# 4

# Eigenvalue Problem and Spectral Decomposition of Second-Order Tensors

## 4.1 Complexification

So far we have considered solely real vectors and real vector spaces. For the purposes of this chapter an introduction of complex vectors is, however, necessary. Indeed, in the following we will see that the existence of a solution of an eigenvalue problem even for real second-order tensors can be guaranteed only within a complex vector space. In order to define the complex vector space let us consider ordered pairs $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ of real vectors $\boldsymbol{x}$ and $\boldsymbol{y} \in \mathbb{E}^n$. The sum of two such pairs is defined by [14]

$$\langle \boldsymbol{x}_1, \boldsymbol{y}_1 \rangle + \langle \boldsymbol{x}_2, \boldsymbol{y}_2 \rangle = \langle \boldsymbol{x}_1 + \boldsymbol{x}_2, \boldsymbol{y}_1 + \boldsymbol{y}_2 \rangle . \tag{4.1}$$

Further, we define the product of a pair $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ by a complex number $\alpha + \mathrm{i}\beta$ by

$$(\alpha + \mathrm{i}\beta) \langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \alpha\boldsymbol{x} - \beta\boldsymbol{y}, \beta\boldsymbol{x} + \alpha\boldsymbol{y} \rangle , \tag{4.2}$$

where $\alpha, \beta \in \mathbb{R}$ and $\mathrm{i} = \sqrt{-1}$. These formulas can easily be recovered assuming that

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x} + \mathrm{i}\boldsymbol{y}. \tag{4.3}$$

The definitions (4.1) and (4.2) enriched by the zero pair $\langle \boldsymbol{0}, \boldsymbol{0} \rangle$ are sufficient to ensure that the axioms (A.1-A.4) and (B.1-B.4) of Chap. 1 are valid. Thus, the set of all pairs $\boldsymbol{z} = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$ characterized by the above properties forms a vector space referred to as complex vector space. Every basis $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ of the underlying Euclidean space $\mathbb{E}^n$ represents simultaneously a basis of the corresponding complexified space. Indeed, for every complex vector within this space

$$\boldsymbol{z} = \boldsymbol{x} + \mathrm{i}\boldsymbol{y}, \tag{4.4}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}^n$ and consequently

$$\boldsymbol{x} = x^i \boldsymbol{g}_i, \quad \boldsymbol{y} = y^i \boldsymbol{g}_i, \tag{4.5}$$

we can write

$$\boldsymbol{z} = \left(x^i + \mathrm{i} y^i\right) \boldsymbol{g}_i. \tag{4.6}$$

Thus, the dimension of the complexified space coincides with the dimension of the original real vector space. Using this fact we will denote the complex vector space based on $\mathbb{E}^n$ by $\mathbb{C}^n$. Clearly, $\mathbb{E}^n$ represents a subset of $\mathbb{C}^n$.

For every vector $\boldsymbol{z} \in \mathbb{C}^n$ given by (4.4) one defines a complex conjugate counterpart by

$$\overline{\boldsymbol{z}} = \boldsymbol{x} - \mathrm{i}\boldsymbol{y}. \tag{4.7}$$

Of special interest is the scalar product of two complex vectors, say $\boldsymbol{z}_1 = \boldsymbol{x}_1 + \mathrm{i}\boldsymbol{y}_1$ and $\boldsymbol{z}_2 = \boldsymbol{x}_2 + \mathrm{i}\boldsymbol{y}_2$, which we define by (see also [4])

$$(\boldsymbol{x}_1 + \mathrm{i}\boldsymbol{y}_1) \cdot (\boldsymbol{x}_2 + \mathrm{i}\boldsymbol{y}_2) = \boldsymbol{x}_1 \cdot \boldsymbol{x}_2 - \boldsymbol{y}_1 \cdot \boldsymbol{y}_2 + \mathrm{i}\,(\boldsymbol{x}_1 \cdot \boldsymbol{y}_2 + \boldsymbol{y}_1 \cdot \boldsymbol{x}_2)\,. \tag{4.8}$$

This scalar product is commutative (C.1), distributive (C.2) and linear in each factor (C.3). Thus, it differs from the classical scalar product of complex vectors given in terms of the complex conjugate (see, e.g., [14]). As a result, the axiom (C.4) does not generally hold. For instance, one can easily imagine a non-zero complex vector whose scalar product with itself is zero. For complex vectors with the scalar product (4.8) the notions of length, orthogonality or parallelity can hardly be interpreted geometrically.

However, for complex vectors the axiom (C.4) can be reformulated by

$$\boldsymbol{z} \cdot \overline{\boldsymbol{z}} \geq 0, \quad \boldsymbol{z} \cdot \overline{\boldsymbol{z}} = 0 \quad \text{if and only if} \quad \boldsymbol{z} = \boldsymbol{0}. \tag{4.9}$$

Indeed, using (4.4), (4.7) and (4.8) we obtain $\boldsymbol{z} \cdot \overline{\boldsymbol{z}} = \boldsymbol{x} \cdot \boldsymbol{x} + \boldsymbol{y} \cdot \boldsymbol{y}$. Bearing in mind that the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ belong to the Euclidean space this immediately implies (4.9).

As we learned in Chap. 1, the Euclidean space $\mathbb{E}^n$ is characterized by the existence of an orthonormal basis (1.8). This can now be postulated for the complex vector space $\mathbb{C}^n$ as well, because $\mathbb{C}^n$ includes $\mathbb{E}^n$ by the very definition. Also Theorem 1.6 remains valid since it has been proved without making use of the property (C.4). Thus, we may state that for every basis in $\mathbb{C}^n$ there exists a unique dual basis.

The last step of the complexification is a generalization of a linear mapping on complex vectors. This can be achieved by setting for every tensor $\mathbf{A} \in \mathbf{Lin}^n$

$$\mathbf{A}\,(\boldsymbol{x} + \mathrm{i}\boldsymbol{y}) = \mathbf{A}\boldsymbol{x} + \mathrm{i}\,(\mathbf{A}\boldsymbol{y})\,. \tag{4.10}$$

## 4.2 Eigenvalue Problem, Eigenvalues and Eigenvectors

Let $\mathbf{A} \in \mathbf{Lin}^n$ be a second-order tensor. The equation

$$\mathbf{A}\boldsymbol{a} = \lambda \boldsymbol{a}, \quad \boldsymbol{a} \neq \boldsymbol{0} \tag{4.11}$$

is referred to as the eigenvalue problem of the tensor $\mathbf{A}$. The non-zero vector $\boldsymbol{a} \in \mathbb{C}^n$ satisfying this equation is called an eigenvector of $\mathbf{A}$; $\lambda$ is called an eigenvalue of $\mathbf{A}$. It is clear that any product of an eigenvector with any (real or complex) scalar is again an eigenvector.

The eigenvalue problem (4.11) and the corresponding eigenvector $\boldsymbol{a}$ can be regarded as the right eigenvalue problem and the right eigenvector, respectively. In contrast, one can define the left eigenvalue problem by

$$\boldsymbol{b}\mathbf{A} = \lambda \boldsymbol{b}, \quad \boldsymbol{b} \neq \boldsymbol{0}, \tag{4.12}$$

where $\boldsymbol{b} \in \mathbb{C}^n$ is the left eigenvector. In view of (1.110), every right eigenvector of $\mathbf{A}$ represents the left eigenvector of $\mathbf{A}^{\mathrm{T}}$ and vice versa. In the following, unless indicated otherwise, we will mean the right eigenvalue problem and the right eigenvector.

Mapping (4.11) by $\mathbf{A}$ several times we obtain

$$\mathbf{A}^k \boldsymbol{a} = \lambda^k \boldsymbol{a}, \quad k = 1, 2, \ldots \tag{4.13}$$

This leads to the following (spectral mapping) theorem.

**Theorem 4.1.** *Let $\lambda$ be an eigenvalue of the tensor $\mathbf{A}$ and let $g(\mathbf{A}) = \sum_{k=0}^{m} a_k \mathbf{A}^k$ be a polynomial of $\mathbf{A}$. Then $g(\lambda) = \sum_{k=0}^{m} a_k \lambda^k$ is the eigenvalue of $g(\mathbf{A})$.*

*Proof.* Let $\boldsymbol{a}$ be an eigenvector of $\mathbf{A}$ associated with $\lambda$. Then, in view of (4.13)

$$g(\mathbf{A})\,\boldsymbol{a} = \sum_{k=0}^{m} a_k \mathbf{A}^k \boldsymbol{a} = \sum_{k=0}^{m} a_k \lambda^k \boldsymbol{a} = \left( \sum_{k=0}^{m} a_k \lambda^k \right) \boldsymbol{a} = g(\lambda)\,\boldsymbol{a}.$$

In order to find the eigenvalues of the tensor $\mathbf{A}$ we consider the following representations:

$$\mathbf{A} = \mathrm{A}^{i}_{\cdot j} \boldsymbol{g}_i \otimes \boldsymbol{g}^j, \quad \boldsymbol{a} = a^i \boldsymbol{g}_i, \quad \boldsymbol{b} = b_i \boldsymbol{g}^i, \tag{4.14}$$

where $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ and $\mathcal{G}' = \{\boldsymbol{g}^1, \boldsymbol{g}^2, \ldots, \boldsymbol{g}^n\}$ are two arbitrary mutually dual bases in $\mathbb{E}^n$ and consequently also in $\mathbb{C}^n$. Note that we prefer here the mixed variant representation of the tensor $\mathbf{A}$. Inserting (4.14) into (4.11) and (4.12) further yields

$$\mathrm{A}^{i}_{\cdot j} a^j \boldsymbol{g}_i = \lambda a^i \boldsymbol{g}_i, \quad \mathrm{A}^{i}_{\cdot j} b_i \boldsymbol{g}^j = \lambda b_j \boldsymbol{g}^j,$$

and therefore

$$\left( \mathrm{A}^{i}_{\cdot j} a^j - \lambda a^i \right) \boldsymbol{g}_i = \boldsymbol{0}, \quad \left( \mathrm{A}^{i}_{\cdot j} b_i - \lambda b_j \right) \boldsymbol{g}^j = \boldsymbol{0}. \tag{4.15}$$

Since the vectors $\boldsymbol{g}_i$ on the one side and $\boldsymbol{g}^i$ $(i = 1, 2, \ldots, n)$ on the other side are linearly independent the associated scalar coefficients in (4.15) must be zero. This results in the following two linear homogeneous equation systems

$$\left(\mathrm{A}^i_{\cdot j} - \lambda \delta^i_j\right) a^j = 0, \quad \left(\mathrm{A}^j_{\cdot i} - \lambda \delta^j_i\right) b_j = 0, \quad i = 1, 2, \ldots, n \tag{4.16}$$

with respect to the components of the right eigenvector $\boldsymbol{a}$ and the left eigenvector $\boldsymbol{b}$, respectively. A non-trivial solution of these equation systems exists if and only if

$$\left|\mathrm{A}^i_{\cdot j} - \lambda \delta^i_j\right| = 0, \tag{4.17}$$

where $|\bullet|$ denotes the determinant of a matrix. Eq. (4.17) is called the characteristic equation of the tensor $\mathbf{A}$. Writing out the determinant on the left hand side of this equation one obtains a polynomial of degree $n$ with respect to the powers of $\lambda$

$$p_{\mathbf{A}}(\lambda) = \lambda^n - \lambda^{n-1} \mathrm{I}^{(1)}_{\mathbf{A}} + \ldots + (-1)^k \lambda^{n-k} \mathrm{I}^{(k)}_{\mathbf{A}} + \ldots + (-1)^n \mathrm{I}^{(n)}_{\mathbf{A}}, \tag{4.18}$$

referred to as the characteristic polynomial of the tensor $\mathbf{A}$. Thereby, it can easily be seen that

$$\mathrm{I}^{(1)}_{\mathbf{A}} = \mathrm{A}^i_{\cdot i} = \mathrm{tr}\mathbf{A}, \quad \mathrm{I}^{(n)}_{\mathbf{A}} = \left|\mathrm{A}^i_{\cdot j}\right|. \tag{4.19}$$

The characteristic equation (4.17) can briefly be written as

$$p_{\mathbf{A}}(\lambda_i) = 0. \tag{4.20}$$

According to the fundamental theorem of algebra, a polynomial of degree $n$ has $n$ complex roots which may be multiple. These roots are the eigenvalues $\lambda_i$ $(i = 1, 2, \ldots, n)$ of the tensor $\mathbf{A}$.

Factorizing the characteristic polynomial (4.18) yields

$$p_{\mathbf{A}}(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i). \tag{4.21}$$

Collecting multiple eigenvalues the polynomial (4.21) can further be rewritten as

$$p_{\mathbf{A}}(\lambda) = \prod_{i=1}^{s} (\lambda - \lambda_i)^{r_i}, \tag{4.22}$$

where $s$ $(1 \le s \le n)$ denotes the number of distinct eigenvalues, while $r_i$ is referred to as an algebraic multiplicity of the eigenvalue $\lambda_i$ $(i = 1, 2, \ldots, s)$. It should formally be distinguished from the so-called geometric multiplicity $t_i$, which represents the number of linearly independent eigenvectors associated with this eigenvalue.

**Example. Eigenvalues and eigenvectors of the deformation gradient in the case of simple shear.** In simple shear, the deformation gradient can be given by $\mathbf{F} = \mathrm{F}^{i}_{\cdot j} \boldsymbol{e}_i \otimes \boldsymbol{e}^j$, where

$$\left[\mathrm{F}^{i}_{\cdot j}\right] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.23}$$

and $\gamma$ denotes the amount of shear. The characteristic equation (4.17) for the tensor $\mathbf{F}$ takes thus the form

$$\begin{vmatrix} 1-\lambda & \gamma & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0.$$

Writing out this determinant we obtain

$$(1-\lambda)^3 = 0,$$

which yields one triple eigenvalue

$$\lambda_1 = \lambda_2 = \lambda_3 = 1.$$

The associated (right) eigenvectors $\boldsymbol{a} = a^i \boldsymbol{e}_i$ can be obtained from the equation system $(4.16)_1$ i.e.

$$\left(\mathrm{F}^{i}_{\cdot j} - \lambda \delta^{i}_{j}\right) a^j = 0, \quad i = 1, 2, 3.$$

In view of (4.23) it reduces to the only non-trivial equation

$$a^2 \gamma = 0.$$

Hence, all eigenvectors of $\mathbf{F}$ can be given by $\boldsymbol{a} = a^1 \boldsymbol{e}_1 + a^3 \boldsymbol{e}_3$. They are linear combinations of the only two linearly independent eigenvectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$. Accordingly, the geometric and algebraic multiplicities of the eigenvalue 1 are $t_1 = 2$ and $r_1 = 3$, respectively.

## 4.3 Characteristic Polynomial

By the very definition of the eigenvalue problem (4.11) the eigenvalues are independent of the choice of the basis. This is also the case for the coefficients $\mathrm{I}^{(i)}_{\mathbf{A}}$ $(i = 1, 2, \ldots, n)$ of the characteristic polynomial (4.18) because they uniquely define the eigenvalues and vice versa. These coefficients are called principal invariants of $\mathbf{A}$. Comparing (4.18) with (4.21) and applying the Vieta theorem we obtain the following relations between the principal invariants and eigenvalues:

$$
\begin{aligned}
\mathrm{I}_{\mathbf{A}}^{(1)} &= \lambda_1 + \lambda_2 + \ldots + \lambda_n, \\
\mathrm{I}_{\mathbf{A}}^{(2)} &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \ldots + \lambda_{n-1}\lambda_n, \\
&\vdots \\
\mathrm{I}_{\mathbf{A}}^{(k)} &= \sum_{o_1<o_2<\ldots<o_k}^{n} \lambda_{o_1}\lambda_{o_2}\ldots\lambda_{o_k}, \\
&\vdots \\
\mathrm{I}_{\mathbf{A}}^{(n)} &= \lambda_1\lambda_2\ldots\lambda_n.
\end{aligned}
\tag{4.24}
$$

The principal invariants can also be expressed in terms of the so-called principal traces $\mathrm{tr}\mathbf{A}^k$ $(k = 1, 2, \ldots, n)$. Indeed, by use of (4.13), $(4.19)_1$ and $(4.24)_1$ we first write

$$
\mathrm{tr}\mathbf{A}^k = \lambda_1^k + \lambda_2^k + \ldots + \lambda_n^k, \quad k = 1, 2, \ldots, n. \tag{4.25}
$$

Then, we apply Newton's formula (see e.g. [9]) relating coefficients of a polynomial to its roots represented by the sum of the powers in the form of the right hand side of (4.25). Taking (4.25) into account, Newton's formula can thus be written as

$$
\begin{aligned}
\mathrm{I}_{\mathbf{A}}^{(1)} &= \mathrm{tr}\mathbf{A}, \\
\mathrm{I}_{\mathbf{A}}^{(2)} &= \frac{1}{2}\left(\mathrm{I}_{\mathbf{A}}^{(1)}\mathrm{tr}\mathbf{A} - \mathrm{tr}\mathbf{A}^2\right), \\
\mathrm{I}_{\mathbf{A}}^{(3)} &= \frac{1}{3}\left(\mathrm{I}_{\mathbf{A}}^{(2)}\mathrm{tr}\mathbf{A} - \mathrm{I}_{\mathbf{A}}^{(1)}\mathrm{tr}\mathbf{A}^2 + \mathrm{tr}\mathbf{A}^3\right), \\
&\vdots \\
\mathrm{I}_{\mathbf{A}}^{(k)} &= \frac{1}{k}\left(\mathrm{I}_{\mathbf{A}}^{(k-1)}\mathrm{tr}\mathbf{A} - \mathrm{I}_{\mathbf{A}}^{(k-2)}\mathrm{tr}\mathbf{A}^2 + \ldots + (-1)^{k-1}\,\mathrm{tr}\mathbf{A}^k\right) \\
&= \frac{1}{k}\sum_{i=1}^{k}(-1)^{i-1}\,\mathrm{I}_{\mathbf{A}}^{(k-i)}\mathrm{tr}\mathbf{A}^i, \\
&\vdots \\
\mathrm{I}_{\mathbf{A}}^{(n)} &= \det\mathbf{A},
\end{aligned}
\tag{4.26}
$$

where we set $\mathrm{I}_{\mathbf{A}}^{(0)} = 1$ and

$$
\det\mathbf{A} = \left|\mathrm{A}^{i}_{\cdot j}\right| = \left|\mathrm{A}_{j\cdot}^{\;i}\right| \tag{4.27}
$$

is called the determinant of the tensor $\mathbf{A}$.

**Example. Three-dimensional space.** For illustration, we consider a second-order tensor $\mathbf{A}$ in three-dimensional space. In this case, the characteristic polynomial (4.18) takes the form

$$
p_{\mathbf{A}}(\lambda) = \lambda^3 - \mathrm{I}_{\mathbf{A}}\lambda^2 + \mathrm{II}_{\mathbf{A}}\lambda - \mathrm{III}_{\mathbf{A}}, \tag{4.28}
$$

where

$$\mathrm{I}_{\mathbf{A}} = \mathrm{I}_{\mathbf{A}}^{(1)} = \mathrm{tr}\mathbf{A},$$

$$\mathrm{II}_{\mathbf{A}} = \mathrm{I}_{\mathbf{A}}^{(2)} = \frac{1}{2}\left[(\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}\mathbf{A}^2\right],$$

$$\mathrm{III}_{\mathbf{A}} = \mathrm{I}_{\mathbf{A}}^{(3)} = \frac{1}{3}\left[\mathrm{tr}\mathbf{A}^3 - \frac{3}{2}\mathrm{tr}\mathbf{A}^2\mathrm{tr}\mathbf{A} + \frac{1}{2}(\mathrm{tr}\mathbf{A})^3\right] = \mathrm{det}\mathbf{A} \tag{4.29}$$

are the principal invariants (4.24) of the tensor $\mathbf{A}$. They can alternatively be expressed in terms of the eigenvalues as follows

$$\mathrm{I}_{\mathbf{A}} = \lambda_1 + \lambda_2 + \lambda_3, \quad \mathrm{II}_{\mathbf{A}} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad \mathrm{III}_{\mathbf{A}} = \lambda_1\lambda_2\lambda_3. \tag{4.30}$$

The roots of the cubic polynomial (4.28) can be obtained in a closed form by means of the Cardano formula (see, e.g. [5]) as

$$\lambda_k = \frac{1}{3}\left\{\mathrm{I}_{\mathbf{A}} + 2\sqrt{\mathrm{I}_{\mathbf{A}}^2 - 3\mathrm{II}_{\mathbf{A}}}\cos\frac{1}{3}\left[\vartheta + 2\pi(k-1)\right]\right\}, \quad k = 1,2,3, \tag{4.31}$$

where

$$\vartheta = \arccos\left[\frac{2\mathrm{I}_{\mathbf{A}}^3 - 9\mathrm{I}_{\mathbf{A}}\mathrm{II}_{\mathbf{A}} + 27\mathrm{III}_{\mathbf{A}}}{2\left(\mathrm{I}_{\mathbf{A}}^2 - 3\mathrm{II}_{\mathbf{A}}\right)^{3/2}}\right], \quad \mathrm{I}_{\mathbf{A}}^2 - 3\mathrm{II}_{\mathbf{A}} \neq 0. \tag{4.32}$$

In the case $\mathrm{I}_{\mathbf{A}}^2 - 3\mathrm{II}_{\mathbf{A}} = 0$, the eigenvalues of $\mathbf{A}$ take another form

$$\lambda_k = \frac{1}{3}\mathrm{I}_{\mathbf{A}} + \frac{1}{3}\left(27\mathrm{III}_{\mathbf{A}} - \mathrm{I}_{\mathbf{A}}^3\right)^{1/3}\left[\cos\left(\tfrac{2}{3}\pi k\right) + \mathrm{i}\sin\left(\tfrac{2}{3}\pi k\right)\right], \tag{4.33}$$

where $k = 1,2,3$.

## 4.4 Spectral Decomposition and Eigenprojections

The spectral decomposition is a powerful tool for the tensor analysis and tensor algebra. It enables to gain a deeper insight into the properties of second-order tensors and to represent various useful tensor operations in a relatively simple form. In the spectral decomposition, eigenvectors represent one of the most important ingredients.

**Theorem 4.2.** *The eigenvectors of a second-order tensor corresponding to pairwise distinct eigenvalues are linearly independent.*

*Proof.* Suppose that these eigenvectors are linearly dependent. Among all possible nontrivial linear relations connecting them we can choose one involving the minimal number, say $r$, of eigenvectors $\boldsymbol{a}_i \neq \boldsymbol{0}$ $(i = 1,2,\ldots,r)$. Obviously, $1 < r \leq n$. Thus,

$$\sum_{i=1}^{r} \alpha_i \boldsymbol{a}_i = \boldsymbol{0}, \tag{4.34}$$

where all $\alpha_i$ $(i = 1, 2, \ldots, r)$ are non-zero. We can also write

$$\mathbf{A}\boldsymbol{a}_i = \lambda_i \boldsymbol{a}_i, \quad i = 1, 2, \ldots, r, \tag{4.35}$$

where $\lambda_i \neq \lambda_j$, $(i \neq j = 1, 2, \ldots, r)$. Mapping both sides of (4.34) by $\mathbf{A}$ and taking (4.35) into account we obtain

$$\sum_{i=1}^{r} \alpha_i \mathbf{A}\boldsymbol{a}_i = \sum_{i=1}^{r} \alpha_i \lambda_i \boldsymbol{a}_i = \boldsymbol{0}. \tag{4.36}$$

Multiplying (4.34) by $\lambda_r$ and subtracting from (4.36) yield

$$\boldsymbol{0} = \sum_{i=1}^{r} \alpha_i \left(\lambda_i - \lambda_r\right) \boldsymbol{a}_i = \sum_{i=1}^{r-1} \alpha_i \left(\lambda_i - \lambda_r\right) \boldsymbol{a}_i.$$

In the latter linear combination none of the coefficients is zero. Thus, we have a linear relation involving only $r - 1$ eigenvectors. This contradicts, however, the earlier assumption that $r$ is the smallest number of eigenvectors satisfying such a relation.

**Theorem 4.3.** *Let $\boldsymbol{b}_i$ be a left and $\boldsymbol{a}_j$ a right eigenvector associated with distinct eigenvalues $\lambda_i \neq \lambda_j$ of a tensor $\mathbf{A}$. Then,*

$$\boldsymbol{b}_i \cdot \boldsymbol{a}_j = 0. \tag{4.37}$$

*Proof.* With the aid of (1.73) and taking (4.11) into account we can write

$$\boldsymbol{b}_i \mathbf{A} \boldsymbol{a}_j = \boldsymbol{b}_i \cdot \left(\mathbf{A}\boldsymbol{a}_j\right) = \boldsymbol{b}_i \cdot \left(\lambda_j \boldsymbol{a}_j\right) = \lambda_j \boldsymbol{b}_i \cdot \boldsymbol{a}_j.$$

On the other hand, in view of (4.12)

$$\boldsymbol{b}_i \mathbf{A} \boldsymbol{a}_j = \left(\boldsymbol{b}_i \mathbf{A}\right) \cdot \boldsymbol{a}_j = \left(\boldsymbol{b}_i \lambda_i\right) \cdot \boldsymbol{a}_j = \lambda_i \boldsymbol{b}_i \cdot \boldsymbol{a}_j.$$

Subtracting one equation from another one we obtain

$$\left(\lambda_i - \lambda_j\right) \boldsymbol{b}_i \cdot \boldsymbol{a}_j = 0.$$

Since $\lambda_i \neq \lambda_j$ this immediately implies (4.37).

Now, we proceed with the spectral decomposition of a second-order tensor $\mathbf{A}$. First, we consider the case of $n$ simple eigenvalues. Solving the equation systems (4.16) one obtains for every simple eigenvalue $\lambda_i$ the components of the right eigenvector $\boldsymbol{a}_i$ and the components of the left eigenvector $\boldsymbol{b}_i$ $(i = 1, 2, \ldots, n)$. $n$ right eigenvectors on the one hand and $n$ left eigenvectors on the other hand are linearly independent and form bases of $\mathbb{C}^n$.

Obviously, $\boldsymbol{b}_i \cdot \boldsymbol{a}_i \neq 0\,(i = 1, 2, \ldots, n)$ because otherwise it would contradict (4.37) (see Exercise 1.8). Normalizing the eigenvectors we can thus write

$$\boldsymbol{b}_i \cdot \boldsymbol{a}_j = \delta_{ij}, \quad i, j = 1, 2, \ldots, n. \tag{4.38}$$

Accordingly, the bases $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ are dual to each other such that $\boldsymbol{a}^i = \boldsymbol{b}_i$ and $\boldsymbol{b}^i = \boldsymbol{a}_i\,(i = 1, 2, \ldots, n)$. Now, representing $\mathbf{A}$ with respect to the basis $\boldsymbol{a}_i \otimes \boldsymbol{b}_j$ $(i, j = 1, 2, \ldots, n)$ as $\mathbf{A} = \mathrm{A}^{ij} \boldsymbol{a}_i \otimes \boldsymbol{b}_j$ we obtain with the aid of (1.83), (4.11) and (4.38)

$$\mathrm{A}^{ij} = \boldsymbol{a}^i \mathbf{A} \boldsymbol{b}^j = \boldsymbol{b}_i \mathbf{A} \boldsymbol{a}_j = \boldsymbol{b}_i \cdot (\mathbf{A} \boldsymbol{a}_j) = \boldsymbol{b}_i \cdot (\lambda_j \boldsymbol{a}_j) = \lambda_j \delta_{ij},$$

where $i, j = 1, 2, \ldots, n$. Thus,

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \boldsymbol{a}_i \otimes \boldsymbol{b}_i. \tag{4.39}$$

Next, we consider second-order tensors with multiple eigenvalues. We assume, however, that the algebraic multiplicity $r_i$ of every eigenvalue $\lambda_i$ coincides with its geometric multiplicity $t_i$. In this case we again have $n$ linearly independent right eigenvectors forming a basis of $\mathbb{C}^n$ (Exercise 4.3). We will denote these eigenvectors by $\boldsymbol{a}_i^{(k)}$ $(i = 1, 2, \ldots, s; k = 1, 2, \ldots, r_i)$ where $s$ is the number of pairwise distinct eigenvalues. Constructing the basis $\boldsymbol{b}_j^{(l)}$ dual to $\boldsymbol{a}_i^{(k)}$ such that

$$\boldsymbol{a}_i^{(k)} \cdot \boldsymbol{b}_j^{(l)} = \delta_{ij} \delta^{kl}, \; i, j = 1, 2, \ldots, s; \; k = 1, 2, \ldots, r_i; \; l = 1, 2, \ldots, r_j \tag{4.40}$$

we can write similarly to (4.39)

$$\mathbf{A} = \sum_{i=1}^{s} \lambda_i \sum_{k=1}^{r_i} \boldsymbol{a}_i^{(k)} \otimes \boldsymbol{b}_i^{(k)}. \tag{4.41}$$

The representations of the form (4.39) or (4.41) are called spectral decomposition in diagonal form or, briefly, spectral decomposition. Note that not every second-order tensor $\mathbf{A} \in \mathbf{Lin}^n$ permits the spectral decomposition. The tensors which can be represented by (4.39) or (4.41) are referred to as diagonalizable tensors. For instance, we will show in the next sections that symmetric, skew-symmetric and orthogonal tensors are always diagonalizable. If, however, the algebraic multiplicity of at least one eigenvalue exceeds its geometric multiplicity, the spectral representation is not possible. Such eigenvalues (for which $r_i > t_i$) are called defective eigenvalues. A tensor that has one or more defective eigenvalues is called defective tensor. In Sect. 4.2 we have seen, for example, that the deformation gradient $\mathbf{F}$ represents in the case of simple shear a defective tensor since its triple eigenvalue 1 is defective. Clearly, a simple eigenvalue ($r_i = 1$) cannot be defective. For this reason, a tensor whose all eigenvalues are simple is diagonalizable.

Now, we look again at the spectral decompositions (4.39) and (4.41). With the aid of the abbreviation

$$\mathbf{P}_i = \sum_{k=1}^{r_i} \boldsymbol{a}_i^{(k)} \otimes \boldsymbol{b}_i^{(k)}, \quad i = 1, 2, \ldots, s \tag{4.42}$$

they can be given in a unified form by

$$\mathbf{A} = \sum_{i=1}^{s} \lambda_i \mathbf{P}_i. \tag{4.43}$$

The generally complex tensors $\mathbf{P}_i$ $(i = 1, 2, \ldots, s)$ defined by (4.42) are called eigenprojections. It follows from (4.40) and (4.42) that (Exercise 4.4)

$$\mathbf{P}_i \mathbf{P}_j = \delta_{ij} \mathbf{P}_i, \quad i, j = 1, 2, \ldots, s \tag{4.44}$$

and consequently

$$\mathbf{P}_i \mathbf{A} = \mathbf{A} \mathbf{P}_i = \lambda_i \mathbf{P}_i, \quad i = 1, 2, \ldots, s. \tag{4.45}$$

Bearing in mind that the eigenvectors $\boldsymbol{a}_i^{(k)}$ $(i = 1, 2, \ldots, s; k = 1, 2, \ldots, r_i)$ form a basis of $\mathbb{C}^n$ and taking (4.40) into account we also obtain (Exercise 4.5)

$$\sum_{i=1}^{s} \mathbf{P}_i = \mathbf{I}. \tag{4.46}$$

Due to these properties of eigenprojections (4.42) the spectral representation (4.43) is very suitable for calculating tensor powers, polynomials and other tensor functions defined in terms of power series. Indeed, in view of (4.44) powers of $\mathbf{A}$ can be expressed by

$$\mathbf{A}^k = \sum_{i=1}^{s} \lambda_i^k \mathbf{P}_i, \quad k = 0, 1, 2, \ldots \tag{4.47}$$

For a tensor polynomial it further yields

$$g(\mathbf{A}) = \sum_{i=1}^{s} g(\lambda_i) \mathbf{P}_i. \tag{4.48}$$

For example, the exponential tensor function (1.109) can thus be represented by

$$\exp(\mathbf{A}) = \sum_{i=1}^{s} \exp(\lambda_i) \mathbf{P}_i. \tag{4.49}$$

With the aid of (4.44) and (4.46) the eigenprojections can be obtained without solving the eigenvalue problem in the general form (4.11). To this end, we first consider $s$ polynomial functions $p_i(\lambda)$ $(i = 1, 2, \ldots, s)$ satisfying the following conditions

$$p_i(\lambda_j) = \delta_{ij}, \quad i, j = 1, 2, \ldots, s. \tag{4.50}$$

Thus, by use of (4.48) we obtain

$$p_i(\mathbf{A}) = \sum_{j=1}^{s} p_i(\lambda_j) \mathbf{P}_j = \sum_{j=1}^{s} \delta_{ij} \mathbf{P}_j = \mathbf{P}_i, \quad i = 1, 2, \ldots, s. \tag{4.51}$$

Using Lagrange's interpolation formula (see, e.g., [5]) and assuming that $s \neq 1$ one can represent the functions $p_i(\lambda)$ (4.50) by the following polynomials of degree $s - 1$:

$$p_i(\lambda) = \prod_{\substack{j=1 \\ j \neq i}}^{s} \frac{\lambda - \lambda_j}{\lambda_i - \lambda_j}, \quad i = 1, 2, \ldots, s > 1. \tag{4.52}$$

Considering these expressions in (4.51) we obtain the so-called Sylvester formula as

$$\mathbf{P}_i = \prod_{\substack{j=1 \\ j \neq i}}^{s} \frac{\mathbf{A} - \lambda_j \mathbf{I}}{\lambda_i - \lambda_j}, \quad i = 1, 2, \ldots, s > 1. \tag{4.53}$$

Note that according to (4.46), $\mathbf{P}_1 = \mathbf{I}$ in the the case of $s = 1$. With this result in hand the above representation can be generalized by

$$\mathbf{P}_i = \delta_{1s} \mathbf{I} + \prod_{\substack{j=1 \\ j \neq i}}^{s} \frac{\mathbf{A} - \lambda_j \mathbf{I}}{\lambda_i - \lambda_j}, \quad i = 1, 2, \ldots, s. \tag{4.54}$$

Writing out the product on the right hand side of (4.54) also delivers (see, e.g., [47])

$$\mathbf{P}_i = \frac{1}{D_i} \sum_{p=0}^{s-1} \iota_{i\, s-p-1} \mathbf{A}^p, \quad i = 1, 2, \ldots, s, \tag{4.55}$$

where $\iota_{i0} = 1$,

$$\iota_{ip} = (-1)^p \sum_{1 \leq o_1 \leq \cdots \leq o_p \leq s} \lambda_{o_1} \cdots \lambda_{o_p} \left(1 - \delta_{io_1}\right) \cdots \left(1 - \delta_{io_p}\right),$$

$$D_i = \delta_{1s} + \prod_{\substack{j=1 \\ j \neq i}}^{s} \left(\lambda_i - \lambda_j\right), \quad p = 1, 2, \ldots, s - 1, \quad i = 1, 2, \ldots, s. \tag{4.56}$$

## 4.5 Spectral Decomposition of Symmetric Second-Order Tensors

We begin with some useful theorems concerning eigenvalues and eigenvectors of symmetric tensors.

**Theorem 4.4.** *The eigenvalues of a symmetric second-order tensor* $\mathbf{M} \in \mathbf{S}\mathrm{ym}^n$ *are real, the eigenvectors belong to* $\mathbb{E}^n$.

*Proof.* Let $\lambda$ be an eigenvalue of $\mathbf{M}$ and $\boldsymbol{a}$ a corresponding eigenvector such that according to (4.11)

$$\mathbf{M}\boldsymbol{a} = \lambda\boldsymbol{a}.$$

The complex conjugate counterpart of this equation is

$$\overline{\mathbf{M}}\,\overline{\boldsymbol{a}} = \overline{\lambda}\,\overline{\boldsymbol{a}}.$$

Taking into account that $\mathbf{M}$ is real and symmetric such that $\overline{\mathbf{M}} = \mathbf{M}$ and $\mathbf{M}^{\mathrm{T}} = \mathbf{M}$ we obtain in view of (1.110)

$$\overline{\boldsymbol{a}}\,\mathbf{M} = \overline{\lambda}\,\overline{\boldsymbol{a}}.$$

Hence, one can write

$$0 = \overline{\boldsymbol{a}}\mathbf{M}\boldsymbol{a} - \overline{\boldsymbol{a}}\mathbf{M}\boldsymbol{a} = \overline{\boldsymbol{a}} \cdot (\mathbf{M}\boldsymbol{a}) - (\overline{\boldsymbol{a}}\mathbf{M}) \cdot \boldsymbol{a}$$
$$= \lambda\,(\overline{\boldsymbol{a}} \cdot \boldsymbol{a}) - \overline{\lambda}\,(\overline{\boldsymbol{a}} \cdot \boldsymbol{a}) = \left(\lambda - \overline{\lambda}\right)(\overline{\boldsymbol{a}} \cdot \boldsymbol{a}).$$

Bearing in mind that $\boldsymbol{a} \neq \boldsymbol{0}$ and taking (4.9) into account we conclude that $\overline{\boldsymbol{a}} \cdot \boldsymbol{a} > 0$. Hence, $\overline{\lambda} = \lambda$. The components of $\boldsymbol{a}$ with respect to a basis $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ in $\mathbb{E}^n$ are real since they represent a solution of the linear equation system $(4.16)_1$ with real coefficients. Therefore, $\boldsymbol{a} \in \mathbb{E}^n$.

**Theorem 4.5.** *Eigenvectors of a symmetric second-order tensor corresponding to distinct eigenvalues are mutually orthogonal.*

*Proof.* According to Theorem 4.3 right and left eigenvectors associated with distinct eigenvalues are mutually orthogonal. However, for a symmetric tensor every right eigenvector represents the left eigenvector associated with the same eigenvalue and vice versa. For this reason, right (left) eigenvectors associated with distinct eigenvalues are mutually orthogonal.

**Theorem 4.6.** *Let* $\lambda_i$ *be an eigenvalue of a symmetric second order tensor* $\mathbf{M}$. *Then, the algebraic and geometric multiplicity of* $\lambda_i$ *coincide.*

*Proof.* Let $\boldsymbol{a}_k \in \mathbb{E}^n$ $(k = 1, 2, \ldots, t_i)$ be all linearly independent eigenvectors associated with $\lambda_i$, while $t_i$ and $r_i$ denote its geometric and algebraic multiplicity, respectively. Every linear combination of $\boldsymbol{a}_k$ is again an eigenvector associated with $\lambda_i$. Indeed,

$$\mathbf{M}\sum_{k=1}^{t_i}\alpha_k\boldsymbol{a}_k=\sum_{k=1}^{t_i}\alpha_k\left(\mathbf{M}\boldsymbol{a}_k\right)=\sum_{k=1}^{t_i}\alpha_k\lambda_i\boldsymbol{a}_k=\lambda_i\sum_{k=1}^{t_i}\alpha_k\boldsymbol{a}_k. \tag{4.57}$$

According to Theorem 1.4 the set of vectors $\boldsymbol{a}_k$ $(k=1,2,\ldots,t_i)$ can be completed to a basis of $\mathbb{E}^n$. With the aid of the Gram-Schmidt procedure described in Chap. 1 (Sect. 1.4) this basis can be transformed to an orthonormal basis $\boldsymbol{e}_l$ $(l=1,2,\ldots,n)$. Since the vectors $\boldsymbol{e}_j$ $(j=1,2,\ldots,t_i)$ are linear combinations of $\boldsymbol{a}_k$ $(k=1,2,\ldots,t_i)$ they likewise represent eigenvectors of $\mathbf{M}$ associated with $\lambda_i$. Further, we represent the tensor $\mathbf{M}$ with respect to the basis $\boldsymbol{e}_l\otimes\boldsymbol{e}_m$ $(l,m=1,2,\ldots,n)$. In view of the identities $\mathbf{M}\boldsymbol{e}_k=\lambda_i\boldsymbol{e}_k$ $(k=1,2,\ldots,t_i)$ and keeping in mind the symmetry of $\mathbf{M}$ we can write:

$$\mathbf{M}=\lambda_i\sum_{k=1}^{t_i}\boldsymbol{e}_k\otimes\boldsymbol{e}_k+\sum_{l,m=t_i+1}^{n}\mathrm{M}'_{lm}\boldsymbol{e}_l\otimes\boldsymbol{e}_m. \tag{4.58}$$

Thus, the characteristic polynomial of $\mathbf{M}$ can be given as

$$p_{\mathbf{M}}\left(\lambda\right)=\left|\mathrm{M}'_{lm}-\lambda\delta_{lm}\right|\left(\lambda_i-\lambda\right)^{t_i}, \tag{4.59}$$

which implies that $r_i\geq t_i$.

Now, we consider the vector space $\mathbb{E}^{n-t_i}$ of all linear combinations of the vectors $\boldsymbol{e}_l$ $(l=t_i+1,\ldots,n)$. The tensor

$$\mathbf{M}'=\sum_{l,m=t_i+1}^{n}\mathrm{M}'_{lm}\boldsymbol{e}_l\otimes\boldsymbol{e}_m$$

represents a linear mapping of this space into itself. The eigenvectors of $\mathbf{M}'$ are linear combinations of $\boldsymbol{e}_l$ $(l=t_i+1,\ldots,n)$ and therefore are linearly independent of $\boldsymbol{e}_k$ $(k=1,2,\ldots,t_i)$. Consequently, $\lambda_i$ is not an eigenvalue of $\mathbf{M}'$. Otherwise, the eigenvector corresponding to this eigenvalue $\lambda_i$ would be linearly independent of $\boldsymbol{e}_k$ $(k=1,2,\ldots,t_i)$ which contradicts the previous assumption. Thus, all the roots of the characteristic polynomial of this tensor

$$p_{\mathbf{M}'}\left(\lambda\right)=\left|\mathrm{M}'_{lm}-\lambda\delta_{lm}\right|$$

differ from $\lambda_i$. In view of (4.59) this implies that $r_i=t_i$.

As a result of this theorem, the spectral decomposition of a symmetric second-order tensor can be given by

$$\mathbf{M}=\sum_{i=1}^{s}\lambda_i\sum_{k=1}^{r_i}\boldsymbol{a}_i^{(k)}\otimes\boldsymbol{a}_i^{(k)}=\sum_{i=1}^{s}\lambda_i\mathbf{P}_i,\quad \mathbf{M}\in\mathbf{Sym}^n, \tag{4.60}$$

in terms of the real symmetric eigenprojections

$$\mathbf{P}_i = \sum_{k=1}^{r_i} \boldsymbol{a}_i^{(k)} \otimes \boldsymbol{a}_i^{(k)}, \tag{4.61}$$

where the eigenvectors $\boldsymbol{a}_i^{(k)}$ form an orthonormal basis in $\mathbb{E}^n$ so that

$$\boldsymbol{a}_i^{(k)} \cdot \boldsymbol{a}_j^{(l)} = \delta_{ij}\delta^{kl}, \tag{4.62}$$

where $i, j = 1, 2, \ldots, s;\ k = 1, 2, \ldots, r_i;\ l = 1, 2, \ldots, r_j$.

## 4.6 Spectral Decomposition of Orthogonal and Skew-Symmetric Second-Order Tensors

We begin with the orthogonal tensors $\mathbf{Q} \in \mathbf{O}\mathrm{rth}^n$ defined by the condition (1.129). For every eigenvector $\boldsymbol{a}$ and the corresponding eigenvalue $\lambda$ we can write

$$\mathbf{Q}\boldsymbol{a} = \lambda\boldsymbol{a}, \quad \mathbf{Q}\overline{\boldsymbol{a}} = \overline{\lambda}\,\overline{\boldsymbol{a}}, \tag{4.63}$$

because $\mathbf{Q}$ is by definition a real tensor such that $\overline{\mathbf{Q}} = \mathbf{Q}$. Mapping both sides of these vector equations by $\mathbf{Q}^{\mathrm{T}}$ and taking (1.110) into account we have

$$\boldsymbol{a}\mathbf{Q} = \lambda^{-1}\boldsymbol{a}, \quad \overline{\boldsymbol{a}}\mathbf{Q} = \overline{\lambda}^{-1}\overline{\boldsymbol{a}}. \tag{4.64}$$

Thus, every right eigenvector of an orthogonal tensor represents its left eigenvector associated with the inverse eigenvalue. Hence, if $\lambda \neq \lambda^{-1}$ or, in other words, $\lambda$ is neither $+1$ nor $-1$, Theorem 4.3 immediately implies the relations

$$\boldsymbol{a} \cdot \boldsymbol{a} = 0, \quad \overline{\boldsymbol{a}} \cdot \overline{\boldsymbol{a}} = 0, \quad \lambda \neq \lambda^{-1} \tag{4.65}$$

indicating that $\boldsymbol{a}$ and consequently $\overline{\boldsymbol{a}}$ are complex (definitely not real) vectors. Using the representation

$$\boldsymbol{a} = \frac{1}{\sqrt{2}}\left(\boldsymbol{p} + \mathrm{i}\boldsymbol{q}\right), \quad \boldsymbol{p}, \boldsymbol{q} \in \mathbb{E}^n \tag{4.66}$$

and applying (4.8) one can write

$$\|\boldsymbol{p}\| = \|\boldsymbol{q}\| = 1, \quad \boldsymbol{p} \cdot \boldsymbol{q} = 0. \tag{4.67}$$

Now, we consider the product $\overline{\boldsymbol{a}}\mathbf{Q}\boldsymbol{a}$. With the aid of $(4.63)_1$ and $(4.64)_2$ we obtain

$$\overline{\boldsymbol{a}}\mathbf{Q}\boldsymbol{a} = \lambda\left(\overline{\boldsymbol{a}} \cdot \boldsymbol{a}\right) = \overline{\lambda}^{-1}\left(\overline{\boldsymbol{a}} \cdot \boldsymbol{a}\right). \tag{4.68}$$

Since, however, $\overline{\boldsymbol{a}} \cdot \boldsymbol{a} = 1/2\left(\boldsymbol{p} \cdot \boldsymbol{p} + \boldsymbol{q} \cdot \boldsymbol{q}\right) = 1$ we infer that

$$\lambda\overline{\lambda} = 1. \tag{4.69}$$

Thus, all eigenvalues of an orthogonal tensor have absolute value 1 so that we can write

$$\lambda = e^{\omega \mathrm{i}} = \cos\omega + \mathrm{i}\sin\omega. \tag{4.70}$$

By virtue of (4.69) one can further rewrite (4.64) as

$$\boldsymbol{a}\mathbf{Q} = \overline{\lambda}\boldsymbol{a}, \quad \overline{\boldsymbol{a}}\mathbf{Q} = \lambda\overline{\boldsymbol{a}}. \tag{4.71}$$

Summarizing these results we conclude that every complex (definitely not real) eigenvalue $\lambda$ of an orthogonal tensor comes in pair with its complex conjugate counterpart $\overline{\lambda} = \lambda^{-1}$. If $\boldsymbol{a}$ is a right eigenvector associated with $\lambda$, then $\overline{\boldsymbol{a}}$ is its left eigenvector. For $\overline{\lambda}$, $\boldsymbol{a}$ is, vice versa, the left eigenvector and $\overline{\boldsymbol{a}}$ the right one.

Next, we show that the algebraic and geometric multiplicities of every eigenvalue of an orthogonal tensor $\mathbf{Q}$ coincide. Let $\tilde{\boldsymbol{a}}_k$ $(k = 1, 2, \ldots, t_i)$ be all linearly independent right eigenvectors associated with an eigenvalue $\lambda_i$. According to Theorem 1.4 these vectors can be completed to a basis of $\mathbb{C}^n$. With the aid of the Gram-Schmidt procedure (see Exercise 4.11) a linear combination of this basis can be constructed in such a way that $\boldsymbol{a}_k \cdot \overline{\boldsymbol{a}}_l = \delta_{kl}$ $(k, l = 1, 2, \ldots, n)$. Since the vectors $\boldsymbol{a}_k$ $(k = 1, 2, \ldots, t_i)$ are linear combinations of $\tilde{\boldsymbol{a}}_k$ $(k = 1, 2, \ldots, t_i)$ they likewise represent eigenvectors of $\mathbf{Q}$ associated with $\lambda_i$. Thus, representing $\mathbf{Q}$ with respect to the basis $\boldsymbol{a}_k \otimes \overline{\boldsymbol{a}}_l$ $(k, l = 1, 2, \ldots, n)$ we can write

$$\mathbf{Q} = \lambda_i \sum_{k=1}^{t_i} \boldsymbol{a}_k \otimes \overline{\boldsymbol{a}}_k + \sum_{l,m=t_i+1}^{n} \mathrm{Q}'_{lm} \boldsymbol{a}_l \otimes \overline{\boldsymbol{a}}_m.$$

Comparing this representation with (4.58) and using the same reasoning as applied for the proof of Theorem 4.6 we infer that $\lambda_i$ cannot be an eigenvalue of $\mathbf{Q}' = \sum_{l,m=t_i+1}^{n} \mathrm{Q}'_{lm} \boldsymbol{a}_l \otimes \overline{\boldsymbol{a}}_m$. This means that the algebraic multiplicity $r_i$ of $\lambda_i$ coincides with its geometric multiplicity $t_i$. Thus, every orthogonal tensor $\mathbf{Q} \in \mathbf{Orth}^n$ is characterized by exactly $n$ linearly independent eigenvectors forming a basis of $\mathbb{C}^n$. Using this fact the spectral decomposition of $\mathbf{Q}$ can be given by

$$\mathbf{Q} = \sum_{k=1}^{r_{+1}} \boldsymbol{a}_{+1}^{(k)} \otimes \boldsymbol{a}_{+1}^{(k)} - \sum_{l=1}^{r_{-1}} \boldsymbol{a}_{-1}^{(l)} \otimes \boldsymbol{a}_{-1}^{(l)}$$
$$+ \sum_{i=1}^{s} \left\{ \lambda_i \sum_{k=1}^{r_i} \boldsymbol{a}_i^{(k)} \otimes \overline{\boldsymbol{a}}_i^{(k)} + \overline{\lambda}_i \sum_{k=1}^{r_i} \overline{\boldsymbol{a}}_i^{(k)} \otimes \boldsymbol{a}_i^{(k)} \right\}, \tag{4.72}$$

where $r_{+1}$ and $r_{-1}$ denote the algebraic multiplicities of real eigenvalues $+1$ and $-1$, respectively, while $\boldsymbol{a}_{+1}^{(k)}$ $(k = 1, 2, \ldots, r_{+1})$ and $\boldsymbol{a}_{-1}^{(l)}$ $(l = 1, 2, \ldots, r_{-1})$

are the corresponding orthonormal real eigenvectors. $s$ is the number of complex conjugate pairs of eigenvalues $\lambda_i = \cos\omega_i \pm \mathrm{i}\sin\omega_i$ with distinct arguments $\omega_i$ and the multiplicities $r_i$. The associated eigenvectors $\boldsymbol{a}_i^{(k)}$ and $\overline{\boldsymbol{a}}_i^{(k)}$ obey the following relations (see also Exercise 4.12)

$$\boldsymbol{a}_i^{(k)} \cdot \boldsymbol{a}_{+1}^{(o)} = 0, \quad \boldsymbol{a}_i^{(k)} \cdot \boldsymbol{a}_{-1}^{(p)} = 0, \quad \boldsymbol{a}_i^{(k)} \cdot \overline{\boldsymbol{a}}_j^{(l)} = \delta_{ij}\delta^{kl}, \quad \boldsymbol{a}_i^{(k)} \cdot \boldsymbol{a}_i^{(m)} = 0, \tag{4.73}$$

where $i, j = 1, 2, \ldots, s$; $k, m = 1, 2, \ldots, r_i$; $l = 1, 2, \ldots, r_j$; $o = 1, 2, \ldots, r_{+1}$; $p = 1, 2, \ldots, r_{-1}$. Using the representations (4.66) and (4.70) the spectral decomposition (4.72) can alternatively be written as

$$\begin{aligned}\mathbf{Q} = &\sum_{k=1}^{r_{+1}} \boldsymbol{a}_{+1}^{(k)} \otimes \boldsymbol{a}_{+1}^{(k)} + \sum_{i=1}^{s} \cos\omega_i \sum_{k=1}^{r_i} \left( \boldsymbol{p}_i^{(k)} \otimes \boldsymbol{p}_i^{(k)} + \boldsymbol{q}_i^{(k)} \otimes \boldsymbol{q}_i^{(k)} \right) \\ &- \sum_{l=1}^{r_{-1}} \boldsymbol{a}_{-1}^{(l)} \otimes \boldsymbol{a}_{-1}^{(l)} + \sum_{i=1}^{s} \sin\omega_i \sum_{k=1}^{r_i} \left( \boldsymbol{p}_i^{(k)} \otimes \boldsymbol{q}_i^{(k)} - \boldsymbol{q}_i^{(k)} \otimes \boldsymbol{p}_i^{(k)} \right).\end{aligned} \tag{4.74}$$

Now, we turn our attention to skew-symmetric tensors $\mathbf{W} \in \mathbf{S}\mathrm{kew}^n$ as defined in (1.148). Instead of (4.64) and (4.68) we have in this case

$$\boldsymbol{a}\mathbf{W} = -\lambda\boldsymbol{a}, \quad \overline{\boldsymbol{a}}\,\mathbf{W} = -\overline{\lambda}\,\overline{\boldsymbol{a}}, \tag{4.75}$$

$$\overline{\boldsymbol{a}}\mathbf{W}\boldsymbol{a} = \lambda\left(\overline{\boldsymbol{a}} \cdot \boldsymbol{a}\right) = -\overline{\lambda}\left(\overline{\boldsymbol{a}} \cdot \boldsymbol{a}\right) \tag{4.76}$$

and consequently

$$\lambda = -\overline{\lambda}. \tag{4.77}$$

Thus, the eigenvalues of $\mathbf{W}$ are either zero or imaginary. The latter ones come in pairs with the complex conjugate like in the case of orthogonal tensors. Similarly to (4.72) and (4.74) we thus obtain

$$\begin{aligned}\mathbf{W} &= \sum_{i=1}^{s} \omega_i \mathrm{i} \sum_{k=1}^{r_i} \left( \boldsymbol{a}_i^{(k)} \otimes \overline{\boldsymbol{a}}_i^{(k)} - \overline{\boldsymbol{a}}_i^{(k)} \otimes \boldsymbol{a}_i^{(k)} \right) \\ &= \sum_{i=1}^{s} \omega_i \sum_{k=1}^{r_i} \left( \boldsymbol{p}_i^{(k)} \otimes \boldsymbol{q}_i^{(k)} - \boldsymbol{q}_i^{(k)} \otimes \boldsymbol{p}_i^{(k)} \right),\end{aligned} \tag{4.78}$$

where $s$ denotes the number of pairwise distinct imaginary eigenvalues $\omega_i\mathrm{i}$ while the associated eigenvectors $\boldsymbol{a}_i^{(k)}$ and $\overline{\boldsymbol{a}}_i^{(k)}$ are subject to the restrictions $(4.73)_{3,4}$.

**Orthogonal tensors in three-dimensional space.** In the three-dimensional case $\mathbf{Q} \in \mathbf{O}\mathrm{rth}^3$, at least one of the eigenvalues is real, since complex eigenvalues of orthogonal tensors appear in pairs with the complex conjugate. Hence, we can write

$$\lambda_1 = \pm 1, \quad \lambda_2 = \mathrm{e}^{\mathrm{i}\omega} = \cos\omega + \mathrm{i}\sin\omega, \quad \lambda_3 = \mathrm{e}^{-\mathrm{i}\omega} = \cos\omega - \mathrm{i}\sin\omega. \tag{4.79}$$

In the case $\sin\omega = 0$ all three eigenvalues become real. The principal invariants (4.30) take thus the form

$$\mathrm{I}_{\mathbf{Q}} = \lambda_1 + 2\cos\omega = \pm 1 + 2\cos\omega,$$
$$\mathrm{II}_{\mathbf{Q}} = 2\lambda_1\cos\omega + 1 = \lambda_1 \mathrm{I}_{\mathbf{Q}} = \pm\mathrm{I}_{\mathbf{Q}},$$
$$\mathrm{III}_{\mathbf{Q}} = \lambda_1 = \pm 1. \tag{4.80}$$

The spectral representation (4.72) takes the form

$$\mathbf{Q} = \pm\boldsymbol{a}_1 \otimes \boldsymbol{a}_1 + (\cos\omega + \mathrm{i}\sin\omega)\,\boldsymbol{a} \otimes \overline{\boldsymbol{a}} + (\cos\omega - \mathrm{i}\sin\omega)\,\overline{\boldsymbol{a}} \otimes \boldsymbol{a}, \tag{4.81}$$

where $\boldsymbol{a}_1 \in \mathbb{E}^3$ and $\boldsymbol{a} \in \mathbb{C}^3$ is given by (4.66) and (4.67). Taking into account that by (4.73)

$$\boldsymbol{a}_1 \cdot \boldsymbol{a} = \boldsymbol{a}_1 \cdot \boldsymbol{p} = \boldsymbol{a}_1 \cdot \boldsymbol{q} = 0 \tag{4.82}$$

we can set

$$\boldsymbol{a}_1 = \boldsymbol{q} \times \boldsymbol{p}. \tag{4.83}$$

Substituting (4.66) into (4.81) we also obtain

$$\mathbf{Q} = \pm\boldsymbol{a}_1 \otimes \boldsymbol{a}_1 + \cos\omega\,(\boldsymbol{p} \otimes \boldsymbol{p} + \boldsymbol{q} \otimes \boldsymbol{q}) + \sin\omega\,(\boldsymbol{p} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{p})\,. \tag{4.84}$$

By means of the vector identity (1.130) and considering (1.65) and (4.83) it finally leads to

$$\mathbf{Q} = \cos\omega\mathbf{I} + \sin\omega\hat{\boldsymbol{a}}_1 + (\pm 1 - \cos\omega)\,\boldsymbol{a}_1 \otimes \boldsymbol{a}_1. \tag{4.85}$$

Comparing this representation with (1.71) we observe that any orthogonal tensor $\mathbf{Q} \in \mathbf{O}\mathrm{rth}^3$ describes a rotation in three-dimensional space if $\mathrm{III}_{\mathbf{Q}} = \lambda_1 = 1$. The eigenvector $\boldsymbol{a}_1$ corresponding to the eigenvalue 1 specifies the rotation axis. In this case, $\mathbf{Q}$ is referred to as a proper orthogonal tensor.

**Skew-symmetric tensors in three-dimensional space.** For a skew-symmetric tensor $\mathbf{W} \in \mathbf{S}\mathrm{kew}^3$ we can write in view of (4.77)

$$\lambda_1 = 0, \quad \lambda_2 = \omega\mathrm{i}, \quad \lambda_3 = -\omega\mathrm{i}. \tag{4.86}$$

Similarly to (4.80) we further obtain (see Exercise 4.13)

$$\mathrm{I}_{\mathbf{W}} = 0, \quad \mathrm{II}_{\mathbf{W}} = \frac{1}{2}\,\|\mathbf{W}\|^2 = \omega^2, \quad \mathrm{III}_{\mathbf{W}} = 0. \tag{4.87}$$

The spectral representation (4.78) takes the form

$$\mathbf{W} = \omega\mathrm{i}\,(\boldsymbol{a} \otimes \overline{\boldsymbol{a}} - \overline{\boldsymbol{a}} \otimes \boldsymbol{a}) = \omega\,(\boldsymbol{p} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{p})\,, \tag{4.88}$$

where $\boldsymbol{a}$, $\boldsymbol{p}$ and $\boldsymbol{q}$ are again related by (4.66) and (4.67). With the aid of the abbreviation

$$\boldsymbol{w} = \omega \boldsymbol{a}_1 = \omega \boldsymbol{q} \times \boldsymbol{p} \tag{4.89}$$

and bearing (1.130) in mind we finally arrive at the representation (1.64)

$$\mathbf{W} = \hat{\boldsymbol{w}}. \tag{4.90}$$

Thus, every skew-symmetric tensor in three-dimensional space describes a cross product by its eigenvector $\boldsymbol{w}$ (4.89) corresponding to the zero eigenvalue. The vector $\boldsymbol{w}$ is called in this case the axial vector of the skew-symmetric tensor $\mathbf{W}$.

## 4.7 Cayley-Hamilton Theorem

**Theorem 4.7.** *Let $p_{\mathbf{A}}(\lambda)$ be the characteristic polynomial of a second-order tensor $\mathbf{A} \in \mathbf{Lin}^n$. Then,*

$$p_{\mathbf{A}}(\mathbf{A}) = \sum_{k=0}^{n} (-1)^k \, \mathrm{I}_{\mathbf{A}}^{(k)} \mathbf{A}^{n-k} = \mathbf{0}. \tag{4.91}$$

*Proof.* As a proof (see, e.g., [11]) we show that

$$p_{\mathbf{A}}(\mathbf{A}) \boldsymbol{x} = \boldsymbol{0}, \quad \forall \boldsymbol{x} \in \mathbb{E}^n. \tag{4.92}$$

For $\boldsymbol{x} = \boldsymbol{0}$ it is trivial, so we suppose that $\boldsymbol{x} \neq \boldsymbol{0}$. Consider the vectors

$$\boldsymbol{y}_i = \mathbf{A}^{i-1} \boldsymbol{x}, \quad i = 1, 2, \ldots. \tag{4.93}$$

Obviously, there is an integer number $k$ such that the vectors $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_k$ are linearly independent, but

$$a_1 \boldsymbol{y}_1 + a_2 \boldsymbol{y}_2 + \ldots + a_k \boldsymbol{y}_k + \mathbf{A}^k \boldsymbol{x} = \boldsymbol{0}. \tag{4.94}$$

Note that $1 \le k \le n$. If $k \neq n$ we can complete the vectors $\boldsymbol{y}_i$ $(i = 1, 2, \ldots, k)$ to a basis $\boldsymbol{y}_i$ $(i = 1, 2, \ldots, n)$ of $\mathbb{E}^n$. Let $\mathbf{A} = \mathrm{A}^{i}_{\cdot j} \boldsymbol{y}_i \otimes \boldsymbol{y}^j$, where the vectors $\boldsymbol{y}^i$ form the basis dual to $\boldsymbol{y}_i$ $(i = 1, 2, \ldots, n)$. By virtue of (4.93) and (4.94) we can write

$$\mathbf{A} \boldsymbol{y}_i = \begin{cases} \boldsymbol{y}_{i+1} & \text{if } i < k, \\ -\sum\limits_{j=1}^{k} a_j \boldsymbol{y}_j & \text{if } i = k. \end{cases} \tag{4.95}$$

The components of $\mathbf{A}$ can thus be given by

$$\left[ \mathrm{A}^{i}_{\cdot j} \right] = \left[ \boldsymbol{y}^i \mathbf{A} \boldsymbol{y}_j \right] = \left[ \begin{array}{ccccc|c} 0 & 0 & \ldots & 0 & -a_1 & \\ 1 & 0 & \ldots & 0 & -a_2 & \\ \vdots & \vdots & & \vdots & \vdots & \mathrm{A}' \\ 0 & 0 & \ldots & 1 & -a_k & \\ \hline & & 0 & & & \mathrm{A}'' \end{array} \right], \tag{4.96}$$

where A′ and A″ denote some submatrices. Therefore, the characteristic polynomial of $\mathbf{A}$ takes the form

$$p_{\mathbf{A}}(\lambda) = p_{\mathbf{A}''}(\lambda) \begin{vmatrix} -\lambda & 0\dots0 & -a_1 \\ 1 & -\lambda\dots0 & -a_2 \\ \vdots & \vdots \quad \vdots & \vdots \\ 0 & 0\dots1 & -a_k-\lambda \end{vmatrix}, \tag{4.97}$$

where $p_{\mathbf{A}''}(\lambda) = \det(\mathbf{A}'' - \lambda\mathbf{I})$. By means of the Laplace expansion rule (see, e.g., [5]) we expand the determinant in (4.97) along the last column, which yields

$$p_{\mathbf{A}}(\lambda) = p_{\mathbf{A}''}(\lambda)(-1)^k \left(a_1 + a_2\lambda + \ldots + a_k\lambda^{k-1} + \lambda^k\right). \tag{4.98}$$

Bearing (4.93) and (4.94) in mind we finally prove (4.92) by

$$\begin{aligned} p_{\mathbf{A}}(\mathbf{A})\boldsymbol{x} &= (-1)^k p_{\mathbf{A}''}(\mathbf{A})\left(a_1\mathbf{I} + a_2\mathbf{A} + \ldots + a_k\mathbf{A}^{k-1} + \mathbf{A}^k\right)\boldsymbol{x} \\ &= (-1)^k p_{\mathbf{A}''}(\mathbf{A})\left(a_1\boldsymbol{x} + a_2\mathbf{A}\boldsymbol{x} + \ldots + a_k\mathbf{A}^{k-1}\boldsymbol{x} + \mathbf{A}^k\boldsymbol{x}\right) \\ &= (-1)^k p_{\mathbf{A}''}(\mathbf{A})\left(a_1\boldsymbol{y}_1 + a_2\boldsymbol{y}_2 + \ldots + a_k\boldsymbol{y}_k + \boldsymbol{y}_{k+1}\right) = \boldsymbol{0}. \end{aligned}$$

## Exercises

**4.1.** Evaluate eigenvalues and eigenvectors of the right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ in the case of simple shear, where $\mathbf{F}$ is defined by (4.23).

**4.2.** Prove identity $(4.29)_3$ using Newton's formula (4.26).

**4.3.** Prove that eigenvectors $\boldsymbol{a}_i^{(k)}$ $(i = 1, 2, \ldots, s; k = 1, 2, \ldots, t_i)$ of a second order tensor $\mathbf{A} \in \mathbf{Lin}^n$ are linearly independent and form a basis of $\mathbb{C}^n$ if for every eigenvalue the algebraic and geometric multiplicities coincide so that $r_i = t_i$ $(i = 1, 2, \ldots, s)$.

**4.4.** Prove identity (4.44) using (4.40) and (4.42).

**4.5.** Prove identity (4.46) taking (4.40) and (4.42) into account and using the results of Exercise 4.3.

**4.6.** Prove the identity $\det[\exp(\mathbf{A})] = \exp(\mathrm{tr}\mathbf{A})$.

**4.7.** Verify the Sylvester formula for $s = 3$ by inserting (4.43) and (4.46) into (4.54).

**4.8.** Calculate eigenvalues and eigenprojections of the tensor $\mathbf{A} = \mathrm{A}^i_j \boldsymbol{e}_i \otimes \boldsymbol{e}^j$, where

$$\left[\mathrm{A}^i_j\right] = \begin{bmatrix} -2 & 2 & 2 \\ 2 & 1 & 4 \\ 2 & 4 & 1 \end{bmatrix}.$$

Apply the Cardano formula (4.31) and Sylvester formula (4.54).

**4.9.** Calculate the exponential of the tensor $\mathbf{A}$ given in Exercise 4.8 using the spectral representation in terms of eigenprojections (4.43).

**4.10.** Calculate eigenvectors of the tensor $\mathbf{A}$ defined in Exercise 4.8. Express eigenprojections by (4.42) and compare the results with those obtained by the Sylvester formula (Exercise 4.8).

**4.11.** Let $\boldsymbol{c}_i$ $(i = 1, 2, \ldots, m) \in \mathbb{C}^n$ be a set of linearly independent complex vectors. Using the (Gram-Schmidt) procedure described in Chap. 1 (Sect. 1.4), construct linear combinations of these vectors, say $\boldsymbol{a}_i$ $(i = 1, 2, \ldots, m)$, again linearly independent, in such a way that $\boldsymbol{a}_i \cdot \overline{\boldsymbol{a}}_j = \delta_{ij}$ $(i, j = 1, 2, \ldots, m)$.

**4.12.** Let $\boldsymbol{a}_i^{(k)}$ $(k = 1, 2, \ldots, t_i)$ be all linearly independent right eigenvectors of an orthogonal tensor associated with a complex (definitely not real) eigenvalue $\lambda_i$. Show that $\boldsymbol{a}_i^{(k)} \cdot \boldsymbol{a}_i^{(l)} = 0$ $(k, l = 1, 2, \ldots, t_i)$.

**4.13.** Evaluate principal invariants of a skew-symmetric tensor in three-dimensional space using (4.29).

**4.14.** Evaluate eigenvalues, eigenvectors and eigenprojections of the tensor describing the rotation by the angle $\alpha$ about the axis $\boldsymbol{e}_3$ (see Exercise 1.22).

**4.15.** Verify the Cayley-Hamilton theorem for the tensor $\mathbf{A}$ defined in Exercise 4.8.

**4.16.** Verify the Cayley-Hamilton theorem for the deformation gradient in the case of simple shear (4.23).