(\frac{1}{4}e^{4x}\right)dx\right]$

$y = \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + C$

$y = \left(\frac{x^2}{4} - \frac{x}{8} + \frac{1}{32}\right)e^{4x} + C$

32. $y = \int x^2 \ln x \, dx$

Let $u = \ln x$ $\qquad dv = x^2 \, dx$

$du = \frac{1}{x} \, dx$ $\qquad v = \frac{1}{3}x^3$

$y = (\ln x)\left(\frac{1}{3}x^3\right) - \int \left(\frac{1}{3}x^3\right)\left(\frac{1}{x} \, dx\right)$

$y = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx$

$y = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

33. Let $w = \sqrt{\theta}$. Then $dw = \frac{d\theta}{2\sqrt{\theta}}$, so $d\theta = 2\sqrt{\theta} \, dw = 2w \, dw$.

$\int \sin \sqrt{\theta} \, d\theta = \int (\sin w)(2w \, dw) = 2\int w \sin w \, dw$

Let $u = w$ $\qquad dv = \sin w \, dw$

$du = dw$ $\qquad v = -\cos w$

$$\int w \sin w \, dw = -w \cos w + \int \cos w \, dw$$
$$= -w \cos w + \sin w + C$$

$$\int \sin \sqrt{\theta} \, d\theta = 2\int w \sin w \, dw$$
$$= -2w \cos w + 2 \sin w + C$$
$$= -2\sqrt{\theta} \cos \sqrt{\theta} + 2 \sin \sqrt{\theta} + C$$

34. $y = \int \theta \sec \theta \tan \theta \, d\theta$

Let $u = \theta$ $\qquad dv = \sec \theta \tan \theta \, d\theta$

$du = d\theta$ $\qquad v = \sec \theta$

$y = \theta \sec \theta - \int \sec \theta \, d\theta$

$y = \theta \sec \theta - \ln|\sec \theta + \tan\theta| + C$

35. (a) $u = x$, $du = dx$; $dv = \sin x \, dx$, $v = -\cos x$;

$$S_1 = \int_0^{\pi} x \sin x \, dx = [-x \cos x]_0^{\pi} + \int_0^{\pi} \cos x \, dx = \pi + [\sin x]_0^{\pi} = \pi$$

(b) $S_2 = -\int_{\pi}^{2\pi} x \sin x \, dx = \left[[-x \cos x]_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \cos x \, dx \right] = -\left[-3\pi + [\sin x]_{\pi}^{2\pi}\right] = 3\pi$

(c) $S_3 = \int_{2\pi}^{3\pi} x \sin x \, dx = [-x \cos x]_{2\pi}^{3\pi} + \int_{2\pi}^{3\pi} \cos x \, dx = 5\pi + [\sin x]_{2\pi}^{3\pi} = 5\pi$

(d) $S_{n+1} = (-1)^{n+1} \int_{n\pi}^{(n+1)\pi} x \sin x \, dx = (-1)^{n+1}\left[[-x \cos x]_{n\pi}^{(n+1)\pi} + [\sin x]_{n\pi}^{(n+1)\pi}\right]$

$= (-1)^{n+1}[-(n+1)\pi(-1)^n + n\pi(-1)^{n+1}] + 0 = (2n+1)\pi$

36. (a) $u = x$, $du = dx$; $dv = \cos x \, dx$, $v = \sin x$;

$S_1 = -\int_{\pi/2}^{3\pi/2} x \cos x \, dx = -\left[[x \sin x]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} \sin x \, dx \right] = -\left(-\frac{3\pi}{2} - \frac{\pi}{2}\right) - [\cos x]_{\pi/2}^{3\pi/2} = 2\pi$

(b) $S_2 = \int_{3\pi/2}^{5\pi/2} x \cos x \, dx = [x \sin x]_{3\pi/2}^{5\pi/2} - \int_{3\pi/2}^{5\pi/2} \sin x \, dx = \left[\frac{5\pi}{2} - \left(-\frac{3\pi}{2}\right)\right] - [\cos x]_{3\pi/2}^{5\pi/2} = 4\pi$

(c) $S_3 = -\int_{5\pi/2}^{7\pi/2} x \cos x \, dx = -\left[[x \sin x]_{5\pi/2}^{7\pi/2} - \int_{5\pi/2}^{7\pi/2} \sin x \, dx \right] = -\left(-\frac{7\pi}{2} - \frac{5\pi}{2}\right) - [\cos x]_{5\pi/2}^{7\pi/2} = 6\pi$

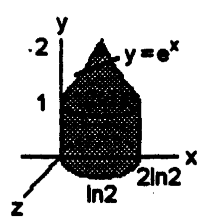
(d) $S_n = (-1)^n \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} x \cos x \, dx = (-1)^n \left[[x \sin x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} - \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} \sin x \, dx \right]$

$= (-1)^n \left[\frac{(2n+1)\pi}{2}(-1)^n - \frac{(2n-1)\pi}{2}(-1)^{n-1}\right] - [\cos x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} = \frac{1}{2}(2n\pi + \pi + 2n\pi - \pi) = 2n\pi$

37. $V = \int_0^{\ln 2} 2\pi(\ln 2 - x) e^x \, dx = 2\pi \ln 2 \int_0^{\ln 2} e^x \, dx - 2\pi \int_0^{\ln 2} xe^x \, dx$

$= (2\pi \ln 2)[e^x]_0^{\ln 2} - 2\pi\left([xe^x]_0^{\ln 2} - \int_0^{\ln 2} e^x \, dx\right)$

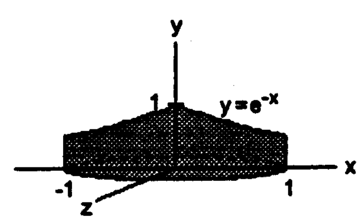
$= 2\pi \ln 2 - 2\pi\left(2 \ln 2 + [e^x]_0^{\ln 2}\right) = -2\pi \ln 2 + 2 = 2\pi(1 - \ln 2)$



38. (a) $V = \int_0^1 2\pi x e^{-x} \, dx = 2\pi\left([-xe^{-x}]_0^1 + \int_0^1 e^{-x} \, dx\right)$

$= 2\pi\left(-\frac{1}{e} + [-e^{-x}]_0^1\right) = 2\pi\left(-\frac{1}{e} - \frac{1}{e} + 1\right)$

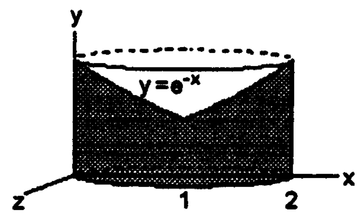
$= 2\pi - \frac{4\pi}{e}$



(b) $V = \int_0^1 2\pi(1-x)e^{-x}\,dx$; $u = 1-x$, $du = -dx$; $dv = e^{-x}\,dx$,

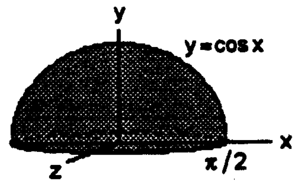
$v = -e^{-x}$; $V = 2\pi\left[\left[(1-x)\left(-e^{-x}\right)\right]_0^1 - \int_0^1 e^{-x}\,dx\right]$

$= 2\pi\left[[0 - 1(-1)] + \left[e^{-x}\right]_0^1\right] = 2\pi\left(1 + \frac{1}{e} - 1\right) = \frac{2\pi}{e}$



39. (a) $V = \int_0^{\pi/2} 2\pi x \cos x\,dx = 2\pi\left([x \sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x\,dx\right)$

$= 2\pi\left(\frac{\pi}{2} + [\cos x]_0^{\pi/2}\right) = 2\pi\left(\frac{\pi}{2} + 0 - 1\right) = \pi(\pi - 2)$



(b) $V = \int_0^{\pi/2} 2\pi\left(\frac{\pi}{2} - x\right)\cos x\,dx$; $u = \frac{\pi}{2} - x$, $du = -dx$; $dv = \cos x\,dx$, $v = \sin x$;

$V = 2\pi\left[\left(\frac{\pi}{2} - x\right)\sin x\right]_0^{\pi/2} + 2\pi\int_0^{\pi/2} \sin x\,dx = 0 + 2\pi[-\cos x]_0^{\pi/2} = 2\pi(0+1) = 2\pi$

40. (a) $V = \int_0^{\pi} 2\pi x(x \sin x)\,dx$;

$$\begin{array}{lcl} & & \sin x \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$

$\Rightarrow V = 2\pi\int_0^{\pi} x^2 \sin x\,dx = 2\pi\left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^{\pi} = 2\pi\left(\pi^2 - 4\right)$

(b) $V = \int_0^{\pi} 2\pi(\pi - x)x \sin x\,dx = 2\pi^2\int_0^{\pi} x \sin x\,dx - 2\pi\int_0^{\pi} x^2 \sin x\,dx = 2\pi^2\left[-x\cos x + \sin x\right]_0^{\pi} - \left(2\pi^3 - 8\pi\right)$

$= 8\pi$

41. $av(y) = \frac{1}{2\pi}\int_0^{2\pi} 2e^{-t}\cos t\,dt = \frac{1}{\pi}\left[e^{-t}\left(\frac{\sin t - \cos t}{2}\right)\right]_0^{2\pi}$

(see Exercise 22) $\Rightarrow av(y) = \frac{1}{2\pi}\left(1 - e^{-2\pi}\right)$

42. $av(y) = \frac{1}{2\pi}\int_0^{2\pi} 4e^{-t}(\sin t - \cos t)\,dt = \frac{2}{\pi}\int_0^{2\pi} e^{-t}\sin t\,dt - \frac{2}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt$

$= \frac{2}{\pi}\left[e^{-t}\left(\frac{-\sin t - \cos t}{2}\right) - e^{-t}\left(\frac{\sin t - \cos t}{2}\right)\right]_0^{2\pi} = \frac{2}{\pi}\left[-e^{-t}\sin t\right]_0^{2\pi} = 0$

43. Let $u = x^n$ $\quad dv = \cos x\, dx$

$du = nx^{n-1}\, dx \quad v = \sin x$

$$\int x^n \cos x\, dx = x^n \sin x - \int (\sin x)(nx^{n-1}\, dx) = x^n \sin x - n\int x^{n-1} \sin x\, dx$$

44. Let $u = x^n$ $\quad dv = \sin x\, dx$

$du = nx^{n-1}\, dx \quad v = -\cos x$

$$\int x^n \sin x\, dx = (x^n)(-\cos x) - \int (-\cos x)(nx^{n-1}\, dx) = -x^n \cos x + n\int x^{n-1} \cos x\, dx$$

45. Let $u = x^n$ $\quad dv = e^{ax}\, dx$

$du = nx^{n-1}\, dx \quad v = \frac{1}{a}e^{ax}$

$$\int x^n e^{ax}\, dx = (x^n)\left(\tfrac{1}{a}e^{ax}\right) - \int \left(\tfrac{1}{a}e^{ax}\right)(nx^{n-1}\, dx) = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\, dx,\ a \neq 0$$

46. Let $u = (\ln x)^n$ $\quad dv = dx$

$du = \frac{n(\ln x)^{n-1}}{x}\, dx \quad v = x$

$$\int (\ln x)^n\, dx = x(\ln x)^n - \int x\left[\frac{n(\ln x)^{n-1}}{x}\right] dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx$$

47. (a) Let $y = f^{-1}(x)$. Then $x = f(y)$, so $dx = f'(y)\, dy$.

Hence, $\int f^{-1}(x)\, dx = \int (y)[f'(y)\, dy] = \int yf'(y)\, dy$

(b) Let $u = y$ $\quad dv = f'(y)\, dy$

$du = dy \quad v = f(y)$

$$\int yf'(y)\, dy = yf(y) - \int f(y)\, dy = f^{-1}(x)(x) - \int f(y)\, dy$$

Hence, $\int f^{-1}(x)\, dx = \int yf'(y)\, dy = xf^{-1}(x) - \int f(y)\, dy.$

48. Let $u = f^{-1}(x)$ $\quad dv = dx$

$du = \left(\frac{d}{dx}f^{-1}(x)\right) dx \quad v = x$

$$\int f^{-1}(x)\, dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right) dx$$

49. (a) Using $y = f^{-1}(x) = \sin^{-1} x$ and $f(y) = \sin y$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, we have:

$$\int \sin^{-1} x\, dx = x \sin^{-1} x - \int \sin y\, dy = x \sin^{-1} x + \cos y + C = x \sin^{-1} x + \cos(\sin^{-1} x) + C$$

(b) $\int \sin^{-1} x\, dx = x \sin^{-1} x - \int x\left(\frac{d}{dx} \sin^{-1} x\right) dx = x \sin^{-1} x - \int x \frac{1}{\sqrt{1-x^2}}\, dx$

$u = 1 - x^2,\ du = -2x\, dx = x \sin^{-1} x + \frac{1}{2} \int u^{-1/2}\, du = x \sin^{-1} x + u^{1/2} + C = x \sin^{-1} x + \sqrt{1-x^2} + C$

(c) $\cos(\sin^{-1} x) = \sqrt{1-x^2}$

50. (a) Using $y = f^{-1}(x) = \tan^{-1} x$ and $f(y) = \tan y,\ -\frac{\pi}{2} < y < \frac{\pi}{2}$, we have:

$\int \tan^{-1} x\, dx = x \tan^{-1} x - \int \tan y\, dy = x \tan^{-1} x - \ln|\sec y| + C = x \tan^{-1} x + \ln|\cos y| + C$

$= x \tan^{-1} x + \ln\left|\cos(\tan^{-1}(x)\right|dx + C$

(b) $\int \tan^{-1} x\, dx = x \tan^{-1} x - \int x\left(\frac{d}{dx} \tan^{-1} x\right) dx = x \tan^{-1} x - \int x\left(\frac{1}{1+x^2}\right) dx$

$u = 1 + x^2,\ du = 2x\, dx = x \tan^{-1} x - \frac{1}{2} \int u^{-1}\, du = x \tan^{-1} x - \frac{1}{2} \ln|u| + C = x \tan^{-1} x - \frac{1}{2} \ln\left(1+x^2\right) + C$

(c) $\ln\left|\cos(\tan^{-1} x)\right| = \ln\left|\frac{1}{\sqrt{1+x^2}}\right| = -\frac{1}{2} \ln\left(1+x^2\right)$

51. (a) Using $y = f^{-1}(x) = \cos^{-1} x$ and $f(y) = \cos x,\ 0 \le x \le \pi$, we have:

$\int \cos^{-1} x\, dx = x \cos^{-1} x - \int \cos y\, dy = x \cos^{-1} x - \sin y + C = x \cos^{-1} x - \sin(\cos^{-1} x) + C$

(b) $\int \cos^{-1} x\, dx = x \cos^{-1} x - \int x\left(\frac{d}{dx} \cos^{-1} x\right) dx = x \cos^{-1} x - \int x\left(-\frac{1}{\sqrt{1-x^2}}\right) dx$

$u = 1 - x^2,\ du = -2x\, dx = x \cos^{-1} x - \frac{1}{2} \int u^{-1/2}\, du = x \cos^{-1} x - u^{1/2} + C = x \cos^{-1} x - \sqrt{1-x^2} + C$

(c) $\sin(\cos^{-1} x) = \sqrt{1-x^2}$

52. (a) Using $y = f^{-1}(x) = \log_2 x$ and $f(y) = 2^y$, we have

$\int \log_2 x\, dx = x \log_2 x - \int 2^y\, dy = x \log_2 x - \frac{2^y}{\ln 2} + C = x \log_2 x - \frac{1}{\ln 2} 2^{\log_2 x}$

(b) $\int \log_2 x\, dx = x \log_2 x - \int x\left(\frac{d}{dx} \log_2 x\right) dx = x \log_2 x - \int x\left(\frac{1}{x \ln 2}\right) dx = x \log_2 x - \int \frac{dx}{\ln 2}$

$= x \log_2 x - \left(\frac{1}{\ln 2}\right) x + C$

(c) $2^{\log_2 x} = x$

7.3 PARTIAL FRACTIONS

1. $\frac{5x-13}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} \Rightarrow 5x - 13 = A(x-2) + B(x-3) = (A+B)x - (2A+3B)$

$\Rightarrow \left.\begin{array}{r} A + B = 5 \\ 2A + 3B = 13 \end{array}\right\} \Rightarrow -B = (10-13) \Rightarrow B = 3 \Rightarrow A = 2$; thus, $\frac{5x-13}{(x-3)(x-2)} = \frac{2}{x-3} + \frac{3}{x-2}$

2. $\frac{5x-7}{x^2-3x+2} = \frac{5x-7}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 5x-7 = A(x-1)+B(x-2) = (A+B)x-(A+2B)$

$\Rightarrow \left.\begin{array}{c} A+B=5 \\ A+2B=7 \end{array}\right\} \Rightarrow B=2 \Rightarrow A=3$; thus, $\frac{5x-7}{x^2-3x+2} = \frac{3}{x-2} + \frac{2}{x-1}$

3. $\frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow x+4 = A(x+1)+B = Ax+(A+B) \Rightarrow \left.\begin{array}{c} A=1 \\ A+B=4 \end{array}\right\} \Rightarrow A=1$ and $B=3$;

thus, $\frac{x+4}{(x+1)^2} = \frac{1}{x+1} + \frac{3}{(x+1)^2}$

4. $\frac{2x+2}{x^2-2x+1} = \frac{2x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 2x+2 = A(x-1)+B = Ax+(-A+B) \Rightarrow \left.\begin{array}{c} A=2 \\ -A+B=2 \end{array}\right\}$

$\Rightarrow A=2$ and $B=4$; thus, $\frac{2x+2}{x^2-2x+1} = \frac{2}{x-1} + \frac{4}{(x-1)^2}$

5. $\frac{z+1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} \Rightarrow z+1 = Az(z-1)+B(z-1)+Cz^2 \Rightarrow z+1 = (A+C)z^2+(-A+B)z-B$

$\Rightarrow \left.\begin{array}{c} A+C=0 \\ -A+B=1 \\ -B=1 \end{array}\right\} \Rightarrow B=-1 \Rightarrow A=-2 \Rightarrow C=2$; thus, $\frac{z+1}{z^2(z-1)} = \frac{-2}{z} + \frac{-1}{z^2} + \frac{2}{z-1}$

6. $\frac{z}{z^3-z^2-6z} = \frac{1}{z^2-z-6} = \frac{1}{(z-3)(z+2)} = \frac{A}{z-3} + \frac{B}{z+2} \Rightarrow 1 = A(z+2)+B(z-3) = (A+B)z+(2A-3B)$

$\Rightarrow \left.\begin{array}{c} A+B=0 \\ 2A-3B=1 \end{array}\right\} \Rightarrow -5B=1 \Rightarrow B=-\frac{1}{5} \Rightarrow A=\frac{1}{5}$; thus, $\frac{z}{z^3-z^2-6z} = \frac{\frac{1}{5}}{z-3} + \frac{-\frac{1}{5}}{z+2}$

7. $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{5t+2}{t^2-5t+6}$ (after long division); $\frac{5t+2}{t^2-5t+6} = \frac{5t+2}{(t-3)(t-2)} = \frac{A}{t-3} + \frac{B}{t-2}$

$\Rightarrow 5t+2 = A(t-2)+B(t-3) = (A+B)t+(-2A-3B) \Rightarrow \left.\begin{array}{c} A+B=5 \\ -2A-3B=2 \end{array}\right\} \Rightarrow -B = (10+2) = 12$

$\Rightarrow B=-12 \Rightarrow A=17$; thus, $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{17}{t-3} + \frac{-12}{t-2}$

8. $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^2(t^2+9)}$ (after long division); $\frac{-9t^2+9}{t^2(t^2+9)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+9}$

$\Rightarrow -9t^2+9 = At(t^2+9)+B(t^2+9)+(Ct+D)t^2 = (A+C)t^3+(B+D)t^2+9At+9B$

$\Rightarrow \left.\begin{array}{c} A+C=0 \\ B+D=-9 \\ 9A=0 \\ 9B=9 \end{array}\right\} \Rightarrow A=0 \Rightarrow C=0;\ B=1 \Rightarrow D=-10$; thus, $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{1}{t^2} + \frac{-10}{t^2+9}$

9. $\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \Rightarrow 1 = A(1+x)+B(1-x);\ x=1 \Rightarrow A=\frac{1}{2};\ x=-1 \Rightarrow B=\frac{1}{2}$;

$\int \frac{dx}{1-x^2} = \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x} = \frac{1}{2}[\ln|1+x| - \ln|1-x|] + C$

10. $\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow 1 = A(x+2) + Bx;\ x = 0 \Rightarrow A = \frac{1}{2};\ x = -2 \Rightarrow B = -\frac{1}{2};$

$\int \frac{dx}{x^2+2x} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2} = \frac{1}{2}[\ln|x| - \ln|x+2|] + C$

11. $\frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1} \Rightarrow x+4 = A(x-1) + B(x+6);\ x = 1 \Rightarrow B = \frac{5}{7};\ x = -6 \Rightarrow A = \frac{-2}{-7} = \frac{2}{7};$

$\int \frac{x+4}{x^2+5x-6}\,dx = \frac{2}{7}\int \frac{dx}{x+6} + \frac{5}{7}\int \frac{dx}{x-1} = \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C = \frac{1}{7}\ln\left|(x+6)^2(x-1)^5\right| + C$

12. $\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3} \Rightarrow 2x+1 = A(x-3) + B(x-4);\ x = 3 \Rightarrow B = \frac{7}{-1} = -7;\ x = 4 \Rightarrow A = \frac{9}{1} = 9;$

$\int \frac{2x+1}{x^2-7x+12}\,dx = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3} = 9\ln|x-4| - 7\ln|x-3| + C = \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C$

13. $\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1} \Rightarrow y = A(y+1) + B(y-3);\ y = -1 \Rightarrow B = \frac{-1}{-4} = \frac{1}{4};\ y = 3 \Rightarrow A = \frac{3}{4};$

$\int_4^8 \frac{y\,dy}{y^2-2y-3} = \frac{3}{4}\int_4^8 \frac{dy}{y-3} + \frac{1}{4}\int_4^8 \frac{dy}{y+1} = \left[\frac{3}{4}\ln|y-3| + \frac{1}{4}\ln|y+1|\right]_4^8 = \left(\frac{3}{4}\ln 5 + \frac{1}{4}\ln 9\right) - \left(\frac{3}{4}\ln 1 + \frac{1}{4}\ln 5\right)$

$= \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3 = \frac{\ln 15}{2}$

14. $\frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow y+4 = A(y+1) + By;\ y = 0 \Rightarrow A = 4;\ y = -1 \Rightarrow B = \frac{3}{-1} = -3;$

$\int_{1/2}^1 \frac{y+4}{y^2+y}\,dy = 4\int_{1/2}^1 \frac{dy}{y} - 3\int_{1/2}^1 \frac{dy}{y+1} = [4\ln|y| - 3\ln|y+1|]_{1/2}^1 = (4\ln 1 - 3\ln 2) - \left(4\ln\frac{1}{2} - 3\ln\frac{3}{2}\right)$

$= \ln\frac{1}{8} - \ln\frac{1}{16} + \ln\frac{27}{8} = \ln\left(\frac{27}{8}\cdot\frac{1}{8}\cdot 16\right) = \ln\frac{27}{4}$

15. $\frac{1}{t^3+t^2-2t} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1} \Rightarrow 1 = A(t+2)(t-1) + Bt(t-1) + Ct(t+2);\ t = 0 \Rightarrow A = -\frac{1}{2};\ t = -2$

$\Rightarrow B = \frac{1}{6};\ t = 1 \Rightarrow C = \frac{1}{3};\ \int \frac{dt}{t^3+t^2-2t} = -\frac{1}{2}\int \frac{dt}{t} + \frac{1}{6}\int \frac{dt}{t+2} + \frac{1}{3}\int \frac{dt}{t-1}$

$= -\frac{1}{2}\ln|t| + \frac{1}{6}\ln|t+2| + \frac{1}{3}\ln|t-1| + C$

16. $\frac{x+3}{2x^3-8x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \Rightarrow \frac{1}{2}(x+3) = A(x+2)(x-2) + Bx(x-2) = Cx(x+2);\ x = 0 \Rightarrow A = \frac{3}{-8};\ x = -2$

$\Rightarrow B = \frac{1}{16};\ x = 2 \Rightarrow C = \frac{5}{16};\ \int \frac{x+3}{2x^3-8x}\,dx = -\frac{3}{8}\int \frac{dx}{x} + \frac{1}{16}\int \frac{dx}{x+2} + \frac{5}{16}\int \frac{dx}{x-2}$

$= -\frac{3}{8}\ln|x| + \frac{1}{16}\ln|x+2| + \frac{5}{16}\ln|x-2| + C = \frac{1}{16}\ln\left|\frac{(x-2)^5(x+2)}{x^6}\right| + C$

17. $\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}$ (after long division); $\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+3 = A(x+1)+B$

$= Ax + (A+B) \Rightarrow A = 3,\ A+B = 2 \Rightarrow A = 3,\ B = -1;\ \int_0^1 \frac{x^3\,dx}{x^2+2x+1}$

$= \int_0^1 (x-2)\,dx + 3\int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2} = \left[\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right]_0^1$

$= \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) = 3\ln 2 - 2$

18. $\frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x+2}{(x-1)^2}$ (after long division); $\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 3x-2 = A(x-1)+B$

$= Ax + (-A+B) \Rightarrow A = 3,\ -A+B = -2 \Rightarrow A = 3,\ B = 1;\ \int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$

$= \int_{-1}^0 (x+2)\,dx + 3\int_{-1}^0 \frac{dx}{x-1} + \int_{-1}^0 \frac{dx}{(x-1)^2} = \left[\frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1}\right]_{-1}^0$

$= \left(0 + 0 + 3\ln 1 - \frac{1}{(-1)}\right) - \left(\frac{1}{2} - 2 + 3\ln 2 - \frac{1}{(-2)}\right) = 2 - 3\ln 2$

19. $\frac{1}{(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+1)^2} + \frac{D}{(x-1)^2} \Rightarrow 1 = A(x+1)(x-1)^2 + B(x-1)(x+1)^2 + C(x-1)^2 + D(x+1)^2;$

$x = -1 \Rightarrow C = \frac{1}{4};\ x = 1 \Rightarrow D = \frac{1}{4}$; coefficient of $x^3 = A + B \Rightarrow A + B = 0$; constant $= A - B + C + D$

$\Rightarrow A - B + C + D = 1 \Rightarrow A - B = \frac{1}{2}$; thus, $A = \frac{1}{4} \Rightarrow B = -\frac{1}{4};\ \int \frac{dx}{(x^2-1)^2}$

$= \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x-1} + \frac{1}{4}\int \frac{dx}{(x+1)^2} + \frac{1}{4}\int \frac{dx}{(x-1)^2} = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2(x^2-1)} + C$

20. $\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \Rightarrow x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1);\ x = -1$

$\Rightarrow C = -\frac{1}{2};\ x = 1 \Rightarrow A = \frac{1}{4}$; coefficient of $x^2 = A + B \Rightarrow A + B = 1 \Rightarrow B = \frac{3}{4};\ \int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$

$= \frac{1}{4}\int \frac{dx}{x-1} + \frac{3}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{(x+1)^2} = \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2(x+1)} + C$

$= \frac{\ln\left|(x-1)(x+1)^3\right|}{4} + \frac{1}{2(x+1)} + C$

21. $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + (Bx+C)(x+1);\ x = -1 \Rightarrow A = \frac{1}{2}$; coefficient of x^2

$= A + B \Rightarrow A + B = 0 \Rightarrow B = -\frac{1}{2}$; constant $= A + C \Rightarrow A + C = 1 \Rightarrow C = \frac{1}{2};\ \int_0^1 \frac{dx}{(x+1)(x^2+1)}$

$= \frac{1}{2}\int_0^1 \frac{dx}{x+1} + \frac{1}{2}\int_0^1 \frac{(-x+1)}{x^2+1}\,dx = \left[\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x\right]_0^1$

$= \left(\frac{1}{2}\ln 2 - \frac{1}{4}\ln 2 + \frac{1}{2}\tan^{-1}1\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{4}\ln 1 + \frac{1}{2}\tan^{-1}0\right) = \frac{1}{4}\ln 2 + \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{(\pi + 2\ln 2)}{8}$

22. $\frac{3t^2+t+4}{t^3+t} = \frac{A}{t} + \frac{Bt+C}{t^2+1} \Rightarrow 3t^2+t+4 = A(t^2+1) + (Bt+C)t$; $t = 0 \Rightarrow A = 4$; coefficient of t^2

$= A + B \Rightarrow A + B = 3 \Rightarrow B = -1$; coefficient of $t = C \Rightarrow C = 1$; $\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt$

$= 4\int_1^{\sqrt{3}} \frac{dt}{t} + \int_1^{\sqrt{3}} \frac{(-t+1)}{t^2+1}\,dt = \left[4\ln|t| - \frac{1}{2}\ln(t^2+1) + \tan^{-1}t\right]_1^{\sqrt{3}}$

$= \left(4\ln\sqrt{3} - \frac{1}{2}\ln 4 + \tan^{-1}\sqrt{3}\right) - \left(4\ln 1 - \frac{1}{2}\ln 2 + \tan^{-1}1\right) = 2\ln 3 - \ln 2 + \frac{\pi}{3} + \frac{1}{2}\ln 2 + \frac{\pi}{4}$

$= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{12} = \ln\left(\frac{9}{\sqrt{2}}\right) + \frac{\pi}{12}$

23. $\frac{y^2+2y+1}{(y^2+1)^2} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2} \Rightarrow y^2+2y+1 = (Ay+B)(y^2+1) + Cy + D$

$= Ay^3 + By^2 + (A+C)y + (B+D) \Rightarrow A = 0,\ B = 1;\ A + C = 2 \Rightarrow C = 2;\ B + D = 1 \Rightarrow D = 0;$

$\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy = \int \frac{1}{y^2+1}\,dy + 2\int \frac{y}{(y^2+1)^2}\,dy = \tan^{-1}y - \frac{1}{y^2+1} + C$

24. $\frac{8x^2+8x+2}{(4x^2+1)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{(4x^2+1)^2} \Rightarrow 8x^2+8x+2 = (Ax+B)(4x^2+1) + Cx + D$

$= 4Ax^3 + 4Bx^2 + (A+C)x + (B+D);\ A = 0,\ B = 2;\ A + C = 8 \Rightarrow C = 8;\ B + D = 2 \Rightarrow D = 0;$

$\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx = 2\int \frac{dx}{4x^2+1} + 8\int \frac{x\,dx}{(4x^2+1)^2} = \tan^{-1}2x - \frac{1}{4x^2+1} + C$

25. $\frac{2s+2}{(s^2+1)(s-1)^3} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{(s-1)^3} \Rightarrow 2s+2$

$= (As+B)(s-1)^3 + C(s^2+1)(s-1)^2 + D(s^2+1)(s-1) + E(s^2+1)$

$= [As^4 + (-3A+B)s^3 + (3A-3B)s^2 + (-A+3B)s - B] + C(s^4 - 2s^3 + 2s^2 - 2s + 1) + D(s^3 - s^2 + s - 1)$

$\quad + E(s^2+1)$

$= (A+C)s^4 + (-3A+B-2C+D)s^3 + (3A-3B+2C-D+E)s^2 + (-A+3B-2C+D)s + (-B+C-D+E)$

$$\Rightarrow \left.\begin{array}{l} A + C = 0 \\ -3A + B - 2C + D = 0 \\ 3A - 3B + 2C - D + E = 0 \\ -A + 3B - 2C + D = 2 \\ -B + C - D + E = 2 \end{array}\right\} \text{summing all equations} \Rightarrow 2E = 4 \Rightarrow E = 2;$$

summing eqs (2) and (3) $\Rightarrow -2B + 2 = 0 \Rightarrow B = 1$; summing eqs (3) and (4) $\Rightarrow 2A + 2 = 2 \Rightarrow A = 0$; $C = 0$ from eq (1); then $-1 + 0 - D + 2 = 2$ from eq (5) $\Rightarrow D = -1$;

$$\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds = \int \frac{ds}{s^2+1} - \int \frac{ds}{(s-1)^2} + 2\int \frac{ds}{(s-1)^3} = -(s-1)^{-2} + (s-1)^{-1} + \tan^{-1} s + C$$

26. $\frac{s^4+81}{s(s^2+9)^2} = \frac{A}{s} + \frac{Bs+C}{s^2+9} + \frac{Ds+E}{(s^2+9)^2} \Rightarrow s^4 + 81 = A(s^2+9)^2 + (Bs+C)s(s^2+9) + (Ds+E)s$

$= A(s^4 + 18s^2 + 81) + (Bs^4 + Cs^3 + 9Bs^2 + 9Cs) + Ds^2 + Es$

$= (A+B)s^4 + Cs^3 + (18A + 9B + D)s^2 + (9C + E)s + 81A \Rightarrow 81A = 81$ or $A = 1$; $A + B = 1 \Rightarrow B = 0$;

$C = 0$; $9C + E = 0 \Rightarrow E = 0$; $18A + 9B + D = 0 \Rightarrow D = -18$; $\int \frac{s^4+81}{s(s^2+9)^2}\,ds = \int \frac{ds}{s} - 18\int \frac{s\,ds}{(s^2+9)^2}$

$= \ln|s| + \frac{9}{(s^2+9)} + C$

27. $\frac{2\theta^3 + 5\theta^2 + 8\theta + 4}{(\theta^2+2\theta+2)^2} = \frac{A\theta+B}{\theta^2+2\theta+2} + \frac{C\theta+D}{(\theta^2+2\theta+2)^2} \Rightarrow 2\theta^3 + 5\theta^2 + 8\theta + 4 = (A\theta+B)(\theta^2+2\theta+2) + C\theta + D$

$= A\theta^3 + (2A+B)\theta^2 + (2A+2B+C)\theta + (2B+D) \Rightarrow A = 2$; $2A + B = 5 \Rightarrow B = 1$; $2A + 2B + C = 8 \Rightarrow C = 2$;

$2B + D = 4 \Rightarrow D = 2$; $\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta = \int \frac{2\theta+1}{(\theta^2+2\theta+2)}\,d\theta + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2}\,d\theta$

$= \int \frac{2\theta+2}{\theta^2+2\theta+2}\,d\theta - \int \frac{d\theta}{\theta^2+2\theta+2} + \int \frac{d(\theta^2+2\theta+2)}{(\theta^2+2\theta+2)^2} = \int \frac{d(\theta^2+2\theta+2)}{\theta^2+2\theta+2} - \int \frac{d\theta}{(\theta+1)^2+1} - \frac{1}{\theta^2+2\theta+2}$

$= \frac{-1}{\theta^2+2\theta+2} + \ln(\theta^2+2\theta+2) - \tan^{-1}(\theta+1) + C$

28. $\frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2+1)^3} = \frac{A\theta+B}{\theta^2+1} + \frac{C\theta+D}{(\theta^2+1)^2} + \frac{E\theta+F}{(\theta^2+1)^3} \Rightarrow \theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1$

$= (A\theta+B)(\theta^2+1)^2 + (C\theta+D)(\theta^2+1) + E\theta + F = (A\theta+B)(\theta^4+2\theta^2+1) + (C\theta^3 + D\theta^2 + C\theta + D) + E\theta + F$

$= (A\theta^5 + B\theta^4 + 2A\theta^3 + 2B\theta^2 + A\theta + B) + (C\theta^3 + D\theta^2 + C\theta + D) + E\theta + F$

$= A\theta^5 + B\theta^4 + (2A+C)\theta^3 + (2B+D)\theta^2 + (A+C+E)\theta + (B+D+F) \Rightarrow A = 0$; $B = 1$; $2A + C = -4$

$\Rightarrow C = -4$; $2B + D = 2 \Rightarrow D = 0$; $A + C + E = -3 \Rightarrow E = 1$; $B + D + F = 1 \Rightarrow F = 0$;

$$\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2+1)^3}\,d\theta = \int \frac{d\theta}{\theta^2+1} - 4\int \frac{\theta\,d\theta}{(\theta^2+1)^2} + \int \frac{\theta\,d\theta}{(\theta^2+1)^3} = \tan^{-1}\theta + 2(\theta^2+1)^{-1} - \tfrac{1}{4}(\theta^2+1)^{-2} + C$$

29. $\frac{2x^3 - 2x^2 + 1}{x^2 - x} = 2x + \frac{1}{x^2 - x} = 2x + \frac{1}{x(x-1)}$; $\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + Bx$; $x = 0 \Rightarrow A = -1$;

$x = 1 \Rightarrow B = 1$; $\int \frac{2x^3 - 2x^2 + 1}{x^2 - x} = \int 2x\,dx - \int \frac{dx}{x} + \int \frac{dx}{x-1} = x^2 - \ln|x| + \ln|x-1| + C = x^2 + \ln\left|\frac{x-1}{x}\right| + C$

30. $\frac{x^4}{x^2-1}=(x^2+1)+\frac{1}{x^2-1}=(x^2+1)+\frac{1}{(x+1)(x-1)}$; $\frac{1}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1} \Rightarrow 1=A(x-1)+B(x+1)$;

$x=-1 \Rightarrow A=-\frac{1}{2}$; $x=1 \Rightarrow B=\frac{1}{2}$; $\int \frac{x^4}{x^2-1}\,dx=\int (x^2+1)\,dx-\frac{1}{2}\int \frac{dx}{x+1}+\frac{1}{2}\int \frac{dx}{x-1}$

$=\frac{1}{3}x^3+x-\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|+C=\frac{x^3}{3}+x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$

31. $\frac{9x^3-3x+1}{x^3-x^2}=9+\frac{9x^2-3x+1}{x^2(x-1)}$ (after long division); $\frac{9x^2-3x+1}{x^2(x-1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}$

$\Rightarrow 9x^2-3x+1=Ax(x-1)+B(x-1)+Cx^2$; $x=1 \Rightarrow C=7$; $x=0 \Rightarrow B=-1$; $A+C=9 \Rightarrow A=2$;

$\int \frac{9x^3-3x+1}{x^3-x^2}\,dx=\int 9\,dx+2\int \frac{dx}{x}-\int \frac{dx}{x^2}+7\int \frac{dx}{x-1}=9x+2\ln|x|+\frac{1}{x}+7\ln|x-1|+C$

32. $\frac{16x^3}{4x^2-4x+1}=(4x+4)+\frac{12x-4}{4x^2-4x+1}$; $\frac{12x-4}{(2x-1)^2}=\frac{A}{2x-1}+\frac{B}{(2x-1)^2} \Rightarrow 12x-4=A(2x-1)+B$

$\Rightarrow A=6$; $-A+B=-4 \Rightarrow B=2$; $\int \frac{16x^3}{4x^2-4x+1}\,dx=4\int (x+1)\,dx+6\int \frac{dx}{2x-1}+2\int \frac{dx}{(2x-1)^2}$

$=2(x+1)^2+3\ln|2x-1|-\frac{1}{2x-1}+C_1=2x^2+4x+3\ln|2x-1|-(2x-1)^{-1}+C$, where $C=2+C_1$

33. $\frac{y^4+y^2-1}{y^3+y}=y-\frac{1}{y(y^2+1)}$; $\frac{1}{y(y^2+1)}=\frac{A}{y}+\frac{By+C}{y^2+1} \Rightarrow 1=A(y^2+1)+(By+C)y=(A+B)y^2+Cy+A$

$\Rightarrow A=1$; $A+B=0 \Rightarrow B=-1$; $C=0$; $\int \frac{y^4+y^2-1}{y^3+y}\,dy=\int y\,dy-\int \frac{dy}{y}+\int \frac{y\,dy}{y^2+1}$

$=\frac{y^2}{2}-\ln|y|+\frac{1}{2}\ln(1+y^2)+C$

34. $\frac{2y^4}{y^3-y^2+y-1}=2y+2+\frac{2}{y^3-y^2+y-1}$; $\frac{2}{y^3-y^2+y-1}=\frac{2}{(y^2+1)(y-1)}=\frac{A}{y-1}+\frac{By+C}{y^2+1}$

$\Rightarrow 2=A(y^2+1)+(By+C)(y-1)=(Ay^2+A)+(By^2+Cy-By-C)=(A+B)y^2+(-B+C)y+(A-C)$

$\Rightarrow A+B=0$, $-B+C=0$ or $C=B$, $A-C=A-B=2 \Rightarrow A=1$, $B=-1$, $C=-1$;

$\int \frac{2y^4}{y^3-y^2+y-1}\,dy=2\int (y+1)\,dy+\int \frac{dy}{y-1}-\int \frac{y}{y^2+1}\,dy-\int \frac{dy}{y^2+1}$

$=(y+1)^2+\ln|y-1|-\frac{1}{2}\ln(y^2+1)-\tan^{-1}y+C_1=y^2+2y+\ln|y-1|-\frac{1}{2}\ln(y^2+1)-\tan^{-1}y+C$,

where $C=C_1+1$

35. $\int \frac{e^t\,dt}{e^{2t}+3e^t+2}=[e^t=y]\int \frac{dy}{y^2+3y+2}=\int \frac{dy}{y+1}-\int \frac{dy}{y+2}=\ln\left|\frac{y+1}{y+2}\right|+C=\ln\left|\frac{e^t+1}{e^t+2}\right|+C$

36. $\int \frac{e^{4t}+2e^{2t}-e^t}{e^{2t}+1}\,dt$; $[e^t=y] \to \int \frac{y^3+2y-1}{y^2+1}\,dy=\int \left(y+\frac{y-1}{y^2+1}\right)dy=\frac{y^2}{2}+\frac{1}{2}\ln(y^2+1)-\tan^{-1}y+C$

$= \frac{1}{2}e^{2t} - \tan^{-1}(e^t) + \frac{1}{2}\ln(e^{2t}+1) + C$

37. $\int \frac{\cos y\, dy}{\sin^2 y + \sin y - 6}$; $[\sin y = t,\ \cos y\, dy = dt] \to \int \frac{dy}{t^2+t-6} = \frac{1}{5}\int\left(\frac{1}{t-2} - \frac{1}{t+3}\right)dt = \frac{1}{5}\ln\left|\frac{t-2}{t+3}\right| + C$

$= \frac{1}{5}\ln\left|\frac{\sin y - 2}{\sin y + 3}\right| + C$

38. $\int \frac{\sin\theta\, d\theta}{\cos^2\theta + \cos\theta - 2}$; $[\cos\theta = y] \to -\int \frac{dy}{y^2+y-2} = \frac{1}{3}\int \frac{dy}{y+2} - \frac{1}{3}\int \frac{dy}{y-1} = \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C = \frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right| + C$

$= \frac{1}{3}\ln\left|\frac{2+\cos\theta}{1-\cos\theta}\right| + C = -\frac{1}{3}\ln\left|\frac{\cos\theta-1}{\cos\theta+2}\right| + C$

39. $\int \frac{(x-2)^2\tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\,dx = \int \frac{\tan^{-1}(2x)}{4x^2+1}\,dx - 3\int \frac{x}{(x-2)^2}\,dx$

$= \frac{1}{2}\int \tan^{-1}(2x)\,d(\tan^{-1}(2x)) - 3\int \frac{dx}{x-2} - 6\int \frac{dx}{(x-2)^2} = \frac{(\tan^{-1}2x)^2}{4} - 3\ln|x-2| + \frac{6}{x-2} + C$

40. $\int \frac{(x+1)^2\tan^{-1}(3x) + 9x^3 + x}{(9x^2+1)(x+1)^2}\,dx = \int \frac{\tan^{-1}(3x)}{9x^2+1}\,dx + \int \frac{x}{(x+1)^2}\,dx$

$= \frac{1}{3}\int \tan^{-1}(3x)\,d(\tan^{-1}(3x)) + \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2} = \frac{(\tan^{-1}3x)^2}{6} + \ln|x+1| + \frac{1}{x+1} + C$

41. $(t^2 - 3t + 2)\frac{dx}{dt} = 1$; $x = \int \frac{dt}{t^2-3t+2} = \int \frac{dt}{t-2} - \int \frac{dt}{t-1} = \ln\left|\frac{t-2}{t-1}\right| + C$; $\frac{t-2}{t-1} = Ce^x$; $t = 3$ and $x = 0$

$\Rightarrow \frac{1}{2} = C \Rightarrow \frac{t-2}{t-1} = \frac{1}{2}e^x \Rightarrow x = \ln\left|2\left(\frac{t-2}{t-1}\right)\right| = \ln|t-2| - \ln|t-1| + \ln 2$

42. $(3t^4 + 4t^2 + 1)\frac{dx}{dt} = 2\sqrt{3}$; $x = 2\sqrt{3}\int \frac{dt}{3t^4+4t^2+1} = \sqrt{3}\int \frac{dt}{t^2+\frac{1}{3}} - \sqrt{3}\int \frac{dt}{t^2+1}$

$= 3\tan^{-1}(\sqrt{3}t) - \sqrt{3}\tan^{-1}t + C$; $t = 1$ and $x = \frac{-\pi\sqrt{3}}{4} \Rightarrow -\frac{\sqrt{3}\pi}{4} = \pi - \frac{\sqrt{3}}{4}\pi + C \Rightarrow C = -\pi$

$\Rightarrow x = 3\tan^{-1}(\sqrt{3}t) - \sqrt{3}\tan^{-1}t - \pi$

43. $(t^2 + 2t)\frac{dx}{dt} = 2x + 2$; $\frac{1}{2}\int \frac{dx}{x+1} = \int \frac{dt}{t^2+2t} \Rightarrow \frac{1}{2}\ln|x+1| = \frac{1}{2}\int \frac{dt}{t} - \frac{1}{2}\int \frac{dt}{t+2} \Rightarrow \ln|x+1| = \ln\left|\frac{t}{t+2}\right| + C$;

$t = 1$ and $x = 1 \Rightarrow \ln 2 = \ln\frac{1}{3} + C \Rightarrow C = \ln 2 + \ln 3 = \ln 6 \Rightarrow \ln|x+1| = \ln 6\left|\frac{t}{t+2}\right| \Rightarrow x + 1 = \frac{6t}{t+2}$

$\Rightarrow x = \frac{6t}{t+2} - 1,\ t > 0$

44. $(t+1)\frac{dx}{dt} = x^2 + 1 \Rightarrow \int \frac{dx}{x^2+1} = \int \frac{dt}{t+1} \Rightarrow \tan^{-1} x = \ln|t+1| + C$; $t = 0$ and $x = \frac{\pi}{4} \Rightarrow \tan^{-1} \frac{\pi}{4} = \ln|1| + C$
$\Rightarrow C = \tan^{-1} \frac{\pi}{4} = 1 \Rightarrow \tan^{-1} x = \ln|t+1| + 1 \Rightarrow x = \tan(\ln(t+1)+1),\ t > -1$

45. $\frac{1}{y^2 - y}\, dy = e^x\, dx \Rightarrow \int \frac{1}{y(y-1)}\, dy = \int e^x\, dx = e^x + C$

$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1} \Rightarrow 1 = A(y-1) + B(y) = (A+B)y - A$

Equating coefficients of like terms gives

$A + B = 0$ and $-A = 1$

Solving the system simultaneously yields $A = -1$, $B = 1$.

$\int \frac{1}{y(y-1)}\, dy = \int -\frac{1}{y}\, dy + \int \frac{1}{y-1}\, dy = -\ln|y| + \ln|y-1| + C_2 \Rightarrow -\ln|y| + \ln|y-1| = e^x + C$

Substitute $x = 0$, $y = 2$.

$-\ln 2 + 0 = 1 + C$ or $C = -1 - \ln 2$

The solution to the initial value problem is

$-\ln|y| + \ln|y-1| = e^x - 1 - \ln 2.$

46. $\frac{1}{(y+1)^2}\, dy = \sin\theta\, d\theta \Rightarrow \int \frac{1}{(y+1)^2}\, dy = \int \sin\theta\, d\theta \Rightarrow -\frac{1}{y+1} = -\cos\theta + C$

Substitute $\theta = \frac{\pi}{2}$, $y = 0 \Rightarrow -1 = 0 + C$ or $C = -1$

The solution to the initial value problem is

$-\frac{1}{y+1} = -\cos\theta - 1 \Rightarrow y + 1 = \frac{1}{\cos\theta + 1} \Rightarrow y = \frac{1}{\cos\theta + 1} - 1$

47. $dy = \frac{dx}{x^2 - 3x + 2}$; $x^2 - 3x + 2 = (x-2)(x-1) \Rightarrow \frac{1}{x^2 - 3x + 2} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x-2)$

$\Rightarrow 1 = (A+B)x - A - 2B$

Equating coefficients of like terms gives

$A + B = 0,\ -A - 2B = 1$

Solving the system simultaneously yields $A = 1$, $B = -1$.

$\int dy = \int \frac{dx}{x^2 - 3x + 2} = \int \frac{dx}{x-2} - \int \frac{dx}{x-1}$

$y = \ln|x-2| - \ln|x-1| + C$

Substitute $x = 3$, $y = 0 \Rightarrow 0 = 0 - \ln 2 + C$ or $C = \ln 2$

The solution to the initial value problems is

$y = \ln|x-2| - \ln|x-1| + \ln 2$

48. $\frac{ds}{2s+2} = \frac{dt}{t^2 + 2t} \Rightarrow \int \frac{ds}{2s+2} = \frac{1}{2} \int \frac{ds}{s+1} = \frac{1}{2} \ln|s+1| + C_1$

$t^2 + 2t = t(t+2) \Rightarrow \frac{1}{t^2+2t} = \frac{A}{t} + \frac{B}{t+2} \Rightarrow 1 = A(t+2) + Bt \Rightarrow 1 = (A+B)t + 2A$

Equating coefficients of like terms gives $A + B = 0$ and $2A = 1$

Solving the system simultaneously yields $A = \frac{1}{2}$, $B = -\frac{1}{2}$.

$\int \frac{dt}{t^2+2t} = \int \frac{1/2}{t}\,dt - \int \frac{1/2}{t+2}\,dt = \frac{1}{2}\ln|t| - \frac{1}{2}\ln|t+2| + C_2 \Rightarrow \frac{1}{2}\ln|s+1| = \frac{1}{2}\ln|t| - \frac{1}{2}\ln|t+2| + C_3$

$\Rightarrow \ln|s+1| = \ln|t| - \ln|t+2| + C$

Substitute $t = 1$, $s = 1 \Rightarrow \ln 2 = 0 - \ln 3 + C$ or $C = \ln 2 + \ln 3 = \ln 6$

The solution to the initial value problem is

$\ln|s+1| = \ln|t| - \ln|t+2| + \ln 6 \Rightarrow \ln|s+1| = \ln\left|\frac{6t}{t+2}\right| \Rightarrow |s+1| = \left|\frac{6t}{t+2}\right|.$

49. $V = \pi \int_{0.5}^{2.5} y^2\,dx = \pi \int_{0.5}^{2.5} \frac{9}{3x - x^2}\,dx = 3\pi\left(\int_{0.5}^{2.5} \left(-\frac{1}{x-3} + \frac{1}{x}\right)\right) dx = \left[3\pi \ln\left|\frac{x}{x-3}\right|\right]_{0.5}^{2.5} = 3\pi \ln 25$

50. $V = 2\pi \int_0^1 xy\,dx = 2\pi \int_0^1 \frac{2x}{(x+1)(2-x)}\,dx = 4\pi \int_0^1 \left(-\frac{1}{3}\left(\frac{1}{x+1}\right) + \frac{2}{3}\left(\frac{1}{2-x}\right)\right) dx$

$= \left[-\frac{4\pi}{3}(\ln|x+1| + 2\ln|2-x|)\right]_0^1 = \frac{4\pi}{3}(\ln 2)$

51. (a) $\frac{dx}{dt} = kx(N-x) \Rightarrow \int \frac{dx}{x(N-x)} = \int k\,dt \Rightarrow \frac{1}{N}\int \frac{dx}{x} + \frac{1}{N}\int \frac{dx}{N-x} = \int k\,dt \Rightarrow \frac{1}{N}\ln\left|\frac{x}{N-x}\right| = kt + C;$

$k = \frac{1}{250}$, $N = 1000$, $t = 0$ and $x = 2 \Rightarrow \frac{1}{1000}\ln\left|\frac{2}{998}\right| = C \Rightarrow \frac{1}{1000}\ln\left|\frac{x}{1000-x}\right| = \frac{t}{250} + \frac{1}{1000}\ln\left(\frac{1}{499}\right)$

$\Rightarrow \ln\left|\frac{499x}{1000-x}\right| = 4t \Rightarrow \frac{499x}{1000-x} = e^{4t} \Rightarrow 499x = e^{4t}(1000-x) \Rightarrow (499 + e^{4t})x = 1000e^{4t} \Rightarrow x = \frac{1000e^{4t}}{499+e^{4t}}$

(b) $x = \frac{1}{2}N = 500 \Rightarrow 500 = \frac{1000e^{4t}}{499+e^{4t}} \Rightarrow 500 \cdot 499 + 500e^{4t} = 1000e^{4t} \Rightarrow e^{4t} = 499 \Rightarrow t = \frac{1}{4}\ln 499 \approx 1.55$ days

52. $\frac{dx}{dt} = k(a-x)(b-x) \Rightarrow \frac{dx}{(a-x)(b-x)} = k\,dt$

(a) $a = b$: $\int \frac{dx}{(a-x)^2} = \int k\,dt \Rightarrow \frac{1}{a-x} = kt + C$; $t = 0$ and $x = 0 \Rightarrow \frac{1}{a} = C \Rightarrow \frac{1}{a-x} = kt + \frac{1}{a}$

$\Rightarrow \frac{1}{a-x} = \frac{akt+1}{a} \Rightarrow a - x = \frac{a}{akt+1} \Rightarrow x = a - \frac{a}{akt+1} = \frac{a^2kt}{akt+1}$

(b) $a \neq b$: $\int \frac{dx}{(a-x)(b-x)} = \int k\,dt \Rightarrow \frac{1}{b-a}\int \frac{dx}{a-x} - \frac{1}{b-a}\int \frac{dx}{b-x} = \int k\,dt \Rightarrow \frac{1}{b-a}\ln\left|\frac{b-x}{a-x}\right| = kt + C;$

$t = 0$ and $x = 0 \Rightarrow \frac{1}{b-a}\ln\frac{b}{a} = C \Rightarrow \ln\left|\frac{b-x}{a-x}\right| = (b-a)kt + \ln\left(\frac{b}{a}\right) \Rightarrow \frac{b-x}{a-x} = \frac{b}{a}e^{(b-a)kt}$

$\Rightarrow x = \frac{ab\left[1 - e^{(a-b)kt}\right]}{a - be^{(a-b)kt}}$

7.4 TRIGONOMETRIC SUBSTITUTIONS

1. $y = 3\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dy = \frac{3\,d\theta}{\cos^2\theta},\ 9 + y^2 = 9(1 + \tan^2\theta) = \frac{9}{\cos^2\theta} \Rightarrow \frac{1}{\sqrt{9+y^2}} = \frac{|\cos\theta|}{3} = \frac{\cos\theta}{3}$

   $\left(\text{because } \cos\theta > 0 \text{ when } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$;

   $\int \frac{dy}{\sqrt{9+y^2}} = 3\int \frac{\cos\theta\, d\theta}{3\cos^2\theta} = \int \frac{d\theta}{\cos\theta} = \ln|\sec\theta + \tan\theta| + C' = \ln\left|\frac{\sqrt{9+y^2}}{3} + \frac{y}{3}\right| + C' = \ln\left|\sqrt{9+y^2} + y\right| + C$

2. $\int \frac{3\,dy}{\sqrt{1+9y^2}};\ [3y = x] \to \int \frac{dx}{\sqrt{1+x^2}};\ x = \tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2},\ dx = \frac{dt}{\cos^2 t},\ \sqrt{1+x^2} = \frac{1}{\cos t};$

   $\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{dt}{\cos^2 t\left(\frac{1}{\cos t}\right)} = \ln|\sec t + \tan t| + C = \ln\left|\sqrt{x^2+1} + x\right| + C = \ln\left|\sqrt{1+9y^2} + 3y\right| + C$

3. $t = 5\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = 5\cos\theta\, d\theta,\ \sqrt{25-t^2} = 5\cos\theta;$

   $\int \sqrt{25-t^2}\, dt = \int (5\cos\theta)(5\cos\theta)\, d\theta = 25\int \cos^2\theta\, d\theta = 25\int \frac{1+\cos 2\theta}{2}\, d\theta = 25\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + C$

   $= \frac{25}{2}(\theta + \sin\theta\cos\theta) + C = \frac{25}{2}\left[\sin^{-1}\left(\frac{t}{5}\right) + \left(\frac{t}{5}\right)\left(\frac{\sqrt{25-t^2}}{5}\right)\right] + C = \frac{25}{2}\sin^{-1}\left(\frac{t}{5}\right) + \frac{t\sqrt{25-t^2}}{2} + C$

4. $t = \frac{1}{3}\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\cos\theta\, d\theta,\ \sqrt{1-9t^2} = \cos\theta;$

   $\int \sqrt{1-9t^2}\, dt = \frac{1}{3}\int (\cos\theta)(\cos\theta)\, d\theta = \frac{1}{3}\int \cos^2\theta\, d\theta = \frac{1}{6}(\theta + \sin\theta\cos\theta) + C = \frac{1}{6}\left[\sin^{-1}(3t) + 3t\sqrt{1-9t^2}\right] + C$

5. $x = \frac{7}{2}\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \frac{7}{2}\sec\theta\tan\theta\, d\theta,\ \sqrt{4x^2-49} = \sqrt{49\sec^2\theta - 49} = 7\tan\theta;$

   $\int \frac{dx}{\sqrt{4x^2-49}} = \int \frac{\left(\frac{7}{2}\sec\theta\tan\theta\right) d\theta}{7\tan\theta} = \frac{1}{2}\int \sec\theta\, d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}\ln\left|\frac{2x}{7} + \frac{\sqrt{4x^2-49}}{7}\right| + C$

6. $x = \frac{3}{5}\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \frac{3}{5}\sec\theta\tan\theta\, d\theta,\ \sqrt{25x^2-9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta;$

   $\int \frac{5\,dx}{\sqrt{25x^2-9}} = \int \frac{5\left(\frac{3}{5}\sec\theta\tan\theta\right) d\theta}{3\tan\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + C$

7. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\,d\theta,\ \sqrt{x^2-1} = \tan\theta;$

$$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\tan\theta} = \int \frac{d\theta}{\sec\theta} = \sin\theta + C = \tan\theta\cos\theta + C = \frac{\sqrt{x^2-1}}{x} + C$$

8. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\,d\theta,\ \sqrt{x^2-1} = \tan\theta;$

$$\int \frac{2\,dx}{x^3\sqrt{x^2-1}} = \int \frac{2\tan\theta\sec\theta\,d\theta}{\sec^3\theta\tan\theta} = 2\int \cos^2\theta\,d\theta = 2\int \left(\frac{1+\cos 2\theta}{2}\right)d\theta = \theta + \sin\theta\cos\theta + C$$

$$= \theta + \tan\theta\cos^2\theta + C = \sec^{-1}x + \sqrt{x^2-1}\left(\frac{1}{x}\right)^2 + C = \sec^{-1}x + \frac{\sqrt{x^2-1}}{x^2} + C$$

9. $x = 2\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{2\,d\theta}{\cos^2\theta},\ \sqrt{x^2+4} = \frac{2}{\cos\theta};$

$$\int \frac{x^3\,dx}{\sqrt{x^2+4}} = \int \frac{(8\tan^3\theta)(\cos\theta)\,d\theta}{\cos^2\theta} = 8\int \frac{\sin^3\theta\,d\theta}{\cos^4\theta} = 8\int \frac{(\cos^2\theta-1)(-\sin\theta)\,d\theta}{\cos^4\theta};$$

$$[t = \cos\theta] \to 8\int \frac{t^2-1}{t^4}\,dt = 8\int\left(\frac{1}{t^2}-\frac{1}{t^4}\right)dt = 8\left(-\frac{1}{t}+\frac{1}{3t^3}\right) + C = 8\left(-\sec\theta + \frac{\sec^3\theta}{3}\right) + C$$

$$= 8\left(-\frac{\sqrt{x^2+4}}{2} + \frac{(x^2+4)^{3/2}}{8\cdot 3}\right) + C = \frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C$$

10. $x = \tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \sec^2\theta\,d\theta,\ \sqrt{x^2+1} = \sec\theta;$

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta\,d\theta}{\tan^2\theta\sec\theta} = \int \frac{\cos\theta\,d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = \frac{-\sqrt{x^2+1}}{x} + C$$

11. $w = 2\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dw = 2\cos\theta\,d\theta,\ \sqrt{4-w^2} = 2\cos\theta;$

$$\int \frac{8\,dw}{w^2\sqrt{4-w^2}} = \int \frac{8\cdot 2\cos\theta\,d\theta}{4\sin^2\theta\cdot 2\cos\theta} = 2\int \frac{d\theta}{\sin^2\theta} = -2\cot\theta + C = \frac{-2\sqrt{4-w^2}}{w} + C$$

12. $w = 3\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dw = 3\cos\theta\,d\theta,\ \sqrt{9-w^2} = 3\cos\theta;$

$$\int \frac{\sqrt{9-w^2}}{w^2}\,dw = \int \frac{3\cos\theta\cdot 3\cos\theta\,d\theta}{9\sin^2\theta} = \int \cot^2\theta\,d\theta = \int\left(\frac{1-\sin^2\theta}{\sin^2\theta}\right)d\theta = \int (\csc^2\theta - 1)\,d\theta$$

$$= -\cot\theta - \theta + C = -\frac{\sqrt{9-w^2}}{w} - \sin^{-1}\left(\frac{w}{3}\right) + C$$

13. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\,d\theta,\ (x^2-1)^{3/2} = \tan^3\theta;$

$$\int \frac{dx}{(x^2-1)^{3/2}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\tan^3\theta} = \int \frac{\cos\theta\,d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = -\left(\frac{1}{\tan\theta}\right)\left(\frac{1}{\cos\theta}\right) + C$$

$$= -\left(\frac{1}{\sqrt{x^2-1}}\right)(x) + C = -\frac{x}{\sqrt{x^2-1}} + C$$

14. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ (x^2-1)^{5/2} = \tan^5\theta;$

$$\int \frac{x^2\,dx}{(x^2-1)^{5/2}} = \int \frac{\sec^2\theta \cdot \sec\theta\tan\theta\,d\theta}{\tan^5\theta} = \int \frac{\cos\theta}{\sin^4\theta}\,d\theta = -\frac{1}{3\sin^3\theta} + C = -\frac{1}{3}\left(\frac{1}{\tan^3\theta}\right)\left(\frac{1}{\cos^3\theta}\right) + C$$

$$= -\frac{1}{3}\left[\frac{1}{(x^2-1)^{3/2}}\right](x^3) + C = -\frac{x^3}{3(x^2-1)^{3/2}} + C$$

15. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\,d\theta,\ (1-x^2)^{3/2} = \cos^3\theta;$

$$\int \frac{(1-x^2)^{3/2}\,dx}{x^6} = \int \frac{\cos^3\theta \cdot \cos\theta\,d\theta}{\sin^6\theta} = \int \cot^4\theta\csc^2\theta\,d\theta = -\frac{\cot^5\theta}{5} + C = -\frac{1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5 + C$$

16. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\,d\theta,\ (1-x^2)^{1/2} = \cos\theta;$

$$\int \frac{(1-x^2)^{1/2}\,dx}{x^4} = \int \frac{\cos\theta \cdot \cos\theta\,d\theta}{\sin^4\theta} = \int \cot^2\theta\csc^2\theta\,d\theta = -\frac{\cot^3\theta}{3} + C = -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C$$

17. $x = \frac{1}{2}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{1}{2}\sec^2\theta\,d\theta,\ (4x^2+1)^2 = \sec^4\theta;$

$$\int \frac{8\,dx}{(4x^2+1)^2} = \int \frac{8\left(\frac{1}{2}\sec^2\theta\right)d\theta}{\sec^4\theta} = 4\int \cos^2\theta\,d\theta = 2(\theta + \sin\theta\cos\theta) + C = 2(\theta + \tan\theta + \cos^2\theta) + C$$

$$= 2\tan^{-1}2x + \frac{4x}{(4x^2+1)} + C$$

18. $t = \frac{1}{3}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\sec^2\theta\,d\theta,\ 9t^2+1 = \sec^2\theta;$

$$\int \frac{6\,dt}{(9t^2+1)^2} = \int \frac{6\left(\frac{1}{3}\sec^2\theta\right)d\theta}{\sec^4\theta} = 2\int \cos^2\theta\,d\theta = \theta + \sin\theta\cos\theta + C = \theta + \tan\theta\cos^2\theta + C$$

$$= \tan^{-1}3t + \frac{3t}{(9t^2+1)} + C$$

19. Let $e^t = 3\tan\theta,\ t = \ln(3\tan\theta),\ dt = \frac{\sec^2\theta}{\tan\theta}\,d\theta,\ \sqrt{e^{2t}+9} = \sqrt{9\tan^2\theta+9} = 3\sec\theta;$

$$\int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3\tan\theta \cdot \sec^2\theta\,d\theta}{\tan\theta \cdot 3\sec\theta} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \sec\theta\,d\theta = \left[\ln|\sec\theta + \tan\theta|\right]_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)}$$

$$= \ln\left(\frac{5}{3} + \frac{4}{3}\right) - \ln\left(\frac{\sqrt{10}}{3} + \frac{1}{3}\right) = \ln 9 - \ln\left(1 + \sqrt{10}\right)$$

20. Let $e^t = \tan\theta$, $t = \ln(\tan\theta)$, $\frac{3}{4} \le \theta \le \frac{4}{3}$, $dt = \frac{\sec^2\theta}{\tan\theta}\, d\theta$, $1 + e^{2t} = 1 + \tan^2\theta = \sec^2\theta$;

$$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\, dt}{\left(1+e^{2t}\right)^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{(\tan\theta)\left(\frac{\sec^2\theta}{\tan\theta}\right) d\theta}{\sec^3\theta} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos\theta\, d\theta = [\sin\theta]_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$$

21. $\int_{1/12}^{1/4} \frac{2\, dt}{\sqrt{t} + 4t\sqrt{t}}$; $\left[u = 2\sqrt{t},\ du = \frac{1}{\sqrt{t}}\, dt\right] \to \int_{1/\sqrt{3}}^{1} \frac{2\, du}{1+u^2}$; $u = \tan\theta$, $\frac{\pi}{6} < \theta < \frac{\pi}{4}$, $du = \sec^2\theta\, d\theta$, $1 + u^2 = \sec^2\theta$;

$$\int_{1/\sqrt{3}}^{1} \frac{4\, du}{u\left(1+u^2\right)} = \int_{\pi/6}^{\pi/4} \frac{2\sec^2\theta\, d\theta}{\sec^2\theta} = [2\theta]_{\pi/6}^{\pi/4} = 2\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\pi}{6}$$

22. $y = e^{\tan\theta}$, $dy = e^{\tan\theta}\sec^2\theta\, d\theta$, $\sqrt{1 + (\ln y)^2} = \sqrt{1 + \tan^2\theta} = \sec\theta$;

$$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan\theta}\sec^2\theta}{e^{\tan\theta}\sec\theta}\, d\theta = \int_0^{\pi/4} \sec\theta\, d\theta = \left[\ln|\sec\theta + \tan\theta|\right]_0^{\pi/4} = \ln\left(1+\sqrt{2}\right)$$

23. $x = \sec\theta$, $dx = \sec\theta\tan\theta\, d\theta$, $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$;

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\tan\theta} = \theta + C = \sec^{-1}|x| + C$$

24. $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, $1 + x^2 = \sec^2\theta$;

$$\int \frac{dx}{x^2+1} = \int \frac{\sec^2\theta\, d\theta}{\sec^2\theta} = \theta + C = \tan^{-1} x + C$$

25. $x = \sec\theta$, $dx = \sec\theta\tan\theta\, d\theta$, $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$;

$$\int \frac{x\, dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta \cdot \sec\theta\tan\theta\, d\theta}{\tan\theta} = \int \sec^2\theta\, d\theta = \tan\theta + C = \sqrt{x^2-1} + C$$

26. $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$;

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\, d\theta}{\cos\theta} = \theta + C = \sin^{-1} x + C$$

27. $x\frac{dy}{dx} = \sqrt{x^2-4}$; $dy = \sqrt{x^2-4}\,\frac{dx}{x}$; $y = \int \frac{\sqrt{x^2-4}}{x}\, dx$; $\left[\begin{array}{c} x = 2\sec\theta,\ 0 < \theta < \frac{\pi}{2} \\ dx = 2\sec\theta\tan\theta\, d\theta \\ \sqrt{x^2-4} = 2\tan\theta \end{array}\right]$

$$\to y = \int \frac{(2\tan\theta)(2\sec\theta\tan\theta)\, d\theta}{2\sec\theta} = 2\int \tan^2\theta\, d\theta = 2\int\left(\sec^2\theta - 1\right) d\theta = 2(\tan\theta - \theta) + C$$

$= 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\frac{x}{2}\right)\right] + C$; $x = 2$ and $y = 0 \Rightarrow 0 = 0 + C \Rightarrow C = 0 \Rightarrow y = 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\frac{x}{2}\right]$

28. $\sqrt{x^2-9}\,\frac{dy}{dx} = 1$, $dy = \frac{dx}{\sqrt{x^2-9}}$; $y = \int \frac{dx}{\sqrt{x^2-9}}$; $\left[\begin{array}{c} x = 3\sec\theta,\ 0 < \theta < \frac{\pi}{2} \\ dx = 3\sec\theta\tan\theta\,d\theta \\ \sqrt{x^2-9} = 3\tan\theta \end{array}\right] \to y = \int \frac{3\sec\theta\tan\theta\,d\theta}{3\tan\theta}$

$= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$; $x = 5$ and $y = \ln 3 \Rightarrow \ln 3 = \ln 3 + C \Rightarrow C = 0$

$\Rightarrow y = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|$

29. $(x^2+4)\frac{dy}{dx} = 3$, $dy = \frac{3\,dx}{x^2+4}$; $y = 3\int \frac{dx}{x^2+4} = \frac{3}{2}\tan^{-1}\frac{x}{2} + C$; $x = 2$ and $y = 0 \Rightarrow 0 = \frac{3}{2}\tan^{-1}1 + C$

$\Rightarrow C = -\frac{3\pi}{8} \Rightarrow y = \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{3\pi}{8}$

30. $(x^2+1)^2\frac{dy}{dx} = \sqrt{x^2+1}$, $dy = \frac{dx}{(x^2+1)^{3/2}}$; $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $(x^2+1)^{3/2} = \sec^3\theta$;

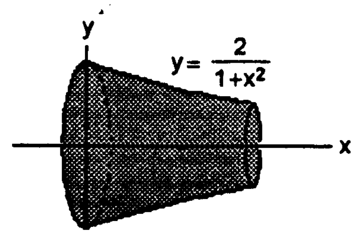
$y = \int \frac{\sec^2\theta\,d\theta}{\sec^3\theta} = \int \cos\theta\,d\theta = \sin\theta + C = \tan\theta\cos\theta + C = \frac{\tan\theta}{\sec\theta} + C = \frac{x}{\sqrt{x^2+1}} + C$; $x = 0$ and $y = 1$

$\Rightarrow 1 = 0 + C \Rightarrow y = \frac{x}{\sqrt{x^2+1}} + 1$

31. $A = \int_0^3 \frac{\sqrt{9-x^2}}{3}\,dx$; $x = 3\sin\theta$, $0 \le \theta \le \frac{\pi}{2}$, $dx = 3\cos\theta\,d\theta$, $\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3\cos\theta$;

$A = \int_0^{\pi/2} \frac{3\cos\theta \cdot 3\cos\theta\,d\theta}{3} = 3\int_0^{\pi/2} \cos^2\theta\,d\theta = \frac{3}{2}\left[\theta + \sin\theta\cos\theta\right]_0^{\pi/2} = \frac{3\pi}{4}$

32. $V = \int_0^1 \pi\left(\frac{2}{1+x^2}\right)^2 dx = 4\pi\int_0^1 \frac{dx}{(x^2+1)^2}$;



$x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $x^2+1 = \sec^2\theta$;

$V = 4\pi\int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{\sec^4\theta} = 4\pi\int_0^{\pi/4} \cos^2\theta\,d\theta = 2\pi\int_0^{\pi/4} (1+\cos 2\theta)\,d\theta = 2\pi\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \pi\left(\frac{\pi}{2} + 1\right)$

33. (a) From the figure, $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$

(b) From part (a), $z = \frac{\sin x}{1+\cos x} \Rightarrow z(1+\cos x) = \sin x \Rightarrow z^2(1+\cos x)^2 = \sin^2 x$

$\Rightarrow z^2(1+\cos x)^2 - (1-\cos x)(1+\cos x) = 0 \Rightarrow (1+\cos x)(z^2 + z^2\cos x - 1 + \cos x) = 0$

$1+\cos x = 0$ or $(z^2+1)\cos x = 1 - z^2$

$\cos x = -1$ $\quad\quad$ $\cos x = \frac{1-z^2}{1+z^2}$

$\cos x = -1$ does not make sense in this case.

(c) From part (b), $\cos x = \frac{1-z^2}{1+z^2} \Rightarrow \sin^2 x = 1 - \frac{(1-z^2)^2}{(1+z^2)^2} = \frac{(1+z^2)^2-(1-z^2)^2}{(1+z^2)^2}$

$= \frac{1+2z^2+z^4-1+2z^2-z^4}{(1+z^2)^2} = \frac{4z^2}{(1+z^2)^2} \Rightarrow \sin x = \pm\frac{2z}{1+z^2}$

Only $\sin x = \frac{2z}{1+z^2}$ makes sense in this case.

(d) $z = \tan\frac{x}{2}$, $dz = \left(\frac{1}{2}\sec^2\frac{x}{2}\right)dx \Rightarrow dz = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)dx \Rightarrow dz = \frac{1}{2}(1+z^2)\,dx \Rightarrow dx = \frac{2\,dz}{1+z^2}$

34. $$\int \frac{dx}{1+\sin x} = \int \frac{\frac{2\,dz}{1+z^2}}{1+\frac{2z}{1+z^2}} = \int \frac{2\,dz}{z^2+2z+1} = \int \frac{2\,dz}{(z+1)^2} = -\frac{2}{z+1} + C = -\frac{2}{\tan\frac{x}{2}+1} + C$$

35. $$\int \frac{dx}{1-\cos x} = \int \frac{\frac{2\,dz}{1+z^2}}{1-\frac{1-z^2}{1+z^2}} = \int \frac{dz}{z^2} = -\frac{1}{z} + C = -\frac{1}{\tan\frac{x}{2}} + C$$

36. $$\int \frac{d\theta}{1-\sin\theta} = \int \frac{\frac{2\,dz}{1+z^2}}{1-\frac{2z}{1+z^2}} = \int \frac{2\,dz}{z^2-2z+1} = \int \frac{2\,dz}{(z-1)^2} = -\frac{2}{z-1} + C = -\frac{2}{\tan\frac{\theta}{2}-1} + C$$

$$= \frac{2}{1-\tan\frac{\theta}{2}} + C$$

37. $$\int \frac{dt}{1+\sin t+\cos t} = \int \frac{\frac{2\,dz}{1+z^2}}{1+\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}} = \int \frac{dz}{z+1} = \ln|z+1| + C = \ln\left|\tan\frac{t}{2}+1\right| + C$$

38. $$\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta} = \int_0^1 \frac{\left(\frac{2\,dz}{1+z^2}\right)}{2+\left(\frac{1-z^2}{1+z^2}\right)} = \int_0^1 \frac{2\,dz}{2+2z^2+1-z^2} = \int_0^1 \frac{2\,dz}{z^2+3} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{z}{\sqrt{3}}\right]_0^1 = \frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}$$

$$= \frac{\pi}{3\sqrt{3}} = \frac{\sqrt{3}\pi}{9}$$

39. $$\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta+\sin\theta} = \int_1^{\sqrt{3}} \frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2\,dz}{1+z^2}\right)}{\left[\frac{2z(1-z^2)}{(1+z^2)^2}+\left(\frac{2z}{1+z^2}\right)\right]} = \int_1^{\sqrt{3}} \frac{2(1-z^2)\,dz}{2z-2z^3+2z+2z^3} = \int_1^{\sqrt{3}} \frac{1-z^2}{2z}\,dz$$

$$= \left[\frac{1}{2}\ln z - \frac{z^2}{4}\right]_1^{\sqrt{3}} = \left(\frac{1}{2}\ \ln\ \sqrt{3} - \frac{3}{4}\right) - \left(0 - \frac{1}{4}\right) = \frac{\ln 3}{4} - \frac{1}{2} = \frac{1}{4}(\ln 3 - 2) = \frac{1}{2}(\ln\ \sqrt{3} - 1)$$

40. $$\int \frac{dt}{\sin t - \cos t} = \int \frac{\left(\frac{2\ dz}{1+z^2}\right)}{\left(\frac{2z}{1+z^2} - \frac{1-z^2}{1+z^2}\right)} = \int \frac{2\ dz}{2z - 1 + z^2} = \int \frac{2\ dz}{(z+1)^2 - 2} = \frac{1}{\sqrt{2}} \ln \left|\frac{z+1-\sqrt{2}}{z+1+\sqrt{2}}\right| + C$$

$$= \frac{1}{\sqrt{2}} \ln \left|\frac{\tan\left(\frac{t}{2}\right) + 1 - \sqrt{2}}{\tan\left(\frac{t}{2}\right) + 1 + \sqrt{2}}\right| + C$$

41. $$\int \frac{\cos t\ dt}{1 - \cos t} = \int \frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2\ dz}{1+z^2}\right)}{1 - \left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{2(1-z^2)\ dz}{(1+z^2)^2 - (1+z^2)(1-z^2)} = \int \frac{2(1-z^2)\ dz}{(1+z^2)(1+z^2-1+z^2)}$$

$$= \int \frac{(1-z^2)\ dz}{(1+z^2)z^2} = \int \frac{dz}{z^2(1+z^2)} - \int \frac{dz}{1+z^2} = \int \frac{dz}{z^2} - 2 \int \frac{dz}{z^2+1} = -\frac{1}{z} - 2\tan^{-1} z + C = -\cot\left(\frac{t}{2}\right) - t + C$$

7.5 INTEGRAL TABLES, COMPUTER ALGEBRA SYSTEMS, AND MONTE CARLO INTEGRATION

1. $$\int \frac{dx}{x\sqrt{x-3}} = \frac{2}{\sqrt{3}} \tan^{-1} \sqrt{\frac{x-3}{3}} + C$$
(We used FORMULA 13(a) with $a = 1$, $b = -3$)

2. $$\int \frac{x\ dx}{\sqrt{x-2}} = \int \frac{(x-2)\ dx}{\sqrt{x-2}} + 2 \int \frac{dx}{\sqrt{x-2}} = \int \left(\sqrt{x-2}\right)^1 dx + 2 \int \left(\sqrt{x-2}\right)^{-1} dx$$

$$= \left(\frac{2}{1}\right)\frac{\left(\sqrt{x-2}\right)^3}{3} + 2\left(\frac{2}{1}\right)\frac{\left(\sqrt{x-2}\right)^1}{1} = \sqrt{x-2}\left[\frac{2(x-2)}{3} + 4\right] + C$$
(We used FORMULA 11 with $a = 1$, $b = -2$, $n = 1$ and $a = 1$, $b = -2$, $n = -1$)

3. $$\int x\sqrt{2x-3}\ dx = \frac{1}{2}\int (2x-3)\sqrt{2x-3}\ dx + \frac{3}{2}\int \sqrt{2x-3}\ dx = \frac{1}{2}\int \left(\sqrt{2x-3}\right)^3 dx + \frac{3}{2}\int \left(\sqrt{2x-3}\right)^1 dx$$

$$= \left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\frac{\left(\sqrt{2x-3}\right)^5}{5} + \left(\frac{3}{2}\right)\left(\frac{2}{2}\right)\frac{\left(\sqrt{2x-3}\right)^3}{3} + C = \frac{(2x-3)^{3/2}}{2}\left[\frac{2x-3}{5} + 1\right] + C = \frac{(2x-3)^{3/2}(x+1)}{5} + C$$
(We used FORMULA 11 with $a = 2$, $b = -3$, $n = 3$ and $a = 2$, $b = -3$, $n = 1$)

4. $$\int \frac{\sqrt{9-4x}}{x^2}\ dx = -\frac{\sqrt{9-4x}}{x} + \frac{(-4)}{2}\int \frac{dx}{x\sqrt{9-4x}} + C$$
(We used FORMULA 14 with $a = -4$, $b = 9$)

$$= -\frac{\sqrt{9-4x}}{x} - 2\left(\frac{1}{\sqrt{9}}\right) \ln\left|\frac{\sqrt{9-4x}-\sqrt{9}}{\sqrt{9-4x}+\sqrt{9}}\right| + C$$

(We used FORMULA 13(b) with $a = -4$, $b = 9$)

$$= \frac{-\sqrt{9-4x}}{x} - \frac{2}{3} \ln\left|\frac{\sqrt{9-4x}-3}{\sqrt{9-4x}+3}\right| + C$$

5. $\int x\sqrt{4x-x^2}\,dx = \int x\sqrt{2\cdot 2x - x^2}\,dx = \frac{(x+2)(2x-3\cdot 2)\sqrt{2\cdot 2\cdot x - x^2}}{6} + \frac{2^3}{2}\sin^{-1}\left(\frac{x-2}{2}\right) + C$

$$= \frac{(x+2)(2x-6)\sqrt{4x-x^2}}{6} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

(We used FORMULA 51 with $a = 2$)

6. $\int \frac{dx}{x\sqrt{7+x^2}} = \int \frac{dx}{x\sqrt{\left(\sqrt{7}\right)^2 + x^2}} = -\frac{1}{\sqrt{7}} \ln\left|\frac{\sqrt{7}+\sqrt{\left(\sqrt{7}\right)^2+x^2}}{x}\right| + C = -\frac{1}{\sqrt{7}} \ln\left|\frac{\sqrt{7}+\sqrt{7+x^2}}{x}\right| + C$

(We used FORMULA 26 with $a = \sqrt{7}$)

7. $\int \frac{\sqrt{4-x^2}}{x}\,dx = \int \frac{\sqrt{2^2-x^2}}{x}\,dx = \sqrt{2^2-x^2} - 2\ln\left|\frac{2+\sqrt{2^2-x^2}}{x}\right| + C = \sqrt{4-x^2} - 2\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right| + C$

(We used FORMULA 31 with $a = 2$)

8. $\int \sqrt{25-p^2}\,dp = \int \sqrt{5^2-p^2}\,dp = \frac{p}{2}\sqrt{5^2-p^2} + \frac{5^2}{2}\sin^{-1}\frac{p}{5} + C = \frac{p}{2}\sqrt{25-p^2} + \frac{25}{2}\sin^{-1}\frac{p}{5} + C$

(We used FORMULA 29 with $a = 5$)

9. $\int \frac{r^2}{\sqrt{4-r^2}}\,dr = \int \frac{r^2}{\sqrt{2^2-r^2}}\,dr = \frac{2^2}{2}\sin^{-1}\left(\frac{r}{2}\right) - \frac{1}{2}r\sqrt{2^2-r^2} + C = 2\sin^{-1}\left(\frac{r}{2}\right) - \frac{1}{2}r\sqrt{4-r^2} + C$

(We used FORMULA 33 with $a = 2$)

10. $\int \frac{d\theta}{5+4\sin 2\theta} = \frac{-2}{2\sqrt{25-16}} \tan^{-1}\left[\sqrt{\frac{5-4}{5+4}} \tan\left(\frac{\pi}{4} - \frac{2\theta}{2}\right)\right] + C = -\frac{1}{3}\tan^{-1}\left[\frac{1}{3}\tan\left(\frac{\pi}{4} - \theta\right)\right] + C$

(We used FORMULA 70 with $b = 5$, $c = 4$, $a = 2$)

11. $\int e^{2t}\cos 3t\,dt = \frac{e^{2t}}{2^2+3^2}(2\cos 3t + 3\sin 3t) + C = \frac{e^{2t}}{13}(2\cos 3t + 3\sin 3t) + C$

(We used FORMULA 108 with $a = 2$, $b = 3$)

12. $\int x\cos^{-1}x\,dx = \int x^1\cos^{-1}x\,dx = \frac{x^{1+1}}{1+1}\cos^{-1}x + \frac{1}{1+1}\int \frac{x^{1+1}\,dx}{\sqrt{1-x^2}} = \frac{x^2}{2}\cos^{-1}x + \frac{1}{2}\int \frac{x^2\,dx}{\sqrt{1-x^2}}$

(We used FORMULA 100 with $a = 1$, $n = 1$)

$= \frac{x^2}{2}\cos^{-1}x + \frac{1}{2}\left(\frac{1}{2}\sin^{-1}x\right) - \frac{1}{2}\left(\frac{1}{2}x\sqrt{1-x^2}\right) + C = \frac{x^2}{2}\cos^{-1}x + \frac{1}{4}\sin^{-1}x - \frac{1}{4}x\sqrt{1-x^2} + C$

(We used FORMULA 33 with a = 1)

13. $\int \frac{ds}{(9-s^2)^2} = \int \frac{ds}{(3^3-s^2)^2} = \frac{s}{2\cdot 3^2\cdot(3^2-s^2)} + \frac{1}{2\cdot 3^2}\int \frac{ds}{3^2-s^2}$

(We used FORMULA 19 with a = 3)

$= \frac{s}{18(9-s^2)} + \frac{1}{18}\left(\frac{1}{2\cdot 3}\ln\left|\frac{s+3}{s-3}\right|\right) + C = \frac{s}{18(9-s^2)} + \frac{1}{108}\ln\left|\frac{s+3}{s-3}\right| + C$

(We used FORMULA 18 with a = 3)

14. $\int \frac{\sqrt{4x+9}}{x^2}\,dx = -\frac{\sqrt{4x+9}}{x} + \frac{4}{2}\int \frac{dx}{x\sqrt{4x+9}}$

(We used FORMULA 14 with a = 4, b = 9)

$= -\frac{\sqrt{4x+9}}{x} + 2\left(\frac{1}{\sqrt{9}}\ln\left|\frac{\sqrt{4x+9}-\sqrt{9}}{\sqrt{4x+9}+\sqrt{9}}\right|\right) + C = -\frac{\sqrt{4x+9}}{x} + \frac{2}{3}\ln\left|\frac{\sqrt{4x+9}-3}{\sqrt{4x+9}+3}\right| + C$

(We used FORMULA 13(b) with a = 4, b = 9)

15. $\int \frac{\sqrt{3t-4}}{t}\,dt = 2\sqrt{3t-4} + (-4)\int \frac{dt}{t\sqrt{3t-4}}$

(We used FORMULA 12 with a = 3, b = −4)

$= 2\sqrt{3t-4} - 4\left(\frac{2}{\sqrt{4}}\tan^{-1}\sqrt{\frac{3t-4}{4}}\right) + C = 2\sqrt{3t-4} - 4\tan^{-1}\sqrt{\frac{3t-4}{4}} + C$

(We used FORMULA 13(a) with a = 3, b = −4)

16. $\int x^2\tan^{-1}x\,dx = \frac{x^{2+1}}{2+1}\tan^{-1}x - \frac{1}{2+1}\int \frac{x^{2+1}}{1+x^2}\,dx = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx$

(We used FORMULA 101 with a = 1, n = 2);

$\int \frac{x^3}{1+x^2}\,dx = \int x\,dx - \int \frac{x\,dx}{1+x^2} = \frac{x^2}{2} - \frac{1}{2}\ln(1+x^2) + C \Rightarrow \int x^2\tan^{-1}x\,dx$

$= \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + \frac{1}{6}\ln(1+x^2) + C$

17. $\int \sin 3x\cos 2x\,dx = -\frac{\cos 5x}{10} - \frac{\cos x}{2} + C$

(We used FORMULA 62(a) with a = 3, b = 2)

18. $\int 8\sin 4t\sin\frac{t}{2}\,dx = \frac{8}{7}\sin\left(\frac{7t}{2}\right) - \frac{8}{9}\sin\left(\frac{9t}{2}\right) + C = 8\left[\frac{\sin\left(\frac{7t}{2}\right)}{7} - \frac{\sin\left(\frac{9t}{2}\right)}{9}\right] + C$

(We used FORMULA 62(b) with a = 4, b = $\frac{1}{2}$)

19. $\int \cos\frac{\theta}{3}\cos\frac{\theta}{4}\,d\theta = 6\sin\left(\frac{\theta}{12}\right)+\frac{6}{7}\sin\left(\frac{7\theta}{12}\right)+C$

(We used FORMULA 62(c) with $a=\frac{1}{3}$, $b=\frac{1}{4}$)

20. $\int \cos\frac{\theta}{2}\cos 7\theta\,d\theta = \frac{1}{13}\sin\left(\frac{13\theta}{2}\right)+\frac{1}{15}\sin\left(\frac{15\theta}{2}\right)+C = \frac{\sin\left(\frac{13\theta}{2}\right)}{13}+\frac{\sin\left(\frac{15\theta}{2}\right)}{15}+C$

(We used FORMULA 62(c) with $a=\frac{1}{2}$, $b=7$)

21. $\int \frac{x^3+x+1}{(x^2+1)^2}\,dx = \int \frac{x\,dx}{x^2+1}+\int\frac{dx}{(x^2+1)^2} = \frac{1}{2}\int\frac{d(x^2+1)}{x^2+1}+\int\frac{dx}{(x^2+1)^2}$

$= \frac{1}{2}\ln|x^2+1|+\frac{x}{2(1+x^2)}+\frac{1}{2}\tan^{-1}x+C$

(For the second integral we used FORMULA 17 with $a=1$)

22. $\int \frac{x^2+6x}{(x^2+3)^2}\,dx = \int\frac{dx}{x^2+3}+\int\frac{6x\,dx}{(x^2+3)^2}-\int\frac{3\,dx}{(x^2+3)^2} = \int\frac{dx}{x^2+(\sqrt{3})^2}+3\int\frac{d(x^2+3)}{(x^2+3)^2}-3\int\frac{dx}{\left[x^2+(\sqrt{3})^2\right]^2}$

$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-\frac{3}{(x^2+3)}-3\left(\frac{x}{2(\sqrt{3})^2\left((\sqrt{3})^2+x^2\right)}+\frac{1}{2(\sqrt{3})^3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right)+C$

(For the first integral we used FORMULA 16 with $a=\sqrt{3}$; for the third integral we used FORMULA 17 with $a=\sqrt{3}$)

$= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-\frac{3}{x^2+3}-\frac{x}{2(x^2+3)}+C$

23. $\int \sin^{-1}\sqrt{x}\,dx;\ \begin{bmatrix} u=\sqrt{x} \\ x=u^2 \\ dx=2u\,du\end{bmatrix} \to 2\int u^1\sin^{-1}u\,du = 2\left(\frac{u^{1+1}}{1+1}\sin^{-1}u-\frac{1}{1+1}\int\frac{u^{1+1}}{\sqrt{1-u^2}}\,du\right)$

$= u^2\sin^{-1}u-\int\frac{u^2\,du}{\sqrt{1-u^2}}$

(We used FORMULA 99 with $a=1$, $n=1$)

$= u^2\sin^{-1}u-\left(\frac{1}{2}\sin^{-1}u-\frac{1}{2}u\sqrt{1-u^2}\right)+C = \left(u^2-\frac{1}{2}\right)\sin^{-1}u+\frac{1}{2}u\sqrt{1-u^2}+C$

(We used FORMULA 33 with $a=1$)

$= \left(x-\frac{1}{2}\right)\sin^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x-x^2}+C$

24. $\int \frac{\cos^{-1}\sqrt{x}}{\sqrt{x}}\,dx;\ \begin{bmatrix} u=\sqrt{x} \\ x=u^2 \\ dx=2u\,du\end{bmatrix} \to \int\frac{\cos^{-1}u}{u}\cdot 2u\,du = 2\int\cos^{-1}u\,du = 2\left(u\cos^{-1}u-\frac{1}{1}\sqrt{1-u^2}\right)+C$

(We used FORMULA 97 with $a=1$)

$= 2\left(\sqrt{x}\cos^{-1}\sqrt{x} - \sqrt{1-x}\right) + C$

25. $\int (\cot t)\sqrt{1-\sin^2 t}\, dt = \int \frac{\sqrt{1-\sin^2 t}\,(\cos t)\, dt}{\sin t}; \begin{bmatrix} u = \sin t \\ du = \cos t\, dt \end{bmatrix} \to \int \frac{\sqrt{1-u^2}\, du}{u}$

$= \sqrt{1-u^2} - \ln\left|\frac{1+\sqrt{1-u^2}}{u}\right| + C$

(We used FORMULA 31 with $a = 1$)

$= \sqrt{1-\sin^2 t} - \ln\left|\frac{1+\sqrt{1-\sin^2 t}}{\sin t}\right| + C$

26. $\int \frac{dt}{(\tan t)\sqrt{4-\sin^2 t}} = \int \frac{\cos t\, dt}{(\sin t)\sqrt{4-\sin^2 t}}; \begin{bmatrix} u = \sin t \\ du = \cos t\, dt \end{bmatrix} \to \int \frac{du}{u\sqrt{4-u^2}} = -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-u^2}}{u}\right| + C$

(We used FORMULA 34 with $a = 2$)

$= -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-\sin^2 t}}{\sin t}\right| + C$

27. $\int \frac{dy}{y\sqrt{3+(\ln y)^2}}; \begin{bmatrix} u = \ln y \\ y = e^u \\ dy = e^u\, du \end{bmatrix} \to \int \frac{e^u\, du}{e^u\sqrt{3+u^2}} = \int \frac{du}{\sqrt{3+u^2}} = \ln\left|u + \sqrt{3+u^2}\right| + C$

$= \ln\left|\ln y + \sqrt{3+(\ln y)^2}\right| + C$

(We used FORMULA 20 with $a = \sqrt{3}$)

28. $\int \frac{\cos\theta\, d\theta}{\sqrt{5+\sin^2\theta}}; \begin{bmatrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \to \int \frac{du}{\sqrt{5+u^2}} = \ln\left|u + \sqrt{5+u^2}\right| + C = \ln\left|\sin\theta + \sqrt{5+\sin^2\theta}\right| + C$

(We used FORMULA 20 with $a = \sqrt{5}$)

29. $\int \frac{3\, dr}{\sqrt{9r^2-1}}; \begin{bmatrix} u = 3r \\ du = 3\, dr \end{bmatrix} \to \int \frac{du}{\sqrt{u^2-1}} = \ln\left|u + \sqrt{u^2-1}\right| + C = \ln\left|3r + \sqrt{9r^2-1}\right| + C$

(We used FORMULA 36 with $a = 1$)

30. $\int \frac{3\, dy}{\sqrt{1+9y^2}}; \begin{bmatrix} u = 3y \\ du = 3\, dy \end{bmatrix} \to \int \frac{du}{\sqrt{1+u^2}} = \ln\left|u + \sqrt{1+u^2}\right| + C = \ln\left|3y + \sqrt{1+9y^2}\right| + C$

(We used FORMULA 20 with $a = 1$)

31. $\int \cos^{-1}\sqrt{x}\,dx; \begin{bmatrix} t=\sqrt{x} \\ x=t^2 \\ dx = 2t\,dt \end{bmatrix} \rightarrow 2\int t\cos^{-1}t\,dt = 2\left(\frac{t^2}{2}\cos^{-1}t + \frac{1}{2}\int \frac{t^2}{\sqrt{1-t^2}}\,dt\right) = t^2\cos^{-1}t + \int \frac{t^2}{\sqrt{1-t^2}}\,dt$

(We used FORMULA 100 with $a = 1$, $n = 1$)

$= t^2\cos^{-1}t + \frac{1}{2}\sin^{-1}t - \frac{1}{2}t\sqrt{1-t^2} + C$

(We used FORMULA 33 with $a = 1$)

$= x\cos^{-1}\sqrt{x} + \frac{1}{2}\sin^{-1}\sqrt{x} - \frac{1}{2}\sqrt{x}\sqrt{1-x} + C = x\cos^{-1}\sqrt{x} + \frac{1}{2}\sin^{-1}\sqrt{x} - \frac{1}{2}\sqrt{x-x^2} + C$

32. $\int \tan^{-1}\sqrt{y}\,dy; \begin{bmatrix} t=\sqrt{y} \\ y=t^2 \\ dy = 2t\,dt \end{bmatrix} \rightarrow 2\int t\tan^{-1}t\,dt = 2\left[\frac{t^2}{2}\tan^{-1}t - \frac{1}{2}\int \frac{t^2}{1+t^2}\,dt\right] = t^2\tan^{-1}t - \int \frac{t^2}{1+t^2}\,dt$

(We used FORMULA 101 with $n = 1$, $a = 1$)

$= t^2\tan^{-1}t - \int \frac{t^2+1}{t^2+1}\,dt + \int \frac{dt}{1+t^2} = t^2\tan^{-1}t - t + \tan^{-1}t + C = y\tan^{-1}\sqrt{y} + \tan^{-1}\sqrt{y} - \sqrt{y} + C$

33. $\int xe^{3x}\,dx = \frac{e^{3x}}{3^2}(3x-1) + C = \frac{e^{3x}}{9}(3x-1) + C$

(We used FORMULA 104 with $a = 3$)

34. $\int x^3e^{x/2}\,dx = 2x^3e^{x/2} - 3\cdot 2\int x^2e^{x/2}\,dx = 2x^3e^{x/2} - 6\left(2x^2e^{x/2} - 2\cdot 2\int xe^{x/2}\,dx\right)$

$= 2x^3e^{x/2} - 12x^2e^{x/2} + 24\cdot 4e^{x/2}\left(\frac{x}{2}-1\right) + C = 2x^3e^{x/2} - 12x^2e^{x/2} + 96e^{x/2}\left(\frac{x}{2}-1\right) + C$

(We used FORMULA 105 with $a = \frac{1}{2}$ twice and FORMULA 104 with $a = \frac{1}{2}$)

35. $\int x^2 2^x\,dx = \frac{x^2 2^x}{\ln 2} - \frac{2}{\ln 2}\int x2^x\,dx = \frac{x^2 2^x}{\ln 2} - \frac{2}{\ln 2}\left(\frac{x2^x}{\ln 2} - \frac{1}{\ln 2}\int 2^x\,dx\right) = \frac{x^2 2^x}{\ln 2} - \frac{2}{\ln 2}\left[\frac{x2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2}\right] + C$

(We used FORMULA 106 with $a = 1$, $b = 2$)

36. $\int x\pi^x\,dx = \frac{x\pi^x}{\ln \pi} - \frac{1}{\ln \pi}\int \pi^x\,dx = \frac{x\pi^x}{\ln \pi} - \frac{1}{\ln \pi}\left(\frac{\pi^x}{\ln \pi}\right) + C = \frac{x\pi^x}{\ln \pi} - \frac{\pi^x}{\ln \pi} - \frac{\pi^x}{(\ln \pi)^2} + C$

(We used FORMULA 106 with $n = 1$, $b = \pi$, $a = 1$)

37. $\int \frac{1}{8}\sinh^5 3x\,dx = \frac{1}{8}\left(\frac{\sinh^4 3x\cosh 3x}{5\cdot 3} - \frac{5-1}{5}\int \sinh^3 3x\,dx\right)$

$= \frac{\sinh^4 3x\cosh 3x}{120} - \frac{1}{10}\left(\frac{\sinh^2 3x\cosh 3x}{3\cdot 3} - \frac{3-1}{3}\int \sinh 3x\,dx\right)$

(We used FORMULA 117 with $a = 3$, $n = 5$ and $a = 1$, $n = 3$)

$= \frac{\sinh^4 3x \cosh 3x}{120} - \frac{\sinh^2 3x \cosh 3x}{90} + \frac{2}{30}\left(\frac{1}{3}\cosh 3x\right) + C$

$= \frac{1}{120}\sinh^4 3x \cosh 3x - \frac{1}{90}\sinh^2 3x \cosh 3x + \frac{2}{90}\cosh 3x + C$

38. $\int \frac{\cosh^4 \sqrt{x}}{\sqrt{x}}\,dx; \begin{bmatrix} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \to 2\int \cosh^4 u\,du = 2\left(\frac{\cosh^3 u \sinh u}{4} + \frac{4-1}{4}\int \cosh^2 u\,du\right)$

$= \frac{\cosh^3 u \sinh u}{2} + \frac{3}{2}\left(\frac{\sinh 2u}{4} + \frac{u}{2}\right) + C$

(We used FORMULA 118 with $a = 1$, $n = 2$ and FORMULA 116 with $a = 1$)

$= \frac{1}{2}\cosh^3 \sqrt{x}\sinh \sqrt{x} + \frac{3}{8}\sinh 2\sqrt{x} + \frac{3}{4}\sqrt{x} + C$

39. $\int x^2 \cosh 3x\,dx = \frac{x^2}{3}\sinh 3x - \frac{2}{3}\int x \sinh 3x\,dx = \frac{x^2}{3}\sinh 3x - \frac{2}{3}\left(\frac{x}{3}\cosh 3x - \frac{1}{3}\int \cosh 3x\,dx\right)$

(We used FORMULA 122 with $a = 3$, $n = 2$ and FORMULA 121 with $a = 3$, $n = 1$)

$= \frac{x^2}{3}\sinh 3x - \frac{2x}{9}\cosh 3x + \frac{2}{27}\sinh 3x + C$

40. $\int x \sinh 5x\,dx = \frac{x}{5}\cosh 5x - \frac{1}{25}\sinh 5x + C$

(We used FORMULA 119 with $a = 5$, $n = 1$)

41. $u = ax + b \Rightarrow x = \frac{u-b}{a} \Rightarrow dx = \frac{du}{a}$;

$\int \frac{x\,dx}{(ax+b)^2} = \int \frac{(u-b)}{au^2}\frac{du}{a} = \frac{1}{a^2}\int\left(\frac{1}{u} - \frac{b}{u^2}\right)du = \frac{1}{a^2}\left[\ln|u| + \frac{b}{u}\right] + C = \frac{1}{a^2}\left[\ln|ax+b| + \frac{b}{ax+b}\right] + C$

42. $x = a\sin\theta \Rightarrow a^2 - x^2 = a^2\cos^2\theta \Rightarrow -2x\,dx = -2a^2\cos\theta\sin\theta\,d\theta \Rightarrow dx = a\cos\theta\,d\theta$;

$\int \sqrt{a^2 - x^2}\,dx = \int a\cos\theta(a\cos\theta)\,d\theta = a^2\int \cos^2\theta\,d\theta = \frac{a^2}{2}\int(1 + \cos 2\theta)\,d\theta = \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$

$= \frac{a^2}{2}(\theta + \cos\theta\sin\theta) + C = \frac{a^2}{2}\left(\theta + \sqrt{1 - \sin^2\theta}\cdot\sin\theta\right) + C = \frac{a^2}{2}\left(\sin^{-1}\frac{x}{a} + \frac{\sqrt{a^2 - x^2}}{a}\cdot\frac{x}{a}\right) + C$

$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C$

43. $\int x^n(\ln ax)^m\,dx = \int (\ln ax)^m\,d\left(\frac{x^{n+1}}{n+1}\right) = \frac{x^{n+1}(\ln ax)^m}{n+1} - \int\left(\frac{x^{n+1}}{n+1}\right)m(\ln ax)^{m-1}\left(\frac{1}{x}\right)dx$

$= \frac{x^{n+1}(\ln ax)^m}{n+1} - \frac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx$

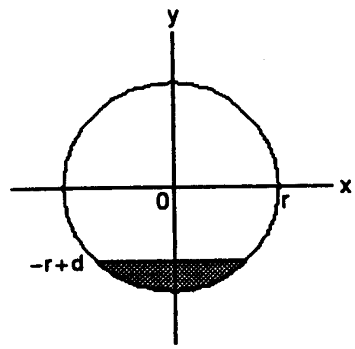
$\left(\text{We used integration by parts } \int u\,dv = uv - \int v\,du \text{ with } u = (\ln ax)^m,\ v = \frac{x^{n+1}}{n+1}\right)$

44. $\int x^n \sin^{-1} ax\, dx = \int \sin^{-1} ax\, d\left(\frac{x^{n+1}}{n+1}\right) = \frac{x^{n+1}}{n+1}\sin^{-1} ax - \int \left(\frac{x^{n+1}}{n+1}\right)\frac{a}{\sqrt{1-(ax)^2}}\, dx$

$= \frac{x^{n+1}}{n+1}\sin^{-1} ax - \frac{a}{n+1}\int \frac{x^{n+1}\, dx}{\sqrt{1-a^2x^2}},\ n \neq -1$

(We used integration by parts $\int u\, dv = uv - \int v\, du$ with $u = \sin^{-1} ax$, $v = \frac{x^{n+1}}{n+1}$)

45. (a) The volume of the filled part equals the length of the tank times the area of the shaded region shown in the accompanying figure. Consider a layer of gasoline of thickness dy located at height y where $-r < y < -r + d$. The width of this layer is $2\sqrt{r^2 - y^2}$. Therefore, $A = 2\int_{-r}^{-r+d} \sqrt{r^2 - y^2}\, dy$ and $V = L \cdot A = 2L\int_{-r}^{-r+d} \sqrt{r^2 - y^2}\, dy$



(b) $2L\int_{-r}^{-r+d} \sqrt{r^2 - y^2}\, dy = 2L\left[\frac{y\sqrt{r^2-y^2}}{2} + \frac{r^2}{2}\sin^{-1}\frac{y}{r}\right]_{-r}^{-r+d}$

(We used FORMULA 29 with a = r)

$= 2L\left[\frac{(d-r)}{2}\sqrt{2rd - d^2} + \frac{r^2}{2}\sin^{-1}\left(\frac{d-r}{r}\right) + \frac{r^2}{2}\left(\frac{\pi}{2}\right)\right] = 2L\left[\left(\frac{d-r}{2}\right)\sqrt{2rd - d^2} + \left(\frac{r^2}{2}\right)\left(\sin^{-1}\left(\frac{d-r}{r}\right) + \frac{\pi}{2}\right)\right]$

46. The integrand $f(x) = \sqrt{x - x^2}$ is nonnegative, so the integral is maximized by integrating over the function's entire domain, which runs from $x = 0$ to $x = 1$

$\Rightarrow \int_0^1 \sqrt{x - x^2}\, dx = \int_0^1 \sqrt{2 \cdot \frac{1}{2}x - x^2}\, dx = \left[\frac{\left(x - \frac{1}{2}\right)}{2}\sqrt{2 \cdot \frac{1}{2}x - x^2} + \frac{\left(\frac{1}{2}\right)^2}{2}\sin^{-1}\left(\frac{x - \frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$

(We used FORMULA 48 with $a = \frac{1}{2}$)

$= \left[\frac{\left(x - \frac{1}{2}\right)}{2}\sqrt{x - x^2} + \frac{1}{8}\sin^{-1}(2x - 1)\right]_0^1 = \frac{1}{8}\cdot\frac{\pi}{2} - \frac{1}{8}\left(-\frac{\pi}{2}\right) = \frac{\pi}{8}$

CAS EXPLORATIONS

For MAPLE use the int(f(x),x) command, and for MATHEMATICA use the command Integrate[f(x),x], as discussed in the text.

47. (e) $\int x^n \ln x\, dx = \frac{x^{n+1}\ln x}{n+1} - \frac{1}{n+1}\int x^n\, dx,\ n \neq -1$

(We used FORMULA 110 with a = 1, m = 1)

$$= \frac{x^{n+1} \ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C$$

48. (e) $\int x^{-n} \ln x \, dx = \frac{x^{-n+1} \ln x}{-n+1} - \frac{1}{(-n)+1} \int x^{-n} \, dx, \; n \neq 1$

(We used FORMULA 110 with $a = 1$, $m = 1$, $n = -n$)

$$= \frac{x^{1-n} \ln x}{1-n} - \frac{1}{1-n}\left(\frac{x^{1-n}}{1-n}\right) + C = \frac{x^{1-n}}{1-n}\left(\ln x - \frac{1}{1-n}\right) + C$$

49. (a) Neither MAPLE nor MATHEMATICA can find this integral for arbitrary n.

(b) MAPLE and MATHEMATICA get stuck at about $n = 5$.

(c) Let $x = \frac{\pi}{2} - u \Rightarrow dx = -du$; $x = 0 \Rightarrow u = \frac{\pi}{2}$, $x = \frac{\pi}{2} \Rightarrow u = 0$;

$$I = \int_0^{\pi/2} \frac{\sin^n x \, dx}{\sin^n x + \cos^n x} = \int_{\pi/2}^{0} \frac{-\sin^n\left(\frac{\pi}{2} - u\right) du}{\sin^n\left(\frac{\pi}{2} - u\right) + \cos^n\left(\frac{\pi}{2} - u\right)} = \int_0^{\pi/2} \frac{\cos^n u \, du}{\cos^n u + \sin^n u} = \int_0^{\pi/2} \frac{\cos^n x \, dx}{\cos^n x + \sin^n x}$$

$$\Rightarrow I + I = \int_0^{\pi/2} \left(\frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x}\right) dx = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

The following *Mathematica* module is used to obtain the Monte Carlo estimates of area in Problems 50 through 55.

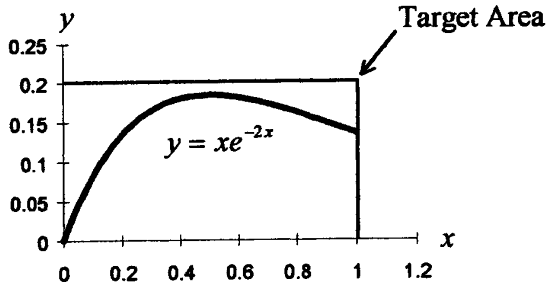
```
monte[f_, indvar_, m_, a_, b_, n_List] :=
 Module[{g, x, xr, yr, area, lim, areaavg, y1, y2},
  g = f/. indvar -> x;
  lim = Length[n];
  area - Table[0, {k, 1, lim}];
  For[k = 1, k <= lim, k++,
    For[counter = 0; i = 1, i <= n[[k]], i++,
     xr = a+ (b - a)*Random[];
     yr = m*Random[];
   If[yr <= g/. x -> xr, counter = counter + 1];];
  area[[k]] = m*(b - a)*counter/n[[k]]];
  areaavg = (Sum[n[[i]]*area[[i]], {i, 1, lim}]) /
    Sum[n[i]], {i, 1, lim}];
  y1 = Integrate[g, {x, a, b}] // N;
  y2 = Integrate[g, {x, a, b}];
  Print[area];
  Print[areaavg];
  Print[y2];
  Print y1  ;
```

The following command executes the preceding module. The arguments are the integrand function, the independent variable, an upper bound on the integrand function, the lower limit of integration, the upper limit of integration, and a list of the numbers of random points to generate in each estimation.

```
monte[z*Sqrt[1 - z], z, 0.5, 0, 1, {100, 200, 300, 400,
 500, 600, 700, 800, 900, 1000, 2000, 3000, 4000,
 5000, 6000, 8000, 10000, 15000, 20000, 30000
```

The preceding command is for Problem 51.

50.



Select M = 0.2

The area approximations will vary depending on the random number generator and seed value that is used

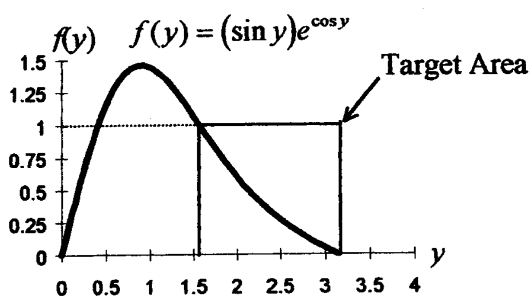
Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.154	2000	0.1492
200	0.151	3000	0.147867
300	0.148	4000	0.1497
400	0.149	5000	0.14712
500	0.1528	6000	0.148433
600	0.151667	8000	0.147925
700	0.149429	10,000	0.14796
800	0.1435	15,000	0.148147
900	0.146444	20,000	0.14824
1000	0.1408	30,000	0.147687

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_0^1 xe^{-2x}\,dx \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area}(i)\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.147987 \text{ by Monte Carlo.}$$

The actual value of the integral is $\frac{(1-3e^2)}{4} \approx 0.148499$.

51.



Select M = 1

The area approximations will vary depending on the random number generator and seed value that is used

Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.722566	2000	0.628319
200	0.628319	3000	0.646121
300	0.586431	4000	0.642456
400	0.581195	5000	0.636487
500	0.637743	6000	0.627533
600	0.560251	8000	0.643437
700	0.583439	10,000	0.62235
800	0.577268	15,000	0.625386
900	0.5621337	20,000	0.635073
1000	0.655022	30,000	0.638895