8

Applications to Continuum Mechanics

8.1 Polar Decomposition of the Deformation Gradient

The deformation gradient $\mathbf{F}$ represents an invertible second-order tensor generally permitting a unique polar decomposition by

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{v}\mathbf{R}, \tag{8.1}$$

where $\mathbf{R}$ is an orthogonal tensor while $\mathbf{U}$ and $\mathbf{v}$ are symmetric tensors. In continuum mechanics, $\mathbf{R}$ is called rotation tensor while $\mathbf{U}$ and $\mathbf{v}$ are referred to as the right and left stretch tensor, respectively. The latter ones have already been introduced in Sect. 7.1 in the context of generalized strain measures.

In order to show that the polar decomposition (8.1) always exists and is unique we first consider the so-called right and left Cauchy-Green tensors respectively by

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}. \tag{8.2}$$

These tensors are symmetric and have principal traces in common. Indeed, in view of (1.144)

$$\mathrm{tr}\left(\mathbf{C}^k\right) = \mathrm{tr}\,\underbrace{\left(\mathbf{F}^{\mathrm{T}}\mathbf{F}\ldots\mathbf{F}^{\mathrm{T}}\mathbf{F}\right)}_{k\ \text{times}} = \mathrm{tr}\,\underbrace{\left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\ldots\mathbf{F}\mathbf{F}^{\mathrm{T}}\right)}_{k\ \text{times}} = \mathrm{tr}\left(\mathbf{b}^k\right). \tag{8.3}$$

For this reason, all scalar-valued isotropic functions of $\mathbf{C}$ and $\mathbf{b}$ such as principal invariants or eigenvalues coincide. Thus, we can write

$$\mathbf{C} = \sum_{i=1}^{s} \Lambda_i \mathbf{P}_i, \quad \mathbf{b} = \sum_{i=1}^{s} \Lambda_i \mathbf{p}_i, \tag{8.4}$$

where eigenvalues Λ_i are positive. Indeed, let $\boldsymbol{a}_i$ be a unit eigenvector associated with the eigenvalue Λ_i. Then, in view of (1.73), (1.99), (1.110) and by Theorem 1.8 one can write

$$\Lambda_i = \boldsymbol{a}_i \cdot (\Lambda_i \boldsymbol{a}_i) = \boldsymbol{a}_i \cdot (\mathbf{C}\boldsymbol{a}_i) = \boldsymbol{a}_i \cdot \left(\mathbf{F}^{\mathrm{T}}\mathbf{F}\boldsymbol{a}_i\right)$$
$$= \left(\boldsymbol{a}_i \mathbf{F}^{\mathrm{T}}\right) \cdot (\mathbf{F}\boldsymbol{a}_i) = (\mathbf{F}\boldsymbol{a}_i) \cdot (\mathbf{F}\boldsymbol{a}_i) > 0.$$

Thus, square roots of $\mathbf{C}$ and $\mathbf{b}$ are unique tensors defined by

$$\mathbf{U} = \sqrt{\mathbf{C}} = \sum_{i=1}^{s} \sqrt{\Lambda_i}\mathbf{P}_i, \quad \mathbf{v} = \sqrt{\mathbf{b}} = \sum_{i=1}^{s} \sqrt{\Lambda_i}\mathbf{p}_i. \tag{8.5}$$

Further, one can show that

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} \tag{8.6}$$

represents an orthogonal tensor. Indeed,

$$\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{F}\mathbf{U}^{-1}\mathbf{U}^{-1}\mathbf{F}^{\mathrm{T}} = \mathbf{F}\mathbf{U}^{-2}\mathbf{F}^{\mathrm{T}} = \mathbf{F}\mathbf{C}^{-1}\mathbf{F}^{\mathrm{T}}$$
$$= \mathbf{F}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F}\right)^{-1}\mathbf{F}^{\mathrm{T}} = \mathbf{F}\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}\mathbf{F}^{\mathrm{T}} = \mathbf{I}.$$

Thus, we can write taking (8.6) into account

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \left(\mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}}\right)\mathbf{R}. \tag{8.7}$$

The tensor

$$\mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}} = \mathbf{F}\mathbf{R}^{\mathrm{T}} \tag{8.8}$$

in (8.7) is symmetric due to symmetry of $\mathbf{U}$ $(8.5)_1$. Thus, one can write

$$\left(\mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}}\right)^2 = \left(\mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}}\right)\left(\mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}}\right)^{\mathrm{T}} = \left(\mathbf{F}\mathbf{R}^{\mathrm{T}}\right)\left(\mathbf{F}\mathbf{R}^{\mathrm{T}}\right)^{\mathrm{T}}$$
$$= \mathbf{F}\mathbf{R}^{\mathrm{T}}\mathbf{R}\mathbf{F}^{\mathrm{T}} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{b}. \tag{8.9}$$

In view of $(8.5)_2$ there exists only one real symmetric tensor whose square is $\mathbf{b}$. Hence,

$$\mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}} = \mathbf{v}, \tag{8.10}$$

which by virtue of (8.7) results in the polar decomposition (8.1).

8.2 Basis-Free Representations for the Stretch and Rotation Tensor

With the aid of the closed-form representations for analytic tensor functions discussed in Chap. 7 the stretch and rotation tensors can be expressed directly in terms of the deformation gradient and Cauchy-Green tensors without any reference to their eigenprojections. First, we deal with the stretch tensors

(8.5). Inserting in (7.61) $g(\Lambda_i) = \sqrt{\Lambda_i} = \lambda_i$ and keeping in mind (7.31) we write

$$\mathbf{U} = \varphi_0 \mathbf{I} + \varphi_1 \mathbf{C} + \varphi_2 \mathbf{C}^2, \quad \mathbf{v} = \varphi_0 \mathbf{I} + \varphi_1 \mathbf{b} + \varphi_2 \mathbf{b}^2, \tag{8.11}$$

where [44]

$$\begin{aligned}
\varphi_0 &= \frac{\lambda_1\lambda_2\lambda_3 (\lambda_1 + \lambda_2 + \lambda_3)}{(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}, \\
\varphi_1 &= \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1}{(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}, \\
\varphi_2 &= -\frac{1}{(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}.
\end{aligned} \tag{8.12}$$

These representations for φ_i are free of singularities and are therefore generally valid for the case of simple as well as repeated eigenvalues of $\mathbf{C}$ and $\mathbf{b}$.

The rotation tensor results from (8.6) where we can again write

$$\mathbf{U}^{-1} = \varsigma_0 \mathbf{I} + \varsigma_1 \mathbf{C} + \varsigma_2 \mathbf{C}^2. \tag{8.13}$$

The representations for ς_p $(p = 0, 1, 2)$ can be obtained either again by (7.61) where $g(\Lambda_i) = \Lambda_i^{-1/2} = \lambda_i^{-1}$ or by applying the Cayley-Hamilton equation (4.91) leading to

$$\begin{aligned}
\mathbf{U}^{-1} &= \mathrm{III}_{\mathbf{U}}^{-1} \left(\mathbf{U}^2 - \mathrm{I}_{\mathbf{U}} \mathbf{U} + \mathrm{II}_{\mathbf{U}} \mathbf{I} \right) \\
&= \mathrm{III}_{\mathbf{U}}^{-1} \left[(\mathrm{II}_{\mathbf{U}} - \varphi_0 \mathrm{I}_{\mathbf{U}}) \mathbf{I} + (1 - \varphi_1 \mathrm{I}_{\mathbf{U}}) \mathbf{C} - \varphi_2 \mathrm{I}_{\mathbf{U}} \mathbf{C}^2 \right],
\end{aligned} \tag{8.14}$$

where

$$\mathrm{I}_{\mathbf{U}} = \lambda_1 + \lambda_2 + \lambda_3, \quad \mathrm{II}_{\mathbf{U}} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad \mathrm{III}_{\mathbf{U}} = \lambda_1\lambda_2\lambda_3. \tag{8.15}$$

Both procedures yield the same representation (8.13) where

$$\begin{aligned}
\varsigma_0 &= \frac{\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1}{\lambda_1\lambda_2\lambda_3} - \frac{(\lambda_1 + \lambda_2 + \lambda_3)^2}{(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}, \\
\varsigma_1 &= \frac{1}{\lambda_1\lambda_2\lambda_3} - \frac{\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1\right)(\lambda_1 + \lambda_2 + \lambda_3)}{\lambda_1\lambda_2\lambda_3 (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}, \\
\varsigma_2 &= \frac{\lambda_1 + \lambda_2 + \lambda_3}{\lambda_1\lambda_2\lambda_3 (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1)}.
\end{aligned} \tag{8.16}$$

Thus, the rotation tensor (8.6) can be given by

$$\mathbf{R} = \mathbf{F} \left(\varsigma_0 \mathbf{I} + \varsigma_1 \mathbf{C} + \varsigma_2 \mathbf{C}^2 \right), \tag{8.17}$$

where the functions ς_i $(i = 0, 1, 2)$ are given by (8.16) in terms of the principal stretches $\lambda_i = \sqrt{\Lambda_i}$, while Λ_i $(i = 1, 2, 3)$ denote the eigenvalues of the right Cauchy-Green tensor $\mathbf{C}$ (8.2).

Example. Stretch and rotation tensor in the case of simple shear. In this loading case the right and left Cauchy-Green tensors take the form (see Exercise 4.1)

$$\mathbf{C} = \mathrm{C}^i_j \boldsymbol{e}_i \otimes \boldsymbol{e}^j, \quad \left[\mathrm{C}^i_j\right] = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{8.18}$$

$$\mathbf{b} = \mathrm{b}^i_j \boldsymbol{e}_i \otimes \boldsymbol{e}^j, \quad \left[\mathrm{b}^i_j\right] = \begin{bmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8.19}$$

with the eigenvalues

$$\Lambda_{1/2} = 1 + \frac{\gamma^2 \pm \sqrt{4\gamma^2 + \gamma^4}}{2} = \left(\frac{\gamma \pm \sqrt{4+\gamma^2}}{2}\right)^2, \quad \Lambda_3 = 1. \tag{8.20}$$

For the principal stretches we thus obtain

$$\lambda_{1/2} = \sqrt{\Lambda_{1/2}} = \frac{\sqrt{4+\gamma^2} \pm \gamma}{2}, \quad \lambda_3 = \sqrt{\Lambda_3} = 1. \tag{8.21}$$

The stretch tensors result from (8.11) where

$$\begin{aligned}
\varphi_0 &= \frac{1+\sqrt{\gamma^2+4}}{2\sqrt{\gamma^2+4}+\gamma^2+4}, \\
\varphi_1 &= \frac{1+\sqrt{\gamma^2+4}}{2+\sqrt{\gamma^2+4}}, \\
\varphi_2 &= -\frac{1}{2\sqrt{\gamma^2+4}+\gamma^2+4}.
\end{aligned} \tag{8.22}$$

This yields the following result (cf. Exercise 7.2)

$$\mathbf{U} = \mathrm{U}^i_j \boldsymbol{e}_i \otimes \boldsymbol{e}^j, \quad \left[\mathrm{U}^i_j\right] = \begin{bmatrix} \frac{2}{\sqrt{\gamma^2+4}} & \frac{\gamma}{\sqrt{\gamma^2+4}} & 0 \\ \frac{\gamma}{\sqrt{\gamma^2+4}} & \frac{\gamma^2+2}{\sqrt{\gamma^2+4}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{8.23}$$

$$\mathbf{v} = \mathrm{v}^i_j \boldsymbol{e}_i \otimes \boldsymbol{e}^j, \quad \left[\mathrm{v}^i_j\right] = \begin{bmatrix} \frac{\gamma^2+2}{\sqrt{\gamma^2+4}} & \frac{\gamma}{\sqrt{\gamma^2+4}} & 0 \\ \frac{\gamma}{\sqrt{\gamma^2+4}} & \frac{2}{\sqrt{\gamma^2+4}} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{8.24}$$

The rotation tensor can be calculated by (8.17) where

$$\varsigma_0 = \sqrt{\gamma^2+4} - \frac{1}{2\sqrt{\gamma^2+4}+\gamma^2+4},$$

$$\varsigma_1 = -\frac{3+\sqrt{\gamma^2+4}+\gamma^2}{2+\sqrt{\gamma^2+4}},$$

$$\varsigma_2 = \frac{1+\sqrt{\gamma^2+4}}{2\sqrt{\gamma^2+4}+\gamma^2+4}. \tag{8.25}$$

By this means we obtain

$$\mathbf{R} = \mathrm{R}^{i}_{\cdot j}\boldsymbol{e}_i \otimes \boldsymbol{e}^j, \quad \left[\mathrm{R}^{i}_{\cdot j}\right] = \begin{bmatrix} \dfrac{2}{\sqrt{\gamma^2+4}} & \dfrac{\gamma}{\sqrt{\gamma^2+4}} & 0 \\ -\dfrac{\gamma}{\sqrt{\gamma^2+4}} & \dfrac{2}{\sqrt{\gamma^2+4}} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{8.26}$$

8.3 The Derivative of the Stretch and Rotation Tensor with Respect to the Deformation Gradient

In continuum mechanics these derivatives are used for the evaluation of the rate of the stretch and rotation tensor. We begin with a very simple representation in terms of eigenprojections of the right and left Cauchy-Green tensors (8.2). Applying the chain rule of differentiation and using (6.125) we first write

$$\mathbf{U},_{\mathbf{F}} = \mathbf{C}^{1/2},_{\mathbf{C}} : \mathbf{C},_{\mathbf{F}} = \mathbf{C}^{1/2},_{\mathbf{C}} : \left[(\mathbf{I}\otimes\mathbf{F})^{\mathrm{t}} + \mathbf{F}^{\mathrm{T}}\otimes\mathbf{I}\right]. \tag{8.27}$$

Further, taking into account the spectral representation of $\mathbf{C}$ $(8.4)_1$ and keeping its symmetry in mind we obtain by virtue of (7.49-7.50)

$$\mathbf{C}^{1/2},_{\mathbf{C}} = \sum_{i,j=1}^{s} (\lambda_i+\lambda_j)^{-1}(\mathbf{P}_i\otimes\mathbf{P}_j)^{\mathrm{s}}. \tag{8.28}$$

Inserting this result into (8.27) delivers by means of (5.33), (5.47), $(5.54)_2$ and (5.55)

$$\mathbf{U},_{\mathbf{F}} = \sum_{i,j=1}^{s} (\lambda_i+\lambda_j)^{-1}\left[(\mathbf{P}_i\otimes\mathbf{F}\mathbf{P}_j)^{\mathrm{t}} + \mathbf{P}_i\mathbf{F}^{\mathrm{T}}\otimes\mathbf{P}_j\right]. \tag{8.29}$$

The same procedure applied to the left stretch tensor yields by virtue of (6.126)

$$\mathbf{v},_{\mathbf{F}} = \sum_{i,j=1}^{s} (\lambda_i + \lambda_j)^{-1} \left[\mathbf{p}_i \otimes \mathbf{F}^{\mathrm{T}} \mathbf{p}_j + (\mathbf{p}_i \mathbf{F} \otimes \mathbf{p}_j)^{\mathrm{t}} \right]. \tag{8.30}$$

Now, applying the product rule of differentiation (6.123) to (8.6) and taking (6.122) into account we write

$$\begin{aligned} \mathbf{R},_{\mathbf{F}} = \left(\mathbf{F}\mathbf{U}^{-1}\right),_{\mathbf{F}} &= \mathbf{I} \otimes \mathbf{U}^{-1} + \mathbf{F}\mathbf{U}^{-1},_{\mathbf{U}} : \mathbf{U},_{\mathbf{F}} \\ &= \mathbf{I} \otimes \mathbf{U}^{-1} - \mathbf{F} \left(\mathbf{U}^{-1} \otimes \mathbf{U}^{-1}\right)^{\mathrm{s}} : \mathbf{U},_{\mathbf{F}} . \end{aligned} \tag{8.31}$$

With the aid of (7.2) and (8.29) this finally leads to

$$\begin{aligned} \mathbf{R},_{\mathbf{F}} &= \mathbf{I} \otimes \left(\sum_{i=1}^{s} \lambda_i^{-1} \mathbf{P}_i \right) \\ &- \mathbf{F} \sum_{i,j=1}^{s} \left[(\lambda_i + \lambda_j) \lambda_i \lambda_j \right]^{-1} \left[(\mathbf{P}_i \otimes \mathbf{F}\mathbf{P}_j)^{\mathrm{t}} + \mathbf{P}_i \mathbf{F}^{\mathrm{T}} \otimes \mathbf{P}_j \right]. \end{aligned} \tag{8.32}$$

Note that the eigenprojections $\mathbf{P}_i$ and $\mathbf{p}_i$ $(i = 1, 2, \ldots, s)$ are uniquely defined by the Sylvester formula (4.54) or its alternative form (7.43) in terms of $\mathbf{C}$ and $\mathbf{b}$, respectively. The functions ρ_{ip} appearing in (7.43) are, in turn, expressed in the unique form by (7.81), (7.83) and (7.85) in terms of the eigenvalues $\Lambda_i = \lambda_i^2$ $(i = 1, 2, \ldots, s)$.

In order to avoid the direct reference to the eigenprojections one can obtain the so-called basis-free solutions for $\mathbf{U},_{\mathbf{F}}$, $\mathbf{v},_{\mathbf{F}}$ and $\mathbf{R},_{\mathbf{F}}$ (see, e.g., [8], [13], [17], [37], [46], [48]). As a rule, they are given in terms of the stretch and rotation tensors themselves and require therefore either the explicit polar decomposition of the deformation gradient or a closed-form representation for $\mathbf{U}$, $\mathbf{v}$ and $\mathbf{R}$ like (8.11) and (8.17). In the following we present the basis-free solutions for $\mathbf{U},_{\mathbf{F}}$, $\mathbf{v},_{\mathbf{F}}$ and $\mathbf{R},_{\mathbf{F}}$ in terms of the Cauchy-Green tensors $\mathbf{C}$ and $\mathbf{b}$ (8.2) and the principal stretches $\lambda_i = \sqrt{\Lambda_i}$ $(i = 1, 2, \ldots, s)$. To this end, we apply the representation (7.38) for the derivative of the square root. Thus, we obtain instead of (8.28)

$$\mathbf{C}^{1/2},_{\mathbf{C}} = \sum_{p,q=0}^{2} \eta_{pq} \left(\mathbf{C}^p \otimes \mathbf{C}^q\right)^{\mathrm{s}}, \quad \mathbf{b}^{1/2},_{\mathbf{b}} = \sum_{p,q=0}^{2} \eta_{pq} \left(\mathbf{b}^p \otimes \mathbf{b}^q\right)^{\mathrm{s}}, \tag{8.33}$$

where the functions η_{pq} result from (7.62) by setting again $g(\Lambda_i) = \sqrt{\Lambda_i} = \lambda_i$. This leads to the following expressions (cf. [17])

$$\begin{aligned} \eta_{00} = \Delta^{-1} &\left[\mathrm{I}_{\mathbf{U}}^5 \mathrm{III}_{\mathbf{U}}^2 - \mathrm{I}_{\mathbf{U}}^4 \mathrm{II}_{\mathbf{U}}^2 \mathrm{III}_{\mathbf{U}} + \mathrm{I}_{\mathbf{U}}^3 \mathrm{II}_{\mathbf{U}}^4 \right. \\ &\left. - \mathrm{I}_{\mathbf{U}}^2 \mathrm{III}_{\mathbf{U}} \left(3\mathrm{II}_{\mathbf{U}}^3 - 2\mathrm{III}_{\mathbf{U}}^2 \right) + 3\mathrm{I}_{\mathbf{U}} \mathrm{II}_{\mathbf{U}}^2 \mathrm{III}_{\mathbf{U}}^2 - \mathrm{II}_{\mathbf{U}} \mathrm{III}_{\mathbf{U}}^3 \right], \end{aligned}$$

$$\begin{aligned} \eta_{01} = \eta_{10} = \Delta^{-1} &\left[\mathrm{I}_{\mathbf{U}}^6 \mathrm{III}_{\mathbf{U}} - \mathrm{I}_{\mathbf{U}}^5 \mathrm{II}_{\mathbf{U}}^2 - \mathrm{I}_{\mathbf{U}}^4 \mathrm{II}_{\mathbf{U}} \mathrm{III}_{\mathbf{U}} \right. \\ &\left. + 2\mathrm{I}_{\mathbf{U}}^3 \left(\mathrm{II}_{\mathbf{U}}^3 + \mathrm{III}_{\mathbf{U}}^2 \right) - 4\mathrm{I}_{\mathbf{U}}^2 \mathrm{II}_{\mathbf{U}}^2 \mathrm{III}_{\mathbf{U}} + 2\mathrm{I}_{\mathbf{U}} \mathrm{II}_{\mathbf{U}} \mathrm{III}_{\mathbf{U}}^2 - \mathrm{III}_{\mathbf{U}}^3 \right], \end{aligned}$$

$$\eta_{02} = \eta_{20} = \Delta^{-1}\left[-\mathrm{I}_{\mathbf{U}}^4\mathrm{III}_{\mathbf{U}} + \mathrm{I}_{\mathbf{U}}^3\mathrm{II}_{\mathbf{U}}^2 - \mathrm{I}_{\mathbf{U}}^2\mathrm{II}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}} - \mathrm{I}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}}^2\right],$$

$$\eta_{11} = \Delta^{-1}\left[\mathrm{I}_{\mathbf{U}}^7 - 4\mathrm{I}_{\mathbf{U}}^5\mathrm{II}_{\mathbf{U}} + 3\mathrm{I}_{\mathbf{U}}^4\mathrm{III}_{\mathbf{U}}\right.$$

$$\left. + 4\mathrm{I}_{\mathbf{U}}^3\mathrm{II}_{\mathbf{U}}^2 - 6\mathrm{I}_{\mathbf{U}}^2\mathrm{II}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}} + \mathrm{I}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}}^2 + \mathrm{II}_{\mathbf{U}}^2\mathrm{III}_{\mathbf{U}}\right],$$

$$\eta_{12} = \eta_{21} = \Delta^{-1}\left[-\mathrm{I}_{\mathbf{U}}^5 + 2\mathrm{I}_{\mathbf{U}}^3\mathrm{II}_{\mathbf{U}} - 2\mathrm{I}_{\mathbf{U}}^2\mathrm{III}_{\mathbf{U}} + \mathrm{II}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}}\right],$$

$$\eta_{22} = \Delta^{-1}\left[\mathrm{I}_{\mathbf{U}}^3 + \mathrm{III}_{\mathbf{U}}\right], \tag{8.34}$$

where

$$\Delta = 2\left(\mathrm{I}_{\mathbf{U}}\mathrm{II}_{\mathbf{U}} - \mathrm{III}_{\mathbf{U}}\right)^3\mathrm{III}_{\mathbf{U}} \tag{8.35}$$

and the principal invariants $\mathrm{I}_{\mathbf{U}}$, $\mathrm{II}_{\mathbf{U}}$ and $\mathrm{III}_{\mathbf{U}}$ are given by (8.15).

Finally, substitution of (8.33) into (8.27) yields

$$\mathbf{U}{,_{\mathbf{F}}} = \sum_{p,q=0}^{2} \eta_{pq}\left[\left(\mathbf{C}^p \otimes \mathbf{F}\mathbf{C}^q\right)^{\mathrm{t}} + \mathbf{C}^p\mathbf{F}^{\mathrm{T}} \otimes \mathbf{C}^q\right]. \tag{8.36}$$

Similar we can also write

$$\mathbf{v}{,_{\mathbf{F}}} = \sum_{p,q=0}^{2} \eta_{pq}\left[\mathbf{b}^p \otimes \mathbf{F}^{\mathrm{T}}\mathbf{b}^q + \left(\mathbf{b}^p\mathbf{F} \otimes \mathbf{b}^q\right)^{\mathrm{t}}\right]. \tag{8.37}$$

Inserting further (8.13) and (8.36) into (8.31) we get

$$\mathbf{R}{,_{\mathbf{F}}} = \mathbf{I} \otimes \sum_{p=0}^{2} \varsigma_p\mathbf{C}^p$$

$$- \mathbf{F}\sum_{p,q,r,t=0}^{2} \varsigma_r\varsigma_t\eta_{pq}\left[\left(\mathbf{C}^{p+r} \otimes \mathbf{F}\mathbf{C}^{q+t}\right)^{\mathrm{t}} + \mathbf{C}^{p+r}\mathbf{F}^{\mathrm{T}} \otimes \mathbf{C}^{q+t}\right], \tag{8.38}$$

where ς_p and η_{pq} $(p, q = 0, 1, 2)$ are given by (8.16) and (8.34), respectively. The third and fourth powers of $\mathbf{C}$ in (8.38) can be expressed by means of the Cayley-Hamilton equation (4.91):

$$\mathbf{C}^3 - \mathrm{I}_{\mathbf{C}}\mathbf{C}^2 + \mathrm{II}_{\mathbf{C}}\mathbf{C} - \mathrm{III}_{\mathbf{C}}\mathbf{I} = \mathbf{0}. \tag{8.39}$$

Composing both sides with $\mathbf{C}$ we can also write

$$\mathbf{C}^4 - \mathrm{I}_{\mathbf{C}}\mathbf{C}^3 + \mathrm{II}_{\mathbf{C}}\mathbf{C}^2 - \mathrm{III}_{\mathbf{C}}\mathbf{C} = \mathbf{0}. \tag{8.40}$$

Thus,

$$\mathbf{C}^3 = \mathrm{I}_{\mathbf{C}}\mathbf{C}^2 - \mathrm{II}_{\mathbf{C}}\mathbf{C} + \mathrm{III}_{\mathbf{C}}\mathbf{I},$$

$$\mathbf{C}^4 = \left(\mathrm{I}_{\mathbf{C}}^2 - \mathrm{II}_{\mathbf{C}}\right)\mathbf{C}^2 + \left(\mathrm{III}_{\mathbf{C}} - \mathrm{I}_{\mathbf{C}}\mathrm{II}_{\mathbf{C}}\right)\mathbf{C} + \mathrm{I}_{\mathbf{C}}\mathrm{III}_{\mathbf{C}}\mathbf{I}. \tag{8.41}$$

Considering these expressions in (8.38) and taking into account that (see, e.g., [43])

$$\mathrm{I}_{\mathbf{C}} = \mathrm{I}_{\mathbf{U}}^2 - 2\mathrm{II}_{\mathbf{U}}, \quad \mathrm{II}_{\mathbf{C}} = \mathrm{II}_{\mathbf{U}}^2 - 2\mathrm{I}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}}, \quad \mathrm{III}_{\mathbf{C}} = \mathrm{III}_{\mathbf{U}}^2 \tag{8.42}$$

we finally obtain

$$\mathbf{R},_{\mathbf{F}} = \mathbf{I} \otimes \sum_{p=0}^{2} \varsigma_p \mathbf{C}^p + \mathbf{F} \sum_{p,q=0}^{2} \mu_{pq} \left[\left(\mathbf{C}^p \otimes \mathbf{F}\mathbf{C}^q\right)^{\mathrm{t}} + \mathbf{C}^p\mathbf{F}^{\mathrm{T}} \otimes \mathbf{C}^q\right], \tag{8.43}$$

where

$$\begin{aligned}
\mu_{00} = \Upsilon^{-1} &\left[\mathrm{I}_{\mathbf{U}}^6\mathrm{III}_{\mathbf{U}}^3 + 2\mathrm{I}_{\mathbf{U}}^5\mathrm{II}_{\mathbf{U}}^2\mathrm{III}_{\mathbf{U}}^2 - 3\mathrm{I}_{\mathbf{U}}^4\mathrm{II}_{\mathbf{U}}^4\mathrm{III}_{\mathbf{U}} - 7\mathrm{I}_{\mathbf{U}}^4\mathrm{II}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}}^3\right.\\
&+\mathrm{I}_{\mathbf{U}}^3\mathrm{II}_{\mathbf{U}}^6 + 8\mathrm{I}_{\mathbf{U}}^3\mathrm{II}_{\mathbf{U}}^3\mathrm{III}_{\mathbf{U}}^2 + 6\mathrm{I}_{\mathbf{U}}^3\mathrm{III}_{\mathbf{U}}^4 - 3\mathrm{I}_{\mathbf{U}}^2\mathrm{II}_{\mathbf{U}}^5\mathrm{III}_{\mathbf{U}}\\
&\left.-6\mathrm{I}_{\mathbf{U}}^2\mathrm{II}_{\mathbf{U}}^2\mathrm{III}_{\mathbf{U}}^3 + 3\mathrm{I}_{\mathbf{U}}\mathrm{II}_{\mathbf{U}}^4\mathrm{III}_{\mathbf{U}}^2 - \mathrm{II}_{\mathbf{U}}^3\mathrm{III}_{\mathbf{U}}^3 + \mathrm{III}_{\mathbf{U}}^5\right],\\
\mu_{01} = \mu_{10} = \Upsilon^{-1} &\left[\mathrm{I}_{\mathbf{U}}^7\mathrm{III}_{\mathbf{U}}^2 + \mathrm{I}_{\mathbf{U}}^6\mathrm{II}_{\mathbf{U}}^2\mathrm{III}_{\mathbf{U}} - \mathrm{I}_{\mathbf{U}}^5\mathrm{II}_{\mathbf{U}}^4 - 6\mathrm{I}_{\mathbf{U}}^5\mathrm{II}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}}^2 + \mathrm{I}_{\mathbf{U}}^4\mathrm{II}_{\mathbf{U}}^3\mathrm{III}_{\mathbf{U}}\right.\\
&+5\mathrm{I}_{\mathbf{U}}^4\mathrm{III}_{\mathbf{U}}^3 + 2\mathrm{I}_{\mathbf{U}}^3\mathrm{II}_{\mathbf{U}}^5 + 4\mathrm{I}_{\mathbf{U}}^3\mathrm{II}_{\mathbf{U}}^2\mathrm{III}_{\mathbf{U}}^2 - 6\mathrm{I}_{\mathbf{U}}^2\mathrm{II}_{\mathbf{U}}^4\mathrm{III}_{\mathbf{U}}\\
&\left.-6\mathrm{I}_{\mathbf{U}}^2\mathrm{II}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}}^3 + 6\mathrm{I}_{\mathbf{U}}\mathrm{II}_{\mathbf{U}}^3\mathrm{III}_{\mathbf{U}}^2 + \mathrm{I}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}}^4 - 2\mathrm{II}_{\mathbf{U}}^2\mathrm{III}_{\mathbf{U}}^3\right],\\
\mu_{02} = \mu_{20} = -\Upsilon^{-1} &\left[\mathrm{I}_{\mathbf{U}}^5\mathrm{III}_{\mathbf{U}}^2 + \mathrm{I}_{\mathbf{U}}^4\mathrm{II}_{\mathbf{U}}^2\mathrm{III}_{\mathbf{U}} - \mathrm{I}_{\mathbf{U}}^3\mathrm{II}_{\mathbf{U}}^4 - 4\mathrm{I}_{\mathbf{U}}^3\mathrm{II}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}}^2\right.\\
&\left.+3\mathrm{I}_{\mathbf{U}}^2\mathrm{II}_{\mathbf{U}}^3\mathrm{III}_{\mathbf{U}} + 4\mathrm{I}_{\mathbf{U}}^2\mathrm{III}_{\mathbf{U}}^3 - 3\mathrm{I}_{\mathbf{U}}\mathrm{II}_{\mathbf{U}}^2\mathrm{III}_{\mathbf{U}}^2 + \mathrm{II}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}}^3\right],\\
\mu_{11} = \Upsilon^{-1} &\left[\mathrm{I}_{\mathbf{U}}^8\mathrm{III}_{\mathbf{U}} + \mathrm{I}_{\mathbf{U}}^7\mathrm{II}_{\mathbf{U}}^2 - 7\mathrm{I}_{\mathbf{U}}^6\mathrm{II}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}} - 4\mathrm{I}_{\mathbf{U}}^5\mathrm{II}_{\mathbf{U}}^3\right.\\
&+5\mathrm{I}_{\mathbf{U}}^5\mathrm{III}_{\mathbf{U}}^2 + 16\mathrm{I}_{\mathbf{U}}^4\mathrm{II}_{\mathbf{U}}^2\mathrm{III}_{\mathbf{U}} + 4\mathrm{I}_{\mathbf{U}}^3\mathrm{II}_{\mathbf{U}}^4 - 16\mathrm{I}_{\mathbf{U}}^3\mathrm{II}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}}^2\\
&\left.-12\mathrm{I}_{\mathbf{U}}^2\mathrm{II}_{\mathbf{U}}^3\mathrm{III}_{\mathbf{U}} + 3\mathrm{I}_{\mathbf{U}}^2\mathrm{III}_{\mathbf{U}}^3 + 12\mathrm{I}_{\mathbf{U}}\mathrm{II}_{\mathbf{U}}^2\mathrm{III}_{\mathbf{U}}^2 - 3\mathrm{II}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}}^3\right],\\
\mu_{12} = \mu_{21} = -\Upsilon^{-1} &\left[\mathrm{I}_{\mathbf{U}}^6\mathrm{III}_{\mathbf{U}} + \mathrm{I}_{\mathbf{U}}^5\mathrm{II}_{\mathbf{U}}^2 - 5\mathrm{I}_{\mathbf{U}}^4\mathrm{II}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}} - 2\mathrm{I}_{\mathbf{U}}^3\mathrm{II}_{\mathbf{U}}^3\right.\\
&\left.+4\mathrm{I}_{\mathbf{U}}^3\mathrm{III}_{\mathbf{U}}^2 + 6\mathrm{I}_{\mathbf{U}}^2\mathrm{II}_{\mathbf{U}}^2\mathrm{III}_{\mathbf{U}} - 6\mathrm{I}_{\mathbf{U}}\mathrm{II}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}}^2 + \mathrm{III}_{\mathbf{U}}^3\right],\\
\mu_{22} = \Upsilon^{-1}\mathrm{I}_{\mathbf{U}} &\left[\mathrm{I}_{\mathbf{U}}^3\mathrm{III}_{\mathbf{U}} + \mathrm{I}_{\mathbf{U}}^2\mathrm{II}_{\mathbf{U}}^2 - 3\mathrm{I}_{\mathbf{U}}\mathrm{II}_{\mathbf{U}}\mathrm{III}_{\mathbf{U}} + 3\mathrm{III}_{\mathbf{U}}^2\right]
\end{aligned} \tag{8.44}$$

and

$$\Upsilon = -2\left(\mathrm{I}_{\mathbf{U}}\mathrm{II}_{\mathbf{U}} - \mathrm{III}_{\mathbf{U}}\right)^3\mathrm{III}_{\mathbf{U}}^3, \tag{8.45}$$

while the principal invariants $\mathrm{I}_{\mathbf{U}}$, $\mathrm{II}_{\mathbf{U}}$ and $\mathrm{III}_{\mathbf{U}}$ are given by (8.15).

The same result for $\mathbf{R},_{\mathbf{F}}$ also follows from

$$\mathbf{R},_{\mathbf{F}} = \left(\mathbf{F}\mathbf{U}^{-1}\right),_{\mathbf{F}} = \mathbf{I} \otimes \mathbf{U}^{-1} + \mathbf{F}\mathbf{U}^{-1},_{\mathbf{C}} : \mathbf{C},_{\mathbf{F}} \tag{8.46}$$

by applying for $\mathbf{U}^{-1},_{\mathbf{C}}$ (7.38) and (7.62) where we set $g(\Lambda_i) = (\Lambda_i)^{-1/2} = \lambda_i^{-1}$. Indeed, this yields

$$\mathbf{C}^{-1/2},_{\mathbf{C}} = \mathbf{U}^{-1},_{\mathbf{C}} = \sum_{p,q=0}^{2} \mu_{pq} \left(\mathbf{C}^p \otimes \mathbf{C}^q\right)^{\mathrm{s}}, \tag{8.47}$$

where μ_{pq} $(p, q = 0, 1, 2)$ are given by (8.44).

8.4 Time Rate of Generalized Strains

Applying the chain rule of differentiation we first write

$$\dot{\mathbf{E}} = \mathbf{E},_{\mathbf{C}} : \dot{\mathbf{C}}, \tag{8.48}$$

where the superposed dot denotes the material time derivative. The derivative $\mathbf{E},_{\mathbf{C}}$ can be expressed in a simple form in terms of the eigenprojections of $\mathbf{E}$ and $\mathbf{C}$. To this end, we apply (7.49-7.50) taking (7.18) and (8.5) into account which yields

$$\mathbf{E},_{\mathbf{C}} = \sum_{i,j=1}^{s} f_{ij} \left(\mathbf{P}_i \otimes \mathbf{P}_j\right)^{\mathrm{s}}, \tag{8.49}$$

where

$$f_{ij} = \begin{cases} \dfrac{f'(\lambda_i)}{2\lambda_i} & \text{if } i = j, \\[2ex] \dfrac{f(\lambda_i) - f(\lambda_j)}{\lambda_i^2 - \lambda_j^2} & \text{if } i \neq j. \end{cases} \tag{8.50}$$

A basis-free representation for $\mathbf{E},_{\mathbf{C}}$ can be obtained either from (8.49) by expressing the eigenprojections by (7.43) with (7.81), (7.83) and (7.85) or directly by using the closed-form solution (7.38) with (7.62), (7.78) and (7.80). Both procedures lead to the same result as follows (cf. [21], [47]).

$$\mathbf{E},_{\mathbf{C}} = \sum_{p,q=0}^{2} \eta_{pq} \left(\mathbf{C}^p \otimes \mathbf{C}^q\right)^{\mathrm{s}}. \tag{8.51}$$

Distinct eigenvalues: $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$,

$$\eta_{00} = \sum_{i=1}^{3} \frac{\lambda_j^4 \lambda_k^4 f'(\lambda_i)}{2\lambda_i \Delta_i^2} - \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{\lambda_i^2 \lambda_j^2 \lambda_k^4 \left[f(\lambda_i) - f(\lambda_j)\right]}{\left(\lambda_i^2 - \lambda_j^2\right)^3 \Delta_k},$$

$$\eta_{01} = \eta_{10} = -\sum_{i=1}^{3} \frac{\left(\lambda_j^2 + \lambda_k^2\right) \lambda_j^2 \lambda_k^2 f'(\lambda_i)}{2\lambda_i \Delta_i^2} + \sum_{\substack{i,j=1\\ i \neq j}}^{3} \frac{\left(\lambda_j^2 + \lambda_k^2\right) \lambda_i^2 \lambda_k^2 \left[f(\lambda_i) - f(\lambda_j)\right]}{\left(\lambda_i^2 - \lambda_j^2\right)^3 \Delta_k},$$

$$\eta_{02} = \eta_{20} = \sum_{i=1}^{3} \frac{\lambda_j^2 \lambda_k^2 f'(\lambda_i)}{2\lambda_i \Delta_i^2} - \sum_{\substack{i,j=1\\ i \neq j}}^{3} \frac{\lambda_i^2 \lambda_k^2 \left[f(\lambda_i) - f(\lambda_j)\right]}{\left(\lambda_i^2 - \lambda_j^2\right)^3 \Delta_k},$$

$$\eta_{11} = \sum_{i=1}^{3} \frac{\left(\lambda_j^2 + \lambda_k^2\right)^2 f'(\lambda_i)}{2\lambda_i \Delta_i^2} - \sum_{\substack{i,j=1\\ i \neq j}}^{3} \frac{\left(\lambda_j^2 + \lambda_k^2\right)\left(\lambda_i^2 + \lambda_k^2\right) \left[f(\lambda_i) - f(\lambda_j)\right]}{\left(\lambda_i^2 - \lambda_j^2\right)^3 \Delta_k},$$

$$\eta_{12} = \eta_{21} = -\sum_{i=1}^{3} \frac{\left(\lambda_j^2 + \lambda_k^2\right) f'(\lambda_i)}{2\lambda_i \Delta_i^2} + \sum_{\substack{i,j=1\\ i \neq j}}^{3} \frac{\left(\lambda_i^2 + \lambda_k^2\right) \left[f(\lambda_i) - f(\lambda_j)\right]}{\left(\lambda_i^2 - \lambda_j^2\right)^3 \Delta_k},$$

$$\eta_{22} = \sum_{i=1}^{3} \frac{f'(\lambda_i)}{2\lambda_i \Delta_i^2} - \sum_{\substack{i,j=1\\ i \neq j}}^{3} \frac{f(\lambda_i) - f(\lambda_j)}{\left(\lambda_i^2 - \lambda_j^2\right)^3 \Delta_k}, \quad i \neq j \neq k \neq i, \tag{8.52}$$

with

$$\Delta_i = \left(\lambda_i^2 - \lambda_j^2\right)\left(\lambda_i^2 - \lambda_k^2\right), \quad i \neq j \neq k \neq i = 1, 2, 3. \tag{8.53}$$

Double coalescence of eigenvalues: $\lambda_i \neq \lambda_j = \lambda_k = \lambda$,

$$\eta_{00} = -2\lambda_i^2 \lambda^2 \frac{f(\lambda_i) - f(\lambda)}{\left(\lambda_i^2 - \lambda^2\right)^3} + \frac{\lambda^5 f'(\lambda_i) + \lambda_i^5 f'(\lambda)}{2\lambda_i \lambda \left(\lambda_i^2 - \lambda^2\right)^2},$$

$$\eta_{01} = \eta_{10} = \left(\lambda_i^2 + \lambda^2\right) \frac{f(\lambda_i) - f(\lambda)}{\left(\lambda_i^2 - \lambda^2\right)^3} - \frac{\lambda^3 f'(\lambda_i) + \lambda_i^3 f'(\lambda)}{2\lambda_i \lambda \left(\lambda_i^2 - \lambda^2\right)^2},$$

$$\eta_{11} = -2 \frac{f(\lambda_i) - f(\lambda)}{\left(\lambda_i^2 - \lambda^2\right)^3} + \frac{\lambda f'(\lambda_i) + \lambda_i f'(\lambda)}{2\lambda_i \lambda \left(\lambda_i^2 - \lambda^2\right)^2},$$

$$\eta_{02} = \eta_{20} = \eta_{12} = \eta_{21} = \eta_{22} = 0. \tag{8.54}$$

Triple coalescence of eigenvalues: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$,

$$\eta_{00} = \frac{f'(\lambda)}{2\lambda}, \quad \eta_{01} = \eta_{10} = \eta_{11} = \eta_{02} = \eta_{20} = \eta_{12} = \eta_{21} = \eta_{22} = 0. \tag{8.55}$$

8.5 Stress Conjugate to a Generalized Strain

Let $\mathbf{E}$ be an arbitrary Lagrangian strain $(7.6)_1$. Assume existence of the so-called strain energy function $\psi(\mathbf{E})$ differentiable with respect to $\mathbf{E}$. The symmetric tensor

$$\mathbf{T} = \psi(\mathbf{E})_{,\mathbf{E}} \tag{8.56}$$

is referred to as stress conjugate to $\mathbf{E}$. With the aid of the chain rule it can be represented by

$$\mathbf{T} = \psi(\mathbf{E})_{,\mathbf{C}} : \mathbf{C}_{,\mathbf{E}} = \frac{1}{2}\mathbf{S} : \mathbf{C}_{,\mathbf{E}}, \tag{8.57}$$

where $\mathbf{S} = 2\psi(\mathbf{E})_{,\mathbf{C}}$ denotes the second Piola-Kirchhoff stress tensor. The latter one is defined in terms of the Cauchy stress $\boldsymbol{\sigma}$ by (see, e.g., [45])

$$\mathbf{S} = \det(\mathbf{F})\,\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}. \tag{8.58}$$

Using (8.56) and (7.7) one can also write

$$\dot{\psi} = \mathbf{T} : \dot{\mathbf{E}} = \mathbf{S} : \frac{1}{2}\dot{\mathbf{C}} = \mathbf{S} : \dot{\mathbf{E}}^{(2)}. \tag{8.59}$$

The fourth-order tensor $\mathbf{C}_{,\mathbf{E}}$ appearing in (8.57) can be expressed in terms of the right Cauchy-Green tensor $\mathbf{C}$ by means of the relation

$$\mathfrak{I}^{\mathrm{s}} = \mathbf{E}_{,\mathbf{E}} = \mathbf{E}_{,\mathbf{C}} : \mathbf{C}_{,\mathbf{E}}, \tag{8.60}$$

where the derivative $\mathbf{E}_{,\mathbf{C}}$ is given by (8.49-8.50). The basis tensors of the latter representation are

$$\mathfrak{P}_{ij} = \begin{cases} (\mathbf{P}_i \otimes \mathbf{P}_i)^{\mathrm{s}} & \text{if } i = j, \\ (\mathbf{P}_i \otimes \mathbf{P}_j + \mathbf{P}_j \otimes \mathbf{P}_i)^{\mathrm{s}} & \text{if } i \neq j. \end{cases} \tag{8.61}$$

In view of (4.44), (5.33) and (5.55) they are pairwise orthogonal (see Exercise 8.2) such that (cf. [47])

$$\mathfrak{P}_{ij} : \mathfrak{P}_{kl} = \begin{cases} \mathfrak{P}_{ij} & \text{if } i = k \text{ and } j = l \text{ or } i = l \text{ and } j = k, \\ \mathfrak{O} & \text{otherwise.} \end{cases} \tag{8.62}$$

By means of (4.46) and (5.84) we can also write

$$\sum_{\substack{i,j=1 \\ j \geq i}}^{s} \mathfrak{P}_{ij} = \left[\left(\sum_{i=1}^{s} \mathbf{P}_i\right) \otimes \left(\sum_{j=1}^{s} \mathbf{P}_j\right)\right]^{\mathrm{s}} = (\mathbf{I} \otimes \mathbf{I})^{\mathrm{s}} = \mathfrak{I}^{\mathrm{s}}. \tag{8.63}$$

Using these properties we thus obtain

$$\mathbf{C}_{,\mathbf{E}} = \sum_{i,j=1}^{s} f_{ij}^{-1} \left(\mathbf{P}_i \otimes \mathbf{P}_j\right)^{\mathrm{s}}, \tag{8.64}$$

where f_{ij} $(i, j = 1, 2, \ldots, s)$ are given by (8.50). Substituting this result into (8.57) and taking $(5.22)_1$, (5.46) and (5.47) into account yields [18]

$$\mathbf{T} = \frac{1}{2} \sum_{i,j=1}^{s} f_{ij}^{-1} \mathbf{P}_i \mathbf{S} \mathbf{P}_j. \tag{8.65}$$

In order to avoid any reference to eigenprojections we can again express them by (7.43) with (7.81), (7.83) and (7.85) or alternatively use the closed-form solution (7.38) with (7.62), (7.78) and (7.80). Both procedures lead to the following result (cf. [47]).

$$\mathbf{T} = \sum_{p,q=0}^{2} \eta_{pq} \mathbf{C}^p \mathbf{S} \mathbf{C}^q. \tag{8.66}$$

Distinct eigenvalues: $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$,

$$\eta_{00} = \sum_{i=1}^{3} \frac{\lambda_j^4 \lambda_k^4 \lambda_i}{f'(\lambda_i) \Delta_i^2} - \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{\lambda_i^2 \lambda_j^2 \lambda_k^4}{2\left(\lambda_i^2 - \lambda_j^2\right)\left[f(\lambda_i) - f(\lambda_j)\right] \Delta_k},$$

$$\eta_{01} = \eta_{10} = -\sum_{i=1}^{3} \frac{\left(\lambda_j^2 + \lambda_k^2\right) \lambda_j^2 \lambda_k^2 \lambda_i}{f'(\lambda_i) \Delta_i^2}$$

$$+ \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{\left(\lambda_j^2 + \lambda_k^2\right) \lambda_i^2 \lambda_k^2}{2\left(\lambda_i^2 - \lambda_j^2\right)\left[f(\lambda_i) - f(\lambda_j)\right] \Delta_k},$$

$$\eta_{02} = \eta_{20} = \sum_{i=1}^{3} \frac{\lambda_j^2 \lambda_k^2 \lambda_i}{f'(\lambda_i) \Delta_i^2} - \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{\lambda_i^2 \lambda_k^2}{2\left(\lambda_i^2 - \lambda_j^2\right)\left[f(\lambda_i) - f(\lambda_j)\right] \Delta_k},$$

$$\eta_{11} = \sum_{i=1}^{3} \frac{\left(\lambda_j^2 + \lambda_k^2\right)^2 \lambda_i}{f'(\lambda_i) \Delta_i^2} - \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{\left(\lambda_j^2 + \lambda_k^2\right)\left(\lambda_i^2 + \lambda_k^2\right)}{2\left(\lambda_i^2 - \lambda_j^2\right)\left[f(\lambda_i) - f(\lambda_j)\right] \Delta_k},$$

$$\eta_{12} = \eta_{21} = -\sum_{i=1}^{3} \frac{\left(\lambda_j^2 + \lambda_k^2\right) \lambda_i}{f'(\lambda_i) \Delta_i^2} + \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{\lambda_i^2 + \lambda_k^2}{2\left(\lambda_i^2 - \lambda_j^2\right)\left[f(\lambda_i) - f(\lambda_j)\right] \Delta_k},$$

$$\eta_{22} = \sum_{i=1}^{3} \frac{\lambda_i}{f'(\lambda_i) \Delta_i^2} - \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{1}{2\left(\lambda_i^2 - \lambda_j^2\right)\left[f(\lambda_i) - f(\lambda_j)\right] \Delta_k}, \tag{8.67}$$

where $i \neq j \neq k \neq i$ and Δ_i are given by (8.53).

Double coalescence of eigenvalues: $\lambda_i \neq \lambda_j = \lambda_k = \lambda$,

$$\eta_{00} = -\frac{\lambda_i^2 \lambda^2}{\left(\lambda_i^2 - \lambda^2\right)\left[f\left(\lambda_i\right) - f\left(\lambda\right)\right]} + \frac{\lambda_i \lambda}{\left(\lambda_i^2 - \lambda^2\right)^2}\left[\frac{\lambda^3}{f'\left(\lambda_i\right)} + \frac{\lambda_i^3}{f'\left(\lambda\right)}\right],$$

$$\eta_{01} = \eta_{10} = \frac{\lambda_i^2 + \lambda^2}{2\left(\lambda_i^2 - \lambda^2\right)\left[f\left(\lambda_i\right) - f\left(\lambda\right)\right]} - \frac{\lambda_i \lambda}{\left(\lambda_i^2 - \lambda^2\right)^2}\left[\frac{\lambda}{f'\left(\lambda_i\right)} + \frac{\lambda_i}{f'\left(\lambda\right)}\right],$$

$$\eta_{11} = -\frac{1}{\left(\lambda_i^2 - \lambda^2\right)\left[f\left(\lambda_i\right) - f\left(\lambda\right)\right]} + \frac{1}{\left(\lambda_i^2 - \lambda^2\right)^2}\left[\frac{\lambda_i}{f'\left(\lambda_i\right)} + \frac{\lambda}{f'\left(\lambda\right)}\right],$$

$$\eta_{02} = \eta_{20} = \eta_{12} = \eta_{21} = \eta_{22} = 0. \tag{8.68}$$

Triple coalescence of eigenvalues: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$,

$$\eta_{00} = \frac{\lambda}{f'\left(\lambda\right)}, \quad \eta_{01} = \eta_{10} = \eta_{11} = \eta_{02} = \eta_{20} = \eta_{12} = \eta_{21} = \eta_{22} = 0. \tag{8.69}$$

8.6 Finite Plasticity Based on the Additive Decomposition of Generalized Strains

Keeping in mind the above results regarding generalized strains we are concerned in this section with a thermodynamically based plasticity theory. The basic kinematic assumption of this theory is the additive decomposition of generalized strains (7.6) into an elastic part $\mathbf{E}_e$ and a plastic part $\mathbf{E}_p$ as

$$\mathbf{E} = \mathbf{E}_e + \mathbf{E}_p. \tag{8.70}$$

The derivation of evolution equations for the plastic strain is based on the second law of thermodynamics and the principle of maximum plastic dissipation. The second law of thermodynamics can be written in the Clausius-Planck form as (see, e.g. [45])

$$\mathcal{D} = \mathbf{T} : \dot{\mathbf{E}} - \dot{\psi} \geq 0, \tag{8.71}$$

where $\mathcal{D}$ denotes the dissipation and $\mathbf{T}$ is again the stress tensor work conjugate to $\mathbf{E}$. Inserting (8.70) into (8.71) we further write

$$\mathcal{D} = \left(\mathbf{T} - \frac{\partial \psi}{\partial \mathbf{E}_e}\right) : \dot{\mathbf{E}}_e + \mathbf{T} : \dot{\mathbf{E}}_p \geq 0, \tag{8.72}$$

where the strain energy is assumed to be a function of the elastic strain as $\psi = \hat{\psi}\left(\mathbf{E}_e\right)$. The first term in the expression of the dissipation (8.72) depends solely on the elastic strain rate $\dot{\mathbf{E}}_e$, while the second one on the plastic strain rate $\dot{\mathbf{E}}_p$. Since the elastic and plastic strain rates are independent of each other the dissipation inequality (8.72) requires that

$$\mathbf{T} = \frac{\partial \psi}{\partial \mathbf{E}_e}. \tag{8.73}$$

This leads to the so-called reduced dissipation inequality

$$\mathcal{D} = \mathbf{T} : \dot{\mathbf{E}}_p \geq 0. \tag{8.74}$$

Among all admissible processes the real one maximizes the dissipation (8.74). This statement is based on the postulate of maximum plastic dissipation (see, e.g., [28]). According to the converse Kuhn-Tucker theorem (see, e.g., [6]) the sufficient conditions of this maximum are written as

$$\dot{\mathbf{E}}_p = \dot{\zeta} \frac{\partial \Phi}{\partial \mathbf{T}}, \quad \dot{\zeta} \geq 0, \quad \dot{\zeta}\Phi = 0, \quad \Phi \leq 0, \tag{8.75}$$

where Φ represents a convex yield function and $\dot{\zeta}$ denotes a consistency parameter. In the following, we will deal with an ideal-plastic isotropic material described by a von Mises-type yield criterion. Written in terms of the stress tensor $\mathbf{T}$ the von Mises yield function takes the form [31]

$$\Phi = \|\mathrm{dev}\mathbf{T}\| - \sqrt{\frac{2}{3}}\sigma_Y, \tag{8.76}$$

where σ_Y denotes the normal yield stress. With the aid of (6.51) and (6.130) the evolution equation $(8.75)_1$ can thus be given by

$$\begin{aligned} \dot{\mathbf{E}}_p &= \dot{\zeta} \left\|\mathrm{dev}\mathbf{T}\right\|,_{\mathbf{T}} \\ &= \dot{\zeta} \left\|\mathrm{dev}\mathbf{T}\right\|,_{\mathrm{dev}\mathbf{T}} : \mathrm{dev}\mathbf{T},_{\mathbf{T}} = \dot{\zeta} \frac{\mathrm{dev}\mathbf{T}}{\|\mathrm{dev}\mathbf{T}\|} : \boldsymbol{\mathcal{P}}_{\mathrm{dev}} = \dot{\zeta} \frac{\mathrm{dev}\mathbf{T}}{\|\mathrm{dev}\mathbf{T}\|}. \end{aligned} \tag{8.77}$$

Taking the quadratic norm on both the right and left hand side of this identity delivers the consistency parameter as $\dot{\zeta} = ||\dot{\mathbf{E}}_p||$. In view of the yield condition $\Phi = 0$ we thus obtain

$$\mathrm{dev}\mathbf{T} = \sqrt{\frac{2}{3}}\sigma_Y \frac{\dot{\mathbf{E}}_p}{\left\|\dot{\mathbf{E}}_p\right\|}, \tag{8.78}$$

which immediately requires that (see Exercise 1.46)

$$\mathrm{tr}\dot{\mathbf{E}}_p = 0. \tag{8.79}$$

In the following, we assume small elastic but large plastic strains and specify the above plasticity model for finite simple shear. In this case all three principal stretches (8.21) are distinct so that we can write by virtue of (7.6)

$$\dot{\mathbf{E}}_p = \dot{\mathbf{E}} = \sum_{i=1}^{3} f'(\lambda_i)\,\dot{\lambda}_i \mathbf{P}_i + \sum_{i=1}^{3} f(\lambda_i)\,\dot{\mathbf{P}}_i. \tag{8.80}$$

By means of the identities $\mathrm{tr}\mathbf{P}_i = 1$ and $\mathrm{tr}\dot{\mathbf{P}}_i = 0$ following from (4.61) and (4.62) where $r_i = 1$ $(i = 1, 2, 3)$ the condition (8.79) requires that

$$\sum_{i=1}^{3} f'(\lambda_i)\,\dot{\lambda}_i = 0. \tag{8.81}$$

In view of (8.21) it leads to the equation

$$f'(\lambda) - f'\left(\lambda^{-1}\right)\lambda^{-2} = 0, \quad \forall\lambda > 0, \tag{8.82}$$

where we set $\lambda_1 = \lambda$ and consequently $\lambda_2 = \lambda^{-1}$. Solutions of this equations can be given by [21]

$$f_a(\lambda) = \begin{cases} \dfrac{1}{2a}\left(\lambda^a - \lambda^{-a}\right) & \text{for } a \neq 0, \\ \ln\lambda & \text{for } a = 0. \end{cases} \tag{8.83}$$

By means of $(7.6)_1$ or $(7.18)_1$ the functions f_a (8.83) yield a set of new generalized strain measures

$$\mathbf{E}^{\langle a\rangle} = \begin{cases} \dfrac{1}{2a}\left(\mathbf{U}^a - \mathbf{U}^{-a}\right) = \dfrac{1}{2a}\left(\mathbf{C}^{a/2} - \mathbf{C}^{-a/2}\right) & \text{for } a \neq 0, \\ \ln\mathbf{U} = \dfrac{1}{2}\ln\mathbf{C} & \text{for } a = 0, \end{cases} \tag{8.84}$$

among which only the logarithmic one $(a = 0)$ belongs to Seth's family (7.7). Henceforth, we will deal only with the generalized strains (8.84) as able to provide the traceless deformation rate (8.79). For these strains eq. (8.78) takes the form

$$\mathrm{dev}\mathbf{T}^{\langle a\rangle} = \sqrt{\frac{2}{3}}\sigma_Y \frac{\dot{\mathbf{E}}^{\langle a\rangle}}{\left\|\dot{\mathbf{E}}^{\langle a\rangle}\right\|}, \tag{8.85}$$

where $\mathbf{T}^{\langle a\rangle}$ denotes the stress tensor work conjugate to $\mathbf{E}^{\langle a\rangle}$. $\mathbf{T}^{\langle a\rangle}$ itself has no physical meaning and should be transformed to the Cauchy stresses. With the aid of (8.57), (8.58) and (8.60) we can write

$$\boldsymbol{\sigma} = \frac{1}{\det\mathbf{F}}\mathbf{F}\mathbf{S}\mathbf{F}^{\mathrm{T}} = \frac{1}{\det\mathbf{F}}\mathbf{F}\left(\mathbf{T}^{\langle a\rangle} : \boldsymbol{\mathcal{P}}_a\right)\mathbf{F}^{\mathrm{T}}, \tag{8.86}$$

where

$$\boldsymbol{\mathcal{P}}_a = 2\mathbf{E}^{\langle a\rangle},_{\mathbf{C}} \tag{8.87}$$

can be expressed either by (8.49-8.50) or by (8.51-8.55). It is seen that this fourth-order tensor is super-symmetric (see Exercise 5.11), so that $\mathbf{T}^{\langle a\rangle} : \boldsymbol{\mathcal{P}}_a = \boldsymbol{\mathcal{P}}_a : \mathbf{T}^{\langle a\rangle}$. Thus, by virtue of (1.152) and (1.153) representation (8.86) can be rewritten as

$$\begin{aligned}\boldsymbol{\sigma} &= \frac{1}{\det\mathbf{F}}\mathbf{F}\left(\boldsymbol{\mathcal{P}}_a : \mathbf{T}^{\langle a\rangle}\right)\mathbf{F}^{\mathrm{T}} \\ &= \frac{1}{\det\mathbf{F}}\mathbf{F}\left[\boldsymbol{\mathcal{P}}_a : \mathrm{dev}\mathbf{T}^{\langle a\rangle} + \frac{1}{3}\mathrm{tr}\mathbf{T}^{\langle a\rangle}\left(\boldsymbol{\mathcal{P}}_a : \mathbf{I}\right)\right]\mathbf{F}^{\mathrm{T}}. \end{aligned} \tag{8.88}$$

With the aid of the relation

$$\begin{aligned}\boldsymbol{\mathcal{P}}_a : \mathbf{I} &= 2\,\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}^{\langle a\rangle}\left(\mathbf{C} + t\mathbf{I}\right)\Big|_{t=0} \\ &= 2\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{3} f_a\left(\sqrt{\lambda_i^2 + t}\right)\mathbf{P}_i\Big|_{t=0} = \sum_{i=1}^{3} f_a'\left(\lambda_i\right)\lambda_i^{-1}\mathbf{P}_i \end{aligned} \tag{8.89}$$

following from (6.111) and taking (8.83) into account one obtains

$$\mathbf{F}\left(\boldsymbol{\mathcal{P}}_a : \mathbf{I}\right)\mathbf{F}^{\mathrm{T}} = \frac{1}{2}\mathbf{F}\left(\mathbf{C}^{a/2-1} + \mathbf{C}^{-a/2-1}\right)\mathbf{F}^{\mathrm{T}} = \frac{1}{2}\left(\mathbf{b}^{a/2} + \mathbf{b}^{-a/2}\right).$$

Inserting this result into (8.88) yields

$$\boldsymbol{\sigma} = \frac{1}{\det\mathbf{F}}\mathbf{F}\left(\boldsymbol{\mathcal{P}}_a : \mathrm{dev}\mathbf{T}^{\langle a\rangle}\right)\mathbf{F}^{\mathrm{T}} + \hat{\boldsymbol{\sigma}} \tag{8.90}$$

with the abbreviation

$$\hat{\boldsymbol{\sigma}} = \frac{\mathrm{tr}\mathbf{T}^{\langle a\rangle}}{6\det\mathbf{F}}\left(\mathbf{b}^{a/2} + \mathbf{b}^{-a/2}\right). \tag{8.91}$$

Using the spectral decomposition of $\mathbf{b}$ by (8.4) and taking into account that in the case of simple shear $\det\mathbf{F} = 1$ we can further write

$$\hat{\boldsymbol{\sigma}} = \frac{1}{6}\mathrm{tr}\mathbf{T}^{\langle a\rangle}\left[\left(\lambda^a + \lambda^{-a}\right)\left(\mathbf{p}_1 + \mathbf{p}_2\right) + 2\mathbf{p}_3\right], \tag{8.92}$$

where λ is given by (8.21). Thus, in the 1-2 shear plane the stress tensor $\hat{\boldsymbol{\sigma}}$ has the double eigenvalue $\frac{1}{6}\mathrm{tr}\mathbf{T}^{\langle a\rangle}\left(\lambda^a + \lambda^{-a}\right)$ and causes equibiaxial tension or compression. Hence, in this plane the component $\hat{\boldsymbol{\sigma}}$ (8.91) is shear free and does not influence the shear stress response. Inserting (8.85) into (8.90) and taking (8.18) and (8.48) into account we finally obtain

$$\boldsymbol{\sigma} = \sqrt{\frac{2}{3}}\sigma_Y\mathbf{F}\left[\frac{\boldsymbol{\mathcal{P}}_a : \boldsymbol{\mathcal{P}}_a : \mathbf{A}}{\|\boldsymbol{\mathcal{P}}_a : \mathbf{A}\|}\right]\mathbf{F}^{\mathrm{T}} + \hat{\boldsymbol{\sigma}}, \tag{8.93}$$

where

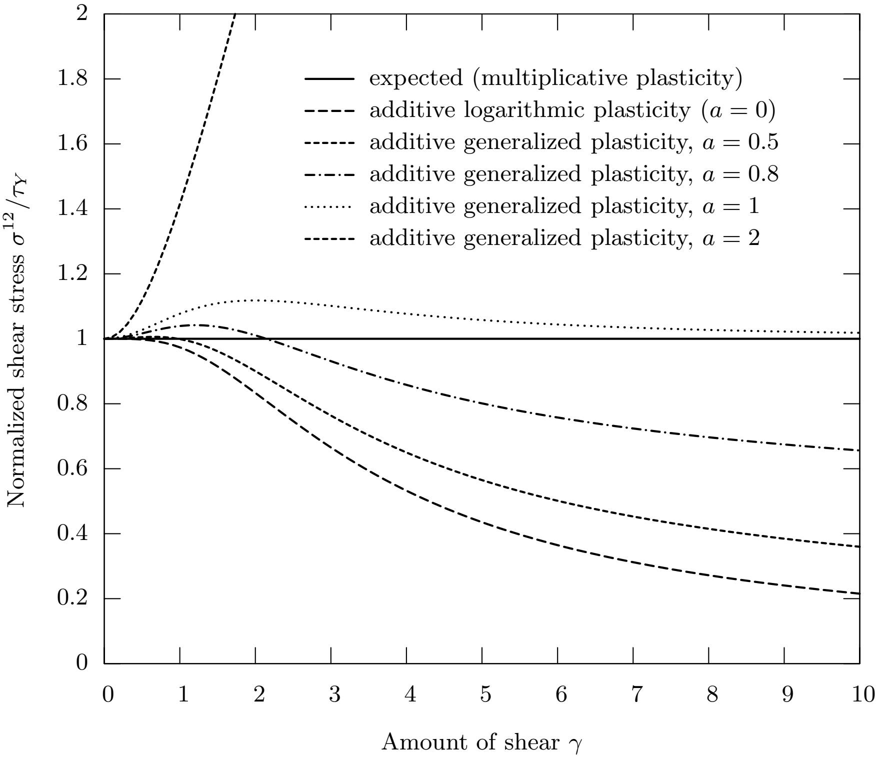

Fig. 8.1. Simple shear of an ideal-plastic material: shear stress responses based on the additive decomposition of generalized strains

$$\mathbf{A} = \frac{1}{2\dot{\gamma}}\dot{\mathbf{C}} = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & \gamma & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j. \tag{8.94}$$

Of particular interest is the shear stress σ^{12} as a function of the amount of shear γ. Inserting (8.51-8.52) and (8.87) into (8.93) we obtain after some algebraic manipulations

$$\frac{\sigma^{12}}{\tau_Y} = \frac{2\sqrt{(4+\gamma^2)\,\mathit{\Gamma}^2 f_a'^2\,(\mathit{\Gamma}) + 4f_a^2\,(\mathit{\Gamma})}}{4+\gamma^2}, \tag{8.95}$$

where

$$\mathit{\Gamma} = \frac{\gamma}{2} + \frac{\sqrt{4+\gamma^2}}{2} \tag{8.96}$$

and $\tau_Y = \sigma_Y/\sqrt{3}$ denotes the shear yield stress. Equation (8.95) is illustrated graphically in Fig. 8.1 for several values of the parameter a. Since the presented plasticity model considers neither softening nor hardening and is restricted to small elastic strains a constant shear stress response even at large plastic deformations is expected. It is also predicted by a plasticity model based on the multiplicative decomposition of the deformation gradient (see, e.g., [21] for more details). The plasticity model based on the additive decomposition of generalized strains exhibits, however, a non-constant shear stress for all examined values of a. This restricts the applicability of this model to moderate plastic shears. Indeed, in the vicinity of the point $\gamma = 0$ the power series expansion of (8.95) takes the form

$$\frac{\sigma^{12}}{\tau_Y} = 1 + \frac{1}{4}a^2\gamma^2 + \left(\frac{1}{16}a^4 - \frac{3}{4}a^2 - 1\right)\gamma^4 + O\left(\gamma^6\right). \tag{8.97}$$

Thus, in the case of simple shear the amount of shear is limited for the logarithmic strain ($a = 0$) by $\gamma^4 \ll 1$ and for other generalized strain measures by $\gamma^2 \ll 1$.

Exercises

8.1. The deformation gradient is given by $\mathbf{F} = \mathrm{F}^i_{\cdot j}\boldsymbol{e}_i \otimes \boldsymbol{e}^j$, where

$$\left[\mathrm{F}^i_{\cdot j}\right] = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Evaluate the stretch tensors $\mathbf{U}$ and $\mathbf{v}$ and the rotation tensor $\mathbf{R}$ using (8.11-8.12) and (8.16-8.17).

8.2. Prove the orthogonality (8.62) of the basis tensors (8.61) using (4.44), (5.33) and (5.55).