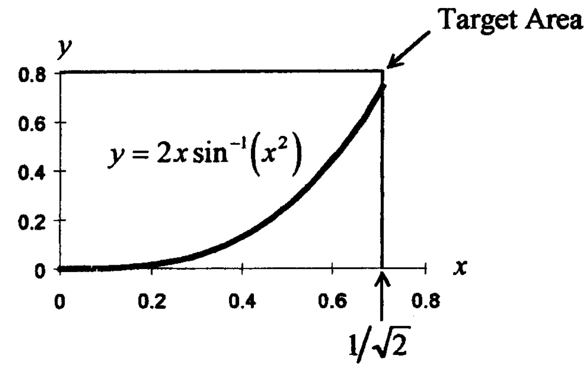
A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_{\pi/2}^{\pi} (\sin y)e^{\cos y}\, dy \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area}(i)\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.63298 \text{ by Monte Carlo.}$$

The actual value of the integral is $1 - \frac{1}{e} \approx 0.632121$.

52.



Select M = 0.8

The area approximations will vary depending on the random number generator and seed value that is used

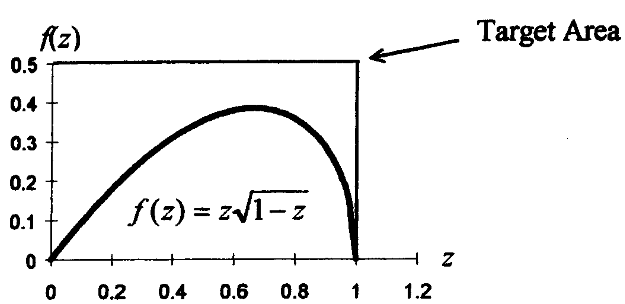
Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.152735	2000	0.129542
200	0.10748	3000	0.133879
300	0.118794	4000	0.125724
400	0.130108	5000	0.123206
500	0.139159	6000	0.130956
600	0.129165	8000	0.128693
700	0.118794	10,000	0.127279
800	0.123744	15,000	0.129844
900	0.121308	20,000	0.129712
1000	0.122188	30,000	0.128335

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_{0}^{1/\sqrt{2}} 2x \sin^{-1}(x^2)\, dx \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area}(i)\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.128523 \text{ by Monte Carlo.}$$

The actual value of the integral is $\frac{\pi - 12 + 6\sqrt{3}}{12} \approx 0.127825$.

53.



Select M = 0.5
The area approximations will vary depending on the random number generator and seed value that is used

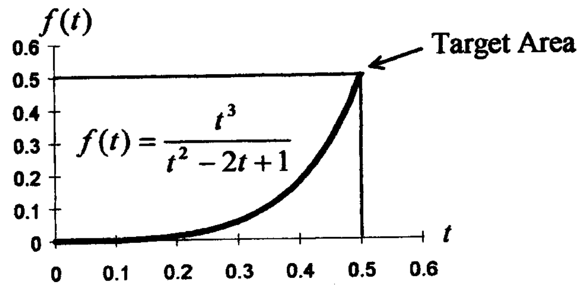
Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.28	2000	0.259
200	0.265	3000	0.262167
300	0.278333	4000	0.259625
400	0.2625	5000	0.2724
500	0.261	6000	0.270583
600	0.27	8000	0.265875
700	0.254286	10,000	0.26495
800	0.270625	15,000	0.2668
900	0.277778	20,000	0.268275
1000	0.2685	30,000	0.265875

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_0^1 z\sqrt{1-z}\,dz \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area(i)}\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.266465 \text{ by Monte Carlo.}$$

The actual value of the integral is $\frac{4}{15} \approx 0.266667$.

54.



Select M = 0.5
The area approximations will vary depending on the random number generator and seed value that is used

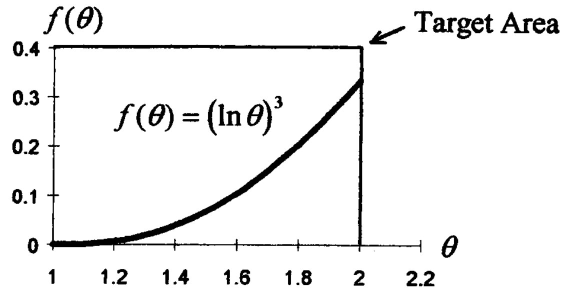
Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.0375	2000	0.04725
200	0.06	3000	0.0435
300	0.06	4000	0.0480625
400	0.0425	5000	0.046
500	0.0435	6000	0.04525
600	0.05125	8000	0.0445937
700	0.0439286	10,000	0.047375
800	0.053125	15,000	0.0449
900	0.0472222	20,000	0.0446375
1000	0.0425	30,000	0.0458

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_0^{1/2} \frac{t^3\,dt}{t^2-2t+1} \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area(i)}\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.0456313 \text{ by Monte Carlo.}$$

The actual value of the integral is $\frac{17}{8} - 3\ln 2 \approx 0.0455585$.

55.



Select M = 0.4
The area approximations will vary depending on the random number generator and seed value that is used

Number of Points	Approximation of Area	Number of Points	Approximation of Area
100	0.096	2000	0.095
200	0.104	3000	0.103467
300	0.0986667	4000	0.0999
400	0.095	5000	0.10096
500	0.0992	6000	0.1048
600	0.096	8000	0.10105
700	0.0908571	10,000	0.104
800	0.0985	15,000	0.0995733
900	0.1	20,000	0.1013
1000	0.104	30,000	0.100707

A weighted average of the areas in the table is used to estimate the integral. Therefore,

$$\int_1^2 (\ln \theta)^3\, d\theta \approx \left(\sum_{i=1}^{20} n_i \cdot \text{area}(i)\right) \Big/ \left(\sum_{i=1}^{20} n(i)\right) = 0.101054 \text{ by Monte Carlo.}$$

The actual value of the integral is $6 + 2\left[(\ln 2)^3 - 3(\ln 2)^2 + 6\ln 2 - 6\right] \approx 0.101097$.

7.6 L'HÔPITAL'S RULE

1. l'Hôpital: $\lim\limits_{x\to 2} \frac{x-2}{x^2-4} = \frac{1}{2x}\Big|_{x=2} = \frac{1}{4}$ or $\lim\limits_{x\to 2} \frac{x-2}{x^2-4} = \lim\limits_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim\limits_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

2. l'Hôpital: $\lim\limits_{x\to 0} \frac{\sin 5x}{x} = \frac{5\cos 5x}{1}\Big|_{x=0} = 5$ or $\lim\limits_{x\to 0} \frac{\sin 5x}{x} = 5 \lim\limits_{\substack{x\to 0\\ 5x\to 0}} \frac{\sin 5x}{5x} = 5 \cdot 1 = 5$

3. l'Hôpital: $\lim\limits_{x\to \infty} \frac{5x^3-3x}{7x^2+1} = \lim\limits_{x\to \infty} \frac{10x-3}{14x} = \lim\limits_{x\to \infty} \frac{10}{14} = \frac{5}{7}$ or $\lim\limits_{x\to \infty} \frac{5x^2-3x}{7x^2+1} = \lim\limits_{x\to \infty} \frac{5-\frac{3}{x}}{7+\frac{1}{x}} = \frac{5}{7}$

4. l'Hôpital: $\lim\limits_{x\to 1} \frac{x^3-1}{4x^3-x-3} = \lim\limits_{x\to 1} \frac{3x^2}{12x^2-1} = \frac{3}{11}$ or $\lim\limits_{x\to 1} \frac{x^3-1}{4x^3-x-3} = \lim\limits_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(4x^2+4x+3)}$

$= \lim\limits_{x\to 1} \frac{(x^2+x+1)}{(4x^2+4x+3)} = \frac{3}{11}$

5. l'Hôpital: $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$ or $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \left[\frac{1-\cos x}{x^2}\left(\frac{1+\cos x}{1+\cos x}\right)\right]$

$= \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)} = \lim_{x\to 0} \left[\left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{x}\right)\left(\frac{1}{1+\cos x}\right)\right] = \frac{1}{2}$

6. l'Hôpital: $\lim_{x\to\infty} \frac{2x^2+3x}{x^3+x+1} = \lim_{x\to\infty} \frac{4x+3}{3x^2+1} = \lim_{x\to\infty} \frac{4}{6x^2} = 0$ or $\lim_{x\to\infty} \frac{2x^2+3x}{x^3+x+1} = \lim_{x\to\infty} \frac{\frac{2}{x}+\frac{3}{x^2}}{1+\frac{1}{x^2}+\frac{1}{x^3}} = \frac{0}{1} = 0$

7. $\lim_{\theta\to 0} \frac{\sin \theta^2}{\theta} = \lim_{\theta\to 0} \frac{2\theta \cos \theta^2}{1} = (2)(0)\cos(0)^2 = 0$

8. $\lim_{\theta\to\pi/2} \frac{1-\sin\theta}{1+\cos 2\theta} = \lim_{\theta\to\pi/2} \frac{-\cos\theta}{-2\sin 2\theta} = \lim_{\theta\to\pi/2} \frac{\sin\theta}{-4\cos 2\theta} = \frac{\sin \pi/2}{-4\cos\pi} = \frac{1}{4}$

9. $\lim_{t\to 0} \frac{\cos t - 1}{e^t - t - 1} = \lim_{t\to 0} \frac{-\sin t}{e^t - 1} = \lim_{t\to 0} \frac{-\cos t}{e^t} = -1$

10. $\lim_{t\to 1} \frac{t-1}{\ln t - \sin \pi t} = \lim_{t\to 1} \frac{1}{\frac{1}{t} - \pi\cos \pi t} = \frac{1}{1-\pi(-1)} = \frac{1}{\pi+1}$

11. $\lim_{x\to\infty} \frac{\ln(x+1)}{\log_2 t} = \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x\ln 2}} = \lim_{x\to\infty} \frac{x\ln 2}{x+1} = \lim_{x\to\infty} \ln 2 = \ln 2$

12. $\lim_{x\to\infty} \frac{\log_2 x}{\log_3(x+3)} = \lim_{x\to\infty} \frac{\frac{1}{x\ln 2}}{\frac{1}{(x+3)\ln 3}} = \lim_{x\to\infty} \frac{(x+3)\ln 3}{x\ln 2} = \lim_{x\to\infty} \frac{x\ln 3 + 3\ln 3}{x\ln 2} = \lim_{x\to\infty} \frac{\ln 3}{\ln 2} = \frac{\ln 3}{\ln 2}$

13. $\lim_{y\to 0^+} \frac{\ln(y^2+2y)}{\ln y} = \lim_{y\to 0^+} \frac{\frac{2y+2}{y^2+2y}}{\frac{1}{y}} = \lim_{y\to 0^+} \frac{y(2y+2)}{y^2+2y} = \lim_{y\to 0^+} \frac{2y^2+2y}{y^2+2y} = \lim_{y\to 0^+} \frac{4y+2}{2y+2} = \frac{4(0)+2}{2(0)+2} = \frac{2}{2} = 1$

14. $\lim_{y\to\pi/2} \left(\frac{\pi}{2}-y\right)\tan y = \lim_{y\to\pi/2} \frac{\left(\frac{\pi}{2}-y\right)\sin y}{\cos y} = \lim_{y\to\pi/2} \frac{\left(\frac{\pi}{2}-y\right)\cos y + (-1)\sin y}{-\sin y} = \frac{\left(\frac{\pi}{2}-\frac{\pi}{2}\right)\cos\frac{\pi}{2} + (-1)\sin\frac{\pi}{2}}{-\sin\frac{\pi}{2}}$

$= \frac{(-1)(1)}{-(1)} = 1$

15. $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{-x^2}{x} = \lim_{x\to 0^+} -x = 0$

16. $\lim\limits_{x\to\infty} x \tan \frac{1}{x} = \lim\limits_{x\to\infty} \frac{\tan \frac{1}{x}}{\frac{1}{x}} = \lim\limits_{x\to\infty} \frac{-\frac{1}{x^2}\sec^2 \frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty} \sec^2 \frac{1}{x} = \sec^2 0 = 1$

17. $\lim\limits_{x\to0^+} (\csc x - \cot x + \cos x) = \lim\limits_{x\to0^+} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} + \cos x\right) = \lim\limits_{x\to0^+} \frac{1 - \cos x + \cos x \sin x}{\sin x}$

$= \lim\limits_{x\to0^+} \frac{\sin x + \cos x \cos x - \sin x \sin x}{\cos x} = 1$

18. $\lim\limits_{x\to\infty} (\ln 2x - \ln(x+1)) = \lim\limits_{x\to\infty} \ln\left(\frac{2x}{x+1}\right)$; Let $f(x) = \frac{2x}{x+1} \Rightarrow \lim\limits_{x\to\infty} \frac{2x}{x+1} = \lim\limits_{x\to\infty} \frac{2}{1} = 2$. Therefore,

$\lim\limits_{x\to\infty} (\ln 2x - \ln \sin x) = \lim\limits_{x\to\infty} \ln f(x) = \ln 2$

19. $\lim\limits_{x\to0^+} (\ln x - \ln \sin x) = \lim\limits_{x\to0^+} \ln \frac{x}{\sin x}$; let $f(x) = \frac{x}{\sin x} \Rightarrow \lim\limits_{x\to0^+} \frac{x}{\sin x} = \lim\limits_{x\to0^+} \frac{1}{\cos x} = 1$. Therefore,

$\lim\limits_{x\to0^+} (\ln x - \ln \sin x) = \lim\limits_{x\to0^+} \ln f(x) = \ln 1 = 0$

20. $\lim\limits_{x\to0^+} \left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right) = \lim\limits_{x\to0^+} \frac{1-\sqrt{x}}{x} = \infty$

21. The limit leads to the indeterminate form 1^∞. Let $f(x) = (e^x + x)^{1/x} \Rightarrow \ln (e^x + x)^{1/x} = \frac{\ln (e^x + x)}{x}$

$\Rightarrow \lim\limits_{x\to0} \frac{\ln (e^x + x)}{x} = \lim\limits_{x\to0} \frac{\frac{e^x+1}{e^x+x}}{1} = 2 \Rightarrow \lim\limits_{x\to0} (e^x + x)^{1/x} \lim\limits_{x\to0} e^{\ln f(x)} = e^2$

22. The limit leads to the indeterminate form ∞^0. Let $f(x) = \left(\frac{1}{x^2}\right)^x \Rightarrow \ln\left(\frac{1}{x^2}\right)^x = x \ln\left(\frac{1}{x^2}\right) = \frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}}$

$\Rightarrow \lim\limits_{x\to0} \frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}} = \lim\limits_{x\to0} \frac{\frac{-2/x^3}{1/x^2}}{-1/x^2} = \lim\limits_{x\to0} 2x = 0 \Rightarrow \lim\limits_{x\to0} \left(\frac{1}{x^2}\right)^x = \lim\limits_{x\to0} e^{\ln f(x)} = e^0 = 1$

23. $\lim\limits_{x\to\pm\infty} \frac{3x-5}{2x^2-x+2} = \lim\limits_{x\to\pm\infty} \frac{3}{4x-1} = 0$

24. $\lim\limits_{x\to0} \frac{\sin 7x}{\tan 11x} = \lim\limits_{x\to0} \frac{7 \cos 7x}{11 \sec^2 11x} = \frac{7}{11}$

25. The limit leads to the indeterminate form ∞^0. Let $f(x) = (\ln x)^{1/x} \Rightarrow \ln (\ln x)^{1/x} = \frac{\ln (\ln x)}{x}$

$\Rightarrow \lim\limits_{x\to\infty} \frac{\ln (\ln x)}{x} = \lim\limits_{x\to\infty} \frac{\frac{1/x}{\ln x}}{1} = \lim\limits_{x\to\infty} \frac{1}{x \ln x} = 0 \Rightarrow \lim\limits_{x\to\infty} (\ln x)^{1/x} = \lim\limits_{x\to\infty} e^{\ln f(x)} = e^0 = 1$

26. The limit leads to the indeterminate form ∞^0. Let $f(x) = (1+2x)^{1/(2\ln x)} \Rightarrow \ln(1+2x)^{1/(2\ln x)} = \frac{\ln(1+2x)}{2\ln x}$

$$\Rightarrow \lim_{x\to\infty} \frac{\ln(1+2x)}{2\ln x} = \lim_{x\to\infty} \frac{\frac{2}{1+2x}}{\frac{2}{x}} = \lim_{x\to\infty} \frac{x}{1+2x} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2} \Rightarrow \lim_{x\to\infty} (1+2x)^{1/(2\ln x)}$$

$$= \lim_{x\to\infty} e^{\ln f(x)} = e^{1/2} = \sqrt{e}$$

27. The limit leads to the indeterminate form 0^0. Let $f(x) = (x^2-2x+1)^{x-1}$

$$\Rightarrow \ln(x^2-2x+1)^{x-1} = (x-1)\ln(x^2-2x+1) = \frac{\ln(x^2-2x+1)}{\frac{1}{x-1}} \Rightarrow \lim_{x\to 1} \frac{\ln(x^2-2x+1)}{\frac{1}{x-1}} = \lim_{x\to 1} \frac{\frac{2x-2}{x^2-2x+1}}{-\frac{1}{(x-1)^2}}$$

$$= \lim_{x\to 1} \frac{\frac{2(x-1)}{(x-1)^2}}{-\frac{1}{(x-1)^2}} = \lim_{x\to 1} -2(x-1) = 0 \Rightarrow \lim_{x\to 1} (x^2-2x+1)^{x-1} = \lim_{x\to 1} e^{\ln f(x)} = e^0 = 1$$

28. The limit leads to the indeterminate form 0^0. Let $f(x) = (\cos x)^{\cos x} \Rightarrow \ln(\cos x)^{\cos x}$

$$= (\cos x)\ln(\cos x) = \frac{\ln(\cos x)}{\sec x} \Rightarrow \lim_{x\to\pi/2^-} \frac{\ln(\cos x)}{\sec x} = \lim_{x\to\pi/2^-} \frac{\frac{-\sin x}{\cos x}}{\sec x\tan x} = \lim_{x\to\pi/2^-} \frac{-\tan x}{\sec x\tan x}$$

$$= \lim_{x\to\pi/2^-} -\cos x = 0 \Rightarrow \lim_{x\to\pi/2^-} (\cos x)^{\cos x} = \lim_{x\to\pi/2^-} e^{\ln f(x)} = e^0 = 1$$

29. The limit leads to the indeterminate form 1^∞. Let $f(x) = (1+x)^{1/x} \Rightarrow \ln(1+x)^{1/x} = \frac{\ln(1+x)}{x}$

$$\Rightarrow \lim_{x\to 0^+} \frac{\ln(1+x)}{x} = \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = 1 \Rightarrow \lim_{x\to 0^+} (1+x)^{1/x} = \lim_{x\to 0^+} e^{\ln f(x)} = e^1 = e$$

30. The limit leads to the indeterminate form 1^∞. Let $f(x) = x^{1/(x-1)} \Rightarrow \ln x^{1/(x-1)} = \frac{\ln x}{x-1}$

$$\Rightarrow \lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = 1 \Rightarrow \lim_{x\to 1} x^{1/(x-1)} = \lim_{x\to 1} e^{\ln f(x)} = e^1 = e$$

31. The limit leads to the indeterminate form 0^0. Let $f(x) = (\sin x)^x \Rightarrow \ln(\sin x)^x = x\ln(\sin x) = \frac{\ln(\sin x)}{\frac{1}{x}}$

$$\Rightarrow \lim_{x\to 0^+} \frac{\ln(\sin x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{-x^2\cos x}{\sin x} = \lim_{x\to 0^+} \frac{x^2\sin x - 2x\cos x}{\cos x} = 0$$

$$\Rightarrow \lim_{x\to 0^+} (\sin x)^x = \lim_{x\to 0^+} e^{\ln f(x)} = e^0 = 1$$

32. The limit leads to the indeterminate form 0^0. Let $f(x) = (\sin x)^{\tan x} \Rightarrow \ln(\sin x)^{\tan x}$

$$= \tan x \ln(\sin x) = \frac{\ln(\sin x)}{\cot x} \Rightarrow \lim_{x\to 0^+} \frac{\ln(\sin x)}{\cot x} = \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{-\csc^2 x} = \lim_{x\to 0^+} (-\sin x \cos x) = 0$$

$$\Rightarrow \lim_{x\to 0^+} (\sin x)^{\tan x} = \lim_{x\to 0^+} e^{\ln f(x)} = e^0 = 1$$

33. The limit leads to the indeterminate form $1^{-\infty}$. Let $f(x) = x^{1/(1-x)} \Rightarrow \ln x^{1/(1-x)} = \frac{\ln x}{1-x}$

$$\Rightarrow \lim_{x\to 1^+} \frac{\ln x}{1-x} = \lim_{x\to 1^+} \frac{\frac{1}{x}}{-1} = -1 \Rightarrow \lim_{x\to 1^+} x^{1/(1-x)} = \lim_{x\to 1^+} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$$

34. $\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$

35. $\lim_{x\to\infty} \int_x^{2x} \frac{1}{t}\,dt = \lim_{x\to\infty} \left[\ln|t|\right]_x^{2x} = \lim_{x\to\infty} \ln\left(\frac{2x}{x}\right) = \ln 2$

36. $\lim_{x\to\infty} \frac{\int_1^x \ln t\,dt}{x \ln x} = \lim_{x\to\infty} \frac{\ln x}{\ln x + 1} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = 1$

37. $\lim_{\theta\to 0} \frac{\cos\theta - 1}{e^\theta - \theta - 1} = \lim_{\theta\to 0} \frac{-\sin\theta}{e^\theta - 1} = \lim_{\theta\to 0} \frac{-\cos\theta}{e^\theta} = -1$

38. $\lim_{t\to\infty} \frac{e^t + t^2}{e^t - 1} = \lim_{t\to\infty} \frac{e^t + 2t}{e^t} = \lim_{t\to\infty} \frac{e^t + 2}{e^t} = \lim_{t\to\infty} \frac{e^t}{e^t} = 1$

39. $\lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}} = \sqrt{\lim_{x\to\infty} \frac{9x+1}{x+1}} = \sqrt{\lim_{x\to\infty} \frac{9}{1}} = \sqrt{9} = 3$

40. $\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}} = \sqrt{\frac{1}{\lim_{x\to 0^+} \frac{\sin x}{x}}} = \sqrt{\frac{1}{1}} = 1$

41. $\lim_{x\to\pi/2^-} \frac{\sec x}{\tan x} = \lim_{x\to\pi/2^-} \left(\frac{1}{\cos x}\right)\left(\frac{\cos x}{\sin x}\right) = \lim_{x\to\pi/2^-} \frac{1}{\sin x} = 1$

42. $\lim_{x\to 0^+} \frac{\cot x}{\csc x} = \lim_{x\to 0^+} \frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{1}{\sin x}\right)} = \lim_{x\to 0^+} \cos x = 1$

43. Part (b) is correct because part (a) is neither in the $\frac{0}{0}$ nor $\frac{\infty}{\infty}$ form and so l'Hôpital's rule may not be used.

44. Answers may vary.

(a) $f(x) = 3x + 1$; $g(x) = x$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{3x+1}{x} = \lim_{x\to\infty} \frac{3}{1} = 3$$

(b) $f(x) = x + 1$; $g(x) = x^2$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x+1}{x^2} = \lim_{x\to\infty} \frac{1}{2x} = 0$$

(c) $f(x) = x^2$; $g(x) = x + 1$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{x+1} = \lim_{x\to\infty} \frac{2x}{1} = \infty$$

45. If f(x) is to be continuous at $x = 0$, then $\lim_{x\to 0} f(x) = f(0) \Rightarrow c = f(0) = \lim_{x\to 0} \frac{9x - 3\sin 3x}{5x^3} = \lim_{x\to 0} \frac{9 - 9\cos 3x}{15x^2}$ $= \lim_{x\to 0} \frac{27\sin 3x}{30x} = \lim_{x\to 0} \frac{81\cos 3x}{30} = \frac{27}{10}$.

46. (a) For $x \neq 0$, $f'(x) = \frac{d}{dx}(x+2) = 1$ and $g'(x) = \frac{d}{dx}(x+1) = 1$. Therefore, $\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \frac{1}{1} = 1$, while

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{x+2}{x+1} = \frac{0+2}{0+1} = 2.$$

(b) This does not contradict l'Hôpital's rule because neither f nor g is differentiable at $x = 0$ (as evidenced by the fact that neither is continuous at $x = 0$), so l'Hôpital's rule does not apply.

47. (a) The limit leads to the indeterminate form 1^∞. Let $f(k) = \left(1 + \frac{r}{k}\right)^{kt} \Rightarrow \ln f(k) = kt \ln\left(1 + \frac{r}{k}\right) = \frac{t\ln\left(1+\frac{r}{k}\right)}{\frac{1}{k}}$

$$\Rightarrow \lim_{k\to\infty} \frac{t\ln\left(1+\frac{r}{k}\right)}{\frac{1}{k}} = \lim_{k\to\infty} \frac{t\left(-\frac{r}{k^2}\right)\left(1+\frac{r}{k}\right)^{-1}}{-\frac{1}{k^2}} = \lim_{k\to\infty} \frac{rt}{1+\frac{r}{k}} = \frac{rt}{1} = rt$$

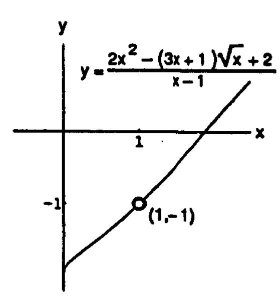
$$\Rightarrow \lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt} = A_0 \lim_{k\to\infty} \left(1+\frac{r}{k}\right)^{kt} = A_0 \lim_{k\to\infty} e^{\ln f(k)} = A_0 e^{rt}$$

(b) Part (a) shows that as the number of compoundings per year increases toward infinity, the limit of interest compounded k times per year is interest compounded continuously.

48. The graph indicates a limit near -1. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x\to 1} \frac{2x^2 - (3x+1)\sqrt{x} + 2}{x-1}$

$$= \lim_{x\to 1} \frac{2x^2 - 3x^{3/2} - x^{1/2} + 2}{x-1} = \lim_{x\to 1} \frac{4x - \frac{9}{2}x^{1/2} - \frac{1}{2}x^{-1/2}}{1}$$

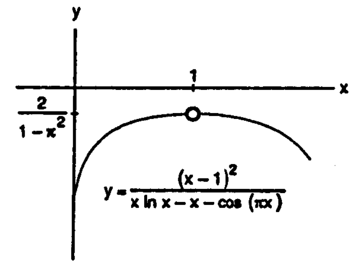
$$= \frac{4 - \frac{9}{2} - \frac{1}{2}}{1} = \frac{4-5}{1} = -1$$



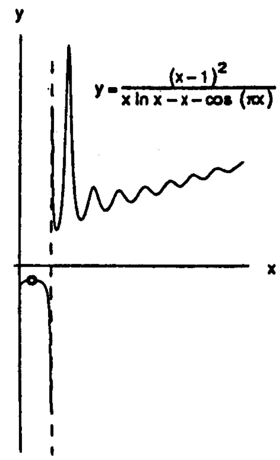
49. (a) The graph indicates a limit near −0.225. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{x\to 1} \dfrac{(x-1)^2}{x \ln x - x - \cos(\pi x)}$

$$= \lim_{x\to 1} \frac{2(x-1)}{\ln x + 1 - 1 + \pi \sin(\pi x)} = \lim_{x\to 1} \frac{2}{\frac{1}{x} + \pi^2 \cos(\pi x)}$$

$$= \frac{2}{1+\pi^2(-1)} = \frac{2}{1-\pi^2}$$



(b) The graph of $y = \dfrac{(x-1)^2}{x \ln x - x - \cos(\pi x)}$ has a vertical asymptote near x = 2.552.



50. (a) $\ln f(x)^{g(x)} = g(x) \ln f(x)$

$$\lim_{x\to c} (g(x) \ln f(x)) = (\lim_{x\to c} g(x))(\lim_{x\to c} \ln f(x)) = \infty(-\infty) = -\infty$$

$$\lim_{x\to c} f(x)^{g(x)} = \lim_{x\to c} e^{\ln f(x)^{g(x)}} = e^{-\infty} = 0$$

(b) $\lim\limits_{x\to c} (g(x) \ln f(x)) = (\lim\limits_{x\to c} g(x))(\lim\limits_{x\to c} \ln f(x)) = (-\infty)(-\infty) = \infty$

$$\lim_{x\to c} f(x)^{g(x)} = \lim_{x\to c} e^{\ln f(x)^{g(x)}} = e^{\infty} = \infty$$

51. (a) Because the difference in the numerator is so small compared to the values being subtracted, any calculator or computer with limited precision will give the incorrect result that $1 - \cos x^6$ is 0 for even moderately small values of x. For example, at x = 0.1, $\cos x^6 \approx 0.9999999999995$ (13 places), so on a 10-place calculator, $\cos x^6 = 1$ and $1 - \cos x^6 = 0$.

(b) Same reason as in part (a) applies.

(c) $\lim\limits_{x\to 0} \dfrac{1-\cos x^6}{x^{12}} = \lim\limits_{x\to 0} \dfrac{6x^5 \sin x^6}{12x^{11}} = \lim\limits_{x\to 0} \dfrac{\sin x^6}{2x^6} = \lim\limits_{x\to 0} \dfrac{6x^5 \cos x^6}{12x^5} = \lim\limits_{x\to 0} \dfrac{\cos x^6}{2} = \dfrac{1}{2}$

(d) The graph and/or table on a grapher shows the value of the function to be 0 for x-values moderately close to 0, but the limit is 1/2. The calculator is giving unreliable information because there is significant round-off error in computing values of this function on a limited precision device.

52. (b) The limit leads to the indeterminate form $\infty - \infty$:

$$\lim_{x\to\infty} \left(x - \sqrt{x^2+x}\right) = \lim_{x\to\infty} \left(x - \sqrt{x^2+x}\right)\left(\frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}}\right) = \lim_{x\to\infty} \left(\frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}}\right) = \lim_{x\to\infty} \left(\frac{-x}{x+\sqrt{x^2+x}}\right)$$

$$= \lim_{x\to\infty} \left(\frac{-1}{1+\sqrt{1+\frac{1}{x}}}\right) = \frac{-1}{1+\sqrt{1+0}} = -\frac{1}{2}$$

53. (a) $f(x) = e^{x \ln(1+1/x)}$

$1 + \frac{1}{x} > 0$ when $x < -1$ or $x > 0$

Domain: $(-\infty, -1) \cup (0, \infty)$

(b) The form is 0^{-1}, so $\lim_{x\to -1} f(x) = \infty$

(c) $$\lim_{x\to -\infty} x \ln\left(1+\tfrac{1}{x}\right) = \lim_{x\to -\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to -\infty} \frac{\left(-\frac{1}{x^2}\right)\left(1+\frac{1}{x}\right)^{-1}}{-\frac{1}{x^2}} = \lim_{x\to -\infty} \frac{1}{1+\frac{1}{x}} = 1$$

$$\Rightarrow \lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} e^{x \ln(1+1/x)} = e$$

54. (a) $y = x^{1/x} \Rightarrow \ln y = \frac{\ln x}{x} \Rightarrow \frac{y'}{y} = \frac{\left(\frac{1}{x}\right)(x) - \ln x}{x^2} \Rightarrow y' = \left(\frac{1-\ln x}{x^2}\right)\left(x^{1/x}\right)$. The sign pattern is

$y' = \underset{0}{|} + + + + + \underset{e}{|} - - - -$ which indicates a maximum value of $y = e^{1/e}$ when $x = e$

(b) $y = x^{1/x^2} \Rightarrow \ln y = \frac{\ln x}{x^2} \Rightarrow \frac{y'}{y} = \frac{\left(\frac{1}{x}\right)(x^2) - 2x \ln x}{x^4} \Rightarrow y' = \left(\frac{1 - 2\ln x}{x^3}\right)\left(x^{1/x^2}\right)$. The sign pattern is

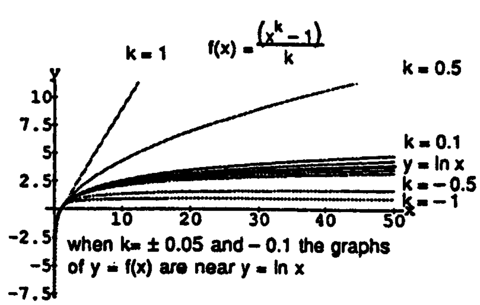
$y' = \underset{0}{|} + + + \underset{\sqrt{e}}{|} - - - -$ which indicates a maximum of $y = e^{1/2e}$ when $x = \sqrt{e}$

(c) $y = x^{1/x^n} \Rightarrow \ln y = \frac{\ln x}{x^n} = \frac{\left(\frac{1}{x}\right)(x^n) - (\ln x)(nx^{n-1})}{x^{2n}} \Rightarrow y' = \frac{x^{n-1}(1 - n \ln x)}{x^{2n}} \cdot x^{1/x^n}$. The sign pattern is

$y' = \underset{0}{|} + + + \underset{\sqrt[n]{e}}{|} - - - -$ which indicates a maximum of $y = e^{1/ne}$ when $x = \sqrt[n]{e}$

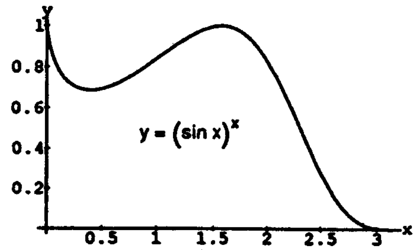
(d) $$\lim_{x\to\infty} x^{1/x^n} = \lim_{x\to\infty} \left(e^{\ln x}\right)^{1/x^n} = \lim_{x\to\infty} e^{(\ln x)/x^n} = \exp\left(\lim_{x\to\infty} \frac{\ln x}{x^n}\right) = \exp\left(\lim_{x\to\infty} \left(\frac{1}{nx^n}\right)\right) = e^0 = 1$$

55. (a)



(b) $\lim_{k\to 0} \frac{x^k - 1}{k} = \lim_{k\to 0} \frac{x^k \ln x}{1} = \ln x$

56. (a) We should assign the value 1 to $f(x) = (\sin x)^x$ to make it continuous at $x = 0$.

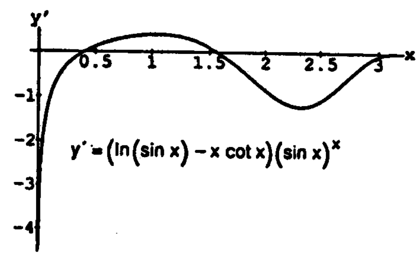


(b) $\ln f(x) = x \ln(\sin x) = \frac{\ln(\sin x)}{\left(\frac{1}{x}\right)} \Rightarrow \lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} \frac{\ln(\sin x)}{\left(\frac{1}{x}\right)} = \lim_{x\to 0^+} \frac{\left(\frac{1}{\sin x}\right)(\cos x)}{\left(-\frac{1}{x^2}\right)}$

$= \lim_{x\to 0} \frac{-x^2}{\tan x} = \lim_{x\to 0} \frac{-2x}{\sec^2 x} = 0 \Rightarrow \lim_{x\to 0} f(x) = e^0 = 1$

(c) The maximum value of f(x) is close to 2 near the point $x \approx 1.55$ (see the graph in part (a)).

(d) The root in question is near 1.57.



(e) $y' = 0 \Rightarrow (\ln(\sin x) - x \cot x)(\sin x)^x = 0 \Rightarrow \ln(\sin x) - x \cot x = 0$. Let $g(x) = \ln(\sin x) - x \cot x$ $\Rightarrow g'(x) = \cot x - \cot x + x \csc^2 x = x \csc^2 x$. Using Newton's method, $g(x) = 0 \Rightarrow x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$ $= x_n - \frac{\ln(\sin x_n) - x_n \cot x_n}{x_n \csc^2 x_n}$. Then $x_1 = 1.55 \Rightarrow x_2 = 1.57093 \Rightarrow x_3 = 1.57080 \Rightarrow x_4 = 1.57080$ $\Rightarrow x_k = 1.57080,\ k \geq 3$.

(f)

x	1.55	1.57	1.57080
$(\sin x)^x$	0.999664854	0.999999502	1

7.7 IMPROPER INTEGRALS

1. (a) The integral is improper because of an infinite limit of integration.

(b) $\int_0^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \left[\tan^{-1} x\right]_0^b = \lim_{b\to\infty} (\tan^{-1} b - 0) = \frac{\pi}{2}$

The integral converges.

(c) $\frac{\pi}{2}$

2. (a) The integral is improper because the integrand has an infinite discontinuity at $x = 0$.

(b) $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{b \to 0^+} \int_b^1 \frac{dx}{\sqrt{x}} = \lim_{b \to 0^+} \left[2\sqrt{x}\right]_b^1 = \lim_{b \to 0^+} \left(2 - 2\sqrt{b}\right) = 2$

The integral converges.

(c) 2

3. (a) The integral involves improper integrals because the integrand has an infinite discontinuity at $x = 0$.

(b) $\int_{-8}^1 \frac{dx}{x^{1/3}} = \int_{-8}^0 \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}}$

$\int_{-8}^0 \frac{dx}{x^{1/3}} = \lim_{b \to 0^-} \int_{-8}^b \frac{dx}{x^{1/3}} = \lim_{b \to 0^-} \left[\frac{3}{2}x^{2/3}\right]_{-8}^b = \lim_{b \to 0^-} \left(\frac{3}{2}b^{2/3} - 6\right) = -6$

$\int_0^1 \frac{dx}{x^{1/3}} = \lim_{b \to 0^+} \int_b^1 \frac{dx}{x^{1/3}} = \lim_{b \to 0^+} \left[\frac{3}{2}x^{2/3}\right]_b^1 = \lim_{b \to 0^+} \left(\frac{3}{2} - \frac{3}{2}b^{2/3}\right) = \frac{3}{2}$

$\int_{-8}^1 \frac{dx}{x^{1/3}} = -6 + \frac{3}{2} = -\frac{9}{2}$

The integral converges.

(c) $-\frac{9}{2}$

4. (a) The integral is improper because of two infinite limits of integration.

(b) $\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2} = \int_{-\infty}^{0} \frac{2x\,dx}{(x^2+1)^2} + \int_{0}^{\infty} \frac{2x\,dx}{(x^2+1)^2}$

$\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2} = \lim_{b \to -\infty} \int_{b}^{0} \frac{2x\,dx}{(x^2+1)^2} = \lim_{b \to -\infty} \left[-(x^2+1)^{-1}\right]_b^0 = \lim_{b \to -\infty} \left[-1 + (b^2+1)^{-1}\right] = -1$

$\int_{0}^{\infty} \frac{2x\,dx}{(x^2+1)^2} = \lim_{b \to \infty} \int_{0}^{b} \frac{2x\,dx}{(x^2+1)^2} = \lim_{b \to \infty} \left[-(x^2+1)^{-1}\right]_0^b$

$\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2} = -1 + 1 = 0$

The integral converges.

(c) 0

5. (a) The integral is improper because the integrand has an infinite discontinuity at 0.

(b) $\int_0^{\ln 2} x^{-2}e^{1/x}\,dx = \lim_{b\to 0^+} \int_b^{\ln 2} x^{-2}e^{1/x}\,dx = \lim_{b\to 0^+} \left[-e^{1/x}\right]_b^{\ln 2} = \lim_{b\to 0^+} \left[-e^{1/\ln 2} + e^{1/b}\right] = \infty$

The integral diverges.

(c) No value

6. (a) The integral is improper because the integrand has an infinite discontinuity at $x = 0$.

(b) $\int_0^{\pi/2} \cot\theta\,d\theta = \lim_{b\to 0^+} \int_b^{\pi/2} \cot\theta\,d\theta = \lim_{b\to 0^+} \int_b^{\pi/2} \frac{\cos\theta\,d\theta}{\sin\theta} = \lim_{b\to 0^+} \left[\ln|\sin\theta|\right]_b^{\pi/2} = \lim_{b\to 0^+} (0 - \ln|\sin b|) = \infty$

The integral diverges.

(c) No value

7. $\int_1^{\infty} \frac{dx}{x^{1.001}} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{x^{1.001}} = \lim_{b\to\infty} \left[-1000x^{-0.001}\right]_1^b = \lim_{b\to\infty} \left(\frac{-1000}{b^{0.001}} + 1000\right) = 1000$

8. $\int_{-1}^{1} \frac{dx}{x^{2/3}} = \int_{-1}^{0} \frac{dx}{x^{2/3}} + \int_{0}^{1} \frac{dx}{x^{2/3}} = \lim_{b\to 0^-} \left[3x^{1/3}\right]_{-1}^{b} + \lim_{c\to 0^+} \left[3x^{1/3}\right]_c^1$

$= \lim_{b\to 0^-} \left[3b^{1/3} - 3(-1)^{1/3}\right] + \lim_{c\to 0^+} \left[3(1)^{1/3} - 3c^{1/3}\right] = (0+3) + (3-0) = 6$

9. $\int_0^4 \frac{dr}{\sqrt{4-r}} = \lim_{b\to 4^-} \left[-2\sqrt{4-r}\right]_0^b = \lim_{b\to 4^-} \left[-2\sqrt{4-b} - (-2\sqrt{4})\right] = 0 + 4 = 4$

10. $\int_0^1 \frac{dr}{r^{0.999}} = \lim_{b\to 0^+} \left[1000r^{0.001}\right]_b^1 = \lim_{b\to 0^+} \left(1000 - 1000b^{0.001}\right) = 1000 - 0 = 1000$

11. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-} \left[\sin^{-1}x\right]_0^b = \lim_{b\to 1^-} \left(\sin^{-1}b - \sin^{-1}0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

12. $\int_{-\infty}^{2} \frac{2\,dx}{x^2+4} = \lim_{b\to -\infty} \left[\tan^{-1}\frac{x}{2}\right]_b^2 = \lim_{b\to -\infty} \left(\tan^{-1}1 - \tan^{-1}b\right) = \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{3\pi}{4}$

13. $\int_{-\infty}^{-2} \frac{2\,dx}{x^2+1} = \int_{-\infty}^{-2} \frac{dx}{x-1} - \int_{-\infty}^{-2} \frac{dx}{x+1} = \lim_{b\to -\infty} \left[\ln|x-1|\right]_b^{-2} - \lim_{b\to -\infty} \left[\ln|x+1|\right]_b^{-2} = \lim_{b\to -\infty} \left[\ln\left|\frac{x-1}{x+1}\right|\right]_b^{-2}$

$= \lim_{b\to -\infty} \left(\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right|\right) = \ln 3 - \ln\left(\lim_{b\to -\infty} \frac{b-1}{b+1}\right) = \ln 3 - \ln 1 = \ln 3$

14. $\int_2^{\infty} \frac{3\,dt}{t^2-1} = \int_2^{\infty} \frac{3\,dt}{t-1} - \int_2^{\infty} \frac{3\,dt}{t} = \lim_{b\to\infty} \left[3\ln(t-1) - 3\ln t\right]_2^b = \lim_{b\to\infty} \left[3\ln\left(\frac{t-1}{t}\right)\right]_2^b$

$= 3\lim_{b\to\infty} \left[\ln\left(\frac{b-1}{b}\right) - \ln\left(\frac{1}{2}\right)\right] = 3\lim_{b\to\infty} \left[\ln\left(\frac{1-\frac{1}{b}}{1}\right) + \ln 2\right] = 3(\ln 1 + \ln 2) = 3\ln 2$

15. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta; \begin{bmatrix} u = \theta^2+2\theta \\ du = 2(\theta+1)\,d\theta \end{bmatrix} \to \int_0^3 \frac{du}{2\sqrt{u}} = \lim_{b\to 0^+} \int_b^3 \frac{du}{2\sqrt{u}} = \lim_{b\to 0^+} \left[\sqrt{u}\right]_b^3 = \lim_{b\to 0^+} \left(\sqrt{3}-\sqrt{b}\right)$

$= \sqrt{3} - 0 = \sqrt{3}$

16. $\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds = \frac{1}{2}\int_0^2 \frac{2s\,ds}{\sqrt{4-s^2}} + \int_0^2 \frac{ds}{\sqrt{4-s^2}}; \begin{bmatrix} u = 4-s^2 \\ du = -2s\,ds \end{bmatrix} \to -\frac{1}{2}\int_4^0 \frac{du}{\sqrt{u}} + \lim_{c\to 2^-} \int_0^c \frac{ds}{\sqrt{4-s^2}}$

$= \lim_{b\to 0^+} \int_b^4 \frac{du}{2\sqrt{u}} + \lim_{c\to 2^-} \int_0^c \frac{ds}{\sqrt{4-s^2}} = \lim_{b\to 0^+} \left[\sqrt{u}\right]_b^4 + \lim_{c\to 2^-} \left[\sin^{-1}\frac{s}{2}\right]_0^c$

$= \lim_{b\to 0^+} \left(2-\sqrt{b}\right) + \lim_{c\to 2^-} \left(\sin^{-1}\frac{c}{2} - \sin^{-1} 0\right) = (2-0) + \left(\frac{\pi}{2} - 0\right) = \frac{4+\pi}{2}$

17. $\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}; \begin{bmatrix} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \to \int_0^{\infty} \frac{2\,du}{u^2+1} = \lim_{b\to\infty} \int_0^b \frac{2\,du}{u^2+1} = \lim_{b\to\infty} \left[2\tan^{-1} u\right]_0^b$

$= \lim_{b\to\infty} \left(2\tan^{-1} b - 1\tan^{-1} 0\right) = 2\left(\frac{\pi}{2}\right) - 2(0) = \pi$

18. $\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \lim_{b\to 1^+} \int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c\to\infty} \int_2^c \frac{dx}{x\sqrt{x^2-1}}$

$= \lim_{b\to 1^+} \left[\sec^{-1}|x|\right]_b^2 + \lim_{c\to\infty} \left[\sec^{-1}|x|\right]_2^c = \lim_{b\to 1^+} \left(\sec^{-1} 2 - \sec^{-1} b\right) + \lim_{c\to\infty} \left(\sec^{-1} c - \sec^{-1} 2\right)$

$= \left(\frac{\pi}{3} - 0\right) + \left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{\pi}{2}$

19. $\int_1^2 \frac{ds}{s\sqrt{s^2-1}} = \lim_{b\to 1^+} \left[\sec^{-1} s\right]_b^2 = \sec^{-1} 2 - \lim_{b\to 1^+} \sec^{-1} b = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

20. $\int_{-1}^{\infty} \frac{d\theta}{\theta^2+5\theta+6} = \lim_{b\to\infty} \left[\ln\left|\frac{\theta+2}{\theta+3}\right|\right]_{-1}^b = \lim_{b\to\infty} \left[\ln\left|\frac{b+2}{b+3}\right|\right] - \ln\left|\frac{-1+2}{-1+3}\right| = 0 - \ln\left(\frac{1}{2}\right) = \ln 2$

21. $\int_2^{\infty} \frac{2\,dv}{v^2-v} = \lim_{b\to\infty} \left[2\ln\left|\frac{v-1}{v}\right|\right]_2^b = \lim_{b\to\infty} \left(2\ln\left|\frac{b-1}{b}\right| - 2\ln\left|\frac{2-1}{2}\right|\right) = 2\ln(1) - 2\ln\left(\frac{1}{2}\right) = 0 + 2\ln 2 = \ln 4$

22. $\int_2^{\infty} \frac{2\,dt}{t^2-1} = \lim_{b\to\infty} \left[\ln\left|\frac{t-1}{t+1}\right|\right]_2^b = \lim_{b\to\infty} \left(\ln\left|\frac{b-1}{b+1}\right| - \ln\left|\frac{2-1}{2+1}\right|\right) = \ln(1) - \ln\left(\frac{1}{3}\right) = 0 + \ln 3 = \ln 3$

23. $\int_0^2 \frac{ds}{\sqrt{4-s^2}} = \lim_{b\to 2^-} \left[\sin^{-1}\frac{s}{2}\right]_0^b = \lim_{b\to 2^-} \left(\sin^{-1}\frac{b}{2}\right) - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

24. $\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}} = \lim_{b\to 1^-} \left[2\sin^{-1}(r^2)\right]_0^b = \lim_{b\to 1^-} \left[2\sin^{-1}(b^2)\right] - 2\sin^{-1} 0 = 2\cdot\frac{\pi}{2} - 0 = \pi$

25. $\int_0^{\infty} \frac{dv}{(1+v^2)(1+\tan^{-1} v)} = \lim_{b\to\infty} \left[\ln\left|1+\tan^{-1} v\right|\right]_0^b = \lim_{b\to\infty} \left[\ln\left|1+\tan^{-1} b\right|\right] - \ln\left|1+\tan^{-1} 0\right|$

$= \ln\left(1+\frac{\pi}{2}\right) - \ln(1+0) = \ln\left(1+\frac{\pi}{2}\right)$

26. $\int_0^{\infty} \frac{16\tan^{-1} x}{1+x^2}\,dx = \lim_{b\to\infty} \left[8\left(\tan^{-1} x\right)^2\right]_0^b = \lim_{b\to\infty} \left[8\left(\tan^{-1} b\right)^2\right] - 8\left(\tan^{-1} 0\right)^2 = 8\left(\frac{\pi}{2}\right)^2 - 8(0) = 2\pi^2$

27. $\int_{-1}^4 \frac{dx}{\sqrt{|x|}} = \lim_{b\to 0^-} \int_{-1}^b \frac{dx}{\sqrt{-x}} + \lim_{c\to 0^+} \int_c^4 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^-} \left[-2\sqrt{-x}\right]_{-1}^b + \lim_{c\to 0^+} \left[2\sqrt{x}\right]_c^4$

$= \lim_{b\to 0^-} \left(-2\sqrt{-b}\right) - \left(-2\sqrt{-(-1)}\right) + 2\sqrt{4} - 6\lim_{c\to 0^+} 2\sqrt{c} = 0 + 2 + 2\cdot 2 - 0 = 6$

28. $\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{b\to 1^-} \left[-2\sqrt{1-x}\right]_0^b + \lim_{c\to 1^+} \left[2\sqrt{x-1}\right]_c^2$

$= \lim_{b\to 1^-} \left(-2\sqrt{1-b}\right) - \left(-2\sqrt{1-0}\right) + 2\sqrt{2-1} - \lim_{c\to 1^+} \left(2\sqrt{c-1}\right) = 0 + 2 + 2 - 0 = 4$

29. $\int_{-\infty}^0 \theta e^{\theta}\,d\theta = \lim_{b\to -\infty} \left[\theta e^{\theta} - e^{\theta}\right]_b^0 = \left(0\cdot e^0 - e^0\right) - \lim_{b\to -\infty} \left[be^b - e^b\right] = -1 - \lim_{b\to -\infty} \left(\frac{b-1}{e^{-b}}\right)$

$= -1 - \lim_{b\to -\infty} \left(\frac{1}{-e^{-b}}\right)$ (l'Hôpital's rule for $\frac{\infty}{\infty}$ form)

$= -1 - 0 = -1$

30. $\int_0^{\infty} 2e^{-\theta}\sin\theta\,d\theta = \lim_{b\to\infty} \int_0^b 2e^{-\theta}\sin\theta\,d\theta$

$= \lim_{b\to\infty} 2\left[\frac{e^{-\theta}}{1+1}\left(-\sin\theta - \cos\theta\right)\right]_0^b$ (FORMULA 107 with $a = -1$, $b = 1$)

$$= \lim_{b\to\infty} \frac{-2(\sin b + \cos b)}{2e^b} + \frac{2(\sin 0 + \cos 0)}{2e^0} = 0 + \frac{2(0+1)}{2} = 1$$

31. $\int_{-\infty}^{0} e^{-|x|}\,dx = \int_{-\infty}^{0} e^x\,dx = \lim_{b\to-\infty} \left[e^x\right]_b^0 = \lim_{b\to-\infty} \left(1 - e^b\right) = (1-0) = 1$

32. $\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx = \int_{-\infty}^{0} 2xe^{-x^2}\,dx + \int_{0}^{\infty} 2xe^{-x^2}\,dx = \lim_{b\to\infty} \left[-e^{-x^2}\right]_b^0 + \lim_{c\to\infty} \left[-e^{-x^2}\right]_0^c$

$$= \lim_{b\to-\infty} \left[-1 - \left(-e^{-b^2}\right)\right] + \lim_{c\to\infty} \left[-e^{-c^2} - (-1)\right] = (-1-0) + (0+1) = 0$$

33. $\int_{0}^{1} x \ln x\,dx = \lim_{b\to 0^+} \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_b^1 = \left(\frac{1}{2}\ln 1 - \frac{1}{4}\right) - \lim_{b\to 0^+} \left(\frac{b^2}{2}\ln b - \frac{b^2}{4}\right) = -\frac{1}{4} - \lim_{b\to 0^+} \frac{\ln b}{\left(\frac{2}{b^2}\right)} + 0$

$$= -\frac{1}{4} - \lim_{b\to 0^+} \frac{\left(\frac{1}{b}\right)}{\left(-\frac{4}{b^3}\right)} = -\frac{1}{4} + \lim_{b\to 0^+} \left(\frac{b^2}{4}\right) = -\frac{1}{4} + 0 = -\frac{1}{4}$$

34. $\int_{0}^{1} (-\ln x)\,dx = \lim_{b\to 0^+} [x - x\ln x]_b^1 = [1 - 1\ln 1] - \lim_{b\to 0^+} [b - b\ln b] = 1 - 0 + \lim_{b\to 0^+} \frac{\ln b}{\left(\frac{1}{b}\right)} = 1 - \lim_{b\to 0^+} \frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^2}\right)}$

$$= 1 + \lim_{b\to 0^+} b = 1 + 0 = 1$$

35. $\int_{0}^{\pi/2} \tan\theta\,d\theta = \lim_{b\to\frac{\pi}{2}^-} \left[-\ln|\cos\theta|\right]_0^b = \lim_{b\to\frac{\pi}{2}^-} \left[-\ln|\cos b|\right] + \ln 1 = \lim_{b\to\frac{\pi}{2}^-} \left[-\ln|\cos b|\right] = -\infty,$

the integral diverges

36. $\int_{0}^{\pi/2} \cot\theta\,d\theta = \lim_{b\to 0^+} \left[\ln|\sin\theta|\right]_b^{\pi/2} = \ln 1 - \lim_{b\to 0^+} \left[\ln|\sin b|\right] = -\lim_{b\to 0^+} \left[\ln|\sin b|\right] = -\infty,$

the integral diverges

37. $\int_{0}^{\pi} \frac{\sin\theta\,d\theta}{\sqrt{\pi-\theta}}$; $[\pi - \theta = x] \to -\int_{\pi}^{0} \frac{\sin x\,dx}{\sqrt{x}} = \int_{0}^{\pi} \frac{\sin x\,dx}{\sqrt{x}}$. Since $0 \le \frac{\sin x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$ for all $0 \le x \le \pi$ and $\int_{0}^{\pi} \frac{dx}{\sqrt{x}}$ converges, then $\int_{0}^{\pi} \frac{\sin x}{\sqrt{x}}\,dx$ converges by the Direct Comparison Test.

38. $\int_{-\pi/2}^{\pi/2} \frac{\cos\theta\,d\theta}{(\pi-2\theta)^{1/3}}$; $\begin{bmatrix} x = \pi - 2\theta \\ \theta = \frac{\pi}{2} - \frac{x}{2} \\ d\theta = -\frac{dx}{2} \end{bmatrix} \to \int_{2\pi}^{0} \frac{-\cos\left(\frac{\pi}{2} - \frac{x}{2}\right)dx}{2x^{1/3}} = \int_{0}^{2\pi} \frac{\sin\left(\frac{x}{2}\right)dx}{2x^{1/3}}$. Since $0 \le \frac{\sin\frac{x}{2}}{x^{1/3}} \le \frac{1}{x^{1/3}}$ for all

$0 \le x \le 2\pi$ and $\int_0^{2\pi} \frac{dx}{2x^{1/3}}$ converges, then $\int_0^{2\pi} \frac{\sin \frac{x}{2}\, dx}{2x^{1/3}}$ converges by the Direct Comparison Test.

39. $\int_0^{\ln 2} x^{-2}e^{-1/x}\, dx; \left[\frac{1}{x} = y\right] \to \int_{\infty}^{1/\ln 2} \frac{y^2e^{-y}\, dy}{-y^2} = \int_{1/\ln 2}^{\infty} e^{-y}\, dy = \lim_{b\to\infty} \left[-e^{-y}\right]_{1/\ln 2}^{b} = \lim_{b\to\infty} \left[-e^{-b}\right] - \left[-e^{-1/\ln 2}\right]$
$= 0 + e^{-1/\ln 2} = e^{-1/\ln 2}$, so the integral converges.

40. $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx; \left[y = \sqrt{x}\right] \to 2\int_0^1 e^{-y}\, dy = 2 - 2e$, so the integral converges.

41. $\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$. Since for $0 \le t \le \pi$, $0 \le \frac{1}{\sqrt{t} + \sin t} \le \frac{1}{\sqrt{t}}$ and $\int_0^{\pi} \frac{dt}{\sqrt{t}}$ converges, then the original integral converges as well by the Direct Comparison Test.

42. $\int_0^1 \frac{dt}{t - \sin t}$; let $f(t) = \frac{1}{t - \sin t}$ and $g(t) = \frac{1}{t^3}$, then $\lim_{t\to 0} \frac{f(t)}{g(t)} = \lim_{t\to 0} \frac{t^3}{t - \sin t} = \lim_{t\to 0} \frac{3t^2}{1 - \cos t} = \lim_{t\to 0} \frac{6t}{\sin t}$
$= \lim_{t\to 0} \frac{6}{\cos t} = 6$. Now, $\int_0^1 \frac{dt}{t^3} = \lim_{b\to 0^+} \left[-\frac{1}{2t^2}\right]_b^1 = -\frac{1}{2} - \lim_{b\to 0^+} \left[-\frac{1}{2b^2}\right] = +\infty$, which diverges $\Rightarrow \int_0^1 \frac{dt}{t - \sin t}$ diverges by the Limit Comparison Test.

43. $\int_0^2 \frac{dx}{1 - x^2} = \int_0^1 \frac{dx}{1 - x^2} + \int_1^2 \frac{dx}{1 - x^2}$ and $\int_0^1 \frac{dx}{1 - x^2} = \lim_{b\to 1^-} \left[\frac{1}{2} \ln\left|\frac{1+x}{1-x}\right|\right]_0^b = \lim_{b\to 1^-} \left[\frac{1}{2} \ln\left|\frac{1+b}{1-b}\right|\right] - 0 = \infty$, which diverges $\Rightarrow \int_0^2 \frac{dx}{1 - x^2}$ diverges as well.

44. $\int_0^2 \frac{dx}{1 - x} = \int_0^1 \frac{dx}{1 - x} + \int_1^2 \frac{dx}{1 - x}$ and $\int_0^1 \frac{dx}{1 - x} = \lim_{b\to 1^-} \left[-\ln(1 - x)\right]_0^b = \lim_{b\to 1^-} \left[-\ln(1 - b)\right] - 0 = \infty$, which diverges $\Rightarrow \int_0^2 \frac{dx}{1 - x}$ diverges as well.

45. $\int_{-1}^1 \ln|x|\, dx = \int_{-1}^0 \ln(-x)\, dx + \int_0^1 \ln x\, dx;\ \int_0^1 \ln x\, dx = \lim_{b\to 0^+} \left[x \ln x - x\right]_b^1 = \left[1 \cdot 0 - 1\right] - \lim_{b\to 0^+} \left[b \ln b - b\right]$
$= -1 - 0 = -1;\ \int_{-1}^0 \ln(-x)\, dx = -1 \Rightarrow \int_{-1}^1 \ln|x|\, dx = -2$ converges.

46. $\int_{-1}^{1} (-x \ln |x|)\, dx = \int_{-1}^{0} [-x \ln(-x)]\, dx + \int_{0}^{1} (-x \ln x)\, dx = \lim_{b\to 0^+} \left[\frac{x^2}{2} \ln x - \frac{x^2}{4}\right]_b^1 - \lim_{c\to 0^+} \left[\frac{x^2}{2} \ln x - \frac{x^2}{4}\right]_c^1$

$= \left[\frac{1}{2} \ln 1 - \frac{1}{4}\right] - \lim_{b\to 0^+} \left[\frac{b^2}{2} \ln b - \frac{b^2}{4}\right] - \left[\frac{1}{2} \ln 1 - \frac{1}{4}\right] + \lim_{c\to 0^+} \left[\frac{c^2}{2} \ln c - \frac{c^2}{4}\right] = -\frac{1}{4} - 0 + \frac{1}{4} + 0 = 0 \Rightarrow$ the integral converges (see Exercise 33 for the limit calculations).

47. $\int_{1}^{\infty} \frac{dx}{1+x^3}$; $0 \le \frac{1}{x^3+1} \le \frac{1}{x^3}$ for $1 \le x < \infty$ and $\int_{1}^{\infty} \frac{dx}{x^3}$ converges $\Rightarrow \int_{1}^{\infty} \frac{dx}{1+x^3}$ converges by the Direct Comparison Test.

48. $\int_{4}^{\infty} \frac{dx}{\sqrt{x}-1}$; $\lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x}-1}\right)}{\left(\frac{1}{\sqrt{x}}\right)} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x}-1} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{\sqrt{x}}} = \frac{1}{1-0} = 1$ and $\int_{4}^{\infty} \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \left[2\sqrt{x}\right]_4^b = \infty$, which diverges $\Rightarrow \int_{4}^{\infty} \frac{dx}{\sqrt{x}-1}$ diverges by the Limit Comparison Test.

49. $\int_{2}^{\infty} \frac{dv}{\sqrt{v-1}}$; $\lim_{v\to\infty} \frac{\left(\frac{1}{\sqrt{v-1}}\right)}{\left(\frac{1}{\sqrt{v}}\right)} = \lim_{v\to\infty} \frac{\sqrt{v}}{\sqrt{v-1}} = \lim_{v\to\infty} \frac{1}{\sqrt{1-\frac{1}{v}}} = \frac{1}{\sqrt{1-0}} = 1$ and $\int_{2}^{\infty} \frac{dv}{\sqrt{v}} = \lim_{b\to\infty} \left[2\sqrt{v}\right]_2^b = \infty$, which diverges $\Rightarrow \int_{2}^{\infty} \frac{dv}{\sqrt{v-1}}$ diverges by the Limit Comparison Test.

50. $\int_{4}^{\infty} \frac{2\, dt}{t^{3/2}+1}$; $\lim_{t\to\infty} \frac{t^{3/2}}{t^{3/2}+1} = 1$ and $\int_{4}^{\infty} \frac{2\, dt}{t^{3/2}} = \lim_{b\to\infty} \left[-4t^{-1/2}\right]_4^b = \lim_{b\to\infty} \left(\frac{-4}{\sqrt{b}} + 2\right) = 2 \Rightarrow \int_{4}^{\infty} \frac{2\, dt}{t^{3/2}}$ converges $\Rightarrow \int_{4}^{\infty} \frac{2\, dt}{t^{3/2}+1}$ converges by the Direct Comparison Test.

51. $\int_{0}^{\infty} \frac{dx}{\sqrt{x^6+1}} = \int_{0}^{1} \frac{dx}{\sqrt{x^6+1}} + \int_{1}^{\infty} \frac{dx}{\sqrt{x^6+1}} < \int_{0}^{1} \frac{dx}{\sqrt{x^6+1}} + \int_{1}^{\infty} \frac{dx}{x^3}$ and $\int_{1}^{\infty} \frac{dx}{x^3} = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_1^b$ $= \lim_{b\to\infty} \left(-\frac{1}{2b^2} + \frac{1}{2}\right) = \frac{1}{2} \Rightarrow \int_{0}^{\infty} \frac{dx}{\sqrt{x^6+1}}$ converges by the Direct Comparison Test.

52. $\int_{2}^{\infty} \frac{dx}{\sqrt{x^2-1}}$; $\lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x^2-1}}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1$; $\int_{2}^{\infty} \frac{1}{x}\, dx = \lim_{b\to\infty} [\ln b]_2^b = \infty$,

which diverges $\Rightarrow \int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ diverges by the Limit Comparison Test.

53. $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$; $\lim_{x\to\infty} \frac{\left(\frac{\sqrt{x}}{x^2}\right)}{\left(\frac{\sqrt{x+1}}{x^2}\right)} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}} = 1$; $\int_1^\infty \frac{\sqrt{x}}{x^2}\,dx = \int_1^\infty \frac{dx}{x^{3/2}}$

$= \lim_{b\to\infty} \left[-2x^{-1/2}\right]_1^b = \lim_{b\to\infty} \left(\frac{-2}{\sqrt{x}}+2\right) = 2 \Rightarrow \int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$ converges by the Limit Comparison Test.

54. $\int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$; $\lim_{x\to\infty} \frac{\left(\frac{x}{\sqrt{x^4-1}}\right)}{\left(\frac{x}{\sqrt{x^4}}\right)} = \lim_{x\to\infty} \frac{\sqrt{x^4}}{\sqrt{x^4-1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^4}}} = 1$; $\int_2^\infty \frac{x\,dx}{\sqrt{x^4}} = \int_2^\infty \frac{dx}{x} = \lim_{b\to\infty} [\ln x]_2^b = \infty$,

which diverges $\Rightarrow \int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$ diverges by the Limit Comparison Test.

55. $\int_\pi^\infty \frac{2+\cos x}{x}\,dx$; $0 < \frac{1}{x} \le \frac{2+\cos x}{x}$ for $x \ge \pi$ and $\int_\pi^\infty \frac{dx}{x} = \lim_{b\to\infty} [\ln x]_\pi^b = \infty$, which diverges

$\Rightarrow \int_\pi^\infty \frac{2+\cos x}{x}\,dx$ diverges by the Direct Comparison Test.

56. $\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$; $0 \le \frac{1+\sin x}{x^2} \le \frac{2}{x^2}$ for $x \ge \pi$ and $\int_\pi^\infty \frac{2}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{2}{x}\right]_\pi^b = \lim_{b\to\infty} \left(-\frac{2}{b}+\frac{2}{\pi}\right) = \frac{2}{\pi}$

$\Rightarrow \int_\pi^\infty \frac{2\,dx}{x^2}$ converges $\Rightarrow \int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$ converges by the Direct Comparison Test.

57. $\int_0^\infty \frac{d\theta}{1+e^\theta}$; $0 \le \frac{1}{1+e^\theta} \le \frac{1}{e^\theta}$ for $0 \le \theta < \infty$ and $\int_0^\infty \frac{d\theta}{e^\theta} = \lim_{b\to 0} \left[-e^{-\theta}\right]_0^b = \lim_{b\to\infty} \left(-e^{-b}+1\right) = 1$

$\Rightarrow \int_0^\infty \frac{d\theta}{e^\theta}$ converges $\Rightarrow \int_0^\infty \frac{d\theta}{1+e^\theta}$ converges by the Direct Comparison Test.

58. $\int_2^\infty \frac{dx}{\ln x}$; $0 < \frac{1}{x} < \frac{1}{\ln x}$ for $x > 2$ and $\int_2^\infty \frac{dx}{x}$ diverges $\Rightarrow \int_2^\infty \frac{dx}{\ln x}$ diverges by the Direct Comparison Test.

59. $\int_1^\infty \frac{e^x}{x}\,dx$; $0 < \frac{1}{x} < \frac{e^x}{x}$ for $x > 1$ and $\int_1^\infty \frac{dx}{x}$ diverges $\Rightarrow \int_1^\infty \frac{e^x\,dx}{x}$ diverges by the Direct Comparison Test.

60. $\int_{e^e}^{\infty} \ln(\ln x)\,dx$; $[x = e^y] \to \int_{e}^{\infty} (\ln y)\,e^y\,dy$; $0 < \ln y < (\ln y)\,e^y$ for $y \ge e$ and $\int_{e}^{\infty} \ln y\,dy = \lim_{b\to\infty} [y \ln y - y]_e^b$

$= \infty$, which diverges $\Rightarrow \int_{e}^{\infty} \ln e^y\,dy$ diverges $\Rightarrow \int_{e^e}^{\infty} \ln(\ln x)\,dx$ diverges by the Direct Comparison Test.

61. $\int_1^{\infty} \frac{dx}{\sqrt{e^x - x}}$; $\lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{e^x - x}}\right)}{\left(\frac{1}{\sqrt{e^x}}\right)} = \lim_{x\to\infty} \frac{\sqrt{e^x}}{\sqrt{e^x - x}} = \lim_{x\to\infty} \frac{1}{\sqrt{1 - \frac{x}{e^x}}} = \frac{1}{\sqrt{1-0}} = 1$; $\int_1^{\infty} \frac{dx}{\sqrt{e^x}} = \int_1^{\infty} e^{-x/2}\,dx$

$= \lim_{b\to\infty} \left[-2e^{-x/2}\right]_1^b = \lim_{b\to\infty} \left(-2e^{-b/2} + 2e^{-1/2}\right) = \frac{2}{\sqrt{e}} \Rightarrow \int_1^{\infty} e^{-x/2}\,dx$ converges $\Rightarrow \int_1^{\infty} \frac{dx}{\sqrt{e^x - x}}$ converges

by the Limit Comparison Test.

62. $\int_1^{\infty} \frac{dx}{e^x - 2^x}$; $\lim_{x\to\infty} \frac{\left(\frac{1}{e^x - 2^x}\right)}{\left(\frac{1}{e^x}\right)} = \lim_{x\to\infty} \frac{e^x}{e^x - 2^x} = \lim_{x\to\infty} \frac{1}{1 - \left(\frac{2}{e}\right)^x} = \frac{1}{1-0} = 1$ and $\int_1^{\infty} \frac{dx}{e^x} = \lim_{b\to\infty} \left[-e^{-x}\right]_1^b$

$= \lim_{b\to\infty} \left(-e^{-b} + e^{-1}\right) = \frac{1}{e} \Rightarrow \int_1^{\infty} \frac{dx}{e^x}$ converges $\Rightarrow \int_1^{\infty} \frac{dx}{e^x - 2^x}$ converges by the Limit Comparison Test.

63. $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}} = 2\int_0^{\infty} \frac{dx}{\sqrt{x^4+1}}$; $\lim_{x\to\infty} \frac{x^2}{\sqrt{x^4+1}} = 1$; $\int_0^{\infty} \frac{dx}{\sqrt{x^4+1}} = \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^{\infty} \frac{dx}{\sqrt{x^4+1}}$

$< \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^{\infty} \frac{dx}{x^2}$ and $\int_1^{\infty} \frac{dx}{x^2} = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}}$ converges by the Direct Comparison Test.

64. $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = 2\int_0^{\infty} \frac{dx}{e^x + e^{-x}}$; $0 < \frac{1}{e^x + e^{-x}} < \frac{1}{e^x}$ for $x > 0$; $\int_0^{\infty} \frac{dx}{e^x}$ converges $\Rightarrow 2\int_0^{\infty} \frac{dx}{e^x + e^{-x}}$ converges by the Direct Comparison Test.

65. (a) $\int_1^2 \frac{dx}{x(\ln x)^p}$; $[t = \ln x] \to \int_0^{\ln 2} \frac{dt}{t^p} = \lim_{b\to 0^+} \left[\frac{1}{-p+1} t^{1-p}\right]_b^{\ln 2} = \lim_{b\to 0^+} \frac{b^{1-p}}{p-1} + \frac{1}{1-p}(\ln 2)^{1-p}$

$\Rightarrow$ the integral converges for $p < 1$ and diverges for $p \ge 1$

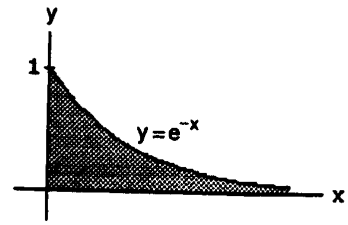
(b) $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$; $[t = \ln x] \to \int_{\ln 2}^{\infty} \frac{dt}{t^p}$ and this integral is essentially the same as in Exercise 67(a): it converges for $p > 1$ and diverges for $p \le 1$

66. $\int_0^\infty \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty} \left[\ln(x^2+1)\right]_0^b = \lim_{b\to\infty} \left[\ln(b^2+1)\right] - 0 = \lim_{b\to\infty} \ln(b^2+1) = \infty \Rightarrow$ the integral $\int_{-\infty}^\infty \frac{2x}{x^2+1}\,dx$ diverges. But $\lim_{b\to\infty} \int_{-b}^{b} \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty} \left[\ln(x^2+1)\right]_{-b}^{b} = \lim_{b\to\infty} \left[\ln(b^2+1) - \ln(b^2+1)\right] = \lim_{b\to\infty} \ln\left(\frac{b^2+1}{b^2+1}\right)$

$= \lim_{b\to\infty} (\ln 1) = 0$

67. $A = \int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]_0^b = \lim_{b\to\infty} \left(-e^{-b}\right) - \left(-e^{-0}\right)$

$= 0 + 1 = 1$



68. $V = \int_0^\infty 2\pi x e^{-x}\,dx = 2\pi \int_0^\infty x e^{-x}\,dx = 2\pi \lim_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_0^b = 2\pi\left[\lim_{b\to\infty} \left(-be^{-b} - e^{-b}\right) - 1\right] = 2\pi$

69. $V = \int_0^\infty \pi\left(e^{-x}\right)^2 dx = \pi \int_0^\infty e^{-2x}\,dx = \pi \lim_{b\to\infty} \left[-\frac{1}{2}e^{-2x}\right]_0^b = \pi \lim_{b\to\infty} \left(-\frac{1}{2}e^{-2b} + \frac{1}{2}\right) = \frac{\pi}{2}$

70. $A = \int_0^{\pi/2} (\sec x - \tan x)\,dx = \lim_{b\to\frac{\pi}{2}^-} \left[\ln|\sec x + \tan x| - \ln|\sec x|\right]_0^b = \lim_{b\to\frac{\pi}{2}^-} \left(\ln\left|1 + \frac{\tan b}{\sec b}\right| - \ln|1+0|\right)$

$= \lim_{b\to\frac{\pi}{2}^-} \ln|1 + \sin b| = \ln 2$

71. $\int_3^\infty \left(\frac{1}{x-2} - \frac{1}{x}\right) dx \neq \int_3^\infty \frac{dx}{x-2} - \int_3^\infty \frac{dx}{x}$, since the left hand integral converges but both of the right hand integrals diverge.

72. (a) The statement is true since $\int_{-\infty}^{b} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^b f(x)\,dx$, $\int_b^\infty f(x)\,dx = \int_a^\infty f(x)\,dx - \int_a^b f(x)\,dx$

and $\int_a^b f(x)\,dx$ exists since f(x) is integrable on every interval [a, b].

(b) $\int_{-\infty}^{a} f(x)\,dx + \int_a^\infty f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^b f(x)\,dx - \int_a^b f(x)\,dx + \int_a^\infty f(x)\,dx$

$= \int_{-\infty}^{b} f(x)\,dx + \int_b^a f(x)\,dx + \int_a^\infty f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_b^\infty f(x)\,dx$

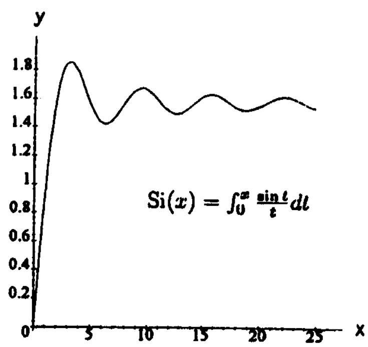
73. (a) $\int_1^{\infty} e^{-3x}\,dx = \lim_{b\to\infty}\left[-\frac{1}{3}e^{-3x}\right]_3^b = \lim_{b\to\infty}\left(-\frac{1}{3}e^{-3b}\right)-\left(-\frac{1}{3}e^{-3\cdot 3}\right) = 0+\frac{1}{3}\cdot e^{-9} = \frac{1}{3}e^{-9}$

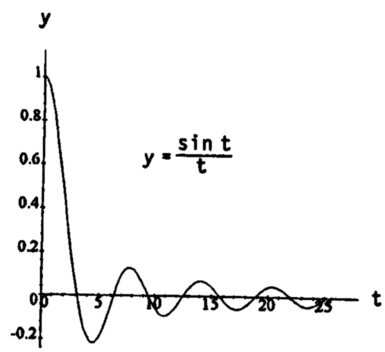
$\approx 0.0000411 < 0.000042$. Since $e^{-x^2} \le e^{-3x}$ for $x > 3$, then $\int_3^{\infty} e^{-x^2}\,dx < 0.000042$ and therefore

$\int_0^{\infty} e^{-x^2}\,dx$ can be replaced by $\int_0^3 e^{-x^2}\,dx$ without introducing an error greater than 0.000042.

(b) $\int_0^3 e^{-x^2}\,dx \cong 0.88621$

74. (a)

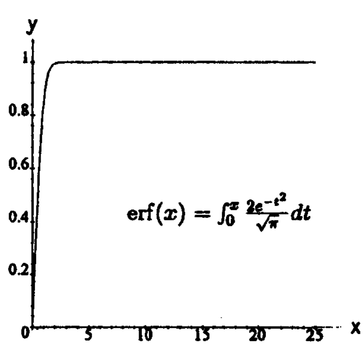




(b) Maple command:

> int((sin(t))/t, t=0..infinity); $\left(\text{answer is } \frac{\pi}{2}\right)$

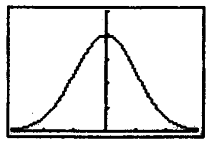
75. (a)



(b) Maple commands:

```
> f:= 2*exp(-t^2)/sqrt(Pi);
> int(f, t=0..infinity); (answer is 1)
```

76. (a) $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$



[−3, 3] by [0, 0.5]

f is increasing on $(-\infty, 0]$. f is decreasing on $[0, \infty)$. f has a local maximum at $(0, f(0)) = \left(0, \frac{1}{\sqrt{2\pi}}\right)$

(b) Maple commands:

```
>f:=exp(-x^2/2)(sqrt(2*pi);
>int(f,x=-1..1);      ≈ 0.683
>int(f,x=-2..2);      ≈ 0.954
>int(f,x=-3..3);      ≈ 0.997
```

(c) Part (b) suggests that as b increases, the integral approaches 1. We can make $\int_{-b}^{b} f(x)\,dx$ as close to 1 as we want by choosing $b > 1$ large enough. Also, we can make $\int_{b}^{\infty} f(x)\,dx$ and $\int_{-\infty}^{-b} f(x)\,dx$ as small as we want by choosing b large enough enough. This is because $0 < f(x) < e^{-x/2}$ for $x > 1$. (Likewise, $0 < f(x) < e^{x/2}$ for $x < -1$.) Thus, $\int_{b}^{\infty} f(x)\,dx < \int_{b}^{\infty} e^{-x/2}\,dx$.

$$\int_{b}^{\infty} e^{-x/2}\,dx = \lim_{c\to\infty} \int_{b}^{c} e^{-x/2}\,dx = \lim_{c\to\infty} \left[-2e^{-x/2}\right]_{b}^{c} = \lim_{c\to\infty} \left[-2e^{-c/2} + 2e^{-b/2}\right] = 2e^{-b/2}$$

As $b \to \infty$, $2e^{-b/2} \to 0$, for for large enough b, $\int_{b}^{\infty} f(x)\,dx$ is as small as we want. Likewise, for large enough b, $\int_{-\infty}^{-b} f(x)\,dx$ is as small as we want.

77-80. Use the MAPLE or MATHEMATICA integration commands, as discussed in the text.

CHAPTER 7 PRACTICE EXERCISES

1. $\int x\sqrt{4x^2-9}\,dx;\ \begin{bmatrix} u=4x^2-9 \\ du=8x\,dx \end{bmatrix} \to \frac{1}{8}\int \sqrt{u}\,du = \frac{1}{8}\cdot\frac{2}{3}u^{3/2}+C = \frac{1}{12}\left(4x^2-9\right)^{3/2}+C$

2. $\int x(2x+1)^{1/2}\,dx;\ \begin{bmatrix} u=2x+1 \\ du=2\,dx \end{bmatrix} \to \frac{1}{2}\int \left(\frac{u-1}{2}\right)\sqrt{u}\,du = \frac{1}{4}\left(\int u^{3/2}\,du - \int u^{1/2}\,du\right) = \frac{1}{4}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)+C$

$= \frac{(2x+1)^{5/2}}{10} - \frac{(2x+1)^{3/2}}{6} + C$

3. $\int \frac{x\,dx}{\sqrt{8x^2+1}};\ \begin{bmatrix} u=8x^2+1 \\ du=16x\,dx \end{bmatrix} \to \frac{1}{16}\int \frac{du}{\sqrt{u}} = \frac{1}{16}\cdot 2u^{1/2}+C = \frac{\sqrt{8x^2+1}}{8}+C$

4. $\int \frac{y\,dy}{25+y^2};\ \begin{bmatrix} u=25+y^2 \\ du=2y\,dy \end{bmatrix} \to \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln\left(25+y^2\right)+C$

5. $\int \frac{t^3\,dt}{\sqrt{9-4t^4}};\ \begin{bmatrix} u=9-4t^4 \\ du=-16t^3\,dt \end{bmatrix} \to -\frac{1}{16}\int \frac{du}{\sqrt{u}} = -\frac{1}{16}\cdot 2u^{1/2}+C = -\frac{\sqrt{9-4t^4}}{8}+C$

6. $\int z^{2/3}\left(z^{5/3}+1\right)^{2/3}\,dz;\ \begin{bmatrix} u=z^{5/3}+1 \\ du=\frac{5}{3}z^{2/3}\,dz \end{bmatrix} \to \frac{3}{5}\int u^{2/3}\,du = \frac{3}{5}\cdot\frac{3}{5}u^{5/3}+C = \frac{9}{25}\left(z^{5/3}+1\right)^{5/3}+C$

7. $\int \frac{\sin 2\theta\,d\theta}{(1-\cos 2\theta)^2};\ \begin{bmatrix} u=1-\cos 2\theta \\ du=2\sin 2\theta\,d\theta \end{bmatrix} \to \frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2u}+C = -\frac{1}{2(1-\cos 2\theta)}+C$

8. $\int \frac{\cos 2t\,dt}{1+\sin 2t};\ \begin{bmatrix} u=1+\sin 2t \\ du=2\cos 2t\,dt \end{bmatrix} \to \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln|1+\sin 2t|+C$

9. $\int (\sin 2x)\,e^{\cos 2x}\,dx;\ \begin{bmatrix} u=\cos 2x \\ du=-2\sin 2x\,dx \end{bmatrix} \to -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u+C = -\frac{1}{2}e^{\cos 2x}+C$

10. $\int e^\theta \sec^2\left(e^\theta\right)d\theta;\ \begin{bmatrix} u=e^\theta \\ du=e^\theta\,d\theta \end{bmatrix} \to \int \sec^2 u\,du = \tan u + C = \tan\left(e^\theta\right)+C$

11. $\int 2^{x-1}\,dx = \frac{2^{x-1}}{\ln 2}+C$

12. $\int \frac{dv}{v\ln v};\ \begin{bmatrix} u=\ln v \\ du=\frac{1}{v}\,dv \end{bmatrix} \to \int \frac{du}{u} = \ln|u|+C = \ln|\ln v|+C$

13. $\int \frac{dx}{(x^2+1)(2+\tan^{-1}x)}$; $\begin{bmatrix} u = 2+\tan^{-1}x \\ du = \frac{dx}{x^2+1} \end{bmatrix} \rightarrow \int \frac{du}{u} = \ln|u| + C = \ln|2+\tan^{-1}x| + C$

14. $\int \frac{2\,dx}{\sqrt{1-4x^2}}$; $\begin{bmatrix} u = 2x \\ du = 2\,dx \end{bmatrix} \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}(2x) + C$

15. $\int \frac{dt}{\sqrt{16-9t^2}} = \frac{1}{4}\int \frac{dt}{\sqrt{1-\left(\frac{3t}{4}\right)^2}}$; $\begin{bmatrix} u = \frac{3}{4}t \\ du = \frac{3}{4}\,dt \end{bmatrix} \rightarrow \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{3}\sin^{-1}u + C = \frac{1}{3}\sin^{-1}\left(\frac{3t}{4}\right) + C$

16. $\int \frac{dt}{9+t^2} = \frac{1}{9}\int \frac{dt}{1+\left(\frac{t}{3}\right)^2}$; $\begin{bmatrix} u = \frac{1}{3}t \\ du = \frac{1}{3}dt \end{bmatrix} \rightarrow \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1}u + C = \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + C$

17. $\int \frac{4\,dx}{5x\sqrt{25x^2-16}} = \frac{4}{25}\int \frac{dx}{x\sqrt{x^2-\frac{16}{25}}} = \frac{1}{5}\sec^{-1}\left|\frac{5x}{4}\right| + C$

18. $\int \frac{dx}{\sqrt{4x-x^2-3}} = \int \frac{d(x-2)}{\sqrt{1-(x-2)^2}} = \sin^{-1}(x-2) + C$

19. $\int \frac{dy}{y^2-4y+8} = \int \frac{d(y-2)}{(y-2)^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{y-2}{2}\right) + C$

20. $\int \frac{dv}{(v+1)\sqrt{v^2+2v}} = \int \frac{d(v+1)}{(v+1)\sqrt{(v+1)^2-1}} = \sec^{-1}|v+1| + C$

21. $\int \cos^2 3x\,dx = \int \frac{1+\cos 6x}{2}\,dx = \frac{x}{2} + \frac{\sin 6x}{12} + C$

22. $\int \sin^3\frac{\theta}{2}\,d\theta = \int \left(1-\cos^2\frac{\theta}{2}\right)\left(\sin\frac{\theta}{2}\right)d\theta$; $\begin{bmatrix} u = \cos\frac{\theta}{2} \\ du = -\frac{1}{2}\sin\frac{\theta}{2}\,d\theta \end{bmatrix} \rightarrow -2\int\left(1-u^2\right)du = \frac{2u^3}{3} - 2u + C$

$= \frac{2}{3}\cos^3\frac{\theta}{2} - 2\cos\frac{\theta}{2} + C$

23. $\int \tan^3 2t\, dt = \int (\tan 2t)(\sec^2 2t - 1)\, dt = \int \tan 2t \sec^2 2t\, dt - \int \tan 2t\, dt; \begin{bmatrix} u = 2t \\ du = 2\, dt \end{bmatrix}$

$\to \frac{1}{2} \int \tan u \sec^2 u\, du - \frac{1}{2} \int \tan u\, du = \frac{1}{4} \tan^2 u + \frac{1}{2} \ln|\cos u| + C = \frac{1}{4} \tan^2 2t + \frac{1}{2} \ln|\cos 2t| + C$

$= \frac{1}{4} \tan^2 2t - \frac{1}{2} \ln|\sec 2t| + C$

24. $\int \frac{dx}{2 \sin x \cos x} = \int \frac{dx}{\sin 2x} = \int \csc 2x\, dx = -\frac{1}{2} \ln|\csc 2x + \cot 2x| + C$

25. $\int \frac{2\, dx}{\cos^2 x - \sin^2 x} = \int \frac{2\, dx}{\cos 2x}; \begin{bmatrix} u = 2x \\ du = 2\, dx \end{bmatrix} \to \int \frac{du}{\cos u} = \int \sec u\, du = \ln|\sec u + \tan u| + C$

$= \ln|\sec 2x + \tan 2x| + C$

26. $\int_{\pi/4}^{\pi/2} \sqrt{\csc^2 y - 1}\, dy = \int_{\pi/4}^{\pi/2} \cot y\, dy = [\ln|\sin y|]_{\pi/4}^{\pi/2} = \ln 1 - \ln \frac{1}{\sqrt{2}} = \ln \sqrt{2}$

27. $\int_{\pi/4}^{3\pi/4} \sqrt{\cot^2 t + 1}\, dt = \int_{\pi/4}^{3\pi/4} \csc t\, dt = [-\ln|\csc t + \cot t|]_{\pi/4}^{3\pi/4} = -\ln\left|\csc \frac{3\pi}{4} + \cot \frac{3\pi}{4}\right| + \ln\left|\csc \frac{\pi}{4} + \cot \frac{\pi}{4}\right|$

$= -\ln|\sqrt{2} - 1| + \ln|\sqrt{2} + 1| = \ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right| = \ln\left|\frac{(\sqrt{2}+1)(\sqrt{2}+1)}{2-1}\right| = \ln(3 + 2\sqrt{2})$

28. $\int_0^{2\pi} \sqrt{1 - \sin^2 \frac{x}{2}}\, dx = \int_0^{2\pi} \left|\cos \frac{x}{2}\right| dx = \int_0^{\pi} \cos \frac{x}{2}\, dx - \int_{\pi}^{2\pi} \cos \frac{x}{2}\, dx = \left[2 \sin \frac{x}{2}\right]_0^{\pi} - \left[2 \sin \frac{x}{2}\right]_{\pi}^{2\pi}$

$= (2 - 0) - (0 - 2) = 4$

29. $\int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos 2t}\, dt = \sqrt{2} \int_{-\pi/2}^{\pi/2} |\sin t|\, dt = 2\sqrt{2} \int_0^{\pi/2} \sin t\, dt = [-2\sqrt{2} \cos t]_0^{\pi/2} = 2\sqrt{2}\, [0 - (-1)] = 2\sqrt{2}$

30. $\int_{\pi}^{2\pi} \sqrt{1 + \cos 2t}\, dt = \sqrt{2} \int_{\pi}^{2\pi} |\cos t|\, dt = -\sqrt{2} \int_{\pi}^{3\pi/2} \cos t\, dt + \sqrt{2} \int_{3\pi/2}^{2\pi} \cos t\, dt$

$= -\sqrt{2}\, [\sin t]_{\pi}^{3\pi/2} + \sqrt{2}\, [\sin t]_{3\pi/2}^{2\pi} = -\sqrt{2}(-1 - 0) + \sqrt{2}\, [0 - (-1)] = 2\sqrt{2}$

31. $\int \frac{x^2\, dx}{x^2 + 4} = x - \int \frac{4\, dx}{x^2 + 4} = x - 2 \tan^{-1}\left(\frac{x}{2}\right) + C$

32. $\int \frac{x^3\, dx}{9 + x^2} = \int \left[\frac{x(x^2 + 9) - 9x}{x^2 + 9}\right] dx = \int \left(x - \frac{9x}{x^2 + 9}\right) dx = \frac{x^2}{2} - \frac{9}{2} \ln(9 + x^2) + C$

33. $\int \frac{2y-1}{y^2+4}\,dy = \int \frac{2y\,dy}{y^2+4} - \int \frac{dy}{y^2+4} = \ln(y^2+4) - \frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) + C$

34. $\int \frac{y+4}{y^2+1}\,dy = \int \frac{y\,dy}{y^2+1} + 4\int \frac{dy}{y^2+1} = \frac{1}{2}\ln(y^2+1) + 4\tan^{-1} y + C$

35. $\int \frac{t+2}{\sqrt{4-t^2}}\,dt = \int \frac{t\,dt}{\sqrt{4-t^2}} + 2\int \frac{dt}{\sqrt{4-t^2}} = -\sqrt{4-t^2} + 2\sin^{-1}\left(\frac{t}{2}\right) + C$

36. $\int \frac{2t^2+\sqrt{1-t^2}}{t\sqrt{1-t^2}}\,dt = \int \frac{2t\,dt}{\sqrt{1-t^2}} + \int \frac{dt}{t} = -2\sqrt{1-t^2} + \ln|t| + C$

37. $\int \frac{\tan x\,dx}{\tan x + \sec x} = \int \frac{\sin x\,dx}{\sin x + 1} = \int \frac{(\sin x)(1-\sin x)}{1-\sin^2 x}\,dx = \int \frac{\sin x - 1 + \cos^2 x}{\cos^2 x}\,dx$

$= -\int \frac{d(\cos x)}{\cos^2 x} - \int \frac{dx}{\cos^2 x} + \int dx = \frac{1}{\cos x} - \tan x + x + C = x - \tan x + \sec x + C$

38. $\int x\csc(x^2+3)\,dx = \frac{1}{2}\int \csc(x^2+3)\,d(x^2+3) = -\frac{1}{2}\ln\left|\csc(x^2+3) + \cot(x^2+3)\right| + C$

39. $\int \cot\left(\frac{x}{4}\right)dx = 4\int \cot\left(\frac{x}{4}\right)d\left(\frac{x}{4}\right) = 4\ln\left|\sin\left(\frac{x}{4}\right)\right| + C$

40. $\int x\sqrt{1-x}\;\;dx;\ \begin{bmatrix} u = 1-x \\ du = -dx \end{bmatrix} \to -\int (1-u)\sqrt{u}\,du = \int \left(u^{3/2} - u^{1/2}\right)du = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + C = -2\left[\frac{\left(\sqrt{1-x}\right)^3}{3} - \frac{\left(\sqrt{1-x}\right)^5}{5}\right] + C$

41. $\int \left(16+z^2\right)^{-3/2}dz;\ \begin{bmatrix} z = 4\tan\theta \\ dz = 4\sec^2\theta\,d\theta \end{bmatrix} \to \int \frac{4\sec^2\theta\,d\theta}{64\sec^3\theta\,d\theta} = \frac{1}{16}\int \cos\theta\,d\theta = \frac{1}{16}\sin\theta + C = \frac{z}{16\sqrt{16+z^2}} + C$

$= \frac{z}{16\left(16+z^2\right)^{1/2}} + C$

42. $\int \frac{dy}{\sqrt{25+y^2}} = \frac{1}{5}\int \frac{dy}{\sqrt{1+\left(\frac{y}{5}\right)^2}} = \int \frac{du}{\sqrt{1+u^2}},\ \left[u = \frac{y}{5}\right];\ \begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\,d\theta \end{bmatrix} \to \int \frac{\sec^2\theta\,d\theta}{\sqrt{1+\tan^2\theta}} = \int \sec\theta\,d\theta$

$= \ln|\sec\theta + \tan\theta| + C_1 = \ln\left|\sqrt{1+u^2} + u\right| + C_1 = \ln\left|\sqrt{1+\left(\frac{y}{5}\right)^2} + \frac{y}{5}\right| + C_1 = \ln\left|\frac{\sqrt{25+y^2}+y}{5}\right| + C_1$

$= \ln\left|y + \sqrt{25+y^2}\right| + C$

43. $\int \frac{dx}{x^2\sqrt{1-x^2}}; \begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{bmatrix} \rightarrow \int \frac{\cos\theta\, d\theta}{\sin^2\theta\cos\theta} = \int \csc^2\theta\, d\theta = -\cot\theta + C = -\frac{\cos\theta}{\sin\theta} + C = \frac{-\sqrt{1-x^2}}{x} + C$

44. $\int \frac{x^2\, dx}{\sqrt{1-x^2}}; \begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{bmatrix} \rightarrow \int \frac{\sin^2\theta\cos\theta\, d\theta}{\cos\theta} = \int \sin^2\theta\, d\theta = \int \frac{1-\cos 2\theta}{2}\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta$

$= \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta = \frac{\sin^{-1} x}{2} - \frac{x\sqrt{1-x^2}}{2} + C$

45. $\int \frac{dx}{\sqrt{x^2-9}}; \begin{bmatrix} x = 3\sec\theta \\ dx = 3\sec\theta\tan\theta\, d\theta \end{bmatrix} \rightarrow \int \frac{3\sec\theta\tan\theta\, d\theta}{\sqrt{9\sec^2\theta - 9}} = \int \frac{3\sec\theta\tan\theta\, d\theta}{3\tan\theta} = \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + C_1 = \ln\left|\frac{x}{3} + \sqrt{\left(\frac{x}{3}\right)^2 - 1}\right| + C_1 = \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| + C_1 = \ln\left|x + \sqrt{x^2-9}\right| + C$

46. $\int \frac{12\, dx}{\left(x^2-1\right)^{3/2}}; \begin{bmatrix} x = \sec\theta \\ dx = \sec\theta\tan\theta\, d\theta \end{bmatrix} \rightarrow \int \frac{12\sec\theta\tan\theta\, d\theta}{\tan^3\theta} = \int \frac{12\cos\theta\, d\theta}{\sin^2\theta}; \begin{bmatrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \rightarrow \int \frac{12\, du}{u^2}$

$= -\frac{12}{u} + C = -\frac{12}{\sin\theta} + C = -\frac{12\sec\theta}{\tan\theta} + C = -\frac{12x}{\sqrt{x^2-1}} + C$

47. $u = \ln(x+1)$, $du = \frac{dx}{x+1}$; $dv = dx$, $v = x$;

$\int \ln(x+1)\, dx = x\ln(x+1) - \int \frac{x}{x+1}\, dx = x\ln(x+1) - \int dx + \int \frac{dx}{x+1} = x\ln(x+1) - x + \ln(x+1) + C_1$

$= (x+1)\ln(x+1) - x + C_1 = (x+1)\ln(x+1) - (x+1) + C$, where $C = C_1 + 1$

48. $u = \ln x$, $du = \frac{dx}{x}$; $dv = x^2\, dx$, $v = \frac{1}{3}x^3$;

$\int x^2 \ln x\, dx = \frac{1}{3}x^3\ln x - \int \frac{1}{3}x^3\left(\frac{1}{x}\right) dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$

49. $u = \tan^{-1} 3x$, $du = \frac{3\, dx}{1+9x^2}$; $dv = dx$, $v = x$;

$\int \tan^{-1} 3x\, dx = x\tan^{-1} 3x - \int \frac{3x\, dx}{1+9x^2}; \begin{bmatrix} y = 1+9x^2 \\ dy = 18x\, dx \end{bmatrix} \rightarrow x\tan^{-1} 3x - \frac{1}{6}\int \frac{dy}{y}$

$= x\tan^{-1}(3x) - \frac{1}{6}\ln\left(1+9x^2\right) + C$

50. $u = \cos^{-1}\left(\frac{x}{2}\right)$, $du = \frac{-dx}{\sqrt{4-x^2}}$; $dv = dx$, $v = x$;

$\int \cos^{-1}\left(\frac{x}{2}\right) dx = x\cos^{-1}\left(\frac{x}{2}\right) + \int \frac{x\, dx}{\sqrt{4-x^2}}; \begin{bmatrix} y = 4-x^2 \\ dy = -2x\, dx \end{bmatrix} \rightarrow x\cos^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\int \frac{dy}{\sqrt{y}}$

$= x\cos^{-1}\left(\frac{x}{2}\right) - \sqrt{4-x^2} + C = x\cos^{-1}\left(\frac{x}{2}\right) - 2\sqrt{1-\left(\frac{x}{2}\right)^2} + C$

51. $\qquad\qquad e^x$

$(x+1)^2 \xrightarrow{(+)} e^x$

$2(x+1) \xrightarrow{(-)} e^x$

$2 \xrightarrow{(+)} e^x$

$0 \qquad \Rightarrow \int (x+1)^2 e^x\,dx = \left[(x+1)^2 - 2(x+1) + 2\right]e^x + C$

52. $\qquad\qquad \sin(1-x)$

$x^2 \xrightarrow{(+)} \cos(1-x)$

$2x \xrightarrow{(-)} -\sin(1-x)$

$2 \xrightarrow{(+)} -\cos(1-x)$

$0 \qquad \Rightarrow \int x^2 \sin(1-x)\,dx = x^2\cos(1-x) + 2x\sin(1-x) - 2\cos(1-x) + C$

53. $u = \cos 2x,\ du = -2\sin 2x\,dx;\ dv = e^x\,dx,\ v = e^x;$

$I = \int e^x \cos 2x\,dx = e^x\cos 2x + 2\int e^x \sin 2x\,dx;$

$u = \sin 2x,\ du = 2\cos 2x\,dx;\ dv = e^x\,dx,\ v = e^x;$

$I = e^x\cos 2x + 2\left[e^x \sin 2x - 2\int e^x\cos 2x\,dx\right] = e^x\cos 2x + 2e^x\sin 2x - 4I \Rightarrow I = \frac{e^x\cos 2x}{5} + \frac{2e^x\sin 2x}{5} + C$

54. $u = \sin 3x,\ du = 3\cos 3x\,dx;\ dv = e^{-2x}\,dx,\ v = -\frac{1}{2}e^{-2x};$

$I = \int e^{-2x}\sin 3x\,dx = -\frac{1}{2}e^{-2x}\sin 3x + \frac{3}{2}\int e^{-2x}\cos 3x\,dx;$

$u = \cos 3x,\ du = -3\sin 3x\,dx;\ dv = e^{-2x}\,dx,\ v = -\frac{1}{2}e^{-2x};$

$I = -\frac{1}{2}e^{-2x}\sin 3x + \frac{3}{2}\left[-\frac{1}{2}e^{-2x}\cos 3x - \frac{3}{2}\int e^{-2x}\sin 3x\,dx\right] = -\frac{1}{2}e^{-2x}\sin 3x - \frac{3}{4}e^{-2x}\cos 3x - \frac{9}{4}I$

$\Rightarrow I = \frac{4}{13}\left(-\frac{1}{2}e^{-2x}\sin 3x - \frac{3}{4}e^{-2x}\cos 3x\right) + C = -\frac{2}{13}e^{-2x}\sin 3x - \frac{3}{13}e^{-2x}\cos 3x + C$

55. $\int \frac{x\,dx}{x^2 - 3x + 2} = \int \frac{2\,dx}{x-2} - \int \frac{dx}{x-1} = 2\ln|x-2| - \ln|x-1| + C$

56. $\int \frac{dx}{x(x+1)^2} = \int \left(\frac{1}{x} - \frac{2}{x+1} + \frac{x}{(x+1)^2}\right)dx = \ln|x| - 2\ln|x+1| + \left(\ln|x+1| + \frac{1}{x+1}\right) + C$

$= \ln|x| - \ln|x+1| + \frac{1}{x+1} + C$

57. $\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2};\ [\cos\theta = y] \to -\int \frac{dy}{y^2 + y - 2} = -\frac{1}{3}\int \frac{dy}{y-1} + \frac{1}{3}\int \frac{dy}{y+2} = \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C$

$= \frac{1}{3}\ln\left|\frac{\cos\theta + 2}{\cos\theta - 1}\right| + C = -\frac{1}{3}\ln\left|\frac{\cos\theta - 1}{\cos\theta + 2}\right| + C$

58. $\int \frac{3x^2+4x+4}{x^3+x}\,dx = \int \frac{4}{x}\,dx - \int \frac{x-4}{x^2+1}\,dx = 4\ln|x| - \frac{1}{2}\ln(x^2+1) + 4\tan^{-1}x + C$

59. $\int \frac{(v+3)\,dv}{2v^3-8v} = \frac{1}{2}\int\left(-\frac{3}{4v}+\frac{5}{8(v-2)}+\frac{1}{8(v+2)}\right)dv = -\frac{3}{8}\ln|v| + \frac{5}{16}\ln|v-2| + \frac{1}{16}\ln|v+2| + C$

$= \frac{1}{16}\ln\left|\frac{(v-2)^5(v+2)}{v^6}\right| + C$

60. $\int \frac{dt}{t^4+4t^2+3} = \frac{1}{2}\int\frac{dt}{t^2+1} - \frac{1}{2}\int\frac{dt}{t^2+3} = \frac{1}{2}\tan^{-1}t - \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C = \frac{1}{2}\tan^{-1}t - \frac{\sqrt{3}}{6}\tan^{-1}\frac{t}{\sqrt{3}} + C$

61. $\int \frac{x^3+x^2}{x^2+x-2}\,dx = \int\left(x+\frac{2x}{x^2+x-2}\right)dx = \int x\,dx + \frac{2}{3}\int\frac{dx}{x-1} + \frac{4}{3}\int\frac{dx}{x+2}$

$= \frac{x^2}{2} + \frac{4}{3}\ln|x+2| + \frac{2}{3}\ln|x-1| + C$

62. $\int \frac{x^3+4x^2}{x^2+4x+3}\,dx = \int\left(x-\frac{3x}{x^2+4x+3}\right)dx = \int x\,dx + \frac{3}{2}\int\frac{dx}{x+1} - \frac{9}{2}\int\frac{dx}{x+3}$

$= \frac{x^2}{2} - \frac{9}{2}\ln|x+3| + \frac{3}{2}\ln|x+1| + C$

63. $\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx = \int\left[(2x-3)+\frac{x}{x^2+2x-8}\right]dx = \int(2x-3)\,dx + \frac{1}{3}\int\frac{dx}{x-2} + \frac{2}{3}\int\frac{dx}{x+4}$

$= x^2 - 3x + \frac{2}{3}\ln|x+4| + \frac{1}{3}\ln|x-2| + C$

64. $\int \frac{dx}{x(3\sqrt{x+1})};\ \left[\begin{array}{c} u=\sqrt{x+1} \\ du=\frac{dx}{2\sqrt{x+1}} \\ dx = 2u\,du \end{array}\right] \to \frac{2}{3}\int\frac{u\,du}{(u^2-1)u} = \frac{1}{3}\int\frac{du}{u-1} - \frac{1}{3}\int\frac{du}{u+1} = \frac{1}{3}\ln|u-1| - \frac{1}{3}\ln|u+1| + C$

$= \frac{1}{3}\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$

65. $\int \frac{ds}{e^s-1};\ \left[\begin{array}{c} u=e^s-1 \\ du=e^s\,ds \\ ds=\frac{du}{u+1} \end{array}\right] \to \int\frac{du}{u(u-1)} = \int\frac{du}{u-1} - \int\frac{du}{u} = \ln\left|\frac{u-1}{u}\right| + C = \ln\left|\frac{e^s-1}{e^s}\right| + C = \ln|1-e^{-s}| + C$

66. $\int \frac{ds}{\sqrt{e^s+1}}; \left[\begin{array}{c} u=\sqrt{e^s+1} \\ du=\frac{e^s\,ds}{2\sqrt{e^s+1}} \\ ds=\frac{2u\,du}{u^2-1} \end{array}\right] \to \int \frac{2u\,du}{u(u^2-1)} = 2\int \frac{du}{(u+1)(u-1)} = \int \frac{du}{u-1} - \int \frac{du}{u+1} = \ln\left|\frac{u-1}{u+1}\right| + C$

$= \ln\left|\frac{\sqrt{e^s+1}-1}{\sqrt{e^s+1}+1}\right| + C$

67. (a) $\int \frac{y\,dy}{\sqrt{16-y^2}} = -\frac{1}{2}\int \frac{d(16-y^2)}{\sqrt{16-y^2}} = -\sqrt{16-y^2} + C$

(b) $\int \frac{y\,dy}{\sqrt{16-y^2}}; [y = 4\sin x] \to 4\int \frac{\sin x \cos x\,dx}{\cos x} = -4\cos x + C = -\frac{4\sqrt{16-y^2}}{4} + C = -\sqrt{16-y^2} + C$

68. (a) $\int \frac{x\,dx}{\sqrt{4+x^2}} = \frac{1}{2}\int \frac{d(4+x^2)}{\sqrt{4+x^2}} = \sqrt{4+x^2} + C$

(b) $\int \frac{x\,dx}{\sqrt{4+x^2}}; [x = 2\tan y] \to \int \frac{2\tan y \cdot 2\sec^2 y\,dy}{2\sec y} = 2\int \sec y \tan y\,dy = 2\sec y + C = \sqrt{4+x^2} + C$

69. (a) $\int \frac{x\,dx}{4-x^2} = -\frac{1}{2}\int \frac{d(4-x^2)}{4-x^2} = -\frac{1}{2}\ln|4-x^2| + C$

(b) $\int \frac{x\,dx}{4-x^2}; [x = 2\sin\theta] \to \int \frac{2\sin\theta \cdot 2\cos\theta\,d\theta}{4\cos^2\theta} = \int \tan\theta\,d\theta = -\ln|\cos\theta| + C = -\ln\sqrt{4-x^2} + C$

$= -\frac{1}{2}\ln|4-x^2| + C$

70. (a) $\int \frac{t\,dt}{\sqrt{4t^2-1}} = \frac{1}{8}\int \frac{d(4t^2-1)}{\sqrt{4t^2-1}} = \frac{1}{4}\sqrt{4t^2-1} + C$

(b) $\int \frac{t\,dt}{\sqrt{4t^2-1}}; \left[t = \frac{1}{2}\sec\theta\right] \to \int \frac{\frac{1}{2}\sec\theta\tan\theta \cdot \frac{1}{2}\sec\theta\,d\theta}{\tan\theta} = \frac{1}{4}\int \sec^2\theta\,d\theta = \frac{\tan\theta}{4} + C = \frac{\sqrt{4t^2-1}}{4} + C$

71. $\int \frac{x\,dx}{9-x^2}; \left[\begin{array}{c} u = 9-x^2 \\ du = -2x\,dx \end{array}\right] \to -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| + C = \ln\frac{1}{\sqrt{u}} + C = \ln\frac{1}{\sqrt{9-x^2}} + C$

72. $\int \frac{dx}{x(9-x^2)} = \frac{1}{9}\int \frac{dx}{x} + \frac{1}{18}\int \frac{dx}{3-x} - \frac{1}{18}\int \frac{dx}{3+x} = \frac{1}{9}\ln|x| - \frac{1}{18}\ln|3-x| - \frac{1}{18}\ln|3+x| + C$

$= \frac{1}{9}\ln|x| - \frac{1}{18}\ln|9-x^2| + C$

73. $\int \frac{dx}{9-x^2} = \frac{1}{6}\int \frac{dx}{3-x} + \frac{1}{6}\int \frac{dx}{3+x} = -\frac{1}{6}\ln|3-x| + \frac{1}{6}\ln|3+x| + C = \frac{1}{6}\ln\left|\frac{x+3}{x-3}\right| + C$

74. $\int \frac{dx}{\sqrt{9-x^2}};\ \begin{bmatrix} x = 3\sin\theta \\ dx = 3\cos\theta\, d\theta \end{bmatrix} \to \int \frac{3\cos\theta}{3\cos\theta}\, d\theta = \int d\theta = \theta + C = \sin^{-1}\frac{x}{3} + C$

75. $\int \frac{x\,dx}{1+\sqrt{x}};\ \begin{bmatrix} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \to \int \frac{u^2 \cdot 2u\,du}{1+u} = \int \left(2u^2 - 2u + 2 - \frac{2}{1+u}\right) du = \frac{2}{3}u^3 - u^2 + 2u - 2\ln|1+u| + C$

$= \frac{2x^{3/2}}{3} - x + 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$

76. $\int \frac{dx}{x\left(x^2+1\right)^2};\ \begin{bmatrix} x = \tan\theta \\ dx = \sec^2\theta\, d\theta \end{bmatrix} \to \int \frac{\sec^2\theta\, d\theta}{\tan\theta\sec^4\theta} = \int \frac{\cos^3\theta\, d\theta}{\sin\theta} = \int \left(\frac{1-\sin^2\theta}{\sin\theta}\right) d(\sin\theta)$

$= \ln|\sin\theta| - \frac{1}{2}\sin^2\theta + C = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| - \frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}}\right)^2 + C$

77. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx;\ \begin{bmatrix} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \to \int \frac{\cos u \cdot 2u\,du}{u} = 2\int \cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C$

78. $\int \frac{dx}{\sqrt{-2x-x^2}} = \int \frac{d(x+1)}{\sqrt{1-(x+1)^2}} = \sin^{-1}(x+1) + C$

79. $\int \frac{du}{\sqrt{1+u^2}};\ [u = \tan\theta] \to \int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \ln|\sec\theta + \tan\theta| + C = \ln\left|\sqrt{1+u^2} + u\right| + C$

80. $\int \frac{2-\cos x+\sin x}{\sin^2 x}\,dx = \int 2\csc^2 x\,dx - \int \frac{\cos x\,dx}{\sin^2 x} + \int \csc x\,dx = -2\cot x + \frac{1}{\sin x} - \ln|\csc x + \cot x| + C$

$= -2\cot x + \csc x - \ln|\csc x + \cot x| + C$

81. $\int \frac{9\,dv}{81-v^4} = \frac{1}{2}\int \frac{dv}{v^2+9} + \frac{1}{12}\int \frac{dv}{3-v} + \frac{1}{12}\int \frac{dv}{3+v} = \frac{1}{12}\ln\left|\frac{3+v}{3-v}\right| + \frac{1}{6}\tan^{-1}\frac{v}{3} + C$

82. $\cos(2\theta+1)$

$\theta \xrightarrow{(+)} \frac{1}{2}\sin(2\theta+1)$

$1 \xrightarrow{(-)} -\frac{1}{4}\cos(2\theta+1)$

0

$\int \theta\cos(2\theta+1)\,d\theta = \frac{\theta}{2}\sin(2\theta+1) + \frac{1}{4}\cos(2\theta+1) + C$

83. $\int \frac{x^3\,dx}{x^2-2x+1} = \int \left(x+2+\frac{3x+2}{x^2-2x+1}\right) dx = \int (x+2)\,dx + 3\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}$

$= \frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1} + C$

84. $\int \frac{d\theta}{\sqrt{1+\sqrt{\theta}}};\ \left[\begin{array}{c} x = 1+\sqrt{\theta} \\ dx = \frac{d\theta}{2\sqrt{\theta}} \\ d\theta = 2(x-1)\,dx \end{array}\right] \to \int \frac{2(x-1)\,dx}{\sqrt{x}} = 2\int \sqrt{x}\,dx - 2\int \frac{dx}{\sqrt{x}} = \frac{4}{3}x^{3/2} - 4x^{1/2} + C$

$= \frac{4}{3}\left(1+\sqrt{\theta}\right)^{3/2} - 4\left(1+\sqrt{\theta}\right)^{1/2} + C = 4\left[\frac{\left(\sqrt{1+\sqrt{\theta}}\right)^3}{3} - \sqrt{1+\sqrt{\theta}}\right] + C$

85. $\int \frac{2\sin\sqrt{x}\,dx}{\sqrt{x}\sec\sqrt{x}};\ \left[\begin{array}{c} y = \sqrt{x} \\ dy = \frac{dx}{2\sqrt{x}} \end{array}\right] \to \int \frac{2\sin y \cdot 2y\,dy}{y\sec y} = \int 2\sin 2y\,dy = -\cos(2y) + C = -\cos\left(2\sqrt{x}\right) + C$

86. $\int \frac{x^5\,dx}{x^4-16} = \int \left(x + \frac{16x}{x^4-16}\right) dx = \frac{x^2}{2} + \int \left(\frac{2x}{x^2-4} - \frac{2x}{x^2+4}\right) dx = \frac{x^2}{2} + \ln\left|\frac{x^2-4}{x^2+4}\right| + C$

87. $\int \frac{d\theta}{\theta^2-2\theta+4} = \int \frac{d\theta}{(\theta-1)^2+3} = \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{\theta-1}{\sqrt{3}}\right) + C$

88. $\int \frac{dr}{(r+1)\sqrt{r^2+2r}} = \int \frac{d(r+1)}{(r+1)\sqrt{(r+1)^2-1}} = \sec^{-1}|r+1| + C$

89. $\int \frac{\sin 2\theta\,d\theta}{(1+\cos 2\theta)^2} = -\frac{1}{2}\int \frac{d(1+\cos 2\theta)}{(1+\cos 2\theta)^2} = \frac{1}{2(1+\cos 2\theta)} + C = \frac{1}{4}\sec^2\theta + C$

90. $\int \frac{dx}{\left(x^2-1\right)^2} = \int \frac{dx}{\left(1-x^2\right)^2} = \frac{x}{2\left(1-x^2\right)} + \frac{1}{2}\int \frac{dx}{1-x^2}$ (FORMULA 19)

$= \frac{x}{2\left(1-x^2\right)} + \frac{1}{4}\int \frac{dx}{1-x} + \frac{1}{4}\int \frac{dx}{1+x} = \frac{x}{2\left(1-x^2\right)} - \frac{1}{4}\ln|1-x| + \frac{1}{4}\ln|1+x| + C = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2\left(x^2-1\right)} + C$

91. $\int \frac{x\,dx}{\sqrt{2-x}};\ \left[\begin{array}{c} y = 2-x \\ dy = -dx \end{array}\right] \to -\int \frac{(2-y)\,dy}{\sqrt{y}} = \frac{2}{3}y^{3/2} - 4y^{1/2} + C = \frac{2}{3}(2-x)^{3/2} - 4(2-x)^{1/2} + C$

$= 2\left[\frac{\left(\sqrt{2-x}\right)^3}{3} - 2\sqrt{2-x}\right] + C$

92. $\int \frac{dy}{y^2-2y+2} = \int \frac{d(y-1)}{(y-1)^2+1} = \tan^{-1}(y-1) + C$

93. $\int \ln \sqrt{x-1}\, dx; \begin{bmatrix} y = \sqrt{x-1} \\ dy = \frac{dx}{2\sqrt{x-1}} \end{bmatrix} \to \int \ln y \cdot 2y\, dy;\ u = \ln y,\ du = \frac{dy}{y};\ dv = 2y\, dy,\ v = y^2$

$\Rightarrow \int 2y \ln y\, dy = y^2 \ln y - \int y\, dy = y^2 \ln y - \frac{1}{2}y^2 + C = (x-1)\ln\sqrt{x-1} - \frac{1}{2}(x-1) + C_1$

$= \frac{1}{2}[(x-1)\ln|x-1| - x] + \left(C_1 + \frac{1}{2}\right) = \frac{1}{2}[x\ln|x-1| - x - \ln|x-1|] + C$

94. $\int \frac{x\, dx}{\sqrt{8-2x^2-x^4}} = \frac{1}{2}\int \frac{d(x^2+1)}{\sqrt{9-(x^2+1)^2}} = \frac{1}{2}\sin^{-1}\left(\frac{x^2+1}{3}\right) + C$

95. $\int \frac{z+1}{z^2(z^2+4)}\, dz = \frac{1}{4}\int\left(\frac{1}{z} + \frac{1}{z^2} - \frac{z+1}{z^2+4}\right) dz = \frac{1}{4}\ln|z| - \frac{1}{4z} - \frac{1}{8}\ln(z^2+4) - \frac{1}{8}\tan^{-1}\frac{z}{2} + C$

96. $\int x^3 e^{x^2}\, dx = \frac{1}{2}\int x^2 e^{x^2}\, d(x^2) = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C = \frac{(x^2-1)e^{x^2}}{2} + C$

97. $u = \tan^{-1} x,\ du = \frac{dx}{1+x^2};\ dv = \frac{dx}{x^2},\ v = -\frac{1}{x};$

$\int \frac{\tan^{-1} x\, dx}{x^2} = -\frac{1}{x}\tan^{-1} x + \int \frac{dx}{x(1+x^2)} = -\frac{1}{x}\tan^{-1} x + \int \frac{dx}{x} - \int \frac{x\, dx}{1+x^2}$

$= -\frac{1}{x}\tan^{-1} x + \ln|x| - \frac{1}{2}\ln(1+x^2) + C = -\frac{\tan^{-1} x}{x} + \ln|x| - \ln\sqrt{1+x^2} + C$

98. $\int \frac{e^t\, dt}{e^{2t}+3e^t+2};\ [e^t = x] \to \int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \ln|x+1| - \ln|x+2| + C$

$= \ln\left|\frac{x+1}{x+2}\right| + C = \ln\left(\frac{e^t+1}{e^t+2}\right) + C$

99. $\int \frac{1-\cos 2x}{1+\cos 2x}\, dx = \int \tan^2 x\, dx = \int \left(\sec^2 x - 1\right) dx = \tan x - x + C$

100. $\int \frac{\cos\left(\sin^{-1} x\right) dx}{\sqrt{1-x^2}}; \begin{bmatrix} u = \sin^{-1} x \\ du = \frac{dx}{\sqrt{1-x^2}} \end{bmatrix} \to \int \cos u\, du = \sin u + C = \sin\left(\sin^{-1} x\right) + C = x + C$

101. $\int \frac{\cos x\, dx}{\sin^3 x - \sin x} = -\int \frac{\cos x\, dx}{(\sin x)(1-\sin^2 x)} = -\int \frac{\cos x\, dx}{(\sin x)(\cos^2 x)} = -\int \frac{2\, dx}{\sin 2x} = -2\int \csc 2x\, dx$

$= \ln|\csc(2x) + \cot(2x)| + C$

102. $\int \frac{e^t\, dt}{1+e^t} = \ln(1+e^t) + C$

103. $\int_1^\infty \frac{\ln y\, dy}{y^3}$; $\begin{bmatrix} x = \ln y \\ dx = \frac{dy}{y} \\ dy = e^x\, dx \end{bmatrix} \to \int_0^\infty \frac{x \cdot e^x}{e^{3x}}\, dx = \int_0^\infty xe^{-2x}\, dx = \lim_{b\to\infty} \left[-\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\right]_0^b$

$= \lim_{b\to\infty} \left(\frac{-b}{2e^{2b}} - \frac{1}{4e^{2b}}\right) - \left(0 - \frac{1}{4}\right) = \frac{1}{4}$

104. $\int \frac{\cot v\, dv}{\ln(\sin v)} = \int \frac{\cos v\, dv}{(\sin v)\ln(\sin v)}$; $\begin{bmatrix} u = \ln(\sin v) \\ du = \frac{\cos v\, dv}{\sin v} \end{bmatrix} \to \int \frac{du}{u} = \ln|u| + C = \ln|\ln(\sin v)| + C$

105. $\int \frac{dx}{(2x-1)\sqrt{x^2-x}} = \int \frac{2\, dx}{(2x-1)\sqrt{4x^2-4x}} = \int \frac{2\, dx}{(2x-1)\sqrt{(2x-1)^2-1}}$; $\begin{bmatrix} u = 2x-1 \\ du = 2\, dx \end{bmatrix} \to \int \frac{du}{u\sqrt{u^2-1}}$

$= \sec^{-1}|u| + C = \sec^{-1}|2x-1| + C$

106. $\int e^{\ln\sqrt{x}}\, dx = \int \sqrt{x}\, dx = \frac{2}{3}x^{3/2} + C$

107. $\int e^\theta\sqrt{3+4e^\theta}\, d\theta$; $\begin{bmatrix} u = 4e^\theta \\ du = 4e^\theta\, d\theta \end{bmatrix} \to \frac{1}{4}\int \sqrt{3+u}\, du = \frac{1}{4}\cdot\frac{2}{3}(3+u)^{3/2} + C = \frac{1}{6}\left(3+4e^\theta\right)^{3/2} + C$

108. $\int \frac{dv}{\sqrt{e^{2v}-1}}$; $\begin{bmatrix} x = e^v \\ dx = e^v\, dv \end{bmatrix} \to \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + C = \sec^{-1}(e^v) + C$

109. $\int (27)^{3\theta+1}\, d\theta = \frac{1}{3}\int (27)^{3\theta+1}\, d(3\theta+1) = \frac{1}{3\ln 27}(27)^{3\theta+1} + C = \frac{1}{3}\left(\frac{27^{3\theta+1}}{\ln 27}\right) + C$

110.

		$\sin x$
x^5	$\xrightarrow{(+)}$	$-\cos x$
$5x^4$	$\xrightarrow{(-)}$	$-\sin x$
$20x^3$	$\xrightarrow{(+)}$	$\cos x$
$60x^2$	$\xrightarrow{(-)}$	$\sin x$
$120x$	$\xrightarrow{(+)}$	$-\cos x$
120	$\xrightarrow{(-)}$	$-\sin x$
0		

$\int x^5 \sin x\, dx = -x^5\cos x + 5x^4\sin x + 20x^3\cos x - 60x^2\sin x - 120x\cos x + 120\sin x + C$

111. $\int \frac{dr}{1+\sqrt{r}}$; $\begin{bmatrix} u = \sqrt{r} \\ du = \frac{dr}{2\sqrt{r}} \end{bmatrix} \to \int \frac{2u\, du}{1+u} = \int \left(2 - \frac{2}{1+u}\right) du = 2u - 2\ln|1+u| + C = 2\sqrt{r} - 2\ln\left(1+\sqrt{r}\right) + C$

112. $\int \frac{8\,dy}{y^3(y+2)} = \int \frac{dy}{y} - \int \frac{2\,dy}{y^2} + \int \frac{4\,dy}{y^3} - \int \frac{dy}{(y+2)} = \ln\left|\frac{y}{y+2}\right| + \frac{2}{y} - \frac{2}{y^2} + C$

113. $\int \frac{8\,dm}{m\sqrt{49m^2-4}} = \frac{8}{7}\int \frac{dm}{m\sqrt{m^2-\left(\frac{2}{7}\right)^2}} = 4\sec^{-1}\left(\frac{7m}{2}\right) + C$

114. $\int \frac{dt}{t(1+\ln t)\sqrt{(\ln t)(2+\ln t)}}; \begin{bmatrix} u = \ln t \\ du = \frac{dt}{t} \end{bmatrix} \to \int \frac{du}{(1+u)\sqrt{u(2+u)}} = \int \frac{du}{(u+1)\sqrt{(u+1)^2-1}}$

$= \sec^{-1}|u+1| + C = \sec^{-1}|\ln t + 1| + C$

115. $\lim\limits_{t\to 0} \frac{t - \ln(1+2t)}{t^2} = \lim\limits_{t\to 0} \frac{1 - \frac{2}{1+2t}}{2t} = \infty$ for $t \to 0^-$ and $-\infty$ for $t \to 0^+$

The limit does not exist.

116. $\lim\limits_{t\to 0} \frac{\tan 3t}{\tan 5t} = \lim\limits_{t\to 0} \frac{3\sec^2 3t}{5\sec^2 5t} = \frac{3}{5}$

117. $\lim\limits_{x\to 0} \frac{x\sin x}{1-\cos x} = \lim\limits_{x\to 0} \frac{x\cos x + \sin x}{\sin x} = \lim\limits_{x\to 0} \frac{-x\sin x + \cos x + \cos x}{\cos x} = 2$

118. The limit leads to the indeterminate form 1^∞. $f(x) = x^{1/(1-x)} \Rightarrow \ln f(x) = \frac{\ln x}{1-x}$

$\Rightarrow \lim\limits_{x\to 1} \frac{\ln x}{1-x} = \lim\limits_{x\to 1} \frac{1/x}{-1} = -1 \Rightarrow \lim\limits_{x\to 1} x^{1/(1-x)} = \lim\limits_{x\to 1} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$

119. The limit leads to the indeterminate form ∞^0. $f(x) = x^{1/x} \Rightarrow \ln f(x) = \frac{\ln x}{x} \Rightarrow \lim\limits_{x\to\infty} \frac{\ln x}{x} = \lim\limits_{x\to\infty} \frac{1/x}{1} = 0$

$\Rightarrow \lim\limits_{x\to\infty} x^{1/x} = \lim\limits_{x\to\infty} e^{\ln f(x)} = e^0 = 1$

120. The limit leads to the indeterminate form 1^∞. $f(x) = \left(1+\frac{3}{x}\right)^x \Rightarrow \ln f(x) = x\ln\left(1+\frac{3}{x}\right) = \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}}$

$\Rightarrow \lim\limits_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}} = \lim\limits_{x\to\infty} \frac{\frac{-3/x^2}{1+3/x}}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty} \frac{3x}{x+3} = 3 \Rightarrow \lim\limits_{x\to\infty} \left(1+\frac{3}{x}\right)^x = \lim\limits_{x\to\infty} e^{\ln f(x)} = e^3$

121. $\lim\limits_{r\to\infty} \frac{\cos r}{\ln r} = 0$ since $|\cos r| \le 1$ and $\ln r \to \infty$ as $r \to \infty$.

122. $\lim\limits_{\theta\to\pi/2} \left(\theta - \frac{\pi}{2}\right)\sec\theta = \lim\limits_{\theta\to\pi/2} \frac{\theta - \frac{\pi}{2}}{\cos\theta} = \lim\limits_{\theta\to\pi/2} \frac{1}{-\sin\theta} = -1$

123. $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1}\left[\frac{\ln x-x+1}{(x-1)\ln x}\right]=\lim_{x\to 1}\frac{\frac{1}{x}-1}{\frac{x-1}{x}+\ln x}=\lim_{x\to 1}\frac{1-x}{x-1+x\ln x}=\lim_{x\to 1}\frac{-1}{1+x/x+\ln x}=-\frac{1}{2}$

124. The limit leads to the indeterminate form ∞^0. $f(x)=\left(1+\frac{1}{x}\right)^x \Rightarrow \ln f(x)=x\ln\left(1+\frac{1}{x}\right)=\frac{\ln(1+1/x)}{1/x}$

$\Rightarrow \lim_{x\to 0^+}\frac{\ln(1+1/x)}{1/x}=\lim_{x\to 0^+}\frac{\frac{-1/x^2}{1+1/x}}{-1/x^2}=\lim_{x\to 0^+}\frac{x}{x+1}=0 \Rightarrow \lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x=\lim_{x\to 0^+}e^{\ln f(x)}=e^0=1$

125. The limit leads to the indeterminate form 0^0. $f(\theta)=(\tan\theta)^\theta \Rightarrow \ln f(\theta)=\theta\ln(\tan\theta)=\frac{\ln(\tan\theta)}{1/\theta}$

$\Rightarrow \lim_{x\to 0^+}\frac{\ln(\tan\theta)}{1/\theta}=\lim_{x\to 0^+}\frac{\frac{\sec^2\theta}{\tan\theta}}{-\frac{1}{\theta^2}}=\lim_{x\to 0^+}-\frac{\theta^2}{\sin\theta\cos\theta}=\lim_{x\to 0^+}\frac{-2\theta}{-\sin^2\theta+\cos^2\theta}=0$

$\Rightarrow \lim_{x\to 0^+}(\tan\theta)^\theta=\lim_{x\to 0^+}e^{\ln f(\theta)}=e^0=1$

126. $\lim_{\theta\to\infty}\theta^2\sin\left(\frac{1}{\theta}\right)=\lim_{t\to 0^+}\frac{\sin t}{t^2}=\lim_{t\to 0^+}\frac{\cos t}{2t}=\infty$

127. $\lim_{x\to\infty}\frac{x^3-3x^2+1}{2x^2+x-3}=\lim_{x\to\infty}\frac{3x^2-6x}{4x+1}=\lim_{x\to\infty}\frac{6x-6}{4}=\infty$

128. $\lim_{x\to\infty}\frac{3x^2-x+1}{x^4-x^3+2}=\lim_{x\to\infty}\frac{6x-1}{4^3-3x^2}=\lim_{x\to\infty}\frac{6}{12x^2-6x}=0$

129. $\int_0^3\frac{dx}{\sqrt{9-x^2}}=\lim_{b\to 3^-}\int_0^b\frac{dx}{\sqrt{9-x^2}}=\lim_{b\to 3^-}\left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^b=\lim_{b\to 3^-}\sin^{-1}\left(\frac{b}{3}\right)-\sin^{-1}\left(\frac{0}{3}\right)=\frac{\pi}{2}-0=\frac{\pi}{2}$

130. $\int_0^1\ln x\,dx=\lim_{b\to 0^+}[x\ln x-x]_b^1=(1\cdot\ln 1-1)-\lim_{b\to 0^+}[b\ln b-b]=-1-\lim_{b\to 0^+}\frac{\ln b}{\left(\frac{1}{b}\right)}=-1-\lim_{b\to 0^+}\frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^2}\right)}$

$=-1+0=-1$

131. $\int_{-1}^1\frac{dy}{y^{2/3}}=\int_{-1}^0\frac{dy}{y^{2/3}}+\int_0^1\frac{dy}{y^{2/3}}=2\int_0^1\frac{dy}{y^{2/3}}=2\cdot 3\lim_{b\to 0^+}\left[y^{1/3}\right]_b^1=6\left(1-\lim_{b\to 0^+}b^{1/3}\right)=6$

132. $\int_{-2}^0\frac{d\theta}{(\theta+1)^{3/5}}=\int_{-2}^{-1}\frac{d\theta}{(\theta+1)^{3/5}}+\int_{-1}^0\frac{d\theta}{(\theta+1)^{3/5}}=\lim_{b\to -1^-}\left[\frac{5}{2}(\theta+1)^{2/5}\right]_{-2}^b+\lim_{b\to -1^+}\left[\frac{5}{2}(\theta+1)^{2/5}\right]_b^0$

$=\lim_{b\to -1^-}\left(\frac{5}{2}(b+1)^{2/5}-\frac{5}{2}\right)+\lim_{b\to -1^+}\left(\frac{5}{2}-\frac{5}{2}(b+1)^{2/5}\right)=-\frac{5}{2}+\frac{5}{2}=0$

133. $\int_3^\infty \frac{2\,du}{u^2-2u} = \int_3^\infty \frac{du}{u-2} - \int_3^\infty \frac{du}{u} = \lim_{b\to\infty} \left[\ln\left|\frac{u-2}{u}\right|\right]_3^b = \lim_{b\to\infty} \left[\ln\left|\frac{b-2}{b}\right|\right] - \ln\left|\frac{3-2}{3}\right| = 0 - \ln\left(\frac{1}{3}\right) = \ln 3$

134. $\int_1^\infty \frac{3v-1}{4v^3-v^2}\,dv = \int_1^\infty \left(\frac{1}{v}+\frac{1}{v^2}-\frac{4}{4v-1}\right)dv = \lim_{b\to\infty} \left[\ln v - \frac{1}{v} - \ln(4v-1)\right]_1^b$

$= \lim_{b\to\infty} \left[\ln\left(\frac{b}{4b-1}\right) - \frac{1}{b}\right] - (\ln 1 - 1 - \ln 3) = \ln \frac{1}{4} + 1 + \ln 3 = 1 + \ln \frac{3}{4}$

135. $\int_0^\infty x^2e^{-x}\,dx = \lim_{b\to\infty} \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^b = \lim_{b\to\infty} \left(-b^2e^{-b} - 2be^{-b} - 2e^{-b}\right) - (-2) = 0 + 2 = 2$

136. $\int_{-\infty}^0 xe^{3x}\,dx = \lim_{b\to-\infty} \left[\frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}\right]_b^0 = -\frac{1}{9} - \lim_{b\to-\infty} \left(\frac{b}{3}e^{3b} - \frac{1}{9}e^{3b}\right) = -\frac{1}{9} - 0 = -\frac{1}{9}$

137. $\int_{-\infty}^\infty \frac{dx}{4x^2+9} = 2\int_0^\infty \frac{dx}{4x^2+9} = \frac{1}{2}\int_0^\infty \frac{dx}{x^2+\frac{9}{4}} = \frac{1}{2}\lim_{b\to\infty} \left[\frac{2}{3}\tan^{-1}\left(\frac{2x}{3}\right)\right]_0^b = \frac{1}{2}\lim_{b\to\infty} \left[\frac{2}{3}\tan^{-1}\left(\frac{2b}{3}\right)\right] - \frac{1}{3}\tan^{-1}(0)$

$= \frac{1}{2}\left(\frac{2}{3}\cdot\frac{\pi}{2}\right) - 0 = \frac{\pi}{6}$

138. $\int_{-\infty}^\infty \frac{4\,dx}{x^2+16} = 2\int_0^\infty \frac{4\,dx}{x^2+16} = 2\lim_{b\to\infty} \left[\tan^{-1}\left(\frac{x}{4}\right)\right]_0^b = 2\lim_{b\to\infty} \left[\tan^{-1}\left(\frac{b}{4}\right)\right] - \tan^{-1}(0) = 2\left(\frac{\pi}{2}\right) - 0 = \pi$

139. $\lim_{\theta\to\infty} \frac{\theta}{\sqrt{\theta^2+1}} = 1$ and $\int_6^\infty \frac{d\theta}{\theta}$ diverges $\Rightarrow \int_6^\infty \frac{d\theta}{\sqrt{\theta^2+1}}$ diverges

140. $I = \int_0^\infty e^{-u}\cos u\,du = \lim_{b\to\infty} \left[-e^{-u}\cos u\right]_0^b - \int_0^\infty e^{-u}\sin u\,du = 1 - \lim_{b\to\infty} \left[e^{-u}\sin u\right]_0^b + \int_0^\infty \left(-e^{-u}\right)\cos u\,du$

$\Rightarrow I = 1 + 0 + I \Rightarrow 2I = 1 \Rightarrow I = \frac{1}{2}$ converges

141. $\int_1^\infty \frac{\ln z}{z}\,dz = \int_1^e \frac{\ln z}{z}\,dz + \int_e^\infty \frac{\ln z}{z}\,dz = \left[(\ln z)^2\right]_1^e + \lim_{b\to\infty} \left[(\ln z)^2\right]_e^b = \left(1^2 - 0\right) + \lim_{b\to\infty} \left[(\ln b)^2 - 1\right]$

$= \infty \Rightarrow$ diverges

142. $0 < \frac{e^{-t}}{\sqrt{t}} \le e^{-t}$ for $t \ge 1$ and $\int_1^\infty e^{-t}\,dt$ converges $\Rightarrow \int_1^\infty \frac{e^{-t}}{\sqrt{t}}\,dt$ converges

143. $0 < \frac{e^{-x}}{3+e^{-2x}} = \frac{1}{3e^x + e^{-x}} < \frac{1}{e^x + e^{-x}}$ and $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = 2\int_0^{\infty} \frac{dx}{e^x + e^{-x}} < \int_0^{\infty} \frac{2\,dx}{e^x}$ converges

$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{-x}}{3+e^{-2x}}\,dx$ converges

144. $\int_{-\infty}^{\infty} \frac{dx}{x^2(1+e^x)} = \int_{-\infty}^{-1} \frac{dx}{x^2(1+e^x)} + \int_{-1}^{0} \frac{dx}{x^2(1+e^x)} + \int_{0}^{1} \frac{dx}{x^2(1+e^x)} + \int_{1}^{\infty} \frac{dx}{x^2(1+e^x)}$;

$\lim_{x\to 0} \frac{\left(\frac{1}{x^2}\right)}{\left[\frac{1}{x^2(1+e^x)}\right]} = \lim_{x\to 0} \frac{x^2(1+e^x)}{x^2} = \lim_{x\to 0} (1+e^x) = 2$ and $\int_0^1 \frac{dx}{x^2}$ diverges $\Rightarrow \int_0^1 \frac{dx}{x^2(1+e^x)}$ diverges

$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2(1+e^x)}$ diverges

145. $\frac{1}{y^2-y}\,dy = e^x\,dx \Rightarrow \int \frac{1}{y(y-1)}\,dy = \int e^x\,dx = e^x + C$; $\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1} \Rightarrow 1 = A(y-1) + B(y)$
$= (A+B)y - A$

Equating coefficients of like terms gives $A+B=0$ and $-A=1$. Solving the system simultaneously yields $A=-1$, $B=1$.

$\int \frac{1}{y(y-1)}\,dy = \int -\frac{1}{y}\,dy + \int \frac{1}{y-1}\,dy = -\ln|y| + \ln|y-1| + C_2 \Rightarrow -\ln|y| + \ln|y-1| = e^x + C$

Substitute $x=0$, $y=2 \Rightarrow -\ln 2 + 0 = 1 + C$ or $C = -1 - \ln 2$.

The solution to the initial value problem is $-\ln|y| + \ln|y-1| = e^x - 1 - \ln 2$.

146. $\frac{1}{(y+1)^2}\,dy = \sin\theta\,d\theta$; $\int \frac{1}{(y+1)^2}\,dy = \int \sin\theta\,d\theta \Rightarrow -\frac{1}{y+1} = -\cos\theta + C$

Substitute $x = \frac{\pi}{2}$, $y = 0 \Rightarrow -1 = 0 + C$ or $C = -1$.

The solution to the initial value problem is $-\frac{1}{y+1} = -\cos\theta - 1 \Rightarrow y+1 = \frac{1}{\cos\theta+1} \Rightarrow y = \frac{1}{\cos\theta+1} - 1$

147. $dy = \frac{dx}{x^2-3x+2}$; $x^2 - 3x + 2 = (x-2)(x-1) \Rightarrow \frac{1}{x^2-3x+2} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x-2)$
$\Rightarrow 1 = (A+B)x - A - 2B$

Equating coefficients of like terms gives $A+B=0$, $-A-2B=1$. Solving the system simultaneously yields $A=1$, $B=-1$.

$\int dy = \int \frac{dx}{x^2-3x+2} = \int \frac{dx}{x-2} - \int \frac{dx}{x-1} \Rightarrow y = \ln|x-2| - \ln|x-1| + C$

Substitute $x=3$, $y=0 \Rightarrow 0 = 0 - \ln 2 + C$ or $C = \ln 2$.

The solution to the initial value problem is $y = \ln|x-2| - \ln|x-1| + \ln 2$.

148. $\frac{ds}{2x+2} = \frac{dt}{t^2+2t}$; $\int \frac{ds}{2x+2} = \frac{1}{2}\int \frac{ds}{s+1} = \frac{1}{2}\ln|s+1| + C_1$; $t^2+2t = t(t+2) \Rightarrow \frac{1}{t^2+2t} = \frac{A}{t} + \frac{B}{t+2}$

$\Rightarrow 1 = A(t+2) + Bt \Rightarrow 1 = (A+B)t + 2A.$

Equating coefficients of like terms gives $A + B = 0$ and $2A = 1$. Solving the system simultaneously yields $A = \frac{1}{2}$, $B = -\frac{1}{2}$.

$$\int \frac{dt}{t^2+2t} = \int \frac{1/2}{t}\,dt - \int \frac{1/2}{t+2}\,dt = \frac{1}{2}\ln|t| - \frac{1}{2}\ln|t+2| + C_2 \Rightarrow \frac{1}{2}\ln|s+1| = \frac{1}{2}\ln|t| - \frac{1}{2}\ln|t+2| + C_3$$

$\Rightarrow \ln|s+1| = \ln|t| - \ln|t+2| + C$

Substitute $t = 1$, $x = 1 \Rightarrow \ln 2 = 0 - \ln 3 + C$ or $C = \ln 2 + \ln 3 = \ln 6$.

The solution to the initial value problem is $\ln|s+1| = \ln|t| - \ln|t+2| + \ln 6 \Rightarrow \ln|s+1| = \ln\left|\frac{6t}{t+2}\right|$

$\Rightarrow |s+1| = \left|\frac{6t}{t+2}\right|$

CHAPTER 7 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. $u = \left(\sin^{-1} x\right)^2$, $du = \frac{2\sin^{-1} x\,dx}{\sqrt{1-x^2}}$; $dv = dx$, $v = x$;

$$\int \left(\sin^{-1} x\right)^2 dx = x\left(\sin^{-1} x\right)^2 - \int \frac{2x\sin^{-1} x\,dx}{\sqrt{1-x^2}};$$

$u = \sin^{-1} x$, $du = \frac{dx}{\sqrt{1-x^2}}$; $dv = -\frac{2x\,dx}{\sqrt{1-x^2}}$, $v = 2\sqrt{1-x^2}$;

$$\int \frac{2x\sin^{-1} x\,dx}{\sqrt{1-x^2}} = 2\left(\sin^{-1} x\right)\sqrt{1-x^2} - \int 2\,dx = 2\left(\sin^{-1} x\right)\sqrt{1-x^2} - 2x + C; \text{ therefore}$$

$$\int \left(\sin^{-1} x\right)^2 dx = x\left(\sin^{-1} x\right)^2 + 2\left(\sin^{-1} x\right)\sqrt{1-x^2} - 2x + C$$

2. $\frac{1}{x} = \frac{1}{x}$,

$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$,

$\frac{1}{x(x+1)(x+2)} = \frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)}$,

$\frac{1}{x(x+1)(x+2)(x+3)} = \frac{1}{6x} - \frac{1}{2(x+1)} + \frac{1}{2(x+2)} - \frac{1}{6(x+3)}$,

$\frac{1}{x(x+1)(x+2)(x+3)(x+4)} = \frac{1}{24x} - \frac{1}{6(x+1)} + \frac{1}{4(x+2)} - \frac{1}{6(x+3)} + \frac{1}{24(x+4)} \Rightarrow$ the following pattern:

$\frac{1}{x(x+1)(x+2)\cdots(x+m)} = \sum_{k=0}^{m} \frac{(-1)^k}{(k!)(m-k)!(x+k)}$; therefore $\int \frac{dx}{x(x+1)(x+2)\cdots(x+m)}$

$$= \sum_{k=0}^{m} \left[\frac{(-1)^k}{(k!)(m-k)!} \ln|x+k| \right] + C$$

3. $u = \sin^{-1} x,\ du = \frac{dx}{\sqrt{1-x^2}};\ dv = x\,dx,\ v = \frac{x^2}{2};$

$$\int x \sin^{-1} x\,dx = \frac{x^2}{2} \sin^{-1} x - \int \frac{x^2\,dx}{2\sqrt{1-x^2}};\ \begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\,d\theta \end{bmatrix} \to \int x \sin^{-1} x\,dx = \frac{x^2}{2}\sin^{-1} x - \int \frac{\sin^2\theta\cos\theta\,d\theta}{2\cos\theta}$$

$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \sin^2\theta\,d\theta = \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + C = \frac{x^2}{2}\sin^{-1} x + \frac{\sin\theta\cos\theta - \theta}{4} + C$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{x\sqrt{1-x^2} - \sin^{-1} x}{4} + C$$

4. $\int \sin^{-1}\sqrt{y}\,dy;\ \begin{bmatrix} z = \sqrt{y} \\ dz = \frac{dy}{2\sqrt{y}} \end{bmatrix} \to \int 2z \sin^{-1} z\,dz$; from Exercise 3, $\int z \sin^{-1} z\,dz$

$$= \frac{z^2 \sin^{-1} z}{2} + \frac{z\sqrt{1-z^2} - \sin^{-1} z}{4} + C \Rightarrow \int \sin^{-1}\sqrt{y}\,dy = y\sin^{-1}\sqrt{y} + \frac{\sqrt{y}\sqrt{1-y} - \sin^{-1}\sqrt{y}}{2} + C$$

$$= y\sin^{-1}\sqrt{y} + \frac{\sqrt{y-y^2}}{2} - \frac{\sin^{-1}\sqrt{y}}{2} + C$$

5. $\int \frac{d\theta}{1-\tan^2\theta} = \int \frac{\cos^2\theta}{\cos^2\theta - \sin^2\theta}\,d\theta = \int \frac{1+\cos 2\theta}{2\cos 2\theta}\,d\theta = \frac{1}{2}\int (\sec 2\theta + 1)\,d\theta = \frac{\ln|\sec 2\theta + \tan 2\theta| + 2\theta}{4} + C$

6. $u = \ln\left(\sqrt{x} + \sqrt{1+x}\right),\ du = \left(\frac{dx}{\sqrt{x}+\sqrt{1+x}}\right)\left(\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{1+x}}\right) = \frac{dx}{2\sqrt{x}\sqrt{1+x}};\ dv = dx,\ v = x;$

$$\int \ln\left(\sqrt{x}+\sqrt{1+x}\right)dx = x\ln\left(\sqrt{x}+\sqrt{1+x}\right) - \frac{1}{2}\int \frac{x\,dx}{\sqrt{x}\sqrt{1+x}} = \frac{1}{2}\int \frac{x\,dx}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}}};$$

$$\begin{bmatrix} x + \frac{1}{2} = \frac{1}{2}\sec\theta \\ dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta \end{bmatrix} \to \frac{1}{4}\int \frac{(\sec\theta - 1)\cdot\sec\theta\tan\theta\,d\theta}{\left(\frac{1}{2}\tan\theta\right)} = \frac{1}{2}\int \left(\sec^2\theta - \sec\theta\right)d\theta$$

$$= \frac{\tan\theta - \ln|\sec\theta + \tan\theta|}{2} + C = \frac{2\sqrt{x^2+x} - \ln\left|2x+1+2\sqrt{x^2+x}\right|}{2} + C$$

$$\Rightarrow \int \ln\left(\sqrt{x}+\sqrt{1+x}\right)dx = x\ln\left(\sqrt{x}+\sqrt{1+x}\right) - \frac{2\sqrt{x^2+x} - \ln\left|2x+1+2\sqrt{x^2+x}\right|}{4} + C$$

7. $$\int \frac{dt}{t-\sqrt{1-t^2}};\ \left[\begin{array}{c} t=\sin\theta \\ dt=\cos\theta\ d\theta \end{array}\right] \to \int \frac{\cos\theta\ d\theta}{\sin\theta-\cos\theta} = \int \frac{d\theta}{\tan\theta-1};\ \left[\begin{array}{c} u=\tan\theta \\ du=\sec^2\theta\ d\theta \\ d\theta=\frac{du}{u^2+1} \end{array}\right] \to \int \frac{du}{(u-1)(u^2+1)}$$

$$=\frac{1}{2}\int \frac{du}{u-1}-\frac{1}{2}\int \frac{du}{u^2+1}-\frac{1}{2}\int \frac{u\ du}{u^2+1}=\frac{1}{2}\ln\left|\frac{u-1}{\sqrt{u^2+1}}\right|-\frac{1}{2}\tan^{-1}u+C=\frac{1}{2}\ln\left|\frac{\tan\theta-1}{\sec\theta}\right|-\frac{1}{2}\theta+C$$

$$=\frac{1}{2}\ln\left(t-\sqrt{1-t^2}\right)-\frac{1}{2}\sin^{-1}t+C$$

8. $$\int \frac{\left(2e^{2x}-e^x\right)dx}{\sqrt{3e^{2x}-6e^x-1}};\ \left[\begin{array}{c} u=e^x \\ du=e^x\ dx \end{array}\right] \to \int \frac{(2u-1)\ du}{\sqrt{3u^2-6u-1}}=\frac{1}{\sqrt{3}}\int \frac{(2u-1)\ du}{\sqrt{(u-1)^2-\frac{4}{3}}};$$

$$\left[\begin{array}{c} u-1=\frac{2}{\sqrt{3}}\sec\theta \\ du=\frac{2}{\sqrt{3}}\sec\theta\tan\theta\ d\theta \end{array}\right] \to \frac{1}{\sqrt{3}}\int\left(\frac{4}{\sqrt{3}}\sec\theta+1\right)(\sec\theta)\ d\theta=\frac{4}{3}\int \sec^2\theta\ d\theta+\frac{1}{\sqrt{3}}\int \sec\theta\ d\theta$$

$$=\frac{4}{3}\tan\theta+\frac{1}{\sqrt{3}}\ln|\sec\theta+\tan\theta|+C_1=\frac{4}{3}\cdot\sqrt{\frac{3}{4}(u-1)^2-1}+\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}}{2}(u-1)+\sqrt{\frac{3}{4}(u-1)^2-1}\right|+C_1$$

$$=\frac{2}{3}\sqrt{3u^2-6u-1}+\frac{1}{\sqrt{3}}\ln\left|u-1+\sqrt{(u-1)^2-\frac{4}{3}}\right|+\left(C_1+\frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}}{2}\right)$$

$$=\frac{1}{\sqrt{3}}\left[2\sqrt{e^{2x}-2e^x-\frac{1}{3}}+\ln\left|e^x-1+\sqrt{e^{2x}-2e^x-\frac{1}{3}}\right|\right]+C$$

9. $$\int \frac{1}{x^4+4}\ dx=\int \frac{1}{\left(x^2+2\right)^2-4x^2}\ dx=\int \frac{1}{\left(x^2+2x+2\right)\left(x^2-2x+2\right)}\ dx$$

$$=\frac{1}{16}\int\left[\frac{2x+2}{x^2+2x+2}+\frac{2}{(x+1)^2+1}-\frac{2x-2}{x^2-2x+2}+\frac{2}{(x-1)^2+1}\right]dx$$

$$=\frac{1}{16}\ln\left|\frac{x^2+2x+2}{x^2-2x+2}\right|+\frac{1}{8}\left[\tan^{-1}(x+1)+\tan^{-1}(x-1)\right]+C$$

10. $$\int \frac{1}{x^6-1}\ dx=\frac{1}{6}\int\left(\frac{1}{x-1}-\frac{1}{x+1}+\frac{x-2}{x^2-x+1}-\frac{x+2}{x^2+x+1}\right)dx$$

$$=\frac{1}{6}\ln\left|\frac{x-1}{x+1}\right|+\frac{1}{12}\int\left[\frac{2x-1}{x^2-x+1}-\frac{3}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}-\frac{2x+1}{x^2+x+1}-\frac{3}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\right]dx$$

$$=\frac{1}{6}\ln\left|\frac{x-1}{x+1}\right|+\frac{1}{12}\left[\ln\left|\frac{x^2-x+1}{x^2+x+1}\right|-2\sqrt{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)-2\sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right]+C$$

11. $\lim_{b\to1^-} \int_0^b \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to1^-} [\sin^{-1} x]_0^b = \lim_{b\to1^-} (\sin^{-1} b - \sin^{-1} 0) = \lim_{b\to1^-} (\sin^{-1} b - 0) = \lim_{b\to1^-} \sin^{-1} b = \frac{\pi}{2}$

12. $\lim_{x\to\infty} \frac{1}{x} \int_0^x \tan^{-1} t\,dt = \lim_{x\to\infty} \frac{\int_0^x \tan^{-1} t\,dt}{x}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$= \lim_{x\to\infty} \frac{\tan^{-1} x}{1} = \frac{\pi}{2}$

13. $y = (\cos\sqrt{x})^{1/x} \Rightarrow \ln y = \frac{1}{x}\ln(\cos\sqrt{x})$ and $\lim_{x\to0^+} \frac{\ln(\cos\sqrt{x})}{x} = \lim_{x\to0^+} \frac{-\sin\sqrt{x}}{2\sqrt{x}\cos\sqrt{x}} = \frac{-1}{2}\lim_{x\to0^+} \frac{\tan\sqrt{x}}{\sqrt{x}}$

$= -\frac{1}{2}\lim_{x\to0^+} \frac{\frac{1}{2}x^{-1/2}\sec^2\sqrt{x}}{\frac{1}{2}x^{-1/2}} = -\frac{1}{2} \Rightarrow \lim_{x\to0^+} (\cos\sqrt{x})^{1/x} = e^{-1/2} = \frac{1}{\sqrt{e}}$

14. $y = (x+e^x)^{2/x} \Rightarrow \ln y = \frac{2\ln(x+e^x)}{x} \Rightarrow \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{2(1+e^x)}{x+e^x} = \lim_{x\to\infty} \frac{2e^x}{1+e^x} = \lim_{x\to\infty} \frac{2e^x}{e^x} = 2$

$\Rightarrow \lim_{x\to\infty} (x+e^x)^{2/x} = \lim_{x\to\infty} e^y = e^2$

15. $\lim_{x\to\infty} \int_{-x}^x \sin t\,dt = \lim_{x\to\infty} [-\cos t]_{-x}^x = \lim_{x\to\infty} [-\cos x + \cos(-x)] = \lim_{x\to\infty} (-\cos x + \cos x) = \lim_{x\to\infty} 0 = 0$

16. $\lim_{x\to0^+} \int_x^1 \frac{\cos t}{t^2}\,dt$; $\lim_{t\to0^+} \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{\cos t}{t^2}\right)} - \lim_{t\to0^+} \frac{1}{\cos t} = 1 \Rightarrow \lim_{x\to0^+} \int_x^1 \frac{\cos t}{t^2}\,dt$ diverges since $\int_0^1 \frac{dt}{t^2}$ diverges; thus

$\lim_{x\to0^+} x\int_x^1 \frac{\cos t}{t^2}\,dt$ is an indeterminate $0\cdot\infty$ form and we apply l'Hôpital's rule:

$$\lim_{x\to0^+} x\int_x^1 \frac{\cos t}{t^2}\,dt = \lim_{x\to0^+} \frac{-\int_1^x \frac{\cos t}{t^2}\,dt}{\frac{1}{x}} = \lim_{x\to0^+} \frac{-\left(\frac{\cos x}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to0^+} \cos x = 1$$

17. $\frac{dy}{dx} = \sqrt{\cos 2x} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \cos 2x = 2\cos^2 x$; $L = \int_0^{\pi/4} \sqrt{1 + (\sqrt{\cos 2t})^2}\,dt = \sqrt{2}\int_0^{\pi/4} \sqrt{\cos^2 t}\,dt$

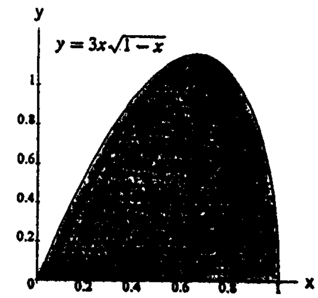
$= \sqrt{2}\,[\sin t]_0^{\pi/4} = 1$

18. $\frac{dy}{dx} = \frac{-2x}{1-x^2} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{(1-x^2)^2 + 4x^2}{(1-x^2)^2} = \frac{1+2x^2+x^4}{(1-x^2)^2} = \left(\frac{1+x^2}{1-x^2}\right)^2$; $L = \int_0^{1/2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= \int_0^{1/2} \left(\frac{1+x^2}{1-x^2}\right) dx = \int_0^{1/2} \left(-1 + \frac{2}{1-x^2}\right) dx = \int_0^{1/2} \left(-1 + \frac{1}{1+x} + \frac{1}{1-x}\right) dx = \left[-x + \ln\left|\frac{1+x}{1-x}\right|\right]_0^{1/2}$$

$$= \left(-\tfrac{1}{2} + \ln 3\right) - (0 + \ln 1) = \ln 3 - \tfrac{1}{2}$$

19. $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx = \int_0^1 2\pi xy \, dx$

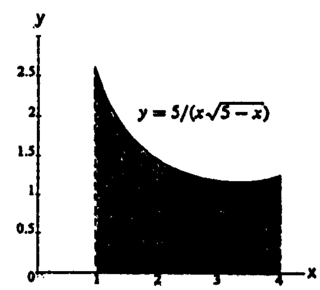


$$= 6\pi \int_0^1 x^2\sqrt{1-x}\, dx; \begin{bmatrix} u = 1-x \\ du = -dx \\ x^2 = (1-u)^2 \end{bmatrix}$$

$$\rightarrow -6\pi \int_1^0 (1-u)^2\sqrt{u}\, du = -6\pi \int_1^0 \left(u^{1/2} - 2u^{3/2} + u^{5/2}\right) du$$

$$= -6\pi\left[\tfrac{2}{3}u^{3/2} - \tfrac{4}{5}u^{5/2} + \tfrac{2}{7}u^{7/2}\right]_1^0 = 6\pi\left(\tfrac{2}{3} - \tfrac{4}{5} + \tfrac{2}{7}\right) = 6\pi\left(\frac{70-84+30}{105}\right) = 6\pi\left(\frac{16}{105}\right) = \frac{32\pi}{35}$$

20. $V = \int_a^b \pi y^2 \, dx = \pi \int_1^4 \frac{25\, dx}{x^2(5-x)} = \pi \int_1^4 \left(\frac{dx}{x} + \frac{5\, dx}{x^2} + \frac{dx}{5-x}\right)$

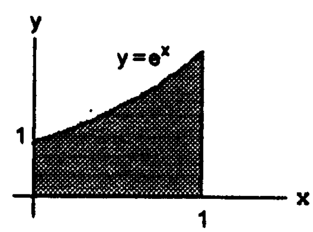


$$= \pi\left[\ln\left|\frac{x}{5-x}\right| - \frac{5}{x}\right]_1^4 = \pi\left(\ln 4 - \tfrac{5}{4}\right) - \pi\left(\ln \tfrac{1}{4} - 5\right)$$

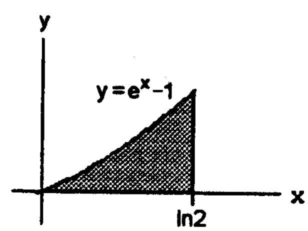
$$= \frac{15\pi}{4} + 2\pi \ln 4$$

21. $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx = \int_0^1 2\pi x e^x \, dx$

$$= 2\pi\left[xe^x - e^x\right]_0^1 = 2\pi$$



22. $V = \int_0^{\ln 2} 2\pi(\ln 2 - x)(e^x - 1)\, dx$

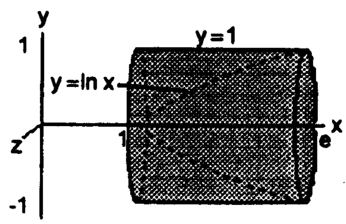


$$= 2\pi \int_0^{\ln 2} \left[(\ln 2)\, e^x - \ln 2 - xe^x + x\right] dx$$

$$= 2\pi\left[(\ln 2)\, e^x - (\ln 2)x - xe^x + e^x + \frac{x^2}{2}\right]_0^{\ln 2}$$

$$= 2\pi\left[2 \ln 2 - (\ln 2)^2 - 2 \ln 2 + 2 + \frac{(\ln 2)^2}{2}\right] - 2\pi(\ln 2 + 1) = 2\pi\left[-\frac{(\ln 2)^2}{2} - \ln 2 + 1\right]$$

23. (a) $V = \int_1^e \pi\left[1 - (\ln x)^2\right] dx$

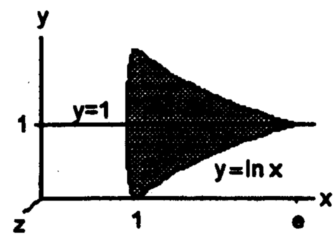


$= \pi\left[x - x(\ln x)^2\right]_1^e - 2\pi \int_1^e \ln x\, dx$ (FORMULA 110)

$= \pi\left[x - x(\ln x)^2 + 2(x \ln x - x)\right]_1^e$

$= \pi\left[-x - x(\ln x)^2 + 2x \ln x\right]_1^e = \pi\left[-e - e + 2e - (-1)\right] = \pi$

(b) $V = \int_1^e \pi(1 - \ln x)^2\, dx = \pi \int_1^e \left[1 - 2\ln x + (\ln x)^2\right] dx$



$= \pi\left[x - 2(x \ln x - x) + x(\ln x)^2\right]_1^e - 2\pi \int_1^e \ln x\, dx$

$= \pi\left[x - 2(x\ln x - x) + x(\ln x)^2 - 2(x \ln x - x)\right]_1^e$

$= \pi\left[5x - 4x \ln x + x(\ln x)^2\right]_1^e = \pi\left[(5e - 4e + e) - (5)\right]$

$= \pi(2e - 5)$

24. (a) $V = \pi \int_0^1 \left[(e^y)^2 - 1\right] dy = \pi \int_0^1 \left(e^{2y} - 1\right) dy = \pi\left[\frac{e^{2y}}{2} - y\right]_0^1 = \pi\left[\frac{e^2}{2} - 1 - \left(\frac{1}{2}\right)\right] = \frac{\pi\left(e^2 - 3\right)}{2}$

(b) $V = \pi \int_0^1 \left(e^y - 1\right)^2 dy = \pi \int_0^1 \left(e^{2y} - 2e^y + 1\right) dy = \pi\left[\frac{e^{2y}}{2} - 2e^y + y\right]_0^1 = \pi\left[\left(\frac{e^2}{2} - 2e + 1\right) - \left(\frac{1}{2} - 2\right)\right]$

$= \pi\left(\frac{e^2}{2} - 2e + \frac{5}{2}\right) = \frac{\pi\left(e^2 - 4e + 5\right)}{2}$

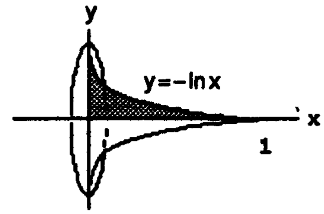
25. (a) $\lim_{x\to 0^+} x \ln x = 0 \Rightarrow \lim_{x \to 0^+} f(x) = 0 = f(0) \Rightarrow$ f is continuous

(b) $V = \int_0^2 \pi x^2(\ln x)^2\, dx;$ $\begin{bmatrix} u = (\ln x)^2 \\ du = (2 \ln x)\frac{dx}{x} \\ dv = x^2 \\ v = \frac{x^3}{3} \end{bmatrix} \to \pi\left(\lim_{b\to 0^+}\left[\frac{x^3}{3}(\ln x)^2\right]_b^2 - \int_0^2 \left(\frac{x^3}{3}\right)(2 \ln x)\frac{dx}{x}\right)$

$= \pi\left[\left(\frac{8}{3}\right)(\ln 2)^2 - \left(\frac{2}{3}\right)\lim_{b\to 0^+}\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_b^2\right] = \pi\left[\frac{8(\ln 2)^2}{3} - \frac{16(\ln 2)}{9} + \frac{16}{27}\right]$

26. $V = \int_0^1 \pi(-\ln x)^2\,dx = \pi\left(\lim_{b\to 0}\left[x(\ln x)^2\right]_b^1 - 2\int_0^1 \ln x\,dx\right)$

$= -2\pi \lim_{b\to 0}\left[x\ln x - x\right]_b^1 = 2\pi$



27. $u = \frac{1}{1+y}$, $du = -\frac{dy}{(1+y)^2}$; $dv = ny^{n-1}\,dy$, $v = y^n$;

$\lim_{n\to\infty}\int_0^1 \frac{ny^{n-1}}{1+y}\,dy = \lim_{n\to\infty}\left(\left[\frac{y^n}{1+y}\right]_0^1 + \int_0^1 \frac{y^n}{1+y^2}\,dy\right) = \frac{1}{2} + \lim_{n\to\infty}\int_0^1 \frac{y^n}{1+y^2}\,dy$. Now, $0 \le \frac{y^n}{1+y^2} \le y^n$

$\Rightarrow 0 \le \lim_{n\to\infty}\int_0^1 \frac{y^n}{1+y^2}\,dy \le \lim_{n\to\infty}\int_0^1 y^n\,dy = \lim_{n\to\infty}\left[\frac{y^{n+1}}{n+1}\right]_0^1 = \lim_{n\to\infty}\frac{1}{n+1} = 0 \Rightarrow \lim_{n\to\infty}\int_0^1 \frac{ny^{n-1}}{1+y}\,dy$

$= \frac{1}{2} + 0 = \frac{1}{2}$

28. $u = x^2 - a^2 \Rightarrow du = 2x\,dx$;

$\int x\left(\sqrt{x^2-a^2}\right)^n dx = \frac{1}{2}\int\left(\sqrt{u}\right)^n du = \frac{1}{2}\int u^{n/2}\,du = \frac{1}{2}\left(\frac{u^{n/2+1}}{\frac{n}{2}+1}\right) + C,\ n \ne -2$

$= \frac{u^{(n+2)/2}}{n+2} + C = \frac{\left(\sqrt{u}\right)^{n+2}}{n+2} + C = \frac{\left(\sqrt{x^2-a^2}\right)^{n+2}}{n+2} + C$

29. $\frac{\pi}{6} = \sin^{-1}\frac{1}{2} = \left[\sin^{-1}\frac{x}{2}\right]_0^1 = \int_0^1 \frac{dx}{\sqrt{4-x^2}} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \int_0^1 \frac{dx}{\sqrt{4-2x^2}} = \frac{1}{\sqrt{2}}\int_0^{\sqrt{2}} \frac{du}{\sqrt{4-u^2}}$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{u}{2}\right]_0^{\sqrt{2}} = \frac{1}{\sqrt{2}}\sin^{-1}\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right) = \frac{\pi\sqrt{2}}{8}$

30. $\int_1^\infty \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right)dx = \lim_{b\to\infty}\int_1^b \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right)dx = \lim_{b\to\infty}\left[\frac{a}{2}\ln(x^2+1) - \frac{1}{2}\ln x\right]_1^b = \lim_{b\to\infty}\left[\frac{1}{2}\ln\frac{(x^2+1)^a}{x}\right]_1^b$

$= \lim_{b\to\infty}\frac{1}{2}\left[\ln\frac{(b^2+1)^a}{b} - \ln 2^a\right]$; $\lim_{b\to\infty}\frac{(b^2+1)^a}{b} > \lim_{b\to\infty}\frac{b^{2a}}{b} = \lim_{b\to\infty} b^{2\left(a-\frac{1}{2}\right)} = \infty$ if $a > \frac{1}{2} \Rightarrow$ the improper

integral diverges if $a > \frac{1}{2}$; for $a = \frac{1}{2}$: $\lim_{b\to\infty}\frac{\sqrt{b^2+1}}{b} = \lim_{b\to\infty}\sqrt{1+\frac{1}{b^2}} = 1 \Rightarrow \lim_{b\to\infty}\frac{1}{2}\left[\ln\frac{(b^2+1)^{1/2}}{b} - \ln 2^{1/2}\right]$

$= \frac{1}{2}\left(\ln 1 - \frac{1}{2}\ln 2\right) = -\frac{\ln 2}{4}$; if $a < \frac{1}{2}$: $0 \le \lim_{b\to\infty}\frac{(b^2+1)^a}{b} < \lim_{b\to\infty}\frac{(b+1)^{2a}}{b+1} = \lim_{b\to\infty}(b+1)^{2a-1} = 0$

$\Rightarrow \lim_{b\to\infty}\ln\frac{(b^2+1)^a}{b} = -\infty \Rightarrow$ the improper integral diverges if $a < \frac{1}{2}$; in summary, the improper integral

$\int_1^\infty \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right)dx$ converges only when $a = \frac{1}{2}$ and has the value $-\frac{\ln 2}{4}$

31. Let $u = f(x) \Rightarrow du = f'(x)\,dx$ and $dv = dx \Rightarrow v = x$;

$$\int_{\pi/2}^{3\pi/2} f(x)\,dx = \left[x\,f(x)\right]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} xf'(x)\,dx = \left[\frac{3\pi}{2} f\left(\frac{3\pi}{2}\right) - \frac{\pi}{2} f\left(\frac{\pi}{2}\right)\right] - \int_{\pi/2}^{3\pi/2} \cos x\,dx$$

$$= \frac{3\pi}{2} b - \frac{\pi}{2} a - [\sin x]_{\pi/2}^{3\pi/2} = \frac{\pi}{2}(3b - a) - [-1 - 1] = \frac{\pi}{2}(3b - a) + 2$$

32. $\int_0^a \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^a = \tan^{-1} a$; $\int_a^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \left[\tan^{-1} x\right]_a^b = \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} a\right) = \frac{\pi}{2} - \tan^{-1} a$;

therefore, $\tan^{-1} a = \frac{\pi}{2} - \tan^{-1} a \Rightarrow \tan^{-1} a = \frac{\pi}{4} \Rightarrow a = 1$ for $a > 0$.

33. $L = 4\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dy$; $x^{2/3} + y^{2/3} = 1 \Rightarrow y = \left(1 - x^{2/3}\right)^{3/2} \Rightarrow \frac{dy}{dx} = -\frac{3}{2}\left(1 - x^{2/3}\right)^{1/2}\left(x^{-1/3}\right)\left(\frac{2}{3}\right)$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1 - x^{2/3}}{x^{2/3}} \Rightarrow L = 4\int_0^1 \sqrt{1 + \left(\frac{1 - x^{2/3}}{x^{2/3}}\right)}\,dx = 4\int_0^1 \frac{dx}{x^{1/3}} = 6\left[x^{2/3}\right]_0^1 = 6$$

34. $\left(\frac{dy}{dx}\right)^2 = \frac{1}{4x} \Rightarrow \frac{dy}{dx} = \frac{\pm 1}{2\sqrt{x}} \Rightarrow y = \pm\sqrt{x},\ 0 \le x \le 4$

35. $P(x) = ax^2 + bx + c$, $P(0) = c = 1$ and $P'(0) = 0 \Rightarrow b = 0 \Rightarrow P(x) = ax^2 + 1$. Next,
$\frac{ax^2 + 1}{x^3(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2}$; for the integral to be a <u>rational</u> function, we must have $A = 0$ and $D = 0$. Thus, $ax^2 + 1 = Bx(x-1)^2 + C(x-1)^2 + Ex^3 = (B+E)x^3 + (C - 2B)x^2 + (B - 2C)x + C$
$\Rightarrow C = 1$; B-2C $= 0 \Rightarrow B = 2$; $C - 2B = a \Rightarrow a = -3$; therefore, $P(x) = -3x^2 + 1$

36. The integral $\int_{-1}^1 \sqrt{1 - x^2}\,dx$ is the area enclosed by the x-axis and the semicircle $y = \sqrt{1 - x^2}$. This area is half the circle's area, or $\frac{\pi}{2}$ and multiplying by 2 gives π. The length of the circular arc $y = \sqrt{1 - x^2}$ from $x = -1$ to $x = 1$ is $L = \int_{-1}^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_{-1}^1 \sqrt{1 + \left(\frac{-x}{\sqrt{1 - x^2}}\right)^2}\,dx = \int_{-1}^1 \frac{dx}{\sqrt{1 - x^2}} = \frac{1}{2}(2\pi) = \pi$ since L is half the circle's circumference. In conclusion, $2\int_{-1}^1 \sqrt{1 - x^2}\,dx = \int_{-1}^1 \frac{dx}{\sqrt{1 - x^2}}$.

37. $A = \int_1^{\infty} \frac{dx}{x^p}$ converges if $p > 1$ and diverges if $p \le 1$ (Exercise 67 in Section 7.6). Thus, $p \le 1$ for infinite area.

The volume of the solid of revolution about the x-axis is $V = \int_1^{\infty} \pi\left(\frac{1}{x^p}\right)^2 dx = \pi \int_1^{\infty} \frac{dx}{x^{2p}}$ which converges if $2p > 1$ and diverges if $2p \le 1$. Thus we want $p > \frac{1}{2}$ for finite volume. In conclusion, the curve $y = x^{-p}$ gives infinite area and finite volume for values of p satisfying $\frac{1}{2} < p \le 1$.

38. The area is given by the integral $A = \int_0^1 \frac{dx}{x^p}$;

$p = 1$: $A = \lim_{b \to 0^+} [\ln x]_b^1 = -\lim_{b \to 0^+} \ln b = \infty$, diverges;

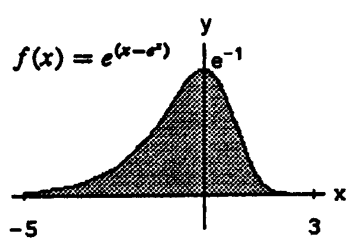
$p > 1$: $A = \lim_{b \to 0^+} \left[x^{1-p}\right]_b^1 = 1 - \lim_{b \to 0^+} b^{1-p} = -\infty$, diverges;

$p < 1$: $A = \lim_{b \to 0^+} \left[x^{1-p}\right]_b^1 = 1 - \lim_{b \to 0^+} b^{1-p} = 1 - 0$, converges; thus, $p \ge 1$ for infinite area.

The volume of the solid of revolution about the x-axis is $V_x = \pi \int_0^1 \frac{dx}{x^{2p}}$ which converges if $2p < 1$ or $p < \frac{1}{2}$, and diverges if $p \ge \frac{1}{2}$. Thus, V_x is infinite whenever the area is infinite ($p \ge 1$).

The volume of the solid of revolution about the y-axis is $V_y = \pi \int_1^{\infty} [R(y)]^2\, dy = \pi \int_1^{\infty} \frac{dy}{y^{2/p}}$ which converges if $\frac{2}{p} > 1 \Leftrightarrow p < 2$ (see Exercise 39). In conclusion, the curve $y = x^{-p}$ gives infinite area and finite volume for values of p satisfying $1 \le p < 2$, as described above.

39. (a)



(b) $\int_{-\infty}^{\infty} e^{(x-e^x)}\, dx = \int_{-\infty}^{\infty} e^{(-e^x)} e^x\, dx$

$$= \lim_{a\to-\infty} \int_a^0 e^{(-e^x)} e^x \, dx + \lim_{b\to+\infty} \int_0^b e^{(-e^x)} e^x \, dx;$$

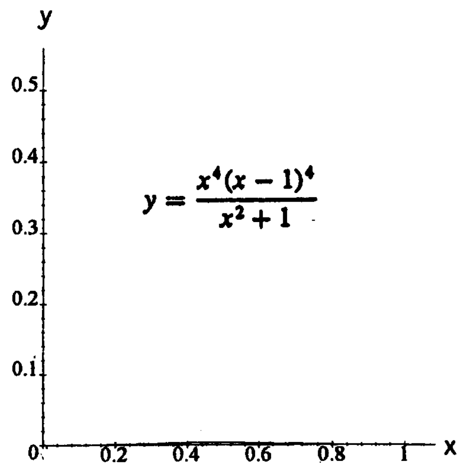
$$\begin{bmatrix} u = e^x \\ du = e^x \, dx \end{bmatrix} \to \lim_{a\to-\infty} \int_{e^a}^1 e^{-u} \, du + \lim_{b\to+\infty} \int_1^{e^b} e^{-u} \, du$$

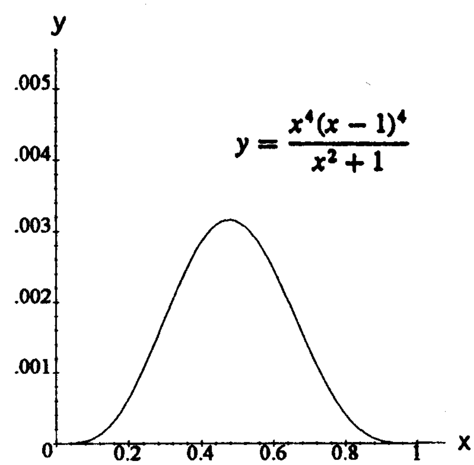
$$= \lim_{a\to-\infty} \left[-e^{-u}\right]_{e^a}^1 + \lim_{b\to-\infty} \left[-e^{-u}\right]_1^{e^b} = \lim_{a\to-\infty} \left[-\tfrac{1}{e} + e^{-(e^a)}\right] + \lim_{b\to+\infty} \left[-e^{-(e^b)} + \tfrac{1}{e}\right] = \left(-\tfrac{1}{e} + e^0\right) + \left(0 + \tfrac{1}{e}\right) = 1$$

40. (a) $\int_0^1 \frac{x^4(x-1)^4}{x^2+1} \, dx = \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}\right) dx = \frac{22}{7} - \pi$

(b) $\frac{\frac{22}{7} - \pi}{\pi} \cdot 100\% \cong 0.04\%$

(c) The area is less than 0.003





41. e^{2x} (+) $\cos 3x$

$2e^{2x}$ (−) $\frac{1}{3}\sin 3x$

$4e^{2x}$ $\underline{\quad(+)\quad}$ $-\frac{1}{9}\cos 3x$

$I = \frac{e^{2x}}{3}\sin 3x + \frac{2e^{2x}}{9}\cos 3x - \frac{4}{9}I \Rightarrow \frac{13}{9}I = \frac{e^{2x}}{9}(3\sin 3x + 2\cos 3x) \Rightarrow I = \frac{e^{2x}}{13}(3\sin 3x + 2\cos 3x) + C$

42. e^{3x} (+) $\sin 4x$

$3e^{3x}$ (−) $-\frac{1}{4}\cos 4x$

$9e^{3x}$ $\underline{\quad(+)\quad}$ $-\frac{1}{16}\sin 4x$

$I = -\frac{e^{3x}}{4}\cos 4x + \frac{3e^{3x}}{16}\sin 4x - \frac{9}{16}I \Rightarrow \frac{25}{16}I = \frac{e^{3x}}{16}(3\sin 4x - 4\cos 4x) \Rightarrow I = \frac{e^{3x}}{25}(3\sin 4x - 4\cos 4x) + C$

43. sin 3x $\quad (+) \quad$ sin x

$3 \cos 3x \quad (-) \quad -\cos x$

$-9 \sin 3x \xrightarrow{(+)} -\sin x$

$I = -\sin 3x \cos x + 3 \cos 3x \sin x + 9I \Rightarrow -8I = -\sin 3x \cos x + 3 \cos 3x \sin x$

$\Rightarrow I = \frac{\sin 3x \cos x - 3 \cos 3x \sin x}{8} + C$

44. cos 5x $\quad (+) \quad$ sin 4x

$-5 \sin 5x \quad (-) \quad -\frac{1}{4} \cos 4x$

$-25 \cos 5x \xrightarrow{(+)} -\frac{1}{16} \sin 4x$

$I = -\frac{1}{4} \cos 5x \cos 4x - \frac{5}{16} \sin 5x \sin 4x + \frac{25}{16} I \Rightarrow -\frac{9}{16} I = -\frac{1}{4} \cos 5x \cos 4x - \frac{5}{16} \sin 5x \sin 4x$

$\Rightarrow I = \frac{1}{9}(4 \cos 5x \cos 4x) \Rightarrow I = \frac{1}{9}(4 \cos 5x \cos 4x + 5 \sin 5x \sin 4x) + C$

45. $e^{ax} \quad (+) \quad \sin bx$

$ae^{ax} \quad (-) \quad -\frac{1}{b} \cos bx$

$a^2 e^{ax} \xrightarrow{(+)} -\frac{1}{b^2} \sin bx$

$I = -\frac{e^{ax}}{b} \cos bx + \frac{ae^{ax}}{b^2} \sin bx - \frac{a^2}{b^2} I \Rightarrow \left(\frac{a^2 + b^2}{b^2}\right) I = \frac{e^{ax}}{b^2}(a \sin bx - b \cos bx)$

$\Rightarrow I = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + C$

46. $e^{ax} \quad (+) \quad \cos bx$

$ae^{ax} \quad (-) \quad \frac{1}{b} \sin bx$

$a^2 e^{ax} \xrightarrow{(+)} -\frac{1}{b^2} \cos bx$

$I = -\frac{e^{ax}}{b} \sin bx + \frac{ae^{ax}}{b^2} \cos bx - \frac{a^2}{b^2} I \Rightarrow \left(\frac{a^2 + b^2}{b^2}\right) I = \frac{e^{ax}}{b^2}(a \cos bx + b \sin bx)$

$\Rightarrow I = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + C$

47. $\ln(ax) \quad (+) \quad 1$

$\frac{1}{x} \xrightarrow{(-)} x$

$I = x \ln(ax) - \int \left(\frac{1}{x}\right) x \, dx = x \ln(ax) - x + C$

48. $\ln(ax)$ (+) x^2

$\frac{1}{x}$ (−) $\frac{1}{3}x^3$

$I = \frac{1}{3}x^3 \ln(ax) - \int \left(\frac{1}{x}\right)\left(\frac{x^3}{3}\right) dx = \frac{1}{3}x^3 \ln(ax) - \frac{1}{9}x^3 + C$

49. (a) $\Gamma(1) = \int_0^\infty e^{-t}\, dt = \lim_{b\to\infty} \int_0^b e^{-t}\, dt = \lim_{b\to\infty} \left[-e^{-t}\right]_0^b = \lim_{b\to\infty} \left[-\frac{1}{e^b} - (-1)\right] = 0 + 1 = 1$

(b) $u = t^x$, $du = xt^{x-1}\, dt$; $dv = e^{-t}\, dt$, $v = -e^{-t}$; x = fixed positive real

$\Rightarrow \Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt = \lim_{b\to\infty} \left[-t^x e^{-t}\right]_0^b + x \int_0^\infty t^{x-1}e^{-t}\, dt = \lim_{b\to\infty} \left(-\frac{b^x}{e^b} + 0^x e^0\right) + x\Gamma(x) = x\Gamma(x)$

(c) $\Gamma(n+1) = n\Gamma(n) = n!$:

$n = 0$: $\Gamma(0+1) = \Gamma(1) = 0!$;

$n = k$: Assume $\Gamma(k+1) = k!$ for some $k > 0$;

$n = k+1$: $\Gamma(k+1+1) = (k+1)\,\Gamma(k+1)$ from part (b)

$= (k+1)k!$ induction hypothesis

$= (k+1)!$ definition of factorial

Thus, $\Gamma(n+1) = n\Gamma(n) = n!$ for every positive integer n.

50. (a) $\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}$ and $n\Gamma(n) = n! \Rightarrow n! \approx n\left(\frac{n}{e}\right)^n \sqrt{\frac{2\pi}{n}} = \left(\frac{n}{e}\right)^n \sqrt{2n\pi}$

(b)

n	$\left(\frac{n}{e}\right)^n \sqrt{2n\pi}$	calculator
10	3598695.619	3628800
20	2.4227868×10^{18}	2.432902×10^{18}
30	2.6451710×10^{32}	2.652528×10^{32}
40	8.1421726×10^{47}	8.1591528×10^{47}
50	3.0363446×10^{64}	3.0414093×10^{64}
60	8.3094383×10^{81}	8.3209871×10^{81}

(c)

n	$\left(\frac{n}{e}\right)^n \sqrt{2n\pi}$	$\left(\frac{n}{e}\right)^n \sqrt{2n\pi}\, e^{1/12n}$	calculator
10	3598695.619	3628810.051	3628800

NOTES: