7

Analytic Tensor Functions

7.1 Introduction

In the previous chapter we discussed isotropic and anisotropic tensor functions and their general representations. Of particular interest in continuum mechanics are isotropic tensor-valued functions of one arbitrary (not necessarily symmetric) tensor. For example, the exponential function of the velocity gradient or other non-symmetric strain rates is very suitable for the formulation of evolution equations in large strain anisotropic plasticity. In this section we focus on a special class of isotropic tensor-valued functions referred here to as analytic tensor functions. In order to specify this class of functions we first deal with the general question how an isotropic tensor-valued function can be defined.

For isotropic functions of diagonalizable tensors the most natural way is the spectral decomposition (4.43)

$$\mathbf{A} = \sum_{i=1}^{s} \lambda_i \mathbf{P}_i, \tag{7.1}$$

so that we may write similarly to (4.48)

$$g(\mathbf{A}) = \sum_{i=1}^{s} g(\lambda_i) \mathbf{P}_i, \tag{7.2}$$

where $g(\lambda_i)$ is an arbitrary (not necessarily polynomial) scalar function defined on the spectrum λ_i $(i = 1, 2, \ldots, s)$ of the tensor $\mathbf{A}$. Obviously, the so-defined function $g(\mathbf{A})$ is isotropic in the sense of the condition (6.73). Indeed,

$$g\left(\mathbf{QAQ}^{\mathrm{T}}\right) = \sum_{i=1}^{s} g(\lambda_i) \mathbf{QP}_i\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}g(\mathbf{A})\mathbf{Q}^{\mathrm{T}}, \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n, \tag{7.3}$$

where we take into account that the spectral decomposition of the tensor $\mathbf{QAQ}^{\mathrm{T}}$ is given by

$$\mathbf{QAQ}^{\mathrm{T}} = \sum_{i=1}^{s} \lambda_i \mathbf{QP}_i \mathbf{Q}^{\mathrm{T}}. \tag{7.4}$$

Example. Generalized strain measures. The so-called generalized strain measures $\mathbf{E}$ and $\mathbf{e}$ (also known as Hill's strains, [15], [16]) play an important role in kinematics of continuum. They are defined by (7.2) as isotropic tensor-valued functions of the symmetric right and left stretch tensor $\mathbf{U}$ and $\mathbf{v}$ and are referred to as Lagrangian (material) and Eulerian (spatial) strains, respectively. The definition of the generalized strains is based on the spectral representations by

$$\mathbf{U} = \sum_{i=1}^{s} \lambda_i \mathbf{P}_i, \quad \mathbf{v} = \sum_{i=1}^{s} \lambda_i \mathbf{p}_i, \tag{7.5}$$

where $\lambda_i > 0$ are the eigenvalues (referred to as principal stretches) while $\mathbf{P}_i$ and $\mathbf{p}_i$ $(i = 1, 2, \ldots, s)$ denote the corresponding eigenprojections. Accordingly,

$$\mathbf{E} = \sum_{i=1}^{s} f(\lambda_i) \mathbf{P}_i, \quad \mathbf{e} = \sum_{i=1}^{s} f(\lambda_i) \mathbf{p}_i, \tag{7.6}$$

where f is a strictly-increasing scalar function satisfying the conditions $f(1) = 0$ and $f'(1) = 1$. A special class of generalized strain measures specified by

$$\mathbf{E}^{(a)} = \begin{cases} \sum\limits_{i=1}^{s} \dfrac{1}{a} \left(\lambda_i^a - 1\right) \mathbf{P}_i & \text{for } a \neq 0, \\ \sum\limits_{i=1}^{s} \ln(\lambda_i) \mathbf{P}_i & \text{for } a = 0, \end{cases} \tag{7.7}$$

$$\mathbf{e}^{(a)} = \begin{cases} \sum\limits_{i=1}^{s} \dfrac{1}{a} \left(\lambda_i^a - 1\right) \mathbf{p}_i & \text{for } a \neq 0, \\ \sum\limits_{i=1}^{s} \ln(\lambda_i) \mathbf{p}_i & \text{for } a = 0 \end{cases} \tag{7.8}$$

are referred to as Seth's strains [39], where a is a real number. For example, the Green-Lagrange strain tensor (6.97) introduced in Chap. 6 belongs to Seth's strains as $\mathbf{E}^{(2)}$.

Since non-symmetric tensors do not generally admit the spectral decomposition in the diagonal form (7.1), it is necessary to search for other approaches for the definition of the isotropic tensor function $g(\mathbf{A}) : \mathbf{Lin}^n \mapsto \mathbf{Lin}^n$. One of these approaches is the tensor power series of the form

$$g(\mathbf{A}) = a_0 \mathbf{I} + a_1 \mathbf{A} + a_2 \mathbf{A}^2 + \ldots = \sum_{r=0}^{\infty} a_r \mathbf{A}^r. \tag{7.9}$$

Indeed, in view of (6.74)

$$g\left(\mathbf{QAQ}^{\mathrm{T}}\right) = \sum_{r=0}^{\infty} a_r \left(\mathbf{QAQ}^{\mathrm{T}}\right)^r$$

$$= \sum_{r=0}^{\infty} a_r \mathbf{QA}^r\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}g\left(\mathbf{A}\right)\mathbf{Q}^{\mathrm{T}}, \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n. \tag{7.10}$$

For example, the exponential tensor function can be defined in terms of the infinite power series (7.9) by (1.109).

One can show that the power series (7.9), provided it converges, represents a generalization of (7.2) to arbitrary (and not necessarily diagonalizable) second-order tensors. Conversely, the isotropic tensor function (7.2) with $g(\lambda)$ analytic on the spectrum of $\mathbf{A}$ can be considered as an extension of infinite power series (7.9) to its non-convergent domain if the latter exists. Indeed, for diagonalizable tensor arguments within the convergence domain of the tensor power series (7.9) both definitions coincide. For example, inserting (7.1) into (7.9) and taking (4.47) into account we have

$$g\left(\mathbf{A}\right) = \sum_{r=0}^{\infty} a_r \left(\sum_{i=1}^{s} \lambda_i \mathbf{P}_i\right)^r = \sum_{r=0}^{\infty} a_r \sum_{i=1}^{s} \lambda_i^r \mathbf{P}_i = \sum_{i=1}^{s} g\left(\lambda_i\right)\mathbf{P}_i \tag{7.11}$$

with the abbreviation

$$g\left(\lambda\right) = \sum_{r=0}^{\infty} a_r \lambda^r, \tag{7.12}$$

so that

$$a_r = \frac{1}{r!} \left.\frac{\partial^r g\left(\lambda\right)}{\partial \lambda^r}\right|_{\lambda=0}. \tag{7.13}$$

The above mentioned convergence requirement vastly restricts the definition domain of many isotropic tensor functions defined in terms of infinite series (7.9). For example, one can show that the power series for the logarithmic tensor function

$$\ln\left(\mathbf{A} + \mathbf{I}\right) = \sum_{r=1}^{\infty} \left(-1\right)^{r+1} \frac{\mathbf{A}^r}{r} \tag{7.14}$$

converges for $|\lambda_i| < 1$ $(i = 1, 2, \ldots, s)$ and diverges if $|\lambda_k| > 1$ at least for some k $(1 \le k \le s)$ (see, e.g., [12]).

In order to avoid this convergence problem we consider a tensor function defined by the so-called Dunford-Taylor integral as (see, for example, [24])

$$g\left(\mathbf{A}\right) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} g\left(\zeta\right)\left(\zeta\mathbf{I} - \mathbf{A}\right)^{-1} \mathrm{d}\zeta \tag{7.15}$$

taken on the complex plane over Γ, where Γ represents a closed curve or consists of simple closed curves, the union interior of which includes all the eigenvalues $\lambda_i \in \mathbb{C}$ $(i = 1, 2, \ldots, s)$ of the tensor argument $\mathbf{A}$. $g(\zeta) : \mathbb{C} \mapsto \mathbb{C}$ is an arbitrary scalar function analytic within and on Γ.

One can easily prove that the tensor function (7.15) is isotropic in the sense of the definition (6.73). Indeed, with the aid of (1.127) and (1.128) we obtain (cf. [33])

$$\begin{aligned} g\left(\mathbf{QAQ}^{\mathrm{T}}\right) &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} g(\zeta) \left(\zeta\mathbf{I} - \mathbf{QAQ}^{\mathrm{T}}\right)^{-1} \mathrm{d}\zeta \\ &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} g(\zeta) \left[\mathbf{Q}(\zeta\mathbf{I} - \mathbf{A})\mathbf{Q}^{\mathrm{T}}\right]^{-1} \mathrm{d}\zeta \\ &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} g(\zeta)\, \mathbf{Q}(\zeta\mathbf{I} - \mathbf{A})^{-1} \mathbf{Q}^{\mathrm{T}} \mathrm{d}\zeta \\ &= \mathbf{Q} g(\mathbf{A}) \mathbf{Q}^{\mathrm{T}}, \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n. \end{aligned} \tag{7.16}$$

It can be verified that for diagonalizable tensors the Dunford-Taylor integral (7.15) reduces to the spectral decomposition (7.2) and represents therefore its generalization. Indeed, inserting (7.1) into (7.15) delivers

$$\begin{aligned} g(\mathbf{A}) &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} g(\zeta) \left(\zeta\mathbf{I} - \sum_{i=1}^{s} \lambda_i \mathbf{P}_i\right)^{-1} \mathrm{d}\zeta \\ &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} g(\zeta) \left[\sum_{i=1}^{s} (\zeta - \lambda_i) \mathbf{P}_i\right]^{-1} \mathrm{d}\zeta \\ &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} g(\zeta) \sum_{i=1}^{s} (\zeta - \lambda_i)^{-1} \mathbf{P}_i \mathrm{d}\zeta \\ &= \sum_{i=1}^{s} \left[\frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} g(\zeta)(\zeta - \lambda_i)^{-1} \mathrm{d}\zeta\right] \mathbf{P}_i = \sum_{i=1}^{s} g(\lambda_i) \mathbf{P}_i, \end{aligned} \tag{7.17}$$

where we keep (4.46) in mind and apply the the Cauchy integral formula (see, e.g. [5]). Using this result we can represent, for example, the generalized strain measures (7.6) by

$$\mathbf{E} = f(\mathbf{U}), \quad \mathbf{e} = f(\mathbf{v}), \tag{7.18}$$

where the tensor functions $f(\mathbf{U})$ and $f(\mathbf{v})$ are defined by (7.15).

Further, one can show that the Dunford-Taylor integral (7.15) also represents a generalization of tensor power series (7.9). For this purpose, it suffices to verify that (7.15) based on a scalar function $g(\zeta) = \zeta^k$ $(k = 0, 1, 2, \ldots)$ results into the monomial $g(\mathbf{A}) = \mathbf{A}^k$. To this end, we consider in (7.15) the following identity [24]

$$g(\zeta)\,\mathbf{I} = (\zeta\mathbf{I})^k = (\zeta\mathbf{I} - \mathbf{A} + \mathbf{A})^k = (\zeta\mathbf{I} - \mathbf{A})^k + \ldots + \mathbf{A}^k. \tag{7.19}$$

Thereby, all terms except of the last one have no pole within Γ and vanish according to the Cauchy theorem (see, e.g., [5]), so that

$$g(\mathbf{A}) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} \left[(\zeta\mathbf{I} - \mathbf{A})^{k-1} + \ldots + \mathbf{A}^k (\zeta\mathbf{I} - \mathbf{A})^{-1} \right] \mathrm{d}\zeta = \mathbf{A}^k. \tag{7.20}$$

Isotropic tensor functions defined by (7.15) will henceforth be referred to as analytic tensor functions. The above discussed properties of analytic tensor functions can be completed by the following relations (Exercise 7.3)

$$\begin{aligned} g(\mathbf{A}) &= \alpha f(\mathbf{A}) + \beta h(\mathbf{A}), && \text{if} \quad g(\lambda) = \alpha f(\lambda) + \beta h(\lambda), \\ g(\mathbf{A}) &= f(\mathbf{A})\, h(\mathbf{A}), && \text{if} \quad g(\lambda) = f(\lambda)\, h(\lambda), \\ g(\mathbf{A}) &= f(h(\mathbf{A})), && \text{if} \quad g(\lambda) = f(h(\lambda)). \end{aligned} \tag{7.21}$$

In the following we will deal with representations for analytic tensor functions and their derivatives.

7.2 Closed-Form Representation for Analytic Tensor Functions and Their Derivatives

Our aim is to obtain the so-called closed form representation for analytic tensor functions and their derivatives. This representation should be given only in terms of finite powers of the tensor argument and its eigenvalues and avoid any reference to the integral over the complex plane or to power series.

We start with the Cayley-Hamilton theorem (4.91) for the tensor $\zeta\mathbf{I} - \mathbf{A}$

$$\sum_{k=0}^{n} (-1)^k \,\mathrm{I}^{(k)}_{\zeta\mathbf{I}-\mathbf{A}} (\zeta\mathbf{I} - \mathbf{A})^{n-k} = \mathbf{0}. \tag{7.22}$$

With the aid of the Vieta theorem (4.24) we can write

$$\mathrm{I}^{(0)}_{\zeta\mathbf{I}-\mathbf{A}} = 1, \quad \mathrm{I}^{(k)}_{\zeta\mathbf{I}-\mathbf{A}} = \sum_{i_1<i_2<\ldots<i_k}^{n} (\zeta - \lambda_{i_1})(\zeta - \lambda_{i_2}) \ldots (\zeta - \lambda_{i_k}), \tag{7.23}$$

where $k = 1, 2, \ldots, n$ and the eigenvalues λ_i $(i = 1, 2, \ldots, n)$ of the tensor $\mathbf{A}$ are counted with their multiplicity.

Composing (7.22) with the so-called resolvent of $\mathbf{A}$

$$\mathbf{R}(\zeta) = (\zeta\mathbf{I} - \mathbf{A})^{-1} \tag{7.24}$$

yields

$$\mathbf{R}(\zeta) = \frac{1}{\mathrm{I}^{(n)}_{\zeta\mathbf{I}-\mathbf{A}}} \sum_{k=0}^{n-1} (-1)^{n-k-1} \mathrm{I}^{(k)}_{\zeta\mathbf{I}-\mathbf{A}} (\zeta\mathbf{I}-\mathbf{A})^{n-k-1}$$

$$= \frac{1}{\mathrm{I}^{(n)}_{\zeta\mathbf{I}-\mathbf{A}}} \sum_{k=0}^{n-1} \mathrm{I}^{(k)}_{\zeta\mathbf{I}-\mathbf{A}} (\mathbf{A}-\zeta\mathbf{I})^{n-k-1} . \tag{7.25}$$

Applying the binomial theorem (see, e.g., [5])

$$(\mathbf{A}-\zeta\mathbf{I})^{l} = \sum_{p=0}^{l} (-1)^{l-p} \binom{l}{p} \zeta^{l-p} \mathbf{A}^{p}, \quad l = 1, 2, \ldots, \tag{7.26}$$

where

$$\binom{l}{p} = \frac{l!}{p!\,(l-p)!}, \tag{7.27}$$

we obtain

$$\mathbf{R}(\zeta) = \frac{1}{\mathrm{I}^{(n)}_{\zeta\mathbf{I}-\mathbf{A}}} \sum_{k=0}^{n-1} \mathrm{I}^{(k)}_{\zeta\mathbf{I}-\mathbf{A}} \sum_{p=0}^{n-k-1} (-1)^{n-k-1-p} \binom{n-k-1}{p} \zeta^{n-k-1-p} \mathbf{A}^{p}. \tag{7.28}$$

Rearranging this expression with respect to the powers of the tensor **A** delivers

$$\mathbf{R}(\zeta) = \sum_{p=0}^{n-1} \alpha_p \mathbf{A}^{p} \tag{7.29}$$

with

$$\alpha_p = \frac{1}{\mathrm{I}^{(n)}_{\zeta\mathbf{I}-\mathbf{A}}} \sum_{k=0}^{n-p-1} (-1)^{n-k-p-1} \binom{n-k-1}{p} \mathrm{I}^{(k)}_{\zeta\mathbf{I}-\mathbf{A}} \zeta^{n-k-p-1}, \tag{7.30}$$

where $p = 0, 1, \ldots, n-1$. Inserting this result into (7.15) we obtain the following closed-form representation for the tensor function $g(\mathbf{A})$ [20]

$$g(\mathbf{A}) = \sum_{p=0}^{n-1} \varphi_p \mathbf{A}^{p}, \tag{7.31}$$

where

$$\varphi_p = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} g(\zeta)\, \alpha_p \mathrm{d}\zeta, \quad p = 0, 1, \ldots, n-1. \tag{7.32}$$

The Cauchy integrals in (7.32) can be calculated with the aid of the residue theorem (see, e.g.,[5]). To this end, we first represent the determinant of the tensor $\zeta\mathbf{I}-\mathbf{A}$ in the form

$$\mathrm{I}^{(n)}_{\zeta\mathbf{I}-\mathbf{A}} = \det\left(\zeta\mathbf{I}-\mathbf{A}\right) = \prod_{i=1}^{s}\left(\zeta-\lambda_i\right)^{r_i}, \tag{7.33}$$

where λ_i denote pairwise distinct eigenvalues with the algebraic multiplicities r_i $(i = 1, 2, \ldots, s)$ such that

$$\sum_{i=1}^{s} r_i = n. \tag{7.34}$$

Thus, inserting (7.30) and (7.33) into (7.32) we obtain

$$\varphi_p = \sum_{i=1}^{s} \frac{1}{(r_i-1)!} \lim_{\zeta\to\lambda_i} \left\{ \frac{\mathrm{d}^{r_i-1}}{\mathrm{d}\zeta^{r_i-1}} \left[g\left(\zeta\right)\alpha_p\left(\zeta\right)\left(\zeta-\lambda_i\right)^{r_i} \right] \right\}, \tag{7.35}$$

where $p = 1, 2, \ldots, n-1$.

The derivative of the tensor function $g\left(\mathbf{A}\right)$ can be obtained by direct differentiation of the Dunfod-Taylor integral (7.15). Thus, by use of (6.122) we can write

$$g\left(\mathbf{A}\right),_{\mathbf{A}} = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} g\left(\zeta\right)\left(\zeta\mathbf{I}-\mathbf{A}\right)^{-1} \otimes \left(\zeta\mathbf{I}-\mathbf{A}\right)^{-1} \mathrm{d}\zeta \tag{7.36}$$

and consequently

$$g\left(\mathbf{A}\right),_{\mathbf{A}} = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} g\left(\zeta\right)\mathbf{R}\left(\zeta\right) \otimes \mathbf{R}\left(\zeta\right) \mathrm{d}\zeta. \tag{7.37}$$

Taking (7.29) into account further yields

$$g\left(\mathbf{A}\right),_{\mathbf{A}} = \sum_{p,q=0}^{n-1} \eta_{pq}\mathbf{A}^p \otimes \mathbf{A}^q, \tag{7.38}$$

where

$$\eta_{pq} = \eta_{qp} = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} g\left(\zeta\right)\alpha_p\left(\zeta\right)\alpha_q\left(\zeta\right)\mathrm{d}\zeta, \quad p, q = 0, 1, \ldots, n-1. \tag{7.39}$$

The residue theorem finally delivers

$$\eta_{pq} = \sum_{i=1}^{s} \frac{1}{(2r_i-1)!} \lim_{\zeta\to\lambda_i} \left\{ \frac{\mathrm{d}^{2r_i-1}}{\mathrm{d}\zeta^{2r_i-1}} \left[g\left(\zeta\right)\alpha_p\left(\zeta\right)\alpha_q\left(\zeta\right)\left(\zeta-\lambda_i\right)^{2r_i} \right] \right\}, \tag{7.40}$$

where $p, q = 0, 1, \ldots, n-1$.

7.3 Special Case: Diagonalizable Tensor Functions

For analytic functions of diagonalizable tensors the definitions in terms of the Dunford-Taylor integral (7.15) on the one side and eigenprojections (7.2) on the other side become equivalent. In this special case, one can obtain alternative closed-form representations for analytic tensor functions and their derivatives. To this end, we first derive an alternative representation of the Sylvester formula (4.54). In Sect. 4.4 we have shown that the eigenprojections can be given by (4.51)

$$\mathbf{P}_i = p_i\left(\mathbf{A}\right), \quad i = 1, 2, \ldots, s, \tag{7.41}$$

where $p_i\,(i = 1, 2, \ldots, s)$ are polynomials satisfying the requirements (4.50). Thus, the eigenprojections of a second-order tensor can be considered as its analytic (isotropic) tensor functions. Applying the Dunford-Taylor integral (7.15) we can thus write

$$\mathbf{P}_i = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} p_i\left(\zeta\right)\left(\zeta \mathbf{I} - \mathbf{A}\right)^{-1} \mathrm{d}\zeta, \quad i = 1, 2, \ldots, s. \tag{7.42}$$

Similarly to (7.31) and (7.35) we further obtain

$$\mathbf{P}_i = \sum_{p=0}^{n-1} \rho_{ip} \mathbf{A}^p, \quad i = 1, 2, \ldots, s, \tag{7.43}$$

where

$$\rho_{ip} = \sum_{k=1}^{s} \frac{1}{\left(r_k - 1\right)!} \lim_{\zeta \to \lambda_k} \left\{ \frac{\mathrm{d}^{r_k - 1}}{\mathrm{d}\zeta^{r_k - 1}} \left[p_i\left(\zeta\right) \alpha_p\left(\zeta\right) \left(\zeta - \lambda_k\right)^{r_k} \right] \right\} \tag{7.44}$$

and $\alpha_p\,(p = 0, 1, \ldots, n-1)$ are given by (7.30). With the aid of polynomial functions $p_i\left(\lambda\right)$ satisfying in addition to (4.50) the following conditions

$$\left.\frac{\mathrm{d}^r}{\mathrm{d}\lambda^r} p_i\left(\lambda\right)\right|_{\lambda = \lambda_j} = 0 \quad i, j = 1, 2, \ldots, s;\ r = 1, 2, \ldots, r_i - 1 \tag{7.45}$$

we can simplify (7.44) by

$$\rho_{ip} = \frac{1}{\left(r_i - 1\right)!} \lim_{\zeta \to \lambda_i} \left\{ \frac{\mathrm{d}^{r_i - 1}}{\mathrm{d}\zeta^{r_i - 1}} \left[\alpha_p\left(\zeta\right) \left(\zeta - \lambda_i\right)^{r_i} \right] \right\}. \tag{7.46}$$

Now, inserting (7.43) into (7.2) delivers

$$g\left(\mathbf{A}\right) = \sum_{i=1}^{s} g\left(\lambda_i\right) \sum_{p=0}^{n-1} \rho_{ip} \mathbf{A}^p. \tag{7.47}$$

In order to obtain an alternative representation for $g(\mathbf{A})_{,\mathbf{A}}$ we again consider the tensor power series (7.9). Direct differentiation of (7.9) with respect to $\mathbf{A}$ delivers with the aid of (6.117)

$$g(\mathbf{A})_{,\mathbf{A}} = \sum_{r=1}^{\infty} a_r \sum_{k=0}^{r-1} \mathbf{A}^{r-1-k} \otimes \mathbf{A}^k. \tag{7.48}$$

Applying the spectral representation (7.1) and taking (4.47) and (7.12) into account we further obtain (see also [18], [48])

$$\begin{aligned} g(\mathbf{A})_{,\mathbf{A}} &= \sum_{r=1}^{\infty} a_r \sum_{k=0}^{r-1} \sum_{i,j=1}^{s} \lambda_i^{r-1-k} \lambda_j^k \mathbf{P}_i \otimes \mathbf{P}_j \\ &= \sum_{i=1}^{s} \sum_{r=1}^{\infty} r a_r \lambda_i^{r-1} \mathbf{P}_i \otimes \mathbf{P}_i + \sum_{\substack{i,j=1 \\ j \neq i}}^{s} \sum_{r=1}^{\infty} a_r \frac{\lambda_i^r - \lambda_j^r}{\lambda_i - \lambda_j} \mathbf{P}_i \otimes \mathbf{P}_j \\ &= \sum_{i=1}^{s} g'(\lambda_i) \mathbf{P}_i \otimes \mathbf{P}_i + \sum_{\substack{i,j=1 \\ j \neq i}}^{s} \frac{g(\lambda_i) - g(\lambda_j)}{\lambda_i - \lambda_j} \mathbf{P}_i \otimes \mathbf{P}_j \\ &= \sum_{i,j=1}^{s} G_{ij} \mathbf{P}_i \otimes \mathbf{P}_j, \end{aligned} \tag{7.49}$$

where

$$G_{ij} = \begin{cases} g'(\lambda_i) & \text{if } i = j, \\ \dfrac{g(\lambda_i) - g(\lambda_j)}{\lambda_i - \lambda_j} & \text{if } i \neq j. \end{cases} \tag{7.50}$$

Inserting into (7.49) the alternative representation for the eigenprojections (7.43) yields

$$g(\mathbf{A})_{,\mathbf{A}} = \sum_{i,j=1}^{s} G_{ij} \sum_{p,q=0}^{n-1} \rho_{ip} \rho_{jq} \mathbf{A}^p \otimes \mathbf{A}^q. \tag{7.51}$$

Thus, we again end up with the representation (7.38)

$$g(\mathbf{A})_{,\mathbf{A}} = \sum_{p,q=0}^{n-1} \eta_{pq} \mathbf{A}^p \otimes \mathbf{A}^q, \tag{7.52}$$

where

$$\eta_{pq} = \eta_{qp} = \sum_{i,j=1}^{s} G_{ij} \rho_{ip} \rho_{jq}, \quad p, q = 0, 1, \ldots, n-1. \tag{7.53}$$

Finally, let us focus on the differentiability of eigenprojections. To this end, we represent them by [24] (Exercise 7.4)

$$\mathbf{P}_i = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_i} (\zeta \mathbf{I} - \mathbf{A})^{-1} \mathrm{d}\zeta, \quad i = 1, 2, \ldots s, \tag{7.54}$$

where the integral is taken on the complex plane over a closed curve Γ_i the interior of which includes only the eigenvalue λ_i. All other eigenvalues of $\mathbf{A}$ lie outside Γ_i. Γ_i does not depend on λ_i as far as this eigenvalue is simple and does not lie directly on Γ_i. Indeed, if λ_i is multiple, a perturbation of $\mathbf{A}$ by $\mathbf{A} + t\mathbf{X}$ can lead to a split of eigenvalues within Γ_i. In this case, (7.54) yields a sum of eigenprojections corresponding to these split eigenvalues which coalesce in λ_i for $t = 0$. Thus, the eigenprojection $\mathbf{P}_i$ corresponding to a simple eigenvalue λ_i is differentiable according to (7.54). Direct differentiation of (7.54) delivers in this case

$$\mathbf{P}_{i,\mathbf{A}} = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_i} (\zeta \mathbf{I} - \mathbf{A})^{-1} \otimes (\zeta \mathbf{I} - \mathbf{A})^{-1} \mathrm{d}\zeta, \quad r_i = 1. \tag{7.55}$$

By analogy with (7.38) we thus obtain

$$\mathbf{P}_{i,\mathbf{A}} = \sum_{p,q=0}^{n-1} \upsilon_{ipq} \mathbf{A}^p \otimes \mathbf{A}^q, \tag{7.56}$$

where

$$\upsilon_{ipq} = \upsilon_{iqp} = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_i} \alpha_p(\zeta)\, \alpha_q(\zeta)\, \mathrm{d}\zeta, \quad p, q = 0, 1, \ldots, n-1. \tag{7.57}$$

By the residue theorem we further write

$$\upsilon_{ipq} = \lim_{\zeta \to \lambda_i} \left\{ \frac{\mathrm{d}}{\mathrm{d}\zeta} \left[\alpha_p(\zeta)\, \alpha_q(\zeta)\, (\zeta - \lambda_i)^2 \right] \right\}, \quad p, q = 0, 1, \ldots, n-1. \tag{7.58}$$

With the aid of (7.49) one can obtain an alternative representation for the derivative of the eigenprojections in terms of the eigenprojections themselves. Indeed, substituting the function g in (7.49) by p_i and taking the properties of the latter function (4.50) and (7.45) into account we have

$$\mathbf{P}_{i,\mathbf{A}} = \sum_{\substack{j=1 \\ j \neq i}}^{s} \frac{\mathbf{P}_i \otimes \mathbf{P}_j + \mathbf{P}_j \otimes \mathbf{P}_i}{\lambda_i - \lambda_j}. \tag{7.59}$$

7.4 Special case: Three-Dimensional Space

First, we specify the closed-form solutions (7.31) and (7.38) for three-dimensional space ($n = 3$). In this case, the functions $\alpha_k(\zeta)$ ($k = 0, 1, 2$) (7.30) take the form

$$\alpha_0(\zeta) = \frac{\zeta^2 - \zeta \mathrm{I}_{\zeta\mathbf{I}-\mathbf{A}} + \mathrm{II}_{\zeta\mathbf{I}-\mathbf{A}}}{\mathrm{III}_{\zeta\mathbf{I}-\mathbf{A}}}$$
$$= \frac{\zeta^2 - \zeta(\lambda_1 + \lambda_2 + \lambda_3) + \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1}{(\zeta - \lambda_1)(\zeta - \lambda_2)(\zeta - \lambda_3)},$$
$$\alpha_1(\zeta) = \frac{\mathrm{I}_{\zeta\mathbf{I}-\mathbf{A}} - 2\zeta}{\mathrm{III}_{\zeta\mathbf{I}-\mathbf{A}}} = \frac{\zeta - \lambda_1 - \lambda_2 - \lambda_3}{(\zeta - \lambda_1)(\zeta - \lambda_2)(\zeta - \lambda_3)},$$
$$\alpha_2(\zeta) = \frac{1}{\mathrm{III}_{\zeta\mathbf{I}-\mathbf{A}}} = \frac{1}{(\zeta - \lambda_1)(\zeta - \lambda_2)(\zeta - \lambda_3)}. \tag{7.60}$$

Inserting these expressions into (7.35) and (7.40) and considering separately cases of distinct and repeated eigenvalues, we obtain the following result [22].

Distinct eigenvalues: $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$,

$$\varphi_0 = \sum_{i=1}^{3} \frac{g(\lambda_i)\lambda_j\lambda_k}{D_i},$$
$$\varphi_1 = -\sum_{i=1}^{3} \frac{g(\lambda_i)(\lambda_j + \lambda_k)}{D_i},$$
$$\varphi_2 = \sum_{i=1}^{3} \frac{g(\lambda_i)}{D_i}, \tag{7.61}$$
$$\eta_{00} = \sum_{i=1}^{3} \frac{\lambda_j^2\lambda_k^2 g'(\lambda_i)}{D_i^2} - \sum_{\substack{i,j=1\\ i\neq j}}^{3} \frac{\lambda_i\lambda_j\lambda_k^2 [g(\lambda_i) - g(\lambda_j)]}{(\lambda_i - \lambda_j)^3 D_k},$$
$$\eta_{01} = \eta_{10} = -\sum_{i=1}^{3} \frac{(\lambda_j + \lambda_k)\lambda_j\lambda_k g'(\lambda_i)}{D_i^2}$$
$$+ \sum_{\substack{i,j=1\\ i\neq j}}^{3} \frac{(\lambda_j + \lambda_k)\lambda_i\lambda_k [g(\lambda_i) - g(\lambda_j)]}{(\lambda_i - \lambda_j)^3 D_k},$$
$$\eta_{02} = \eta_{20} = \sum_{i=1}^{3} \frac{\lambda_j\lambda_k g'(\lambda_i)}{D_i^2} - \sum_{\substack{i,j=1\\ i\neq j}}^{3} \frac{\lambda_i\lambda_k [g(\lambda_i) - g(\lambda_j)]}{(\lambda_i - \lambda_j)^3 D_k},$$
$$\eta_{11} = \sum_{i=1}^{3} \frac{(\lambda_j + \lambda_k)^2 g'(\lambda_i)}{D_i^2} - \sum_{\substack{i,j=1\\ i\neq j}}^{3} \frac{(\lambda_j + \lambda_k)(\lambda_i + \lambda_k)[g(\lambda_i) - g(\lambda_j)]}{(\lambda_i - \lambda_j)^3 D_k},$$

$$\eta_{12} = \eta_{21} = -\sum_{i=1}^{3} \frac{(\lambda_j + \lambda_k)\, g'(\lambda_i)}{D_i^2} + \sum_{\substack{i,j=1\\ i\neq j}}^{3} \frac{(\lambda_i + \lambda_k)\,[g(\lambda_i) - g(\lambda_j)]}{(\lambda_i - \lambda_j)^3 D_k},$$

$$\eta_{22} = \sum_{i=1}^{3} \frac{g'(\lambda_i)}{D_i^2} - \sum_{\substack{i,j=1\\ i\neq j}}^{3} \frac{g(\lambda_i) - g(\lambda_j)}{(\lambda_i - \lambda_j)^3 D_k}, \quad i \neq j \neq k \neq i, \tag{7.62}$$

where

$$D_i = (\lambda_i - \lambda_j)(\lambda_i - \lambda_k), \quad i \neq j \neq k \neq i = 1, 2, 3. \tag{7.63}$$

Double coalescence of eigenvalues: $\lambda_i \neq \lambda_j = \lambda_k = \lambda$, $j \neq k$,

$$\varphi_0 = \lambda \frac{\lambda g(\lambda_i) - \lambda_i g(\lambda)}{(\lambda_i - \lambda)^2} + \frac{\lambda_i g(\lambda)}{(\lambda_i - \lambda)} - \frac{\lambda \lambda_i g'(\lambda)}{(\lambda_i - \lambda)},$$

$$\varphi_1 = -2\lambda \frac{g(\lambda_i) - g(\lambda)}{(\lambda_i - \lambda)^2} + \frac{g'(\lambda)(\lambda_i + \lambda)}{(\lambda_i - \lambda)},$$

$$\varphi_2 = \frac{g(\lambda_i) - g(\lambda)}{(\lambda_i - \lambda)^2} - \frac{g'(\lambda)}{(\lambda_i - \lambda)}, \tag{7.64}$$

$$\eta_{00} = \frac{\left(2\lambda^2\lambda_i^2 - 6\lambda^3\lambda_i\right)[g(\lambda_i) - g(\lambda)]}{(\lambda_i - \lambda)^5}$$
$$+ \frac{\lambda^4 g'(\lambda_i) + \left(2\lambda^3\lambda_i + 4\lambda^2\lambda_i^2 - 4\lambda\lambda_i^3 + \lambda_i^4\right) g'(\lambda)}{(\lambda_i - \lambda)^4}$$
$$+ \frac{\left(2\lambda^2\lambda_i^2 - \lambda_i^3\lambda\right) g''(\lambda)}{(\lambda_i - \lambda)^3} + \frac{\lambda^2\lambda_i^2 g'''(\lambda)}{6(\lambda_i - \lambda)^2},$$

$$\eta_{01} = \eta_{10} = \frac{\left(3\lambda^3 + 7\lambda_i\lambda^2 - 2\lambda_i^2\lambda\right)[g(\lambda_i) - g(\lambda)]}{(\lambda_i - \lambda)^5}$$
$$- \frac{2\lambda^3 g'(\lambda_i) + \left(\lambda^3 + 7\lambda_i\lambda^2 - 2\lambda_i^2\lambda\right) g'(\lambda)}{(\lambda_i - \lambda)^4}$$
$$- \frac{\left(4\lambda^2\lambda_i + \lambda_i^2\lambda - \lambda_i^3\right) g''(\lambda)}{2(\lambda_i - \lambda)^3} - \frac{\lambda_i\lambda(\lambda_i + \lambda)\, g'''(\lambda)}{6(\lambda_i - \lambda)^2},$$

$$\eta_{02} = \eta_{20} = \frac{\left(\lambda_i^2 - 3\lambda_i\lambda - 2\lambda^2\right)\left[g\left(\lambda_i\right) - g\left(\lambda\right)\right]}{\left(\lambda_i - \lambda\right)^5}$$
$$+ \frac{\lambda^2 g'\left(\lambda_i\right) + \left(\lambda^2 + 3\lambda_i\lambda - \lambda_i^2\right) g'\left(\lambda\right)}{\left(\lambda_i - \lambda\right)^4}$$
$$+ \frac{\left(3\lambda\lambda_i - \lambda_i^2\right) g''\left(\lambda\right)}{2\left(\lambda_i - \lambda\right)^3} + \frac{\lambda_i\lambda g'''\left(\lambda\right)}{6\left(\lambda_i - \lambda\right)^2},$$

$$\eta_{11} = -4\frac{\lambda\left(\lambda_i + 3\lambda\right)\left[g\left(\lambda_i\right) - g\left(\lambda\right)\right]}{\left(\lambda_i - \lambda\right)^5} + 4\frac{\lambda^2 g'\left(\lambda_i\right) + \lambda\left(\lambda_i + 2\lambda\right) g'\left(\lambda\right)}{\left(\lambda_i - \lambda\right)^4}$$
$$+ \frac{2\lambda\left(\lambda_i + \lambda\right) g''\left(\lambda\right)}{\left(\lambda_i - \lambda\right)^3} + \frac{\left(\lambda_i + \lambda\right)^2 g'''\left(\lambda\right)}{6\left(\lambda_i - \lambda\right)^2},$$

$$\eta_{12} = \eta_{21} = \frac{\left(\lambda_i + 7\lambda\right)\left[g\left(\lambda_i\right) - g\left(\lambda\right)\right]}{\left(\lambda_i - \lambda\right)^5} - \frac{2\lambda g'\left(\lambda_i\right) + \left(\lambda_i + 5\lambda\right) g'\left(\lambda\right)}{\left(\lambda_i - \lambda\right)^4}$$
$$- \frac{\left(\lambda_i + 3\lambda\right) g''\left(\lambda\right)}{2\left(\lambda_i - \lambda\right)^3} - \frac{\left(\lambda_i + \lambda\right) g'''\left(\lambda\right)}{6\left(\lambda_i - \lambda\right)^2},$$
$$\eta_{22} = -4\frac{g\left(\lambda_i\right) - g\left(\lambda\right)}{\left(\lambda_i - \lambda\right)^5} + \frac{g'\left(\lambda_i\right) + 3g'\left(\lambda\right)}{\left(\lambda_i - \lambda\right)^4} + \frac{g''\left(\lambda\right)}{\left(\lambda_i - \lambda\right)^3} + \frac{g'''\left(\lambda\right)}{6\left(\lambda_i - \lambda\right)^2}. \quad (7.65)$$

Triple coalescence of eigenvalues: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$,

$$\varphi_0 = g\left(\lambda\right) - \lambda g'\left(\lambda\right) + \frac{1}{2}\lambda^2 g''\left(\lambda\right),$$
$$\varphi_1 = g'\left(\lambda\right) - \lambda g''\left(\lambda\right),$$
$$\varphi_2 = \frac{1}{2} g''\left(\lambda\right), \quad (7.66)$$
$$\eta_{00} = g'\left(\lambda\right) - \lambda g''\left(\lambda\right) + \frac{\lambda^2 g'''\left(\lambda\right)}{2} - \frac{\lambda^3 g^{IV}\left(\lambda\right)}{12} + \frac{\lambda^4 g^{V}\left(\lambda\right)}{120},$$
$$\eta_{01} = \eta_{10} = \frac{g''\left(\lambda\right)}{2} - \frac{\lambda g'''\left(\lambda\right)}{2} + \frac{\lambda^2 g^{IV}\left(\lambda\right)}{8} - \frac{\lambda^3 g^{V}\left(\lambda\right)}{60},$$
$$\eta_{02} = \eta_{20} = \frac{g'''\left(\lambda\right)}{6} - \frac{\lambda g^{IV}\left(\lambda\right)}{24} + \frac{\lambda^2 g^{V}\left(\lambda\right)}{120},$$
$$\eta_{11} = \frac{g'''\left(\lambda\right)}{6} - \frac{\lambda g^{IV}\left(\lambda\right)}{6} + \frac{\lambda^2 g^{V}\left(\lambda\right)}{30},$$
$$\eta_{12} = \eta_{21} = \frac{g^{IV}\left(\lambda\right)}{24} - \frac{\lambda g^{V}\left(\lambda\right)}{60},$$

$$\eta_{22} = \frac{g^{V}(\lambda)}{120}, \tag{7.67}$$

where superposed Roman numerals denote the order of the derivative.

Example. To illustrate the application of the above closed-form solution we consider the exponential function of the velocity gradient under simple shear. The velocity gradient is defined as the material time derivative of the deformation gradient by $\mathbf{L} = \dot{\mathbf{F}}$. Using the representation of $\mathbf{F}$ in the case of simple shear (4.23) we can write

$$\mathbf{L} = \mathrm{L}^{i}_{\cdot j} \boldsymbol{e}_i \otimes \boldsymbol{e}^j, \quad \text{where} \quad \left[\mathrm{L}^{i}_{\cdot j}\right] = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7.68}$$

We observe that $\mathbf{L}$ has a triple ($r_1 = 3$) zero eigenvalue

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda = 0. \tag{7.69}$$

This eigenvalue is, however, defect since it is associated with only two ($t_1 = 2$) linearly independent (right) eigenvectors

$$\boldsymbol{a}_1 = \boldsymbol{e}_1, \quad \boldsymbol{a}_2 = \boldsymbol{e}_3. \tag{7.70}$$

Therefore, $\mathbf{L}$ (7.68) is not diagonalizable and admits no spectral decomposition in the form (7.1). For this reason, isotropic functions of $\mathbf{L}$ as well as their derivative cannot be obtained on the basis of eigenprojections. Instead, we exploit the closed-form solution (7.31), (7.38) with the coefficients calculated for the case of triple coalescence of eigenvalues by (7.66) and (7.67). Thus, we can write

$$\exp(\mathbf{L}) = \exp(\lambda) \left[\left(\frac{1}{2}\lambda^2 - \lambda + 1 \right) \mathbf{I} + (1 - \lambda)\mathbf{L} + \frac{1}{2}\mathbf{L}^2 \right], \tag{7.71}$$

$$\begin{aligned} \exp(\mathbf{L})_{,\mathbf{L}} = \exp(\lambda) \Bigg[& \left(1 - \lambda + \frac{\lambda^2}{2} - \frac{\lambda^3}{12} + \frac{\lambda^4}{120} \right) \mathfrak{I} \\ & + \left(\frac{1}{2} - \frac{\lambda}{2} + \frac{\lambda^2}{8} - \frac{\lambda^3}{60} \right) (\mathbf{L} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{L}) \\ & + \left(\frac{1}{6} - \frac{\lambda}{6} + \frac{\lambda^2}{30} \right) \mathbf{L} \otimes \mathbf{L} \\ & + \left(\frac{1}{6} - \frac{\lambda}{24} + \frac{\lambda^2}{120} \right) \left(\mathbf{L}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{L}^2 \right) \\ & + \left(\frac{1}{24} - \frac{\lambda}{60} \right) \left(\mathbf{L}^2 \otimes \mathbf{L} + \mathbf{L} \otimes \mathbf{L}^2 \right) + \frac{1}{120}\mathbf{L}^2 \otimes \mathbf{L}^2 \Bigg]. \end{aligned} \tag{7.72}$$

On use of (7.69) this finally leads to the following expressions

$$\exp(\mathbf{L}) = \mathbf{I} + \mathbf{L} + \frac{1}{2}\mathbf{L}^2, \tag{7.73}$$

$$\exp(\mathbf{L})_{,\mathbf{L}} = \mathfrak{I} + \frac{1}{2}(\mathbf{L}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{L}) + \frac{1}{6}\mathbf{L}\otimes\mathbf{L} + \frac{1}{6}\left(\mathbf{L}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{L}^2\right)$$
$$+ \frac{1}{24}\left(\mathbf{L}^2\otimes\mathbf{L} + \mathbf{L}\otimes\mathbf{L}^2\right) + \frac{1}{120}\mathbf{L}^2\otimes\mathbf{L}^2. \tag{7.74}$$

Taking into account a special property of $\mathbf{L}$ (7.68):

$$\mathbf{L}^k = \mathbf{0}, \quad k = 2, 3, \ldots \tag{7.75}$$

the same results can also be obtained directly from the power series (1.109) and its derivative. By virtue of (6.117) the latter one can be given by

$$\exp(\mathbf{L})_{,\mathbf{L}} = \sum_{r=1}^{\infty}\frac{1}{r!}\sum_{k=0}^{r-1}\mathbf{L}^{r-1-k}\otimes\mathbf{L}^k. \tag{7.76}$$

For diagonalizable tensor functions the representations (7.31) and (7.38) can be simplified in the cases of repeated eigenvalues where the coefficients φ_p and η_{pq} are given by (7.64-7.67). To this end, we use the identities $\mathbf{A}^2 = (\lambda_i + \lambda)\mathbf{A} - \lambda_i\lambda\mathbf{I}$ for the case of double coalescence of eigenvalues $(\lambda_i \neq \lambda_j = \lambda_k = \lambda)$ and $\mathbf{A} = \lambda\mathbf{I}$, $\mathbf{A}^2 = \lambda^2\mathbf{I}$ for the case of triple coalescence of eigenvalues $(\lambda_1 = \lambda_2 = \lambda_3 = \lambda)$. Thus, we obtain the following result well-known for symmetric isotropic tensor functions [7].

Double coalescence of eigenvalues: $\lambda_i \neq \lambda_j = \lambda_k = \lambda$, $\mathbf{A}^2 = (\lambda_i + \lambda)\mathbf{A} - \lambda_i\lambda\mathbf{I}$,

$$\varphi_0 = \frac{\lambda_i g(\lambda) - \lambda g(\lambda_i)}{\lambda_i - \lambda}, \quad \varphi_1 = \frac{g(\lambda_i) - g(\lambda)}{\lambda_i - \lambda}, \quad \varphi_2 = 0, \tag{7.77}$$

$$\eta_{00} = -2\lambda_i\lambda\frac{g(\lambda_i) - g(\lambda)}{(\lambda_i - \lambda)^3} + \frac{\lambda^2 g'(\lambda_i) + \lambda_i^2 g'(\lambda)}{(\lambda_i - \lambda)^2},$$
$$\eta_{01} = \eta_{10} = (\lambda_i + \lambda)\frac{g(\lambda_i) - g(\lambda)}{(\lambda_i - \lambda)^3} - \frac{\lambda g'(\lambda_i) + \lambda_i g'(\lambda)}{(\lambda_i - \lambda)^2},$$
$$\eta_{11} = -2\frac{g(\lambda_i) - g(\lambda)}{(\lambda_i - \lambda)^3} + \frac{g'(\lambda_i) + g'(\lambda)}{(\lambda_i - \lambda)^2},$$
$$\eta_{02} = \eta_{20} = \eta_{12} = \eta_{21} = \eta_{22} = 0. \tag{7.78}$$

Triple coalescence of eigenvalues: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, $\mathbf{A} = \lambda\mathbf{I}$, $\mathbf{A}^2 = \lambda^2\mathbf{I}$,

$$\varphi_0 = g(\lambda), \quad \varphi_1 = \varphi_2 = 0, \tag{7.79}$$

$$\eta_{00} = g'(\lambda), \quad \eta_{01} = \eta_{10} = \eta_{11} = \eta_{02} = \eta_{20} = \eta_{12} = \eta_{21} = \eta_{22} = 0. \quad (7.80)$$

Finally, we specify the representations for eigenprojections (7.43) and their derivative (7.56) for three-dimensional space. The expressions for the functions ρ_{ip} (7.46) and υ_{ipq} (7.58) can be obtained from the representations for φ_p (7.61), (7.77), (7.79) and η_{pq} (7.62), (7.78), respectively. To this end, we set there $g(\lambda_i) = 1$, $g(\lambda_j) = g(\lambda_k) = g'(\lambda_i) = g'(\lambda_j) = g'(\lambda_k) = 0$. Accordingly, we obtain the following representations.

Distinct eigenvalues: $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$,

$$\rho_{i0} = \frac{\lambda_j \lambda_k}{D_i}, \quad \rho_{i1} = -\frac{\lambda_j + \lambda_k}{D_i}, \quad \rho_{i2} = \frac{1}{D_i}, \quad (7.81)$$

$$\upsilon_{i00} = -2\lambda_i\lambda_j\lambda_k \left[\frac{\lambda_k}{(\lambda_i - \lambda_j)^3 D_k} + \frac{\lambda_j}{(\lambda_i - \lambda_k)^3 D_j} \right],$$

$$\upsilon_{i01} = \upsilon_{i10} = \lambda_k \frac{\lambda_i(\lambda_j + \lambda_k) + \lambda_j(\lambda_i + \lambda_k)}{(\lambda_i - \lambda_j)^3 D_k} + \lambda_j \frac{\lambda_i(\lambda_j + \lambda_k) + \lambda_k(\lambda_i + \lambda_j)}{(\lambda_i - \lambda_k)^3 D_j},$$

$$\upsilon_{i02} = \upsilon_{i20} = -\lambda_k \frac{\lambda_i + \lambda_j}{(\lambda_i - \lambda_j)^3 D_k} - \lambda_j \frac{\lambda_i + \lambda_k}{(\lambda_i - \lambda_k)^3 D_j},$$

$$\upsilon_{i11} = -2(\lambda_j + \lambda_k) \left[\frac{\lambda_i + \lambda_k}{(\lambda_i - \lambda_j)^3 D_k} + \frac{\lambda_i + \lambda_j}{(\lambda_i - \lambda_k)^3 D_j} \right],$$

$$\upsilon_{i12} = \upsilon_{i21} = \frac{\lambda_i + \lambda_j + 2\lambda_k}{(\lambda_i - \lambda_j)^3 D_k} + \frac{\lambda_i + 2\lambda_j + \lambda_k}{(\lambda_i - \lambda_k)^3 D_j},$$

$$\upsilon_{i22} = -\frac{2}{(\lambda_i - \lambda_j)^3 D_k} - \frac{2}{(\lambda_i - \lambda_k)^3 D_j}, \quad i \neq j \neq k \neq i = 1, 2, 3. \quad (7.82)$$

Double coalescence of eigenvalues: $\lambda_i \neq \lambda_j = \lambda_k = \lambda$, $j \neq k$,

$$\rho_{i0} = -\frac{\lambda}{\lambda_i - \lambda}, \quad \rho_{i1} = \frac{1}{\lambda_i - \lambda}, \quad \rho_{i2} = 0, \quad (7.83)$$

$$\upsilon_{i00} = -\frac{2\lambda\lambda_i}{(\lambda_i - \lambda)^3}, \quad \upsilon_{i01} = \upsilon_{i10} = \frac{\lambda_i + \lambda}{(\lambda_i - \lambda)^3}, \quad \upsilon_{i11} = -\frac{2}{(\lambda_i - \lambda)^3},$$

$$\upsilon_{i02} = \upsilon_{i20} = \upsilon_{i12} = \upsilon_{i21} = \upsilon_{i22} = 0. \quad (7.84)$$

Triple coalescence of eigenvalues: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$,

$$\rho_{10} = 1, \quad \rho_{11} = \rho_{12} = 0. \quad (7.85)$$

The functions $\upsilon_{1pq}\,(p, q = 0, 1, 2)$ are in this case undefined since the only eigenprojection $\mathbf{P}_1$ is not differentiable.

7.5 Recurrent Calculation of Tensor Power Series and Their Derivatives

In numerical calculations with a limited number of digits the above presented closed-form solutions especially those ones for the derivative of analytic tensor functions can lead to inexact results if at least two eigenvalues of the tensor argument are close to each other but do not coincide (see [19]). In this case, a numerical calculation of the derivative of an analytic tensor function on the basis of the corresponding power series expansion might be advantageous provided this series converges very fast so that only a relatively small number of terms are sufficient in order to ensure a desired precision. This numerical calculation can be carried out by means of a recurrent procedure presented below.

The recurrent procedure is based on the sequential application of the Cayley-Hamilton equation (4.91). Accordingly, we can write for an arbitrary second-order tensor $\mathbf{A} \in \mathbf{L}\mathrm{in}^n$

$$\mathbf{A}^n = \sum_{k=0}^{n-1} (-1)^{n-k+1} \, \mathrm{I}_{\mathbf{A}}^{(n-k)} \mathbf{A}^k. \tag{7.86}$$

With the aid of this relation any non-negative integer power of $\mathbf{A}$ can be represented by

$$\mathbf{A}^r = \sum_{k=0}^{n-1} \omega_k^{(r)} \mathbf{A}^k, \quad r = 0, 1, 2, \ldots \tag{7.87}$$

Indeed, for $r \leq n$ one obtains directly from (7.86)

$$\omega_k^{(r)} = \delta_{rk}, \quad \omega_k^{(n)} = (-1)^{n-k+1} \, \mathrm{I}_{\mathbf{A}}^{(n-k)}, \quad r, k = 0, 1, \ldots, n-1. \tag{7.88}$$

Further powers of $\mathbf{A}$ can be expressed by composing (7.87) with $\mathbf{A}$ and representing $\mathbf{A}^n$ by (7.86) as

$$\begin{aligned}
\mathbf{A}^{r+1} &= \sum_{k=0}^{n-1} \omega_k^{(r)} \mathbf{A}^{k+1} = \sum_{k=1}^{n-1} \omega_{k-1}^{(r)} \mathbf{A}^k + \omega_{n-1}^{(r)} \mathbf{A}^n \\
&= \sum_{k=1}^{n-1} \omega_{k-1}^{(r)} \mathbf{A}^k + \omega_{n-1}^{(r)} \sum_{k=0}^{n-1} (-1)^{n-k-1} \, \mathrm{I}_{\mathbf{A}}^{(n-k)} \mathbf{A}^k.
\end{aligned}$$

Comparing with (7.87) we obtain the following recurrent relations (see also [38])

$$\begin{aligned}
\omega_0^{(r+1)} &= \omega_{n-1}^{(r)} (-1)^{n-1} \, \mathrm{I}_{\mathbf{A}}^{(n)}, \\
\omega_k^{(r+1)} &= \omega_{k-1}^{(r)} + \omega_{n-1}^{(r)} (-1)^{n-k-1} \, \mathrm{I}_{\mathbf{A}}^{(n-k)}, \quad k = 1, 2, \ldots, n-1.
\end{aligned} \tag{7.89}$$

With the aid of representation (7.87) the infinite power series (7.9) can thus be expressed by (7.31)

$$g(\mathbf{A}) = \sum_{p=0}^{n-1} \varphi_p \mathbf{A}^p, \tag{7.90}$$

where

$$\varphi_p = \sum_{r=0}^{\infty} a_r \omega_p^{(r)}. \tag{7.91}$$

Thus, the infinite power series (7.9) with the coefficients (7.13) results in the same representation as the corresponding analytic tensor function (7.15) provided the infinite series (7.91) converges.

Further, inserting (7.87) into (7.48) we obtain again the representation (7.38)

$$g(\mathbf{A})_{,\mathbf{A}} = \sum_{p,q=0}^{n-1} \eta_{pq} \mathbf{A}^p \otimes \mathbf{A}^q, \tag{7.92}$$

where

$$\eta_{pq} = \eta_{qp} = \sum_{r=1}^{\infty} a_r \sum_{k=0}^{r-1} \omega_p^{(r-1-k)} \omega_q^{(k)}, \quad p, q = 0, 1, \ldots, n-1. \tag{7.93}$$

The procedure computing the coefficients η_{pq} (7.93) can be simplified by means of the following recurrent identity (see also [30])

$$\begin{aligned}\sum_{k=0}^{r} \mathbf{A}^{r-k} \otimes \mathbf{A}^k &= \mathbf{A}^r \otimes \mathbf{I} + \left[\sum_{k=0}^{r-1} \mathbf{A}^{r-1-k} \otimes \mathbf{A}^k\right] \mathbf{A} \\ &= \mathbf{A} \left[\sum_{k=0}^{r-1} \mathbf{A}^{r-1-k} \otimes \mathbf{A}^k\right] + \mathbf{I} \otimes \mathbf{A}^r, \quad r = 1, 2 \ldots,\end{aligned} \tag{7.94}$$

where

$$\sum_{k=0}^{r-1} \mathbf{A}^{r-1-k} \otimes \mathbf{A}^k = \sum_{p,q=0}^{n-1} \xi_{pq}^{(r)} \mathbf{A}^p \otimes \mathbf{A}^q, \quad r = 1, 2 \ldots \tag{7.95}$$

Thus, we obtain

$$\eta_{pq} = \sum_{r=1}^{\infty} a_r \xi_{pq}^{(r)}, \tag{7.96}$$

where [19]

$$\begin{aligned}
&\xi_{pq}^{(1)} = \xi_{qp}^{(1)} = \omega_p^{(0)}\omega_q^{(0)} = \delta_{0p}\delta_{0q}, \quad p \le q;\ p,q = 0,1,\ldots,n-1,\\
&\xi_{00}^{(r)} = \xi_{0\,n-1}^{(r-1)}\omega_0^{(n)} + \omega_0^{(r-1)},\\
&\xi_{0q}^{(r)} = \xi_{q0}^{(r)} = \xi_{0\,q-1}^{(r-1)} + \xi_{0\,n-1}^{(r-1)}\omega_q^{(n)} = \xi_{n-1\,q}^{(r-1)}\omega_0^{(n)} + \omega_q^{(r-1)},\\
&\xi_{pq}^{(r)} = \xi_{qp}^{(r)} = \xi_{p\,q-1}^{(r-1)} + \xi_{p\,n-1}^{(r-1)}\omega_q^{(n)} = \xi_{p-1\,q}^{(r-1)} + \xi_{n-1\,q}^{(r-1)}\omega_p^{(n)},\\
&p \le q;\ p,q = 1,2,\ldots,n-1, \quad r = 2,3,\ldots
\end{aligned} \tag{7.97}$$

The calculation of coefficient series (7.89) and (7.97) can be finished as soon as for some r

$$\left|a_r\omega_p^{(r)}\right| \le \varepsilon \left|\sum_{t=0}^{r} a_t\omega_p^{(t)}\right|,$$

$$\left|a_r\xi_{pq}^{(r)}\right| \le \varepsilon \left|\sum_{t=1}^{r} a_t\xi_{pq}^{(t)}\right|, \quad p,q = 0,1,\ldots,n-1, \tag{7.98}$$

where $\varepsilon > 0$ denotes a precision parameter.

Example. To illustrate the application of the above recurrent procedure we consider again the exponential function of the velocity gradient under simple shear (7.68). In view of (7.69) we can write

$$\mathrm{I}_{\mathbf{L}}^{(1)} = \mathrm{I}_{\mathbf{L}}^{(2)} = \mathrm{I}_{\mathbf{L}}^{(3)} = 0. \tag{7.99}$$

With this result in hand the coefficients $\omega_p^{(r)}$ and $\xi_{pq}^{(r)}$ $(p,q = 0,1,2)$ appearing in the representation of the analytic tensor function (7.90), (7.91) and its derivative (7.92), (7.96) can easily be calculated by means of the above recurrent formulas (7.88), (7.89) and (7.97). The results of the calculation are summarized in Table 7.1.

Considering these results in (7.90), (7.91), (7.92) and (7.96) we obtain the representations (7.73) and (7.74). Note that the recurrent procedure delivers an exact result only in some special cases like this where the argument tensor is characterized by the property (7.75).

Exercises

7.1. Let $\mathbf{R}(\omega)$ be a proper orthogonal tensor describing a rotation about some axis $\boldsymbol{e} \in \mathbb{E}^3$ by the angle ω. Prove that $\mathbf{R}^a(\omega) = \mathbf{R}(a\omega)$ for any real number a.

7.2. Specify the right stretch tensor $\mathbf{U}$ $(7.5)_1$ for simple shear utilizing the results of Exercise 4.1.

7.3. Prove the properties of analytic tensor functions (7.21).

Table 7.1. Recurrent calculation of the coefficients $\omega_p^{(r)}$ and $\xi_{pq}^{(r)}$

r	$\omega_0^{(r)}$	$\omega_1^{(r)}$	$\omega_2^{(r)}$	$\xi_{00}^{(r)}$	$\xi_{01}^{(r)}$	$\xi_{02}^{(r)}$	$\xi_{11}^{(r)}$	$\xi_{12}^{(r)}$	$\xi_{22}^{(r)}$	a_r
0	1	0	0							1
1	0	1	0	1	0	0	0	0	0	1
2	0	0	1	0	1	0	0	0	0	1/2
3	0	0	0	0	0	1	1	0	0	1/6
4	0	0	0	0	0	0	0	1	0	1/24
5	0	0	0	0	0	0	0	0	1	1/120
6	0	0	0	0	0	0	0	0	0	1/720
$\sum\limits_{r=0} a_r \omega_p^{(r)}$	1	1	$\frac{1}{2}$							
$\sum\limits_{r=1} a_r \xi_{pq}^{(r)}$				1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{120}$	

7.4. Prove representation (7.54) for eigenprojections of diagonalizable second-order tensors.

7.5. Calculate eigenprojections and their derivatives for the tensor $\mathbf{A}$ (Exercise 4.8) using representations (7.81-7.85).

7.6. Calculate by means of the closed-form solution $\exp(\mathbf{A})$ and $\exp(\mathbf{A})_{,\mathbf{A}}$, where the tensor $\mathbf{A}$ is defined in Exercise 4.8. Compare the results for $\exp(\mathbf{A})$ with those of Exercise 4.9.

7.7. Compute $\exp(\mathbf{A})$ and $\exp(\mathbf{A})_{,\mathbf{A}}$ by means of the recurrent procedure with the precision parameter $\varepsilon = 1 \cdot 10^{-6}$, where the tensor $\mathbf{A}$ is defined in Exercise 4.8. Compare the results with those of Exercise 7.6.