5

Fourth-Order Tensors

5.1 Fourth-Order Tensors as a Linear Mapping

Fourth-order tensors play an important role in continuum mechanics where they appear as elasticity and compliance tensors. In this section we define fourth-order tensors and learn some basic operations with them. To this end, we consider a set $\mathscr{L}\text{in}^n$ of all linear mappings of one second-order tensor into another one within $\mathbf{L}\text{in}^n$. Such mappings are denoted by a colon as

$$\mathbf{Y} = \mathscr{A} : \mathbf{X}, \quad \mathscr{A} \in \mathscr{L}\text{in}^n,\ \mathbf{Y} \in \mathbf{L}\text{in}^n,\ \forall \mathbf{X} \in \mathbf{L}\text{in}^n. \tag{5.1}$$

The elements of $\mathscr{L}\text{in}^n$ are called fourth-order tensors.

Example. Elasticity and compliance tensors. A constitutive law of a linearly elastic material establishes a linear relationship between the Cauchy stress tensor $\boldsymbol{\sigma}$ and Cauchy strain tensor $\boldsymbol{\epsilon}$. Since these tensors are of the second-order a linear relation between them can be expressed by fourth-order tensors like

$$\boldsymbol{\sigma} = \mathscr{C} : \boldsymbol{\epsilon} \quad \text{or} \quad \boldsymbol{\epsilon} = \mathscr{H} : \boldsymbol{\sigma}. \tag{5.2}$$

The fourth-order tensors $\mathscr{C}$ and $\mathscr{H}$ describe properties of the elastic material and are called the elasticity and compliance tensor, respectively.

Linearity of the mapping (5.1) implies that

$$\mathscr{A} : (\mathbf{X} + \mathbf{Y}) = \mathscr{A} : \mathbf{X} + \mathscr{A} : \mathbf{Y}, \tag{5.3}$$

$$\mathscr{A} : (\alpha \mathbf{X}) = \alpha (\mathscr{A} : \mathbf{X}), \quad \forall \mathbf{X}, \mathbf{Y} \in \mathbf{L}\text{in}^n,\ \forall \alpha \in \mathbb{R},\ \mathscr{A} \in \mathscr{L}\text{in}^n. \tag{5.4}$$

Similarly to second-order tensors one defines the product of a fourth-order tensor with a scalar

$$(\alpha \mathscr{A}) : \mathbf{X} = \alpha (\mathscr{A} : \mathbf{X}) = \mathscr{A} : (\alpha \mathbf{X}) \tag{5.5}$$

and the sum of two fourth-order tensors by

$$(\mathcal{A} + \mathcal{B}) : \mathbf{X} = \mathcal{A} : \mathbf{X} + \mathcal{B} : \mathbf{X}, \quad \forall \mathbf{X} \in \mathbf{Lin}^n. \tag{5.6}$$

Further, we define the zero-tensor $\mathcal{O}$ of the fourth-order by

$$\mathcal{O} : \mathbf{X} = \mathbf{0}, \quad \forall \mathbf{X} \in \mathbf{Lin}^n. \tag{5.7}$$

Thus, summarizing the properties of fourth-order tensors one can write similarly to second-order tensors

$$\mathcal{A} + \mathcal{B} = \mathcal{B} + \mathcal{A}, \quad \text{(addition is commutative)}, \tag{5.8}$$

$$\mathcal{A} + (\mathcal{B} + \mathcal{C}) = (\mathcal{A} + \mathcal{B}) + \mathcal{C}, \quad \text{(addition is associative)}, \tag{5.9}$$

$$\mathcal{O} + \mathcal{A} = \mathcal{A}, \tag{5.10}$$

$$\mathcal{A} + (-\mathcal{A}) = \mathcal{O}, \tag{5.11}$$

$$\alpha (\beta \mathcal{A}) = (\alpha\beta) \mathcal{A}, \quad \text{(multiplication by scalars is associative)}, \tag{5.12}$$

$$1\mathcal{A} = \mathcal{A}, \tag{5.13}$$

$$\alpha (\mathcal{A} + \mathcal{B}) = \alpha\mathcal{A} + \alpha\mathcal{B}, \quad \text{(multiplication by scalars is distributive with respect to tensor addition)}, \tag{5.14}$$

$$(\alpha + \beta) \mathcal{A} = \alpha\mathcal{A} + \beta\mathcal{A}, \quad \text{(multiplication by scalars is distributive with respect to scalar addition)}, \quad \forall \mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{L}\text{in}^n, \ \forall \alpha, \beta \in \mathbb{R}. \tag{5.15}$$

Thus, the set of fourth-order tensors $\mathcal{L}\text{in}^n$ forms a vector space.

On the basis of the "right" mapping (5.1) and the scalar product of two second-order tensors (1.136) we can also define the "left" mapping by

$$(\mathbf{Y} : \mathcal{A}) : \mathbf{X} = \mathbf{Y} : (\mathcal{A} : \mathbf{X}), \quad \forall \mathbf{X}, \mathbf{Y} \in \mathbf{Lin}^n. \tag{5.16}$$

5.2 Tensor Products, Representation of Fourth-Order Tensors with Respect to a Basis

For the construction of fourth-order tensors from second-order ones we introduce two tensor products as follows

$$\mathbf{A} \otimes \mathbf{B} : \mathbf{X} = \mathbf{AXB}, \quad \mathbf{A} \odot \mathbf{B} : \mathbf{X} = \mathbf{A} (\mathbf{B} : \mathbf{X}), \quad \forall \mathbf{A}, \mathbf{B}, \mathbf{X} \in \mathbf{Lin}^n. \tag{5.17}$$

Note, that the tensor product "$\otimes$" $(5.17)_1$ applied to second-order tensors differs from the tensor product of vectors (1.75). One can easily show that the mappings described by (5.17) are linear and therefore represent fourth-order tensors. Indeed, we have, for example, for the tensor product "$\otimes$" $(5.17)_1$

$$\mathbf{A}\otimes\mathbf{B}:(\mathbf{X}+\mathbf{Y})=\mathbf{A}\,(\mathbf{X}+\mathbf{Y})\,\mathbf{B}$$
$$=\mathbf{AXB}+\mathbf{AYB}=\mathbf{A}\otimes\mathbf{B}:\mathbf{X}+\mathbf{A}\otimes\mathbf{B}:\mathbf{Y},\tag{5.18}$$

$$\mathbf{A}\otimes\mathbf{B}:(\alpha\mathbf{X})=\mathbf{A}\,(\alpha\mathbf{X})\,\mathbf{B}=\alpha\,(\mathbf{AXB})$$
$$=\alpha\,(\mathbf{A}\otimes\mathbf{B}:\mathbf{X})\,,\quad\forall\mathbf{X},\mathbf{Y}\in\mathbf{L}\mathrm{in}^n,\ \forall\alpha\in\mathbb{R}.\tag{5.19}$$

With definitions (5.17) in hand one can easily prove the following identities

$$\mathbf{A}\otimes(\mathbf{B}+\mathbf{C})=\mathbf{A}\otimes\mathbf{B}+\mathbf{A}\otimes\mathbf{C},\quad(\mathbf{B}+\mathbf{C})\otimes\mathbf{A}=\mathbf{B}\otimes\mathbf{A}+\mathbf{C}\otimes\mathbf{A},\tag{5.20}$$

$$\mathbf{A}\odot(\mathbf{B}+\mathbf{C})=\mathbf{A}\odot\mathbf{B}+\mathbf{A}\odot\mathbf{C},\quad(\mathbf{B}+\mathbf{C})\odot\mathbf{A}=\mathbf{B}\odot\mathbf{A}+\mathbf{C}\odot\mathbf{A}.\tag{5.21}$$

For the left mapping (5.16) the tensor products (5.17) yield

$$\mathbf{Y}:\mathbf{A}\otimes\mathbf{B}=\mathbf{A}^{\mathrm{T}}\mathbf{Y}\mathbf{B}^{\mathrm{T}},\quad\mathbf{Y}:\mathbf{A}\odot\mathbf{B}=(\mathbf{Y}:\mathbf{A})\,\mathbf{B}.\tag{5.22}$$

As fourth-order tensors represent vectors they can be given with respect to a basis in $\mathfrak{L}\mathrm{in}^n$.

Theorem 5.1. *Let* $\mathcal{F}=\{\mathbf{F}_1,\mathbf{F}_2,\ldots,\mathbf{F}_{n^2}\}$ *and* $\mathcal{G}=\{\mathbf{G}_1,\mathbf{G}_2,\ldots,\mathbf{G}_{n^2}\}$ *be two arbitrary (not necessarily distinct) bases of* $\mathbf{L}\mathit{in}^n$*. Then, fourth-order tensors* $\mathbf{F}_i\odot\mathbf{G}_j$ $\left(i,j=1,2,\ldots,n^2\right)$ *form a basis of* $\mathfrak{L}\mathit{in}^n$*. The dimension of* $\mathfrak{L}\mathit{in}^n$ *is thus* n^4*.*

Proof. See the proof of Theorem 1.6.

A basis in $\mathfrak{L}\mathrm{in}^n$ can be represented in another way as by the tensors $\mathbf{F}_i\odot\mathbf{G}_j$ $\left(i,j=1,2,\ldots,n^2\right)$. To this end, we prove the following identity

$$(\boldsymbol{a}\otimes\boldsymbol{d})\odot(\boldsymbol{b}\otimes\boldsymbol{c})=\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c}\otimes\boldsymbol{d},\tag{5.23}$$

where we set

$$(\boldsymbol{a}\otimes\boldsymbol{b})\otimes(\boldsymbol{c}\otimes\boldsymbol{d})=\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c}\otimes\boldsymbol{d}.\tag{5.24}$$

Indeed, let $\mathbf{X}\in\mathbf{L}\mathrm{in}^n$ be an arbitrary second-order tensor. Then, in view of (1.135) and $(5.17)_2$

$$(\boldsymbol{a}\otimes\boldsymbol{d})\odot(\boldsymbol{b}\otimes\boldsymbol{c}):\mathbf{X}=(\boldsymbol{b}\mathbf{X}\boldsymbol{c})\,(\boldsymbol{a}\otimes\boldsymbol{d})\,.\tag{5.25}$$

For the right hand side of (5.23) we obtain the same result using $(5.17)_1$ and (5.24)

$$\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c}\otimes\boldsymbol{d}:\mathbf{X}=(\boldsymbol{a}\otimes\boldsymbol{b})\otimes(\boldsymbol{c}\otimes\boldsymbol{d}):\mathbf{X}=(\boldsymbol{b}\mathbf{X}\boldsymbol{c})\,(\boldsymbol{a}\otimes\boldsymbol{d})\,.\tag{5.26}$$

For the left mapping (5.16) it thus holds

$$\mathbf{Y}:\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c}\otimes\boldsymbol{d}=(\boldsymbol{a}\mathbf{Y}\boldsymbol{d})\,(\boldsymbol{b}\otimes\boldsymbol{c})\,.\tag{5.27}$$

Now, we are in a position to prove the following theorem.

Theorem 5.2. *Let* $\mathcal{E} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$, $\mathcal{F} = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n\}$, $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ *and finally* $\mathcal{H} = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_n\}$ *be four arbitrary (not necessarily distinct) bases of* $\mathbb{E}^n$. *Then, fourth-order tensors* $\boldsymbol{e}_i \otimes \boldsymbol{f}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{h}_l$ $(i, j, k, l = 1, 2, \ldots, n)$ *represent a basis of* $\boldsymbol{\mathfrak{L}}\mathrm{in}^n$.

Proof. In view of (5.23)

$$\boldsymbol{e}_i \otimes \boldsymbol{f}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{h}_l = (\boldsymbol{e}_i \otimes \boldsymbol{h}_l) \odot (\boldsymbol{f}_j \otimes \boldsymbol{g}_k) .$$

According to Theorem 1.6 the second-order tensors $\boldsymbol{e}_i \otimes \boldsymbol{h}_l$ $(i, l = 1, 2, \ldots, n)$ on the one hand and $\boldsymbol{f}_j \otimes \boldsymbol{g}_k$ $(j, k = 1, 2, \ldots, n)$ on the other hand form bases of $\mathbf{Lin}^n$. According to Theorem 5.1 the fourth-order tensors $(\boldsymbol{e}_i \otimes \boldsymbol{h}_l) \odot (\boldsymbol{f}_j \otimes \boldsymbol{g}_k)$ and consequently $\boldsymbol{e}_i \otimes \boldsymbol{f}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{h}_l$ $(i, j, k, l = 1, 2, \ldots, n)$ represent thus a basis of $\boldsymbol{\mathfrak{L}}\mathrm{in}^n$.

As a result of this Theorem any fourth-order tensor can be represented by

$$\begin{aligned} \boldsymbol{\mathcal{A}} &= \mathcal{A}^{ijkl} \boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}_l = \mathcal{A}_{ijkl} \boldsymbol{g}^i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}^k \otimes \boldsymbol{g}^l \\ &= \mathcal{A}^{ij}_{\cdot \cdot kl} \boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}^k \otimes \boldsymbol{g}^l = \ldots \end{aligned} \tag{5.28}$$

The components of $\boldsymbol{\mathcal{A}}$ appearing in (5.28) can be expressed by

$$\begin{aligned} &\mathcal{A}^{ijkl} = \boldsymbol{g}^i \otimes \boldsymbol{g}^l : \boldsymbol{\mathcal{A}} : \boldsymbol{g}^j \otimes \boldsymbol{g}^k, \quad \mathcal{A}_{ijkl} = \boldsymbol{g}_i \otimes \boldsymbol{g}_l : \boldsymbol{\mathcal{A}} : \boldsymbol{g}_j \otimes \boldsymbol{g}_k, \\ &\mathcal{A}^{ij}_{\cdot \cdot kl} = \boldsymbol{g}^i \otimes \boldsymbol{g}_l : \boldsymbol{\mathcal{A}} : \boldsymbol{g}^j \otimes \boldsymbol{g}_k, \quad i, j, k, l = 1, 2, \ldots, n. \end{aligned} \tag{5.29}$$

By virtue of (1.104), $(5.17)_1$ and $(5.22)_1$ the right and left mappings with a second-order tensor (5.1) and (5.16) can thus be represented by

$$\begin{aligned} \boldsymbol{\mathcal{A}} : \mathbf{X} &= \left(\mathcal{A}^{ijkl} \boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}_l\right) : \left(\mathrm{X}_{qp} \boldsymbol{g}^q \otimes \boldsymbol{g}^p\right) = \mathcal{A}^{ijkl} \mathrm{X}_{jk} \boldsymbol{g}_i \otimes \boldsymbol{g}_l, \\ \mathbf{X} : \boldsymbol{\mathcal{A}} &= \left(\mathrm{X}_{qp} \boldsymbol{g}^q \otimes \boldsymbol{g}^p\right) : \left(\mathcal{A}^{ijkl} \boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}_l\right) = \mathcal{A}^{ijkl} \mathrm{X}_{il} \boldsymbol{g}_j \otimes \boldsymbol{g}_k. \end{aligned} \tag{5.30}$$

We observe that the basis vectors of the second-order tensor are scalarly multiplied either by the “inner” (right mapping) or “outer” (left mapping) basis vectors of the fourth-order tensor.

5.3 Special Operations with Fourth-Order Tensors

Similarly to second-order tensors one defines also for fourth-order tensors some specific operations which are not generally applicable to conventional vectors in the Euclidean space.

Composition. In analogy with second-order tensors we define the composition of two fourth-order tensors $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ denoted by $\boldsymbol{\mathcal{A}} : \boldsymbol{\mathcal{B}}$ as

$$(\mathcal{A} : \mathcal{B}) : \mathbf{X} = \mathcal{A} : (\mathcal{B} : \mathbf{X}), \quad \forall \mathbf{X} \in \mathbf{Lin}^n. \tag{5.31}$$

For the left mapping (5.16) one can thus write

$$\mathbf{Y} : (\mathcal{A} : \mathcal{B}) = (\mathbf{Y} : \mathcal{A}) : \mathcal{B}, \quad \forall \mathbf{Y} \in \mathbf{Lin}^n. \tag{5.32}$$

For the tensor products (5.17) the composition (5.31) further yields

$$(\mathbf{A} \otimes \mathbf{B}) : (\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{DB}), \tag{5.33}$$

$$(\mathbf{A} \otimes \mathbf{B}) : (\mathbf{C} \odot \mathbf{D}) = (\mathbf{ACB}) \odot \mathbf{D}, \tag{5.34}$$

$$(\mathbf{A} \odot \mathbf{B}) : (\mathbf{C} \otimes \mathbf{D}) = \mathbf{A} \odot \left(\mathbf{C}^{\mathrm{T}}\mathbf{B}\mathbf{D}^{\mathrm{T}}\right), \tag{5.35}$$

$$(\mathbf{A} \odot \mathbf{B}) : (\mathbf{C} \odot \mathbf{D}) = (\mathbf{B} : \mathbf{C})\, \mathbf{A} \odot \mathbf{D}, \quad \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathbf{Lin}^n. \tag{5.36}$$

For example, the identity (5.33) can be proved within the following steps

$$(\mathbf{A} \otimes \mathbf{B}) : (\mathbf{C} \otimes \mathbf{D}) : \mathbf{X} = (\mathbf{A} \otimes \mathbf{B}) : (\mathbf{CXD})$$
$$= \mathbf{ACXDB} = (\mathbf{AC}) \otimes (\mathbf{DB}) : \mathbf{X}, \quad \forall \mathbf{X} \in \mathbf{Lin}^n,$$

where we again take into account the definition of the tensor product (5.17).

For the component representation (5.28) we further obtain

$$\mathcal{A} : \mathcal{B} = \left(\mathcal{A}^{ijkl}\boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}_l\right) : \left(\mathcal{B}_{pqrt}\boldsymbol{g}^p \otimes \boldsymbol{g}^q \otimes \boldsymbol{g}^r \otimes \boldsymbol{g}^t\right)$$
$$= \mathcal{A}^{ijkl}\mathcal{B}_{jqrk}\boldsymbol{g}_i \otimes \boldsymbol{g}^q \otimes \boldsymbol{g}^r \otimes \boldsymbol{g}_l. \tag{5.37}$$

Note that the "inner" basis vectors of the left tensor $\mathcal{A}$ are scalarly multiplied with the "outer" basis vectors of the right tensor $\mathcal{B}$.

The composition of fourth-order tensors also gives rise to the definition of powers as

$$\mathcal{A}^k = \underbrace{\mathcal{A} : \mathcal{A} : \ldots : \mathcal{A}}_{k \text{ times}}, \quad k = 1, 2, \ldots, \quad \mathcal{A}^0 = \mathfrak{J}, \tag{5.38}$$

where $\mathfrak{J}$ stands for the fourth-order identity tensor to be defined in the next section. By means of (5.33) and (5.36) powers of tensor products (5.17) take the following form

$$(\mathbf{A} \otimes \mathbf{B})^k = \mathbf{A}^k \otimes \mathbf{B}^k, \;\; (\mathbf{A} \odot \mathbf{B})^k = (\mathbf{A} : \mathbf{B})^{k-1}\, \mathbf{A} \odot \mathbf{B}, \;\; k = 1, 2, \ldots \tag{5.39}$$

Simple composition with second-order tensors. Let $\mathfrak{D}$ be a fourth-order tensor and $\mathbf{A}$, $\mathbf{B}$ two second-order tensors. One defines a fourth-order tensor $\mathbf{A}\mathfrak{D}\mathbf{B}$ by

$$(\mathbf{A}\mathfrak{D}\mathbf{B}) : \mathbf{X} = \mathbf{A}\,(\mathfrak{D} : \mathbf{X})\,\mathbf{B}, \quad \forall \mathbf{X} \in \mathbf{Lin}^n. \tag{5.40}$$

Thus, we can also write

$$\mathbf{A}\mathfrak{D}\mathbf{B} = (\mathbf{A} \otimes \mathbf{B}) : \mathfrak{D}. \tag{5.41}$$

This operation is very useful for the formulation of tensor differentiation rules to be discussed in the next chapter.

For the tensor products (5.17) we further obtain

$$\mathbf{A}\left(\mathbf{B}\otimes\mathbf{C}\right)\mathbf{D} = \left(\mathbf{AB}\right)\otimes\left(\mathbf{CD}\right) = \left(\mathbf{A}\otimes\mathbf{D}\right):\left(\mathbf{B}\otimes\mathbf{C}\right), \tag{5.42}$$

$$\mathbf{A}\left(\mathbf{B}\odot\mathbf{C}\right)\mathbf{D} = \left(\mathbf{ABD}\right)\odot\mathbf{C} = \left(\mathbf{A}\otimes\mathbf{D}\right):\left(\mathbf{B}\odot\mathbf{C}\right). \tag{5.43}$$

With respect to a basis the simple composition can be given by

$$\begin{aligned}\mathbf{A}\mathfrak{D}\mathbf{B} &= \left(\mathrm{A}_{pq}\boldsymbol{g}^p\otimes\boldsymbol{g}^q\right)\left(\mathcal{D}^{ijkl}\boldsymbol{g}_i\otimes\boldsymbol{g}_j\otimes\boldsymbol{g}_k\otimes\boldsymbol{g}_l\right)\left(\mathrm{B}_{rs}\boldsymbol{g}^r\otimes\boldsymbol{g}^s\right)\\ &= \mathrm{A}_{pi}\mathcal{D}^{ijkl}\mathrm{B}_{ls}\boldsymbol{g}^p\otimes\boldsymbol{g}_j\otimes\boldsymbol{g}_k\otimes\boldsymbol{g}^s.\end{aligned} \tag{5.44}$$

It is seen that expressed in component form the simple composition of second-order tensors with a fourth-order tensor represents the so-called simple contraction of the classical tensor algebra (see, e.g., [41]).

Transposition. In contrast to second-order tensors allowing for the unique transposition operation one can define for fourth-order tensors various transpositions. We confine our attention here to the following two operations $(\bullet)^{\mathrm{T}}$ and $(\bullet)^{\mathrm{t}}$ defined by

$$\mathcal{A}^{\mathrm{T}}:\mathbf{X} = \mathbf{X}:\mathcal{A}, \quad \mathcal{A}^{\mathrm{t}}:\mathbf{X} = \mathcal{A}:\mathbf{X}^{\mathrm{T}}, \quad \forall\mathbf{X}\in\mathbf{Lin}^n. \tag{5.45}$$

Thus we can also write

$$\mathbf{Y}:\mathcal{A}^{\mathrm{t}} = \left(\mathbf{Y}:\mathcal{A}\right)^{\mathrm{T}}. \tag{5.46}$$

Indeed, a scalar product with an arbitrary second order tensor $\mathbf{X}$ yields in view of (1.140) and (5.16)

$$\begin{aligned}\left(\mathbf{Y}:\mathcal{A}^{\mathrm{t}}\right):\mathbf{X} &= \mathbf{Y}:\left(\mathcal{A}^{\mathrm{t}}:\mathbf{X}\right) = \mathbf{Y}:\left(\mathcal{A}:\mathbf{X}^{\mathrm{T}}\right)\\ &= \left(\mathbf{Y}:\mathcal{A}\right):\mathbf{X}^{\mathrm{T}} = \left(\mathbf{Y}:\mathcal{A}\right)^{\mathrm{T}}:\mathbf{X}, \quad \forall\mathbf{X}\in\mathbf{Lin}^n.\end{aligned}$$

Of special importance is also the following symmetrization operation resulting from the transposition $(\bullet)^{\mathrm{t}}$:

$$\mathfrak{F}^{\mathrm{s}} = \frac{1}{2}\left(\mathfrak{F}+\mathfrak{F}^{\mathrm{t}}\right). \tag{5.47}$$

In view of $(1.146)_1$, $(5.45)_2$ and (5.46) we thus write

$$\mathfrak{F}^{\mathrm{s}}:\mathbf{X} = \mathfrak{F}:\mathrm{sym}\mathbf{X}, \quad \mathbf{Y}:\mathfrak{F}^{\mathrm{s}} = \mathrm{sym}\left(\mathbf{Y}:\mathfrak{F}\right). \tag{5.48}$$

Applying the transposition operations to the tensor products (5.17) we have

$$\left(\mathbf{A}\otimes\mathbf{B}\right)^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\otimes\mathbf{B}^{\mathrm{T}}, \quad \left(\mathbf{A}\odot\mathbf{B}\right)^{\mathrm{T}} = \mathbf{B}\odot\mathbf{A}, \tag{5.49}$$

$$\left(\mathbf{A}\odot\mathbf{B}\right)^{\mathrm{t}} = \mathbf{A}\odot\mathbf{B}^{\mathrm{T}}, \quad \mathbf{A},\mathbf{B}\in\mathbf{Lin}^n. \tag{5.50}$$

With the aid of (5.26) and (5.27) we further obtain

$$(\boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c} \otimes \boldsymbol{d})^{\mathrm{T}} = \boldsymbol{b} \otimes \boldsymbol{a} \otimes \boldsymbol{d} \otimes \boldsymbol{c}, \tag{5.51}$$

$$(\boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c} \otimes \boldsymbol{d})^{\mathrm{t}} = \boldsymbol{a} \otimes \boldsymbol{c} \otimes \boldsymbol{b} \otimes \boldsymbol{d}. \tag{5.52}$$

It can also easily be proved that

$$\boldsymbol{\mathcal{A}}^{\mathrm{TT}} = \boldsymbol{\mathcal{A}}, \quad \boldsymbol{\mathcal{A}}^{\mathrm{tt}} = \boldsymbol{\mathcal{A}}, \quad \forall \boldsymbol{\mathcal{A}} \in \boldsymbol{\mathcal{L}}\mathrm{in}^n. \tag{5.53}$$

Note, however, that the transposition operations (5.45) are not commutative with each other so that generally $\boldsymbol{\mathfrak{D}}^{\mathrm{Tt}} \neq \boldsymbol{\mathfrak{D}}^{\mathrm{tT}}$.

Applied to the composition of fourth-order tensors these transposition operations yield (Exercise 5.6):

$$(\boldsymbol{\mathcal{A}} : \boldsymbol{\mathfrak{B}})^{\mathrm{T}} = \boldsymbol{\mathfrak{B}}^{\mathrm{T}} : \boldsymbol{\mathcal{A}}^{\mathrm{T}}, \quad (\boldsymbol{\mathcal{A}} : \boldsymbol{\mathfrak{B}})^{\mathrm{t}} = \boldsymbol{\mathcal{A}} : \boldsymbol{\mathfrak{B}}^{\mathrm{t}}. \tag{5.54}$$

For the tensor products (5.17) we also obtain the following relations (see Exercise 5.7)

$$(\mathbf{A} \otimes \mathbf{B})^{\mathrm{t}} : (\mathbf{C} \otimes \mathbf{D}) = \left[\left(\mathbf{A}\mathbf{D}^{\mathrm{T}}\right) \otimes \left(\mathbf{C}^{\mathrm{T}}\mathbf{B}\right)\right]^{\mathrm{t}}, \tag{5.55}$$

$$(\mathbf{A} \otimes \mathbf{B})^{\mathrm{t}} : (\mathbf{C} \odot \mathbf{D}) = \left(\mathbf{A}\mathbf{C}^{\mathrm{T}}\mathbf{B}\right) \odot \mathbf{D}. \tag{5.56}$$

Scalar product. Similarly to second-order tensors the scalar product of fourth-order tensors can be defined in terms of the basis vectors or tensors. To this end, let us consider two fourth-order tensors $\mathbf{A} \odot \mathbf{B}$ and $\mathbf{C} \odot \mathbf{D}$, where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathbf{L}\mathrm{in}^n$. Then, we set

$$(\mathbf{A} \odot \mathbf{B}) :: (\mathbf{C} \odot \mathbf{D}) = (\mathbf{A} : \mathbf{C})(\mathbf{B} : \mathbf{D}). \tag{5.57}$$

As a result of this definition we also obtain in view of (1.134) and (5.23)

$$(\boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c} \otimes \boldsymbol{d}) :: (\boldsymbol{e} \otimes \boldsymbol{f} \otimes \boldsymbol{g} \otimes \boldsymbol{h}) = (\boldsymbol{a} \cdot \boldsymbol{e})(\boldsymbol{b} \cdot \boldsymbol{f})(\boldsymbol{c} \cdot \boldsymbol{g})(\boldsymbol{d} \cdot \boldsymbol{h}). \tag{5.58}$$

For the component representation of fourth-order tensors it finally yields

$$\begin{aligned} \boldsymbol{\mathcal{A}} :: \boldsymbol{\mathfrak{B}} &= \left(\mathcal{A}^{ijkl}\boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}_l\right) \\ &:: \left(\mathcal{B}_{pqrt}\boldsymbol{g}^p \otimes \boldsymbol{g}^q \otimes \boldsymbol{g}^r \otimes \boldsymbol{g}^t\right) = \mathcal{A}^{ijkl}\mathcal{B}_{ijkl}. \end{aligned} \tag{5.59}$$

Using the latter relation one can easily prove that the properties of the scalar product (D.1-D.4) hold for fourth-order tensors as well.

5.4 Super-Symmetric Fourth-Order Tensors

On the basis of the transposition operations one defines symmetric and super-symmetric fourth-order tensors. Accordingly, a fourth-order tensor $\boldsymbol{\mathfrak{C}}$ is said to be symmetric if (major symmetry)

$$\mathbf{\mathcal{C}}^{\mathrm{T}} = \mathbf{\mathcal{C}} \tag{5.60}$$

and super-symmetric if additionally (minor symmetry)

$$\mathbf{\mathcal{C}}^{\mathrm{t}} = \mathbf{\mathcal{C}}. \tag{5.61}$$

In this section we focus on the properties of super-symmetric fourth-order tensors. They constitute a subspace of $\mathbf{\mathcal{L}}\mathrm{in}^n$ denoted in the following by $\mathbf{\mathcal{S}}\mathrm{sym}^n$. First, we prove that every super-symmetric fourth-order tensor maps an arbitrary (not necessarily symmetric) second-order tensor into a symmetric one so that

$$(\mathbf{\mathcal{C}} : \mathbf{X})^{\mathrm{T}} = \mathbf{\mathcal{C}} : \mathbf{X}, \quad \forall \mathbf{\mathcal{C}} \in \mathbf{\mathcal{S}}\mathrm{sym}^n, \quad \forall \mathbf{X} \in \mathbf{L}\mathrm{in}^n. \tag{5.62}$$

Indeed, in view of (5.45), (5.46), (5.60) and (5.61) we have

$$(\mathbf{\mathcal{C}} : \mathbf{X})^{\mathrm{T}} = \left(\mathbf{X} : \mathbf{\mathcal{C}}^{\mathrm{T}}\right)^{\mathrm{T}} = (\mathbf{X} : \mathbf{\mathcal{C}})^{\mathrm{T}} = \mathbf{X} : \mathbf{\mathcal{C}}^{\mathrm{t}} = \mathbf{X} : \mathbf{\mathcal{C}} = \mathbf{X} : \mathbf{\mathcal{C}}^{\mathrm{T}} = \mathbf{\mathcal{C}} : \mathbf{X}.$$

Next, we deal with representations of super-symmetric fourth-order tensors and study the properties of their components. Let $\mathcal{F} = \{\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_{n^2}\}$ be an arbitrary basis of $\mathbf{L}\mathrm{in}^n$ and $\mathcal{F}' = \left\{\mathbf{F}^1, \mathbf{F}^2, \ldots, \mathbf{F}^{n^2}\right\}$ the corresponding dual basis such that

$$\mathbf{F}_p : \mathbf{F}^q = \delta_p^q, \quad p, q = 1, 2, \ldots, n^2. \tag{5.63}$$

According to Theorem 5.1 we first write

$$\mathbf{\mathcal{C}} = \mathcal{C}^{pq} \mathbf{F}_p \odot \mathbf{F}_q. \tag{5.64}$$

Taking (5.60) into account and in view of $(5.49)_2$ we infer that

$$\mathcal{C}^{pq} = \mathcal{C}^{qp}, \quad p \neq q; \; p, q = 1, 2, \ldots, n^2. \tag{5.65}$$

Mapping (5.64) with the dual tensors $\mathbf{F}^r$ further yields

$$\mathbf{\mathcal{C}} : \mathbf{F}^r = \left(\mathcal{C}^{pq} \mathbf{F}_p \odot \mathbf{F}_q\right) : \mathbf{F}^r = \mathcal{C}^{pr} \mathbf{F}_p, \quad r = 1, 2, \ldots, n^2. \tag{5.66}$$

Let now $\mathbf{F}_p = \mathbf{M}_p$ $(p = 1, 2, \ldots, m)$ and $\mathbf{F}_q = \mathbf{W}_{q-m}$ $\left(q = m+1, \ldots, n^2\right)$ be bases of $\mathbf{S}\mathrm{ym}^n$ and $\mathbf{S}\mathrm{kew}^n$ (Sect. 1.9), respectively, where $m = \frac{1}{2} n (n+1)$. In view of $(5.45)_2$, (5.61), (5.62) and (5.66) we conclude that

$$\mathcal{C}^{pr} = \mathcal{C}^{rp} = 0, \quad p = 1, 2, \ldots, n^2; \; r = m+1, \ldots, n^2 \tag{5.67}$$

and consequently

$$\mathbf{\mathcal{C}} = \sum_{p,q=1}^{m} \mathcal{C}^{pq} \mathbf{M}_p \odot \mathbf{M}_q, \quad m = \frac{1}{2} n (n+1). \tag{5.68}$$

Keeping (5.65) in mind we can also write by analogy with (1.149)

$$\mathcal{C} = \sum_{p=1}^{m} \mathcal{C}^{pp} \mathbf{M}_p \odot \mathbf{M}_p + \sum_{\substack{p,q=1 \\ p>q}}^{m} \mathcal{C}^{pq} \left(\mathbf{M}_p \odot \mathbf{M}_q + \mathbf{M}_q \odot \mathbf{M}_p \right). \tag{5.69}$$

Therefore, every super-symmetric fourth-order tensor can be represented with respect to the basis $\frac{1}{2}\left(\mathbf{M}_p \odot \mathbf{M}_q + \mathbf{M}_q \odot \mathbf{M}_p\right)$, where $\mathbf{M}_q \in \mathbf{Sym}^n$ and $p \geq q = 1, 2, \ldots, \frac{1}{2}n(n+1)$. Thus, we infer that the dimension of $\mathcal{S}\mathrm{sym}^n$ is $\frac{1}{2}m(m+1) = \frac{1}{8}n^2(n+1)^2 + \frac{1}{4}n(n+1)$. We also observe that $\mathcal{S}\mathrm{sym}^n$ can be considered as the set of all linear mappings within $\mathbf{Sym}^n$.

Applying Theorem 5.2 we can also represent a super-symmetric tensor by $\mathcal{C} = \mathcal{C}^{ijkl} \boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}_l$. In this case, (5.51) and (5.52) require that (Exercise 5.8)

$$\mathcal{C}^{ijkl} = \mathcal{C}^{jilk} = \mathcal{C}^{ikjl} = \mathcal{C}^{ljki} = \mathcal{C}^{klij}. \tag{5.70}$$

Thus, we can also write

$$\begin{aligned} \mathcal{C} &= \mathcal{C}^{ijkl} \left(\boldsymbol{g}_i \otimes \boldsymbol{g}_l \right) \odot \left(\boldsymbol{g}_j \otimes \boldsymbol{g}_k \right) \\ &= \frac{1}{4} \mathcal{C}^{ijkl} \left(\boldsymbol{g}_i \otimes \boldsymbol{g}_l + \boldsymbol{g}_l \otimes \boldsymbol{g}_i \right) \odot \left(\boldsymbol{g}_j \otimes \boldsymbol{g}_k + \boldsymbol{g}_k \otimes \boldsymbol{g}_j \right) \\ &= \frac{1}{4} \mathcal{C}^{ijkl} \left(\boldsymbol{g}_j \otimes \boldsymbol{g}_k + \boldsymbol{g}_k \otimes \boldsymbol{g}_j \right) \odot \left(\boldsymbol{g}_i \otimes \boldsymbol{g}_l + \boldsymbol{g}_l \otimes \boldsymbol{g}_i \right). \end{aligned} \tag{5.71}$$

Finally, we briefly consider the eigenvalue problem for super-symmetric fourth-order tensors. It is defined as

$$\mathcal{C} : \mathbf{M} = \Lambda \mathbf{M}, \quad \mathcal{C} \in \mathcal{S}\mathrm{sym}^n, \tag{5.72}$$

where Λ and $\mathbf{M} \in \mathbf{Sym}^n$ denote the eigenvalue and the corresponding eigentensor, respectively. The spectral decomposition of $\mathcal{C}$ can be given similarly to symmetric second-order tensors (4.60) by

$$\mathcal{C} = \sum_{p=1}^{m} \Lambda_p \mathbf{M}_p \odot \mathbf{M}_p, \tag{5.73}$$

where again $m = \frac{1}{2}n(n+1)$ and

$$\mathbf{M}_p : \mathbf{M}_q = \delta_{pq}, \quad p, q = 1, 2, \ldots, m. \tag{5.74}$$

5.5 Special Fourth-Order Tensors

Identity tensor. The fourth-order identity tensor $\mathcal{I}$ is defined by

$$\mathcal{I} : \mathbf{X} = \mathbf{X}, \quad \forall \mathbf{X} \in \mathbf{Lin}^n. \tag{5.75}$$

It is seen that $\mathfrak{J}$ is a symmetric (but not super-symmetric) fourth-order tensor such that $\mathfrak{J}^{\mathrm{T}} = \mathfrak{J}$. Indeed,

$$\mathbf{X} : \mathfrak{J} = \mathbf{X}, \quad \forall \mathbf{X} \in \mathbf{L}\mathrm{in}^n. \tag{5.76}$$

With the aid of (1.86), $(5.17)_1$ or alternatively by using (5.29) the fourth-order identity tensor can be represented by

$$\mathfrak{J} = \mathbf{I} \otimes \mathbf{I} = \boldsymbol{g}_i \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}^j. \tag{5.77}$$

For the composition with other fourth-order tensors we can also write

$$\mathfrak{J} : \mathcal{A} = \mathcal{A} : \mathfrak{J} = \mathcal{A}, \quad \forall \mathcal{A} \in \mathfrak{L}\mathrm{in}^n. \tag{5.78}$$

Transposition tensor. The transposition of second-order tensors represents a linear mapping and can therefore be expressed in terms of a fourth-order tensor. This tensor denoted by $\mathfrak{T}$ is referred to as the transposition tensor. Thus,

$$\mathfrak{T} : \mathbf{X} = \mathbf{X}^{\mathrm{T}}, \quad \forall \mathbf{X} \in \mathbf{L}\mathrm{in}^n. \tag{5.79}$$

One can easily show that (Exercise 5.9)

$$\mathbf{Y} : \mathfrak{T} = \mathbf{Y}^{\mathrm{T}}, \quad \forall \mathbf{Y} \in \mathbf{L}\mathrm{in}^n. \tag{5.80}$$

Hence, the transposition tensor is symmetric such that $\mathfrak{T} = \mathfrak{T}^{\mathrm{T}}$. By virtue of $(5.45)_2$ and (5.75), $\mathfrak{T}$ can further be expressed in terms of the identity tensor by

$$\mathfrak{T} = \mathfrak{J}^{\mathrm{t}}. \tag{5.81}$$

Indeed,

$$\mathfrak{J}^{\mathrm{t}} : \mathbf{X} = \mathfrak{J} : \mathbf{X}^{\mathrm{T}} = \mathbf{X}^{\mathrm{T}} = \mathfrak{T} : \mathbf{X}, \quad \forall \mathbf{X} \in \mathbf{L}\mathrm{in}^n.$$

Considering (5.52) and (5.77) in (5.81) we thus obtain

$$\mathfrak{T} = (\mathbf{I} \otimes \mathbf{I})^{\mathrm{t}} = \boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}^i \otimes \boldsymbol{g}^j. \tag{5.82}$$

The transposition tensor can further be characterized by the following identities (see Exercise 5.10)

$$\mathcal{A} : \mathfrak{T} = \mathcal{A}^{\mathrm{t}}, \quad \mathfrak{T} : \mathcal{A} = \mathcal{A}^{\mathrm{TtT}}, \quad \mathfrak{T} : \mathfrak{T} = \mathfrak{J}, \quad \forall \mathcal{A} \in \mathfrak{L}\mathrm{in}^n. \tag{5.83}$$

Super-symmetric identity tensor. The identity tensor (5.77) is symmetric but not super-symmetric. For this reason, it is useful to define a special identity tensor within $\mathfrak{S}\mathrm{sym}^n$. This super-symmetric tensor maps every symmetric second-order tensor into itself like the identity tensor (5.77). It can be expressed by

$$\mathfrak{I}^{\mathrm{s}} = \frac{1}{2}(\mathfrak{I} + \mathfrak{T}) = (\mathbf{I} \otimes \mathbf{I})^{\mathrm{s}}. \tag{5.84}$$

However, in contrast to the identity tensor $\mathfrak{I}$ (5.77), the super-symmetric identity tensor $\mathfrak{I}^{\mathrm{s}}$ (5.84) maps any arbitrary (not necessarily symmetric) second-order tensor into its symmetric part so that in view of (5.48)

$$\mathfrak{I}^{\mathrm{s}} : \mathbf{X} = \mathbf{X} : \mathfrak{I}^{\mathrm{s}} = \mathrm{sym}\mathbf{X}, \quad \forall \mathbf{X} \in \mathbf{Lin}^n. \tag{5.85}$$

Spherical, deviatoric and trace projection tensors. The spherical and deviatoric part of a second-order tensor are defined as a linear mapping (1.153) and can thus be expressed by

$$\mathrm{sph}\mathbf{A} = \mathfrak{P}_{\mathrm{sph}} : \mathbf{A}, \quad \mathrm{dev}\mathbf{A} = \mathfrak{P}_{\mathrm{dev}} : \mathbf{A}, \tag{5.86}$$

where the fourth-order tensors $\mathfrak{P}_{\mathrm{sph}}$ and $\mathfrak{P}_{\mathrm{dev}}$ are called the spherical and deviatoric projection tensors, respectively. In view of (1.153) they are given by

$$\mathfrak{P}_{\mathrm{sph}} = \frac{1}{n}\mathbf{I} \odot \mathbf{I}, \quad \mathfrak{P}_{\mathrm{dev}} = \mathfrak{I} - \frac{1}{n}\mathbf{I} \odot \mathbf{I}, \tag{5.87}$$

where $\mathbf{I} \odot \mathbf{I}$ represents the so-called trace projection tensor. Indeed,

$$\mathbf{I} \odot \mathbf{I} : \mathbf{X} = \mathbf{I}\mathrm{tr}\mathbf{X}, \quad \forall \mathbf{X} \in \mathbf{Lin}^n. \tag{5.88}$$

According to $(5.49)_2$ and (5.50), the spherical and trace projection tensors are super-symmetric. The spherical and deviatoric projection tensors are furthermore characterized by the properties:

$$\mathfrak{P}_{\mathrm{dev}} : \mathfrak{P}_{\mathrm{dev}} = \mathfrak{P}_{\mathrm{dev}}, \quad \mathfrak{P}_{\mathrm{sph}} : \mathfrak{P}_{\mathrm{sph}} = \mathfrak{P}_{\mathrm{sph}},$$
$$\mathfrak{P}_{\mathrm{dev}} : \mathfrak{P}_{\mathrm{sph}} = \mathfrak{P}_{\mathrm{sph}} : \mathfrak{P}_{\mathrm{dev}} = \mathfrak{O}. \tag{5.89}$$

Example. Elasticity tensor for the generalized Hooke's law. The generalized Hooke's law is written as

$$\boldsymbol{\sigma} = 2G\boldsymbol{\epsilon} + \lambda\mathrm{tr}(\boldsymbol{\epsilon})\mathbf{I} = 2G\mathrm{dev}\boldsymbol{\epsilon} + \left(\lambda + \frac{2}{3}G\right)\mathrm{tr}(\boldsymbol{\epsilon})\mathbf{I}, \tag{5.90}$$

where G and λ denote the so-called Lamé constants. The corresponding super-symmetric elasticity tensor takes the form

$$\mathfrak{C} = 2G\mathfrak{I}^{\mathrm{s}} + \lambda\mathbf{I} \odot \mathbf{I} = 2G\mathfrak{P}^{\mathrm{s}}_{\mathrm{dev}} + (3\lambda + 2G)\,\mathfrak{P}_{\mathrm{sph}}. \tag{5.91}$$

Exercises

5.1. Prove relations (5.20) and (5.21).

5.2. Prove relations (5.22).

5.3. Prove relations (5.42) and (5.43).

5.4. Prove relations (5.49-5.52).

5.5. Prove that $\boldsymbol{\mathcal{A}}^{\mathrm{Tt}} \neq \boldsymbol{\mathcal{A}}^{\mathrm{tT}}$ for $\boldsymbol{\mathcal{A}} = \boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c} \otimes \boldsymbol{d}$.

5.6. Prove identities (5.54).

5.7. Verify relations (5.55) and (5.56).

5.8. Prove relations (5.70) for the components of a super-symmetric fourth-order tensor using (5.51) and (5.52).

5.9. Prove relation (5.80) using (5.16) and (5.79).

5.10. Verify the properties of the transposition tensor (5.83).

5.11. Prove that the fourth-order tensor of the form

$$\boldsymbol{\mathfrak{C}} = (\mathbf{M}_1 \otimes \mathbf{M}_2 + \mathbf{M}_2 \otimes \mathbf{M}_1)^{\mathrm{s}}$$

is super-symmetric if $\mathbf{M}_1, \mathbf{M}_2 \in \mathbf{S}\mathrm{ym}^n$.

5.12. Calculate eigenvalues and eigentensors of the following super-symmetric fourth-order tensors for $n = 3$: (a) $\boldsymbol{\mathfrak{I}}^{\mathrm{s}}$(5.84), (b) $\boldsymbol{\mathfrak{P}}_{\mathrm{sph}}$ $(5.87)_1$, (c) $\boldsymbol{\mathfrak{P}}^{\mathrm{s}}_{\mathrm{dev}}$ $(5.87)_2$, (d) $\boldsymbol{\mathfrak{C}}$ (5.91).