6

Analysis of Tensor Functions

6.1 Scalar-Valued Isotropic Tensor Functions

Let us consider a real scalar-valued function $f(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_l)$ of second-order tensors $\mathbf{A}_k \in \mathbf{L}\mathrm{in}^n$ $(k = 1, 2, \ldots, l)$. The function f is said to be isotropic if

$$f\left(\mathbf{Q}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_2\mathbf{Q}^{\mathrm{T}}, \ldots, \mathbf{Q}\mathbf{A}_l\mathbf{Q}^{\mathrm{T}}\right) = f(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_l), \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n. \tag{6.1}$$

Example. Consider the function $f(\mathbf{A}, \mathbf{B}) = \mathrm{tr}(\mathbf{AB})$. Since in view of (1.129) and (1.144)

$$\begin{aligned} f\left(\mathbf{QAQ}^{\mathrm{T}}, \mathbf{QBQ}^{\mathrm{T}}\right) &= \mathrm{tr}\left(\mathbf{QAQ}^{\mathrm{T}}\mathbf{QBQ}^{\mathrm{T}}\right) \\ &= \mathrm{tr}\left(\mathbf{QABQ}^{\mathrm{T}}\right) = \mathrm{tr}\left(\mathbf{ABQ}^{\mathrm{T}}\mathbf{Q}\right) \\ &= \mathrm{tr}(\mathbf{AB}) = f(\mathbf{A}, \mathbf{B}), \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n, \end{aligned}$$

this function is isotropic according to the definition (6.1). In contrast, the function $f(\mathbf{A}) = \mathrm{tr}(\mathbf{AL})$, where $\mathbf{L}$ denotes a second-order tensor, is not isotropic. Indeed,

$$f\left(\mathbf{QAQ}^{\mathrm{T}}\right) = \mathrm{tr}\left(\mathbf{QAQ}^{\mathrm{T}}\mathbf{L}\right) \neq \mathrm{tr}(\mathbf{AL}).$$

Scalar-valued isotropic tensor functions are also called isotropic invariants of the tensors $\mathbf{A}_k$ $(k = 1, 2, \ldots, l)$. For such a tensor system one can construct, in principle, an unlimited number of isotropic invariants. However, for every finite system of tensors one can find a finite number of isotropic invariants in terms of which all other isotropic invariants can be expressed (Hilbert's theorem). This system of invariants is called functional basis of the tensors $\mathbf{A}_k$ $(k = 1, 2, \ldots, l)$. For one and the same system of tensors there exist many functional bases. A functional basis is called irreducible if none of its elements can be expressed in a unique form in terms of the remaining invariants.

First, we focus on isotropic functions of one second-order tensor

$$f\left(\mathbf{QAQ}^{\mathrm{T}}\right) = f\left(\mathbf{A}\right), \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n, \quad \mathbf{A} \in \mathbf{L}\mathrm{in}^n. \tag{6.2}$$

One can show that the principal traces $\mathrm{tr}\mathbf{A}^k$, principal invariants $\mathrm{I}_{\mathbf{A}}^{(k)}$ and eigenvalues λ_k, $(k = 1, 2, \ldots, n)$ of the tensor $\mathbf{A}$ represent its isotropic tensor functions. Indeed, for the principal traces we can write by virtue of (1.144)

$$\mathrm{tr}\left(\mathbf{QAQ}^{\mathrm{T}}\right)^k = \mathrm{tr}\left(\underbrace{\mathbf{QAQ}^{\mathrm{T}}\mathbf{QAQ}^{\mathrm{T}}\ldots\mathbf{QAQ}^{\mathrm{T}}}_{k\ \text{times}}\right) = \mathrm{tr}\left(\mathbf{QA}^k\mathbf{Q}^{\mathrm{T}}\right)$$

$$= \mathrm{tr}\left(\mathbf{A}^k\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\right) = \mathrm{tr}\mathbf{A}^k, \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n. \tag{6.3}$$

The principal invariants are uniquely expressed in terms of the principal traces by means of Newton's formula (4.26), while the eigenvalues are, in turn, defined by the principal invariants as solutions of the characteristic equation (4.20) with the characteristic polynomial given by (4.18).

Further, we prove that both the eigenvalues λ_k, principal invariants $\mathrm{I}_{\mathbf{M}}^{(k)}$ and principal traces $\mathrm{tr}\mathbf{M}^k$ $(k = 1, 2, \ldots, n)$ of one symmetric tensor $\mathbf{M} \in \mathbf{S}\mathrm{ym}^n$ form its functional bases (see also [45]). To this end, we consider two arbitrary symmetric second-order tensors $\mathbf{M}_1, \mathbf{M}_2 \in \mathbf{S}\mathrm{ym}^n$ with the same eigenvalues. Then, the spectral representation (4.60) takes the form

$$\mathbf{M}_1 = \sum_{i=1}^{n} \lambda_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i, \quad \mathbf{M}_2 = \sum_{i=1}^{n} \lambda_i \boldsymbol{m}_i \otimes \boldsymbol{m}_i, \tag{6.4}$$

where according to (4.62) both the eigenvectors $\boldsymbol{n}_i$ and $\boldsymbol{m}_i$ form orthonormal bases such that $\boldsymbol{n}_i \cdot \boldsymbol{n}_j = \delta_{ij}$ and $\boldsymbol{m}_i \cdot \boldsymbol{m}_j = \delta_{ij}$ $(i, j = 1, 2, \ldots, n)$. Now, we consider the orthogonal tensor

$$\mathbf{Q} = \sum_{i=1}^{n} \boldsymbol{m}_i \otimes \boldsymbol{n}_i. \tag{6.5}$$

Indeed,

$$\mathbf{QQ}^{\mathrm{T}} = \left(\sum_{i=1}^{n} \boldsymbol{m}_i \otimes \boldsymbol{n}_i\right)\left(\sum_{j=1}^{n} \boldsymbol{n}_j \otimes \boldsymbol{m}_j\right)$$

$$= \sum_{i,j=1}^{n} \delta_{ij} \boldsymbol{m}_i \otimes \boldsymbol{m}_j = \sum_{i=1}^{n} \boldsymbol{m}_i \otimes \boldsymbol{m}_i = \mathbf{I}.$$

By use of (1.116), (6.4) and (6.5) we further obtain

$$\mathbf{Q}\mathbf{M}_1\mathbf{Q}^{\mathrm{T}} = \left(\sum_{i=1}^{n} \boldsymbol{m}_i \otimes \boldsymbol{n}_i\right) \left(\sum_{j=1}^{n} \lambda_j \boldsymbol{n}_j \otimes \boldsymbol{n}_j\right) \left(\sum_{k=1}^{n} \boldsymbol{n}_k \otimes \boldsymbol{m}_k\right)$$

$$= \sum_{i,j,k=1}^{n} \delta_{ij}\delta_{jk}\lambda_j \boldsymbol{m}_i \otimes \boldsymbol{m}_k = \sum_{i=1}^{n} \lambda_i \boldsymbol{m}_i \otimes \boldsymbol{m}_i = \mathbf{M}_2. \tag{6.6}$$

Hence,

$$f(\mathbf{M}_1) = f\left(\mathbf{Q}\mathbf{M}_1\mathbf{Q}^{\mathrm{T}}\right) = f(\mathbf{M}_2). \tag{6.7}$$

Thus, f takes the same value for all symmetric tensors with pairwise equal eigenvalues. This means that an isotropic tensor function of a symmetric tensor is uniquely defined in terms of its eigenvalues, principal invariants or principal traces because the latter ones are, in turn, uniquely defined by the eigenvalues according to (4.24) and (4.25). This implies the following representations

$$f(\mathbf{M}) = \widehat{f}\left(\mathrm{I}_{\mathbf{M}}^{(1)}, \mathrm{I}_{\mathbf{M}}^{(2)}, \ldots, \mathrm{I}_{\mathbf{M}}^{(n)}\right) = \hat{f}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

$$= \tilde{f}\left(\mathrm{tr}\mathbf{M}, \mathrm{tr}\mathbf{M}^2, \ldots, \mathrm{tr}\mathbf{M}^n\right), \quad \mathbf{M} \in \mathbf{S}\mathrm{ym}^n. \tag{6.8}$$

Example. Strain energy function of an isotropic hyperelastic material. A material is said to be hyperelastic if it is characterized by the existence of a strain energy ψ defined as a function, for example, of the right Cauchy-Green tensor $\mathbf{C}$. For isotropic materials this strain energy function obeys the condition

$$\psi\left(\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}\right) = \psi(\mathbf{C}), \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^3. \tag{6.9}$$

By means of (6.8) this function can be expressed by

$$\psi(\mathbf{C}) = \widehat{\psi}(\mathrm{I}_{\mathbf{C}}, \mathrm{II}_{\mathbf{C}}, \mathrm{III}_{\mathbf{C}}) = \hat{\psi}(\lambda_1, \lambda_2, \lambda_3) = \tilde{\psi}\left(\mathrm{tr}\mathbf{C}, \mathrm{tr}\mathbf{C}^2, \mathrm{tr}\mathbf{C}^3\right), \tag{6.10}$$

where λ_i denote the so-called principal stretches. They are expressed in terms of the eigenvalues Λ_i $(i = 1, 2, 3)$ of the right Cauchy-Green tensor $\mathbf{C} = \sum_{i=1}^{3} \Lambda_i \mathbf{P}_i$ as $\lambda_i = \sqrt{\Lambda_i}$. For example, the strain energy function of the so-called Mooney-Rivlin material is given in terms of the first and second principal invariants by

$$\psi(\mathbf{C}) = c_1(\mathrm{I}_{\mathbf{C}} - 3) + c_2(\mathrm{II}_{\mathbf{C}} - 3), \tag{6.11}$$

where c_1 and c_2 represent material constants. In contrast, the strain energy function of the Ogden material [29] is defined in terms of the principal stretches by

$$\psi(\mathbf{C}) = \sum_{r=1}^{m} \frac{\mu_r}{\alpha_r}\left(\lambda_1^{\alpha_r} + \lambda_2^{\alpha_r} + \lambda_3^{\alpha_r} - 3\right), \tag{6.12}$$

where μ_r, α_r $(r = 1, 2, \ldots, m)$ denote material constants.

For isotropic functions (6.1) of a finite number l of arbitrary second-order tensors the functional basis is obtained only for three-dimensional space. In order to represent this basis, the tensor arguments are split according to (1.145) into a symmetric and a skew-symmetric part respectively as follows:

$$\mathbf{M}_i = \mathrm{sym}\mathbf{A}_i = \frac{1}{2}\left(\mathbf{A}_i + \mathbf{A}_i^{\mathrm{T}}\right), \quad \mathbf{W}_i = \mathrm{skew}\mathbf{A}_i = \frac{1}{2}\left(\mathbf{A}_i - \mathbf{A}_i^{\mathrm{T}}\right). \tag{6.13}$$

In this manner, every isotropic tensor function can be given in terms of a finite number of symmetric tensors $\mathbf{M}_i \in \mathbf{S}\mathrm{ym}^3$ $(i = 1, 2, \ldots, m)$ and skew-symmetric tensors $\mathbf{W}_i \in \mathbf{S}\mathrm{kew}^3$ $(i = 1, 2, \ldots, w)$ as

$$f = \hat{f}\left(\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_m, \mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_w\right). \tag{6.14}$$

An irreducible functional basis of such a system of tensors is proved to be given by (see [2], [32], [40])

$$\begin{aligned}
&\mathrm{tr}\mathbf{M}_i,\ \mathrm{tr}\mathbf{M}_i^2,\ \mathrm{tr}\mathbf{M}_i^3,\\
&\mathrm{tr}\left(\mathbf{M}_i\mathbf{M}_j\right),\ \mathrm{tr}\left(\mathbf{M}_i^2\mathbf{M}_j\right),\ \mathrm{tr}\left(\mathbf{M}_i\mathbf{M}_j^2\right),\ \mathrm{tr}\left(\mathbf{M}_i^2\mathbf{M}_j^2\right),\ \mathrm{tr}\left(\mathbf{M}_i\mathbf{M}_j\mathbf{M}_k\right),\\
&\mathrm{tr}\mathbf{W}_p^2,\ \mathrm{tr}\left(\mathbf{W}_p\mathbf{W}_q\right),\ \mathrm{tr}\left(\mathbf{W}_p\mathbf{W}_q\mathbf{W}_r\right),\\
&\mathrm{tr}\left(\mathbf{M}_i\mathbf{W}_p^2\right),\ \mathrm{tr}\left(\mathbf{M}_i^2\mathbf{W}_p^2\right),\ \mathrm{tr}\left(\mathbf{M}_i^2\mathbf{W}_p^2\mathbf{M}_i\mathbf{W}_p\right),\ \mathrm{tr}\left(\mathbf{M}_i\mathbf{W}_p\mathbf{W}_q\right),\\
&\mathrm{tr}\left(\mathbf{M}_i\mathbf{W}_p^2\mathbf{W}_q\right),\ \mathrm{tr}\left(\mathbf{M}_i\mathbf{W}_p\mathbf{W}_q^2\right),\ \mathrm{tr}\left(\mathbf{M}_i\mathbf{M}_j\mathbf{W}_p\right),\\
&\mathrm{tr}\left(\mathbf{M}_i\mathbf{W}_p^2\mathbf{M}_j\mathbf{W}_p\right),\ \mathrm{tr}\left(\mathbf{M}_i^2\mathbf{M}_j\mathbf{W}_p\right),\ \mathrm{tr}\left(\mathbf{M}_i\mathbf{M}_j^2\mathbf{W}_p\right),\\
&i < j < k = 1, 2, \ldots, m, \quad p < q < r = 1, 2, \ldots, w.
\end{aligned} \tag{6.15}$$

For illustration of this result we consider some examples.

Example 1. Functional basis of one skew-symmetric second-order tensor $\mathbf{W} \in \mathbf{S}\mathrm{kew}^3$. With the aid of (6.15) and (4.87) we obtain the basis consisting of only one invariant

$$\mathrm{tr}\mathbf{W}^2 = -2\mathrm{II}_{\mathbf{W}} = -\left\|\mathbf{W}\right\|^2. \tag{6.16}$$

Example 2. Functional basis of an arbitrary second-order tensor $\mathbf{A} \in \mathbf{L}\mathrm{in}^3$. By means of (6.15) one can write the following functional basis of $\mathbf{A}$

$$\begin{aligned}
&\mathrm{tr}\mathbf{M},\ \mathrm{tr}\mathbf{M}^2,\ \mathrm{tr}\mathbf{M}^3,\\
&\mathrm{tr}\mathbf{W}^2,\ \mathrm{tr}\left(\mathbf{M}\mathbf{W}^2\right),\ \mathrm{tr}\left(\mathbf{M}^2\mathbf{W}^2\right),\ \mathrm{tr}\left(\mathbf{M}^2\mathbf{W}^2\mathbf{M}\mathbf{W}\right),
\end{aligned} \tag{6.17}$$

where $\mathbf{M} = \mathrm{sym}\mathbf{A}$ and $\mathbf{W} = \mathrm{skew}\mathbf{A}$. Inserting representations (6.13) into (6.17) the functional basis of $\mathbf{A}$ can be rewritten as (see Exercise 6.2)

$$\mathrm{tr}\mathbf{A},\ \mathrm{tr}\mathbf{A}^2,\ \mathrm{tr}\mathbf{A}^3,\ \mathrm{tr}\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right),\ \mathrm{tr}\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right)^2,\ \mathrm{tr}\left(\mathbf{A}^2\mathbf{A}^{\mathrm{T}}\right),$$
$$\mathrm{tr}\left[\left(\mathbf{A}^{\mathrm{T}}\right)^2\mathbf{A}^2\mathbf{A}^{\mathrm{T}}\mathbf{A}-\mathbf{A}^2\left(\mathbf{A}^{\mathrm{T}}\right)^2\mathbf{A}\mathbf{A}^{\mathrm{T}}\right]. \tag{6.18}$$

Example 3. Functional basis of two symmetric second-order tensors $\mathbf{M}_1, \mathbf{M}_2 \in \mathbf{S}\mathrm{ym}^3$. According to (6.15) the functional basis includes in this case the following ten invariants

$$\mathrm{tr}\mathbf{M}_1,\ \mathrm{tr}\mathbf{M}_1^2,\ \mathrm{tr}\mathbf{M}_1^3,\ \mathrm{tr}\mathbf{M}_2,\ \mathrm{tr}\mathbf{M}_2^2,\ \mathrm{tr}\mathbf{M}_2^3,$$
$$\mathrm{tr}\left(\mathbf{M}_1\mathbf{M}_2\right),\ \mathrm{tr}\left(\mathbf{M}_1^2\mathbf{M}_2\right),\ \mathrm{tr}\left(\mathbf{M}_1\mathbf{M}_2^2\right),\ \mathrm{tr}\left(\mathbf{M}_1^2\mathbf{M}_2^2\right). \tag{6.19}$$

6.2 Scalar-Valued Anisotropic Tensor Functions

A real scalar-valued function $f\left(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_l\right)$ of second-order tensors $\mathbf{A}_k \in \mathbf{L}\mathrm{in}^n$ $(k = 1, 2, \ldots, l)$ is said to be anisotropic if it is invariant only with respect to a subset of all orthogonal transformations:

$$f\left(\mathbf{Q}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_2\mathbf{Q}^{\mathrm{T}}, \ldots, \mathbf{Q}\mathbf{A}_l\mathbf{Q}^{\mathrm{T}}\right)$$
$$= f\left(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_l\right), \quad \forall \mathbf{Q} \in \mathbf{S}\mathrm{orth}^n \subset \mathbf{O}\mathrm{rth}^n. \tag{6.20}$$

The subset $\mathbf{S}\mathrm{orth}^n$ represents a group called symmetry group. In continuum mechanics, anisotropic properties of materials are characterized by their symmetry group. The largest symmetry group $\mathbf{O}\mathrm{rth}^3$ (in three-dimensional space) includes all orthogonal transformations and is referred to as isotropic. In contrast, the smallest symmetry group consists of only two elements $\mathbf{I}$ and $-\mathbf{I}$ and is called triclinic.

Example. Transversely isotropic material symmetry. In this case the material is characterized by symmetry with respect to one selected direction referred to as principal material direction. Properties of a transversely isotropic material remain unchanged by rotations about, and reflections from the planes orthogonal or parallel to, this direction. Introducing a unit vector $\boldsymbol{l}$ in the principal direction we can write

$$\mathbf{Q}\boldsymbol{l} = \pm\boldsymbol{l}, \quad \forall \mathbf{Q} \in \mathfrak{g}_t, \tag{6.21}$$

where $\mathfrak{g}_t \subset \mathbf{O}\mathrm{rth}^3$ denotes the transversely isotropic symmetry group. With the aid of a special tensor

$$\mathbf{L} = \boldsymbol{l} \otimes \boldsymbol{l}, \tag{6.22}$$

called structural tensor, condition (6.21) can be represented as

$$\mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathrm{T}} = \mathbf{L}, \quad \forall \mathbf{Q} \in \mathfrak{g}_t. \tag{6.23}$$

Hence, the transversely isotropic symmetry group can be defined by

$$\mathfrak{g}_t = \left\{ \mathbf{Q} \in \mathbf{O}\mathrm{rth}^3 : \mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathrm{T}} = \mathbf{L} \right\}. \tag{6.24}$$

A strain energy function ψ of a transversely isotropic material is invariant with respect to all orthogonal transformations within $\mathfrak{g}_t$. Using a representation in terms of the right Cauchy-Green tensor $\mathbf{C}$ this leads to the following condition:

$$\psi\left(\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}\right) = \psi\left(\mathbf{C}\right), \quad \forall \mathbf{Q} \in \mathfrak{g}_t. \tag{6.25}$$

It can be shown that this condition is a priori satisfied if the strain energy function can be represented as an isotropic function of both $\mathbf{C}$ and $\mathbf{L}$ so that

$$\hat{\psi}\left(\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathrm{T}}\right) = \hat{\psi}\left(\mathbf{C}, \mathbf{L}\right), \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^3. \tag{6.26}$$

Indeed,

$$\hat{\psi}\left(\mathbf{C}, \mathbf{L}\right) = \hat{\psi}\left(\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathrm{T}}\right) = \hat{\psi}\left(\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}, \mathbf{L}\right), \quad \forall \mathbf{Q} \in \mathfrak{g}_t. \tag{6.27}$$

With the aid of the functional basis (6.19) and taking into account the identities

$$\mathbf{L}^k = \mathbf{L}, \quad \mathrm{tr}\mathbf{L}^k = 1, \quad k = 1, 2, \ldots \tag{6.28}$$

resulting from (6.22) we can thus represent the transversely isotropic function in terms of the five invariants by (see also [42])

$$\psi = \hat{\psi}\left(\mathbf{C}, \mathbf{L}\right) = \tilde{\psi}\left[\mathrm{tr}\mathbf{C}, \mathrm{tr}\mathbf{C}^2, \mathrm{tr}\mathbf{C}^3, \mathrm{tr}\left(\mathbf{C}\mathbf{L}\right), \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}\right)\right]. \tag{6.29}$$

The above procedure can be generalized for an arbitrary anisotropic symmetry group $\mathfrak{g}$. Let $\mathbf{L}_i$ $(i = 1, 2, \ldots, m)$ be a set of second-order tensors which uniquely define $\mathfrak{g}$ by

$$\mathfrak{g} = \left\{ \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n : \mathbf{Q}\mathbf{L}_i\mathbf{Q}^{\mathrm{T}} = \mathbf{L}_i,\ i = 1, 2, \ldots, m \right\}. \tag{6.30}$$

In continuum mechanics the tensors $\mathbf{L}_i$ are called structural tensors since they lay down the material or structural symmetry.

It is seen that the isotropic tensor function

$$f\left(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{L}_j\mathbf{Q}^{\mathrm{T}}\right) = f\left(\mathbf{A}_i, \mathbf{L}_j\right), \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n, \tag{6.31}$$

where we use the abbreviated notation

$$f\left(\mathbf{A}_i, \mathbf{L}_j\right) = f\left(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_l, \mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_m\right), \tag{6.32}$$

is anisotropic with respect to the arguments $\mathbf{A}_i$ $(i = 1, 2, \ldots, l)$ so that

$$f\left(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}\right) = f\left(\mathbf{A}_i\right), \quad \forall \mathbf{Q} \in \mathfrak{g}. \tag{6.33}$$

Indeed, by virtue of (6.30) and (6.31) we have

$$f\left(\mathbf{A}_i, \mathbf{L}_j\right) = f\left(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{L}_j\mathbf{Q}^{\mathrm{T}}\right) = f\left(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}, \mathbf{L}_j\right), \quad \forall \mathbf{Q} \in \mathfrak{g}. \tag{6.34}$$

Thus, every isotropic invariant of the tensor system $\mathbf{A}_i$ $(i = 1, 2, \ldots, l)$, $\mathbf{L}_j$ $(j = 1, 2, \ldots, m)$ represents an anisotropic invariant of the tensors $\mathbf{A}_i$ $(i = 1, 2, \ldots, l)$ in the sense of definition (6.20). Conversely, one can show that for every anisotropic function (6.33) there exists an equivalent isotropic function of the tensor system $\mathbf{A}_i$ $(i = 1, 2, \ldots, l)$, $\mathbf{L}_j$ $(j = 1, 2, \ldots, m)$. In order to prove this statement we consider a new tensor function defined by

$$\hat{f}\left(\mathbf{A}_i, \mathbf{X}_j\right) = f\left(\mathbf{Q}'\mathbf{A}_i\mathbf{Q}'^{\mathrm{T}}\right), \tag{6.35}$$

where the tensor $\mathbf{Q}' \in \mathbf{O}\mathrm{rth}^n$ results from the condition:

$$\mathbf{Q}'\mathbf{X}_j\mathbf{Q}'^{\mathrm{T}} = \mathbf{L}_j, \quad j = 1, 2, \ldots, m. \tag{6.36}$$

Thus, the function $\hat{f}$ is defined only over such tensors $\mathbf{X}_j$ that can be obtained from the structural tensors $\mathbf{L}_j$ $(j = 1, 2, \ldots, m)$ by the transformation

$$\mathbf{X}_j = \mathbf{Q}'^{\mathrm{T}}\mathbf{L}_j\mathbf{Q}', \quad j = 1, 2, \ldots, m, \tag{6.37}$$

where $\mathbf{Q}'$ is an arbitrary orthogonal tensor.

Further, one can show that the so-defined function (6.35) is isotropic. Indeed,

$$\hat{f}\left(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{X}_j\mathbf{Q}^{\mathrm{T}}\right) = f\left(\mathbf{Q}''\mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}\mathbf{Q}''^{\mathrm{T}}\right), \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n, \tag{6.38}$$

where according to (6.36)

$$\mathbf{Q}''\mathbf{Q}\mathbf{X}_j\mathbf{Q}^{\mathrm{T}}\mathbf{Q}''^{\mathrm{T}} = \mathbf{L}_j, \quad \mathbf{Q}'' \in \mathbf{O}\mathrm{rth}^n. \tag{6.39}$$

Inserting (6.37) into (6.39) yields

$$\mathbf{Q}''\mathbf{Q}\mathbf{Q}'^{\mathrm{T}}\mathbf{L}_j\mathbf{Q}'\mathbf{Q}^{\mathrm{T}}\mathbf{Q}''^{\mathrm{T}} = \mathbf{L}_j, \tag{6.40}$$

so that

$$\mathbf{Q}^* = \mathbf{Q}''\mathbf{Q}\mathbf{Q}'^{\mathrm{T}} \in \mathfrak{g}. \tag{6.41}$$

Hence, we can write

$$f\left(\mathbf{Q}''\mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}\mathbf{Q}''^{\mathrm{T}}\right) = f\left(\mathbf{Q}^*\mathbf{Q}'\mathbf{A}_i\mathbf{Q}'^{\mathrm{T}}\mathbf{Q}^{*\mathrm{T}}\right)$$
$$= f\left(\mathbf{Q}'\mathbf{A}_i\mathbf{Q}'^{\mathrm{T}}\right) = \hat{f}\left(\mathbf{A}_i, \mathbf{X}_j\right)$$

and consequently in view of (6.38)

$$\hat{f}\left(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{X}_j\mathbf{Q}^{\mathrm{T}}\right) = \hat{f}\left(\mathbf{A}_i, \mathbf{X}_j\right), \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n. \tag{6.42}$$

Thus, we have proved the following theorem [49].

Theorem 6.1. *A scalar-valued function* $f(\mathbf{A}_i)$ *is invariant within the symmetry group* $\mathfrak{g}$ *defined by* (6.30) *if and only if there exists an isotropic function* $\hat{f}(\mathbf{A}_i, \mathbf{L}_j)$ *such that*

$$f(\mathbf{A}_i) = \hat{f}(\mathbf{A}_i, \mathbf{L}_j). \tag{6.43}$$

6.3 Derivatives of Scalar-Valued Tensor Functions

Let us again consider a scalar-valued tensor function $f(\mathbf{A}) : \mathbf{Lin}^n \mapsto \mathbb{R}$. This function is said to be differentiable in a neighborhood of $\mathbf{A}$ if there exists a tensor $f(\mathbf{A})_{,\mathbf{A}} \in \mathbf{Lin}^n$, such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{A} + t\mathbf{X})\right|_{t=0} = f(\mathbf{A})_{,\mathbf{A}} : \mathbf{X}, \quad \forall \mathbf{X} \in \mathbf{Lin}^n. \tag{6.44}$$

This definition implies that the directional derivative (also called Gateaux derivative) $\left.\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{A} + t\mathbf{X})\right|_{t=0}$ exists and is continuous at $\mathbf{A}$. The tensor $f(\mathbf{A})_{,\mathbf{A}}$ is referred to as the derivative or the gradient of the tensor function $f(\mathbf{A})$.

In order to obtain a direct expression for $f(\mathbf{A})_{,\mathbf{A}}$ we represent the tensors $\mathbf{A}$ and $\mathbf{X}$ in (6.44) with respect to an arbitrary basis, say $\boldsymbol{g}_i \otimes \boldsymbol{g}^j$ $(i, j = 1, 2, \ldots, n)$. Then, using the chain rule one can write

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{A} + t\mathbf{X})\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t} f\left[\left(\mathrm{A}^i_{\cdot j} + t\mathrm{X}^i_{\cdot j}\right) \boldsymbol{g}_i \otimes \boldsymbol{g}^j\right]\right|_{t=0} = \frac{\partial f}{\partial \mathrm{A}^i_{\cdot j}} \mathrm{X}^i_{\cdot j}.$$

Comparing this result with (6.44) yields

$$f(\mathbf{A})_{,\mathbf{A}} = \frac{\partial f}{\partial \mathrm{A}^i_{\cdot j}} \boldsymbol{g}^i \otimes \boldsymbol{g}_j = \frac{\partial f}{\partial \mathrm{A}_{ij}} \boldsymbol{g}_i \otimes \boldsymbol{g}_j = \frac{\partial f}{\partial \mathrm{A}^{ij}} \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \frac{\partial f}{\partial \mathrm{A}_{i\cdot}^{\ j}} \boldsymbol{g}_i \otimes \boldsymbol{g}^j. \tag{6.45}$$

If the function $f(\mathbf{A})$ is defined not on all linear transformations but only on a subset $\mathbf{Slin}^n \subset \mathbf{Lin}^n$, the directional derivative (6.44) does not, however, yield a unique result for $f(\mathbf{A})_{,\mathbf{A}}$. In this context, let us consider for example scalar-valued functions of symmetric tensors: $f(\mathbf{M}) : \mathbf{Sym}^n \mapsto \mathbb{R}$. In this case, the directional derivative (6.44) defines $f(\mathbf{M})_{,\mathbf{M}}$ only up to an arbitrary skew-symmetric component $\mathbf{W}$. Indeed,

$$f(\mathbf{M})_{,\mathbf{M}} : \mathbf{X} = \left[f(\mathbf{M})_{,\mathbf{M}} + \mathbf{W}\right] : \mathbf{X}, \quad \forall \mathbf{W} \in \mathbf{Skew}^n, \ \forall \mathbf{X} \in \mathbf{Sym}^n. \tag{6.46}$$

In this relation, $\mathbf{X}$ is restricted to symmetric tensors because the tensor $\mathbf{M} + t\mathbf{X}$ appearing in the directional derivative (6.44) must belong to the definition domain of the function f for all real values of t.

To avoid this non-uniqueness we will assume that the derivative $f(\mathbf{A})_{,\mathbf{A}}$ belongs to the same subset $\mathbf{S}\mathrm{lin}^n \subset \mathbf{L}\mathrm{in}^n$ as its argument $\mathbf{A} \in \mathbf{S}\mathrm{lin}^n$. In particular, for symmetric tensor functions it implies that

$$f(\mathbf{M})_{,\mathbf{M}} \in \mathbf{S}\mathrm{ym}^n \quad \text{for} \quad \mathbf{M} \in \mathbf{S}\mathrm{ym}^n. \tag{6.47}$$

In order to calculate the derivative of a symmetric tensor function satisfying the condition (6.47) one can apply the following procedure. First, the definition domain of the function f is notionally extended to all linear transformations $\mathbf{L}\mathrm{in}^n$. Applying then the directional derivative (6.44) one obtains a unique result for the tensor $f_{,\mathbf{M}}$ which is finally symmetrized. For the derivative with respect to a symmetric part (1.146) of a tensor argument this procedure can be written by

$$f(\mathbf{A})_{,\mathrm{sym}\mathbf{A}} = \mathrm{sym}\left[f(\mathbf{A})_{,\mathbf{A}}\right], \quad \mathbf{A} \in \mathbf{L}\mathrm{in}^n. \tag{6.48}$$

The problem with the non-uniqueness appears likewise by using the component representation (6.45) for the gradient of symmetric tensor functions. Indeed, in this case $\mathrm{M}^{ij} = \mathrm{M}^{ji}$ $(i \neq j = 1, 2, \ldots, n)$, so that only $n(n+1)/2$ among all n^2 components of the tensor argument $\mathbf{M} \in \mathbf{S}\mathrm{ym}^n$ are independent. Thus, according to (1.149)

$$\mathbf{M} = \sum_{i=1}^{n} \mathrm{M}^{ii} \boldsymbol{g}_i \otimes \boldsymbol{g}_i + \sum_{\substack{i,j=1 \\ j<i}}^{n} \mathrm{M}^{ij} \left(\boldsymbol{g}_i \otimes \boldsymbol{g}_j + \boldsymbol{g}_j \otimes \boldsymbol{g}_i\right), \quad \mathbf{M} \in \mathbf{S}\mathrm{ym}^n. \tag{6.49}$$

Hence, instead of (6.45) we obtain

$$\begin{aligned} f(\mathbf{M})_{,\mathbf{M}} &= \frac{1}{2} \sum_{\substack{i,j=1 \\ j\leq i}}^{n} \frac{\partial f}{\partial \mathrm{M}^{ij}} \left(\boldsymbol{g}^i \otimes \boldsymbol{g}^j + \boldsymbol{g}^j \otimes \boldsymbol{g}^i\right) \\ &= \frac{1}{2} \sum_{\substack{i,j=1 \\ j\leq i}}^{n} \frac{\partial f}{\partial \mathrm{M}_{ij}} \left(\boldsymbol{g}_i \otimes \boldsymbol{g}_j + \boldsymbol{g}_j \otimes \boldsymbol{g}_i\right), \quad \mathbf{M} \in \mathbf{S}\mathrm{ym}^n. \end{aligned} \tag{6.50}$$

It is seen that the derivative is taken here only with respect to the independent components of the symmetric tensor argument; the resulting tensor is then symmetrized.

Example 1. Derivative of the quadratic norm $\|\mathbf{A}\| = \sqrt{\mathbf{A} : \mathbf{A}}$:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \left[(\mathbf{A} + t\mathbf{X}) : (\mathbf{A} + t\mathbf{X})\right]^{1/2} \Big|_{t=0} \\ &\qquad = \frac{\mathrm{d}}{\mathrm{d}t} \left[\mathbf{A} : \mathbf{A} + 2t\mathbf{A} : \mathbf{X} + t^2 \mathbf{X} : \mathbf{X}\right]^{1/2} \Big|_{t=0} \\ &\qquad = \frac{2\mathbf{A} : \mathbf{X} + 2t\mathbf{X} : \mathbf{X}}{2\left[\mathbf{A} : \mathbf{A} + 2t\mathbf{A} : \mathbf{X} + t^2 \mathbf{X} : \mathbf{X}\right]^{1/2}} \Bigg|_{t=0} = \frac{\mathbf{A}}{\|\mathbf{A}\|} : \mathbf{X}. \end{aligned}$$

Thus,

$$\|\mathbf{A}\|,_{\mathbf{A}} = \frac{\mathbf{A}}{\|\mathbf{A}\|}. \tag{6.51}$$

The same result can also be obtained using (6.45). Indeed, let $\mathbf{A} = \mathrm{A}_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j$. Then,

$$\|\mathbf{A}\| = \sqrt{\mathbf{A} : \mathbf{A}} = \sqrt{\left(\mathrm{A}_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j\right) : \left(\mathrm{A}_{kl}\boldsymbol{g}^k \otimes \boldsymbol{g}^l\right)} = \sqrt{\mathrm{A}_{ij}\mathrm{A}_{kl}g^{ik}g^{jl}}.$$

Utilizing the identity

$$\frac{\partial \mathrm{A}_{ij}}{\partial \mathrm{A}_{pq}} = \delta_i^p \delta_j^q, \quad i, j, p, q = 1, 2, \ldots, n$$

we further write

$$\begin{aligned}
\|\mathbf{A}\|,_{\mathbf{A}} &= \frac{\partial \sqrt{\mathrm{A}_{ij}\mathrm{A}_{kl}g^{ik}g^{jl}}}{\partial \mathrm{A}_{pq}} \boldsymbol{g}_p \otimes \boldsymbol{g}_q \\
&= \frac{1}{2\|\mathbf{A}\|} \left(\mathrm{A}_{kl}g^{ik}g^{jl}\boldsymbol{g}_i \otimes \boldsymbol{g}_j + \mathrm{A}_{ij}g^{ik}g^{jl}\boldsymbol{g}_k \otimes \boldsymbol{g}_l\right) \\
&= \frac{1}{2\|\mathbf{A}\|} 2\mathrm{A}_{kl}g^{ik}g^{jl}\boldsymbol{g}_i \otimes \boldsymbol{g}_j = \frac{1}{\|\mathbf{A}\|}\mathrm{A}_{kl}\boldsymbol{g}^k \otimes \boldsymbol{g}^l = \frac{\mathbf{A}}{\|\mathbf{A}\|}.
\end{aligned}$$

Example 2. Derivatives of the principal traces $\mathrm{tr}\mathbf{A}^k$ $(k = 1, 2, \ldots)$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{tr}\left(\mathbf{A} + t\mathbf{X}\right)^k\right]\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\mathbf{A} + t\mathbf{X}\right)^k : \mathbf{I}\right]\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\mathbf{A} + t\mathbf{X}\right)^k\right]\Big|_{t=0} : \mathbf{I} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[\underbrace{\left(\mathbf{A} + t\mathbf{X}\right)\left(\mathbf{A} + t\mathbf{X}\right)\ldots\left(\mathbf{A} + t\mathbf{X}\right)}_{k \text{ times}}\right]\Bigg|_{t=0} : \mathbf{I} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{A}^k + t\sum_{i=0}^{k-1}\mathbf{A}^i\mathbf{X}\mathbf{A}^{k-1-i} + t^2\ldots\right]\Bigg|_{t=0} : \mathbf{I} \\
&= \sum_{i=0}^{k-1}\mathbf{A}^i\mathbf{X}\mathbf{A}^{k-1-i} : \mathbf{I} = k\left(\mathbf{A}^{k-1}\right)^{\mathrm{T}} : \mathbf{X}.
\end{aligned}$$

Thus,

$$\left(\mathrm{tr}\mathbf{A}^k\right),_{\mathbf{A}} = k\left(\mathbf{A}^{k-1}\right)^{\mathrm{T}}. \tag{6.52}$$

In the special case $k = 1$ we obtain

$$\left(\mathrm{tr}\mathbf{A}\right),_{\mathbf{A}} = \mathbf{I}. \tag{6.53}$$

Example 3. Derivatives of $\mathrm{tr}\left(\mathbf{A}^k\mathbf{L}\right)$ $(k = 1, 2, \ldots)$ with respect to $\mathbf{A}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(\mathbf{A}+t\mathbf{X})^k : \mathbf{L}^{\mathrm{T}}\right]\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left[(\mathbf{A}+t\mathbf{X})^k\right]\bigg|_{t=0} : \mathbf{L}^{\mathrm{T}}$$
$$= \sum_{i=0}^{k-1} \mathbf{A}^i\mathbf{X}\mathbf{A}^{k-1-i} : \mathbf{L}^{\mathrm{T}} = \sum_{i=0}^{k-1} \left(\mathbf{A}^{\mathrm{T}}\right)^i \mathbf{L}^{\mathrm{T}} \left(\mathbf{A}^{\mathrm{T}}\right)^{k-1-i} : \mathbf{X}.$$

Hence,

$$\mathrm{tr}\left(\mathbf{A}^k\mathbf{L}\right),_{\mathbf{A}} = \sum_{i=0}^{k-1}\left(\mathbf{A}^i\mathbf{L}\mathbf{A}^{k-1-i}\right)^{\mathrm{T}}. \tag{6.54}$$

In the special case $k = 1$ we have

$$\mathrm{tr}\left(\mathbf{A}\mathbf{L}\right),_{\mathbf{A}} = \mathbf{L}^{\mathrm{T}}. \tag{6.55}$$

It is seen that the derivative of $\mathrm{tr}\left(\mathbf{A}^k\mathbf{L}\right)$ is not in general symmetric even if the tensor argument $\mathbf{A}$ is. Applying (6.48) we can write in this case

$$\mathrm{tr}\left(\mathbf{M}^k\mathbf{L}\right),_{\mathbf{M}} = \mathrm{sym}\left[\sum_{i=0}^{k-1}\left(\mathbf{M}^i\mathbf{L}\mathbf{M}^{k-1-i}\right)^{\mathrm{T}}\right] = \sum_{i=0}^{k-1}\mathbf{M}^i\left(\mathrm{sym}\mathbf{L}\right)\mathbf{M}^{k-1-i}, \tag{6.56}$$

where $\mathbf{M} \in \mathbf{S}\mathrm{ym}^n$.

Example 4. Derivatives of the principal invariants $\mathrm{I}_{\mathbf{A}}^{(k)}$ $(k = 1, 2, \ldots, n)$ of a second-order tensor $\mathbf{A} \in \mathbf{L}\mathrm{in}^n$. By virtue of the representations (4.26) and using (6.52) we obtain

$$\mathrm{I}_{\mathbf{A}}^{(1)},_{\mathbf{A}} = \left(\mathrm{tr}\mathbf{A}\right),_{\mathbf{A}} = \mathbf{I},$$

$$\mathrm{I}_{\mathbf{A}}^{(2)},_{\mathbf{A}} = \frac{1}{2}\left(\mathrm{I}_{\mathbf{A}}^{(1)}\mathrm{tr}\mathbf{A} - \mathrm{tr}\mathbf{A}^2\right),_{\mathbf{A}} = \mathrm{I}_{\mathbf{A}}^{(1)}\mathbf{I} - \mathbf{A}^{\mathrm{T}},$$

$$\begin{aligned}\mathrm{I}_{\mathbf{A}}^{(3)},_{\mathbf{A}} &= \frac{1}{3}\left(\mathrm{I}_{\mathbf{A}}^{(2)}\mathrm{tr}\mathbf{A} - \mathrm{I}_{\mathbf{A}}^{(1)}\mathrm{tr}\mathbf{A}^2 + \mathrm{tr}\mathbf{A}^3\right),_{\mathbf{A}}\\ &= \frac{1}{3}\left[\mathrm{tr}\mathbf{A}\left(\mathrm{I}_{\mathbf{A}}^{(1)}\mathbf{I} - \mathbf{A}^{\mathrm{T}}\right) + \mathrm{I}_{\mathbf{A}}^{(2)}\mathbf{I} - \left(\mathrm{tr}\mathbf{A}^2\right)\mathbf{I} - 2\mathrm{I}_{\mathbf{A}}^{(1)}\mathbf{A}^{\mathrm{T}} + 3\left(\mathbf{A}^{\mathrm{T}}\right)^2\right]\\ &= \left[\mathbf{A}^2 - \mathrm{I}_{\mathbf{A}}^{(1)}\mathbf{A} + \mathrm{I}_{\mathbf{A}}^{(2)}\mathbf{I}\right]^{\mathrm{T}}, \quad \ldots\end{aligned} \tag{6.57}$$

Herein, one can observe the following regularity

$$\mathrm{I}_{\mathbf{A}}^{(k)},_{\mathbf{A}} = \sum_{i=0}^{k-1}(-1)^i\,\mathrm{I}_{\mathbf{A}}^{(k-1-i)}\left(\mathbf{A}^{\mathrm{T}}\right)^i = -\mathrm{I}_{\mathbf{A}}^{(k-1)},_{\mathbf{A}}\mathbf{A}^{\mathrm{T}} + \mathrm{I}_{\mathbf{A}}^{(k-1)}\mathbf{I}, \tag{6.58}$$

where we again set $\mathrm{I}_{\mathbf{A}}^{(0)} = 1$. The above identity can be proved by mathematical induction (see also [7]). Indeed, according to (4.26) and (6.52)

$$\mathrm{I}_{\mathbf{A}}^{(k)},_{\mathbf{A}} = \frac{1}{k}\left[\sum_{i=1}^{k}(-1)^{i-1}\,\mathrm{I}_{\mathbf{A}}^{(k-i)}\mathrm{tr}\mathbf{A}^i\right],_{\mathbf{A}}$$

$$= \frac{1}{k}\sum_{i=1}^{k}(-1)^{i-1}\,i\mathrm{I}_{\mathbf{A}}^{(k-i)}\left(\mathbf{A}^{\mathrm{T}}\right)^{i-1} + \frac{1}{k}\sum_{i=1}^{k-1}(-1)^{i-1}\,\mathrm{I}_{\mathbf{A}}^{(k-i)},_{\mathbf{A}}\,\mathrm{tr}\mathbf{A}^i. \quad (6.59)$$

Now, let

$$\mathbf{Y}_{k+1} = \sum_{i=0}^{k}(-1)^i\,\mathrm{I}_{\mathbf{A}}^{(k-i)}\left(\mathbf{A}^{\mathrm{T}}\right)^i = -\mathrm{I}_{\mathbf{A}}^{(k)},_{\mathbf{A}}\,\mathbf{A}^{\mathrm{T}} + \mathrm{I}_{\mathbf{A}}^{(k)}\mathbf{I}. \quad (6.60)$$

Inserting (4.26) and (6.59) into the latter expression (6.60) delivers

$$\mathbf{Y}_{k+1} = -\,\frac{1}{k}\sum_{i=1}^{k}(-1)^{i-1}\,i\mathrm{I}_{\mathbf{A}}^{(k-i)}\left(\mathbf{A}^{\mathrm{T}}\right)^i - \left[\sum_{i=1}^{k-1}(-1)^{i-1}\,\mathrm{I}_{\mathbf{A}}^{(k-i)},_{\mathbf{A}}\,\mathrm{tr}\mathbf{A}^i\right]\frac{\mathbf{A}^{\mathrm{T}}}{k}$$
$$+\,\frac{\mathbf{I}}{k}\left[\sum_{i=1}^{k}(-1)^{i-1}\,\mathrm{I}_{\mathbf{A}}^{(k-i)}\mathrm{tr}\mathbf{A}^i\right].$$

Adding $\mathbf{Y}_{k+1}/k$ to both sides of this equality and using for $\mathbf{Y}_{k+1}$ the first expression in (6.60) we further obtain

$$\frac{k+1}{k}\mathbf{Y}_{k+1} = \frac{1}{k}\sum_{i=1}^{k}(-1)^i\,i\mathrm{I}_{\mathbf{A}}^{(k-i)}\left(\mathbf{A}^{\mathrm{T}}\right)^i + \frac{1}{k}\sum_{i=0}^{k}(-1)^i\,\mathrm{I}_{\mathbf{A}}^{(k-i)}\left(\mathbf{A}^{\mathrm{T}}\right)^i$$
$$+\,\frac{1}{k}\left[\sum_{i=1}^{k}(-1)^{i-1}\left(-\mathrm{I}_{\mathbf{A}}^{(k-i)},_{\mathbf{A}}\,\mathbf{A}^{\mathrm{T}} + \mathrm{I}_{\mathbf{A}}^{(k-i)}\mathbf{I}\right)\mathrm{tr}\mathbf{A}^i\right].$$

Now, let us assume that representation (6.58) holds at least until the number k. Then, taking (6.59) again into account we can write

$$\frac{k+1}{k}\mathbf{Y}_{k+1} = \frac{1}{k}\sum_{i=0}^{k}(-1)^i\,(i+1)\,\mathrm{I}_{\mathbf{A}}^{(k-i)}\left(\mathbf{A}^{\mathrm{T}}\right)^i$$
$$+\,\frac{1}{k}\left[\sum_{i=1}^{k}(-1)^{i-1}\,\mathrm{I}_{\mathbf{A}}^{(k+1-i)},_{\mathbf{A}}\,\mathrm{tr}\mathbf{A}^i\right] = \frac{k+1}{k}\mathrm{I}_{\mathbf{A}}^{(k+1)},_{\mathbf{A}}\,.$$

Hence,

$$\mathbf{Y}_{k+1} = \mathrm{I}_{\mathbf{A}}^{(k+1)},_{\mathbf{A}}\,,$$

which immediately implies that (6.58) holds for $k+1$ as well. Thereby, representation (6.58) is proved.

For invertible tensors one can get a simpler representation for the derivative of the last invariant $\mathrm{I}_{\mathbf{A}}^{(n)}$. This representation results from the Cayley-Hamilton theorem (4.91) as follows

$$\mathrm{I}_{\mathbf{A}}^{(n)},_{\mathbf{A}} \mathbf{A}^{\mathrm{T}} = \left[\sum_{i=0}^{n-1} (-1)^i \mathrm{I}_{\mathbf{A}}^{(n-1-i)} \left(\mathbf{A}^{\mathrm{T}}\right)^i \right] \mathbf{A}^{\mathrm{T}} = \sum_{i=1}^{n} (-1)^{i-1} \mathrm{I}_{\mathbf{A}}^{(n-i)} \left(\mathbf{A}^{\mathrm{T}}\right)^i$$

$$= \sum_{i=0}^{n} (-1)^{i-1} \mathrm{I}_{\mathbf{A}}^{(n-i)} \left(\mathbf{A}^{\mathrm{T}}\right)^i + \mathrm{I}_{\mathbf{A}}^{(n)} \mathbf{I} = \mathrm{I}_{\mathbf{A}}^{(n)} \mathbf{I}.$$

Thus,

$$\mathrm{I}_{\mathbf{A}}^{(n)},_{\mathbf{A}} = \mathrm{I}_{\mathbf{A}}^{(n)} \mathbf{A}^{-\mathrm{T}}, \quad \mathbf{A} \in \mathbf{I}\mathrm{nv}^n. \tag{6.61}$$

Example 5. Derivatives of the eigenvalues λ_i. First, we show that simple eigenvalues of a second-order tensor $\mathbf{A}$ are differentiable. To this end, we consider the directional derivative (6.44) of an eigenvalue λ:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} \lambda \left(\mathbf{A} + t\mathbf{X}\right)\right|_{t=0}. \tag{6.62}$$

Herein, $\lambda(t)$ represents an implicit function defined through the characteristic equation

$$\det \left(\mathbf{A} + t\mathbf{X} - \lambda \mathbf{I}\right) = p\left(\lambda, t\right) = 0. \tag{6.63}$$

This equation can be written out in the polynomial form (4.18) with respect to powers of λ. The coefficients of this polynomial are principal invariants of the tensor $\mathbf{A} + t\mathbf{X}$. According to the results of the previous example these invariants are differentiable with respect to $\mathbf{A} + t\mathbf{X}$ and therefore also with respect to t. For this reason, the function $p(\lambda, t)$ is differentiable both with respect to λ and t. For a simple eigenvalue $\lambda_0 = \lambda(0)$ we can further write (see also [26])

$$p\left(\lambda_0, 0\right) = 0, \quad \left.\frac{\partial p\left(\lambda, 0\right)}{\partial \lambda}\right|_{\lambda=\lambda_0} \neq 0. \tag{6.64}$$

According to the implicit function theorem (see, e.g., [5]), the above condition ensures the differentiability of the function $\lambda(t)$ at $t = 0$. Thus, the directional derivative (6.62) exists and is continuous at $\mathbf{A}$.

In order to represent the derivative $\lambda_i,_{\mathbf{A}}$ we first consider the spectral representation (4.43) of the tensor $\mathbf{A}$ with pairwise distinct eigenvalues

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{P}_i, \tag{6.65}$$

where $\mathbf{P}_i$ $(i = 1, 2, \ldots, n)$ denote the eigenprojections. They can uniquely be determined from the equation system

$$\mathbf{A}^k = \sum_{i=1}^{n} \lambda_i^k \mathbf{P}_i, \quad k = 0, 1, \ldots, n-1 \tag{6.66}$$

resulting from (4.47). Applying the Vieta theorem to the tensor $\mathbf{A}^l$ $(l = 1, 2, \ldots, n)$ we further obtain relation (4.25) written as

$$\mathrm{tr}\mathbf{A}^l = \sum_{i=1}^{n} \lambda_i^l, \quad l = 1, 2, \ldots, n. \tag{6.67}$$

The derivative of (6.67) with respect to $\mathbf{A}$ further yields by virtue of (6.52)

$$l \left(\mathbf{A}^{\mathrm{T}}\right)^{l-1} = l \sum_{i=1}^{n} \lambda_i^{l-1} \lambda_i,_{\mathbf{A}}, \quad l = 1, 2, \ldots, n$$

and consequently

$$\mathbf{A}^k = \sum_{i=1}^{n} \lambda_i^k \left(\lambda_i,_{\mathbf{A}}\right)^{\mathrm{T}}, \quad k = 0, 1, \ldots, n-1. \tag{6.68}$$

Comparing the linear equation systems (6.66) and (6.68) we notice that

$$\lambda_i,_{\mathbf{A}} = \mathbf{P}_i^{\mathrm{T}}. \tag{6.69}$$

Finally, the Sylvester formula (4.54) results in the expression

$$\lambda_i,_{\mathbf{A}} = \delta_{1n}\mathbf{I} + \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{\mathbf{A}^{\mathrm{T}} - \lambda_j \mathbf{I}}{\lambda_i - \lambda_j}. \tag{6.70}$$

It is seen that the solution (6.70) holds even if the remainder eigenvalues λ_j $(j = 1, 2, \ldots, i-1, i+1, \ldots, n)$ of the tensor $\mathbf{A}$ are not simple. In this case (6.70) transforms to

$$\lambda_i,_{\mathbf{A}} = \delta_{1n}\mathbf{I} + \prod_{\substack{j=1\\ j\neq i}}^{s} \frac{\mathbf{A}^{\mathrm{T}} - \lambda_j \mathbf{I}}{\lambda_i - \lambda_j}, \tag{6.71}$$

where s denotes the number of pairwise distinct eigenvalues λ_i $(i = 1, 2, \ldots, s)$.

6.4 Tensor-Valued Isotropic and Anisotropic Tensor Functions

A tensor-valued function $g(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_l) \in \mathbf{Lin}^n$ of a tensor system $\mathbf{A}_k \in \mathbf{Lin}^n$ $(k = 1, 2, \ldots, l)$ is called anisotropic if

$$g\left(\mathbf{QA}_1\mathbf{Q}^{\mathrm{T}}, \mathbf{QA}_2\mathbf{Q}^{\mathrm{T}}, \ldots, \mathbf{QA}_l\mathbf{Q}^{\mathrm{T}}\right)$$
$$= \mathbf{Q}g\left(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_l\right)\mathbf{Q}^{\mathrm{T}}, \quad \forall \mathbf{Q} \in \mathbf{S}\mathrm{orth}^n \subset \mathbf{O}\mathrm{rth}^n. \tag{6.72}$$

For isotropic tensor-valued tensor functions the above identity holds for all orthogonal transformations so that $\mathbf{S}\mathrm{orth}^n = \mathbf{O}\mathrm{rth}^n$.

As a starting point for the discussion of tensor-valued tensor functions we again consider isotropic functions of one argument. In this case,

$$g\left(\mathbf{QAQ}^{\mathrm{T}}\right) = \mathbf{Q}g\left(\mathbf{A}\right)\mathbf{Q}^{\mathrm{T}}, \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n. \tag{6.73}$$

For example, one can easily show that the polynomial function (1.108) and the exponential function (1.109) introduced in Chap. 1 are isotropic. Indeed, for a tensor polynomial $g\left(\mathbf{A}\right) = \sum_{k=0}^{m} a_k \mathbf{A}^k$ we have (see also Exercise 1.32)

$$g\left(\mathbf{QAQ}^{\mathrm{T}}\right) = \sum_{k=0}^{m} a_k \left(\mathbf{QAQ}^{\mathrm{T}}\right)^k = \sum_{k=0}^{m} a_k \left(\underbrace{\mathbf{QAQ}^{\mathrm{T}}\mathbf{QAQ}^{\mathrm{T}}\ldots\mathbf{QAQ}^{\mathrm{T}}}_{k \text{ times}}\right)$$
$$= \sum_{k=0}^{m} a_k \left(\mathbf{QA}^k\mathbf{Q}^{\mathrm{T}}\right) = \mathbf{Q}\left(\sum_{k=0}^{m} a_k \mathbf{A}^k\right)\mathbf{Q}^{\mathrm{T}}$$
$$= \mathbf{Q}g\left(\mathbf{A}\right)\mathbf{Q}^{\mathrm{T}}, \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n. \tag{6.74}$$

Of special interest are isotropic functions of a symmetric tensor. First, we prove that the tensors $g\left(\mathbf{M}\right)$ and $\mathbf{M} \in \mathbf{S}\mathrm{ym}^n$ are coaxial i.e. have the eigenvectors in common. To this end, we represent $\mathbf{M}$ in the spectral form (4.60) by

$$\mathbf{M} = \sum_{i=1}^{n} \lambda_i \boldsymbol{b}_i \otimes \boldsymbol{b}_i, \tag{6.75}$$

where $\boldsymbol{b}_i \cdot \boldsymbol{b}_j = \delta_{ij}$ $(i, j = 1, 2, \ldots, n)$. Further, we choose an arbitrary eigenvector, say $\boldsymbol{b}_k$, and show that it simultaneously represents an eigenvector of $g\left(\mathbf{M}\right)$. Indeed, let

$$\mathbf{Q} = 2\boldsymbol{b}_k \otimes \boldsymbol{b}_k - \mathbf{I} = \boldsymbol{b}_k \otimes \boldsymbol{b}_k + \sum_{\substack{i=1\\ i\neq k}}^{n} (-1)\, \boldsymbol{b}_i \otimes \boldsymbol{b}_i \tag{6.76}$$

bearing in mind that $\mathbf{I} = \sum_{i=1}^{n} \boldsymbol{b}_i \otimes \boldsymbol{b}_i$ in accordance with (1.87). The tensor $\mathbf{Q}$ (6.76) is orthogonal since

$$\mathbf{QQ}^{\mathrm{T}} = \left(2\boldsymbol{b}_k \otimes \boldsymbol{b}_k - \mathbf{I}\right)\left(2\boldsymbol{b}_k \otimes \boldsymbol{b}_k - \mathbf{I}\right) = 4\boldsymbol{b}_k \otimes \boldsymbol{b}_k - 2\boldsymbol{b}_k \otimes \boldsymbol{b}_k - 2\boldsymbol{b}_k \otimes \boldsymbol{b}_k + \mathbf{I} = \mathbf{I}$$

and symmetric as well. One of its eigenvalues is equal to 1 while all the other ones are -1. Thus, we can write

$$\mathbf{QM} = (2\boldsymbol{b}_k \otimes \boldsymbol{b}_k - \mathbf{I})\,\mathbf{M} = 2\lambda_k \boldsymbol{b}_k \otimes \boldsymbol{b}_k - \mathbf{M} = \mathbf{M}\,(2\boldsymbol{b}_k \otimes \boldsymbol{b}_k - \mathbf{I}) = \mathbf{MQ}$$

and consequently

$$\mathbf{QMQ}^{\mathrm{T}} = \mathbf{M}. \tag{6.77}$$

Since the function $g(\mathbf{M})$ is isotropic

$$g(\mathbf{M}) = g\left(\mathbf{QMQ}^{\mathrm{T}}\right) = \mathbf{Q}g(\mathbf{M})\,\mathbf{Q}^{\mathrm{T}}$$

and therefore

$$\mathbf{Q}g(\mathbf{M}) = g(\mathbf{M})\,\mathbf{Q}. \tag{6.78}$$

Mapping the vector $\boldsymbol{b}_k$ by both sides of this identity yields in view of (6.76)

$$\mathbf{Q}g(\mathbf{M})\,\boldsymbol{b}_k = g(\mathbf{M})\,\boldsymbol{b}_k. \tag{6.79}$$

It is seen that the vector $g(\mathbf{M})\,\boldsymbol{b}_k$ is an eigenvector of $\mathbf{Q}$ (6.76) associated with the eigenvalue 1. Since it is the simple eigenvalue

$$g(\mathbf{M})\,\boldsymbol{b}_k = \gamma_k \boldsymbol{b}_k, \tag{6.80}$$

where γ_k is some real number. Hence, $\boldsymbol{b}_k$ represents the right eigenvector of $g(\mathbf{M})$. Forming the left mapping of $\boldsymbol{b}_k$ by (6.78) one can similarly show that $\boldsymbol{b}_k$ is also the left eigenvector of $g(\mathbf{M})$, which implies the symmetry of the tensor $g(\mathbf{M})$.

Now, we are in a position to prove the following representation theorem [35], [45].

Theorem 6.2. *A tensor-valued tensor function $g(\mathbf{M})$, $\mathbf{M} \in \mathbf{Sym}^n$ is isotropic if and only if it allows the following representation*

$$g(\mathbf{M}) = \varphi_0 \mathbf{I} + \varphi_1 \mathbf{M} + \varphi_2 \mathbf{M}^2 + \ldots + \varphi_{n-1}\mathbf{M}^{n-1} = \sum_{i=0}^{n-1} \varphi_i \mathbf{M}^i, \tag{6.81}$$

where φ_i are isotropic invariants (isotropic scalar functions) of $\mathbf{M}$ and can therefore be expressed as functions of its principal invariants by

$$\varphi_i = \widehat{\varphi}_i\left(\mathrm{I}_{\mathbf{M}}^{(1)}, \mathrm{I}_{\mathbf{M}}^{(2)}, \ldots, \mathrm{I}_{\mathbf{M}}^{(n)}\right), \quad i = 0, 1, \ldots, n-1. \tag{6.82}$$

Proof. We have already proved that the tensors $g(\mathbf{M})$ and $\mathbf{M}$ have eigenvectors in common. Thus, according to (6.75)

$$g(\mathbf{M}) = \sum_{i=1}^{n} \gamma_i \boldsymbol{b}_i \otimes \boldsymbol{b}_i, \tag{6.83}$$

where $\gamma_i = \gamma_i(\mathbf{M})$. Hence (see Exercise 6.1(e)),

$$g\left(\mathbf{QMQ}^{\mathrm{T}}\right) = \sum_{i=1}^{n} \gamma_i \left(\mathbf{QMQ}^{\mathrm{T}}\right) \mathbf{Q} \left(\boldsymbol{b}_i \otimes \boldsymbol{b}_i\right) \mathbf{Q}^{\mathrm{T}}. \tag{6.84}$$

Since the function $g(\mathbf{M})$ is isotropic we have

$$g\left(\mathbf{QMQ}^{\mathrm{T}}\right) = \mathbf{Q} g(\mathbf{M}) \mathbf{Q}^{\mathrm{T}}$$
$$= \sum_{i=1}^{n} \gamma_i(\mathbf{M}) \mathbf{Q} \left(\boldsymbol{b}_i \otimes \boldsymbol{b}_i\right) \mathbf{Q}^{\mathrm{T}}, \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n. \tag{6.85}$$

Comparing (6.84) with (6.85) we conclude that

$$\gamma_i \left(\mathbf{QMQ}^{\mathrm{T}}\right) = \gamma_i(\mathbf{M}), \quad i = 1, \ldots, n, \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n. \tag{6.86}$$

Thus, the eigenvalues of the tensor $g(\mathbf{M})$ represent isotropic (scalar-valued) functions of $\mathbf{M}$. Collecting repeated eigenvalues of $g(\mathbf{M})$ we can further rewrite (6.83) in terms of the eigenprojections $\mathbf{P}_i$ $(i = 1, 2, \ldots, s)$ by

$$g(\mathbf{M}) = \sum_{i=1}^{s} \gamma_i \mathbf{P}_i, \tag{6.87}$$

where s $(1 \leq s \leq n)$ denotes the number of pairwise distinct eigenvalues of $g(\mathbf{M})$. Using the representation of the eigenprojections (4.55) based on the Sylvester formula (4.54) we can write

$$\mathbf{P}_i = \sum_{r=0}^{s-1} \alpha_i^{(r)} \left(\lambda_1, \lambda_2, \ldots, \lambda_s\right) \mathbf{M}^r, \quad i = 1, 2, \ldots, s. \tag{6.88}$$

Inserting this result into (6.87) yields the representation (sufficiency):

$$g(\mathbf{M}) = \sum_{i=0}^{s-1} \varphi_i \mathbf{M}^i, \tag{6.89}$$

where the functions φ_i $(i = 0, 1, 2, \ldots, s-1)$ are given according to (6.8) and (6.86) by (6.82). The necessity is evident. Indeed, the function (6.81) is isotropic since in view of (6.74)

$$g\left(\mathbf{QMQ}^{\mathrm{T}}\right) = \sum_{i=0}^{n-1} \varphi_i \left(\mathbf{QMQ}^{\mathrm{T}}\right) \mathbf{QM}^i\mathbf{Q}^{\mathrm{T}}$$
$$= \mathbf{Q} \left[\sum_{i=0}^{n-1} \varphi_i(\mathbf{M}) \mathbf{M}^i\right] \mathbf{Q}^{\mathrm{T}} = \mathbf{Q} g(\mathbf{M}) \mathbf{Q}^{\mathrm{T}}, \ \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n. \tag{6.90}$$

Example. Constitutive relations for isotropic materials. For isotropic materials the second Piola-Kirchhoff stress tensor $\mathbf{S}$ represents an isotropic function of the right Cauchy-Green tensor $\mathbf{C}$ so that

$$\mathbf{S}\left(\mathbf{QCQ}^{\mathrm{T}}\right) = \mathbf{QS}\left(\mathbf{C}\right)\mathbf{Q}^{\mathrm{T}}, \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^3. \tag{6.91}$$

Thus, according to the representation theorem

$$\mathbf{S}\left(\mathbf{C}\right) = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{C} + \alpha_2 \mathbf{C}^2, \tag{6.92}$$

where $\alpha_i = \alpha_i\left(\mathbf{C}\right)$ $(i = 0, 1, 2)$ are some scalar-valued isotropic functions of $\mathbf{C}$. The same result can be obtained for isotropic hyperelastic materials by considering the representation of the strain energy function (6.10) in the relation (see, e.g., [29])

$$\mathbf{S} = 2\frac{\partial \psi}{\partial \mathbf{C}}. \tag{6.93}$$

Indeed, using the chain rule of differentiation and keeping in mind that the tensor $\mathbf{C}$ is symmetric we obtain by means of (6.52)

$$\mathbf{S} = 2\sum_{k=1}^{3} \frac{\partial \tilde{\psi}}{\partial \mathrm{tr}\mathbf{C}^k} \frac{\partial \mathrm{tr}\mathbf{C}^k}{\partial \mathbf{C}} = 2\sum_{k=1}^{3} k \frac{\partial \tilde{\psi}}{\partial \mathrm{tr}\mathbf{C}^k} \mathbf{C}^{k-1}, \tag{6.94}$$

so that $\alpha_i\left(\mathbf{C}\right) = 2\left(i+1\right)\partial\tilde{\psi}/\partial \mathrm{tr}\mathbf{C}^{i+1}$ $(i = 0, 1, 2)$.

Let us further consider a linearly elastic material characterized by a linear stress-strain response. In this case, the relation (6.92) reduces to

$$\mathbf{S}\left(\mathbf{C}\right) = \varphi\left(\mathbf{C}\right)\mathbf{I} + c\mathbf{C}, \tag{6.95}$$

where c is a material constant and $\varphi\left(\mathbf{C}\right)$ represents an isotropic scalar-valued function linear in $\mathbf{C}$. In view of (6.15) this function can be expressed by

$$\varphi\left(\mathbf{C}\right) = a + b\mathrm{tr}\mathbf{C}, \tag{6.96}$$

where a and b are again material constants. Assuming that the reference configuration, in which $\mathbf{C} = \mathbf{I}$, is stress free, yields $a + 3b + c = 0$ and consequently

$$\mathbf{S}\left(\mathbf{C}\right) = \left(-c - 3b + b\mathrm{tr}\mathbf{C}\right)\mathbf{I} + c\mathbf{C} = b\left(\mathrm{tr}\mathbf{C} - 3\right)\mathbf{I} + c\left(\mathbf{C} - \mathbf{I}\right).$$

Introducing further the so-called Green-Lagrange strain tensor defined by

$$\tilde{\mathbf{E}} = \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right) \tag{6.97}$$

we finally obtain

$$\mathbf{S}\left(\tilde{\mathbf{E}}\right) = 2b\left(\mathrm{tr}\tilde{\mathbf{E}}\right)\mathbf{I} + 2c\tilde{\mathbf{E}}. \tag{6.98}$$

The material described by the linear constitutive relation (6.98) is referred to as St.Venant-Kirchhoff material. The corresponding material constants $2b$ and $2c$ are called Lamé constants. The strain energy function resulting in the constitutive law (6.98) by (6.93) or equivalently by $\mathbf{S} = \partial\psi/\partial\tilde{\mathbf{E}}$ is of the form

$$\psi\left(\tilde{\mathbf{E}}\right) = b\mathrm{tr}^2\tilde{\mathbf{E}} + c\mathrm{tr}\tilde{\mathbf{E}}^2. \tag{6.99}$$

For isotropic functions of an arbitrary tensor system $\mathbf{A}_k \in \mathbf{Lin}^n$ $(k = 1, 2, \ldots, l)$ the representations are obtained only for the three-dimensional space. One again splits tensor arguments into symmetric $\mathbf{M}_i \in \mathbf{Sym}^3$ $(i = 1, 2, \ldots, m)$ and skew-symmetric tensors $\mathbf{W}_j \in \mathbf{Skew}^3$ $(j = 1, 2, \ldots, w)$ according to (6.13). Then, all isotropic tensor-valued functions of these tensors can be represented as linear combinations of the following terms (see [32], [40]), where the coefficients represent scalar-valued isotropic functions of the same tensor arguments.

Symmetric generators:

$$\begin{aligned}
&\mathbf{I},\\
&\mathbf{M}_i, \quad \mathbf{M}_i^2, \quad \mathbf{M}_i\mathbf{M}_j + \mathbf{M}_j\mathbf{M}_i, \quad \mathbf{M}_i^2\mathbf{M}_j + \mathbf{M}_j\mathbf{M}_i^2, \quad \mathbf{M}_i\mathbf{M}_j^2 + \mathbf{M}_j^2\mathbf{M}_i,\\
&\mathbf{W}_p^2, \quad \mathbf{W}_p\mathbf{W}_q + \mathbf{W}_q\mathbf{W}_p, \quad \mathbf{W}_p^2\mathbf{W}_q - \mathbf{W}_q\mathbf{W}_p^2, \quad \mathbf{W}_p\mathbf{W}_q^2 - \mathbf{W}_q^2\mathbf{W}_p,\\
&\mathbf{M}_i\mathbf{W}_p - \mathbf{W}_p\mathbf{M}_i, \quad \mathbf{W}_p\mathbf{M}_i\mathbf{W}_p, \quad \mathbf{M}_i^2\mathbf{W}_p - \mathbf{W}_p\mathbf{M}_i^2,\\
&\mathbf{W}_p\mathbf{M}_i\mathbf{W}_p^2 - \mathbf{W}_p^2\mathbf{M}_i\mathbf{W}_p.
\end{aligned} \tag{6.100}$$

Skew-symmetric generators:

$$\begin{aligned}
&\mathbf{W}_p, \quad \mathbf{W}_p\mathbf{W}_q - \mathbf{W}_q\mathbf{W}_p,\\
&\mathbf{M}_i\mathbf{M}_j - \mathbf{M}_j\mathbf{M}_i, \quad \mathbf{M}_i^2\mathbf{M}_j - \mathbf{M}_j\mathbf{M}_i^2, \quad \mathbf{M}_i\mathbf{M}_j^2 - \mathbf{M}_j^2\mathbf{M}_i,\\
&\mathbf{M}_i\mathbf{M}_j\mathbf{M}_i^2 - \mathbf{M}_i^2\mathbf{M}_j\mathbf{M}_i, \quad \mathbf{M}_j\mathbf{M}_i\mathbf{M}_j^2 - \mathbf{M}_j^2\mathbf{M}_i\mathbf{M}_j,\\
&\mathbf{M}_i\mathbf{M}_j\mathbf{M}_k + \mathbf{M}_j\mathbf{M}_k\mathbf{M}_i + \mathbf{M}_k\mathbf{M}_i\mathbf{M}_j - \mathbf{M}_j\mathbf{M}_i\mathbf{M}_k - \mathbf{M}_k\mathbf{M}_j\mathbf{M}_i - \mathbf{M}_i\mathbf{M}_k\mathbf{M}_j,\\
&\mathbf{M}_i\mathbf{W}_p + \mathbf{W}_p\mathbf{M}_i, \quad \mathbf{M}_i\mathbf{W}_p^2 - \mathbf{W}_p^2\mathbf{M}_i,\\
&i < j = 1, 2, \ldots, m, \quad p < q = 1, 2, \ldots, w.
\end{aligned} \tag{6.101}$$

For anisotropic tensor-valued tensor functions one utilizes the procedure applied for scalar-valued functions. It is based on the following theorem [49] (cf. Theorem 6.1).

Theorem 6.3. (Rychlewski's theorem) *A tensor-valued function* $g(\mathbf{A}_i)$ *is anisotropic with the symmetry group* $\mathbf{Sorth}^n = \mathfrak{g}$ *defined by* (6.30) *if and only if there exists an isotropic tensor-valued function* $\hat{g}(\mathbf{A}_i, \mathbf{L}_j)$ *such that*

$$g(\mathbf{A}_i) = \hat{g}(\mathbf{A}_i, \mathbf{L}_j). \tag{6.102}$$

Proof. Let us define a new tensor-valued function by

$$\hat{g}\left(\mathbf{A}_i,\mathbf{X}_j\right)=\mathbf{Q}'^{\mathrm{T}}g\left(\mathbf{Q}'\mathbf{A}_i\mathbf{Q}'^{\mathrm{T}}\right)\mathbf{Q}', \tag{6.103}$$

where the tensor $\mathbf{Q}'\in\mathbf{O}\mathrm{rth}^n$ results from the condition (6.36). The further proof is similar to Theorem 6.1 (Exercise 6.12).

Example. Constitutive relations for a transversely isotropic elastic material. For illustration of the above results we construct a general constitutive equation for an elastic transversely isotropic material. The transversely isotropic material symmetry is defined by one structural tensor $\mathbf{L}$ (6.22) according to (6.24). The second Piola-Kirchhoff stress tensor $\mathbf{S}$ is a transversely isotropic function of the right Cauchy-Green tensor $\mathbf{C}$. According to Rychlewski's theorem $\mathbf{S}$ can be represented as an isotropic tensor function of $\mathbf{C}$ and $\mathbf{L}$ by

$$\mathbf{S}=\mathbf{S}\left(\mathbf{C},\mathbf{L}\right), \tag{6.104}$$

such that

$$\mathbf{S}\left(\mathbf{QCQ}^{\mathrm{T}},\mathbf{QLQ}^{\mathrm{T}}\right)=\mathbf{QS}\left(\mathbf{C},\mathbf{L}\right)\mathbf{Q}^{\mathrm{T}},\quad\forall\mathbf{Q}\in\mathbf{O}\mathrm{rth}^3. \tag{6.105}$$

This ensures that the condition of the material symmetry is fulfilled a priori since

$$\mathbf{S}\left(\mathbf{QCQ}^{\mathrm{T}},\mathbf{L}\right)=\mathbf{S}\left(\mathbf{QCQ}^{\mathrm{T}},\mathbf{QLQ}^{\mathrm{T}}\right)=\mathbf{QS}\left(\mathbf{C},\mathbf{L}\right)\mathbf{Q}^{\mathrm{T}},\ \forall\mathbf{Q}\in\mathfrak{g}_t. \tag{6.106}$$

Keeping in mind that $\mathbf{S}$, $\mathbf{C}$ and $\mathbf{L}$ are symmetric tensors we can write by virtue of $(6.28)_1$ and (6.100)

$$\begin{aligned}\mathbf{S}\left(\mathbf{C},\mathbf{L}\right)&=\alpha_0\mathbf{I}+\alpha_1\mathbf{L}+\alpha_2\mathbf{C}\\&\quad+\alpha_3\mathbf{C}^2+\alpha_4\left(\mathbf{CL}+\mathbf{LC}\right)+\alpha_5\left(\mathbf{C}^2\mathbf{L}+\mathbf{LC}^2\right).\end{aligned} \tag{6.107}$$

The coefficients α_i $(i=0,1,\ldots,5)$ represent scalar-valued isotropic tensor functions of $\mathbf{C}$ and $\mathbf{L}$ so that similar to (6.29)

$$\alpha_i\left(\mathbf{C},\mathbf{L}\right)=\hat{\alpha}_i\left[\mathrm{tr}\mathbf{C},\mathrm{tr}\mathbf{C}^2,\mathrm{tr}\mathbf{C}^3,\mathrm{tr}\left(\mathbf{CL}\right),\mathrm{tr}\left(\mathbf{C}^2\mathbf{L}\right)\right]. \tag{6.108}$$

For comparison we derive the constitutive equations for a hyperelastic transversely isotropic material. To this end, we utilize the general representation for the transversely isotropic strain energy function (6.29). By the chain rule of differentiation and with the aid of (6.52) and (6.54) we obtain

$$\begin{aligned}\mathbf{S}&=2\frac{\partial\tilde{\psi}}{\partial\mathrm{tr}\mathbf{C}}\mathbf{I}+4\frac{\partial\tilde{\psi}}{\partial\mathrm{tr}\mathbf{C}^2}\mathbf{C}+6\frac{\partial\tilde{\psi}}{\partial\mathrm{tr}\mathbf{C}^3}\mathbf{C}^2\\&\quad+2\frac{\partial\tilde{\psi}}{\partial\mathrm{tr}\left(\mathbf{CL}\right)}\mathbf{L}+2\frac{\partial\tilde{\psi}}{\partial\mathrm{tr}\left(\mathbf{C}^2\mathbf{L}\right)}\left(\mathbf{CL}+\mathbf{LC}\right)\end{aligned} \tag{6.109}$$

and finally

$$\mathbf{S} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{L} + \alpha_2 \mathbf{C} + \alpha_3 \mathbf{C}^2 + \alpha_4 (\mathbf{CL} + \mathbf{LC}). \tag{6.110}$$

Comparing (6.107) and (6.110) we observe that the representation for the hyperelastic transversely isotropic material does not include the last term in (6.107) with $\mathbf{C}^2\mathbf{L}+\mathbf{LC}^2$. Thus, the constitutive equations containing this term correspond to an elastic but not hyperelastic transversely isotropic material. The latter material cannot be described by a strain energy function.

6.5 Derivatives of Tensor-Valued Tensor Functions

The derivative of a tensor-valued tensor function can be defined in a similar fashion to (6.44). A function $g(\mathbf{A}) : \mathbf{Lin}^n \mapsto \mathbf{Lin}^n$ is said to be differentiable in a neighborhood of $\mathbf{A}$ if there exists a fourth-order tensor $g(\mathbf{A})_{,\mathbf{A}} \in \mathfrak{L}\mathrm{in}^n$ (called the derivative), such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} g(\mathbf{A} + t\mathbf{X})\right|_{t=0} = g(\mathbf{A})_{,\mathbf{A}} : \mathbf{X}, \quad \forall \mathbf{X} \in \mathbf{Lin}^n. \tag{6.111}$$

The above definition implies that the directional derivative $\left.\frac{\mathrm{d}}{\mathrm{d}t} g(\mathbf{A} + t\mathbf{X})\right|_{t=0}$ exists and is continuous at $\mathbf{A}$.

Similarly to (6.45) we can obtain a direct relation for the fourth-order tensor $g(\mathbf{A})_{,\mathbf{A}}$. To this end, we represent the tensors $\mathbf{A}$, $\mathbf{X}$ and $\mathbf{G} = g(\mathbf{A})$ with respect to an arbitrary basis in $\mathbf{Lin}^n$, say $\boldsymbol{g}_i \otimes \boldsymbol{g}^j$ $(i, j = 1, 2, \ldots, n)$. Applying the chain rule of differentiation we can write

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t} g(\mathbf{A} + t\mathbf{X})\right|_{t=0} &= \left.\frac{\mathrm{d}}{\mathrm{d}t} \left\{ \mathrm{G}^i_{\cdot j} \left[\left(\mathrm{A}^k_{\cdot l} + t\mathrm{X}^k_{\cdot l} \right) \boldsymbol{g}_k \otimes \boldsymbol{g}^l \right] \boldsymbol{g}_i \otimes \boldsymbol{g}^j \right\}\right|_{t=0} \\
&= \frac{\partial \mathrm{G}^i_{\cdot j}}{\partial \mathrm{A}^k_{\cdot l}} \mathrm{X}^k_{\cdot l} \boldsymbol{g}_i \otimes \boldsymbol{g}^j .
\end{aligned} \tag{6.112}$$

In view of $(5.30)_1$ and (6.111) this results in the following representations

$$\begin{aligned}
g_{,\mathbf{A}} &= \frac{\partial \mathrm{G}^i_{\cdot j}}{\partial \mathrm{A}^k_{\cdot l}} \boldsymbol{g}_i \otimes \boldsymbol{g}^k \otimes \boldsymbol{g}_l \otimes \boldsymbol{g}^j = \frac{\partial \mathrm{G}^i_{\cdot j}}{\partial \mathrm{A}_{k\cdot}^{\ l}} \boldsymbol{g}_i \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}^l \otimes \boldsymbol{g}^j \\
&= \frac{\partial \mathrm{G}^i_{\cdot j}}{\partial \mathrm{A}^{kl}} \boldsymbol{g}_i \otimes \boldsymbol{g}^k \otimes \boldsymbol{g}^l \otimes \boldsymbol{g}^j = \frac{\partial \mathrm{G}^i_{\cdot j}}{\partial \mathrm{A}_{kl}} \boldsymbol{g}_i \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}_l \otimes \boldsymbol{g}^j .
\end{aligned} \tag{6.113}$$

For functions defined only on a subset $\mathbf{Slin}^n \subset \mathbf{Lin}^n$ the directional derivative (6.111) again does not deliver a unique result. Similarly to scalar-valued functions this problem can be avoided defining the fourth-order tensor $g(\mathbf{A})_{,\mathbf{A}}$

as a linear mapping on $\mathbf{S}\mathrm{lin}^n$. Of special interest in this context are symmetric tensor functions. In this case, using (5.47) and applying the procedure described in Sect. 6.3 we can write

$$g(\mathbf{A})\,,_{\mathrm{sym}\mathbf{A}} = \left[g(\mathbf{A})\,,_{\mathbf{A}}\right]^{\mathrm{s}}, \quad \mathbf{A} \in \mathbf{L}\mathrm{in}^n. \tag{6.114}$$

The component representation (6.113) can be given for symmetric tensor functions by

$$\begin{aligned} g(\mathbf{M})\,,_{\mathbf{M}} &= \frac{1}{2} \sum_{\substack{k,l=1 \\ l \le k}}^{n} \frac{\partial \mathrm{G}^i_{\cdot j}}{\partial \mathrm{M}^{kl}} \boldsymbol{g}_i \otimes \left(\boldsymbol{g}^k \otimes \boldsymbol{g}^l + \boldsymbol{g}^l \otimes \boldsymbol{g}^k\right) \otimes \boldsymbol{g}^j \\ &= \frac{1}{2} \sum_{\substack{k,l=1 \\ l \le k}}^{n} \frac{\partial \mathrm{G}^i_{\cdot j}}{\partial \mathrm{M}_{kl}} \boldsymbol{g}_i \otimes \left(\boldsymbol{g}_k \otimes \boldsymbol{g}_l + \boldsymbol{g}_l \otimes \boldsymbol{g}_k\right) \otimes \boldsymbol{g}^j, \end{aligned} \tag{6.115}$$

where $\mathbf{M} \in \mathbf{S}\mathrm{ym}^n$.

Example 1. Derivative of the power function $\mathbf{A}^k$ $(k = 1, 2, \ldots)$. The directional derivative (6.111) of the power function yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} (\mathbf{A} + t\mathbf{X})^k \Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathbf{A}^k + t \sum_{i=0}^{k-1} \mathbf{A}^i \mathbf{X} \mathbf{A}^{k-1-i} + t^2 \ldots \right) \Bigg|_{t=0} \\ &= \sum_{i=0}^{k-1} \mathbf{A}^i \mathbf{X} \mathbf{A}^{k-1-i}. \end{aligned} \tag{6.116}$$

Bearing $(5.17)_1$ and (6.111) in mind we finally obtain

$$\mathbf{A}^k,_{\mathbf{A}} = \sum_{i=0}^{k-1} \mathbf{A}^i \otimes \mathbf{A}^{k-1-i}, \quad \mathbf{A} \in \mathbf{L}\mathrm{in}^n. \tag{6.117}$$

In the special case $k = 1$ it leads to the identity

$$\mathbf{A},_{\mathbf{A}} = \mathfrak{I}, \quad \mathbf{A} \in \mathbf{L}\mathrm{in}^n. \tag{6.118}$$

For power functions of symmetric tensors application of (6.114) yields

$$\mathbf{M}^k,_{\mathbf{M}} = \sum_{i=0}^{k-1} \left(\mathbf{M}^i \otimes \mathbf{M}^{k-1-i}\right)^{\mathrm{s}}, \quad \mathbf{M} \in \mathbf{S}\mathrm{ym}^n \tag{6.119}$$

and consequently

$$\mathbf{M},_{\mathbf{M}} = \mathfrak{I}^{\mathrm{s}}, \quad \mathbf{M} \in \mathbf{S}\mathrm{ym}^n. \tag{6.120}$$

Example 2. Derivative of the transposed tensor $\mathbf{A}^{\mathrm{T}}$. In this case, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{A}+t\mathbf{X}\right)^{\mathrm{T}}\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{A}^{\mathrm{T}}+t\mathbf{X}^{\mathrm{T}}\right)\Big|_{t=0} = \mathbf{X}^{\mathrm{T}}.$$

On use of (5.79) this yields

$$\mathbf{A}^{\mathrm{T}},_{\mathbf{A}} = \mathfrak{T}. \tag{6.121}$$

Example 3. Derivative of the inverse tensor $\mathbf{A}^{-1}$, where $\mathbf{A} \in \mathbf{Inv}^n$. Consider the directional derivative of the identity $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. It delivers:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{A}+t\mathbf{X}\right)^{-1}\left(\mathbf{A}+t\mathbf{X}\right)\Big|_{t=0} = \mathbf{0}.$$

Applying the product rule of differentiation (2.9) and using (6.116) we further write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{A}+t\mathbf{X}\right)^{-1}\Big|_{t=0}\mathbf{A} + \mathbf{A}^{-1}\mathbf{X} = \mathbf{0}$$

and finally

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{A}+t\mathbf{X}\right)^{-1}\Big|_{t=0} = -\mathbf{A}^{-1}\mathbf{X}\mathbf{A}^{-1}.$$

Hence, in view of $(5.17)_1$

$$\mathbf{A}^{-1},_{\mathbf{A}} = -\mathbf{A}^{-1}\otimes\mathbf{A}^{-1}. \tag{6.122}$$

The calculation of the derivative of tensor functions can be simplified by means of differentiation rules. One of them is the following composition rule. Let $\mathbf{G} = g\left(\mathbf{A}\right)$ and $\mathbf{H} = h\left(\mathbf{A}\right)$ be two arbitrary differentiable tensor-valued functions of $\mathbf{A}$. Then,

$$\left(\mathbf{GH}\right),_{\mathbf{A}} = \mathbf{G},_{\mathbf{A}}\mathbf{H} + \mathbf{G}\mathbf{H},_{\mathbf{A}}. \tag{6.123}$$

For the proof we again apply the directional derivative (6.111) taking (2.9) and (5.40) into account

$$\begin{aligned}
\left(\mathbf{GH}\right),_{\mathbf{A}} : \mathbf{X} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[g\left(\mathbf{A}+t\mathbf{X}\right)h\left(\mathbf{A}+t\mathbf{X}\right)\right]\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}g\left(\mathbf{A}+t\mathbf{X}\right)\Big|_{t=0}\mathbf{H} + \mathbf{G}\frac{\mathrm{d}}{\mathrm{d}t}h\left(\mathbf{A}+t\mathbf{X}\right)\Big|_{t=0} \\
&= \left(\mathbf{G},_{\mathbf{A}} : \mathbf{X}\right)\mathbf{H} + \mathbf{G}\left(\mathbf{H},_{\mathbf{A}} : \mathbf{X}\right) \\
&= \left(\mathbf{G},_{\mathbf{A}}\mathbf{H} + \mathbf{G}\mathbf{H},_{\mathbf{A}}\right) : \mathbf{X}, \quad \forall\mathbf{X} \in \mathbf{Lin}^n.
\end{aligned}$$

Example 4. The right and left Cauchy-Green tensors are given in terms of the deformation gradient $\mathbf{F}$ respectively by

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}. \tag{6.124}$$

Of special interest in continuum mechanics is the derivative of these tensors with respect to $\mathbf{F}$. With the aid of the product rule (6.123) and using (5.42), (5.77), (5.82), $(5.83)_1$, (6.118) and (6.121) we obtain

$$\mathbf{C},_{\mathbf{F}} = \mathbf{F}^{\mathrm{T}},_{\mathbf{F}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\mathbf{F},_{\mathbf{F}} = \mathfrak{T}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\mathfrak{I} = (\mathbf{I} \otimes \mathbf{F})^{\mathrm{t}} + \mathbf{F}^{\mathrm{T}} \otimes \mathbf{I}, \tag{6.125}$$

$$\mathbf{b},_{\mathbf{F}} = \mathbf{F},_{\mathbf{F}}\mathbf{F}^{\mathrm{T}} + \mathbf{F}\mathbf{F}^{\mathrm{T}},_{\mathbf{F}} = \mathfrak{I}\mathbf{F}^{\mathrm{T}} + \mathbf{F}\mathfrak{T} = \mathbf{I} \otimes \mathbf{F}^{\mathrm{T}} + (\mathbf{F} \otimes \mathbf{I})^{\mathrm{t}}. \tag{6.126}$$

Further product rules of differentiation of tensor functions can be written as

$$(f\mathbf{G}),_{\mathbf{A}} = \mathbf{G} \odot f,_{\mathbf{A}} + f\mathbf{G},_{\mathbf{A}}, \tag{6.127}$$

$$(\mathbf{G} : \mathbf{H}),_{\mathbf{A}} = \mathbf{H} : \mathbf{G},_{\mathbf{A}} + \mathbf{G} : \mathbf{H},_{\mathbf{A}}, \tag{6.128}$$

where $f = \hat{f}(\mathbf{A})$, $\mathbf{G} = g(\mathbf{A})$ and $\mathbf{H} = h(\mathbf{A})$ are again a scalar-valued and two tensor-valued differentiable tensor functions, respectively. The proof is similar to (6.123) (see Exercise 6.14).

Example 5. With the aid of the above differentiation rules we can easily express the derivatives of the spherical and deviatoric parts (1.153) of a second-order tensor by

$$\mathrm{sph}\mathbf{A},_{\mathbf{A}} = \left[\frac{1}{n}\mathrm{tr}(\mathbf{A})\mathbf{I}\right],_{\mathbf{A}} = \frac{1}{n}\mathbf{I} \odot \mathbf{I} = \mathfrak{P}_{\mathrm{sph}}, \tag{6.129}$$

$$\mathrm{dev}\mathbf{A},_{\mathbf{A}} = \left[\mathbf{A} - \frac{1}{n}\mathrm{tr}(\mathbf{A})\mathbf{I}\right],_{\mathbf{A}} = \mathfrak{I} - \frac{1}{n}\mathbf{I} \odot \mathbf{I} = \mathfrak{P}_{\mathrm{dev}}. \tag{6.130}$$

Example 6. Tangent moduli of hyperelastic isotropic and transversely isotropic materials. The tangent moduli are defined by (see, e.g., [29])

$$\mathfrak{C} = \frac{\partial \mathbf{S}}{\partial \tilde{\mathbf{E}}} = 2\frac{\partial \mathbf{S}}{\partial \mathbf{C}}, \tag{6.131}$$

where $\tilde{\mathbf{E}}$ denotes the Green-Lagrange strain tensor defined in (6.97). For hyperelastic materials this definition implies in view of (6.93) the representation

$$\mathfrak{C} = \frac{\partial^2 \psi}{\partial \tilde{\mathbf{E}} \partial \tilde{\mathbf{E}}} = 4\frac{\partial^2 \psi}{\partial \mathbf{C} \partial \mathbf{C}}. \tag{6.132}$$

For a hyperelastic isotropic material we thus obtain by virtue of (6.119), (6.127), (6.10) or (6.94)

$$\mathfrak{C} = 4 \sum_{k,l=1}^{3} kl \frac{\partial^2 \tilde{\psi}}{\partial \mathrm{tr}\mathbf{C}^k \partial \mathrm{tr}\mathbf{C}^l} \mathbf{C}^{k-1} \odot \mathbf{C}^{l-1}$$

$$+8 \frac{\partial \tilde{\psi}}{\partial \mathrm{tr}\mathbf{C}^2} \mathfrak{J}^{\mathrm{s}} + 12 \frac{\partial \tilde{\psi}}{\partial \mathrm{tr}\mathbf{C}^3} \left(\mathbf{C} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{C}\right)^{\mathrm{s}} . \tag{6.133}$$

For a hyperelastic transversely isotropic material the above procedure yields with the aid of (6.109)

$$\mathfrak{C} = 4 \sum_{k,l=1}^{3} kl \frac{\partial^2 \tilde{\psi}}{\partial \mathrm{tr}\mathbf{C}^k \partial \mathrm{tr}\mathbf{C}^l} \mathbf{C}^{k-1} \odot \mathbf{C}^{l-1} + 4 \frac{\partial^2 \tilde{\psi}}{\partial \mathrm{tr}\left(\mathbf{CL}\right) \partial \mathrm{tr}\left(\mathbf{CL}\right)} \mathbf{L} \odot \mathbf{L}$$

$$+ 4 \frac{\partial^2 \tilde{\psi}}{\partial \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}\right) \partial \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}\right)} \left(\mathbf{CL} + \mathbf{LC}\right) \odot \left(\mathbf{CL} + \mathbf{LC}\right)$$

$$+ 4 \sum_{k}^{3} k \frac{\partial^2 \tilde{\psi}}{\partial \mathrm{tr}\mathbf{C}^k \partial \mathrm{tr}\left(\mathbf{CL}\right)} \left(\mathbf{C}^{k-1} \odot \mathbf{L} + \mathbf{L} \odot \mathbf{C}^{k-1}\right)$$

$$+ 4 \sum_{k}^{3} k \frac{\partial^2 \tilde{\psi}}{\partial \mathrm{tr}\mathbf{C}^k \partial \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}\right)} \left[\mathbf{C}^{k-1} \odot \left(\mathbf{CL} + \mathbf{LC}\right) + \left(\mathbf{CL} + \mathbf{LC}\right) \odot \mathbf{C}^{k-1}\right]$$

$$+ 4 \frac{\partial^2 \tilde{\psi}}{\partial \mathrm{tr}\left(\mathbf{CL}\right) \partial \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}\right)} \left[\mathbf{L} \odot \left(\mathbf{CL} + \mathbf{LC}\right) + \left(\mathbf{CL} + \mathbf{LC}\right) \odot \mathbf{L}\right] + 8 \frac{\partial \tilde{\psi}}{\partial \mathrm{tr}\mathbf{C}^2} \mathfrak{J}^{\mathrm{s}}$$

$$+ 12 \frac{\partial \tilde{\psi}}{\partial \mathrm{tr}\mathbf{C}^3} \left(\mathbf{C} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{C}\right)^{\mathrm{s}} + 4 \frac{\partial \tilde{\psi}}{\partial \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}\right)} \left(\mathbf{L} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{L}\right)^{\mathrm{s}} . \tag{6.134}$$

6.6 Generalized Rivlin's Identities

The Cayley-Hamilton equation (4.91)

$$\mathbf{A}^n - \mathrm{I}_{\mathbf{A}}^{(1)} \mathbf{A}^{n-1} + \mathrm{I}_{\mathbf{A}}^{(2)} \mathbf{A}^{n-2} + \ldots + (-1)^n \, \mathrm{I}_{\mathbf{A}}^{(n)} \mathbf{I} = \mathbf{0} \tag{6.135}$$

represents a universal relation connecting powers of a second-order tensor $\mathbf{A}$ with its principal invariants. Similar universal relations connecting several second-order tensors might also be useful for example for the representation of isotropic tensor functions or for the solution of tensor equations. Such relations are generally called Rivlin's identities.

In order to formulate the Rivlin identities we first differentiate the Cayley-Hamilton equation (6.135) with respect to $\mathbf{A}$. With the aid of (6.58), (6.117) and (6.127) we can write

$$\mathfrak{O} = \left[\sum_{k=0}^{n} (-1)^k \mathrm{I}_{\mathbf{A}}^{(k)} \mathbf{A}^{n-k}\right]_{,\mathbf{A}}$$

$$= \sum_{k=1}^{n} (-1)^k \mathbf{A}^{n-k} \odot \left[\sum_{i=1}^{k} (-1)^{i-1} \mathrm{I}_{\mathbf{A}}^{(k-i)} \left(\mathbf{A}^{\mathrm{T}}\right)^{i-1}\right]$$

$$+ \sum_{k=0}^{n-1} (-1)^k \mathrm{I}_{\mathbf{A}}^{(k)} \left[\sum_{i=1}^{n-k} \mathbf{A}^{n-k-i} \otimes \mathbf{A}^{i-1}\right].$$

Substituting in the last row the summation index $k+i$ by k and using (5.42) and (5.43) we further obtain

$$\sum_{k=1}^{n} \mathbf{A}^{n-k} \sum_{i=1}^{k} (-1)^{k-i} \mathrm{I}_{\mathbf{A}}^{(k-i)} \left[\mathbf{I} \odot \left(\mathbf{A}^{\mathrm{T}}\right)^{i-1} - \mathbf{I} \otimes \mathbf{A}^{i-1}\right] = \mathfrak{O}. \tag{6.136}$$

Mapping an arbitrary second-order tensor $\mathbf{B}$ by both sides of this equation yields an identity written in terms of second-order tensors [10]

$$\sum_{k=1}^{n} \mathbf{A}^{n-k} \sum_{i=1}^{k} (-1)^{k-i} \mathrm{I}_{\mathbf{A}}^{(k-i)} \left[\mathrm{tr}\left(\mathbf{A}^{i-1}\mathbf{B}\right) \mathbf{I} - \mathbf{B}\mathbf{A}^{i-1}\right] = \mathbf{0}. \tag{6.137}$$

This relation is referred to as the generalized Rivlin's identity. Indeed, in the special case of three-dimensional space ($n = 3$) it takes the form

$$\mathbf{ABA} + \mathbf{A}^2\mathbf{B} + \mathbf{BA}^2 - \mathrm{tr}\left(\mathbf{A}\right)\left(\mathbf{AB} + \mathbf{BA}\right) - \mathrm{tr}\left(\mathbf{B}\right)\mathbf{A}^2$$

$$- \left[\mathrm{tr}\left(\mathbf{AB}\right) - \mathrm{tr}\mathbf{A}\mathrm{tr}\mathbf{B}\right]\mathbf{A} + \frac{1}{2}\left[\mathrm{tr}^2\mathbf{A} - \mathrm{tr}\mathbf{A}^2\right]\mathbf{B}$$

$$- \left\{\mathrm{tr}\left(\mathbf{A}^2\mathbf{B}\right) - \mathrm{tr}\mathbf{A}\mathrm{tr}\left(\mathbf{AB}\right) + \frac{1}{2}\mathrm{tr}\mathbf{B}\left[\mathrm{tr}^2\mathbf{A} - \mathrm{tr}\mathbf{A}^2\right]\right\}\mathbf{I} = \mathbf{0}, \tag{6.138}$$

originally obtained by Rivlin [34] by means of matrix calculations.

Differentiating (6.137) again with respect to $\mathbf{A}$ delivers

$$\mathfrak{O} = \sum_{k=1}^{n-1}\sum_{j=1}^{n-k} \left(\mathbf{A}^{n-k-j} \otimes \mathbf{A}^{j-1}\right) \sum_{i=1}^{k} (-1)^{k-i} \mathrm{I}_{\mathbf{A}}^{(k-i)} \left[\mathrm{tr}\left(\mathbf{A}^{i-1}\mathbf{B}\right)\mathbf{I} - \mathbf{BA}^{i-1}\right]$$

$$+ \sum_{k=2}^{n}\sum_{i=1}^{k-1} (-1)^{k-i} \mathbf{A}^{n-k} \left[\mathrm{tr}\left(\mathbf{A}^{i-1}\mathbf{B}\right)\mathbf{I} - \mathbf{BA}^{i-1}\right]$$

$$\odot \left[\sum_{j=1}^{k-i} (-1)^{j-1} \mathrm{I}_{\mathbf{A}}^{(k-i-j)} \left(\mathbf{A}^{\mathrm{T}}\right)^{j-1}\right]$$

$$+\sum_{k=2}^{n}\sum_{i=2}^{k}(-1)^{k-i}\,\mathrm{I}_{\mathbf{A}}^{(k-i)}\mathbf{A}^{n-k}\odot\left[\sum_{j=1}^{i-1}\left(\mathbf{A}^{j-1}\mathbf{B}\mathbf{A}^{i-1-j}\right)^{\mathrm{T}}\right]$$

$$-\sum_{k=2}^{n}\sum_{i=2}^{k}(-1)^{k-i}\,\mathrm{I}_{\mathbf{A}}^{(k-i)}\mathbf{A}^{n-k}\mathbf{B}\left[\sum_{j=1}^{i-1}\left(\mathbf{A}^{i-j-1}\otimes\mathbf{A}^{j-1}\right)\right].$$

Changing the summation indices and summation order we obtain

$$\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}\sum_{j=1}^{k-i}(-1)^{k-i-j}\,\mathrm{I}_{\mathbf{A}}^{(k-i-j)}\mathbf{A}^{n-k}\Big\{\mathbf{I}\otimes\left[\mathrm{tr}\left(\mathbf{A}^{j-1}\mathbf{B}\right)\mathbf{A}^{i-1}\right.$$
$$\left.-\,\mathbf{A}^{i-1}\mathbf{B}\mathbf{A}^{j-1}\right]-\left[\mathrm{tr}\left(\mathbf{A}^{i-1}\mathbf{B}\right)\mathbf{I}-\mathbf{B}\mathbf{A}^{i-1}\right]\odot\left(\mathbf{A}^{\mathrm{T}}\right)^{j-1}$$
$$+\,\mathbf{I}\odot\left(\mathbf{A}^{i-1}\mathbf{B}\mathbf{A}^{j-1}\right)^{\mathrm{T}}-\mathbf{B}\mathbf{A}^{j-1}\otimes\mathbf{A}^{i-1}\Big\}=\mathbf{\mathfrak{O}}. \tag{6.139}$$

The second-order counterpart of this relation can be obtained by mapping another arbitrary second-order tensor $\mathbf{C}\in\mathbf{L}\mathrm{in}^n$ as [10]

$$\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}\sum_{j=1}^{k-i}(-1)^{k-i-j}\,\mathrm{I}_{\mathbf{A}}^{(k-i-j)}\mathbf{A}^{n-k}\Big\{\mathrm{tr}\left(\mathbf{A}^{j-1}\mathbf{B}\right)\mathbf{C}\mathbf{A}^{i-1}$$
$$-\mathbf{C}\mathbf{A}^{i-1}\mathbf{B}\mathbf{A}^{j-1}-\left[\mathrm{tr}\left(\mathbf{A}^{i-1}\mathbf{B}\right)\mathbf{I}-\mathbf{B}\mathbf{A}^{i-1}\right]\mathrm{tr}\left(\mathbf{A}^{j-1}\mathbf{C}\right)$$
$$+\mathrm{tr}\left(\mathbf{A}^{i-1}\mathbf{B}\mathbf{A}^{j-1}\mathbf{C}\right)\mathbf{I}-\mathbf{B}\mathbf{A}^{j-1}\mathbf{C}\mathbf{A}^{i-1}\Big\}=\mathbf{0}. \tag{6.140}$$

In the special case of three-dimensional space ($n=3$) equation (6.140) leads to the well-known identity (see [27], [34], [36])

$$\mathbf{ABC}+\mathbf{ACB}+\mathbf{BCA}+\mathbf{BAC}+\mathbf{CAB}+\mathbf{CBA}-\mathrm{tr}\left(\mathbf{A}\right)\left(\mathbf{BC}+\mathbf{CB}\right)$$
$$-\mathrm{tr}\left(\mathbf{B}\right)\left(\mathbf{CA}+\mathbf{AC}\right)-\mathrm{tr}\left(\mathbf{C}\right)\left(\mathbf{AB}+\mathbf{BA}\right)+\left[\mathrm{tr}\left(\mathbf{B}\right)\mathrm{tr}\left(\mathbf{C}\right)-\mathrm{tr}\left(\mathbf{BC}\right)\right]\mathbf{A}$$
$$+\left[\mathrm{tr}\left(\mathbf{C}\right)\mathrm{tr}\left(\mathbf{A}\right)-\mathrm{tr}\left(\mathbf{CA}\right)\right]\mathbf{B}+\left[\mathrm{tr}\left(\mathbf{A}\right)\mathrm{tr}\left(\mathbf{B}\right)-\mathrm{tr}\left(\mathbf{AB}\right)\right]\mathbf{C}$$
$$-\left[\mathrm{tr}\left(\mathbf{A}\right)\mathrm{tr}\left(\mathbf{B}\right)\mathrm{tr}\left(\mathbf{C}\right)-\mathrm{tr}\left(\mathbf{A}\right)\mathrm{tr}\left(\mathbf{BC}\right)-\mathrm{tr}\left(\mathbf{B}\right)\mathrm{tr}\left(\mathbf{CA}\right)\right.$$
$$\left.-\mathrm{tr}\left(\mathbf{C}\right)\mathrm{tr}\left(\mathbf{AB}\right)+\mathrm{tr}\left(\mathbf{ABC}\right)+\mathrm{tr}\left(\mathbf{ACB}\right)\right]\mathbf{I}=\mathbf{0}. \tag{6.141}$$

Exercises

6.1. Check isotropy of the following tensor functions:

(a) $f(\mathbf{A})=\boldsymbol{a}\mathbf{A}\boldsymbol{b}$, where $\boldsymbol{a},\boldsymbol{b}\in\mathbb{E}^n$,

(b) $f(\mathbf{A})=\mathrm{A}^{11}+\mathrm{A}^{22}+\mathrm{A}^{33}$,

(c) $f(\mathbf{A})=\mathrm{A}^{11}+\mathrm{A}^{12}+\mathrm{A}^{13}$, where A^{ij} represent the components of $\mathbf{A}\in\mathbf{L}\mathrm{in}^3$

with respect to an orthonormal basis $\boldsymbol{e}_i$ $(i = 1, 2, 3)$, so that $\mathbf{A} = \mathrm{A}^{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$,
(d) $f(\mathbf{A}) = \det\mathbf{A}$,
(e) $f(\mathbf{A}) = \lambda_{\max}$, where $\lambda_{\max}$ denotes the maximal (in the sense of the norm $\sqrt{\lambda\bar{\lambda}}$) eigenvalue of $\mathbf{A} \in \mathbf{Lin}^n$.

6.2. Prove the alternative representation (6.18) for the functional basis of an arbitrary second-order tensor $\mathbf{A}$.

6.3. An orthotropic symmetry group $\mathfrak{g}_o$ is described in terms of three structural tensors defined by $\mathbf{L}_i = \boldsymbol{l}_i \otimes \boldsymbol{l}_i$, where $\boldsymbol{l}_i \cdot \boldsymbol{l}_j = \delta_{ij}$ $(i, j = 1, 2, 3)$ are unit vectors along mutually orthogonal principal material directions. Represent the general orthotropic strain energy function

$$\psi\left(\mathbf{QCQ}^{\mathrm{T}}\right) = \psi(\mathbf{C}), \quad \forall \mathbf{Q} \in \mathfrak{g}_o \tag{6.142}$$

in terms of the orthotropic invariants.

6.4. Using the results of Exercise 6.3, derive the constitutive relation for the second Piola-Kirchhoff stress tensor $\mathbf{S}$ (6.93) and the tangent moduli $\mathfrak{C}$ (6.131) for the general hyperelastic orthotropic material.

6.5. Represent the general constitutive relation for an orthotropic elastic material as a function $\mathbf{S}(\mathbf{C})$.

6.6. A symmetry group $\mathfrak{g}_f$ of a fiber reinforced material with an isotropic matrix is described in terms of structural tensors defined by $\mathbf{L}_i = \boldsymbol{l}_i \otimes \boldsymbol{l}_i$, where the unit vectors $\boldsymbol{l}_i$ $(i = 1, 2, \ldots, k)$ define the directions of fiber families and are not necessarily orthogonal to each other. Represent the strain energy function

$$\psi\left(\mathbf{QCQ}^{\mathrm{T}}\right) = \psi(\mathbf{C}), \quad \forall \mathbf{Q} \in \mathfrak{g}_f \tag{6.143}$$

of a fiber reinforced material with two families of fibers $(k = 2)$.

6.7. Derive the constitutive relation $\mathbf{S} = 2\partial\psi/\partial\mathbf{C} + p\mathbf{C}^{-1}$ and the tangent moduli $\mathfrak{C} = 2\partial\mathbf{S}/\partial\mathbf{C}$ for the Mooney-Rivlin material represented by the strain energy function (6.11).

6.8. Derive the constitutive relation for the Ogden model (6.12) in terms of the second Piola-Kirchhoff stress tensor using expression (6.93).

6.9. Show that $\mathrm{tr}(\mathbf{CL}_i\mathbf{CL}_j)$, where $\mathbf{L}_i$ $(i = 1, 2, 3)$ are structural tensors defined in Exercise 6.3, represents an orthotropic tensor function (orthotropic invariant) of $\mathbf{C}$. Express this function in terms of the orthotropic functional basis obtained in Exercise 6.3.

6.10. The strain energy function of the orthotropic St.Venant-Kirchhoff material is given by

$$\psi\left(\tilde{\mathbf{E}}\right) = \frac{1}{2}\sum_{i,j=1}^{3} a_{ij}\operatorname{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_i\right)\operatorname{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_j\right) + \sum_{\substack{i,j=1\\ i\neq j}}^{3} G_{ij}\operatorname{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_i\tilde{\mathbf{E}}\mathbf{L}_j\right), \qquad (6.144)$$

where $\tilde{\mathbf{E}}$ denotes the Green-Lagrange strain tensor (6.97) and $\mathbf{L}_i$ $(i = 1,2,3)$ are the structural tensors defined in Exercise 6.3. $a_{ij} = a_{ji}$ $(i,j = 1,2,3)$ and $G_{ij} = G_{ji}$ $(i \neq j = 1,2,3)$ represent material constants. Derive the constitutive relation for the second Piola-Kirchhoff stress tensor $\mathbf{S}$ (6.93) and the tangent moduli $\mathfrak{C}$ (6.131).

6.11. Show that the function $\psi(\tilde{\mathbf{E}})$ (6.144) becomes transversely isotropic if

$$a_{22} = a_{33}, \quad a_{12} = a_{13}, \quad G_{12} = G_{13}, \quad G_{23} = \frac{1}{2}\left(a_{22} - a_{23}\right) \qquad (6.145)$$

and isotropic of the form (6.99) if

$$a_{12} = a_{13} = a_{23} = \lambda, \quad G_{12} = G_{13} = G_{23} = G,$$
$$a_{11} = a_{22} = a_{33} = \lambda + 2G. \qquad (6.146)$$

6.12. Complete the proof of Theorem 6.3.

6.13. Express $\mathbf{A}^{-k}{}_{,\mathbf{A}}$, where $k = 1, 2, \ldots$.

6.14. Prove the product rules of differentiation (6.127) and (6.128).

6.15. Write out Rivlin's identity (6.137) for $n = 2$.