1

Vectors and Tensors in a Finite-Dimensional Space

1.1 Notion of the Vector Space

We start with the definition of the vector space over the field of real numbers $\mathbb{R}$.

Definition 1.1. *A vector space is a set* $\mathbb{V}$ *of elements called vectors satisfying the following axioms.*

A. To every pair, $\boldsymbol{x}$ and $\boldsymbol{y}$ of vectors in $\mathbb{V}$ there corresponds a vector $\boldsymbol{x}+\boldsymbol{y}$, called the sum of $\boldsymbol{x}$ and $\boldsymbol{y}$, such that

(A.1) $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x}$ (addition is commutative),

(A.2) $(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z}=\boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})$ (addition is associative),

(A.3) there exists in $\mathbb{V}$ a unique vector zero $\boldsymbol{0}$, such that $\boldsymbol{0}+\boldsymbol{x}=\boldsymbol{x}$, $\forall\boldsymbol{x}\in\mathbb{V}$,

(A.4) to every vector $\boldsymbol{x}$ in $\mathbb{V}$ there corresponds a unique vector $-\boldsymbol{x}$ such that $\boldsymbol{x}+(-\boldsymbol{x})=\boldsymbol{0}$.

B. To every pair α and $\boldsymbol{x}$, where α is a scalar real number and $\boldsymbol{x}$ is a vector in $\mathbb{V}$, there corresponds a vector $\alpha\boldsymbol{x}$, called the product of α and $\boldsymbol{x}$, such that

(B.1) $\alpha(\beta\boldsymbol{x})=(\alpha\beta)\boldsymbol{x}$ (multiplication by scalars is associative),

(B.2) $1\boldsymbol{x}=\boldsymbol{x}$,

(B.3) $\alpha(\boldsymbol{x}+\boldsymbol{y})=\alpha\boldsymbol{x}+\alpha\boldsymbol{y}$ (multiplication by scalars is distributive with respect to vector addition),

(B.4) $(\alpha+\beta)\boldsymbol{x}=\alpha\boldsymbol{x}+\beta\boldsymbol{x}$ (multiplication by scalars is distributive with respect to scalar addition),

$\forall\alpha,\beta\in\mathbb{R}$, $\forall\boldsymbol{x},\boldsymbol{y}\in\mathbb{V}$.

Examples of vector spaces.

1) The set of all real numbers $\mathbb{R}$.

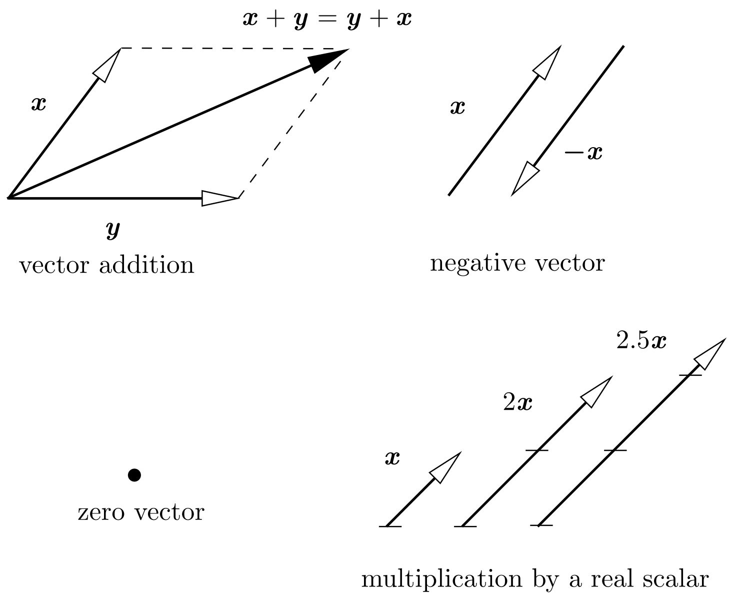

Fig. 1.1. Geometric illustration of vector axioms in two dimensions

2) The set of all directional arrows in two or three dimensions. Applying the usual definitions for summation, multiplication by a scalar, the negative and zero vector (Fig. 1.1) one can easily see that the above axioms hold for directional arrows.
3) The set of all n-tuples of real numbers $\mathbb{R}$:

$$\boldsymbol{a} = \left\{ \begin{array}{c} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{array} \right\}.$$

Indeed, the axioms (A) and (B) apply to the n-tuples if one defines addition, multiplication by a scalar and finally the zero tuple by

$$\alpha \boldsymbol{a} = \left\{ \begin{array}{c} \alpha a_1 \\ \alpha a_2 \\ \cdot \\ \cdot \\ \alpha a_n \end{array} \right\}, \quad \boldsymbol{a} + \boldsymbol{b} = \left\{ \begin{array}{c} a_1 + b_1 \\ a_2 + b_2 \\ \cdot \\ \cdot \\ a_n + b_n \end{array} \right\}, \quad \boldsymbol{0} = \left\{ \begin{array}{c} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{array} \right\}.$$

4) The set of all real-valued functions defined on a real line.

1.2 Basis and Dimension of the Vector Space

Definition 1.2. *A set of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ is called linearly dependent if there exists a set of corresponding scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, not all zero, such that*

$$\sum_{i=1}^{n} \alpha_i \boldsymbol{x}_i = \boldsymbol{0}. \tag{1.1}$$

Otherwise, the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are called linearly independent. In this case, none of the vectors $\boldsymbol{x}_i$ is the zero vector (Exercise 1.2).

Definition 1.3. *The vector*

$$\boldsymbol{x} = \sum_{i=1}^{n} \alpha_i \boldsymbol{x}_i \tag{1.2}$$

is called linear combination of the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$, where $\alpha_i \in \mathbb{R}$ ($i = 1, 2, \ldots, n$).

Theorem 1.1. *The set of n non-zero vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ is linearly dependent if and only if some vector $\boldsymbol{x}_k$ $(2 \le k \le n)$ is a linear combination of the preceding ones $\boldsymbol{x}_i$ $(i = 1, \ldots, k-1)$.*

Proof. If the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are linearly dependent, then

$$\sum_{i=1}^{n} \alpha_i \boldsymbol{x}_i = \boldsymbol{0},$$

where not all α_i are zero. Let α_k $(2 \le k \le n)$ be the last non-zero number, so that $\alpha_i = 0$ $(i = k+1, \ldots, n)$. Then,

$$\sum_{i=1}^{k} \alpha_i \boldsymbol{x}_i = \boldsymbol{0} \;\Rightarrow\; \boldsymbol{x}_k = \sum_{i=1}^{k-1} \frac{-\alpha_i}{\alpha_k} \boldsymbol{x}_i.$$

Thereby, the case $k = 1$ is avoided because $\alpha_1 \boldsymbol{x}_1 = \boldsymbol{0}$ implies that $\boldsymbol{x}_1 = \boldsymbol{0}$ (Exercise 1.1). Thus, the sufficiency is proved. The necessity is evident.

Definition 1.4. *A basis of a vector space $\mathbb{V}$ is a set $\mathcal{G}$ of linearly independent vectors such that every vector in $\mathbb{V}$ is a linear combination of elements of $\mathcal{G}$. A vector space $\mathbb{V}$ is finite-dimensional if it has a finite basis.*

Within this book, we restrict our attention to finite-dimensional vector spaces. Although one can find for a finite-dimensional vector space an infinite number of bases, they all have the same number of vectors.

Theorem 1.2. *All the bases of a finite-dimensional vector space* $\mathbb{V}$ *contain the same number of vectors.*

Proof. Let $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ and $\mathcal{F} = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_m\}$ be two arbitrary bases of $\mathbb{V}$ with different numbers of elements, say $m > n$. Then, every vector in $\mathbb{V}$ is a linear combination of the following vectors:

$$\boldsymbol{f}_1, \boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n. \tag{1.3}$$

These vectors are non-zero and linearly dependent. Thus, according to Theorem 1.1 we can find such a vector $\boldsymbol{g}_k$, which is a linear combination of the preceding ones. Excluding this vector we obtain the set $\mathcal{G}'$ by

$$\boldsymbol{f}_1, \boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_{k-1}, \boldsymbol{g}_{k+1}, \ldots, \boldsymbol{g}_n$$

again with the property that every vector in $\mathbb{V}$ is a linear combination of the elements of $\mathcal{G}'$. Now, we consider the following vectors

$$\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_{k-1}, \boldsymbol{g}_{k+1}, \ldots, \boldsymbol{g}_n$$

and repeat the excluding procedure just as before. We see that none of the vectors $\boldsymbol{f}_i$ can be eliminated in this way because they are linearly independent. As soon as all $\boldsymbol{g}_i$ $(i = 1, 2, \ldots, n)$ are exhausted we conclude that the vectors

$$\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_{n+1}$$

are linearly dependent. This contradicts, however, the previous assumption that they belong to the basis $\mathcal{F}$.

Definition 1.5. *The dimension of a finite-dimensional vector space* $\mathbb{V}$ *is the number of elements in a basis of* $\mathbb{V}$.

Theorem 1.3. *Every set* $\mathcal{F} = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n\}$ *of linearly independent vectors in an* n*-dimensional vectors space* $\mathbb{V}$ *forms a basis of* $\mathbb{V}$. *Every set of more than* n *vectors is linearly dependent.*

Proof. The proof of this theorem is similar to the preceding one. Let $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ be a basis of $\mathbb{V}$. Then, the vectors (1.3) are linearly dependent and non-zero. Excluding a vector $\boldsymbol{g}_k$ we obtain a set of vectors, say $\mathcal{G}'$, with the property that every vector in $\mathbb{V}$ is a linear combination of the elements of $\mathcal{G}'$. Repeating this procedure we finally end up with the set $\mathcal{F}$ with the same property. Since the vectors $\boldsymbol{f}_i$ $(i = 1, 2, \ldots, n)$ are linearly independent they form a basis of $\mathbb{V}$. Any further vectors in $\mathbb{V}$, say $\boldsymbol{f}_{n+1}, \boldsymbol{f}_{n+2}, \ldots$ are thus linear combinations of $\mathcal{F}$. Hence, any set of more than n vectors is linearly dependent.

Theorem 1.4. *Every set* $\mathcal{F} = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_m\}$ *of linearly independent vectors in an* n*-dimensional vector space* $\mathbb{V}$ *can be extended to a basis.*

Proof. If $m = n$, then $\mathcal{F}$ is already a basis according to Theorem 1.3. If $m < n$, then we try to find $n - m$ vectors $\boldsymbol{f}_{m+1}, \boldsymbol{f}_{m+2}, \ldots, \boldsymbol{f}_n$, such that all the vectors $\boldsymbol{f}_i$, that is, $\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_m, \boldsymbol{f}_{m+1}, \ldots, \boldsymbol{f}_n$ are linearly independent and consequently form a basis. Let us assume, on the contrary, that only $k < n - m$ such vectors can be found. In this case, for all $\boldsymbol{x} \in \mathbb{V}$ there exist scalars $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_{m+k}$, not all zero, such that

$$\alpha \boldsymbol{x} + \alpha_1 \boldsymbol{f}_1 + \alpha_2 \boldsymbol{f}_2 + \ldots + \alpha_{m+k} \boldsymbol{f}_{m+k} = \boldsymbol{0},$$

where $\alpha \neq 0$ since otherwise the vectors $\boldsymbol{f}_i$ $(i = 1, 2, \ldots, m+k)$ would be linearly dependent. Thus, all the vectors $\boldsymbol{x}$ of $\mathbb{V}$ are linear combinations of $\boldsymbol{f}_i$ $(i = 1, 2, \ldots, m+k)$. Then, the dimension of $\mathbb{V}$ is $m + k < n$, which contradicts the assumption of this theorem.

1.3 Components of a Vector, Summation Convention

Let $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ be a basis of an n-dimensional vector space $\mathbb{V}$. Then,

$$\boldsymbol{x} = \sum_{i=1}^{n} x^i \boldsymbol{g}_i, \quad \forall \boldsymbol{x} \in \mathbb{V}. \tag{1.4}$$

Theorem 1.5. *The representation* (1.4) *with respect to a given basis* $\mathcal{G}$ *is unique.*

Proof. Let

$$\boldsymbol{x} = \sum_{i=1}^{n} x^i \boldsymbol{g}_i \quad \text{and} \quad \boldsymbol{x} = \sum_{i=1}^{n} y^i \boldsymbol{g}_i$$

be two different representations of a vector $\boldsymbol{x}$, where not all scalar coefficients x^i and y^i $(i = 1, 2, \ldots, n)$ are pairwise identical. Then,

$$\boldsymbol{0} = \boldsymbol{x} + (-\boldsymbol{x}) = \boldsymbol{x} + (-1)\,\boldsymbol{x} = \sum_{i=1}^{n} x^i \boldsymbol{g}_i + \sum_{i=1}^{n} \left(-y^i\right) \boldsymbol{g}_i = \sum_{i=1}^{n} \left(x^i - y^i\right) \boldsymbol{g}_i,$$

where we use the identity $-\boldsymbol{x} = (-1)\,\boldsymbol{x}$ (Exercise 1.1). Thus, either the numbers x^i and y^i are pairwise equal $x^i = y^i$ $(i = 1, 2, \ldots, n)$ or the vectors $\boldsymbol{g}_i$ are linearly dependent. The latter one is likewise impossible because these vectors form a basis of $\mathbb{V}$.

The scalar numbers x^i $(i = 1, 2, \ldots, n)$ in the representation (1.4) are called components of the vector $\boldsymbol{x}$ with respect to the basis $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$.

The summation of the form (1.4) is often used in tensor analysis. For this reason it is usually represented without the summation symbol in a short form by

$$\boldsymbol{x} = \sum_{i=1}^{n} x^i \boldsymbol{g}_i = x^i \boldsymbol{g}_i \tag{1.5}$$

referred to as Einstein's summation convention. Accordingly, the summation is implied if an index appears twice in a multiplicative term, once as a superscript and once as a subscript. Such a repeated index (called dummy index) takes the values from 1 to n (the dimension of the vector space in consideration). The sense of the index changes (from superscript to subscript or vice versa) if it appears under the fraction bar.

1.4 Scalar Product, Euclidean Space, Orthonormal Basis

The scalar product plays an important role in vector and tensor algebra. The properties of the vector space essentially depend on whether and how the scalar product is defined in this space.

Definition 1.6. *The scalar (inner) product is a real-valued function* $\boldsymbol{x} \cdot \boldsymbol{y}$ *of two vectors* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *in a vector space* $\mathbb{V}$*, satisfying the following conditions.*

C. (C.1) $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}$ (commutative rule),

(C.2) $\boldsymbol{x} \cdot (\boldsymbol{y} + \boldsymbol{z}) = \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{x} \cdot \boldsymbol{z}$ (distributive rule),

(C.3) $\alpha (\boldsymbol{x} \cdot \boldsymbol{y}) = (\alpha \boldsymbol{x}) \cdot \boldsymbol{y} = \boldsymbol{x} \cdot (\alpha \boldsymbol{y})$ (associative rule for the multiplication by a scalar), $\forall \alpha \in \mathbb{R}$, $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{V}$,

(C.4) $\boldsymbol{x} \cdot \boldsymbol{x} \geq 0 \quad \forall \boldsymbol{x} \in \mathbb{V}, \quad \boldsymbol{x} \cdot \boldsymbol{x} = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$.

An n-dimensional vector space furnished by the scalar product with properties (C.1-C.4) is called Euclidean space $\mathbb{E}^n$. On the basis of this scalar product one defines the Euclidean length (also called norm) of a vector $\boldsymbol{x}$ by

$$\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}. \tag{1.6}$$

A vector whose length is equal to 1 is referred to as unit vector.

Definition 1.7. *Two vectors* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are called orthogonal (perpendicular), denoted by* $\boldsymbol{x} \perp \boldsymbol{y}$*, if*

$$\boldsymbol{x} \cdot \boldsymbol{y} = 0. \tag{1.7}$$

Of special interest is the so-called orthonormal basis of the Euclidean space.

Definition 1.8. *A basis* $\mathcal{E} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ *of an n-dimensional Euclidean space* $\mathbb{E}^n$ *is called orthonormal if*

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}, \quad i, j = 1, 2, \ldots, n, \tag{1.8}$$

where

$$\delta_{ij} = \delta^{ij} = \delta^i_j = \begin{cases} 1 \text{ for } i = j, \\ 0 \text{ for } i \neq j \end{cases} \tag{1.9}$$

denotes the Kronecker delta.

Thus, the elements of an orthonormal basis represent pairwise orthogonal unit vectors. Of particular interest is the question of the existence of an orthonormal basis. Now, we are going to demonstrate that every set of $m \leq n$ linearly independent vectors in $\mathbb{E}^n$ can be orthogonalized and normalized by means of a linear transformation (Gram-Schmidt procedure). In other words, starting from linearly independent vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$ one can always construct their linear combinations $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_m$ such that $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}\,(i, j = 1, 2, \ldots, m)$. Indeed, since the vectors $\boldsymbol{x}_i\,(i = 1, 2, \ldots, m)$ are linearly independent they are all non-zero (see Exercise 1.2). Thus, we can define the first unit vector by

$$\boldsymbol{e}_1 = \frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|}. \tag{1.10}$$

Next, we consider the vector

$$\boldsymbol{e}'_2 = \boldsymbol{x}_2 - (\boldsymbol{x}_2 \cdot \boldsymbol{e}_1)\,\boldsymbol{e}_1 \tag{1.11}$$

orthogonal to $\boldsymbol{e}_1$. This holds for the unit vector $\boldsymbol{e}_2 = \boldsymbol{e}'_2/\|\boldsymbol{e}'_2\|$ as well. It is also seen that $\|\boldsymbol{e}'_2\| = \sqrt{\boldsymbol{e}'_2 \cdot \boldsymbol{e}'_2} \neq 0$ because otherwise $\boldsymbol{e}'_2 = \boldsymbol{0}$ and thus $\boldsymbol{x}_2 = (\boldsymbol{x}_2 \cdot \boldsymbol{e}_1)\,\boldsymbol{e}_1 = (\boldsymbol{x}_2 \cdot \boldsymbol{e}_1)\,\|\boldsymbol{x}_1\|^{-1}\,\boldsymbol{x}_1$. However, the latter result contradicts the fact that the vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are linearly independent.

Further, we proceed to construct the vectors

$$\boldsymbol{e}'_3 = \boldsymbol{x}_3 - (\boldsymbol{x}_3 \cdot \boldsymbol{e}_2)\,\boldsymbol{e}_2 - (\boldsymbol{x}_3 \cdot \boldsymbol{e}_1)\,\boldsymbol{e}_1, \quad \boldsymbol{e}_3 = \frac{\boldsymbol{e}'_3}{\|\boldsymbol{e}'_3\|} \tag{1.12}$$

orthogonal to $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. Repeating this procedure we finally obtain the set of orthonormal vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_m$. Since these vectors are non-zero and mutually orthogonal, they are linearly independent (see Exercise 1.6). In the case $m = n$, this set represents, according to Theorem 1.3, the orthonormal basis (1.8) in $\mathbb{E}^n$.

With respect to an orthonormal basis the scalar product of two vectors $\boldsymbol{x} = x^i \boldsymbol{e}_i$ and $\boldsymbol{y} = y^i \boldsymbol{e}_i$ in $\mathbb{E}^n$ takes the form

$$\boldsymbol{x} \cdot \boldsymbol{y} = x^1 y^1 + x^2 y^2 + \ldots + x^n y^n. \tag{1.13}$$

For the length of the vector $\boldsymbol{x}$ (1.6) we thus obtain the Pythagoras formula

$$\|\boldsymbol{x}\| = \sqrt{x^1 x^1 + x^2 x^2 + \ldots + x^n x^n}, \quad \boldsymbol{x} \in \mathbb{E}^n. \tag{1.14}$$

1.5 Dual Bases

Definition 1.9. *Let* $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ *be a basis in the* n*-dimensional Euclidean space* $\mathbb{E}^n$*. Then, a basis* $\mathcal{G}' = \left\{\boldsymbol{g}^1, \boldsymbol{g}^2, \ldots, \boldsymbol{g}^n\right\}$ *of* $\mathbb{E}^n$ *is called dual to* $\mathcal{G}$*, if*

$$\boldsymbol{g}_i \cdot \boldsymbol{g}^j = \delta_i^j, \quad i, j = 1, 2, \ldots, n. \tag{1.15}$$

In the following we show that a set of vectors $\mathcal{G}' = \left\{\boldsymbol{g}^1, \boldsymbol{g}^2, \ldots, \boldsymbol{g}^n\right\}$ satisfying the conditions (1.15) always exists, is unique and forms a basis in $\mathbb{E}^n$.

Let $\mathcal{E} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ be an orthonormal basis in $\mathbb{E}^n$. Since $\mathcal{G}$ also represents a basis, we can write

$$\boldsymbol{e}_i = \alpha_i^j \boldsymbol{g}_j, \quad \boldsymbol{g}_i = \beta_i^j \boldsymbol{e}_j, \quad i = 1, 2, \ldots, n, \tag{1.16}$$

where α_i^j and β_i^j $(i = 1, 2, \ldots, n)$ denote the components of $\boldsymbol{e}_i$ and $\boldsymbol{g}_i$, respectively. Inserting the first relation (1.16) into the second one yields

$$\boldsymbol{g}_i = \beta_i^j \alpha_j^k \boldsymbol{g}_k, \quad \Rightarrow \quad \boldsymbol{0} = \left(\beta_i^j \alpha_j^k - \delta_i^k\right) \boldsymbol{g}_k, \quad i = 1, 2, \ldots, n. \tag{1.17}$$

Since the vectors $\boldsymbol{g}_i$ are linearly independent we obtain

$$\beta_i^j \alpha_j^k = \delta_i^k, \quad i, k = 1, 2, \ldots, n. \tag{1.18}$$

Let further

$$\boldsymbol{g}^i = \alpha_j^i \boldsymbol{e}^j, \quad i = 1, 2, \ldots, n, \tag{1.19}$$

where and henceforth we set $\boldsymbol{e}^j = \boldsymbol{e}_j$ $(j = 1, 2, \ldots, n)$ in order to take the advantage of Einstein's summation convention. By virtue of (1.8), (1.16) and (1.18) one finally finds

$$\boldsymbol{g}_i \cdot \boldsymbol{g}^j = \left(\beta_i^k \boldsymbol{e}_k\right) \cdot \left(\alpha_l^j \boldsymbol{e}^l\right) = \beta_i^k \alpha_l^j \delta_k^l = \beta_i^k \alpha_k^j = \delta_i^j, \quad i, j = 1, 2, \ldots, n. \tag{1.20}$$

Next, we show that the vectors $\boldsymbol{g}^i$ $(i = 1, 2, \ldots, n)$ (1.19) are linearly independent and for this reason form a basis of $\mathbb{E}^n$. Assume on the contrary that

$$a_i \boldsymbol{g}^i = \boldsymbol{0},$$

where not all scalars a_i $(i = 1, 2, \ldots, n)$ are zero. The scalar product of this relation with the vectors $\boldsymbol{g}_j$ $(j = 1, 2, \ldots, n)$ leads to a contradiction. Indeed, we obtain in this case

$$0 = a_i \boldsymbol{g}^i \cdot \boldsymbol{g}_j = a_i \delta_j^i = a_j, \quad j = 1, 2, \ldots, n.$$

The next important question is whether the dual basis is unique. Let $\mathcal{G}' = \left\{\boldsymbol{g}^1, \boldsymbol{g}^2, \ldots, \boldsymbol{g}^n\right\}$ and $\mathcal{H}' = \left\{\boldsymbol{h}^1, \boldsymbol{h}^2, \ldots, \boldsymbol{h}^n\right\}$ be two arbitrary non-coinciding bases in $\mathbb{E}^n$, both dual to $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$. Then,

$$\boldsymbol{h}^i = h^i_j \boldsymbol{g}^j, \quad i = 1, 2, \ldots, n.$$

Forming the scalar product with the vectors $\boldsymbol{g}_j$ $(j = 1, 2, \ldots, n)$ we can conclude that the bases $\mathcal{G}'$ and $\mathcal{H}'$ coincide:

$$\delta^i_j = \boldsymbol{h}^i \cdot \boldsymbol{g}_j = \left(h^i_k \boldsymbol{g}^k\right) \cdot \boldsymbol{g}_j = h^i_k \delta^k_j = h^i_j \quad \Rightarrow \quad \boldsymbol{h}^i = \boldsymbol{g}^i, \quad i = 1, 2, \ldots, n.$$

Thus, we have proved the following theorem.

Theorem 1.6. *To every basis in an Euclidean space* $\mathbb{E}^n$ *there exists a unique dual basis.*

Relation (1.19) enables to determine the dual basis. However, it can also be obtained without any orthonormal basis. Indeed, let $\boldsymbol{g}^i$ be a basis dual to $\boldsymbol{g}_i$ $(i = 1, 2, \ldots, n)$. Then

$$\boldsymbol{g}^i = g^{ij} \boldsymbol{g}_j, \quad \boldsymbol{g}_i = g_{ij} \boldsymbol{g}^j, \quad i = 1, 2, \ldots, n. \tag{1.21}$$

Inserting the second relation (1.21) into the first one yields

$$\boldsymbol{g}^i = g^{ij} g_{jk} \boldsymbol{g}^k, \quad i = 1, 2, \ldots, n. \tag{1.22}$$

Multiplying scalarly with the vectors $\boldsymbol{g}_l$ we have by virtue of (1.15)

$$\delta^i_l = g^{ij} g_{jk} \delta^k_l = g^{ij} g_{jl}, \quad i, l = 1, 2, \ldots, n. \tag{1.23}$$

Thus, we see that the matrices $[g_{kj}]$ and $\left[g^{kj}\right]$ are inverse to each other such that

$$\left[g^{kj}\right] = [g_{kj}]^{-1}. \tag{1.24}$$

Now, multiplying scalarly the first and second relation (1.21) by the vectors $\boldsymbol{g}^j$ and $\boldsymbol{g}_j$ $(j = 1, 2, \ldots, n)$, respectively, we obtain with the aid of (1.15) the following important identities:

$$g^{ij} = g^{ji} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j, \quad g_{ij} = g_{ji} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j, \quad i, j = 1, 2, \ldots, n. \tag{1.25}$$

By definition (1.8) the orthonormal basis in $\mathbb{E}^n$ is self-dual, so that

$$\boldsymbol{e}_i = \boldsymbol{e}^i, \quad \boldsymbol{e}_i \cdot \boldsymbol{e}^j = \delta^j_i, \quad i, j = 1, 2, \ldots, n. \tag{1.26}$$

With the aid of the dual bases one can represent an arbitrary vector in $\mathbb{E}^n$ by

$$\boldsymbol{x} = x^i \boldsymbol{g}_i = x_i \boldsymbol{g}^i, \quad \forall \boldsymbol{x} \in \mathbb{E}^n, \tag{1.27}$$

where

$$x^i = \boldsymbol{x} \cdot \boldsymbol{g}^i, \quad x_i = \boldsymbol{x} \cdot \boldsymbol{g}_i, \quad i = 1, 2, \ldots, n. \tag{1.28}$$

Indeed, using (1.15) we can write

$$\boldsymbol{x} \cdot \boldsymbol{g}^i = \left(x^j \boldsymbol{g}_j\right) \cdot \boldsymbol{g}^i = x^j \delta_j^i = x^i,$$

$$\boldsymbol{x} \cdot \boldsymbol{g}_i = \left(x_j \boldsymbol{g}^j\right) \cdot \boldsymbol{g}_i = x_j \delta_i^j = x_i, \quad i = 1, 2, \ldots, n.$$

The components of a vector with respect to the dual bases are suitable for calculating the scalar product. For example, for two arbitrary vectors $\boldsymbol{x} = x^i \boldsymbol{g}_i = x_i \boldsymbol{g}^i$ and $\boldsymbol{y} = y^i \boldsymbol{g}_i = y_i \boldsymbol{g}^i$ we obtain

$$\boldsymbol{x} \cdot \boldsymbol{y} = x^i y^j g_{ij} = x_i y_j g^{ij} = x^i y_i = x_i y^i. \tag{1.29}$$

The length of the vector $\boldsymbol{x}$ can thus be written by

$$\|x\| = \sqrt{x_i x_j g^{ij}} = \sqrt{x^i x^j g_{ij}} = \sqrt{x_i x^i}. \tag{1.30}$$

Example. Dual basis in $\mathbb{E}^3$. Let $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$ be a basis of the three-dimensional Euclidean space and

$$g = [\boldsymbol{g}_1 \boldsymbol{g}_2 \boldsymbol{g}_3], \tag{1.31}$$

where $[\bullet \bullet \bullet]$ denotes the mixed product of vectors. It is defined by

$$[\boldsymbol{abc}] = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = (\boldsymbol{b} \times \boldsymbol{c}) \cdot \boldsymbol{a} = (\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{b}, \tag{1.32}$$

where "$\times$" denotes the vector (also called cross or outer) product of vectors. Consider the following set of vectors:

$$\boldsymbol{g}^1 = g^{-1} \boldsymbol{g}_2 \times \boldsymbol{g}_3, \quad \boldsymbol{g}^2 = g^{-1} \boldsymbol{g}_3 \times \boldsymbol{g}_1, \quad \boldsymbol{g}^3 = g^{-1} \boldsymbol{g}_1 \times \boldsymbol{g}_2. \tag{1.33}$$

It seen that the vectors (1.33) satisfy conditions (1.15), are linearly independent (Exercise 1.11) and consequently form the basis dual to $\boldsymbol{g}_i$ $(i = 1, 2, 3)$. Further, it can be shown that

$$g^2 = |g_{ij}|, \tag{1.34}$$

where $|\bullet|$ denotes the determinant of the matrix $[\bullet]$. Indeed, with the aid of $(1.16)_2$ we obtain

$$g = [\boldsymbol{g}_1 \boldsymbol{g}_2 \boldsymbol{g}_3] = \left[\beta_1^i \boldsymbol{e}_i \beta_2^j \boldsymbol{e}_j \beta_3^k \boldsymbol{e}_k\right]$$
$$= \beta_1^i \beta_2^j \beta_3^k [\boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_k] = \beta_1^i \beta_2^j \beta_3^k e_{ijk} = \left|\beta_j^i\right|, \tag{1.35}$$

where e_{ijk} denotes the permutation symbol (also called Levi-Civita symbol). It is defined by

$$e_{ijk} = e^{ijk} = [\boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_k]$$
$$= \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of 123,} \\ -1 & \text{if } ijk \text{ is an odd permutation of 123,} \\ 0 & \text{otherwise,} \end{cases} \tag{1.36}$$

where the orthonormal vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are numerated in such a way that they form a right-handed system. In this case, $[\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3] = 1$.

On the other hand, we can write again using $(1.16)_2$

$$g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j = \sum_{k=1}^{3} \beta_i^k \beta_j^k .$$

The latter sum can be represented as a product of two matrices so that

$$[g_{ij}] = \left[\beta_i^j\right] \left[\beta_i^j\right]^{\mathrm{T}} . \tag{1.37}$$

Since the determinant of the matrix product is equal to the product of the matrix determinants we finally have

$$|g_{ij}| = \left|\beta_i^j\right|^2 = g^2 . \tag{1.38}$$

With the aid of the permutation symbol (1.36) one can represent the identities (1.33) by

$$\boldsymbol{g}_i \times \boldsymbol{g}_j = e_{ijk}\, g\, \boldsymbol{g}^k, \quad i,j = 1,2,3, \tag{1.39}$$

which also delivers

$$[\boldsymbol{g}_i \boldsymbol{g}_j \boldsymbol{g}_k] = e_{ijk}\, g, \quad i,j,k = 1,2,3. \tag{1.40}$$

Similarly to (1.34) one can also show that (see Exercise 1.12)

$$\left[\boldsymbol{g}^1 \boldsymbol{g}^2 \boldsymbol{g}^3\right]^2 = \left|g^{ij}\right| = g^{-2} . \tag{1.41}$$

Thus,

$$\boldsymbol{g}^i \times \boldsymbol{g}^j = \frac{e^{ijk}}{g} \boldsymbol{g}_k, \quad i,j = 1,2,3, \tag{1.42}$$

which yields by analogy with (1.40)

$$\left[\boldsymbol{g}^i \boldsymbol{g}^j \boldsymbol{g}^k\right] = \frac{e^{ijk}}{g}, \quad i,j,k = 1,2,3. \tag{1.43}$$

Relations (1.39) and (1.42) permit a useful representation of the vector product. Indeed, let $\boldsymbol{a} = a^i \boldsymbol{g}_i = a_i \boldsymbol{g}^i$ and $\boldsymbol{b} = b^j \boldsymbol{g}_j = b_j \boldsymbol{g}^j$ be two arbitrary vectors in $\mathbb{E}^3$. Then,

$$\boldsymbol{a} \times \boldsymbol{b} = \left(a^i \boldsymbol{g}_i\right) \times \left(b^j \boldsymbol{g}_j\right) = a^i b^j e_{ijk} g \boldsymbol{g}^k = g \begin{vmatrix} a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \\ \boldsymbol{g}^1 & \boldsymbol{g}^2 & \boldsymbol{g}^3 \end{vmatrix} ,$$

$$\boldsymbol{a} \times \boldsymbol{b} = \left(a_i \boldsymbol{g}^i\right) \times \left(b_j \boldsymbol{g}^j\right) = a_i b_j e^{ijk} g^{-1} \boldsymbol{g}_k = \frac{1}{g} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \boldsymbol{g}_1 & \boldsymbol{g}_2 & \boldsymbol{g}_3 \end{vmatrix}. \tag{1.44}$$

For the orthonormal basis in $\mathbb{E}^3$ relations (1.39) and (1.42) reduce to

$$\boldsymbol{e}_i \times \boldsymbol{e}_j = e_{ijk} \boldsymbol{e}^k = e^{ijk} \boldsymbol{e}_k, \quad i, j = 1, 2, 3, \tag{1.45}$$

so that the vector product (1.44) can be written by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{vmatrix}, \tag{1.46}$$

where $\boldsymbol{a} = a_i \boldsymbol{e}^i$ and $\boldsymbol{b} = b_j \boldsymbol{e}^j$.

1.6 Second-Order Tensor as a Linear Mapping

Let us consider a set $\mathbf{Lin}^n$ of all linear mappings of one vector into another one within $\mathbb{E}^n$. Such a mapping can be written as

$$\boldsymbol{y} = \mathbf{A}\boldsymbol{x}, \quad \boldsymbol{y} \in \mathbb{E}^n, \quad \forall \boldsymbol{x} \in \mathbb{E}^n, \quad \forall \mathbf{A} \in \mathbf{Lin}^n. \tag{1.47}$$

Elements of the set $\mathbf{Lin}^n$ are called second-order tensors or simply tensors. Linearity of the mapping (1.47) is expressed by the following relations:

$$\mathbf{A}\left(\boldsymbol{x} + \boldsymbol{y}\right) = \mathbf{A}\boldsymbol{x} + \mathbf{A}\boldsymbol{y}, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}^n, \quad \forall \mathbf{A} \in \mathbf{Lin}^n, \tag{1.48}$$

$$\mathbf{A}\left(\alpha \boldsymbol{x}\right) = \alpha \left(\mathbf{A}\boldsymbol{x}\right), \quad \forall \boldsymbol{x} \in \mathbb{E}^n, \quad \forall \alpha \in \mathbb{R}, \quad \forall \mathbf{A} \in \mathbf{Lin}^n. \tag{1.49}$$

Further, we define the product of a tensor by a scalar number $\alpha \in \mathbb{R}$ as

$$\left(\alpha \mathbf{A}\right) \boldsymbol{x} = \alpha \left(\mathbf{A}\boldsymbol{x}\right) = \mathbf{A}\left(\alpha \boldsymbol{x}\right), \quad \forall \boldsymbol{x} \in \mathbb{E}^n \tag{1.50}$$

and the sum of two tensors $\mathbf{A}$ and $\mathbf{B}$ as

$$\left(\mathbf{A} + \mathbf{B}\right) \boldsymbol{x} = \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x}, \quad \forall \boldsymbol{x} \in \mathbb{E}^n. \tag{1.51}$$

Thus, properties (A.1), (A.2) and (B.1-B.4) apply to the set $\mathbf{Lin}^n$. Setting in (1.50) $\alpha = -1$ we obtain the negative tensor by

$$-\mathbf{A} = \left(-1\right) \mathbf{A}. \tag{1.52}$$

Further, we define a zero tensor $\mathbf{0}$ in the following manner

$$\mathbf{0}\boldsymbol{x} = \boldsymbol{0}, \quad \forall \boldsymbol{x} \in \mathbb{E}^n, \tag{1.53}$$

so that the elements of the set $\mathbf{Lin}^n$ also fulfill conditions (A.3) and (A.4) and accordingly form a vector space.

The properties of second-order tensors can thus be summarized by

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \quad \text{(addition is commutative)}, \tag{1.54}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}, \quad \text{(addition is associative)}, \tag{1.55}$$

$$\mathbf{0} + \mathbf{A} = \mathbf{A}, \tag{1.56}$$

$$\mathbf{A} + (-\mathbf{A}) = \mathbf{0}, \tag{1.57}$$

$$\alpha (\beta \mathbf{A}) = (\alpha\beta) \mathbf{A}, \quad \text{(multiplication by scalars is associative)}, \tag{1.58}$$

$$1\mathbf{A} = \mathbf{A}, \tag{1.59}$$

$$\alpha (\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}, \quad \text{(multiplication by scalars is distributive with respect to tensor addition)}, \tag{1.60}$$

$$(\alpha + \beta) \mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}, \quad \text{(multiplication by scalars is distributive with respect to scalar addition)}, \quad \forall \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbf{Lin}^n, \ \forall \alpha, \beta \in \mathbb{R}. \tag{1.61}$$

Example. Vector product in $\mathbb{E}^3$. The vector product of two vectors in $\mathbb{E}^3$ represents again a vector in $\mathbb{E}^3$

$$\boldsymbol{z} = \boldsymbol{w} \times \boldsymbol{x}, \quad \boldsymbol{z} \in \mathbb{E}^3, \quad \forall \boldsymbol{w}, \boldsymbol{x} \in \mathbb{E}^3. \tag{1.62}$$

According to (1.44) the mapping $\boldsymbol{x} \to \boldsymbol{z}$ is linear so that

$$\boldsymbol{w} \times (\alpha\boldsymbol{x}) = \alpha (\boldsymbol{w} \times \boldsymbol{x}),$$

$$\boldsymbol{w} \times (\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{w} \times \boldsymbol{x} + \boldsymbol{w} \times \boldsymbol{y}, \quad \forall \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}^3, \quad \forall \alpha \in \mathbb{R}. \tag{1.63}$$

Thus, it can be described by means of a tensor of the second order by

$$\boldsymbol{w} \times \boldsymbol{x} = \mathbf{W}\boldsymbol{x}, \quad \mathbf{W} \in \mathbf{Lin}^3, \quad \forall \boldsymbol{x} \in \mathbb{E}^3. \tag{1.64}$$

The tensor which forms the vector product by a vector $\boldsymbol{w}$ according to (1.64) will be denoted in the following by $\hat{\boldsymbol{w}}$. Thus, we write

$$\boldsymbol{w} \times \boldsymbol{x} = \hat{\boldsymbol{w}}\boldsymbol{x}. \tag{1.65}$$

Example. Representation of a rotation by a second-order tensor. A rotation of a vector $\boldsymbol{a}$ in $\mathbb{E}^3$ about an axis yields another vector $\boldsymbol{r}$ in $\mathbb{E}^3$. It can be shown that the mapping $\boldsymbol{a} \to \boldsymbol{r}(\boldsymbol{a})$ is linear such that

$$\boldsymbol{r}(\alpha\boldsymbol{a}) = \alpha\boldsymbol{r}(\boldsymbol{a}), \ \boldsymbol{r}(\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{r}(\boldsymbol{a}) + \boldsymbol{r}(\boldsymbol{b}), \ \forall \alpha \in \mathbb{R}, \ \forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}^3. \tag{1.66}$$

Thus, it can again be described by a second-order tensor as

$$\boldsymbol{r}(\boldsymbol{a}) = \mathbf{R}\boldsymbol{a}, \quad \forall \boldsymbol{a} \in \mathbb{E}^3, \quad \mathbf{R} \in \mathbf{Lin}^3. \tag{1.67}$$

This tensor $\mathbf{R}$ is referred to as rotation tensor.

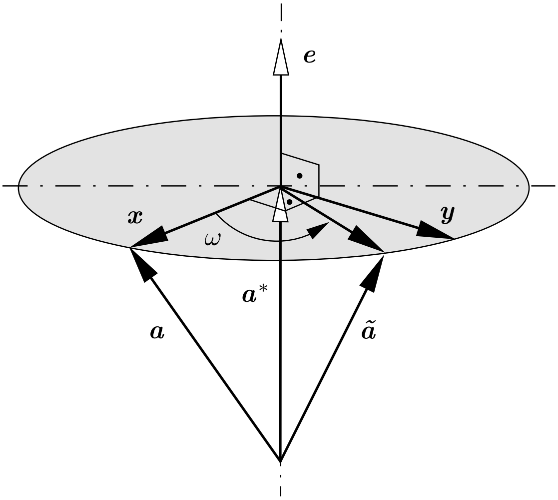

Fig. 1.2. Finite rotation of a vector in $\mathbb{E}^3$

Let us construct the rotation tensor which rotates an arbitrary vector $\boldsymbol{a} \in \mathbb{E}^3$ about an axis specified by a unit vector $\boldsymbol{e} \in \mathbb{E}^3$ (see Fig. 1.2). Decomposing the vector $\boldsymbol{a}$ by $\boldsymbol{a} = \boldsymbol{a}^* + \boldsymbol{x}$ in two vectors along and perpendicular to the rotation axis we can write

$$\tilde{\boldsymbol{a}} = \boldsymbol{a}^* + \boldsymbol{x}\cos\omega + \boldsymbol{y}\sin\omega = \boldsymbol{a}^* + (\boldsymbol{a} - \boldsymbol{a}^*)\cos\omega + \boldsymbol{y}\sin\omega, \tag{1.68}$$

where ω denotes the rotation angle. By virtue of the geometric identities

$$\boldsymbol{a}^* = (\boldsymbol{a} \cdot \boldsymbol{e})\,\boldsymbol{e} = (\boldsymbol{e} \otimes \boldsymbol{e})\,\boldsymbol{a}, \quad \boldsymbol{y} = \boldsymbol{e}\times\boldsymbol{x} = \boldsymbol{e}\times(\boldsymbol{a} - \boldsymbol{a}^*) = \boldsymbol{e}\times\boldsymbol{a} = \hat{\boldsymbol{e}}\boldsymbol{a}, \tag{1.69}$$

where "⊗" denotes the so-called tensor product (1.75) (see Sect. 1.7), we obtain

$$\tilde{\boldsymbol{a}} = \cos\omega\boldsymbol{a} + \sin\omega\hat{\boldsymbol{e}}\boldsymbol{a} + (1 - \cos\omega)\,(\boldsymbol{e} \otimes \boldsymbol{e})\,\boldsymbol{a}. \tag{1.70}$$

Thus the rotation tensor can be given by

$$\mathbf{R} = \cos\omega\mathbf{I} + \sin\omega\hat{\boldsymbol{e}} + (1 - \cos\omega)\,\boldsymbol{e} \otimes \boldsymbol{e}, \tag{1.71}$$

where $\mathbf{I}$ denotes the so-called identity tensor (1.84) (see Sect. 1.7).

Example. The Cauchy stress tensor as a linear mapping of the unit surface normal into the Cauchy stress vector. Let us consider a body B in the current configuration at a time t. In order to define the stress in some point P let us further imagine a smooth surface going through P and separating B into two parts (Fig. 1.3). Then, one can define a force $\Delta\boldsymbol{p}$ and a couple $\Delta\boldsymbol{m}$ resulting from the forces exerted by the (hidden) material on

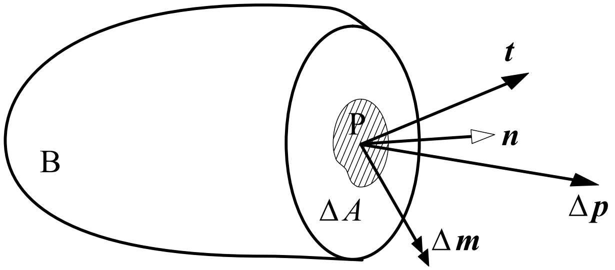

Fig. 1.3. Cauchy stress vector

one side of the surface ΔA and acting on the material on the other side of this surface. Let the area ΔA tend to zero keeping P as inner point. A basic postulate of continuum mechanics is that the limit

$$\boldsymbol{t} = \lim_{\Delta A \to 0} \frac{\Delta \boldsymbol{p}}{\Delta A}$$

exists and is final. The so-defined vector $\boldsymbol{t}$ is called Cauchy stress vector. Cauchy's fundamental postulate states that the vector $\boldsymbol{t}$ depends on the surface only through the outward unit normal $\boldsymbol{n}$. In other words, the Cauchy stress vector is the same for all surfaces through P which have $\boldsymbol{n}$ as the normal in P. Further, according to Cauchy's theorem the mapping $\boldsymbol{n} \to \boldsymbol{t}$ is linear provided $\boldsymbol{t}$ is a continuous function of the position vector $\boldsymbol{x}$ at P. Hence, this mapping can be described by a second-order tensor $\boldsymbol{\sigma}$ called the Cauchy stress tensor so that

$$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}. \tag{1.72}$$

On the basis of the "right" mapping (1.47) we can also define the "left" one by the following condition

$$(\boldsymbol{y}\mathbf{A}) \cdot \boldsymbol{x} = \boldsymbol{y} \cdot (\mathbf{A}\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{E}^n, \quad \mathbf{A} \in \mathbf{Lin}^n. \tag{1.73}$$

First, it should be shown that for all $\boldsymbol{y} \in \mathbb{E}^n$ there exists a unique vector $\boldsymbol{y}\mathbf{A} \in \mathbb{E}^n$ satisfying the condition (1.73) for all $\boldsymbol{x} \in \mathbb{E}^n$. Let $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ and $\mathcal{G}' = \left\{\boldsymbol{g}^1, \boldsymbol{g}^2, \ldots, \boldsymbol{g}^n\right\}$ be dual bases in $\mathbb{E}^n$. Then, we can represent two arbitrary vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}^n$, by $\boldsymbol{x} = x_i \boldsymbol{g}^i$ and $\boldsymbol{y} = y_i \boldsymbol{g}^i$. Now, consider the vector

$$\boldsymbol{y}\mathbf{A} = y_i \left[\boldsymbol{g}^i \cdot \left(\mathbf{A}\boldsymbol{g}^j\right)\right] \boldsymbol{g}_j.$$

It holds: $(\boldsymbol{y}\mathbf{A}) \cdot \boldsymbol{x} = y_i x_j \left[\boldsymbol{g}^i \cdot \left(\mathbf{A}\boldsymbol{g}^j\right)\right]$. On the other hand, we obtain the same result also by

$$\boldsymbol{y} \cdot (\mathbf{A}\boldsymbol{x}) = \boldsymbol{y} \cdot \left(x_j \mathbf{A}\boldsymbol{g}^j\right) = y_i x_j \left[\boldsymbol{g}^i \cdot \left(\mathbf{A}\boldsymbol{g}^j\right)\right].$$

Further, we show that the vector $\boldsymbol{y}\mathbf{A}$, satisfying condition (1.73) for all $\boldsymbol{x} \in \mathbb{E}^n$, is unique. Conversely, let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}^n$ be two such vectors. Then, we have

$$\boldsymbol{a} \cdot \boldsymbol{x} = \boldsymbol{b} \cdot \boldsymbol{x} \;\Rightarrow\; (\boldsymbol{a} - \boldsymbol{b}) \cdot \boldsymbol{x} = 0,\ \forall \boldsymbol{x} \in \mathbb{E}^n \;\Rightarrow\; (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b}) = 0,$$

which by axiom (C.4) implies that $\boldsymbol{a} = \boldsymbol{b}$.

Since the order of mappings in (1.73) is irrelevant we can write them without brackets and dots as follows

$$\boldsymbol{y} \cdot (\mathbf{A}\boldsymbol{x}) = (\boldsymbol{y}\mathbf{A}) \cdot \boldsymbol{x} = \boldsymbol{y}\mathbf{A}\boldsymbol{x}. \tag{1.74}$$

1.7 Tensor Product, Representation of a Tensor with Respect to a Basis

The tensor product plays an important role since it enables to construct a second-order tensor from two vectors. In order to define the tensor product we consider two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}^n$. An arbitrary vector $\boldsymbol{x} \in \mathbb{E}^n$ can be mapped into another vector $\boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{x}) \in \mathbb{E}^n$. This mapping is denoted by symbol "$\otimes$" as $\boldsymbol{a} \otimes \boldsymbol{b}$. Thus,

$$(\boldsymbol{a} \otimes \boldsymbol{b})\, \boldsymbol{x} = \boldsymbol{a}\,(\boldsymbol{b} \cdot \boldsymbol{x}), \quad \boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}^n,\ \forall \boldsymbol{x} \in \mathbb{E}^n. \tag{1.75}$$

It can be shown that the mapping (1.75) fulfills the conditions (1.48-1.50) and for this reason is linear. Indeed, by virtue of (B.1), (B.4), (C.2) and (C.3) we can write

$$\begin{aligned}(\boldsymbol{a} \otimes \boldsymbol{b})\,(\boldsymbol{x} + \boldsymbol{y}) &= \boldsymbol{a}\,[\boldsymbol{b} \cdot (\boldsymbol{x} + \boldsymbol{y})] = \boldsymbol{a}\,(\boldsymbol{b} \cdot \boldsymbol{x} + \boldsymbol{b} \cdot \boldsymbol{y}) \\ &= (\boldsymbol{a} \otimes \boldsymbol{b})\, \boldsymbol{x} + (\boldsymbol{a} \otimes \boldsymbol{b})\, \boldsymbol{y},\end{aligned} \tag{1.76}$$

$$\begin{aligned}(\boldsymbol{a} \otimes \boldsymbol{b})\,(\alpha \boldsymbol{x}) &= \boldsymbol{a}\,[\boldsymbol{b} \cdot (\alpha \boldsymbol{x})] = \alpha\,(\boldsymbol{b} \cdot \boldsymbol{x})\, \boldsymbol{a} \\ &= \alpha\,(\boldsymbol{a} \otimes \boldsymbol{b})\, \boldsymbol{x}, \quad \boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}^n,\ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}^n,\ \forall \alpha \in \mathbb{R}.\end{aligned} \tag{1.77}$$

Thus, the tensor product of two vectors represents a second-order tensor. Further, it holds

$$\boldsymbol{c} \otimes (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{c} \otimes \boldsymbol{a} + \boldsymbol{c} \otimes \boldsymbol{b}, \quad (\boldsymbol{a} + \boldsymbol{b}) \otimes \boldsymbol{c} = \boldsymbol{a} \otimes \boldsymbol{c} + \boldsymbol{b} \otimes \boldsymbol{c}, \tag{1.78}$$

$$(\alpha \boldsymbol{a}) \otimes (\beta \boldsymbol{b}) = \alpha\beta\,(\boldsymbol{a} \otimes \boldsymbol{b}), \quad \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{E}^n,\ \forall \alpha, \beta \in \mathbb{R}. \tag{1.79}$$

Indeed, mapping an arbitrary vector $\boldsymbol{x} \in \mathbb{E}^n$ by both sides of these relations and using (1.51) and (1.75) we obtain

$$\begin{aligned}\boldsymbol{c} \otimes (\boldsymbol{a} + \boldsymbol{b})\, \boldsymbol{x} &= \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{x} + \boldsymbol{b} \cdot \boldsymbol{x}) = \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{x}) + \boldsymbol{c}\,(\boldsymbol{b} \cdot \boldsymbol{x}) \\ &= (\boldsymbol{c} \otimes \boldsymbol{a})\, \boldsymbol{x} + (\boldsymbol{c} \otimes \boldsymbol{b})\, \boldsymbol{x} = (\boldsymbol{c} \otimes \boldsymbol{a} + \boldsymbol{c} \otimes \boldsymbol{b})\, \boldsymbol{x},\end{aligned}$$

$$[(\boldsymbol{a}+\boldsymbol{b}) \otimes \boldsymbol{c}]\, \boldsymbol{x} = (\boldsymbol{a}+\boldsymbol{b})\,(\boldsymbol{c}\cdot\boldsymbol{x}) = \boldsymbol{a}\,(\boldsymbol{c}\cdot\boldsymbol{x}) + \boldsymbol{b}\,(\boldsymbol{c}\cdot\boldsymbol{x})$$
$$= (\boldsymbol{a}\otimes\boldsymbol{c})\,\boldsymbol{x} + (\boldsymbol{b}\otimes\boldsymbol{c})\,\boldsymbol{x} = (\boldsymbol{a}\otimes\boldsymbol{c} + \boldsymbol{b}\otimes\boldsymbol{c})\,\boldsymbol{x},$$

$$(\alpha\boldsymbol{a}) \otimes (\beta\boldsymbol{b})\,\boldsymbol{x} = (\alpha\boldsymbol{a})\,(\beta\boldsymbol{b}\cdot\boldsymbol{x})$$
$$= \alpha\beta\boldsymbol{a}\,(\boldsymbol{b}\cdot\boldsymbol{x}) = \alpha\beta\,(\boldsymbol{a}\otimes\boldsymbol{b})\,\boldsymbol{x}, \quad \forall\boldsymbol{x}\in\mathbb{E}^n.$$

For the "left" mapping by the tensor $\boldsymbol{a}\otimes\boldsymbol{b}$ we obtain from (1.73) (see Exercise 1.19)

$$\boldsymbol{y}\,(\boldsymbol{a}\otimes\boldsymbol{b}) = (\boldsymbol{y}\cdot\boldsymbol{a})\,\boldsymbol{b}, \quad \forall\boldsymbol{y}\in\mathbb{E}^n. \tag{1.80}$$

We have already seen that the set of all second-order tensors $\mathbf{Lin}^n$ represents a vector space. In the following, we show that a basis of $\mathbf{Lin}^n$ can be constructed with the aid of the tensor product (1.75).

Theorem 1.7. *Let* $\mathcal{F} = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n\}$ *and* $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ *be two arbitrary bases of* $\mathbb{E}^n$*. Then, the tensors* $\boldsymbol{f}_i \otimes \boldsymbol{g}_j$ $(i, j = 1, 2, \ldots, n)$ *represent a basis of* $\mathbf{Lin}^n$*. The dimension of the vector space* $\mathbf{Lin}^n$ *is thus* n^2*.*

Proof. First, we prove that every tensor in $\mathbf{Lin}^n$ represents a linear combination of the tensors $\boldsymbol{f}_i\otimes\boldsymbol{g}_j$ $(i, j = 1, 2, \ldots, n)$. Indeed, let $\mathbf{A}\in\mathbf{Lin}^n$ be an arbitrary second-order tensor. Consider the following linear combination

$$\mathbf{A}' = \left(\boldsymbol{f}^i\mathbf{A}\boldsymbol{g}^j\right)\boldsymbol{f}_i\otimes\boldsymbol{g}_j,$$

where the vectors $\boldsymbol{f}^i$ and $\boldsymbol{g}^i$ $(i = 1, 2, \ldots, n)$ form the bases dual to $\mathcal{F}$ and $\mathcal{G}$, respectively. The tensors $\mathbf{A}$ and $\mathbf{A}'$ coincide if and only if

$$\mathbf{A}'\boldsymbol{x} = \mathbf{A}\boldsymbol{x}, \quad \forall\boldsymbol{x}\in\mathbb{E}^n. \tag{1.81}$$

Let $\boldsymbol{x} = x_j\boldsymbol{g}^j$. Then

$$\mathbf{A}'\boldsymbol{x} = \left(\boldsymbol{f}^i\mathbf{A}\boldsymbol{g}^j\right)\boldsymbol{f}_i\otimes\boldsymbol{g}_j\left(x_k\boldsymbol{g}^k\right) = \left(\boldsymbol{f}^i\mathbf{A}\boldsymbol{g}^j\right)\boldsymbol{f}_i x_k\delta_j^k = x_j\left(\boldsymbol{f}^i\mathbf{A}\boldsymbol{g}^j\right)\boldsymbol{f}_i.$$

On the other hand, $\mathbf{A}\boldsymbol{x} = x_j\mathbf{A}\boldsymbol{g}^j$. By virtue of (1.27-1.28) we can represent the vectors $\mathbf{A}\boldsymbol{g}^j$ $(j = 1, 2, \ldots, n)$ with respect to the basis $\mathcal{F}$ by $\mathbf{A}\boldsymbol{g}^j = \left[\boldsymbol{f}^i\cdot\left(\mathbf{A}\boldsymbol{g}^j\right)\right]\boldsymbol{f}_i = \left(\boldsymbol{f}^i\mathbf{A}\boldsymbol{g}^j\right)\boldsymbol{f}_i$ $(j = 1, 2, \ldots, n)$. Hence,

$$\mathbf{A}\boldsymbol{x} = x_j\left(\boldsymbol{f}^i\mathbf{A}\boldsymbol{g}^j\right)\boldsymbol{f}_i.$$

Thus, it is seen that condition (1.81) is satisfied for all $\boldsymbol{x}\in\mathbb{E}^n$. Finally, we show that the tensors $\boldsymbol{f}_i\otimes\boldsymbol{g}_j$ $(i, j = 1, 2, \ldots, n)$ are linearly independent. Otherwise, there would exist scalars α^{ij} $(i, j = 1, 2, \ldots, n)$, not all zero, such that

$$\alpha^{ij}\boldsymbol{f}_i\otimes\boldsymbol{g}_j = \mathbf{0}.$$

The right mapping of $\boldsymbol{g}^k$ $(k = 1, 2, \ldots, n)$ by this tensor equality yields then: $\alpha^{ik}\boldsymbol{f}_i = \boldsymbol{0}$ $(k = 1, 2, \ldots, n)$. This contradicts, however, the fact that the vectors $\boldsymbol{f}_k$ $(k = 1, 2, \ldots, n)$ form a basis and are therefore linearly independent.

For the representation of second-order tensors we will in the following use primarily the bases $\boldsymbol{g}_i \otimes \boldsymbol{g}_j$, $\boldsymbol{g}^i \otimes \boldsymbol{g}^j$, $\boldsymbol{g}^i \otimes \boldsymbol{g}_j$ or $\boldsymbol{g}_i \otimes \boldsymbol{g}^j$ $(i, j = 1, 2, \ldots, n)$. With respect to these bases a tensor $\mathbf{A} \in \mathbf{Lin}^n$ is written as

$$\mathbf{A} = \mathrm{A}^{ij} \boldsymbol{g}_i \otimes \boldsymbol{g}_j = \mathrm{A}_{ij} \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \mathrm{A}^{i}_{\cdot j} \boldsymbol{g}_i \otimes \boldsymbol{g}^j = \mathrm{A}_{i\cdot}^{\ j} \boldsymbol{g}^i \otimes \boldsymbol{g}_j \tag{1.82}$$

with the components (see Exercise 1.20)

$$\mathrm{A}^{ij} = \boldsymbol{g}^i \mathbf{A} \boldsymbol{g}^j, \quad \mathrm{A}_{ij} = \boldsymbol{g}_i \mathbf{A} \boldsymbol{g}_j,$$

$$\mathrm{A}^{i}_{\cdot j} = \boldsymbol{g}^i \mathbf{A} \boldsymbol{g}_j, \quad \mathrm{A}_{i\cdot}^{\ j} = \boldsymbol{g}_i \mathbf{A} \boldsymbol{g}^j, \quad i, j = 1, 2, \ldots, n. \tag{1.83}$$

Note, that the subscript dot indicates the position of the above index. For example, for the components $\mathrm{A}^{i}_{\cdot j}$, i is the first index while for the components $\mathrm{A}_{j\cdot}^{\ i}$, i is the second index.

Of special importance is the so-called identity tensor $\mathbf{I}$. It is defined by

$$\mathbf{I}\boldsymbol{x} = \boldsymbol{x}, \quad \forall \boldsymbol{x} \in \mathbb{E}^n. \tag{1.84}$$

With the aid of (1.25), (1.82) and (1.83) the components of the identity tensor can be expressed by

$$\mathrm{I}^{ij} = \boldsymbol{g}^i \mathbf{I} \boldsymbol{g}^j = \boldsymbol{g}^i \cdot \boldsymbol{g}^j = g^{ij}, \quad \mathrm{I}_{ij} = \boldsymbol{g}_i \mathbf{I} \boldsymbol{g}_j = \boldsymbol{g}_i \cdot \boldsymbol{g}_j = g_{ij},$$

$$\mathrm{I}^{i}_{\cdot j} = \mathrm{I}_{i\cdot}^{\ j} = \mathrm{I}^{i}_{j} = \boldsymbol{g}^i \mathbf{I} \boldsymbol{g}_j = \boldsymbol{g}_i \mathbf{I} \boldsymbol{g}^j = \boldsymbol{g}^i \cdot \boldsymbol{g}_j = \boldsymbol{g}_i \cdot \boldsymbol{g}^j = \delta^i_j, \tag{1.85}$$

where $i, j = 1, 2, \ldots, n$. Thus,

$$\mathbf{I} = g_{ij} \boldsymbol{g}^i \otimes \boldsymbol{g}^j = g^{ij} \boldsymbol{g}_i \otimes \boldsymbol{g}_j = \boldsymbol{g}^i \otimes \boldsymbol{g}_i = \boldsymbol{g}_i \otimes \boldsymbol{g}^i. \tag{1.86}$$

It is seen that the components $(1.85)_{1,2}$ of the identity tensor are given by relation (1.25). In view of (1.30) they characterize metric properties of the Euclidean space and are referred to as metric coefficients. For this reason, the identity tensor is frequently called metric tensor. With respect to an orthonormal basis relation (1.86) reduces to

$$\mathbf{I} = \sum_{i=1}^{n} \boldsymbol{e}_i \otimes \boldsymbol{e}_i. \tag{1.87}$$

1.8 Change of the Basis, Transformation Rules

Now, we are going to clarify how the vector and tensor components transform with the change of the basis. Let $\boldsymbol{x}$ be a vector and $\mathbf{A}$ a second-order tensor. According to (1.27) and (1.82)

$$\boldsymbol{x} = x^i \boldsymbol{g}_i = x_i \boldsymbol{g}^i, \tag{1.88}$$

$$\mathbf{A} = \mathrm{A}^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j = \mathrm{A}_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j = \mathrm{A}^i_{\cdot j}\boldsymbol{g}_i \otimes \boldsymbol{g}^j = \mathrm{A}_{i\cdot}^{\ j}\boldsymbol{g}^i \otimes \boldsymbol{g}_j. \tag{1.89}$$

With the aid of (1.21) and (1.28) we can write

$$x^i = \boldsymbol{x} \cdot \boldsymbol{g}^i = \boldsymbol{x} \cdot \left(g^{ij}\boldsymbol{g}_j\right) = x_j g^{ji}, \quad x_i = \boldsymbol{x} \cdot \boldsymbol{g}_i = \boldsymbol{x} \cdot \left(g_{ij}\boldsymbol{g}^j\right) = x^j g_{ji}, \tag{1.90}$$

where $i = 1, 2, \ldots, n$. Similarly we obtain by virtue of (1.83)

$$\begin{aligned} \mathrm{A}^{ij} = \boldsymbol{g}^i \mathbf{A} \boldsymbol{g}^j &= \boldsymbol{g}^i \mathbf{A} \left(g^{jk}\boldsymbol{g}_k\right) \\ &= \left(g^{il}\boldsymbol{g}_l\right) \mathbf{A} \left(g^{jk}\boldsymbol{g}_k\right) = \mathrm{A}^i_{\cdot k} g^{kj} = g^{il}\mathrm{A}_{lk}g^{kj}, \end{aligned} \tag{1.91}$$

$$\begin{aligned} \mathrm{A}_{ij} = \boldsymbol{g}_i \mathbf{A} \boldsymbol{g}_j &= \boldsymbol{g}_i \mathbf{A} \left(g_{jk}\boldsymbol{g}^k\right) \\ &= \left(g_{il}\boldsymbol{g}^l\right) \mathbf{A} \left(g_{jk}\boldsymbol{g}^k\right) = \mathrm{A}_{i\cdot}^{\ k} g_{kj} = g_{il}\mathrm{A}^{lk}g_{kj}, \end{aligned} \tag{1.92}$$

where $i, j = 1, 2, \ldots, n$. The transformation rules (1.90-1.92) hold not only for dual bases. Indeed, let $\boldsymbol{g}_i$ and $\bar{\boldsymbol{g}}_i$ $(i = 1, 2, \ldots, n)$ be two arbitrary bases in $\mathbb{E}^n$, so that

$$\boldsymbol{x} = x^i \boldsymbol{g}_i = \bar{x}^i \bar{\boldsymbol{g}}_i, \tag{1.93}$$

$$\mathbf{A} = \mathrm{A}^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j = \bar{\mathrm{A}}^{ij}\bar{\boldsymbol{g}}_i \otimes \bar{\boldsymbol{g}}_j. \tag{1.94}$$

By means of the relations

$$\boldsymbol{g}_i = a_i^j \bar{\boldsymbol{g}}_j, \quad i = 1, 2, \ldots, n \tag{1.95}$$

one thus obtains

$$\boldsymbol{x} = x^i \boldsymbol{g}_i = x^i a_i^j \bar{\boldsymbol{g}}_j \quad \Rightarrow \quad \bar{x}^j = x^i a_i^j, \quad j = 1, 2, \ldots, n, \tag{1.96}$$

$$\begin{aligned} \mathbf{A} = \mathrm{A}^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j = \ & \mathrm{A}^{ij}\left(a_i^k \bar{\boldsymbol{g}}_k\right) \otimes \left(a_j^l \bar{\boldsymbol{g}}_l\right) = \mathrm{A}^{ij} a_i^k a_j^l \bar{\boldsymbol{g}}_k \otimes \bar{\boldsymbol{g}}_l \\ & \Rightarrow \ \bar{\mathrm{A}}^{kl} = \mathrm{A}^{ij} a_i^k a_j^l, \quad k, l = 1, 2, \ldots, n. \end{aligned} \tag{1.97}$$

1.9 Special Operations with Second-Order Tensors

In Sect. 1.6 we have seen that the set $\mathbf{Lin}^n$ represents a finite-dimensional vector space. Its elements are second-order tensors that can be treated as vectors in $\mathbb{E}^{n^2}$ with all the operations specific for vectors such as summation, multiplication by a scalar or a scalar product (the latter one will be defined for second-order tensors in Sect. 1.10). However, in contrast to conventional vectors in the Euclidean space, for second-order tensors one can additionally define some special operations as for example composition, transposition or inversion.

Composition (simple contraction). Let $\mathbf{A}, \mathbf{B} \in \mathbf{L}\mathrm{in}^n$ be two second-order tensors. The tensor $\mathbf{C} = \mathbf{AB}$ is called composition of $\mathbf{A}$ and $\mathbf{B}$ if

$$\mathbf{C}\boldsymbol{x} = \mathbf{A}(\mathbf{B}\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{E}^n. \tag{1.98}$$

For the left mapping (1.73) one can write

$$\boldsymbol{y}(\mathbf{AB}) = (\boldsymbol{y}\mathbf{A})\mathbf{B}, \quad \forall \boldsymbol{y} \in \mathbb{E}^n. \tag{1.99}$$

In order to prove the last relation we use again (1.73) and (1.98):

$$\begin{aligned}\boldsymbol{y}(\mathbf{AB})\boldsymbol{x} &= \boldsymbol{y} \cdot [(\mathbf{AB})\boldsymbol{x}] = \boldsymbol{y} \cdot [\mathbf{A}(\mathbf{B}\boldsymbol{x})] \\ &= (\boldsymbol{y}\mathbf{A}) \cdot (\mathbf{B}\boldsymbol{x}) = [(\boldsymbol{y}\mathbf{A})\mathbf{B}] \cdot \boldsymbol{x}, \quad \forall \boldsymbol{x} \in \mathbb{E}^n.\end{aligned}$$

The composition of tensors (1.98) is generally not commutative so that $\mathbf{AB} \neq \mathbf{BA}$. Two tensors $\mathbf{A}$ and $\mathbf{B}$ are called commutative if on the contrary $\mathbf{AB} = \mathbf{BA}$. Besides, the composition of tensors is characterized by the following properties (see Exercise 1.24):

$$\mathbf{A0} = \mathbf{0A} = \mathbf{0}, \quad \mathbf{AI} = \mathbf{IA} = \mathbf{A}, \tag{1.100}$$

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}, \quad (\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}, \tag{1.101}$$

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}. \tag{1.102}$$

For example, the distributive rule $(1.101)_1$ can be proved as follows

$$\begin{aligned}[\mathbf{A}(\mathbf{B} + \mathbf{C})]\boldsymbol{x} &= \mathbf{A}[(\mathbf{B} + \mathbf{C})\boldsymbol{x}] = \mathbf{A}(\mathbf{B}\boldsymbol{x} + \mathbf{C}\boldsymbol{x}) = \mathbf{A}(\mathbf{B}\boldsymbol{x}) + \mathbf{A}(\mathbf{C}\boldsymbol{x}) \\ &= (\mathbf{AB})\boldsymbol{x} + (\mathbf{AC})\boldsymbol{x} = (\mathbf{AB} + \mathbf{AC})\boldsymbol{x}, \quad \forall \boldsymbol{x} \in \mathbb{E}^n.\end{aligned}$$

For the tensor product (1.75) the composition (1.98) yields

$$(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a} \otimes \boldsymbol{d}, \quad \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{E}^n. \tag{1.103}$$

Indeed, by virtue of (1.75), (1.77) and (1.98)

$$\begin{aligned}(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d})\boldsymbol{x} &= (\boldsymbol{a} \otimes \boldsymbol{b})[(\boldsymbol{c} \otimes \boldsymbol{d})\boldsymbol{x}] = (\boldsymbol{d} \cdot \boldsymbol{x})(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} \\ &= (\boldsymbol{d} \cdot \boldsymbol{x})(\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a} = (\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{a} \otimes \boldsymbol{d})\boldsymbol{x} \\ &= [(\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a} \otimes \boldsymbol{d}]\boldsymbol{x}, \quad \forall \boldsymbol{x} \in \mathbb{E}^n.\end{aligned}$$

Thus, we can write

$$\begin{aligned}\mathbf{AB} &= \mathrm{A}^{ik}\mathrm{B}_{k\cdot}^{\cdot j}\boldsymbol{g}_i \otimes \boldsymbol{g}_j = \mathrm{A}_{ik}\mathrm{B}^{kj}\boldsymbol{g}^i \otimes \boldsymbol{g}_j \\ &= \mathrm{A}_{\cdot k}^{i\cdot}\mathrm{B}_{\cdot j}^{k\cdot}\boldsymbol{g}_i \otimes \boldsymbol{g}^j = \mathrm{A}_{i\cdot}^{\cdot k}\mathrm{B}_{kj}\boldsymbol{g}^i \otimes \boldsymbol{g}^j,\end{aligned} \tag{1.104}$$

where $\mathbf{A}$ and $\mathbf{B}$ are given in the form (1.82).

Powers, polynomials and functions of second-order tensors. On the basis of the composition (1.98) one defines by

$$\mathbf{A}^m = \underbrace{\mathbf{AA}\ldots\mathbf{A}}_{m \text{ times}}, \quad m = 1, 2, 3\ldots, \quad \mathbf{A}^0 = \mathbf{I} \tag{1.105}$$

powers (monomials) of second-order tensors characterized by the following evident properties

$$\mathbf{A}^k\mathbf{A}^l = \mathbf{A}^{k+l}, \quad \left(\mathbf{A}^k\right)^l = \mathbf{A}^{kl}, \tag{1.106}$$

$$(\alpha\mathbf{A})^k = \alpha^k\mathbf{A}^k, \quad k, l = 0, 1, 2\ldots \tag{1.107}$$

With the aid of the tensor powers a polynomial of $\mathbf{A}$ can be defined by

$$g(\mathbf{A}) = a_0\mathbf{I} + a_1\mathbf{A} + a_2\mathbf{A}^2 + \ldots + a_m\mathbf{A}^m = \sum_{k=0}^{m} a_k\mathbf{A}^k. \tag{1.108}$$

$g(\mathbf{A})$: $\mathbf{Lin}^n \mapsto \mathbf{Lin}^n$ represents a tensor function mapping one second-order tensor into another one within $\mathbf{Lin}^n$. By this means one can define various tensor functions. Of special interest is the exponential one

$$\exp(\mathbf{A}) = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} \tag{1.109}$$

given by the infinite power series.

Transposition. The transposed tensor $\mathbf{A}^{\mathrm{T}}$ is defined by:

$$\mathbf{A}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}\mathbf{A}, \quad \forall\boldsymbol{x} \in \mathbb{E}^n, \tag{1.110}$$

so that one can also write

$$\mathbf{A}\boldsymbol{y} = \boldsymbol{y}\mathbf{A}^{\mathrm{T}}, \quad \boldsymbol{x}\mathbf{A}\boldsymbol{y} = \boldsymbol{y}\mathbf{A}^{\mathrm{T}}\boldsymbol{x}, \quad \forall\boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}^n. \tag{1.111}$$

Indeed,

$$\boldsymbol{x} \cdot (\mathbf{A}\boldsymbol{y}) = (\boldsymbol{x}\mathbf{A}) \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \left(\mathbf{A}^{\mathrm{T}}\boldsymbol{x}\right) = \boldsymbol{y}\mathbf{A}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x} \cdot \left(\boldsymbol{y}\mathbf{A}^{\mathrm{T}}\right), \ \forall\boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}^n.$$

Consequently,

$$\left(\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{A}. \tag{1.112}$$

Transposition represents a linear operation over a second-order tensor since

$$(\mathbf{A} + \mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}} \tag{1.113}$$

and

$$(\alpha\mathbf{A})^{\mathrm{T}} = \alpha\mathbf{A}^{\mathrm{T}}, \quad \forall\alpha \in \mathbb{R}. \tag{1.114}$$

The composition of second-order tensors is transposed by

$$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}. \tag{1.115}$$

Indeed, in view of (1.99) and (1.110)

$$(\mathbf{AB})^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}(\mathbf{AB}) = (\boldsymbol{x}\mathbf{A})\mathbf{B} = \mathbf{B}^{\mathrm{T}}(\boldsymbol{x}\mathbf{A}) = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\boldsymbol{x}, \quad \forall\boldsymbol{x} \in \mathbb{E}^n.$$

For the tensor product of two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}^n$ we further obtain by use of (1.75) and (1.80)

$$(\boldsymbol{a} \otimes \boldsymbol{b})^{\mathrm{T}} = \boldsymbol{b} \otimes \boldsymbol{a}. \tag{1.116}$$

This ensures the existence and uniqueness of the transposed tensor. Indeed, every tensor $\mathbf{A}$ in $\mathbf{Lin}^n$ can be represented with respect to the tensor product of the basis vectors in $\mathbb{E}^n$ in the form (1.82). Hence, considering (1.116) we have

$$\mathbf{A}^{\mathrm{T}} = \mathrm{A}^{ij}\boldsymbol{g}_j \otimes \boldsymbol{g}_i = \mathrm{A}_{ij}\boldsymbol{g}^j \otimes \boldsymbol{g}^i = \mathrm{A}^{i}_{\cdot j}\boldsymbol{g}^j \otimes \boldsymbol{g}_i = \mathrm{A}_{i\cdot}^{\ j}\boldsymbol{g}_j \otimes \boldsymbol{g}^i, \tag{1.117}$$

or

$$\mathbf{A}^{\mathrm{T}} = \mathrm{A}^{ji}\boldsymbol{g}_i \otimes \boldsymbol{g}_j = \mathrm{A}_{ji}\boldsymbol{g}^i \otimes \boldsymbol{g}^j = \mathrm{A}^{j}_{\cdot i}\boldsymbol{g}^i \otimes \boldsymbol{g}_j = \mathrm{A}_{j\cdot}^{\ i}\boldsymbol{g}_i \otimes \boldsymbol{g}^j. \tag{1.118}$$

Comparing the latter result with the original representation (1.82) one observes that the components of the transposed tensor can be expressed by

$$\left(\mathbf{A}^{\mathrm{T}}\right)_{ij} = \mathrm{A}_{ji}, \quad \left(\mathbf{A}^{\mathrm{T}}\right)^{ij} = \mathrm{A}^{ji}, \tag{1.119}$$

$$\left(\mathbf{A}^{\mathrm{T}}\right)_{i\cdot}^{\ j} = \mathrm{A}^{j}_{\cdot i} = g^{jk}\mathrm{A}_{k\cdot}^{\ l}g_{li}, \quad \left(\mathbf{A}^{\mathrm{T}}\right)^{i}_{\cdot j} = \mathrm{A}_{j\cdot}^{\ i} = g_{jk}\mathrm{A}^{k}_{\cdot l}g^{li}. \tag{1.120}$$

For example, the last relation results from (1.83) and (1.111) within the following steps

$$\left(\mathbf{A}^{\mathrm{T}}\right)^{i}_{\cdot j} = \boldsymbol{g}^i\mathbf{A}^{\mathrm{T}}\boldsymbol{g}_j = \boldsymbol{g}_j\mathbf{A}\boldsymbol{g}^i = \boldsymbol{g}_j\left(\mathrm{A}^{k}_{\cdot l}\boldsymbol{g}_k \otimes \boldsymbol{g}^l\right)\boldsymbol{g}^i = g_{jk}\mathrm{A}^{k}_{\cdot l}g^{li}.$$

According to (1.119) the homogeneous (covariant or contravariant) components of the transposed tensor can simply be obtained by reflecting the matrix of the original components from the main diagonal. It does not, however, hold for the mixed components (1.120).

The transposition operation (1.110) gives rise to the definition of symmetric $\mathbf{M}^{\mathrm{T}} = \mathbf{M}$ and skew-symmetric second-order tensors $\mathbf{W}^{\mathrm{T}} = -\mathbf{W}$.

Obviously, the identity tensor is symmetric

$$\mathbf{I}^{\mathrm{T}} = \mathbf{I}. \tag{1.121}$$

Indeed,

$$\boldsymbol{x}\mathbf{I}\boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x} = \boldsymbol{y}\mathbf{I}\boldsymbol{x} = \boldsymbol{x}\mathbf{I}^{\mathrm{T}}\boldsymbol{y}, \quad \forall\boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}^n.$$

Inversion. Let

$$\boldsymbol{y} = \mathbf{A}\boldsymbol{x}. \tag{1.122}$$

A tensor $\mathbf{A} \in \mathbf{Lin}^n$ is referred to as invertible if there exists a tensor $\mathbf{A}^{-1} \in \mathbf{Lin}^n$ satisfying the condition

$$\boldsymbol{x} = \mathbf{A}^{-1}\boldsymbol{y}, \quad \forall \boldsymbol{x} \in \mathbb{E}^n. \tag{1.123}$$

The tensor $\mathbf{A}^{-1}$ is called inverse of $\mathbf{A}$. The set of all invertible tensors $\mathbf{Inv}^n = \left\{\mathbf{A} \in \mathbf{Lin}^n : \exists \mathbf{A}^{-1}\right\}$ forms a subset of all second-order tensors $\mathbf{Lin}^n$.

Inserting (1.122) into (1.123) yields

$$\boldsymbol{x} = \mathbf{A}^{-1}\boldsymbol{y} = \mathbf{A}^{-1}(\mathbf{A}\boldsymbol{x}) = \left(\mathbf{A}^{-1}\mathbf{A}\right)\boldsymbol{x}, \quad \forall \boldsymbol{x} \in \mathbb{E}^n$$

and consequently

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}. \tag{1.124}$$

Theorem 1.8. *A tensor* $\mathbf{A}$ *is invertible if and only if* $\mathbf{A}\boldsymbol{x} = \boldsymbol{0}$ *implies that* $\boldsymbol{x} = \boldsymbol{0}$.

Proof. First we prove the sufficiency. To this end, we map the vector equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{0}$ by $\mathbf{A}^{-1}$. According to (1.124) it yields: $\boldsymbol{0} = \mathbf{A}^{-1}\mathbf{A}\boldsymbol{x} = \mathbf{I}\boldsymbol{x} = \boldsymbol{x}$. To prove the necessity we consider a basis $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ in $\mathbb{E}^n$. It can be shown that the vectors $\boldsymbol{h}_i = \mathbf{A}\boldsymbol{g}_i \,(i = 1, 2, \ldots, n)$ form likewise a basis of $\mathbb{E}^n$. Conversely, let these vectors be linearly dependent so that $a^i\boldsymbol{h}_i = \boldsymbol{0}$, where not all scalars $a^i \,(i = 1, 2, \ldots, n)$ are zero. Then, $\boldsymbol{0} = a^i\boldsymbol{h}_i = a^i\mathbf{A}\boldsymbol{g}_i = \mathbf{A}\boldsymbol{a}$, where $\boldsymbol{a} = a^i\boldsymbol{g}_i \neq \boldsymbol{0}$, which contradicts the assumption of the theorem. Now, consider the tensor $\mathbf{A}' = \boldsymbol{g}_i \otimes \boldsymbol{h}^i$, where the vectors $\boldsymbol{h}^i$ are dual to $\boldsymbol{h}_i \,(i = 1, 2, \ldots, n)$. One can show that this tensor is inverse to $\mathbf{A}$, such that $\mathbf{A}' = \mathbf{A}^{-1}$. Indeed, let $\boldsymbol{x} = x^i\boldsymbol{g}_i$ be an arbitrary vector in $\mathbb{E}^n$. Then, $\boldsymbol{y} = \mathbf{A}\boldsymbol{x} = x^i\mathbf{A}\boldsymbol{g}_i = x^i\boldsymbol{h}_i$ and therefore $\mathbf{A}'\boldsymbol{y} = \boldsymbol{g}_i \otimes \boldsymbol{h}^i\left(x^j\boldsymbol{h}_j\right) = \boldsymbol{g}_i x^j \delta^i_j = x^i\boldsymbol{g}_i = \boldsymbol{x}$.

Conversely, it can be shown that an invertible tensor $\mathbf{A}$ is inverse to $\mathbf{A}^{-1}$ and consequently

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}. \tag{1.125}$$

For the proof we again consider the bases $\boldsymbol{g}_i$ and $\mathbf{A}\boldsymbol{g}_i \,(i = 1, 2, \ldots, n)$. Let $\boldsymbol{y} = y^i\mathbf{A}\boldsymbol{g}_i$ be an arbitrary vector in $\mathbb{E}^n$. Let further $\boldsymbol{x} = \mathbf{A}^{-1}\boldsymbol{y} = y^i\boldsymbol{g}_i$ in view of (1.124). Then, $\mathbf{A}\boldsymbol{x} = y^i\mathbf{A}\boldsymbol{g}_i = \boldsymbol{y}$ which implies that the tensor $\mathbf{A}$ is inverse to $\mathbf{A}^{-1}$.

Relation (1.125) implies the uniqueness of the inverse. Indeed, if $\mathbf{A}^{-1}$ and $\widetilde{\mathbf{A}^{-1}}$ are two distinct tensors both inverse to $\mathbf{A}$ then there exists at least one vector $\boldsymbol{y} \in \mathbb{E}^n$ such that $\mathbf{A}^{-1}\boldsymbol{y} \neq \widetilde{\mathbf{A}^{-1}}\boldsymbol{y}$. Mapping both sides of this vector inequality by $\mathbf{A}$ and taking (1.125) into account we immediately come to the contradiction.

By means of (1.115), (1.121) and (1.125) we can write (see Exercise 1.37)

$$\left(\mathbf{A}^{-1}\right)^{\mathrm{T}} = \left(\mathbf{A}^{\mathrm{T}}\right)^{-1} = \mathbf{A}^{-\mathrm{T}}. \tag{1.126}$$

The composition of two arbitrary invertible tensors **A** and **B** is inverted by

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}. \tag{1.127}$$

Indeed, let

$$\boldsymbol{y} = \mathbf{AB}\boldsymbol{x}.$$

Mapping both sides of this vector identity by $\mathbf{A}^{-1}$ and then by $\mathbf{B}^{-1}$, we obtain with the aid of (1.124)

$$\boldsymbol{x} = \mathbf{B}^{-1}\mathbf{A}^{-1}\boldsymbol{y}, \quad \forall \boldsymbol{x} \in \mathbb{E}^n.$$

On the basis of transposition and inversion one defines the so-called orthogonal tensors. They do not change after consecutive transposition and inversion and form the following subset of $\mathbf{Lin}^n$:

$$\mathbf{O}\mathrm{rth}^n = \left\{\mathbf{Q} \in \mathbf{Lin}^n : \mathbf{Q} = \mathbf{Q}^{-\mathrm{T}}\right\}. \tag{1.128}$$

For orthogonal tensors we can write in view of (1.124) and (1.125)

$$\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}, \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n. \tag{1.129}$$

For example, one can show that the rotation tensor (1.71) is orthogonal. To this end, we complete the vector $\boldsymbol{e}$ defining the rotation axis (Fig. 1.2) to an orthonormal basis $\{\boldsymbol{e}, \boldsymbol{q}, \boldsymbol{p}\}$ such that $\boldsymbol{e} = \boldsymbol{q} \times \boldsymbol{p}$. Then, using the vector identity (see Exercise 1.15)

$$\boldsymbol{p}\left(\boldsymbol{q} \cdot \boldsymbol{x}\right) - \boldsymbol{q}\left(\boldsymbol{p} \cdot \boldsymbol{x}\right) = \left(\boldsymbol{q} \times \boldsymbol{p}\right) \times \boldsymbol{x}, \quad \forall \boldsymbol{x} \in \mathbb{E}^3 \tag{1.130}$$

we can write

$$\hat{\boldsymbol{e}} = \boldsymbol{p} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{p}. \tag{1.131}$$

The rotation tensor (1.71) takes thus the form

$$\mathbf{R} = \cos\omega\mathbf{I} + \sin\omega\left(\boldsymbol{p} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{p}\right) + \left(1 - \cos\omega\right)\left(\boldsymbol{e} \otimes \boldsymbol{e}\right). \tag{1.132}$$

Hence,

$$\begin{aligned}
\mathbf{R}\mathbf{R}^{\mathrm{T}} &= \left[\cos\omega\mathbf{I} + \sin\omega\left(\boldsymbol{p} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{p}\right) + \left(1 - \cos\omega\right)\left(\boldsymbol{e} \otimes \boldsymbol{e}\right)\right] \\
&\quad \left[\cos\omega\mathbf{I} - \sin\omega\left(\boldsymbol{p} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{p}\right) + \left(1 - \cos\omega\right)\left(\boldsymbol{e} \otimes \boldsymbol{e}\right)\right] \\
&= \cos^2\omega\mathbf{I} + \sin^2\omega\left(\boldsymbol{e} \otimes \boldsymbol{e}\right) + \sin^2\omega\left(\boldsymbol{p} \otimes \boldsymbol{p} + \boldsymbol{q} \otimes \boldsymbol{q}\right) = \mathbf{I}.
\end{aligned}$$

It is interesting that the exponential function (1.109) of a skew-symmetric tensors represents an orthogonal tensor. Indeed, keeping in mind that a skew-symmetric tensor $\mathbf{W}$ commutes with its transposed counterpart $\mathbf{W}^{\mathrm{T}} = -\mathbf{W}$ and using the identities $\exp(\mathbf{A}+\mathbf{B}) = \exp(\mathbf{A})\exp(\mathbf{B})$ for commutative tensors (Exercise 1.27) and $\left(\mathbf{A}^k\right)^{\mathrm{T}} = \left(\mathbf{A}^{\mathrm{T}}\right)^k$ for integer k (Exercise 1.35) we can write

$$\begin{aligned}\mathbf{I} &= \exp(\mathbf{0}) = \exp(\mathbf{W}-\mathbf{W}) = \exp\left(\mathbf{W}+\mathbf{W}^{\mathrm{T}}\right)\\ &= \exp(\mathbf{W})\exp\left(\mathbf{W}^{\mathrm{T}}\right) = \exp(\mathbf{W})\left[\exp(\mathbf{W})\right]^{\mathrm{T}}, \quad \forall\mathbf{W}\in\mathbf{S}\mathrm{kew}^n. \end{aligned} \qquad (1.133)$$

1.10 Scalar Product of Second-Order Tensors

Consider two second-order tensors $\boldsymbol{a}\otimes\boldsymbol{b}$ and $\boldsymbol{c}\otimes\boldsymbol{d}$ given in terms of the tensor product (1.75). Their scalar product can be defined in the following manner:

$$(\boldsymbol{a}\otimes\boldsymbol{b}):(\boldsymbol{c}\otimes\boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}), \quad \boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}\in\mathbb{E}^n. \qquad (1.134)$$

It leads to the following identity (Exercise 1.39):

$$\boldsymbol{c}\otimes\boldsymbol{d}:\mathbf{A} = \boldsymbol{c}\mathbf{A}\boldsymbol{d} = \boldsymbol{d}\mathbf{A}^{\mathrm{T}}\boldsymbol{c}. \qquad (1.135)$$

For two arbitrary tensors $\mathbf{A}$ and $\mathbf{B}$ given in the form (1.82) we thus obtain

$$\mathbf{A}:\mathbf{B} = \mathrm{A}_{ij}\mathrm{B}^{ij} = \mathrm{A}^{ij}\mathrm{B}_{ij} = \mathrm{A}^{i}_{\cdot j}\mathrm{B}_{i\cdot}^{\ j} = \mathrm{A}_{i\cdot}^{\ j}\mathrm{B}^{i}_{\cdot j}. \qquad (1.136)$$

Similar to vectors the scalar product of tensors is a real function characterized by the following properties (see Exercise 1.40)

D. (D.1) $\mathbf{A}:\mathbf{B} = \mathbf{B}:\mathbf{A}$ (commutative rule),

(D.2) $\mathbf{A}:(\mathbf{B}+\mathbf{C}) = \mathbf{A}:\mathbf{B}+\mathbf{A}:\mathbf{C}$ (distributive rule),

(D.3) $\alpha(\mathbf{A}:\mathbf{B}) = (\alpha\mathbf{A}):\mathbf{B} = \mathbf{A}:(\alpha\mathbf{B})$ (associative rule for multiplication by a scalar), $\forall\mathbf{A},\mathbf{B}\in\mathbf{L}\mathrm{in}^n$, $\forall\alpha\in\mathbb{R}$,

(D.4) $\mathbf{A}:\mathbf{A}\geq 0 \ \ \forall\mathbf{A}\in\mathbf{L}\mathrm{in}^n$, $\mathbf{A}:\mathbf{A}=0$ if and only if $\mathbf{A}=\mathbf{0}$.

We prove for example the property (D.4). To this end, we represent the tensor $\mathbf{A}$ with respect to an orthonormal basis (1.8) in $\mathbb{E}^n$ as: $\mathbf{A} = \mathrm{A}^{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j = \mathrm{A}_{ij}\boldsymbol{e}^i\otimes\boldsymbol{e}^j$, where $\mathrm{A}^{ij} = \mathrm{A}_{ij}$, $(i,j=1,2,\ldots,n)$, since $\boldsymbol{e}^i = \boldsymbol{e}_i$ $(i=1,2,\ldots,n)$. Keeping (1.136) in mind we then obtain:

$$\mathbf{A}:\mathbf{A} = \mathrm{A}^{ij}\mathrm{A}_{ij} = \sum_{i,j=1}^{n}\mathrm{A}^{ij}\mathrm{A}^{ij} = \sum_{i,j=1}^{n}\left(\mathrm{A}^{ij}\right)^2 \geq 0.$$

Using this important property one can define the norm of a second-order tensor by:

$$\|\mathbf{A}\| = (\mathbf{A} : \mathbf{A})^{1/2}, \quad \mathbf{A} \in \mathbf{L}\mathrm{in}^n. \tag{1.137}$$

For the scalar product of tensors one of which is given by a composition we can write

$$\mathbf{A} : (\mathbf{BC}) = \left(\mathbf{B}^{\mathrm{T}}\mathbf{A}\right) : \mathbf{C} = \left(\mathbf{A}\mathbf{C}^{\mathrm{T}}\right) : \mathbf{B}. \tag{1.138}$$

We prove this identity first for the tensor products:

$$\begin{aligned}(\boldsymbol{a}\otimes\boldsymbol{b}) : [(\boldsymbol{c}\otimes\boldsymbol{d})(\boldsymbol{e}\otimes\boldsymbol{f})] &= (\boldsymbol{d}\cdot\boldsymbol{e})[(\boldsymbol{a}\otimes\boldsymbol{b}) : (\boldsymbol{c}\otimes\boldsymbol{f})] \\ &= (\boldsymbol{d}\cdot\boldsymbol{e})(\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{f}),\end{aligned}$$

$$\begin{aligned}\left[(\boldsymbol{c}\otimes\boldsymbol{d})^{\mathrm{T}}(\boldsymbol{a}\otimes\boldsymbol{b})\right] : (\boldsymbol{e}\otimes\boldsymbol{f}) &= [(\boldsymbol{d}\otimes\boldsymbol{c})(\boldsymbol{a}\otimes\boldsymbol{b})] : (\boldsymbol{e}\otimes\boldsymbol{f}) \\ &= (\boldsymbol{a}\cdot\boldsymbol{c})[(\boldsymbol{d}\otimes\boldsymbol{b}) : (\boldsymbol{e}\otimes\boldsymbol{f})] \\ &= (\boldsymbol{d}\cdot\boldsymbol{e})(\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{f}),\end{aligned}$$

$$\begin{aligned}\left[(\boldsymbol{a}\otimes\boldsymbol{b})(\boldsymbol{e}\otimes\boldsymbol{f})^{\mathrm{T}}\right] : (\boldsymbol{c}\otimes\boldsymbol{d}) &= [(\boldsymbol{a}\otimes\boldsymbol{b})(\boldsymbol{f}\otimes\boldsymbol{e})] : (\boldsymbol{c}\otimes\boldsymbol{d}) \\ &= (\boldsymbol{b}\cdot\boldsymbol{f})[(\boldsymbol{a}\otimes\boldsymbol{e}) : (\boldsymbol{c}\otimes\boldsymbol{d})] \\ &= (\boldsymbol{d}\cdot\boldsymbol{e})(\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{f}).\end{aligned}$$

For three arbitrary tensors **A**, **B** and **C** given in the form (1.82) we can write in view of (1.120) and (1.136)

$$\begin{aligned}\mathrm{A}^{i}_{\cdot j}\left(\mathrm{B}_{i\cdot}^{\ k}\mathrm{C}_{k\cdot}^{\ j}\right) &= \left(\mathrm{B}_{i\cdot}^{\ k}\mathrm{A}^{i}_{\cdot j}\right)\mathrm{C}_{k\cdot}^{\ j} = \left[\left(\mathbf{B}^{\mathrm{T}}\right)^{k}_{\cdot i}\mathrm{A}^{i}_{\cdot j}\right]\mathrm{C}_{k\cdot}^{\ j}, \\ \mathrm{A}^{i}_{\cdot j}\left(\mathrm{B}_{i\cdot}^{\ k}\mathrm{C}_{k\cdot}^{\ j}\right) &= \left(\mathrm{A}^{i}_{\cdot j}\mathrm{C}_{k\cdot}^{\ j}\right)\mathrm{B}_{i\cdot}^{\ k} = \left[\mathrm{A}^{i}_{\cdot j}\left(\mathbf{C}^{\mathrm{T}}\right)^{j}_{\cdot k}\right]\mathrm{B}_{i\cdot}^{\ k}.\end{aligned} \tag{1.139}$$

Similarly we can prove that

$$\mathbf{A} : \mathbf{B} = \mathbf{A}^{\mathrm{T}} : \mathbf{B}^{\mathrm{T}}. \tag{1.140}$$

On the basis of the scalar product one defines the trace of second-order tensors by:

$$\mathrm{tr}\mathbf{A} = \mathbf{A} : \mathbf{I}. \tag{1.141}$$

For the tensor product (1.75) the trace (1.141) yields in view of (1.135)

$$\mathrm{tr}\,(\boldsymbol{a}\otimes\boldsymbol{b}) = \boldsymbol{a}\cdot\boldsymbol{b}. \tag{1.142}$$

With the aid of the relation (1.138) we further write

$$\mathrm{tr}\,(\mathbf{AB}) = \mathbf{A} : \mathbf{B}^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} : \mathbf{B}. \tag{1.143}$$

In view of (D.1) this also implies that

$$\mathrm{tr}\,(\mathbf{AB}) = \mathrm{tr}\,(\mathbf{BA}). \tag{1.144}$$

1.11 Decompositions of Second-Order Tensors

Additive decomposition into a symmetric and a skew-symmetric part. Every second-order tensor can be decomposed additively into a symmetric and a skew-symmetric part by

$$\mathbf{A} = \mathrm{sym}\mathbf{A} + \mathrm{skew}\mathbf{A}, \tag{1.145}$$

where

$$\mathrm{sym}\mathbf{A} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right), \quad \mathrm{skew}\mathbf{A} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathrm{T}}\right). \tag{1.146}$$

Symmetric and skew-symmetric tensors form subsets of $\mathbf{Lin}^n$ defined by

$$\mathbf{Sym}^n = \left\{\mathbf{M} \in \mathbf{Lin}^n : \mathbf{M} = \mathbf{M}^{\mathrm{T}}\right\}, \tag{1.147}$$

$$\mathbf{Skew}^n = \left\{\mathbf{W} \in \mathbf{Lin}^n : \mathbf{W} = -\mathbf{W}^{\mathrm{T}}\right\}. \tag{1.148}$$

One can easily show that these subsets represent vector spaces and can be referred to as subspaces of $\mathbf{Lin}^n$. Indeed, the axioms (A.1-A.4) and (B.1-B.4) including operations with the zero tensor are valid both for symmetric and skew-symmetric tensors. The zero tensor is the only linear mapping that is both symmetric and skew-symmetric such that $\mathbf{Sym}^n \cap \mathbf{Skew}^n = \mathbf{0}$.

For every symmetric tensor $\mathbf{M} = \mathrm{M}^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j$ it follows from (1.119) that $\mathrm{M}^{ij} = \mathrm{M}^{ji}$ $(i \neq j,\ i,j = 1,2,\ldots,n)$. Thus, we can write

$$\mathbf{M} = \sum_{i=1}^{n} \mathrm{M}^{ii}\boldsymbol{g}_i \otimes \boldsymbol{g}_i + \sum_{\substack{i,j=1\\ i>j}}^{n} \mathrm{M}^{ij}\left(\boldsymbol{g}_i \otimes \boldsymbol{g}_j + \boldsymbol{g}_j \otimes \boldsymbol{g}_i\right), \quad \mathbf{M} \in \mathbf{Sym}^n. \tag{1.149}$$

Similarly we can write for a skew-symmetric tensor

$$\mathbf{W} = \sum_{\substack{i,j=1\\ i>j}}^{n} \mathrm{W}^{ij}\left(\boldsymbol{g}_i \otimes \boldsymbol{g}_j - \boldsymbol{g}_j \otimes \boldsymbol{g}_i\right), \quad \mathbf{W} \in \mathbf{Skew}^n \tag{1.150}$$

taking into account that $\mathrm{W}^{ii} = 0$ and $\mathrm{W}^{ij} = -\mathrm{W}^{ji}$ $(i \neq j,\ i,j = 1,2,\ldots,n)$. Therefore, the basis of $\mathbf{Sym}^n$ is formed by n tensors $\boldsymbol{g}_i \otimes \boldsymbol{g}_i$ and $\frac{1}{2}n(n-1)$ tensors $\boldsymbol{g}_i \otimes \boldsymbol{g}_j + \boldsymbol{g}_j \otimes \boldsymbol{g}_i$, while the basis of $\mathbf{Skew}^n$ consists of $\frac{1}{2}n(n-1)$ tensors $\boldsymbol{g}_i \otimes \boldsymbol{g}_j - \boldsymbol{g}_j \otimes \boldsymbol{g}_i$, where $i > j = 1,2,\ldots,n$. Thus, the dimensions of $\mathbf{Sym}^n$ and $\mathbf{Skew}^n$ are $\frac{1}{2}n(n+1)$ and $\frac{1}{2}n(n-1)$, respectively. It follows from (1.145) that any basis of $\mathbf{Skew}^n$ complements any basis of $\mathbf{Sym}^n$ to a basis of $\mathbf{Lin}^n$.

Obviously, symmetric and skew-symmetric tensors are mutually orthogonal such that (see Exercise 1.43)

$$\mathbf{M} : \mathbf{W} = 0, \quad \forall \mathbf{M} \in \mathbf{Sym}^n,\ \forall \mathbf{W} \in \mathbf{Skew}^n. \tag{1.151}$$

Spaces characterized by this property are called orthogonal.

Additive decomposition into a spherical and a deviatoric part. For every second-order tensor $\mathbf{A}$ we can write

$$\mathbf{A} = \mathrm{sph}\mathbf{A} + \mathrm{dev}\mathbf{A}, \tag{1.152}$$

where

$$\mathrm{sph}\mathbf{A} = \frac{1}{n}\mathrm{tr}\left(\mathbf{A}\right)\mathbf{I}, \quad \mathrm{dev}\mathbf{A} = \mathbf{A} - \frac{1}{n}\mathrm{tr}\left(\mathbf{A}\right)\mathbf{I} \tag{1.153}$$

denote its spherical and deviatoric part, respectively. Thus, every spherical tensor $\mathbf{S}$ can be represented by $\mathbf{S} = \alpha\mathbf{I}$, where α is a scalar number. In turn, every deviatoric tensor $\mathbf{D}$ is characterized by the condition $\mathrm{tr}\mathbf{D} = 0$. Just like symmetric and skew-symmetric tensors, spherical and deviatoric tensors form orthogonal subspaces of $\mathbf{Lin}^n$.

1.12 Tensors of Higher Orders

Similarly to second-order tensors we can define tensors of higher orders. For example, a third-order tensor can be defined as a linear mapping from $\mathbb{E}^n$ to $\mathbf{Lin}^n$. Thus, we can write

$$\mathbf{Y} = \mathsf{A}\boldsymbol{x}, \quad \mathbf{Y} \in \mathbf{Lin}^n, \quad \forall \boldsymbol{x} \in \mathbb{E}^n, \quad \forall \mathsf{A} \in \mathsf{Lin}^n, \tag{1.154}$$

where Lin^n denotes the set of all linear mappings of vectors in $\mathbb{E}^n$ into second-order tensors in $\mathbf{Lin}^n$. The tensors of the third order can likewise be represented with respect to a basis in Lin^n e.g. by

$$\mathsf{A} = \mathsf{A}^{ijk}\boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k = \mathsf{A}_{ijk}\boldsymbol{g}^i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}^k$$
$$= \mathsf{A}^{i}_{\cdot jk}\boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}^k = \mathsf{A}_{i\cdot k}^{\ j}\boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}^k. \tag{1.155}$$

For the components of the tensor A (1.155) we can thus write by analogy with (1.139)

$$\mathsf{A}^{ijk} = \mathsf{A}^{ij}_{\cdot\cdot s}g^{sk} = \mathsf{A}^{i}_{\cdot st}g^{sj}g^{tk} = \mathsf{A}_{rst}g^{ri}g^{sj}g^{tk},$$
$$\mathsf{A}_{ijk} = \mathsf{A}^{r}_{\cdot jk}g_{ri} = \mathsf{A}^{rs}_{\cdot\cdot k}g_{ri}g_{sj} = \mathsf{A}^{rst}g_{ri}g_{sj}g_{tk}. \tag{1.156}$$

Exercises

1.1. Prove that if $\boldsymbol{x} \in \mathbb{V}$ is a vector and $\alpha \in \mathbb{R}$ is a scalar, then the following identities hold.
(a) $-\boldsymbol{0} = \boldsymbol{0}$, (b) $\alpha\boldsymbol{0} = \boldsymbol{0}$, (c) $0\boldsymbol{x} = \boldsymbol{0}$, (d) $-\boldsymbol{x} = (-1)\,\boldsymbol{x}$, (e) if $\alpha\boldsymbol{x} = \boldsymbol{0}$, then either $\alpha = 0$ or $\boldsymbol{x} = \boldsymbol{0}$ or both.

1.2. Prove that $\boldsymbol{x}_i \neq \boldsymbol{0}\,(i = 1, 2, \ldots, n)$ for linearly independent vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$. In other words, linearly independent vectors are all non-zero.

1.3. Prove that any non-empty subset of linearly independent vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ is also linearly independent.

1.4. Write out in full the following expressions for $n = 3$: (a) $\delta^i_j a^j$, (b) $\delta_{ij} x^i x^j$, (c) δ^i_i, (d) $\dfrac{\partial f_i}{\partial x^j} dx^j$.

1.5. Prove that $\boldsymbol{0} \cdot \boldsymbol{x} = 0$, $\forall \boldsymbol{x} \in \mathbb{E}^n$.

1.6. Prove that a set of mutually orthogonal non-zero vectors is always linearly independent.

1.7. Prove the so-called parallelogram law: $\|\boldsymbol{x} + \boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 + 2\boldsymbol{x} \cdot \boldsymbol{y} + \|\boldsymbol{y}\|^2$.

1.8. Let $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ be a basis in $\mathbb{E}^n$ and $\boldsymbol{a} \in \mathbb{E}^n$ be a vector. Prove that $\boldsymbol{a} \cdot \boldsymbol{g}_i = 0$ $(i = 1, 2, \ldots, n)$ if and only if $\boldsymbol{a} = \boldsymbol{0}$.

1.9. Prove that $\boldsymbol{a} = \boldsymbol{b}$ if and only if $\boldsymbol{a} \cdot \boldsymbol{x} = \boldsymbol{b} \cdot \boldsymbol{x}$, $\forall \boldsymbol{x} \in \mathbb{E}^n$.

1.10. (a) Construct an orthonormal set of vectors orthogonalizing and normalizing (with the aid of the procedure described in Sect. 1.4) the following linearly independent vectors:

$$\boldsymbol{g}_1 = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}, \quad \boldsymbol{g}_2 = \begin{Bmatrix} 2 \\ 1 \\ -2 \end{Bmatrix}, \quad \boldsymbol{g}_3 = \begin{Bmatrix} 4 \\ 2 \\ 1 \end{Bmatrix},$$

where the components are given with respect to an orthonormal basis.
(b) Construct a basis in $\mathbb{E}^3$ dual to the given above by means of $(1.21)_1$, (1.24) and $(1.25)_2$.
(c) Calculate again the vectors $\boldsymbol{g}^i$ dual to $\boldsymbol{g}_i$ $(i = 1, 2, 3)$ by using relations (1.33) and (1.35). Compare the result with the solution of problem (b).

1.11. Verify that the vectors (1.33) are linearly independent.

1.12. Prove identity (1.41) by means of (1.18), (1.19) and (1.36).

1.13. Prove relations (1.39) and (1.42) using (1.33), (1.36) and (1.41).

1.14. Verify the following identities involving the permutation symbol (1.36) for $n = 3$: (a) $\delta^{ij} e_{ijk} = 0$, (b) $e^{ikm} e_{jkm} = 2\delta^i_j$, (c) $e^{ijk} e_{ijk} = 6$, (d) $e^{ijm} e_{klm} = \delta^i_k \delta^j_l - \delta^i_l \delta^j_k$.

1.15. Prove the identity

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\,\boldsymbol{a}, \quad \forall \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{E}^3. \tag{1.157}$$

1.16. Prove that $\mathbf{A}\boldsymbol{0} = \boldsymbol{0}\mathbf{A} = \boldsymbol{0}$, $\forall \mathbf{A} \in \mathbf{Lin}^n$.

1.17. Prove that $0\mathbf{A} = \mathbf{0}$, $\forall \mathbf{A} \in \mathbf{Lin}^n$.

1.18. Prove formula (1.57), where the negative tensor $-\mathbf{A}$ is defined by (1.52).

1.19. Prove relation (1.80).

1.20. Prove (1.83) using (1.82) and (1.15).

1.21. Evaluate the tensor $\mathbf{W} = \hat{\boldsymbol{w}} = \boldsymbol{w}\times$, where $\boldsymbol{w} = w^i \boldsymbol{g}_i$.

1.22. Evaluate components of the tensor describing a rotation about the axis $\boldsymbol{e}_3$ by the angle α.

1.23. Let $\mathbf{A} = \mathrm{A}^{ij} \boldsymbol{g}_i \otimes \boldsymbol{g}_j$, where

$$\left[\mathrm{A}^{ij}\right] = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and the vectors $\boldsymbol{g}_i$ are given in Exercise 1.10. Evaluate the components A_{ij}, $\mathrm{A}^{i}_{\cdot j}$ and $\mathrm{A}_{i\cdot}^{\ j}$.

1.24. Prove identities (1.100) and (1.102).

1.25. Let $\mathbf{A} = \mathrm{A}^{i}_{\cdot j} \boldsymbol{g}_i \otimes \boldsymbol{g}^j$, $\mathbf{B} = \mathrm{B}^{i}_{\cdot j} \boldsymbol{g}_i \otimes \boldsymbol{g}^j$, $\mathbf{C} = \mathrm{C}^{i}_{\cdot j} \boldsymbol{g}_i \otimes \boldsymbol{g}^j$ and $\mathbf{D} = \mathrm{D}^{i}_{\cdot j} \boldsymbol{g}_i \otimes \boldsymbol{g}^j$, where

$$\left[\mathrm{A}^{i}_{\cdot j}\right] = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \left[\mathrm{B}^{i}_{\cdot j}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \left[\mathrm{C}^{i}_{\cdot j}\right] = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\left[\mathrm{D}^{i}_{\cdot j}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 10 \end{bmatrix}.$$

Find commutative pairs of tensors.

1.26. Let $\mathbf{A}$ and $\mathbf{B}$ be two commutative tensors. Write out in full $(\mathbf{A} + \mathbf{B})^k$, where $k = 2, 3, \ldots$

1.27. Prove that

$$\exp(\mathbf{A} + \mathbf{B}) = \exp(\mathbf{A}) \exp(\mathbf{B}), \tag{1.158}$$

where $\mathbf{A}$ and $\mathbf{B}$ commute.

1.28. Prove that $\exp(k\mathbf{A}) = [\exp(\mathbf{A})]^k$, where $k = 2, 3, \ldots$

1.29. Evaluate $\exp(\mathbf{0})$ and $\exp(\mathbf{I})$.

1.30. Prove that $\exp(-\mathbf{A})\exp(\mathbf{A}) = \exp(\mathbf{A})\exp(-\mathbf{A}) = \mathbf{I}$.

1.31. Prove that $\exp(\mathbf{A}+\mathbf{B}) = \exp(\mathbf{A}) + \exp(\mathbf{B})$ if $\mathbf{AB} = \mathbf{BA} = \mathbf{0}$.

1.32. Prove that $\exp\left(\mathbf{QAQ}^{\mathrm{T}}\right) = \mathbf{Q}\exp(\mathbf{A})\mathbf{Q}^{\mathrm{T}}, \ \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n$.

1.33. Compute the exponential of the tensors $\mathbf{D} = \mathrm{D}^i_{\cdot j}\boldsymbol{g}_i \otimes \boldsymbol{g}^j$, $\mathbf{E} = \mathrm{E}^i_{\cdot j}\boldsymbol{g}_i \otimes \boldsymbol{g}^j$ and $\mathbf{F} = \mathrm{F}^i_{\cdot j}\boldsymbol{g}_i \otimes \boldsymbol{g}^j$, where

$$\left[\mathrm{D}^i_{\cdot j}\right] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \left[\mathrm{E}^i_{\cdot j}\right] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \left[\mathrm{F}^i_{\cdot j}\right] = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1.34. Prove that $(\mathbf{ABCD})^{\mathrm{T}} = \mathbf{D}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

1.35. Verify that $\left(\mathbf{A}^k\right)^{\mathrm{T}} = \left(\mathbf{A}^{\mathrm{T}}\right)^k$, where $k = 1, 2, 3, \ldots$

1.36. Evaluate the components B^{ij}, B_{ij}, $\mathrm{B}^i_{\cdot j}$ and $\mathrm{B}_{i\cdot}^{\ j}$ of the tensor $\mathbf{B} = \mathbf{A}^{\mathrm{T}}$, where $\mathbf{A}$ is defined in Exercise 1.23.

1.37. Prove relation (1.126).

1.38. Verify that $\left(\mathbf{A}^{-1}\right)^k = \left(\mathbf{A}^k\right)^{-1} = \mathbf{A}^{-k}$, where $k = 1, 2, 3, \ldots$

1.39. Prove identity (1.135) using (1.82) and (1.134).

1.40. Prove by means of (1.134-1.136) the properties of the scalar product (D.1-D.3).

1.41. Verify that $[(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d})] : \mathbf{I} = (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c})$.

1.42. Express $\mathrm{tr}\mathbf{A}$ in terms of the components $\mathrm{A}^i_{\cdot j}$, A_{ij}, A^{ij}.

1.43. Prove that $\mathbf{M} : \mathbf{W} = 0$, where $\mathbf{M}$ is a symmetric tensor and $\mathbf{W}$ a skew-symmetric tensor.

1.44. Evaluate $\mathrm{tr}\mathbf{W}^k$, where $\mathbf{W}$ is a skew-symmetric tensor and $k = 1, 3, 5, \ldots$

1.45. Verify that $\mathrm{sym}(\mathrm{skew}\mathbf{A}) = \mathrm{skew}(\mathrm{sym}\mathbf{A}) = \mathbf{0}, \ \forall \mathbf{A} \in \mathbf{L}\mathrm{in}^n$.

1.46. Prove that $\mathrm{sph}(\mathrm{dev}\mathbf{A}) = \mathrm{dev}(\mathrm{sph}\mathbf{A}) = \mathbf{0}, \ \forall \mathbf{A} \in \mathbf{L}\mathrm{in}^n$.