3

Curves and Surfaces in Three-Dimensional Euclidean Space

3.1 Curves in Three-Dimensional Euclidean Space

A curve in three-dimensional space is defined by a vector function

$$\boldsymbol{r} = \boldsymbol{r}(t), \quad \boldsymbol{r} \in \mathbb{E}^3, \tag{3.1}$$

where the real variable t belongs to some interval: $t_1 \le t \le t_2$. Henceforth, we assume that the function $\boldsymbol{r}(t)$ is sufficiently differentiable and

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \neq \boldsymbol{0} \tag{3.2}$$

over the whole definition domain. Specifying an arbitrary coordinate system (2.15) as

$$\theta^i = \theta^i(\boldsymbol{r}), \quad i = 1, 2, 3, \tag{3.3}$$

the curve (3.1) can alternatively be defined by

$$\theta^i = \theta^i(t), \quad i = 1, 2, 3. \tag{3.4}$$

Example. Straight line. A straight line can be defined by

$$\boldsymbol{r}(t) = \boldsymbol{a} + \boldsymbol{b}t, \quad \boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}^3. \tag{3.5}$$

With respect to linear coordinates related to a basis $\mathcal{H} = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3\}$ it is equivalent to

$$r^i(t) = a^i + b^i t, \quad i = 1, 2, 3, \tag{3.6}$$

where $\boldsymbol{r} = r^i \boldsymbol{h}_i$, $\boldsymbol{a} = a^i \boldsymbol{h}_i$ and $\boldsymbol{b} = b^i \boldsymbol{h}_i$.

Example. Circular helix. The circular helix (Fig. 3.1) is defined by

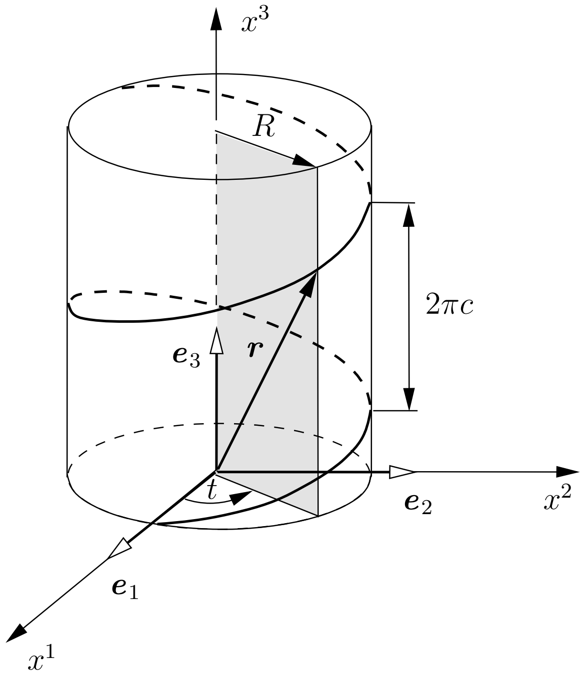

Fig. 3.1. Circular helix

$$\boldsymbol{r}(t) = R\cos(t)\,\boldsymbol{e}_1 + R\sin(t)\,\boldsymbol{e}_2 + ct\boldsymbol{e}_3, \quad c \neq 0, \tag{3.7}$$

where $\boldsymbol{e}_i$ $(i = 1, 2, 3)$ form an orthonormal basis in $\mathbb{E}^3$. For the definition of the circular helix the cylindrical coordinates (2.16) appear to be very suitable. Indeed, alternatively to (3.7) we can write

$$r = R, \quad \varphi = t, \quad z = ct. \tag{3.8}$$

In the previous chapter we defined tangent vectors to the coordinate lines. By analogy one can also define a vector tangent to the curve (3.1) as

$$\boldsymbol{g}_t = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}. \tag{3.9}$$

It is advantageous to parametrize the curve (3.1) in terms of the so-called arc length. To this end, we first calculate the length of a curve segment between the points corresponding to parameters t_1 and t as

$$s(t) = \int\limits_{\boldsymbol{r}(t_1)}^{\boldsymbol{r}(t)} \sqrt{\mathrm{d}\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{r}}. \tag{3.10}$$

With the aid of (3.9) we can write $\mathrm{d}\boldsymbol{r} = \boldsymbol{g}_t \mathrm{d}t$ and consequently

$$s(t) = \int_{t_1}^{t} \sqrt{\boldsymbol{g}_t \cdot \boldsymbol{g}_t} \mathrm{d}t = \int_{t_1}^{t} \|\boldsymbol{g}_t\| \, \mathrm{d}t = \int_{t_1}^{t} \sqrt{g_{tt}(t)} \mathrm{d}t. \tag{3.11}$$

Using this equation and keeping in mind assumption (3.2) we have

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{g_{tt}(t)} \neq 0. \tag{3.12}$$

This implies that the function $s = s(t)$ is invertible and

$$t(s) = \int_{s(t_1)}^{s} \|\boldsymbol{g}_t\|^{-1} \, \mathrm{d}s = \int_{s(t_1)}^{s} \frac{\mathrm{d}s}{\sqrt{g_{tt}(t)}}. \tag{3.13}$$

Thus, the curve (3.1) can be redefined in terms of the arc length s as

$$\boldsymbol{r} = \boldsymbol{r}(t(s)) = \widehat{\boldsymbol{r}}(s). \tag{3.14}$$

In analogy with (3.9) one defines the vector tangent to the curve $\widehat{\boldsymbol{r}}(s)$ (3.14) as

$$\boldsymbol{a}_1 = \frac{\mathrm{d}\widehat{\boldsymbol{r}}}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\boldsymbol{g}_t}{\|\boldsymbol{g}_t\|} \tag{3.15}$$

being a unit vector: $\|\boldsymbol{a}_1\| = 1$. Differentiation of this vector with respect to s further yields

$$\boldsymbol{a}_{1,s} = \frac{\mathrm{d}\boldsymbol{a}_1}{\mathrm{d}s} = \frac{\mathrm{d}^2\widehat{\boldsymbol{r}}}{\mathrm{d}s^2}. \tag{3.16}$$

It can be shown that the tangent vector $\boldsymbol{a}_1$ is orthogonal to $\boldsymbol{a}_{1,s}$ provided the latter one is not zero. Indeed, differentiating the identity $\boldsymbol{a}_1 \cdot \boldsymbol{a}_1 = 1$ with respect to s we have

$$\boldsymbol{a}_1 \cdot \boldsymbol{a}_{1,s} = 0. \tag{3.17}$$

The length of the vector $\boldsymbol{a}_{1,s}$

$$\varkappa(s) = \|\boldsymbol{a}_{1,s}\| \tag{3.18}$$

plays an important role in the theory of curves and is called curvature. The inverse value

$$\rho(s) = \frac{1}{\varkappa(s)} \tag{3.19}$$

is referred to as the radius of curvature of the curve at the point $\widehat{\boldsymbol{r}}(s)$. Henceforth, we focus on curves with non-zero curvature. The case of zero curvature corresponds to a straight line (see Exercise 3.1) and is trivial.

Next, we define the unit vector in the direction of $\boldsymbol{a}_{1},_{s}$

$$\boldsymbol{a}_2 = \frac{\boldsymbol{a}_{1},_{s}}{\|\boldsymbol{a}_{1},_{s}\|} = \frac{\boldsymbol{a}_{1},_{s}}{\varkappa(s)} \tag{3.20}$$

called the principal normal vector to the curve. The orthogonal vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ can further be completed to an orthonormal basis in $\mathbb{E}^3$ by the vector

$$\boldsymbol{a}_3 = \boldsymbol{a}_1 \times \boldsymbol{a}_2 \tag{3.21}$$

called the unit binormal vector. The triplet of vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ is referred to as the moving trihedron of the curve.

In order to study the rotation of the trihedron along the curve we again consider the arc length s as a coordinate. In this case, we can write similarly to (2.66)

$$\boldsymbol{a}_{i},_{s} = \Gamma^k_{is}\boldsymbol{a}_k, \quad i = 1, 2, 3, \tag{3.22}$$

where $\Gamma^k_{is} = \boldsymbol{a}_{i},_{s} \cdot \boldsymbol{a}_k$ $(i, k = 1, 2, 3)$. From (3.17), (3.20) and (3.21) we immediately observe that $\Gamma^2_{1s} = \varkappa$ and $\Gamma^1_{1s} = \Gamma^3_{1s} = 0$. Further, differentiating the identities

$$\boldsymbol{a}_3 \cdot \boldsymbol{a}_3 = 1, \qquad \boldsymbol{a}_1 \cdot \boldsymbol{a}_3 = 0 \tag{3.23}$$

with respect to s yields

$$\boldsymbol{a}_3 \cdot \boldsymbol{a}_{3},_{s} = 0, \qquad \boldsymbol{a}_{1},_{s} \cdot \boldsymbol{a}_3 + \boldsymbol{a}_1 \cdot \boldsymbol{a}_{3},_{s} = 0. \tag{3.24}$$

Taking into account (3.20) this results in the following identity

$$\boldsymbol{a}_1 \cdot \boldsymbol{a}_{3},_{s} = -\boldsymbol{a}_{1},_{s} \cdot \boldsymbol{a}_3 = -\varkappa \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 = 0. \tag{3.25}$$

Relations (3.24) and (3.25) suggest that

$$\boldsymbol{a}_{3},_{s} = -\tau(s)\,\boldsymbol{a}_2, \tag{3.26}$$

where the function

$$\tau(s) = -\boldsymbol{a}_{3},_{s} \cdot \boldsymbol{a}_2 \tag{3.27}$$

is called torsion of the curve at the point $\widehat{\boldsymbol{r}}(s)$. Thus, $\Gamma^2_{3s} = -\tau$ and $\Gamma^1_{3s} = \Gamma^3_{3s} = 0$. The sign of the torsion (3.27) has a geometric meaning and remains unaffected by the change of the positive sense of the curve, i.e. by the transformation $s = -s'$ (see Exercise 3.2). Accordingly, one distinguishes right-handed curves with a positive torsion and left-handed curves with a negative torsion. In the case of zero torsion the curve is referred to as a plane curve.

Finally, differentiating the identities

$$\boldsymbol{a}_2 \cdot \boldsymbol{a}_1 = 0, \quad \boldsymbol{a}_2 \cdot \boldsymbol{a}_2 = 1, \quad \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 = 0$$

with respect to s and using (3.20) and (3.27) we get

$$\boldsymbol{a}_{2,s} \cdot \boldsymbol{a}_1 = -\boldsymbol{a}_2 \cdot \boldsymbol{a}_{1,s} = -\varkappa \boldsymbol{a}_2 \cdot \boldsymbol{a}_2 = -\varkappa, \tag{3.28}$$

$$\boldsymbol{a}_2 \cdot \boldsymbol{a}_{2,s} = 0, \quad \boldsymbol{a}_{2,s} \cdot \boldsymbol{a}_3 = -\boldsymbol{a}_2 \cdot \boldsymbol{a}_{3,s} = \tau, \tag{3.29}$$

so that $\Gamma^1_{2s} = -\varkappa$, $\Gamma^2_{2s} = 0$ and $\Gamma^3_{2s} = \tau$. Summarizing the above results we can write

$$\left[\Gamma^j_{is}\right] = \begin{bmatrix} 0 & \varkappa & 0 \\ -\varkappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \tag{3.30}$$

and

$$\begin{aligned} \boldsymbol{a}_{1,s} &= && \varkappa \boldsymbol{a}_2, && \\ \boldsymbol{a}_{2,s} &= -\varkappa \boldsymbol{a}_1 && && +\tau \boldsymbol{a}_3, \\ \boldsymbol{a}_{3,s} &= && -\tau \boldsymbol{a}_2. && \end{aligned} \tag{3.31}$$

Relations (3.31) are known as the Frenet formulas.

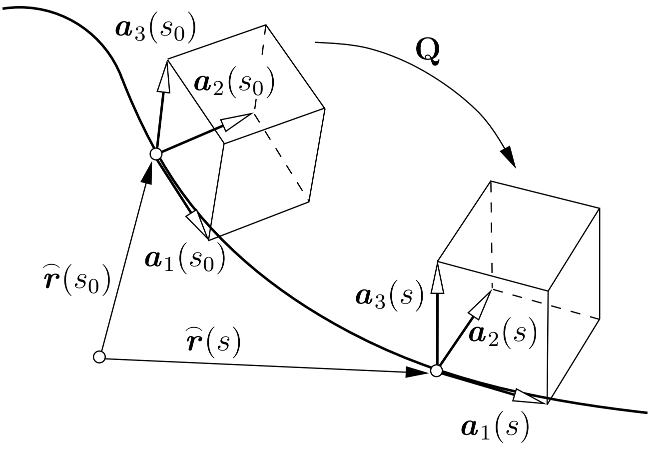

Fig. 3.2. Rotation of the moving trihedron

A useful illustration of the Frenet formulas can be gained with the aid of a skew-symmetric tensor. To this end, we consider the rotation of the trihedron from some initial position at s_0 to the actual state at s. This rotation can be described by an orthogonal tensor $\mathbf{Q}(s)$ as (Fig. 3.2)

$$\boldsymbol{a}_i(s) = \mathbf{Q}(s)\,\boldsymbol{a}_i(s_0), \quad i = 1, 2, 3. \tag{3.32}$$

Differentiating this relation with respect to s yields

$$\boldsymbol{a}_i,_s(s) = \mathbf{Q},_s(s)\,\boldsymbol{a}_i(s_0), \quad i = 1, 2, 3. \tag{3.33}$$

Mapping both sides of (3.32) by $\mathbf{Q}^{\mathrm{T}}(s)$ and inserting the result into (3.33) we further obtain

$$\boldsymbol{a}_i,_s(s) = \mathbf{Q},_s(s)\,\mathbf{Q}^{\mathrm{T}}(s)\,\boldsymbol{a}_i(s), \quad i = 1, 2, 3. \tag{3.34}$$

Differentiating the identity (1.129) $\mathbf{Q}(s)\,\mathbf{Q}^{\mathrm{T}}(s) = \mathbf{I}$ with respect to s we have $\mathbf{Q},_s(s)\,\mathbf{Q}^{\mathrm{T}}(s) + \mathbf{Q}(s)\,\mathbf{Q}^{\mathrm{T}},_s(s) = \mathbf{0}$, which implies that the tensor $\mathbf{W}(s) = \mathbf{Q},_s(s)\,\mathbf{Q}^{\mathrm{T}}(s)$ is skew-symmetric. Hence, eq. (3.34) can be rewritten as (see also [3])

$$\boldsymbol{a}_i,_s(s) = \mathbf{W}(s)\,\boldsymbol{a}_i(s), \quad \mathbf{W} \in \mathbf{S}\mathrm{kew}^3, \quad i = 1, 2, 3, \tag{3.35}$$

where according to (3.31)

$$\mathbf{W}(s) = \tau(s)\,(\boldsymbol{a}_3 \otimes \boldsymbol{a}_2 - \boldsymbol{a}_2 \otimes \boldsymbol{a}_3) + \varkappa(s)\,(\boldsymbol{a}_2 \otimes \boldsymbol{a}_1 - \boldsymbol{a}_1 \otimes \boldsymbol{a}_2). \tag{3.36}$$

By virtue of (1.130) and (1.131) we further obtain

$$\mathbf{W} = \tau\hat{\boldsymbol{a}}_1 + \varkappa\hat{\boldsymbol{a}}_3 \tag{3.37}$$

and consequently

$$\boldsymbol{a}_i,_s = \boldsymbol{d} \times \boldsymbol{a}_i = \hat{\boldsymbol{d}}\boldsymbol{a}_i, \quad i = 1, 2, 3, \tag{3.38}$$

where

$$\boldsymbol{d} = \tau\boldsymbol{a}_1 + \varkappa\boldsymbol{a}_3 \tag{3.39}$$

is referred to as the Darboux vector.

Example. Curvature, torsion, moving trihedron and Darboux vector for a circular helix. Inserting (3.7) into (3.9) delivers

$$\boldsymbol{g}_t = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = -R\sin(t)\,\boldsymbol{e}_1 + R\cos(t)\,\boldsymbol{e}_2 + c\boldsymbol{e}_3, \tag{3.40}$$

so that

$$g_{tt} = \boldsymbol{g}_t \cdot \boldsymbol{g}_t = R^2 + c^2 = const. \tag{3.41}$$

Thus, using (3.13) we may set

$$t(s) = \frac{s}{\sqrt{R^2 + c^2}}. \tag{3.42}$$

Using this result, the circular helix (3.7) can be parametrized in terms of the arc length s by

$$\widehat{\boldsymbol{r}}(s) = R\cos\left(\frac{s}{\sqrt{R^2+c^2}}\right)\boldsymbol{e}_1 + R\sin\left(\frac{s}{\sqrt{R^2+c^2}}\right)\boldsymbol{e}_2 + \frac{cs}{\sqrt{R^2+c^2}}\boldsymbol{e}_3. \tag{3.43}$$

With the aid of (3.15) we further write

$$\boldsymbol{a}_1 = \frac{\mathrm{d}\widehat{\boldsymbol{r}}}{\mathrm{d}s} = \frac{1}{\sqrt{R^2+c^2}}\left[-R\sin\left(\frac{s}{\sqrt{R^2+c^2}}\right)\boldsymbol{e}_1 + R\cos\left(\frac{s}{\sqrt{R^2+c^2}}\right)\boldsymbol{e}_2 + c\boldsymbol{e}_3\right], \tag{3.44}$$

$$\boldsymbol{a}_1,_s = -\frac{R}{R^2+c^2}\left[\cos\left(\frac{s}{\sqrt{R^2+c^2}}\right)\boldsymbol{e}_1 + \sin\left(\frac{s}{\sqrt{R^2+c^2}}\right)\boldsymbol{e}_2\right]. \tag{3.45}$$

According to (3.18) the curvature of the helix is thus

$$\varkappa = \frac{R}{R^2+c^2}. \tag{3.46}$$

By virtue of (3.20), (3.21) and (3.27) we have

$$\boldsymbol{a}_2 = \frac{\boldsymbol{a}_1,_s}{\varkappa} = -\cos\left(\frac{s}{\sqrt{R^2+c^2}}\right)\boldsymbol{e}_1 - \sin\left(\frac{s}{\sqrt{R^2+c^2}}\right)\boldsymbol{e}_2, \tag{3.47}$$

$$\boldsymbol{a}_3 = \boldsymbol{a}_1 \times \boldsymbol{a}_2 = \frac{1}{\sqrt{R^2+c^2}}\left[c\sin\left(\frac{s}{\sqrt{R^2+c^2}}\right)\boldsymbol{e}_1 - c\cos\left(\frac{s}{\sqrt{R^2+c^2}}\right)\boldsymbol{e}_2 + R\boldsymbol{e}_3\right]. \tag{3.48}$$

$$\boldsymbol{a}_3,_s = \frac{c}{R^2+c^2}\left[\cos\left(\frac{s}{\sqrt{R^2+c^2}}\right)\boldsymbol{e}_1 + \sin\left(\frac{s}{\sqrt{R^2+c^2}}\right)\boldsymbol{e}_2\right], \tag{3.49}$$

$$\tau = \frac{c}{R^2+c^2}. \tag{3.50}$$

It is seen that the circular helix is right-handed for $c > 0$, left-handed for $c < 0$ and becomes a circle for $c = 0$. For the Darboux vector (3.39) we finally obtain

$$\boldsymbol{d} = \tau\boldsymbol{a}_1 + \varkappa\boldsymbol{a}_3 = \frac{1}{\sqrt{R^2+c^2}}\boldsymbol{e}_3. \tag{3.51}$$

3.2 Surfaces in Three-Dimensional Euclidean Space

A surface in three-dimensional Euclidean space is defined by a vector function

$$\boldsymbol{r} = \boldsymbol{r}\left(t^1, t^2\right), \quad \boldsymbol{r} \in \mathbb{E}^3, \tag{3.52}$$

of two real variables t^1 and t^2 referred to as Gauss coordinates. With the aid of the coordinate system (3.3) one can alternatively write

$$\theta^i = \theta^i\left(t^1, t^2\right), \quad i = 1, 2, 3. \tag{3.53}$$

In the following, we assume that the function $\boldsymbol{r}\left(t^1, t^2\right)$ is sufficiently differentiable with respect to both arguments and

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t^\alpha} \neq \boldsymbol{0}, \quad \alpha = 1, 2 \tag{3.54}$$

over the whole definition domain.

Example 1. Plane. Let us consider three linearly independent vectors $\boldsymbol{x}_i$ $(i = 0, 1, 2)$ specifying three points in three-dimensional space. The plane going through these points can be defined by

$$\boldsymbol{r}\left(t^1, t^2\right) = \boldsymbol{x}_0 + t^1\left(\boldsymbol{x}_1 - \boldsymbol{x}_0\right) + t^2\left(\boldsymbol{x}_2 - \boldsymbol{x}_0\right). \tag{3.55}$$

Example 2. Cylinder. A cylinder of radius R with the axis parallel to $\boldsymbol{e}_3$ is defined by

$$\boldsymbol{r}\left(t^1, t^2\right) = R\cos t^1 \boldsymbol{e}_1 + R\sin t^1 \boldsymbol{e}_2 + t^2 \boldsymbol{e}_3, \tag{3.56}$$

where $\boldsymbol{e}_i$ $(i = 1, 2, 3)$ again form an orthonormal basis in $\mathbb{E}^3$. With the aid of the cylindrical coordinates (2.16) we can alternatively write

$$\varphi = t^1, \quad z = t^2, \quad r = R. \tag{3.57}$$

Example 3. Sphere. A sphere of radius R with the center at $\boldsymbol{r} = \boldsymbol{0}$ is defined by

$$\boldsymbol{r}\left(t^1, t^2\right) = R\sin t^1 \sin t^2 \boldsymbol{e}_1 + R\cos t^2 \boldsymbol{e}_2 + R\cos t^1 \sin t^2 \boldsymbol{e}_3, \tag{3.58}$$

or by means of spherical coordinates (2.144) as

$$\varphi = t^1, \quad \phi = t^2, \quad r = R. \tag{3.59}$$

Using a parametric representation (see, e.g., [25])

$$t^1 = t^1(t), \quad t^2 = t^2(t) \tag{3.60}$$

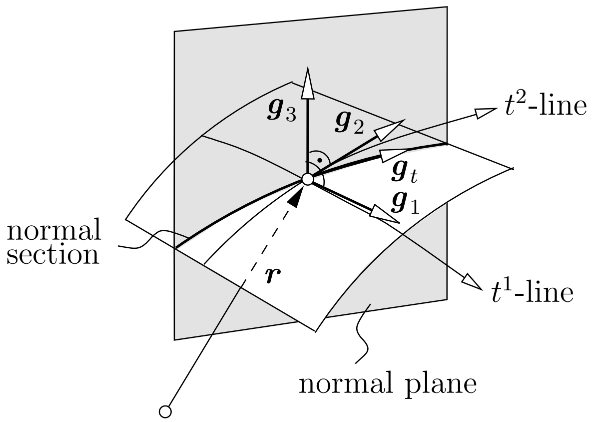

Fig. 3.3. Coordinate lines on the surface, normal section and tangent vectors

one defines a curve on the surface (3.52). In particular, the curves $t^1 = const$ and $t^2 = const$ are called t^2 and t^1 coordinate lines, respectively (Fig. 3.3). The vector tangent to the curve (3.60) can be expressed by

$$\boldsymbol{g}_t = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\partial \boldsymbol{r}}{\partial t^1}\frac{\mathrm{d}t^1}{\mathrm{d}t} + \frac{\partial \boldsymbol{r}}{\partial t^2}\frac{\mathrm{d}t^2}{\mathrm{d}t} = \boldsymbol{g}_1\frac{\mathrm{d}t^1}{\mathrm{d}t} + \boldsymbol{g}_2\frac{\mathrm{d}t^2}{\mathrm{d}t}, \tag{3.61}$$

where

$$\boldsymbol{g}_\alpha = \frac{\partial \boldsymbol{r}}{\partial t^\alpha} = \boldsymbol{r}_{,\alpha}\,, \quad \alpha = 1, 2 \tag{3.62}$$

represent tangent vectors to the coordinate lines. For the length of an infinitesimal element of the curve (3.60) we thus write

$$(\mathrm{d}s)^2 = \mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r} = (\boldsymbol{g}_t\mathrm{d}t)\cdot(\boldsymbol{g}_t\mathrm{d}t) = \left(\boldsymbol{g}_1\mathrm{d}t^1 + \boldsymbol{g}_2\mathrm{d}t^2\right)\cdot\left(\boldsymbol{g}_1\mathrm{d}t^1 + \boldsymbol{g}_2\mathrm{d}t^2\right). \tag{3.63}$$

With the aid of the abbreviation

$$g_{\alpha\beta} = g_{\beta\alpha} = \boldsymbol{g}_\alpha \cdot \boldsymbol{g}_\beta, \quad \alpha, \beta = 1, 2, \tag{3.64}$$

it delivers the quadratic form

$$(\mathrm{d}s)^2 = g_{11}\left(\mathrm{d}t^1\right)^2 + 2g_{12}\mathrm{d}t^1\mathrm{d}t^2 + g_{22}\left(\mathrm{d}t^2\right)^2 \tag{3.65}$$

referred to as the first fundamental form of the surface. The latter result can briefly be written as

$$(\mathrm{d}s)^2 = g_{\alpha\beta}\mathrm{d}t^\alpha\mathrm{d}t^\beta, \tag{3.66}$$

where and henceforth within this chapter the summation convention is implied for repeated Greek indices taking the values from 1 to 2. Similar to the metric coefficients $(1.85)_{1,2}$ in n-dimensional Euclidean space $g_{\alpha\beta}$ (3.64) describe the metric on a surface. Generally, the metric described by a differential quadratic form like (3.66) is referred to as Riemannian metric.

The tangent vectors (3.62) can be completed to a basis in $\mathbb{E}^3$ by the unit vector

$$\boldsymbol{g}_3 = \frac{\boldsymbol{g}_1 \times \boldsymbol{g}_2}{\|\boldsymbol{g}_1 \times \boldsymbol{g}_2\|} \tag{3.67}$$

called principal normal vector to the surface.

In the following, we focus on a special class of surface curves called normal sections. These are curves passing through a point of the surface $\boldsymbol{r}\left(t^1, t^2\right)$ and obtained by intersection of this surface with a plane involving the principal normal vector. Such a plane is referred to as the normal plane.

In order to study curvature properties of normal sections we first express the derivatives of the basis vectors $\boldsymbol{g}_i$ $(i = 1, 2, 3)$ with respect to the surface coordinates. Using the formalism of Christoffel symbols we can write

$$\boldsymbol{g}_{i,\alpha} = \frac{\partial \boldsymbol{g}_i}{\partial t^\alpha} = \Gamma_{i\alpha k} \boldsymbol{g}^k = \Gamma^k_{i\alpha} \boldsymbol{g}_k, \quad i = 1, 2, 3, \tag{3.68}$$

where

$$\Gamma_{i\alpha k} = \boldsymbol{g}_{i,\alpha} \cdot \boldsymbol{g}_k, \quad \Gamma^k_{i\alpha} = \boldsymbol{g}_{i,\alpha} \cdot \boldsymbol{g}^k, \quad i = 1, 2, 3, \quad \alpha = 1, 2. \tag{3.69}$$

Taking into account the identity $\boldsymbol{g}_3 = \boldsymbol{g}^3$ resulting from (3.67) we immediately observe that

$$\Gamma_{i\alpha 3} = \Gamma^3_{i\alpha}, \quad i = 1, 2, 3, \quad \alpha = 1, 2. \tag{3.70}$$

Differentiating the relations

$$\boldsymbol{g}_\alpha \cdot \boldsymbol{g}_3 = 0, \quad \boldsymbol{g}_3 \cdot \boldsymbol{g}_3 = 1 \tag{3.71}$$

with respect to the Gauss coordinates we further obtain

$$\boldsymbol{g}_{\alpha,\beta} \cdot \boldsymbol{g}_3 = -\boldsymbol{g}_\alpha \cdot \boldsymbol{g}_{3,\beta}\,, \quad \boldsymbol{g}_{3,\alpha} \cdot \boldsymbol{g}^3 = 0, \quad \alpha, \beta = 1, 2 \tag{3.72}$$

and consequently

$$\Gamma^3_{\alpha\beta} = -\Gamma_{3\beta\alpha}, \quad \Gamma^3_{3\alpha} = 0, \quad \alpha, \beta = 1, 2. \tag{3.73}$$

Using in (3.68) the abbreviation

$$b_{\alpha\beta} = b_{\beta\alpha} = \Gamma^3_{\alpha\beta} = -\Gamma_{3\alpha\beta} = \boldsymbol{g}_{\alpha,\beta} \cdot \boldsymbol{g}_3, \quad \alpha, \beta = 1, 2, \tag{3.74}$$

we arrive at the relations

$$\boldsymbol{g}_{\alpha,\beta} = \Gamma^\rho_{\alpha\beta} \boldsymbol{g}_\rho + b_{\alpha\beta} \boldsymbol{g}_3, \quad \alpha, \beta = 1, 2 \tag{3.75}$$

called the Gauss formulas.

Similarly to a coordinate system one can notionally define the covariant derivative also on the surface. To this end, relations (2.84), (2.86) and (2.87) are specified to the two-dimensional space in a straight forward manner as

$$f^{\alpha}|_{\beta}= f^{\alpha},_{\beta} + f^{\rho}\Gamma^{\alpha}_{\rho\beta}, \quad f_{\alpha}|_{\beta}= f_{\alpha},_{\beta} - f_{\rho}\Gamma^{\rho}_{\alpha\beta}, \tag{3.76}$$

$$\mathrm{F}^{\alpha\beta}|_{\gamma}= \mathrm{F}^{\alpha\beta},_{\gamma} + \mathrm{F}^{\rho\beta}\Gamma^{\alpha}_{\rho\gamma} + \mathrm{F}^{\alpha\rho}\Gamma^{\beta}_{\rho\gamma}, \quad \mathrm{F}_{\alpha\beta}|_{\gamma}= \mathrm{F}_{\alpha\beta},_{\gamma} - \mathrm{F}_{\rho\beta}\Gamma^{\rho}_{\alpha\gamma} - \mathrm{F}_{\alpha\rho}\Gamma^{\rho}_{\beta\gamma},$$

$$\mathrm{F}^{\alpha}_{\cdot\beta}|_{\gamma}= \mathrm{F}^{\alpha}_{\cdot\beta},_{\gamma} + \mathrm{F}^{\rho}_{\cdot\beta}\Gamma^{\alpha}_{\rho\gamma} - \mathrm{F}^{\alpha}_{\cdot\rho}\Gamma^{\rho}_{\beta\gamma}, \quad \alpha,\beta,\gamma = 1,2. \tag{3.77}$$

Thereby, with the aid of $(3.76)_2$ the Gauss formulas (3.75) can alternatively be given by (cf. (2.89))

$$\boldsymbol{g}_{\alpha}|_{\beta}= b_{\alpha\beta}\boldsymbol{g}_3, \quad \alpha,\beta = 1,2. \tag{3.78}$$

Further, we can write

$$b^{\beta}_{\alpha} = b_{\alpha\rho}g^{\rho\beta} = -\Gamma_{3\alpha\rho}g^{\rho\beta} = -\Gamma^{\beta}_{3\alpha}, \quad \alpha,\beta = 1,2. \tag{3.79}$$

Inserting the latter relation into (3.68) and considering $(3.73)_2$, this yields the identities

$$\boldsymbol{g}_3,_{\alpha} = \boldsymbol{g}_3|_{\alpha}= -b^{\rho}_{\alpha}\boldsymbol{g}_{\rho}, \quad \alpha = 1,2 \tag{3.80}$$

referred to as the Weingarten formulas.

Now, we are in a position to express the curvature of a normal section. It is called normal curvature and denoted in the following by $\varkappa_n$. At first, we observe that the principal normals of the surface and of the normal section coincide in the sense that $\boldsymbol{a}_2 = \pm\boldsymbol{g}_3$. Using (3.13), (3.28), (3.61), $(3.72)_1$ and (3.74) and assuming for the moment that $\boldsymbol{a}_2 = \boldsymbol{g}_3$ we get

$$\varkappa_n = -\boldsymbol{a}_2,_s \cdot\boldsymbol{a}_1 = -\boldsymbol{g}_3,_s \cdot\frac{\boldsymbol{g}_t}{\|\boldsymbol{g}_t\|} = -\left(\boldsymbol{g}_3,_t \frac{\mathrm{d}t}{\mathrm{d}s}\right)\cdot\frac{\boldsymbol{g}_t}{\|\boldsymbol{g}_t\|} = -\boldsymbol{g}_3,_t \cdot\frac{\boldsymbol{g}_t}{\|\boldsymbol{g}_t\|^2}$$

$$= -\left(\boldsymbol{g}_3,_{\alpha} \frac{\mathrm{d}t^{\alpha}}{\mathrm{d}t}\right)\cdot\left(\boldsymbol{g}_{\beta}\frac{\mathrm{d}t^{\beta}}{\mathrm{d}t}\right)\|\boldsymbol{g}_t\|^{-2} = b_{\alpha\beta}\frac{\mathrm{d}t^{\alpha}}{\mathrm{d}t}\frac{\mathrm{d}t^{\beta}}{\mathrm{d}t}\|\boldsymbol{g}_t\|^{-2}.$$

By virtue of (3.63) and (3.66) this leads to the following result

$$\varkappa_n = \frac{b_{\alpha\beta}\mathrm{d}t^{\alpha}\mathrm{d}t^{\beta}}{g_{\alpha\beta}\mathrm{d}t^{\alpha}\mathrm{d}t^{\beta}}, \tag{3.81}$$

where the quadratic form

$$b_{\alpha\beta}\mathrm{d}t^{\alpha}\mathrm{d}t^{\beta} = -\mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{g}_3 \tag{3.82}$$

is referred to as the second fundamental form of the surface. In the case $\boldsymbol{a}_2 = -\boldsymbol{g}_3$ the sign of the expression for $\varkappa_n$ (3.81) must be changed. Instead of that, we assume that the normal curvature can, in contrast to the curvature of space curves (3.18), be negative. However, the sign of $\varkappa_n$ (3.81) has no geometrical meaning. Indeed, it depends on the orientation of $\boldsymbol{g}_3$ with respect to $\boldsymbol{a}_2$ which is immaterial. For example, $\boldsymbol{g}_3$ changes the sign in coordinate transformations like $\bar{t}^1 = t^2$, $\bar{t}^2 = t^1$.

Of special interest is the dependence of the normal curvature $\varkappa_n$ on the direction of the normal section. For example, for the normal sections passing through the coordinate lines we have

$$\varkappa_n\big|_{t^2=const} = \frac{b_{11}}{g_{11}}, \quad \varkappa_n\big|_{t^1=const} = \frac{b_{22}}{g_{22}}. \tag{3.83}$$

In the following, we are going to find the directions of the maximal and minimal curvature. Necessary conditions for the extremum of the normal curvature (3.81) are given by

$$\frac{\partial \varkappa_n}{\partial t^\alpha} = 0, \quad \alpha = 1, 2. \tag{3.84}$$

Rewriting (3.81) as

$$(b_{\alpha\beta} - \varkappa_n g_{\alpha\beta})\, \mathrm{d}t^\alpha \mathrm{d}t^\beta = 0 \tag{3.85}$$

and differentiating with respect to t^α we obtain

$$(b_{\alpha\beta} - \varkappa_n g_{\alpha\beta})\, \mathrm{d}t^\beta = 0, \quad \alpha = 1, 2. \tag{3.86}$$

Multiplying both sides of this equation system by $g^{\alpha\rho}$ and summing up over α we have with the aid of (3.79)

$$\left(b^\rho_{\cdot\beta} - \varkappa_n \delta^\rho_\beta\right) \mathrm{d}t^\beta = 0, \quad \rho = 1, 2. \tag{3.87}$$

A nontrivial solution of this homogeneous equation system exists if and only if

$$\begin{vmatrix} b^1_{\cdot 1} - \varkappa_n & b^1_{\cdot 2} \\ b^2_{\cdot 1} & b^2_{\cdot 2} - \varkappa_n \end{vmatrix} = 0. \tag{3.88}$$

Writing out the above determinant we can also write

$$\varkappa_n^2 - b^\alpha_{\cdot\alpha} \varkappa_n + \left|b^\alpha_{\cdot\beta}\right| = 0. \tag{3.89}$$

The maximal and minimal curvatures $\varkappa_1$ and $\varkappa_2$ resulting from this quadratic equation are called the principal curvatures. One can show that directions of principal curvatures are mutually orthogonal (see Theorem 4.5, Sect. 4). These directions are called principal directions of normal curvature or curvature directions (see also [25]).

According to the Vieta theorem the product of principal curvatures can be expressed by

$$K = \varkappa_1 \varkappa_2 = \left| b_{\beta}^{\alpha} \right| = \frac{b}{g^2}, \tag{3.90}$$

where

$$b = |b_{\alpha\beta}| = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = b_{11}b_{22} - (b_{12})^2, \tag{3.91}$$

$$g^2 = [\boldsymbol{g}_1 \boldsymbol{g}_2 \boldsymbol{g}_3]^2 = \begin{vmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{vmatrix} = g_{11}g_{22} - (g_{12})^2. \tag{3.92}$$

For the arithmetic mean of the principal curvatures we further obtain

$$H = \frac{1}{2}(\varkappa_1 + \varkappa_2) = \frac{1}{2} b_{\alpha}^{\alpha}. \tag{3.93}$$

The values K (3.90) and H (3.93) do not depend on the direction of the normal section and are called the Gaussian and mean curvatures, respectively. In terms of K and H the solutions of the quadratic equation (3.89) can simply be given by

$$\varkappa_{1,2} = H \pm \sqrt{H^2 - K}. \tag{3.94}$$

One recognizes that the sign of the Gaussian curvature K (3.90) is defined by the sign of b (3.91). For positive b both $\varkappa_1$ and $\varkappa_2$ are positive or negative so that $\varkappa_n$ has the same sign for all directions of the normal sections at $\boldsymbol{r}\left(t^1, t^2\right)$. In other words, the orientation of $\boldsymbol{a}_2$ with respect to $\boldsymbol{g}_3$ remains constant. Such a point of the surface is called elliptic.

For $b < 0$, principal curvatures are of different signs so that different normal sections are characterized by different orientations of $\boldsymbol{a}_2$ with respect to $\boldsymbol{g}_3$. There are two directions of the normal sections with zero curvature. Such normal sections are referred to as asymptotic directions. The corresponding point of the surface is called hyperbolic or saddle point.

In the intermediate case $b = 0$, $\varkappa_n$ does not change sign. There is only one asymptotic direction which coincides with one of the principal directions (of $\varkappa_1$ or $\varkappa_2$). The corresponding point of the surface is called parabolic point.

Example. Torus. A torus is a surface obtained by rotating a circle about a coplanar axis (see Fig. 3.4). Additionally we assume that the rotation axis lies outside of the circle. Accordingly, the torus can be defined by

$$\boldsymbol{r}\left(t^1, t^2\right) = \left(R_0 + R\cos t^2\right)\cos t^1 \boldsymbol{e}_1 + \left(R_0 + R\cos t^2\right)\sin t^1 \boldsymbol{e}_2 + R\sin t^2 \boldsymbol{e}_3, \tag{3.95}$$

where R_0 is the radius of the circle and R is the distance between its center and the rotation axis. By means of (3.62) and (3.67) we obtain

$$\boldsymbol{g}_1 = -\left(R_0 + R\cos t^2\right)\sin t^1 \boldsymbol{e}_1 + \left(R_0 + R\cos t^2\right)\cos t^1 \boldsymbol{e}_2,$$

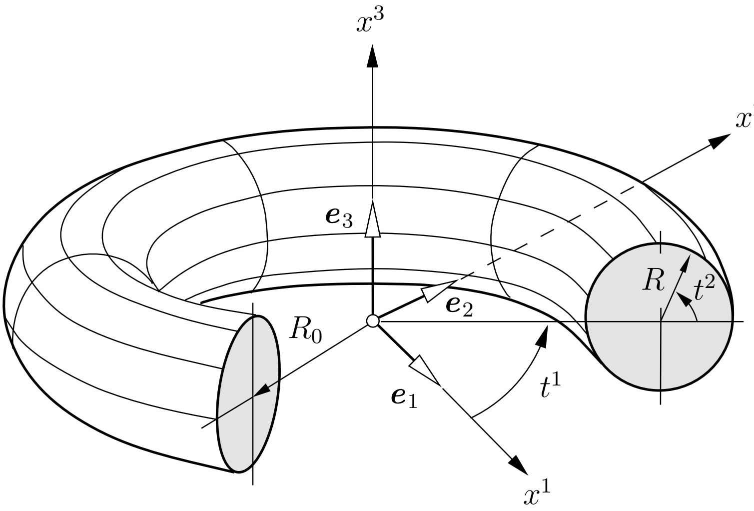

Fig. 3.4. Torus

$$\boldsymbol{g}_2 = -R\cos t^1 \sin t^2 \boldsymbol{e}_1 - R\sin t^1 \sin t^2 \boldsymbol{e}_2 + R\cos t^2 \boldsymbol{e}_3,$$
$$\boldsymbol{g}_3 = \cos t^1 \cos t^2 \boldsymbol{e}_1 + \sin t^1 \cos t^2 \boldsymbol{e}_2 + \sin t^2 \boldsymbol{e}_3. \tag{3.96}$$

Thus, the coefficients (3.64) of the first fundamental form (3.65) are given by

$$g_{11} = \left(R_0 + R\cos t^2\right)^2, \quad g_{12} = 0, \quad g_{22} = R^2. \tag{3.97}$$

In order to express coefficients (3.74) of the second fundamental form (3.82) we first calculate derivatives of the tangent vectors $(3.96)_{1,2}$

$$\boldsymbol{g}_{1,1} = -\left(R_0 + R\cos t^2\right)\cos t^1 \boldsymbol{e}_1 - \left(R_0 + R\cos t^2\right)\sin t^1 \boldsymbol{e}_2,$$
$$\boldsymbol{g}_{1,2} = \boldsymbol{g}_{2,1} = R\sin t^1 \sin t^2 \boldsymbol{e}_1 - R\cos t^1 \sin t^2 \boldsymbol{e}_2,$$
$$\boldsymbol{g}_{2,2} = -R\cos t^1 \cos t^2 \boldsymbol{e}_1 - R\sin t^1 \cos t^2 \boldsymbol{e}_2 - R\sin t^2 \boldsymbol{e}_3. \tag{3.98}$$

Inserting these expressions as well as $(3.96)_3$ into (3.74) we obtain

$$b_{11} = -\left(R_0 + R\cos t^2\right)\cos t^2, \quad b_{12} = b_{21} = 0, \quad b_{22} = -R. \tag{3.99}$$

In view of (3.79) and (3.97) $b_1^2 = b_2^1 = 0$. Thus, the solution of the equation system (3.88) delivers

$$\varkappa_1 = b_1^1 = \frac{b_{11}}{g_{11}} = -\frac{\cos t^2}{R_0 + R\cos t^2}, \quad \varkappa_2 = b_2^2 = \frac{b_{22}}{g_{22}} = -R^{-1}. \tag{3.100}$$

Comparing this result with (3.83) we see that the coordinate lines of the torus (3.95) coincide with the principal directions of the normal curvature. Hence, by (3.90)

$$K = \varkappa_1 \varkappa_2 = \frac{\cos t^2}{R\left(R_0 + R\cos t^2\right)}. \qquad (3.101)$$

Thus, points of the torus for which $-\pi/2 < t^2 < \pi/2$ are elliptic while points for which $\pi/2 < t^2 < 3\pi/2$ are hyperbolic. Points of the coordinates lines $t^2 = -\pi/2$ and $t^2 = \pi/2$ are parabolic.

3.3 Application to Shell Theory

Geometry of the shell continuum. Let us consider a surface in the three-dimensional Euclidean space defined by (3.52) as

$$\boldsymbol{r} = \boldsymbol{r}\left(t^1, t^2\right), \quad \boldsymbol{r} \in \mathbb{E}^3 \qquad (3.102)$$

and bounded by a closed curve C (Fig. 3.5). The shell continuum can then be described by a vector function

$$\boldsymbol{r}^* = \boldsymbol{r}^*\left(t^1, t^2, t^3\right) = \boldsymbol{r}\left(t^1, t^2\right) + \boldsymbol{g}_3 t^3, \qquad (3.103)$$

where the unit vector $\boldsymbol{g}_3$ is defined by (3.62) and (3.67) while $-h/2 \le t^3 \le h/2$. The surface (3.102) is referred to as the middle surface of the shell. The thickness of the shell h is assumed to be small in comparison to its other dimensions as for example the minimal curvature radius of the middle surface.

Every fixed value of the thickness coordinate t^3 defines a surface $\boldsymbol{r}^*\left(t^1, t^2\right)$ whose geometrical variables are obtained according to (1.40), (3.62), (3.64), (3.79), (3.80), (3.90), (3.93) and (3.103) as follows.

$$\boldsymbol{g}^*_\alpha = \boldsymbol{r}^*{,_\alpha} = \boldsymbol{g}_\alpha + t^3 \boldsymbol{g}_3{,_\alpha} = \left(\delta^\rho_\alpha - t^3 b^\rho_\alpha\right)\boldsymbol{g}_\rho, \quad \alpha = 1, 2, \qquad (3.104)$$

$$\boldsymbol{g}^*_3 = \frac{\boldsymbol{g}^*_1 \times \boldsymbol{g}^*_2}{\left\|\boldsymbol{g}^*_1 \times \boldsymbol{g}^*_2\right\|} = \boldsymbol{r}^*{,_3} = \boldsymbol{g}_3, \qquad (3.105)$$

$$g^*_{\alpha\beta} = \boldsymbol{g}^*_\alpha \cdot \boldsymbol{g}^*_\beta = g_{\alpha\beta} - 2t^3 b_{\alpha\beta} + \left(t^3\right)^2 b_{\alpha\rho} b^\rho_\beta, \quad \alpha, \beta = 1, 2, \qquad (3.106)$$

$$\begin{aligned} g^* &= \left[\boldsymbol{g}^*_1 \boldsymbol{g}^*_2 \boldsymbol{g}^*_3\right] = \left[\left(\delta^\rho_1 - t^3 b^\rho_1\right)\boldsymbol{g}_\rho \, \left(\delta^\gamma_2 - t^3 b^\gamma_2\right)\boldsymbol{g}_\gamma \boldsymbol{g}_3\right] \\ &= \left(\delta^\rho_1 - t^3 b^\rho_1\right)\left(\delta^\gamma_2 - t^3 b^\gamma_2\right) g e_{\rho\gamma 3} = g\left|\delta^\alpha_\beta - t^3 b^\alpha_\beta\right| \\ &= g\left[1 - 2t^3 H + \left(t^3\right)^2 K\right]. \end{aligned} \qquad (3.107)$$

The factor in brackets in the latter expression

$$\mu = \frac{g^*}{g} = 1 - 2t^3 H + \left(t^3\right)^2 K \qquad (3.108)$$

is called the shell shifter.

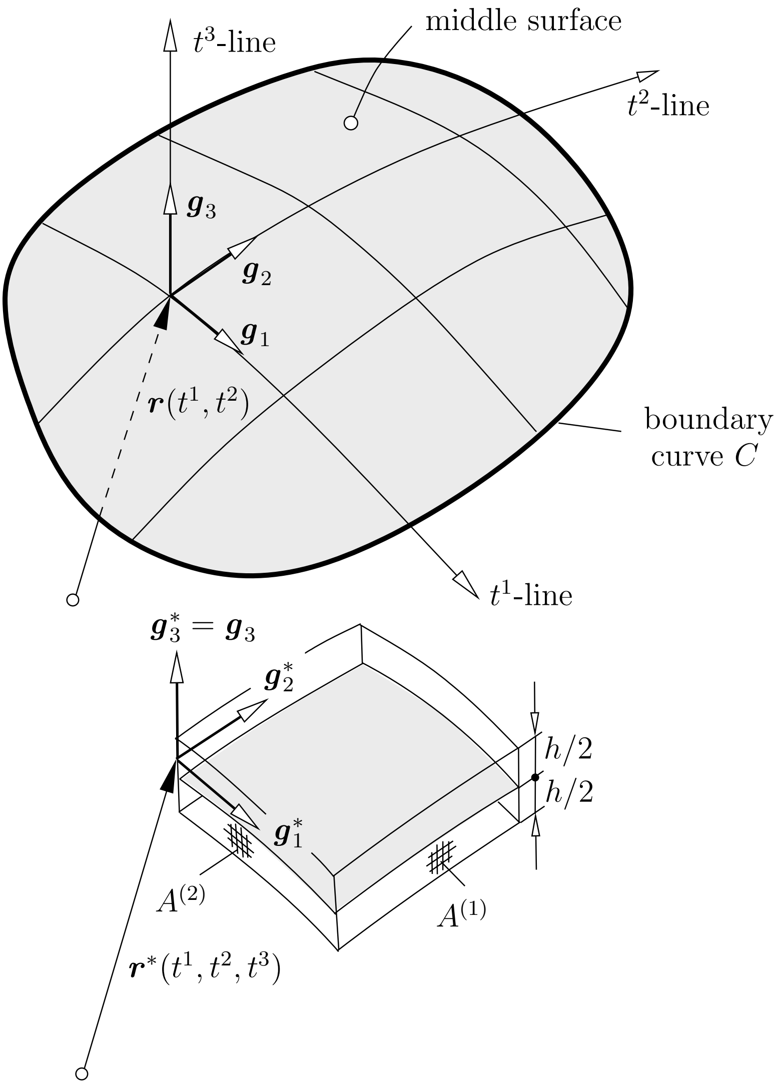

Fig. 3.5. Geometry of the shell continuum

Internal force variables. Let us consider an element of the shell continuum (see Fig. 3.6) bounded by the coordinate lines t^α and $t^\alpha + \Delta t^\alpha$ $(\alpha = 1, 2)$. One defines the force vector $\boldsymbol{f}^\alpha$ and the couple vector $\boldsymbol{m}^\alpha$ relative to the middle surface of the shell, respectively, by

$$\boldsymbol{f}^\alpha = \int_{-h/2}^{h/2} \mu \boldsymbol{\sigma} \boldsymbol{g}^{*\alpha} \mathrm{d}t^3, \quad \boldsymbol{m}^\alpha = \int_{-h/2}^{h/2} \mu \boldsymbol{r}^* \times (\boldsymbol{\sigma} \boldsymbol{g}^{*\alpha}) \, \mathrm{d}t^3, \quad \alpha = 1, 2, \quad (3.109)$$

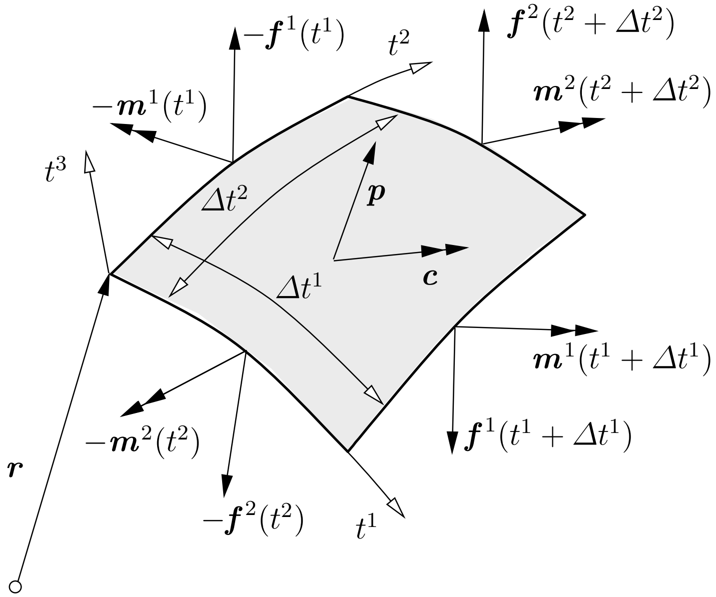

Fig. 3.6. Force variables related to the middle surface of the shell

where $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor on the boundary surface $A^{(\alpha)}$ spanned on the coordinate lines t^3 and t^β $(\beta \neq \alpha)$. The unit normal to this boundary surface is given by

$$\boldsymbol{n}^{(\alpha)} = \frac{\boldsymbol{g}^{*\alpha}}{\|\boldsymbol{g}^{*\alpha}\|} = \frac{\boldsymbol{g}^{*\alpha}}{\sqrt{g^{*\alpha\alpha}}} = \frac{g^*}{\sqrt{g^*_{\beta\beta}}}\boldsymbol{g}^{*\alpha}, \quad \beta \neq \alpha = 1, 2, \tag{3.110}$$

where we keep in mind that $\boldsymbol{g}^{*\alpha} \cdot \boldsymbol{g}^*_\beta = \boldsymbol{g}^{*\alpha} \cdot \boldsymbol{g}_3 = 0$ and (see Exercise 3.7)

$$g^{*\alpha\alpha} = \frac{g^*_{\beta\beta}}{g^{*2}}, \quad \beta \neq \alpha = 1, 2. \tag{3.111}$$

Applying the Cauchy theorem (1.72) and bearing (3.108) in mind we obtain

$$\boldsymbol{f}^\alpha = \frac{1}{g} \int\limits_{-h/2}^{h/2} \sqrt{g^*_{\beta\beta}}\boldsymbol{t}\mathrm{d}t^3, \quad \boldsymbol{m}^\alpha = \frac{1}{g} \int\limits_{-h/2}^{h/2} \sqrt{g^*_{\beta\beta}}\,(\boldsymbol{r}^* \times \boldsymbol{t})\,\mathrm{d}t^3, \tag{3.112}$$

where again $\beta \neq \alpha = 1, 2$ and $\boldsymbol{t}$ denotes the Cauchy stress vector. The force and couple resulting on the whole boundary surface can thus be expressed respectively by

$$\int\limits_{A^{(\alpha)}} \boldsymbol{t}\mathrm{d}A^{(\alpha)} = \int\limits_{t^\beta}^{t^\beta+\Delta t^\beta} \int\limits_{-h/2}^{h/2} \boldsymbol{t}\sqrt{g^*_{\beta\beta}}\mathrm{d}t^3\mathrm{d}t^\beta = \int\limits_{t^\beta}^{t^\beta+\Delta t^\beta} g\boldsymbol{f}^\alpha\mathrm{d}t^\beta, \tag{3.113}$$

$$\int\limits_{A^{(\alpha)}} (\boldsymbol{r}^* \times \boldsymbol{t})\, \mathrm{d}A^{(\alpha)} = \int\limits_{t^\beta}^{t^\beta+\Delta t^\beta} \int\limits_{-h/2}^{h/2} (\boldsymbol{r}^* \times \boldsymbol{t}) \sqrt{g^*_{\beta\beta}} \mathrm{d}t^3 \mathrm{d}t^\beta$$

$$= \int\limits_{t^\beta}^{t^\beta+\Delta t^\beta} g\boldsymbol{m}^\alpha \mathrm{d}t^\beta, \quad \beta \neq \alpha = 1, 2, \tag{3.114}$$

where we make use of the relation

$$\mathrm{d}A^{(\alpha)} = g^* \sqrt{g^{*\alpha\alpha}} \mathrm{d}t^\beta \mathrm{d}t^3 = \sqrt{g^*_{\beta\beta}} \mathrm{d}t^\beta \mathrm{d}t^3, \quad \beta \neq \alpha = 1, 2 \tag{3.115}$$

following immediately from (2.96) and (3.111).

The force and couple vectors (3.109) are usually represented with respect to the basis related to the middle surface as (see also [1])

$$\boldsymbol{f}^\alpha = f^{\alpha\beta} \boldsymbol{g}_\beta + q^\alpha \boldsymbol{g}_3, \quad \boldsymbol{m}^\alpha = m^{\alpha\beta} \boldsymbol{g}_3 \times \boldsymbol{g}_\beta = g\, e_{3\beta\rho} m^{\alpha\beta} \boldsymbol{g}^\rho. \tag{3.116}$$

In shell theory, their components are denoted as follows.

$f^{\alpha\beta}$ - components of the stress resultant tensor,

q^α - components of the transverse shear stress vector,

$m^{\alpha\beta}$ - components of the moment tensor.

External force variables. One defines the load force vector and the load moment vector related to a unit area of the middle surface, respectively by

$$\boldsymbol{p} = p^i \boldsymbol{g}_i, \quad \boldsymbol{c} = c^\rho \boldsymbol{g}_3 \times \boldsymbol{g}_\rho. \tag{3.117}$$

The load moment vector $\boldsymbol{c}$ is thus assumed to be tangential to the middle surface. The resulting force and couple can be expressed respectively by

$$\int\limits_{t^2}^{t^2+\Delta t^2} \int\limits_{t^1}^{t^1+\Delta t^1} \boldsymbol{p} g \mathrm{d}t^1 \mathrm{d}t^2, \qquad \int\limits_{t^2}^{t^2+\Delta t^2} \int\limits_{t^1}^{t^1+\Delta t^1} \boldsymbol{c} g \mathrm{d}t^1 \mathrm{d}t^2. \tag{3.118}$$

Equilibrium conditions. Taking (3.113) and $(3.118)_1$ into account the force equilibrium condition of the shell element can be expressed as

$$\sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{2} \int\limits_{t^\beta}^{t^\beta+\Delta t^\beta} \left[g\left(t^\alpha + \Delta t^\alpha\right) \boldsymbol{f}^\alpha \left(t^\alpha + \Delta t^\alpha\right) - g\left(t^\alpha\right) \boldsymbol{f}^\alpha \left(t^\alpha\right) \right] \mathrm{d}t^\beta$$

$$+\int_{t^2}^{t^2+\Delta t^2}\int_{t^1}^{t^1+\Delta t^1} \boldsymbol{p} g \mathrm{d}t^1\mathrm{d}t^2 = \boldsymbol{0}. \tag{3.119}$$

Rewriting the first integral in (3.119) we further obtain

$$\int_{t^2}^{t^2+\Delta t^2}\int_{t^1}^{t^1+\Delta t^1} \left[\left(g\boldsymbol{f}^{\alpha}\right),_{\alpha} + g\boldsymbol{p}\right]\mathrm{d}t^1\mathrm{d}t^2 = \boldsymbol{0}. \tag{3.120}$$

Since the latter condition holds for all shell elements we infer that

$$\left(g\boldsymbol{f}^{\alpha}\right),_{\alpha} + g\boldsymbol{p} = \boldsymbol{0}, \tag{3.121}$$

which leads by virtue of (2.98) and $(3.73)_2$ to

$$\boldsymbol{f}^{\alpha}|_{\alpha} + \boldsymbol{p} = \boldsymbol{0}, \tag{3.122}$$

where the covariant derivative is formally applied to the vectors $\boldsymbol{f}^{\alpha}$ according to $(3.76)_1$.

In a similar fashion we can treat the moment equilibrium. In this case, we obtain instead of (3.121) the following condition

$$\left[g\left(\boldsymbol{m}^{\alpha} + \boldsymbol{r}\times\boldsymbol{f}^{\alpha}\right)\right],_{\alpha} + g\boldsymbol{r}\times\boldsymbol{p} + g\boldsymbol{c} = \boldsymbol{0}. \tag{3.123}$$

With the aid of (3.62) and keeping (3.122) in mind, it finally delivers

$$\boldsymbol{m}^{\alpha}|_{\alpha} + \boldsymbol{g}_{\alpha}\times\boldsymbol{f}^{\alpha} + \boldsymbol{c} = \boldsymbol{0}. \tag{3.124}$$

In order to rewrite the equilibrium conditions (3.122) and (3.124) in component form we further utilize representations (3.116), (3.117) and apply the product rule of differentiation for the covariant derivative (see, e.g., (2.92-2.94)). By virtue of (3.78) and (3.80) it delivers

$$\left(f^{\alpha\rho}|_{\alpha} - b^{\rho}_{\alpha}q^{\alpha} + p^{\rho}\right)\boldsymbol{g}_{\rho} + \left(f^{\alpha\beta}b_{\alpha\beta} + q^{\alpha}|_{\alpha} + p^3\right)\boldsymbol{g}_3 = \boldsymbol{0}, \tag{3.125}$$

$$\left(m^{\alpha\rho}|_{\alpha} - q^{\rho} + c^{\rho}\right)\boldsymbol{g}_3\times\boldsymbol{g}_{\rho} + g\, e_{\alpha\beta3}\tilde{f}^{\alpha\beta}\boldsymbol{g}_3 = \boldsymbol{0} \tag{3.126}$$

with a new variable

$$\tilde{f}^{\alpha\beta} = f^{\alpha\beta} + b^{\beta}_{\gamma}m^{\gamma\alpha}, \quad \alpha,\beta = 1,2 \tag{3.127}$$

called pseudo-stress resultant. Keeping in mind that the vectors $\boldsymbol{g}_i$ $(i = 1,2,3)$ are linearly independent we thus obtain the following scalar force equilibrium conditions

$$f^{\alpha\rho}|_{\alpha} - b^{\rho}_{\alpha}q^{\alpha} + p^{\rho} = 0, \quad \rho = 1,2, \tag{3.128}$$

$$b_{\alpha\beta}f^{\alpha\beta} + q^{\alpha}|_{\alpha} + p^3 = 0 \tag{3.129}$$

and moment equilibrium conditions

$$m^{\alpha\rho}|_\alpha - q^\rho + c^\rho = 0, \quad \rho = 1, 2, \tag{3.130}$$

$$\tilde{f}^{\alpha\beta} = \tilde{f}^{\beta\alpha}, \quad \alpha, \beta = 1, 2, \quad \alpha \neq \beta. \tag{3.131}$$

With the aid of (3.127) one can finally eliminate the components of the stress resultant tensor $f^{\alpha\beta}$ from (3.128) and (3.129). This leads to the following equation system

$$\tilde{f}^{\alpha\rho}|_\alpha - \left(b^\rho_\gamma m^{\gamma\alpha}\right)|_\alpha - b^\rho_\alpha q^\alpha + p^\rho = 0, \quad \rho = 1, 2, \tag{3.132}$$

$$b_{\alpha\beta}\tilde{f}^{\alpha\beta} - b_{\alpha\beta} b^\beta_\gamma m^{\gamma\alpha} + q^\alpha|_\alpha + p^3 = 0, \tag{3.133}$$

$$m^{\alpha\rho}|_\alpha - q^\rho + c^\rho = 0, \quad \rho = 1, 2, \tag{3.134}$$

where the latter relation is repeated from (3.130) for completeness.

Example. Equilibrium equations of plate theory. In this case, the middle surface of the shell is a plane (3.55) for which

$$b_{\alpha\beta} = b^\alpha_\beta = 0, \quad \alpha, \beta = 1, 2. \tag{3.135}$$

Thus, the equilibrium equations (3.132-3.134) simplify to

$$f^{\alpha\rho}{}_{,\alpha} + p^\rho = 0, \quad \rho = 1, 2, \tag{3.136}$$

$$q^\alpha{}_{,\alpha} + p^3 = 0, \tag{3.137}$$

$$m^{\alpha\rho}{}_{,\alpha} - q^\rho + c^\rho = 0, \quad \rho = 1, 2, \tag{3.138}$$

where in view of (3.127) and (3.131) $f^{\alpha\beta} = f^{\beta\alpha}$ $(\alpha \neq \beta = 1, 2)$.

Example. Equilibrium equations of membrane theory. The membrane theory assumes that the shell is moment free so that

$$m^{\alpha\beta} = 0, \quad c^\beta = 0, \quad \alpha, \beta = 1, 2. \tag{3.139}$$

In this case, the equilibrium equations (3.132-3.134) reduce to

$$f^{\alpha\rho}|_\alpha + p^\rho = 0, \quad \rho = 1, 2, \tag{3.140}$$

$$b_{\alpha\beta} f^{\alpha\beta} + p^3 = 0, \tag{3.141}$$

$$q^\rho = 0, \quad \rho = 1, 2, \tag{3.142}$$

where again $f^{\alpha\beta} = f^{\beta\alpha}$ $(\alpha \neq \beta = 1, 2)$.

Exercises

3.1. Show that a curve $\boldsymbol{r}(s)$ is a straight line if $\varkappa(s) \equiv 0$ for any s .

3.2. Show that the curves $\boldsymbol{r}(s)$ and $\boldsymbol{r}'(s) = \boldsymbol{r}(-s)$ have the same curvature and torsion.

3.3. Show that a curve $\boldsymbol{r}(s)$ characterized by zero torsion $\tau(s) \equiv 0$ for any s lies in a plane.

3.4. Evaluate the Christoffel symbols of the second kind, the coefficients of the first and second fundamental forms, the Gaussian and mean curvatures for the cylinder (3.56).

3.5. Evaluate the Christoffel symbols of the second kind, the coefficients of the first and second fundamental forms, the Gaussian and mean curvatures for the sphere (3.58).

3.6. For the so-called hyperbolic paraboloidal surface defined by

$$\boldsymbol{r}\left(t^1, t^2\right) = t^1 \boldsymbol{e}_1 + t^2 \boldsymbol{e}_2 + \frac{t^1 t^2}{c} \boldsymbol{e}_3, \quad c > 0, \tag{3.143}$$

evaluate the tangent vectors to the coordinate lines, the coefficients of the first and second fundamental forms, the Gaussian and mean curvatures.

3.7. Verify relation (3.111).

3.8. Write out equilibrium equations (3.140-3.141) of the membrane theory for a cylindrical shell and a spherical shell.