Solutions

Exercises of Chapter 1

1.1 (a) (A.4), (A.3):

$$\boldsymbol{0} = \boldsymbol{0} + (-\boldsymbol{0}) = -\boldsymbol{0}.$$

(b) (A.2-A.4), (B.3):

$$\begin{aligned} \alpha\boldsymbol{0} = \boldsymbol{0} + \alpha\boldsymbol{0} &= \alpha\boldsymbol{x} + (-\alpha\boldsymbol{x}) + \alpha\boldsymbol{0} \\ &= \alpha\,(\boldsymbol{0} + \boldsymbol{x}) + (-\alpha\boldsymbol{x}) = \alpha\boldsymbol{x} + (-\alpha\boldsymbol{x}) = \boldsymbol{0}. \end{aligned}$$

(c) (A.2-A.4), (B.4):

$$0\boldsymbol{x} = 0\boldsymbol{x} + \boldsymbol{0} = 0\boldsymbol{x} + 0\boldsymbol{x} + (-0\boldsymbol{x}) = 0\boldsymbol{x} + (-0\boldsymbol{x}) = \boldsymbol{0}, \quad \forall\boldsymbol{x} \in \mathbb{V}.$$

(d) (A.2-A.4), (B.2), (B.4), (c):

$$\begin{aligned} (-1)\,\boldsymbol{x} &= (-1)\,\boldsymbol{x} + \boldsymbol{0} = (-1)\,\boldsymbol{x} + \boldsymbol{x} + (-\boldsymbol{x}) \\ &= (-1+1)\,\boldsymbol{x} + (-\boldsymbol{x}) = 0\boldsymbol{x} + (-\boldsymbol{x}) = \boldsymbol{0} + (-\boldsymbol{x}) = -\boldsymbol{x}, \quad \forall\boldsymbol{x} \in \mathbb{V}. \end{aligned}$$

(e) If, on the contrary, $\alpha \neq 0$ and $\boldsymbol{x} \neq \boldsymbol{0}$, then according to (b), (B.1), (B.2):

$$\boldsymbol{0} = \alpha^{-1}\boldsymbol{0} = \alpha^{-1}\,(\alpha\boldsymbol{x}) = \boldsymbol{x}.$$

1.2 Let, on the contrary, $\boldsymbol{x}_k = \boldsymbol{0}$ for some k. Then, $\sum_{i=1}^{n} \alpha_i \boldsymbol{x}_i = \boldsymbol{0}$, where $\alpha_k = 1, \alpha_i = 0, i = 1, \ldots, k-1, k+1, \ldots, n$.

1.3 If, on the contrary, for some $k < n$: $\sum_{i=1}^{k} \alpha_i \boldsymbol{x}_i = \boldsymbol{0}$, where not all $\alpha_i, (i = 1, 2, \ldots, k)$ are zero, then we can also write: $\sum_{i=1}^{n} \alpha_i \boldsymbol{x}_i = \boldsymbol{0}$, where $\alpha_i = 0$, for $i = k+1, \ldots, n$.

1.4 (a) $\delta^i_j a^j = \delta^i_1 a^1 + \delta^i_2 a^2 + \delta^i_3 a^3 = a^i$,

(b) $\delta_{ij} x^i x^j = \delta_{11} x^1 x^1 + \delta_{12} x^1 x^2 + \ldots + \delta_{33} x^3 x^3 = x^1 x^1 + x^2 x^2 + x^3 x^3$,

(c) $\delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3 = 3$,

(d) $\dfrac{\partial f_i}{\partial x^j}dx^j = \dfrac{\partial f_i}{\partial x^1}dx^1 + \dfrac{\partial f_i}{\partial x^2}dx^2 + \dfrac{\partial f_i}{\partial x^3}dx^3$.

1.5 (A.4), (C.2), (C.3), Ex. 1.1 (d):

$$\boldsymbol{0} \cdot \boldsymbol{x} = [\boldsymbol{x} + (-\boldsymbol{x})] \cdot \boldsymbol{x} = [\boldsymbol{x} + (-1)\,\boldsymbol{x}] \cdot \boldsymbol{x} = \boldsymbol{x} \cdot \boldsymbol{x} - \boldsymbol{x} \cdot \boldsymbol{x} = 0.$$

1.6 Let on the contrary $\sum_{i=1}^{m} \alpha_i \boldsymbol{g}_i = \boldsymbol{0}$, where not all α_i $(i = 1, 2, \ldots, m)$ are zero. Multiplying scalarly by $\boldsymbol{g}_j$ we obtain: $0 = \boldsymbol{g}_j \cdot (\sum_{i=1}^{m} \alpha_i \boldsymbol{g}_i)$. Since $\boldsymbol{g}_i \cdot \boldsymbol{g}_j = 0$ for $i \neq j$, we can write: $\alpha_j \boldsymbol{g}_j \cdot \boldsymbol{g}_j = 0\,(j = 1, 2, \ldots, m)$. The fact that the vectors $\boldsymbol{g}_j$ are non-zero leads in view of (C.4) to the conclusion that $\alpha_j = 0\,(j = 1, 2, \ldots, m)$ which contradicts the earlier assumption.

1.7 (1.6), (C.1), (C.2):

$$\begin{aligned}\|\boldsymbol{x} + \boldsymbol{y}\|^2 &= (\boldsymbol{x} + \boldsymbol{y}) \cdot (\boldsymbol{x} + \boldsymbol{y}) \\ &= \boldsymbol{x} \cdot \boldsymbol{x} + \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{y} \cdot \boldsymbol{x} + \boldsymbol{y} \cdot \boldsymbol{y} = \|\boldsymbol{x}\|^2 + 2\boldsymbol{x} \cdot \boldsymbol{y} + \|\boldsymbol{y}\|^2 .\end{aligned}$$

1.8 Since $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ is a basis we can write $\boldsymbol{a} = a^i \boldsymbol{g}_i$. Then, $\boldsymbol{a} \cdot \boldsymbol{a} = a^i\,(\boldsymbol{g}_i \cdot \boldsymbol{a})$. Thus, if $\boldsymbol{a} \cdot \boldsymbol{g}_i = 0$ $(i = 1, 2, \ldots, n)$, then $\boldsymbol{a} \cdot \boldsymbol{a} = \boldsymbol{0}$ and according to (C.4) $\boldsymbol{a} = \boldsymbol{0}$ (sufficiency). Conversely, if $\boldsymbol{a} = \boldsymbol{0}$, then (see Exercise 1.5) $\boldsymbol{a} \cdot \boldsymbol{g}_i = 0$ $(i = 1, 2, \ldots, n)$ (necessity).

1.9 Necessity. (C.2): $\boldsymbol{a} \cdot \boldsymbol{x} = \boldsymbol{b} \cdot \boldsymbol{x} \Rightarrow \boldsymbol{a} \cdot \boldsymbol{x} - \boldsymbol{b} \cdot \boldsymbol{x} = (\boldsymbol{a} - \boldsymbol{b}) \cdot \boldsymbol{x} = 0$, $\forall \boldsymbol{x} \in \mathbb{E}^n$. Let $\boldsymbol{x} = \boldsymbol{a} - \boldsymbol{b}$, then $(\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b}) = 0$ and according to (C.4) $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{0}$. This implies that $\boldsymbol{a} = \boldsymbol{b}$. The sufficiency is evident.

1.10 (a) Orthonormal vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ can be calculated by means of the Gram-Schmidt procedure (1.10-1.12) as follows

$$\boldsymbol{e}_1 = \frac{\boldsymbol{g}_1}{\|\boldsymbol{g}_1\|} = \begin{Bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{Bmatrix},$$

$$\boldsymbol{e}_2' = \boldsymbol{g}_2 - (\boldsymbol{g}_2 \cdot \boldsymbol{e}_1)\,\boldsymbol{e}_1 = \begin{Bmatrix} 1/2 \\ -1/2 \\ -2 \end{Bmatrix}, \quad \boldsymbol{e}_2 = \frac{\boldsymbol{e}_2'}{\|\boldsymbol{e}_2'\|} = \begin{Bmatrix} \sqrt{2}/6 \\ -\sqrt{2}/6 \\ -2\sqrt{2}/3 \end{Bmatrix},$$

$$\boldsymbol{e}_3' = \boldsymbol{g}_3 - (\boldsymbol{g}_3 \cdot \boldsymbol{e}_2)\,\boldsymbol{e}_2 - (\boldsymbol{g}_3 \cdot \boldsymbol{e}_1)\,\boldsymbol{e}_1 = \begin{Bmatrix} 10/9 \\ -10/9 \\ 5/9 \end{Bmatrix}, \boldsymbol{e}_3 = \frac{\boldsymbol{e}_3'}{\|\boldsymbol{e}_3'\|} = \begin{Bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{Bmatrix}.$$

(b) First, we calculate the matrices $[g_{ij}]$ and $\left[g^{ij}\right]$ by $(1.25)_2$ and (1.24)

$$[g_{ij}] = [\boldsymbol{g}_i \cdot \boldsymbol{g}_j] = \begin{bmatrix} 2 & 3 & 6 \\ 3 & 9 & 8 \\ 6 & 8 & 21 \end{bmatrix}, \quad \left[g^{ij}\right] = [g_{ij}]^{-1} = \begin{bmatrix} 5 & -\frac{3}{5} & -\frac{6}{5} \\ -\frac{3}{5} & \frac{6}{25} & \frac{2}{25} \\ -\frac{6}{5} & \frac{2}{25} & \frac{9}{25} \end{bmatrix},$$

With the aid of (1.21) we thus obtain

$$\boldsymbol{g}^1 = g^{11}\boldsymbol{g}_1 + g^{12}\boldsymbol{g}_2 + g^{13}\boldsymbol{g}_3 = \left\{ \begin{array}{c} -1 \\ 2 \\ 0 \end{array} \right\},$$

$$\boldsymbol{g}^2 = g^{21}\boldsymbol{g}_1 + g^{22}\boldsymbol{g}_2 + g^{23}\boldsymbol{g}_3 = \left\{ \begin{array}{c} 1/5 \\ -1/5 \\ -2/5 \end{array} \right\},$$

$$\boldsymbol{g}^3 = g^{31}\boldsymbol{g}_1 + g^{32}\boldsymbol{g}_2 + g^{33}\boldsymbol{g}_3 = \left\{ \begin{array}{c} 2/5 \\ -2/5 \\ 1/5 \end{array} \right\}.$$

(c) By virtue of (1.35) we write

$$g = \left|\beta^i_j\right| = \left| \begin{array}{ccc} 1 & 1 & 0 \\ 2 & 1 & -2 \\ 4 & 2 & 1 \end{array} \right| = -5.$$

Applying (1.33) we further obtain with the aid of (1.46)

$$\boldsymbol{g}^1 = g^{-1}\boldsymbol{g}_2 \times \boldsymbol{g}_3 = -\frac{1}{5} \left| \begin{array}{ccc} 2 & 1 & -2 \\ 4 & 2 & 1 \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{array} \right| = -\boldsymbol{a}_1 + 2\boldsymbol{a}_2,$$

$$\boldsymbol{g}^2 = g^{-1}\boldsymbol{g}_3 \times \boldsymbol{g}_1 = -\frac{1}{5} \left| \begin{array}{ccc} 4 & 2 & 1 \\ 1 & 1 & 0 \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{array} \right| = \frac{1}{5}\left(\boldsymbol{a}_1 - \boldsymbol{a}_2 - 2\boldsymbol{a}_3\right),$$

$$\boldsymbol{g}^3 = g^{-1}\boldsymbol{g}_1 \times \boldsymbol{g}_2 = -\frac{1}{5} \left| \begin{array}{ccc} 1 & 1 & 0 \\ 2 & 1 & -2 \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{array} \right| = \frac{1}{5}\left(2\boldsymbol{a}_1 - 2\boldsymbol{a}_2 + \boldsymbol{a}_3\right),$$

where $\boldsymbol{a}_i$ denote the orthonormal basis the components of the original vectors $\boldsymbol{g}_i$ $(i = 1, 2, 3)$ are related to.

1.11 Let on the contrary $\sum_{i=1}^{3} \alpha_i \boldsymbol{g}^i = \boldsymbol{0}$, where not all α_i are zero. Multiplying scalarly by $\boldsymbol{g}_j$ we obtain by virtue of (1.15): $0 = \boldsymbol{g}_j \cdot \left(\sum_{i=1}^{3} \alpha_i \boldsymbol{g}^i\right) = \sum_{i=1}^{3} \alpha_i \delta^i_j = \alpha_j$ $(j = 1, 2, 3)$.

1.12 Similarly to (1.35) we write using also (1.18), (1.19) and (1.36)

$$\left[\boldsymbol{g}^1\boldsymbol{g}^2\boldsymbol{g}^3\right] = \left[\alpha^1_i \boldsymbol{e}^i \alpha^2_j \boldsymbol{e}^j \alpha^3_k \boldsymbol{e}^k\right] = \alpha^1_i \alpha^2_j \alpha^3_k \left[\boldsymbol{e}^i \boldsymbol{e}^j \boldsymbol{e}^k\right]$$

$$= \alpha^1_i \alpha^2_j \alpha^3_k e^{ijk} = \left|\alpha^i_j\right| = \left|\beta^i_j\right|^{-1} = g^{-1}.$$

1.13 In view of (1.33), (1.36) and (1.38) the left and right hand side of (1.39) yield for the case

$i = j$:

$$\boldsymbol{g}_i \times \boldsymbol{g}_i = \boldsymbol{0}, \quad e_{iik} g \boldsymbol{g}^k = \boldsymbol{0},$$

$\exists k$ such that ijk is an even permutation of 123:

$$\boldsymbol{g}_i \times \boldsymbol{g}_j = [\boldsymbol{g}_1 \boldsymbol{g}_2 \boldsymbol{g}_3]\, \boldsymbol{g}^k, \quad e_{ijl} g \boldsymbol{g}^l = [\boldsymbol{g}_1 \boldsymbol{g}_2 \boldsymbol{g}_3]\, \boldsymbol{g}^k,$$

$\exists k$ such that ijk is an odd permutation of 123:

$$\boldsymbol{g}_i \times \boldsymbol{g}_j = -\boldsymbol{g}_j \times \boldsymbol{g}_i = -\left[\boldsymbol{g}_1 \boldsymbol{g}_2 \boldsymbol{g}_3\right] \boldsymbol{g}^k, \quad e_{ijl} g \boldsymbol{g}^l = -\left[\boldsymbol{g}_1 \boldsymbol{g}_2 \boldsymbol{g}_3\right] \boldsymbol{g}^k.$$

Using the result of the previous exercise one proves in the same manner (1.42).

1.14 (a) $\delta^{ij} e_{ijk} = \delta^{11} e_{11k} + \delta^{12} e_{12k} + \ldots + \delta^{33} e_{33k} = 0$.

(b) Writing out the term $e^{ikm} e_{jkm}$ we first obtain

$$\begin{aligned} e^{ikm} e_{jkm} &= e^{i11} e_{j11} + e^{i12} e_{j12} + \ldots + e^{i33} e_{j33} \\ &= e^{i12} e_{j12} + e^{i21} e_{j21} + e^{i13} e_{j13} + e^{i31} e_{j31} + e^{i32} e_{j32} + e^{i23} e_{j23}. \end{aligned}$$

For $i \neq j$ each term in this sum is equal to zero. Let further $i = j = 1$. Then we obtain $e^{i12} e_{j12} + e^{i21} e_{j21} + e^{i13} e_{j13} + e^{i31} e_{j31} + e^{i32} e_{j32} + e^{i23} e_{j23} = e^{132} e_{132} + e^{123} e_{123} = (-1)(-1) + 1 \cdot 1 = 2$. The same result also holds for the cases $i = j = 2$ and $i = j = 3$. Thus, we can write $e^{ikm} e_{jkm} = 2\delta^i_j$.

(c) By means of the previous result (b) we can write: $e^{ijk} e_{ijk} = 2\delta^i_i = 2(\delta^1_1 + \delta^2_2 + \delta^3_3) = 6$. This can also be shown directly by

$$\begin{aligned} e^{ijk} e_{ijk} &= e^{123} e_{123} + e^{132} e_{132} + e^{213} e_{213} + e^{231} e_{231} + e^{312} e_{312} + e^{321} e_{321} \\ &= 1 \cdot 1 + (-1) \cdot (-1) + (-1) \cdot (-1) + 1 \cdot 1 + 1 \cdot 1 + (-1) \cdot (-1) = 6. \end{aligned}$$

(d) $e^{ijm} e_{klm} = e^{ij1} e_{kl1} + e^{ij2} e_{kl2} + e^{ij3} e_{kl3}$. It is seen that in the case $i = j$ or $k = l$ this sum as well as the right hand side $\delta^i_k \delta^j_l - \delta^i_l \delta^j_k$ are both zero. Let further $i \neq j$. Then, only one term in the above sum is non-zero if $k = i$ and $l = j$ or vice versa $l = i$ and $k = j$. In the first case the left hand side is 1 and in the last case -1. The same holds also for the right side $\delta^i_k \delta^j_l - \delta^i_l \delta^j_k$. Indeed, we obtain for $k = i \neq l = j$: $\delta^i_k \delta^j_l - \delta^i_l \delta^j_k = 1 \cdot 1 - 0 = 1$ and for $l = i \neq k = j$: $\delta^i_k \delta^j_l - \delta^i_l \delta^j_k = 0 - 1 \cdot 1 = -1$.

1.15 Using the representations $\boldsymbol{a} = a^i \boldsymbol{g}_i$, $\boldsymbol{b} = b^j \boldsymbol{g}_j$ and $\boldsymbol{c} = c_l \boldsymbol{g}^l$ we can write by virtue of (1.39) and (1.42)

$$\begin{aligned} (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} &= \left[\left(a^i \boldsymbol{g}_i\right) \times \left(b^j \boldsymbol{g}_j\right)\right] \times \boldsymbol{c} = \left(a^i b^j e_{ijk} g \boldsymbol{g}^k\right) \times \left(c_l \boldsymbol{g}^l\right) \\ &= a^i b^j c_l e_{ijk} e^{klm} \boldsymbol{g}_m = a^i b^j c_l e_{ijk} e^{lmk} \boldsymbol{g}_m. \end{aligned}$$

With the aid of the identity $e^{ijm} e_{klm} = \delta^i_k \delta^j_l - \delta^i_l \delta^j_k$ (Exercise 1.14) we finally obtain

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = a^i b^j c_l \left(\delta_i^l \delta_j^m - \delta_i^m \delta_j^l\right) \boldsymbol{g}_m = a^i b^j c_l \delta_i^l \delta_j^m \boldsymbol{g}_m - a^i b^j c_l \delta_i^m \delta_j^l \boldsymbol{g}_m$$

$$= a^i b^j c_i \boldsymbol{g}_j - a^i b^j c_j \boldsymbol{g}_i = (\boldsymbol{a} \cdot \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\, \boldsymbol{a}.$$

1.16 (A.2-A.4), (1.48):

$$\boldsymbol{0} = \mathbf{A}\boldsymbol{x} + (-\mathbf{A}\boldsymbol{x}) = \mathbf{A}\,(\boldsymbol{x} + \boldsymbol{0}) + (-\mathbf{A}\boldsymbol{x}) = \mathbf{A}\boldsymbol{x} + \mathbf{A}\boldsymbol{0} + (-\mathbf{A}\boldsymbol{x}) = \mathbf{A}\boldsymbol{0}.$$

1.17 (1.49), Exercises 1.1(c), 1.16: $(0\mathbf{A})\,\boldsymbol{x} = \mathbf{A}\,(0\boldsymbol{x}) = \mathbf{A}\boldsymbol{0} = \boldsymbol{0}$, $\forall \boldsymbol{x} \in \mathbb{E}^n$.

1.18 (1.61), Ex. 1.17: $\mathbf{A} + (-\mathbf{A}) = \mathbf{A} + (-1)\,\mathbf{A} = (1-1)\,\mathbf{A} = 0\mathbf{A} = \mathbf{0}$.

1.19 Indeed, a scalar product of the right-hand side of (1.80) with an arbitrary vector $\boldsymbol{x}$ yields $[(\boldsymbol{y} \cdot \boldsymbol{a})\, \boldsymbol{b}] \cdot \boldsymbol{x} = (\boldsymbol{y} \cdot \boldsymbol{a})\,(\boldsymbol{b} \cdot \boldsymbol{x})$. The same result follows also from $\boldsymbol{y} \cdot [(\boldsymbol{a} \otimes \boldsymbol{b})\, \boldsymbol{x}] = (\boldsymbol{y} \cdot \boldsymbol{a})\,(\boldsymbol{b} \cdot \boldsymbol{x})\,, \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}^n$ for the left-hand side. This implies that the identity (1.80) is true (see Exercise 1.9).

1.20 For $(1.83)_1$ we have for example

$$\boldsymbol{g}^i \mathbf{A} \boldsymbol{g}^j = \boldsymbol{g}^i \left(\mathrm{A}^{kl} \boldsymbol{g}_k \otimes \boldsymbol{g}_l\right) \boldsymbol{g}^j = \mathrm{A}^{kl} \left(\boldsymbol{g}^i \cdot \boldsymbol{g}_k\right) \left(\boldsymbol{g}_l \cdot \boldsymbol{g}^j\right) = \mathrm{A}^{kl} \delta_k^i \delta_l^j = \mathrm{A}^{ij}.$$

1.21 For an arbitrary vector $\boldsymbol{x} = x^i \boldsymbol{g}_i \in \mathbb{E}^3$ we can write using (1.28), (1.39) and (1.75)

$$\mathbf{W}\boldsymbol{x} = \boldsymbol{w} \times \boldsymbol{x} = \left(w^i \boldsymbol{g}_i\right) \times \left(x^j \boldsymbol{g}_j\right)$$
$$= e_{ijk} g w^i x^j \boldsymbol{g}^k = e_{ijk} g w^i \left(\boldsymbol{x} \cdot \boldsymbol{g}^j\right) \boldsymbol{g}^k = e_{ijk} g w^i \left(\boldsymbol{g}^k \otimes \boldsymbol{g}^j\right) \boldsymbol{x}.$$

Comparing the left and right hand side of this equality we obtain

$$\mathbf{W} = e_{ijk} g w^i \boldsymbol{g}^k \otimes \boldsymbol{g}^j, \tag{S.1}$$

so that the components of $\mathbf{W} = \mathrm{W}_{kj} \boldsymbol{g}^k \otimes \boldsymbol{g}^j$ can be given by $\mathrm{W}_{kj} = e_{ijk} g w^i$ or in the matrix form as

$$[\mathrm{W}_{kj}] = g \left[e_{ijk} w^i\right] = g \begin{bmatrix} 0 & -w^3 & w^2 \\ w^3 & 0 & -w^1 \\ -w^2 & w^1 & 0 \end{bmatrix}.$$

This yields also an alternative representation for $\mathbf{W}\boldsymbol{x}$ as follows

$$\mathbf{W}\boldsymbol{x} = g \left[\left(w^2 x^3 - w^3 x^2\right) \boldsymbol{g}^1 + \left(w^3 x^1 - w^1 x^3\right) \boldsymbol{g}^2 + \left(w^1 x^2 - w^2 x^1\right) \boldsymbol{g}^3\right].$$

It is seen that the tensor $\mathbf{W}$ is skew-symmetric because $\mathbf{W}^{\mathrm{T}} = -\mathbf{W}$.

1.22 According to (1.71) we can write

$$\mathbf{R} = \cos\alpha \mathbf{I} + \sin\alpha \hat{\boldsymbol{e}}_3 + (1 - \cos\alpha)\,(\boldsymbol{e}_3 \otimes \boldsymbol{e}_3)\,.$$

Thus, an arbitrary vector $\boldsymbol{a} = a^i \boldsymbol{e}_i$ in $\mathbb{E}^3$ is rotated to $\mathbf{R}\boldsymbol{a} = \cos\alpha \left(a^i \boldsymbol{e}_i\right) + \sin\alpha \boldsymbol{e}_3 \times \left(a^i \boldsymbol{e}_i\right) + (1 - \cos\alpha)\, a^3 \boldsymbol{e}_3$. By virtue of (1.45) we can further write

$$\mathbf{R}\boldsymbol{a} = \cos\alpha\left(a^i\boldsymbol{e}_i\right) + \sin\alpha\left(a^1\boldsymbol{e}_2 - a^2\boldsymbol{e}_1\right) + (1-\cos\alpha)\,a^3\boldsymbol{e}_3$$
$$= \left(a^1\cos\alpha - a^2\sin\alpha\right)\boldsymbol{e}_1 + \left(a^1\sin\alpha + a^2\cos\alpha\right)\boldsymbol{e}_2 + a^3\boldsymbol{e}_3.$$

Thus, the rotation tensor can be given by $\mathbf{R} = \mathrm{R}^{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, where

$$\left[\mathrm{R}^{ij}\right] = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1.23 With the aid of (1.83) and (1.92) we obtain

$$\left[\mathrm{A}^i_{\cdot j}\right] = \left[\mathrm{A}^{ik}g_{kj}\right] = \left[\mathrm{A}^{ik}\right]\left[g_{kj}\right] = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 & 6 \\ 3 & 9 & 8 \\ 6 & 8 & 21 \end{bmatrix} = \begin{bmatrix} -3 & -9 & -8 \\ 0 & 0 & 0 \\ 2 & 3 & 6 \end{bmatrix},$$

$$\left[\mathrm{A}_{i\cdot}^{\ j}\right] = \left[g_{ik}\mathrm{A}^{kj}\right] = \left[g_{ik}\right]\left[\mathrm{A}^{kj}\right] = \begin{bmatrix} 2 & 3 & 6 \\ 3 & 9 & 8 \\ 6 & 8 & 21 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 0 \\ 8 & -3 & 0 \\ 21 & -6 & 0 \end{bmatrix},$$

$$\left[\mathrm{A}_{ij}\right] = \left[g_{ik}\mathrm{A}^k_{\cdot j}\right] = \left[g_{ik}\right]\left[\mathrm{A}^k_{\cdot j}\right] = \left[\mathrm{A}_{i\cdot}^{\ k}g_{kj}\right] = \left[\mathrm{A}_{i\cdot}^{\ k}\right]\left[g_{kj}\right]$$
$$= \begin{bmatrix} 2 & 3 & 6 \\ 3 & 9 & 8 \\ 6 & 8 & 21 \end{bmatrix}\begin{bmatrix} -3 & -9 & -8 \\ 0 & 0 & 0 \\ 2 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 0 \\ 8 & -3 & 0 \\ 21 & -6 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 & 6 \\ 3 & 9 & 8 \\ 6 & 8 & 21 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 20 \\ 7 & -3 & 24 \\ 24 & 9 & 78 \end{bmatrix}.$$

1.24 By means of (1.53), (1.84), (1.98), Exercise 1.16 we can write

$$(\mathbf{A}\mathbf{0})\,\boldsymbol{x} = \mathbf{A}\,(\mathbf{0}\boldsymbol{x}) = \mathbf{A}\boldsymbol{0} = \boldsymbol{0}, \quad (\mathbf{0}\mathbf{A})\,\boldsymbol{x} = \mathbf{0}\,(\mathbf{A}\boldsymbol{x}) = \boldsymbol{0},$$

$$(\mathbf{A}\mathbf{I})\,\boldsymbol{x} = \mathbf{A}\,(\mathbf{I}\boldsymbol{x}) = \mathbf{A}\boldsymbol{x}, \quad (\mathbf{I}\mathbf{A})\,\boldsymbol{x} = \mathbf{I}\,(\mathbf{A}\boldsymbol{x}) = \mathbf{A}\boldsymbol{x},$$

$$\mathbf{A}\,(\mathbf{B}\mathbf{C})\,\boldsymbol{x} = \mathbf{A}\,[\mathbf{B}\,(\mathbf{C}\boldsymbol{x})] = (\mathbf{A}\mathbf{B})\,(\mathbf{C}\boldsymbol{x}) = [(\mathbf{A}\mathbf{B})\,\mathbf{C}]\,\boldsymbol{x}, \quad \forall\boldsymbol{x} \in \mathbb{E}^n.$$

1.25 To check the commutativeness of the tensors **A** and **B** we compute the components of the tensor **AB** − **BA**:

$$\left[(\mathbf{AB}-\mathbf{BA})^i_{\cdot j}\right] = \left[\mathrm{A}^i_{\cdot k}\mathrm{B}^k_{\cdot j} - \mathrm{B}^i_{\cdot k}\mathrm{A}^k_{\cdot j}\right] = \left[\mathrm{A}^i_{\cdot k}\right]\left[\mathrm{B}^k_{\cdot j}\right] - \left[\mathrm{B}^i_{\cdot k}\right]\left[\mathrm{A}^k_{\cdot j}\right]$$
$$= \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similar we also obtain

$$\left[(\mathbf{AC}-\mathbf{CA})^i_{\cdot j}\right] = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \left[(\mathbf{AD}-\mathbf{DA})^i_{\cdot j}\right] = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\left[(\mathbf{BC} - \mathbf{CB})^{i}_{\cdot j} \right] = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \left[(\mathbf{BD} - \mathbf{DB})^{i}_{\cdot j} \right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\left[(\mathbf{CD} - \mathbf{DC})^{i}_{\cdot j} \right] = \begin{bmatrix} 0 & 1 & -27 \\ 0 & 0 & 0 \\ 0 & 19/2 & 0 \end{bmatrix}.$$

Thus, $\mathbf{A}$ commutes with $\mathbf{B}$ while $\mathbf{B}$ also commutes with $\mathbf{D}$.

1.26 Taking into account commutativeness of $\mathbf{A}$ and $\mathbf{B}$ we obtain for example for $k = 2$

$$(\mathbf{A} + \mathbf{B})^2 = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2.$$

Generalizing this result for $k = 2, 3, \ldots$ we obtain using the Newton formula

$$(\mathbf{A} + \mathbf{B})^k = \sum_{i=0}^{k} \binom{k}{i} \mathbf{A}^{k-i}\mathbf{B}^i, \quad \text{where} \quad \binom{k}{i} = \frac{k!}{i!\,(k-i)!}. \tag{S.2}$$

1.27 Using the result of the previous Exercise we first write out the left hand side of (1.158) by

$$\exp(\mathbf{A} + \mathbf{B}) = \sum_{k=0}^{\infty} \frac{(\mathbf{A} + \mathbf{B})^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} \mathbf{A}^{k-i}\mathbf{B}^i$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!} \mathbf{A}^{k-i}\mathbf{B}^i = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{\mathbf{A}^{k-i}\mathbf{B}^i}{i!\,(k-i)!}.$$

Changing the summation order as shown in Fig. S.1 and using the abbreviation $l = k - i$ it yields

$$\exp(\mathbf{A} + \mathbf{B}) = \sum_{l=0}^{\infty} \sum_{i=0}^{\infty} \frac{\mathbf{A}^l\mathbf{B}^i}{i!l!}.$$

The same expression can alternatively be obtained by applying formally the Cauchy product of infinite series (see e.g. [23]). For the right hand side of (1.158) we finally get the same result as above:

$$\exp(\mathbf{A})\exp(\mathbf{B}) = \left(\sum_{l=0}^{\infty} \frac{\mathbf{A}^l}{l!} \right) \left(\sum_{i=0}^{\infty} \frac{\mathbf{B}^i}{i!} \right) = \sum_{l=0}^{\infty} \sum_{i=0}^{\infty} \frac{\mathbf{A}^l\mathbf{B}^i}{l!i!}.$$

1.28 Using the result of the previous Exercise we can write

$$\exp(k\mathbf{A}) = \exp[(k-1)\mathbf{A} + \mathbf{A}] = \exp[(k-1)\mathbf{A}]\exp(\mathbf{A})$$

$$= \exp[(k-2)\mathbf{A}][\exp(\mathbf{A})]^2 = \ldots$$

$$= \exp(\mathbf{A})[\exp(\mathbf{A})]^{k-1} = [\exp(\mathbf{A})]^k.$$

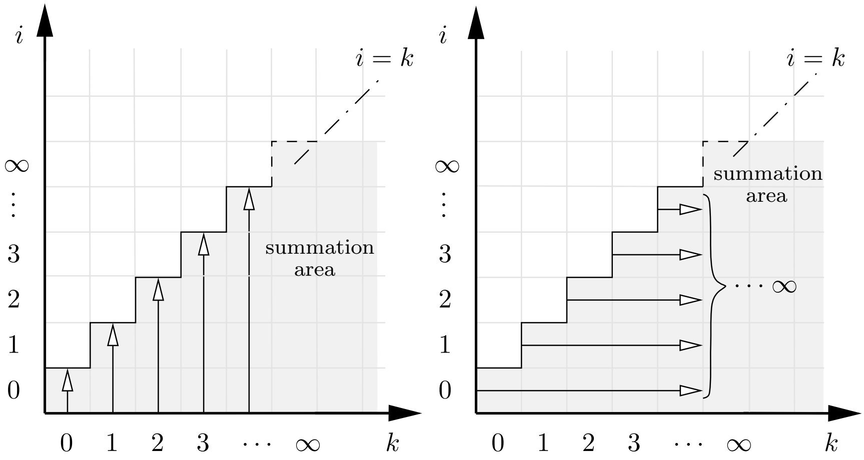

Fig. S.1. Geometric illustration of the summation order

1.29 Using the definition of the exponential tensor function (1.109) we get

$$\exp(\mathbf{0}) = \sum_{k=0}^{\infty} \frac{\mathbf{0}^k}{k!} = \mathbf{I} + \mathbf{0} + \mathbf{0} + \ldots = \mathbf{I},$$

$$\exp(\mathbf{I}) = \sum_{k=0}^{\infty} \frac{\mathbf{I}^k}{k!} = \sum_{k=0}^{\infty} \frac{\mathbf{I}}{k!} = \mathbf{I} \sum_{k=0}^{\infty} \frac{1}{k!} = \exp(1)\,\mathbf{I} = \mathrm{e}\mathbf{I}.$$

1.30 Since the tensors $\mathbf{A}$ and $-\mathbf{A}$ commute we can write

$$\exp(\mathbf{A})\exp(-\mathbf{A}) = \exp(-\mathbf{A})\exp(\mathbf{A}) = \exp[\mathbf{A} + (-\mathbf{A})] = \exp(\mathbf{0}) = \mathbf{I}.$$

1.31 (1.109), (S.2):

$$\exp(\mathbf{A} + \mathbf{B}) = \sum_{k=0}^{\infty} \frac{(\mathbf{A} + \mathbf{B})^k}{k!} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k + \mathbf{B}^k}{k!} = \exp(\mathbf{A}) + \exp(\mathbf{B}).$$

1.32 (1.109), (1.129):

$$\exp\left(\mathbf{QAQ}^{\mathrm{T}}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\mathbf{QAQ}^{\mathrm{T}}\right)^k = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\mathbf{QAQ}^{\mathrm{T}}\mathbf{QAQ}^{\mathrm{T}} \ldots \mathbf{QAQ}^{\mathrm{T}}}_{k \text{ times}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{QA}^k\mathbf{Q}^{\mathrm{T}} = \mathbf{Q} \left(\sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{A}^k \right) \mathbf{Q}^{\mathrm{T}} = \mathbf{Q} \exp(\mathbf{A}) \mathbf{Q}^{\mathrm{T}}.$$

1.33 We begin with the power of the tensor $\mathbf{D}$.

$$\mathbf{D}^2 = \mathbf{D}\mathbf{D} = \left(\mathrm{D}^i_{\cdot j}\boldsymbol{g}_i \otimes \boldsymbol{g}^j\right)\left(\mathrm{D}^k_{\cdot l}\boldsymbol{g}_k \otimes \boldsymbol{g}^l\right)$$
$$= \mathrm{D}^i_{\cdot j}\mathrm{D}^k_{\cdot l}\delta^j_k\boldsymbol{g}_i \otimes \boldsymbol{g}^l = \mathrm{D}^i_{\cdot j}\mathrm{D}^j_{\cdot l}\boldsymbol{g}_i \otimes \boldsymbol{g}^l = \left(\mathbf{D}^2\right)^i_{\cdot j}\boldsymbol{g}_i \otimes \boldsymbol{g}^j,$$

where $\left[\left(\mathbf{D}^2\right)^i_{\cdot j}\right] = \left[\mathrm{D}^i_{\cdot j}\right]\left[\mathrm{D}^i_{\cdot j}\right]$. Generalizing this results for an arbitrary integer exponent yields

$$\left[(\mathbf{D}^m)^i_{\cdot j}\right] = \underbrace{\left[\mathrm{D}^i_{\cdot j}\right]\ldots\left[\mathrm{D}^i_{\cdot j}\right]}_{m \text{ times}} = \begin{bmatrix} 2^m & 0 & 0 \\ 0 & 3^m & 0 \\ 0 & 0 & 1^m \end{bmatrix}.$$

We observe that the composition of tensors represented by mixed components related to the same mixed basis can be expressed in terms of the product of the component matrices. With this result in hand we thus obtain

$$\exp(\mathbf{D}) = \sum_{m=0}^{\infty} \frac{\mathbf{D}^m}{m!} = \exp(\mathbf{D})^i_{\cdot j}\,\boldsymbol{g}_i \otimes \boldsymbol{g}^j,$$

where

$$\left[\exp(\mathbf{D})^i_{\cdot j}\right] = \begin{bmatrix} \sum\limits_{m=0}^{\infty} \frac{2^m}{m!} & 0 & 0 \\ 0 & \sum\limits_{m=0}^{\infty} \frac{3^m}{m!} & 0 \\ 0 & 0 & \sum\limits_{m=0}^{\infty} \frac{1^m}{m!} \end{bmatrix} = \begin{bmatrix} \mathrm{e}^2 & 0 & 0 \\ 0 & \mathrm{e}^3 & 0 \\ 0 & 0 & \mathrm{e} \end{bmatrix}.$$

For the powers of the tensor $\mathbf{E}$ we further obtain

$$\mathbf{E}^k = \mathbf{0},\ k = 2, 3 \ldots$$

Hence,

$$\exp(\mathbf{E}) = \sum_{m=0}^{\infty} \frac{\mathbf{E}^m}{m!} = \mathbf{I} + \mathbf{E} + \mathbf{0} + \mathbf{0} + \ldots = \mathbf{I} + \mathbf{E},$$

so that

$$\left[\exp(\mathbf{E})^i_{\cdot j}\right] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

To express the exponential of the tensor $\mathbf{F}$ we first decompose it by $\mathbf{F} = \mathbf{X} + \mathbf{Y}$, where

$$\left[\mathrm{X}^i_{\cdot j}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \left[\mathrm{Y}^i_{\cdot j}\right] = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\mathbf{X}$ and $\mathbf{Y}$ are commutative since $\mathbf{XY} = \mathbf{YX} = \mathbf{0}$. Hence,

$$\exp(\mathbf{F}) = \exp(\mathbf{X} + \mathbf{Y}) = \exp(\mathbf{X})\exp(\mathbf{Y}).$$

Noticing that $\mathbf{X}$ has the form of $\mathbf{D}$ and $\mathbf{Y}$ that of $\mathbf{E}$ we can write

$$\left[\exp(\mathbf{X})^{i}_{\cdot j}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e} \end{bmatrix}, \quad \left[\exp(\mathbf{Y})^{i}_{\cdot j}\right] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally, we obtain

$$\left[\exp(\mathbf{F})^{i}_{\cdot j}\right] = \left[\exp(\mathbf{X})^{i}_{\cdot j}\right]\left[\exp(\mathbf{Y})^{i}_{\cdot j}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e} \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e} \end{bmatrix}.$$

1.34 (1.115): $(\mathbf{ABCD})^{\mathrm{T}} = (\mathbf{CD})^{\mathrm{T}}(\mathbf{AB})^{\mathrm{T}} = \mathbf{D}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

1.35 Using the result of the previous Exercise we can write

$$(\underbrace{\mathbf{AA}\ldots\mathbf{A}}_{k\ \text{times}})^{\mathrm{T}} = \underbrace{\mathbf{A}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\ldots\mathbf{A}^{\mathrm{T}}}_{k\ \text{times}} = \left(\mathbf{A}^{\mathrm{T}}\right)^{k}.$$

1.36 According to (1.119) and (1.120) $\mathrm{B}^{ij} = \mathrm{A}^{ji}$, $\mathrm{B}_{ij} = \mathrm{A}_{ji}$, $\mathrm{B}_{i\cdot}^{\ j} = \mathrm{A}^{j}_{\cdot i}$ and $\mathrm{B}^{i}_{\cdot j} = \mathrm{A}_{j\cdot}^{\ i}$ so that (see Exercise 1.23)

$$\left[\mathrm{B}^{ij}\right] = \left[\mathrm{A}^{ij}\right]^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \left[\mathrm{B}_{ij}\right] = \left[\mathrm{A}_{ij}\right]^{\mathrm{T}} = \begin{bmatrix} 6 & 7 & 24 \\ 0 & -3 & 9 \\ 20 & 24 & 78 \end{bmatrix},$$

$$\left[\mathrm{B}_{i\cdot}^{\ j}\right] = \left[\mathrm{A}^{i}_{\cdot j}\right]^{\mathrm{T}} = \begin{bmatrix} -3 & 0 & 2 \\ -9 & 0 & 3 \\ -8 & 0 & 6 \end{bmatrix}, \quad \left[\mathrm{B}^{i}_{\cdot j}\right] = \left[\mathrm{A}_{i\cdot}^{\ j}\right]^{\mathrm{T}} = \begin{bmatrix} 6 & 8 & 21 \\ -2 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}.$$

1.37 (1.115), (1.121), (1.125):

$$\mathbf{I} = \mathbf{I}^{\mathrm{T}} = \left(\mathbf{AA}^{-1}\right)^{\mathrm{T}} = \left(\mathbf{A}^{-1}\right)^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}.$$

1.38 $\left(\mathbf{A}^{k}\right)^{-1}$ is the tensor satisfying the identity $\left(\mathbf{A}^{k}\right)^{-1}\mathbf{A}^{k} = \mathbf{I}$. On the other hand, $\left(\mathbf{A}^{-1}\right)^{k}\mathbf{A}^{k} = \underbrace{\mathbf{A}^{-1}\mathbf{A}^{-1}\ldots\mathbf{A}^{-1}}_{k\ \text{times}}\underbrace{\mathbf{AA}\ldots\mathbf{A}}_{k\ \text{times}} = \mathbf{I}$. Thus, $\left(\mathbf{A}^{-1}\right)^{k} = \left(\mathbf{A}^{k}\right)^{-1}$.

1.39 An arbitrary tensor $\mathbf{A} \in \mathbf{Lin}^{n}$ can be represented with respect to a basis for example by $\mathbf{A} = \mathrm{A}^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j$. Thus, by virtue of (1.134) we obtain:

$$\begin{aligned}\boldsymbol{c} \otimes \boldsymbol{d} : \mathbf{A} &= \boldsymbol{c} \otimes \boldsymbol{d} : \left(\mathrm{A}^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j\right) = \mathrm{A}^{ij}\left(\boldsymbol{c} \cdot \boldsymbol{g}_i\right)\left(\boldsymbol{g}_j \cdot \boldsymbol{d}\right) \\ &= \boldsymbol{c}\left(\mathrm{A}^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j\right)\boldsymbol{d} = \boldsymbol{c}\mathbf{A}\boldsymbol{d} = \boldsymbol{d}\mathbf{A}^{\mathrm{T}}\boldsymbol{c}.\end{aligned}$$

1.40 The properties (D.1) and (D.3) directly follow from (1.134) and (1.136). Further, for three arbitrary tensors $\mathbf{A}$, $\mathbf{B} = \mathrm{B}^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j$ and $\mathbf{C} = \mathrm{C}^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j$ we have with the aid of (1.135)

$$\begin{aligned}\mathbf{A} : (\mathbf{B} + \mathbf{C}) &= \mathbf{A} : \left[\left(\mathrm{B}^{ij} + \mathrm{C}^{ij}\right)(\boldsymbol{g}_i \otimes \boldsymbol{g}_j)\right] = \left(\mathrm{B}^{ij} + \mathrm{C}^{ij}\right)(\boldsymbol{g}_i \mathbf{A} \boldsymbol{g}_j) \\ &= \mathrm{B}^{ij}(\boldsymbol{g}_i \mathbf{A} \boldsymbol{g}_j) + \mathrm{C}^{ij}(\boldsymbol{g}_i \mathbf{A} \boldsymbol{g}_j) \\ &= \mathbf{A} : \left(\mathrm{B}^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j\right) + \mathbf{A} : \left(\mathrm{C}^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j\right) = \mathbf{A} : \mathbf{B} + \mathbf{A} : \mathbf{C},\end{aligned}$$

which implies (D.2).

1.41 By virtue of (1.103), (1.84) and (1.135) we obtain

$$\begin{aligned}\left[(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d})\right] : \mathbf{I} &= \left[(\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{a} \otimes \boldsymbol{d})\right] : \mathbf{I} \\ &= (\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{a}\mathbf{I}\boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}).\end{aligned}$$

1.42 By virtue of (1.23), (1.82), (1.86), (1.91) and (1.134) we can write

$$\begin{aligned}\mathrm{tr}\mathbf{A} = \mathbf{A} : \mathbf{I} &= \left(\mathrm{A}^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j\right) : \left(\boldsymbol{g}_k \otimes \boldsymbol{g}^k\right) = \mathrm{A}^{ij}(\boldsymbol{g}_i \cdot \boldsymbol{g}_k)\left(\boldsymbol{g}_j \cdot \boldsymbol{g}^k\right) \\ &= \mathrm{A}^{ij} g_{ik}\delta_j^k = \mathrm{A}^{ij} g_{ij} = \mathrm{A}^{i}_{\cdot l} g^{lj} g_{ij} = \mathrm{A}^{i}_{\cdot l}\delta_i^l = \mathrm{A}^{i}_{\cdot i} = \mathrm{A}_{ji} g^{ji}.\end{aligned}$$

1.43 (1.140): $\mathbf{M} : \mathbf{W} = \mathbf{M}^{\mathrm{T}} : \mathbf{W}^{\mathrm{T}} = \mathbf{M} : (-\mathbf{W}) = -(\mathbf{M} : \mathbf{W}) = 0$.

1.44 $\mathbf{W}^k$ is skew-symmetric for odd k. Indeed, $\left(\mathbf{W}^k\right)^{\mathrm{T}} = \left(\mathbf{W}^{\mathrm{T}}\right)^k = (-\mathbf{W})^k = (-1)^k\mathbf{W}^k = -\mathbf{W}^k$. Thus, using the result of the previous Exercise we can write: $\mathrm{tr}\mathbf{W}^k = \mathbf{W}^k : \mathbf{I} = 0$.

1.45 By means of the definition (1.146) we obtain

$$\begin{aligned}\mathrm{sym}(\mathrm{skew}\mathbf{A}) &= \frac{1}{2}\left[\mathrm{skew}\mathbf{A} + (\mathrm{skew}\mathbf{A})^{\mathrm{T}}\right] \\ &= \frac{1}{2}\left[\frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathrm{T}}\right) + \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}}\right] \\ &= \frac{1}{2}\left[\frac{1}{2}\mathbf{A} - \frac{1}{2}\mathbf{A}^{\mathrm{T}} + \frac{1}{2}\mathbf{A}^{\mathrm{T}} - \frac{1}{2}\mathbf{A}\right] = \mathbf{0}.\end{aligned}$$

The same procedure leads to the identity $\mathrm{skew}(\mathrm{sym}\mathbf{A}) = \mathbf{0}$.

1.46 On use of (1.153) we can write

$$\mathrm{sph}(\mathrm{dev}\mathbf{A}) = \mathrm{sph}\left[\mathbf{A} - \frac{1}{n}\mathrm{tr}(\mathbf{A})\mathbf{I}\right] = \frac{1}{n}\mathrm{tr}\left[\mathbf{A} - \frac{1}{n}\mathrm{tr}(\mathbf{A})\mathbf{I}\right]\mathbf{I} = \mathbf{0},$$

where we take into account that $\mathrm{tr}\mathbf{I} = n$. In the same way, one proves that $\mathrm{dev}(\mathrm{sph}\mathbf{A}) = \mathbf{0}$.

Exercises of Chapter 2

2.1 The tangent vectors take the form:

$$\boldsymbol{g}_1 = \frac{\partial \boldsymbol{r}}{\partial \varphi} = r\cos\varphi\sin\phi\boldsymbol{e}_1 - r\sin\varphi\sin\phi\boldsymbol{e}_3,$$
$$\boldsymbol{g}_2 = \frac{\partial \boldsymbol{r}}{\partial \phi} = r\sin\varphi\cos\phi\boldsymbol{e}_1 - r\sin\phi\boldsymbol{e}_2 + r\cos\varphi\cos\phi\boldsymbol{e}_3,$$
$$\boldsymbol{g}_3 = \frac{\partial \boldsymbol{r}}{\partial r} = \sin\varphi\sin\phi\boldsymbol{e}_1 + \cos\phi\boldsymbol{e}_2 + \cos\varphi\sin\phi\boldsymbol{e}_3. \tag{S.3}$$

For the metrics coefficients we can further write:

$$\begin{aligned}\boldsymbol{g}_1\cdot\boldsymbol{g}_1 = &\ (r\cos\varphi\sin\phi\boldsymbol{e}_1 - r\sin\varphi\sin\phi\boldsymbol{e}_3)\\ &\cdot(r\cos\varphi\sin\phi\boldsymbol{e}_1 - r\sin\varphi\sin\phi\boldsymbol{e}_3) = r^2\sin^2\phi,\end{aligned}$$

$$\begin{aligned}\boldsymbol{g}_1\cdot\boldsymbol{g}_2 = &\ (r\cos\varphi\sin\phi\boldsymbol{e}_1 - r\sin\varphi\sin\phi\boldsymbol{e}_3)\\ &\cdot(r\sin\varphi\cos\phi\boldsymbol{e}_1 - r\sin\phi\boldsymbol{e}_2 + r\cos\varphi\cos\phi\boldsymbol{e}_3)\\ = &\ r^2(\sin\varphi\cos\varphi\sin\phi\cos\phi - \sin\varphi\cos\varphi\sin\phi\cos\phi) = 0,\end{aligned}$$

$$\begin{aligned}\boldsymbol{g}_1\cdot\boldsymbol{g}_3 = &\ (r\cos\varphi\sin\phi\boldsymbol{e}_1 - r\sin\varphi\sin\phi\boldsymbol{e}_3)\\ &\cdot(\sin\varphi\sin\phi\boldsymbol{e}_1 + \cos\phi\boldsymbol{e}_2 + \cos\varphi\sin\phi\boldsymbol{e}_3)\\ = &\ r\left(\sin\varphi\cos\varphi\sin^2\phi - \sin\varphi\cos\varphi\sin^2\phi\right) = 0,\end{aligned}$$

$$\begin{aligned}\boldsymbol{g}_2\cdot\boldsymbol{g}_2 = &\ (r\sin\varphi\cos\phi\boldsymbol{e}_1 - r\sin\phi\boldsymbol{e}_2 + r\cos\varphi\cos\phi\boldsymbol{e}_3)\\ &\cdot(r\sin\varphi\cos\phi\boldsymbol{e}_1 - r\sin\phi\boldsymbol{e}_2 + r\cos\varphi\cos\phi\boldsymbol{e}_3)\\ = &\ r^2\left(\sin^2\varphi\cos^2\phi + \sin^2\phi + \cos^2\varphi\cos^2\phi\right) = r^2,\end{aligned}$$

$$\begin{aligned}\boldsymbol{g}_2\cdot\boldsymbol{g}_3 = &\ (r\sin\varphi\cos\phi\boldsymbol{e}_1 - r\sin\phi\boldsymbol{e}_2 + r\cos\varphi\cos\phi\boldsymbol{e}_3)\\ &\cdot(\sin\varphi\sin\phi\boldsymbol{e}_1 + \cos\phi\boldsymbol{e}_2 + \cos\varphi\sin\phi\boldsymbol{e}_3)\\ = &\ r\left(\sin^2\varphi\sin\phi\cos\phi - \sin\phi\cos\phi + \cos^2\varphi\sin\phi\cos\phi\right) = 0,\end{aligned}$$

$$\begin{aligned}\boldsymbol{g}_3\cdot\boldsymbol{g}_3 = &\ (\sin\varphi\sin\phi\boldsymbol{e}_1 + \cos\phi\boldsymbol{e}_2 + \cos\varphi\sin\phi\boldsymbol{e}_3)\\ &\cdot(\sin\varphi\sin\phi\boldsymbol{e}_1 + \cos\phi\boldsymbol{e}_2 + \cos\varphi\sin\phi\boldsymbol{e}_3)\\ = &\ \sin^2\varphi\sin^2\phi + \cos^2\phi + \cos^2\varphi\sin^2\phi = 1.\end{aligned}$$

Thus,

$$[g_{ij}] = [\boldsymbol{g}_i\cdot\boldsymbol{g}_j] = \begin{bmatrix} r^2\sin^2\phi & 0 & 0\\ 0 & r^2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

and consequently

$$\left[g^{ij}\right] = [g_{ij}]^{-1} = \begin{bmatrix} \dfrac{1}{r^2\sin^2\phi} & 0 & 0\\ 0 & \dfrac{1}{r^2} & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{S.4}$$

Finally, we calculate the dual basis by $(1.21)_1$:

$$\boldsymbol{g}^1 = \frac{1}{r^2\sin^2\phi}\boldsymbol{g}_1 = r^{-1}\frac{\cos\varphi}{\sin\phi}\boldsymbol{e}_1 - r^{-1}\frac{\sin\varphi}{\sin\phi}\boldsymbol{e}_3,$$

$$\boldsymbol{g}^2 = \frac{1}{r^2}\boldsymbol{g}_2 = r^{-1}\sin\varphi\cos\phi\boldsymbol{e}_1 - r^{-1}\sin\phi\boldsymbol{e}_2 + r^{-1}\cos\varphi\cos\phi\boldsymbol{e}_3,$$

$$\boldsymbol{g}^3 = \boldsymbol{g}_3 = \sin\varphi\sin\phi\boldsymbol{e}_1 + \cos\phi\boldsymbol{e}_2 + \cos\varphi\sin\phi\boldsymbol{e}_3.$$

2.2 The connection between the linear and spherical coordinates (2.144) can be expressed by

$$x^1 = r\sin\varphi\sin\phi, \quad x^2 = r\cos\phi, \quad x^3 = r\cos\varphi\sin\phi.$$

Thus, we obtain

$$\frac{\partial x^1}{\partial\varphi} = r\cos\varphi\sin\phi, \qquad \frac{\partial x^1}{\partial\phi} = r\sin\varphi\cos\phi, \qquad \frac{\partial x^1}{\partial r} = \sin\varphi\sin\phi,$$

$$\frac{\partial x^2}{\partial\varphi} = 0, \qquad \frac{\partial x^2}{\partial\phi} = -r\sin\phi, \qquad \frac{\partial x^2}{\partial r} = \cos\phi,$$

$$\frac{\partial x^3}{\partial\varphi} = -r\sin\varphi\sin\phi, \qquad \frac{\partial x^3}{\partial\phi} = r\cos\varphi\cos\phi, \qquad \frac{\partial x^3}{\partial r} = \cos\varphi\sin\phi.$$

Inverting the so-constructed matrix $\left[\frac{\partial x^i}{\partial\varphi}\ \frac{\partial x^i}{\partial\phi}\ \frac{\partial x^i}{\partial r}\right]$ further yields

$$\frac{\partial\varphi}{\partial x^1} = \frac{\cos\varphi}{r\sin\phi}, \qquad \frac{\partial\varphi}{\partial x^2} = 0, \qquad \frac{\partial\varphi}{\partial x^3} = -\frac{\sin\varphi}{r\sin\phi},$$

$$\frac{\partial\phi}{\partial x^1} = \frac{\sin\varphi\cos\phi}{r}, \qquad \frac{\partial\phi}{\partial x^2} = -\frac{\sin\phi}{r}, \qquad \frac{\partial\phi}{\partial x^3} = \frac{\cos\varphi\cos\phi}{r},$$

$$\frac{\partial r}{\partial x^1} = \sin\varphi\sin\phi, \qquad \frac{\partial r}{\partial x^2} = \cos\phi, \qquad \frac{\partial r}{\partial x^3} = \cos\varphi\sin\phi.$$

2.3 Applying the directional derivative we have

$$\text{(a)}: \ \frac{\mathrm{d}}{\mathrm{d}s}\left\|\boldsymbol{r}+s\boldsymbol{a}\right\|^{-1}\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\left[(\boldsymbol{r}+s\boldsymbol{a})\cdot(\boldsymbol{r}+s\boldsymbol{a})\right]^{-1/2}\Big|_{s=0}$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\left[\boldsymbol{r}\cdot\boldsymbol{r} + 2s\boldsymbol{r}\cdot\boldsymbol{a} + s^2\boldsymbol{a}\cdot\boldsymbol{a}\right]^{-1/2}\Big|_{s=0}$$

$$= -\frac{1}{2}\frac{2\boldsymbol{r}\cdot\boldsymbol{a} + 2s\boldsymbol{a}\cdot\boldsymbol{a}}{\left[(\boldsymbol{r}+s\boldsymbol{a})\cdot(\boldsymbol{r}+s\boldsymbol{a})\right]^{3/2}}\Big|_{s=0} = -\frac{\boldsymbol{r}\cdot\boldsymbol{a}}{\left\|\boldsymbol{r}\right\|^3}.$$

Comparing with (2.51) finally yields

$$\operatorname{grad} \|\boldsymbol{r}\|^{-1} = -\frac{\boldsymbol{r}}{\|\boldsymbol{r}\|^3}.$$

$$(\mathrm{b}): \frac{\mathrm{d}}{\mathrm{d}s} (\boldsymbol{r} + s\boldsymbol{a}) \cdot \boldsymbol{w}\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s} (\boldsymbol{r} \cdot \boldsymbol{w} + s\boldsymbol{a} \cdot \boldsymbol{w})\Big|_{s=0} = \boldsymbol{a} \cdot \boldsymbol{w}.$$

Hence, $\operatorname{grad} (\boldsymbol{r} \cdot \boldsymbol{w}) = \boldsymbol{w}$.

$$(\mathrm{c}): \frac{\mathrm{d}}{\mathrm{d}s} (\boldsymbol{r} + s\boldsymbol{a}) \mathbf{A} (\boldsymbol{r} + s\boldsymbol{a})\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s} \left(\boldsymbol{r}\mathbf{A}\boldsymbol{r} + s\boldsymbol{a}\mathbf{A}\boldsymbol{r} + s\boldsymbol{r}\mathbf{A}\boldsymbol{a} + s^2\boldsymbol{a}\mathbf{A}\boldsymbol{a}\right)\Big|_{s=0}$$

$$= \boldsymbol{a}\mathbf{A}\boldsymbol{r} + \boldsymbol{r}\mathbf{A}\boldsymbol{a} = (\mathbf{A}\boldsymbol{r}) \cdot \boldsymbol{a} + (\boldsymbol{r}\mathbf{A}) \cdot \boldsymbol{a} = (\mathbf{A}\boldsymbol{r} + \boldsymbol{r}\mathbf{A}) \cdot \boldsymbol{a},$$

Thus, applying (1.110) and $(1.146)_1$ we can write

$$\operatorname{grad} (\boldsymbol{r}\mathbf{A}\boldsymbol{r}) = \mathbf{A}\boldsymbol{r} + \boldsymbol{r}\mathbf{A} = \left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right) \boldsymbol{r} = 2 (\operatorname{sym}\mathbf{A}) \boldsymbol{r}.$$

$$(\mathrm{d}): \frac{\mathrm{d}}{\mathrm{d}s} \mathbf{A} (\boldsymbol{r} + s\boldsymbol{a})\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s} (\mathbf{A}\boldsymbol{r} + s\mathbf{A}\boldsymbol{a})\Big|_{s=0} = \mathbf{A}\boldsymbol{a}.$$

Comparing with (2.53) we then have

$$\operatorname{grad} (\mathbf{A}\boldsymbol{r}) = \mathbf{A}.$$

(e) : In view of (1.64) and using the results of (d) we obtain

$$\operatorname{grad} (\boldsymbol{w} \times \boldsymbol{r}) = \operatorname{grad} (\mathbf{W}\boldsymbol{r}) = \mathbf{W}.$$

With the aid of the representation $\boldsymbol{w} = w^i \boldsymbol{g}_i$ we can further write (see Exercise 1.21)

$$\mathbf{W} = \mathrm{W}_{ij} \boldsymbol{g}^i \otimes \boldsymbol{g}^j, \quad [\mathrm{W}_{ij}] = g \begin{bmatrix} 0 & -w^3 & w^2 \\ w^3 & 0 & -w^1 \\ -w^2 & w^1 & 0 \end{bmatrix}.$$

2.4 We begin with the derivative of the metrics coefficients obtained in Exercise 2.1:

$$[g_{ij},_1] = \left[\frac{\partial g_{ij}}{\partial \varphi}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; [g_{ij},_2] = \left[\frac{\partial g_{ij}}{\partial \phi}\right] = \begin{bmatrix} 2r^2 \sin\phi \cos\phi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[g_{ij},_3] = \left[\frac{\partial g_{ij}}{\partial r}\right] = \begin{bmatrix} 2r \sin^2\phi & 0 & 0 \\ 0 & 2r & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, according to (2.74)

$$[\Gamma_{ij1}] = \left[\frac{1}{2}\left(g_{1i},_j + g_{1j},_i - g_{ij},_1\right)\right]$$

$$= \begin{bmatrix} 0 & r^2\sin\phi\cos\phi & r\sin^2\phi \\ r^2\sin\phi\cos\phi & 0 & 0 \\ r\sin^2\phi & 0 & 0 \end{bmatrix},$$

$$[\Gamma_{ij2}] = \left[\frac{1}{2}\left(g_{2i},_j + g_{2j},_i - g_{ij},_2\right)\right] = \begin{bmatrix} -r^2\sin\phi\cos\phi & 0 & 0 \\ 0 & 0 & r \\ 0 & r & 0 \end{bmatrix},$$

$$[\Gamma_{ij3}] = \left[\frac{1}{2}\left(g_{3i},_j + g_{3j},_i - g_{ij},_3\right)\right] = \begin{bmatrix} -r\sin^2\phi & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

With the aid of (2.67) we further obtain

$$\Gamma^1_{ij} = g^{1l}\Gamma_{ijl} = g^{11}\Gamma_{ij1} + g^{12}\Gamma_{ij2} + g^{13}\Gamma_{ij3} = \frac{\Gamma_{ij1}}{r^2\sin^2\phi}, \quad i,j = 1,2,3,$$

$$\left[\Gamma^1_{ij}\right] = \begin{bmatrix} 0 & \cot\phi & r^{-1} \\ \cot\phi & 0 & 0 \\ r^{-1} & 0 & 0 \end{bmatrix}, \tag{S.5}$$

$$\Gamma^2_{ij} = g^{2l}\Gamma_{ijl} = g^{21}\Gamma_{ij1} + g^{22}\Gamma_{ij2} + g^{23}\Gamma_{ij3} = \frac{\Gamma_{ij2}}{r^2}, \quad i,j = 1,2,3,$$

$$\left[\Gamma^2_{ij}\right] = \begin{bmatrix} -\sin\phi\cos\phi & 0 & 0 \\ 0 & 0 & r^{-1} \\ 0 & r^{-1} & 0 \end{bmatrix}, \tag{S.6}$$

$$\Gamma^3_{ij} = g^{3l}\Gamma_{ijl} = g^{31}\Gamma_{ij1} + g^{32}\Gamma_{ij2} + g^{33}\Gamma_{ij3} = \Gamma_{ij3}, \quad i,j = 1,2,3,$$

$$\left[\Gamma^3_{ij}\right] = \begin{bmatrix} -r\sin^2\phi & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{S.7}$$

2.5 Relations (2.87) can be obtained in the same manner as (2.85). Indeed, using the representation $\mathbf{A} = \mathrm{A}_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j$ and by virtue of (2.72) we have for example for $(2.87)_1$:

$$\begin{aligned}
\mathbf{A},_k &= \left(\mathrm{A}_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j\right),_k \\
&= \mathrm{A}_{ij},_k\, \boldsymbol{g}^i \otimes \boldsymbol{g}^j + \mathrm{A}_{ij}\boldsymbol{g}^i,_k \otimes \boldsymbol{g}^j + \mathrm{A}_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j,_k \\
&= \mathrm{A}_{ij},_k\, \boldsymbol{g}^i \otimes \boldsymbol{g}^j + \mathrm{A}_{ij}\left(-\Gamma^i_{lk}\boldsymbol{g}^l\right) \otimes \boldsymbol{g}^j + \mathrm{A}_{ij}\boldsymbol{g}^i \otimes \left(-\Gamma^j_{lk}\boldsymbol{g}^l\right) \\
&= \left(\mathrm{A}_{ij},_k - \mathrm{A}_{lj}\Gamma^l_{ik} - \mathrm{A}_{il}\Gamma^l_{jk}\right)\boldsymbol{g}^i \otimes \boldsymbol{g}^j.
\end{aligned}$$

2.6 Using $(2.87)_1$ we write for example for the left hand side of (2.92)

$$\mathrm{A}_{ij}|_k = \mathrm{A}_{ij},_k - \mathrm{A}_{lj}\Gamma^l_{ik} - \mathrm{A}_{il}\Gamma^l_{jk}.$$

In view of $(2.84)_2$ the same result holds for the right hand side of (2.92) as well. Indeed,

$$\begin{aligned} a_i|_k\, b_j + a_i b_j|_k &= \left(a_i,_k - a_l\Gamma^l_{ik}\right) b_j + a_i \left(b_j,_k - b_l\Gamma^l_{jk}\right) \\ &= a_i,_k\, b_j + a_i b_j,_k - a_l b_j \Gamma^l_{ik} - a_i b_l \Gamma^l_{jk} \\ &= \mathrm{A}_{ij},_k - \mathrm{A}_{lj}\Gamma^l_{ik} - \mathrm{A}_{il}\Gamma^l_{jk}. \end{aligned}$$

2.7 By analogy with (S.1)

$$\hat{\boldsymbol{t}} = e^{ijk} g^{-1} t_i \boldsymbol{g}_k \otimes \boldsymbol{g}_j.$$

Inserting this expression into (2.116) and taking (2.103) into account we further write

$$\mathrm{curl}\boldsymbol{t} = -\mathrm{div}\hat{\boldsymbol{t}} = -\left(e^{ijk} g^{-1} t_i \boldsymbol{g}_k \otimes \boldsymbol{g}_j\right),_l \boldsymbol{g}^l.$$

With the aid of the identities $\left(g^{-1}\boldsymbol{g}_j\right),_l \cdot \boldsymbol{g}^l = 0\,(j = 1, 2, 3)$ following from (2.66) and (2.98) and applying the product rule of differentiation we finally obtain

$$\begin{aligned} \mathrm{curl}\boldsymbol{t} &= -e^{ijk} g^{-1} t_i,_j\, \boldsymbol{g}_k - e^{ijk} g^{-1} t_i \boldsymbol{g}_k,_j \\ &= -e^{ijk} g^{-1} t_i,_j\, \boldsymbol{g}_k = -e^{ijk} g^{-1} t_i|_j\, \boldsymbol{g}_k = e^{jik} g^{-1} t_i|_j\, \boldsymbol{g}_k \end{aligned}$$

keeping (1.36), (2.68) and $(2.84)_2$ in mind.

2.8 We begin with the covariant derivative of the Cauchy stress components (2.109). Using the results of Exercise 2.4 concerning the Christoffel symbols for the spherical coordinates we get

$$\sigma^{1j}|_j = \sigma^{1j},_j + \sigma^{lj}\Gamma^1_{lj} + \sigma^{1l}\Gamma^j_{lj} = \sigma^{11},_1 + \sigma^{12},_2 + \sigma^{13},_3 + 3\sigma^{12}\cot\phi + 4\frac{\sigma^{13}}{r},$$

$$\begin{aligned} \sigma^{2j}|_j &= \sigma^{2j},_j + \sigma^{lj}\Gamma^2_{lj} + \sigma^{2l}\Gamma^j_{lj} \\ &= \sigma^{21},_1 + \sigma^{22},_2 + \sigma^{23},_3 - \sigma^{11}\sin\phi\cos\phi + \sigma^{22}\cot\phi + 4\frac{\sigma^{23}}{r}, \end{aligned}$$

$$\begin{aligned} \sigma^{3j}|_j &= \sigma^{3j},_j + \sigma^{lj}\Gamma^3_{lj} + \sigma^{3l}\Gamma^j_{lj} \\ &= \sigma^{31},_1 + \sigma^{32},_2 + \sigma^{33},_3 - \sigma^{11} r\sin^2\phi - \sigma^{22} r + \sigma^{32}\cot\phi + 2\frac{\sigma^{33}}{r}. \end{aligned}$$

The balance equations (2.107) take thus the form

$$\rho\ddot{x}^1 = \sigma^{11},_1 + \sigma^{12},_2 + \sigma^{13},_3 + 3\sigma^{12}\cot\phi + 4\frac{\sigma^{13}}{r} + f^1,$$

$$\rho\ddot{x}^2 = \sigma^{21},_1 + \sigma^{22},_2 + \sigma^{23},_3 - \sigma^{11}\sin\phi\cos\phi + \sigma^{22}\cot\phi + 4\frac{\sigma^{23}}{r} + f^2,$$

$$\rho\ddot{x}^3 = \sigma^{31},_1 + \sigma^{32},_2 + \sigma^{33},_3 - \sigma^{11}r\sin^2\phi - \sigma^{22}r + \sigma^{32}\cot\phi + 2\frac{\sigma^{33}}{r} + f^3.$$

2.9 The tangent vectors take the form:

$$\boldsymbol{g}_1 = \frac{\partial \boldsymbol{r}}{\partial r} = \left(\cos\frac{s}{r} + \frac{s}{r}\sin\frac{s}{r}\right)\boldsymbol{e}_1 + \left(\sin\frac{s}{r} - \frac{s}{r}\cos\frac{s}{r}\right)\boldsymbol{e}_2,$$

$$\boldsymbol{g}_2 = \frac{\partial \boldsymbol{r}}{\partial s} = -\sin\frac{s}{r}\boldsymbol{e}_1 + \cos\frac{s}{r}\boldsymbol{e}_2, \quad \boldsymbol{g}_3 = \frac{\partial \boldsymbol{r}}{\partial z} = \boldsymbol{e}_3.$$

The metrics coefficients can further be written by

$$[g_{ij}] = [\boldsymbol{g}_i \cdot \boldsymbol{g}_j] = \begin{bmatrix} 1+\frac{s^2}{r^2} & -\frac{s}{r} & 0 \\ -\frac{s}{r} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \left[g^{ij}\right] = [g_{ij}]^{-1} = \begin{bmatrix} 1 & \frac{s}{r} & 0 \\ \frac{s}{r} & 1+\frac{s^2}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For the dual basis we use $(1.21)_1$:

$$\boldsymbol{g}^1 = \boldsymbol{g}_1 + \frac{s}{r}\boldsymbol{g}_2 = \cos\frac{s}{r}\boldsymbol{e}_1 + \sin\frac{s}{r}\boldsymbol{e}_2,$$

$$\boldsymbol{g}^2 = \frac{s}{r}\boldsymbol{g}_1 + \left(1+\frac{s^2}{r^2}\right)\boldsymbol{g}_2$$

$$= \left(-\sin\frac{s}{r} + \frac{s}{r}\cos\frac{s}{r}\right)\boldsymbol{e}_1 + \left(\cos\frac{s}{r} + \frac{s}{r}\sin\frac{s}{r}\right)\boldsymbol{e}_2,$$

$$\boldsymbol{g}_3 = \boldsymbol{g}^3 = \boldsymbol{e}_3.$$

The derivatives of the metrics coefficients become

$$[g_{ij},_1] = \begin{bmatrix} -2\frac{s^2}{r^3} & \frac{s}{r^2} & 0 \\ \frac{s}{r^2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [g_{ij},_2] = \begin{bmatrix} \frac{2s}{r^2} & -\frac{1}{r} & 0 \\ -\frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [g_{ij},_3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For the Christoffel symbols we thus obtain by means of (2.74) and (2.67):

$$[\Gamma_{ij1}] = \begin{bmatrix} -\frac{s^2}{r^3} & \frac{s}{r^2} & 0 \\ \frac{s}{r^2} & -\frac{1}{r} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\Gamma_{ij2}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\Gamma_{ij3}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[\Gamma^1_{ij}] = \begin{bmatrix} -\frac{s^2}{r^3} & \frac{s}{r^2} & 0 \\ \frac{s}{r^2} & -\frac{1}{r} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ [\Gamma^2_{ij}] = \begin{bmatrix} -\frac{s^3}{r^4} & \frac{s^2}{r^3} & 0 \\ \frac{s^2}{r^3} & -\frac{s}{r^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ [\Gamma^3_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2.10 First, we express the covariant derivative of the Cauchy stress components by (2.109) using the results of the previous exercise:

$$\sigma^{1j}|_j = \sigma^{11},_r + \sigma^{12},_s + \sigma^{13},_z - \sigma^{11}\frac{s^2}{r^3} - \frac{\sigma^{22}}{r} + 2\sigma^{12}\frac{s}{r^2},$$

$$\sigma^{2j}|_j = \sigma^{21},_r + \sigma^{22},_s + \sigma^{23},_z - \sigma^{11}\frac{s^3}{r^4} - \sigma^{22}\frac{s}{r^2} + 2\sigma^{12}\frac{s^2}{r^3},$$

$$\sigma^{3j}|_j = \sigma^{31},_r + \sigma^{32},_s + \sigma^{33},_z .$$

The balance equations (2.107) become

$$\rho\ddot{x}^1 = \sigma^{11},_r + \sigma^{12},_s + \sigma^{13},_z - \sigma^{11}\frac{s^2}{r^3} - \frac{\sigma^{22}}{r} + 2\sigma^{12}\frac{s}{r^2} + f^1,$$

$$\rho\ddot{x}^2 = \sigma^{21},_r + \sigma^{22},_s + \sigma^{23},_z - \sigma^{11}\frac{s^3}{r^4} - \sigma^{22}\frac{s}{r^2} + 2\sigma^{12}\frac{s^2}{r^3} + f^2,$$

$$\rho\ddot{x}^3 = \sigma^{31},_r + \sigma^{32},_s + \sigma^{33},_z + f^3.$$

2.11 (2.117), (2.119), (1.32), (2.72):

$$\begin{aligned}\operatorname{div}\operatorname{curl}\boldsymbol{t} &= \left(\boldsymbol{g}^i \times \boldsymbol{t},_i\right),_j \cdot \boldsymbol{g}^j = \left(-\Gamma^i_{kj}\boldsymbol{g}^k \times \boldsymbol{t},_i + \boldsymbol{g}^i \times \boldsymbol{t},_{ij}\right) \cdot \boldsymbol{g}^j \\ &= -\left(\Gamma^i_{kj}\boldsymbol{g}^j \times \boldsymbol{g}^k\right) \cdot \boldsymbol{t},_i + \left(\boldsymbol{g}^j \times \boldsymbol{g}^i\right) \cdot \boldsymbol{t},_{ij} = 0,\end{aligned}$$

where we take into consideration that $\boldsymbol{t},_{ij} = \boldsymbol{t},_{ji}$, $\Gamma^l_{ij} = \Gamma^l_{ji}$ and $\boldsymbol{g}^i \times \boldsymbol{g}^j = -\boldsymbol{g}^j \times \boldsymbol{g}^i$ $(i \neq j,\ i,j = 1,2,3)$.

(2.117), (2.119), (1.32):

$$\begin{aligned}\operatorname{div}(\boldsymbol{u} \times \boldsymbol{v}) &= (\boldsymbol{u} \times \boldsymbol{v}),_i \cdot \boldsymbol{g}^i = (\boldsymbol{u},_i \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{v},_i) \cdot \boldsymbol{g}^i \\ &= \left(\boldsymbol{g}^i \times \boldsymbol{u},_i\right) \cdot \boldsymbol{v} + \left(\boldsymbol{v},_i \times \boldsymbol{g}^i\right) \cdot \boldsymbol{u} = \boldsymbol{v} \cdot \operatorname{curl}\boldsymbol{u} - \boldsymbol{u} \cdot \operatorname{curl}\boldsymbol{v}.\end{aligned}$$

(2.6), $(2.65)_1$, (2.117):

$$\operatorname{grad}\operatorname{div}\boldsymbol{t} = \left(\boldsymbol{t},_i \cdot \boldsymbol{g}^i\right),_j \boldsymbol{g}^j = \left(\boldsymbol{t},_{ij} \cdot \boldsymbol{g}^i\right)\boldsymbol{g}^j + \left(\boldsymbol{t},_i \cdot \boldsymbol{g}^i,_j\right)\boldsymbol{g}^j.$$

Using the relation

$$\begin{aligned}\left(\boldsymbol{t},_i \cdot \boldsymbol{g}^i,_j\right)\boldsymbol{g}^j &= \left[\boldsymbol{t},_i \cdot \left(-\Gamma^i_{jk}\boldsymbol{g}^k\right)\right]\boldsymbol{g}^j \\ &= \left(\boldsymbol{t},_i \cdot \boldsymbol{g}^k\right)\left(-\Gamma^i_{jk}\boldsymbol{g}^j\right) = \left(\boldsymbol{t},_i \cdot \boldsymbol{g}^k\right)\boldsymbol{g}^i,_k = \left(\boldsymbol{t},_i \cdot \boldsymbol{g}^j\right)\boldsymbol{g}^i,_j \end{aligned} \qquad \text{(S.8)}$$

following from (2.72) we thus write

$$\operatorname{grad}\operatorname{div}\boldsymbol{t} = \left(\boldsymbol{t},_{ij}\cdot\boldsymbol{g}^i\right)\boldsymbol{g}^j + \left(\boldsymbol{t},_i\cdot\boldsymbol{g}^j\right)\boldsymbol{g}^i,_j .$$

(2.119), (1.157):

$$\operatorname{curl}\operatorname{curl}\boldsymbol{t} = \boldsymbol{g}^j\times\left(\boldsymbol{g}^i\times\boldsymbol{t},_i\right),_j = \boldsymbol{g}^j\times\left(\boldsymbol{g}^i,_j\times\boldsymbol{t},_i\right) + \boldsymbol{g}^j\times\left(\boldsymbol{g}^i\times\boldsymbol{t},_{ij}\right)$$
$$= \left(\boldsymbol{g}^j\cdot\boldsymbol{t},_i\right)\boldsymbol{g}^i,_j - \left(\boldsymbol{g}^j\cdot\boldsymbol{g}^i,_j\right)\boldsymbol{t},_i + \left(\boldsymbol{g}^j\cdot\boldsymbol{t},_{ij}\right)\boldsymbol{g}^i - g^{ij}\boldsymbol{t},_{ij} .$$

(2.8), $(2.59)_1$, (2.117), (1.116), (S.8):

$$\operatorname{div}\operatorname{grad}\boldsymbol{t} = \left(\boldsymbol{t},_i\otimes\boldsymbol{g}^i\right),_j\cdot\boldsymbol{g}^j = \left(\boldsymbol{t},_{ij}\otimes\boldsymbol{g}^i\right)\cdot\boldsymbol{g}^j + \left(\boldsymbol{t},_i\otimes\boldsymbol{g}^i,_j\right)\cdot\boldsymbol{g}^j$$
$$= g^{ij}\boldsymbol{t},_{ij} + \left(\boldsymbol{g}^i,_j\cdot\boldsymbol{g}^j\right)\boldsymbol{t},_i . \tag{S.9}$$

$$\operatorname{div}\left(\operatorname{grad}\boldsymbol{t}\right)^{\mathrm{T}} = \left(\boldsymbol{t},_i\otimes\boldsymbol{g}^i\right)^{\mathrm{T}},_j\cdot\boldsymbol{g}^j = \left(\boldsymbol{g}^i\otimes\boldsymbol{t},_{ij}\right)\cdot\boldsymbol{g}^j + \left(\boldsymbol{g}^i,_j\otimes\boldsymbol{t},_i\right)\cdot\boldsymbol{g}^j$$
$$= \left(\boldsymbol{t},_{ij}\cdot\boldsymbol{g}^j\right)\boldsymbol{g}^i + \left(\boldsymbol{t},_i\cdot\boldsymbol{g}^j\right)\boldsymbol{g}^i,_j .$$

The latter four relations immediately imply (2.126) and (2.127).

(2.5), (1.135), (2.103), (2.117), $(2.59)_1$:

$$\operatorname{div}\left(\boldsymbol{t}\mathbf{A}\right) = \left(\boldsymbol{t}\mathbf{A}\right),_i\cdot\boldsymbol{g}^i = \left(\boldsymbol{t},_i\mathbf{A}\right)\cdot\boldsymbol{g}^i + \left(\boldsymbol{t}\mathbf{A},_i\right)\cdot\boldsymbol{g}^i$$
$$= \mathbf{A}:\boldsymbol{t},_i\otimes\boldsymbol{g}^i + \boldsymbol{t}\cdot\left(\mathbf{A},_i\boldsymbol{g}^i\right) = \mathbf{A}:\operatorname{grad}\boldsymbol{t} + \boldsymbol{t}\cdot\operatorname{div}\mathbf{A}.$$

(2.3), (2.58), (2.117):

$$\operatorname{div}\left(\Phi\boldsymbol{t}\right) = \left(\Phi\boldsymbol{t}\right),_i\cdot\boldsymbol{g}^i = \left(\Phi,_i\boldsymbol{t}\right)\cdot\boldsymbol{g}^i + \left(\Phi\boldsymbol{t},_i\right)\cdot\boldsymbol{g}^i$$
$$= \boldsymbol{t}\cdot\left(\Phi,_i\boldsymbol{g}^i\right) + \Phi\left(\boldsymbol{t},_i\cdot\boldsymbol{g}^i\right) = \boldsymbol{t}\cdot\operatorname{grad}\Phi + \Phi\operatorname{div}\boldsymbol{t}.$$

(2.4), (2.58), (2.103):

$$\operatorname{div}\left(\Phi\mathbf{A}\right) = \left(\Phi\mathbf{A}\right),_i\boldsymbol{g}^i = \left(\Phi,_i\mathbf{A}\right)\boldsymbol{g}^i + \left(\Phi\mathbf{A},_i\right)\boldsymbol{g}^i$$
$$= \mathbf{A}\left(\Phi,_i\boldsymbol{g}^i\right) + \Phi\left(\mathbf{A},_i\boldsymbol{g}^i\right) = \mathbf{A}\operatorname{grad}\Phi + \Phi\operatorname{div}\mathbf{A}.$$

2.12 Cylindrical coordinates, $(2.65)_2$, (2.84), (2.81):

$$\operatorname{grad}\boldsymbol{t} = t_i|_j\,\boldsymbol{g}^i\otimes\boldsymbol{g}^j = \left(t_i,_j - t_k\Gamma^k_{ij}\right)\boldsymbol{g}^i\otimes\boldsymbol{g}^j$$
$$= t_i,_j\,\boldsymbol{g}^i\otimes\boldsymbol{g}^j + rt_3\boldsymbol{g}^1\otimes\boldsymbol{g}^1 - r^{-1}t_1\left(\boldsymbol{g}^1\otimes\boldsymbol{g}^3 + \boldsymbol{g}^3\otimes\boldsymbol{g}^1\right),$$

or alternatively

$$\operatorname{grad}\boldsymbol{t} = t^i|_j\,\boldsymbol{g}_i\otimes\boldsymbol{g}^j = \left(t^i,_j + t^k\Gamma^i_{kj}\right)\boldsymbol{g}_i\otimes\boldsymbol{g}^j$$
$$= t^i,_j\,\boldsymbol{g}_i\otimes\boldsymbol{g}^j + r^{-1}t^3\boldsymbol{g}_1\otimes\boldsymbol{g}^1 + t^1\left(r^{-1}\boldsymbol{g}_1\otimes\boldsymbol{g}^3 - r\boldsymbol{g}_3\otimes\boldsymbol{g}^1\right).$$

(2.28), (2.118):

$$\operatorname{div}\boldsymbol{t} = \operatorname{tr}\operatorname{grad}\boldsymbol{t} = t_i,_j\,g^{ij} + rt_3g^{11} - 2r^{-1}t_1g^{13}$$
$$= r^{-2}t_1,_1 + t_2,_2 + t_3,_3 + r^{-1}t_3,$$

or alternatively

$$\mathrm{div}\boldsymbol{t} = \mathrm{tr}\,\mathrm{grad}\boldsymbol{t} = t^i{,}_i + r^{-1}t^3 = t^i{,}_i + r^{-1}t_3 = t^1{,}_1 + t^2{,}_2 + t^3{,}_3 + r^{-1}t_3.$$

(2.84), (2.120):

$$\begin{aligned}\mathrm{curl}\boldsymbol{t} &= e^{jik}\frac{1}{g}t_i|_j\, \boldsymbol{g}_k \\ &= g^{-1}\left[(t_3|_2 - t_2|_3)\,\boldsymbol{g}_1 + (t_1|_3 - t_3|_1)\,\boldsymbol{g}_2 + (t_2|_1 - t_1|_2)\,\boldsymbol{g}_3\right] \\ &= r^{-1}\left[(t_3{,}_2 - t_2{,}_3)\,\boldsymbol{g}_1 + (t_1{,}_3 - t_3{,}_1)\,\boldsymbol{g}_2 + (t_2{,}_1 - t_1{,}_2)\,\boldsymbol{g}_3\right].\end{aligned}$$

Spherical coordinates, (S.5-S.7):

$$\begin{aligned}\mathrm{grad}\boldsymbol{t} &= \left(t_i{,}_j - t_k\Gamma^k_{ij}\right)\boldsymbol{g}^i \otimes \boldsymbol{g}^j \\ &= \left(t_1{,}_1 + t_2 \sin\phi\cos\phi + t_3 r \sin^2\phi\right)\boldsymbol{g}^1 \otimes \boldsymbol{g}^1 + (t_2{,}_2 + t_3 r)\,\boldsymbol{g}^2 \otimes \boldsymbol{g}^2 \\ &+ t_3{,}_3\,\boldsymbol{g}^3 \otimes \boldsymbol{g}^3 + (t_1{,}_2 - t_1\cot\phi)\,\boldsymbol{g}^1 \otimes \boldsymbol{g}^2 + (t_2{,}_1 - t_1\cot\phi)\,\boldsymbol{g}^2 \otimes \boldsymbol{g}^1 \\ &+ \left(t_1{,}_3 - t_1 r^{-1}\right)\boldsymbol{g}^1 \otimes \boldsymbol{g}^3 + \left(t_3{,}_1 - t_1 r^{-1}\right)\boldsymbol{g}^3 \otimes \boldsymbol{g}^1 \\ &+ \left(t_2{,}_3 - t_2 r^{-1}\right)\boldsymbol{g}^2 \otimes \boldsymbol{g}^3 + \left(t_3{,}_2 - t_2 r^{-1}\right)\boldsymbol{g}^3 \otimes \boldsymbol{g}^2,\end{aligned}$$

or alternatively

$$\begin{aligned}\mathrm{grad}\boldsymbol{t} &= \left(t^i{,}_j + t^k\Gamma^i_{kj}\right)\boldsymbol{g}_i \otimes \boldsymbol{g}^j = \left(t^1{,}_1 + t^2\cot\phi + t^3 r^{-1}\right)\boldsymbol{g}_1 \otimes \boldsymbol{g}^1 \\ &+ \left(t^2{,}_2 + t^3 r^{-1}\right)\boldsymbol{g}_2 \otimes \boldsymbol{g}^2 + t^3{,}_3\,\boldsymbol{g}_3 \otimes \boldsymbol{g}^3 \\ &+ \left(t^1{,}_2 + t^1\cot\phi\right)\boldsymbol{g}_1 \otimes \boldsymbol{g}^2 + \left(t^2{,}_1 - t^1\sin\phi\cos\phi\right)\boldsymbol{g}_2 \otimes \boldsymbol{g}^1 \\ &+ \left(t^1{,}_3 + t^1 r^{-1}\right)\boldsymbol{g}_1 \otimes \boldsymbol{g}^3 + \left(t^3{,}_1 - t^1 r \sin^2\phi\right)\boldsymbol{g}_3 \otimes \boldsymbol{g}^1 \\ &+ \left(t^2{,}_3 + t^2 r^{-1}\right)\boldsymbol{g}_2 \otimes \boldsymbol{g}^3 + \left(t^3{,}_2 - t^2 r\right)\boldsymbol{g}_3 \otimes \boldsymbol{g}^2,\end{aligned}$$

(2.84), (2.118), (2.120), (S.3-S.7):

$$\begin{aligned}\mathrm{div}\boldsymbol{t} &= \left(t_i{,}_j - t_k\Gamma^k_{ij}\right)g^{ij} = \frac{t_1{,}_1}{r^2\sin^2\phi} + r^{-2}t_2{,}_2 + t_3{,}_3 + r^{-2}\cot\phi\, t_2 + 2r^{-1}t_3 \\ &= t^i{,}_i + t^k\Gamma^i_{ki} = t^1{,}_1 + t^2{,}_2 + t^3{,}_3 + \cot\phi\, t^2 + 2r^{-1}t^3,\end{aligned}$$

$$\begin{aligned}\mathrm{curl}\boldsymbol{t} &= g^{-1}\left[(t_3|_2 - t_2|_3)\,\boldsymbol{g}_1 + (t_1|_3 - t_3|_1)\,\boldsymbol{g}_2 + (t_2|_1 - t_1|_2)\,\boldsymbol{g}_3\right] \\ &= -\frac{1}{r^2\sin\phi}\left[(t_3{,}_2 - t_2{,}_3)\,\boldsymbol{g}_1 + (t_1{,}_3 - t_3{,}_1)\,\boldsymbol{g}_2 + (t_2{,}_1 - t_1{,}_2)\,\boldsymbol{g}_3\right].\end{aligned}$$

2.13 According to the result (S.9) of Exercise 2.11

$$\Delta\boldsymbol{t} = \mathrm{div}\,\mathrm{grad}\boldsymbol{t} = g^{ij}\boldsymbol{t}{,}_{ij} + \left(\boldsymbol{g}^i{,}_j \cdot \boldsymbol{g}^j\right)\boldsymbol{t}{,}_i\,.$$

By virtue of (2.63), (2.72) and $(2.84)_2$ we further obtain

$$\Delta \boldsymbol{t} = g^{ij} \boldsymbol{t},_{ij} - \Gamma^k_{ij} g^{ij} \boldsymbol{t},_k = g^{ij} \left(\boldsymbol{t},_{ij} - \Gamma^k_{ij} \boldsymbol{t},_k \right) = g^{ij} \boldsymbol{t},_i|_j = \boldsymbol{t},_i|^i \ .$$

In Cartesian coordinates it leads to the well-known relation

$$\operatorname{div} \operatorname{grad} \boldsymbol{t} = \boldsymbol{t},_{11} + \boldsymbol{t},_{22} + \boldsymbol{t},_{33} \ .$$

2.14 Specifying the result of Exercise 2.13 to scalar functions we can write

$$\Delta \Phi = g^{ij} \left(\Phi,_{ij} - \Gamma^k_{ij} \Phi,_k \right) = \Phi,_i|^i \ .$$

For the cylindrical coordinates it takes in view of (2.28) and (2.81) the following form

$$\Delta \Phi = r^{-2} \Phi,_{11} + \Phi,_{22} + \Phi,_{33} + r^{-1} \Phi,_3 = \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r}.$$

For the spherical coordinates we use (S.4-S.7). Thus,

$$\begin{aligned} \Delta \Phi &= \frac{1}{r^2 \sin^2 \varphi} \Phi,_{11} + r^{-2} \Phi,_{22} + \Phi,_{33} + \frac{\cos \phi}{r^2 \sin \phi} \Phi,_2 + 2 r^{-1} \Phi,_3 \\ &= \frac{1}{r^2 \sin^2 \varphi} \frac{\partial^2 \Phi}{\partial \varphi^2} + r^{-2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial r^2} + r^{-2} \cot \phi \frac{\partial \Phi}{\partial \phi} + 2 r^{-1} \frac{\partial \Phi}{\partial r}. \end{aligned}$$

2.15 According to the solution of Exercise 2.13

$$\Delta \boldsymbol{t} = g^{ij} \left(\boldsymbol{t},_{ij} - \Gamma^m_{ij} \boldsymbol{t},_m \right), \tag{S.10}$$

where in view of $(2.62)_1$

$$\boldsymbol{t},_i = t^k|_i \, \boldsymbol{g}_k, \qquad \boldsymbol{t},_{ij} = t^k|_{ij} \, \boldsymbol{g}_k.$$

By virtue of $(2.84)_1$ we further write $t^k|_i = t^k,_i + \Gamma^k_{li} t^l$ and consequently

$$t^k|_{ij} = t^k|_i,_j + \Gamma^k_{mj} t^m|_i = t^k,_{ij} + \Gamma^k_{li},_j t^l + \Gamma^k_{li} t^l,_j + \Gamma^k_{mj} t^m,_i + \Gamma^k_{mj} \Gamma^m_{li} t^l.$$

Substituting these results into the expression of the Laplacian (S.10) finally yields

$$\Delta \boldsymbol{t} = g^{ij} \left(t^k,_{ij} + 2 \Gamma^k_{li} t^l,_j - \Gamma^m_{ij} t^k,_m + \Gamma^k_{li},_j t^l + \Gamma^k_{mj} \Gamma^m_{li} t^l - \Gamma^m_{ij} \Gamma^k_{lm} t^l \right) \boldsymbol{g}_k.$$

Taking (S.4-S.7) into account we thus obtain for the spherical coordinates (2.144)

$$\begin{aligned}\Delta \boldsymbol{t} = &\left(\frac{t^1{,_{\varphi\varphi}}}{r^2 \sin^2 \phi} + \frac{t^1{,_{\phi\phi}}}{r^2} + t^1{,_{rr}} \right.\\ &\left. + \frac{3 \cot \phi}{r^2} t^1{,_\phi} + \frac{2 \cos \phi}{r^2 \sin^3 \phi} t^2{,_\varphi} + \frac{4t^1{,_r}}{r} + \frac{2t^3{,_\varphi}}{r^3 \sin^2 \phi} \right) \boldsymbol{g}_1\\ &+ \left(\frac{t^2{,_{\varphi\varphi}}}{r^2 \sin^2 \phi} + \frac{t^2{,_{\phi\phi}}}{r^2} + t^2{,_{rr}} \right.\\ &\left. - \frac{2 \cot \phi}{r^2} t^1{,_\varphi} + \frac{\cot \phi}{r^2} t^2{,_\phi} + \frac{4t^2{,_r}}{r} + \frac{2t^3{,_\phi}}{r^3} + \frac{1 - \cot^2 \phi}{r^2} t^2 \right) \boldsymbol{g}_2\\ &+ \left(\frac{t^3{,_{\varphi\varphi}}}{r^2 \sin^2 \phi} + \frac{t^3{,_{\phi\phi}}}{r^2} + t^3{,_{rr}} \right.\\ &\left. - \frac{2t^1{,_\varphi}}{r} - \frac{2t^2{,_\phi}}{r} + \frac{\cot \phi}{r^2} t^3{,_\phi} + \frac{2t^3{,_r}}{r} - \frac{2 \cot \phi}{r} t^2 - \frac{2t^3}{r^2} \right) \boldsymbol{g}_3 .\end{aligned}$$

Exercises of Chapter 3

3.1 (C.4), (3.18): $\boldsymbol{a}_1 = \mathrm{d}\boldsymbol{r}/\mathrm{d}s = const.$ Hence, $\boldsymbol{r}(s) = \boldsymbol{b} + s\boldsymbol{a}_1$.

3.2 Using the fact that $\mathrm{d}/\mathrm{d}(-s) = -\mathrm{d}/\mathrm{d}s$ we can write by means of (3.15), (3.18), (3.20), (3.21) and (3.27): $\boldsymbol{a}_1'(s) = -\boldsymbol{a}_1(s)$, $\boldsymbol{a}_2'(s) = \boldsymbol{a}_2(s)$, $\boldsymbol{a}_3'(s) = -\boldsymbol{a}_3(s)$, $\varkappa'(s) = \varkappa(s)$ and $\tau'(s) = \tau(s)$.

3.3 Let us show that the curve $\boldsymbol{r}(s)$ with the zero torsion $\tau(s) \equiv 0$ belongs to the plane $\boldsymbol{p}(t^1, t^2) = \boldsymbol{r}(s_0) + t^1\boldsymbol{a}_1(s_0) + t^2\boldsymbol{a}_2(s_0)$, where $\boldsymbol{a}_1(s_0)$ and $\boldsymbol{a}_2(s_0)$ are, respectively, the unit tangent vector and the principal normal vector at a point s_0. For any arbitrary point we can write using (3.15)

$$\boldsymbol{r}(s) = \boldsymbol{r}(s_0) + \int_{\boldsymbol{r}(s_0)}^{\boldsymbol{r}(s)} \mathrm{d}\boldsymbol{r} = \boldsymbol{r}(s_0) + \int_{s_0}^{s} \boldsymbol{a}_1(s)\,\mathrm{d}s. \tag{S.11}$$

The vector $\boldsymbol{a}_1(s)$ can further be represented with respect to the trihedron at s_0 as $\boldsymbol{a}_1(s) = \alpha^i(s)\,\boldsymbol{a}_i(s_0)$. Taking (3.26) into account we observe that $\boldsymbol{a}_3{,_s} = \boldsymbol{0}$ and consequently $\boldsymbol{a}_3(s) \equiv \boldsymbol{a}_3(s_0)$. In view of $(3.23)_2$ it yields $\boldsymbol{a}_1(s) \cdot \boldsymbol{a}_3(s_0) = 0$, so that $\boldsymbol{a}_1(s) = \alpha^1(s)\,\boldsymbol{a}_1(s_0) + \alpha^2(s)\,\boldsymbol{a}_2(s_0)$. Inserting this result into (S.11) we have

$$\begin{aligned}\boldsymbol{r}(s) &= \boldsymbol{r}(s_0) + \boldsymbol{a}_1(s_0) \int_{s_0}^{s} \alpha^1(s)\,\mathrm{d}s + \boldsymbol{a}_2(s_0) \int_{s_0}^{s} \alpha^2(s)\,\mathrm{d}s\\ &= \boldsymbol{r}(s_0) + t^1\boldsymbol{a}_1(s_0) + t^2\boldsymbol{a}_2(s_0),\end{aligned}$$

where we set $t^i = \int_{s_0}^{s} \alpha^i(s)\,\mathrm{d}s\ (i = 1, 2)$.

3.4 Setting in (2.28) $r = R$ yields

$$[g_{\alpha\beta}] = \begin{bmatrix} R^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

By means of (2.81), (3.74), (3.79), (3.90) and (3.93) we further obtain

$$[b_{\alpha\beta}] = \begin{bmatrix} -R & 0 \\ 0 & 0 \end{bmatrix}, \quad [b_\beta^\alpha] = \begin{bmatrix} -R^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \Gamma^1_{\alpha\beta} = \Gamma^2_{\alpha\beta} = 0, \quad \alpha, \beta = 1, 2,$$

$$K = \left|b_\beta^\alpha\right| = 0, \quad H = \frac{1}{2} b_\alpha^\alpha = -\frac{1}{2} R^{-1}. \tag{S.12}$$

3.5 Keeping in mind the results of Exercise 2.1 and using (S.5-S.7), (3.58), (3.62), (3.67), (3.74), (3.79), (3.90) and (3.93) we write

$$\begin{aligned}
\boldsymbol{g}_1 &= R\cos t^1 \sin t^2 \boldsymbol{e}_1 - R\sin t^1 \sin t^2 \boldsymbol{e}_3, \\
\boldsymbol{g}_2 &= R\sin t^1 \cos t^2 \boldsymbol{e}_1 - R\sin t^2 \boldsymbol{e}_2 + R\cos t^1 \cos t^2 \boldsymbol{e}_3, \\
\boldsymbol{g}_3 &= -\sin t^1 \sin t^2 \boldsymbol{e}_1 - \cos t^2 \boldsymbol{e}_2 - \cos t^1 \sin t^2 \boldsymbol{e}_3,
\end{aligned}$$

$$[g_{\alpha\beta}] = \begin{bmatrix} R^2\sin^2 t^2 & 0 \\ 0 & R^2 \end{bmatrix}, \quad [b_{\alpha\beta}] = \begin{bmatrix} R\sin^2 t^2 & 0 \\ 0 & R \end{bmatrix}, \quad [b_\beta^\alpha] = \begin{bmatrix} R^{-1} & 0 \\ 0 & R^{-1} \end{bmatrix},$$

$$\left[\Gamma^1_{\alpha\beta}\right] = \begin{bmatrix} 0 & \cot t^2 \\ \cot t^2 & 0 \end{bmatrix}, \quad \left[\Gamma^2_{\alpha\beta}\right] = \begin{bmatrix} -\sin t^2 \cos t^2 & 0 \\ 0 & 0 \end{bmatrix},$$

$$K = \left|b_\beta^\alpha\right| = R^{-2}, \quad H = \frac{1}{2} b_\alpha^\alpha = R^{-1}. \tag{S.13}$$

3.6 (3.62), (3.67), (3.143):

$$\boldsymbol{g}_1 = \frac{\partial \boldsymbol{r}}{\partial t^1} = \boldsymbol{e}_1 + \bar{t}^2 \boldsymbol{e}_3, \quad \boldsymbol{g}_2 = \frac{\partial \boldsymbol{r}}{\partial t^2} = \boldsymbol{e}_2 + \bar{t}^1 \boldsymbol{e}_3,$$

$$\boldsymbol{g}_3 = \frac{\boldsymbol{g}_1 \times \boldsymbol{g}_2}{\|\boldsymbol{g}_1 \times \boldsymbol{g}_2\|} = \frac{1}{\sqrt{1 + \left(\bar{t}^1\right)^2 + \left(\bar{t}^2\right)^2}} \left(-\bar{t}^2 \boldsymbol{e}_1 - \bar{t}^1 \boldsymbol{e}_2 + \boldsymbol{e}_3\right),$$

where $\bar{t}^i = \dfrac{t^i}{c}$ $(i = 1, 2)$. Thus, the coefficients of the first fundamental form are

$$g_{11} = \boldsymbol{g}_1 \cdot \boldsymbol{g}_1 = 1 + \left(\bar{t}^2\right)^2, \ g_{12} = g_{21} = \boldsymbol{g}_1 \cdot \boldsymbol{g}_2 = \bar{t}^1 \bar{t}^2, \ g_{22} = \boldsymbol{g}_2 \cdot \boldsymbol{g}_2 = 1 + \left(\bar{t}^1\right)^2.$$

For the coefficients of the inversed matrix $\left[g^{\alpha\beta}\right]$ we have

$$\left[g^{\alpha\beta}\right] = \frac{1}{1 + \left(\bar{t}^1\right)^2 + \left(\bar{t}^2\right)^2} \begin{bmatrix} 1 + \left(\bar{t}^1\right)^2 & -\bar{t}^1 \bar{t}^2 \\ -\bar{t}^1 \bar{t}^2 & 1 + \left(\bar{t}^2\right)^2 \end{bmatrix}.$$

The derivatives of the tangent vectors result in

$$\boldsymbol{g}_1,_1 = \boldsymbol{0}, \quad \boldsymbol{g}_1,_2 = \boldsymbol{g}_2,_1 = \frac{1}{c}\boldsymbol{e}_3, \quad \boldsymbol{g}_2,_2 = \boldsymbol{0}.$$

By (3.74), (3.79), (3.90) and (3.93) we further obtain

$$b_{11} = 0, \quad b_{12} = b_{21} = \frac{1}{c\sqrt{1 + (\bar{t}^1)^2 + (\bar{t}^2)^2}}, \quad b_{22} = 0,$$

$$\left[b_{\alpha\cdot}^{\beta}\right] = \frac{1}{c\left[1 + (\bar{t}^1)^2 + (\bar{t}^2)^2\right]^{3/2}} \begin{bmatrix} -\bar{t}^1\bar{t}^2 & 1 + \left(\bar{t}^2\right)^2 \\ 1 + \left(\bar{t}^1\right)^2 & -\bar{t}^1\bar{t}^2 \end{bmatrix},$$

$$K = \left|b_{\beta}^{\alpha}\right| = -\frac{1}{c^2\left[1 + (\bar{t}^1)^2 + (\bar{t}^2)^2\right]^2} = -\left[c^2 + \left(t^1\right)^2 + \left(t^2\right)^2\right]^{-2},$$

$$H = \frac{1}{2}b_{\alpha}^{\alpha} = -\frac{\bar{t}^1\bar{t}^2}{c\left[1 + (\bar{t}^1)^2 + (\bar{t}^2)^2\right]^{3/2}}.$$

3.7 Taking (3.105) into account we can write

$$\left[g_{ij}^*\right] = \begin{bmatrix} g_{11}^* & g_{12}^* & 0 \\ g_{21}^* & g_{22}^* & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \left[g^{*ij}\right] = \left[g_{ij}^*\right]^{-1} = \frac{1}{\left|g_{ij}^*\right|} \begin{bmatrix} g_{22}^* & -g_{21}^* & 0 \\ -g_{12}^* & g_{11}^* & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which immediately implies (3.111).

3.8 For a cylindrical shell equilibrium equations (3.140-3.141) take by means of $(3.77)_1$ and (S.12) the form

$$f^{11},_1 + f^{12},_2 + p^1 = 0, \quad f^{12},_1 + f^{22},_2 + p^2 = 0, \quad -Rf^{11} + p^3 = 0.$$

For a spherical shell we further obtain by virtue of (S.13)

$$f^{11},_1 + f^{12},_2 + 3\cot t^2 f^{12} + p^1 = 0,$$

$$f^{12},_1 + f^{22},_2 - \sin t^2 \cos t^2 f^{11} + \cot t^2 f^{22} + p^2 = 0,$$

$$R\sin^2 t^2 f^{11} + Rf^{22} + p^3 = 0.$$

Exercises of Chapter 4

4.1 In the case of simple shear the right Cauchy-Green tensor $\mathbf{C}$ has the form:

$$\mathbf{C} = \mathrm{C}_j^i \boldsymbol{e}_i \otimes \boldsymbol{e}^j, \quad \left[\mathrm{C}_j^i\right] = \left[\mathrm{C}_{ij}\right] = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1 + \gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The characteristic equation can then be written by

$$\begin{vmatrix} 1-\Lambda & \gamma & 0 \\ \gamma & 1+\gamma^2-\Lambda & 0 \\ 0 & 0 & 1-\Lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (1-\Lambda)\left\{\Lambda^2 - \Lambda\left(2+\gamma^2\right)+1\right\} = 0.$$

Solving the latter equation with respect to Λ we obtain the eigenvalues of $\mathbf{C}$ as

$$\Lambda_{1/2} = 1 + \frac{\gamma^2 \pm \sqrt{4\gamma^2+\gamma^4}}{2} = \left(\frac{\gamma \pm \sqrt{4+\gamma^2}}{2}\right)^2, \quad \Lambda_3 = 1. \tag{S.14}$$

The eigenvectors $\boldsymbol{a} = a^i \boldsymbol{e}_i$ corresponding to the first two eigenvalues result from the equation system $(4.16)_1$

$$\begin{cases} \dfrac{-\gamma^2 \mp \sqrt{4\gamma^2+\gamma^4}}{2} a^1 & +\gamma a^2 & & = 0, \\ \gamma a^1 & + \dfrac{\gamma^2 \mp \sqrt{4\gamma^2+\gamma^4}}{2} a^2 & & = 0, \\ & & \dfrac{-\gamma^2 \mp \sqrt{4\gamma^2+\gamma^4}}{2} a^3 & = 0. \end{cases}$$

Since the first and second equation are equivalent we only obtain

$$a^2 = \frac{\gamma \pm \sqrt{4+\gamma^2}}{2} a^1, \quad a^3 = 0,$$

so that $a^2 = \sqrt{\Lambda_1} a^1$ or $a^2 = -\sqrt{\Lambda_2} a^1$. In order to ensure the unit length of the eigenvectors we also require that

$$\left(a^1\right)^2 + \left(a^2\right)^2 + \left(a^3\right)^2 = 1.$$

This yields

$$\boldsymbol{a}_1 = \frac{1}{\sqrt{1+\Lambda_1}} \boldsymbol{e}_1 + \sqrt{\frac{\Lambda_1}{1+\Lambda_1}} \boldsymbol{e}_2, \quad \boldsymbol{a}_2 = \frac{1}{\sqrt{1+\Lambda_2}} \boldsymbol{e}_1 - \sqrt{\frac{\Lambda_2}{1+\Lambda_2}} \boldsymbol{e}_2. \tag{S.15}$$

Applying the above procedure for the third eigenvector corresponding to the eigenvalue $\Lambda_3 = 1$ we easily obtain: $\boldsymbol{a}_3 = \boldsymbol{e}_3$.

4.2 Using $(4.26)_{1-3}$ we write

$$\mathrm{I}_{\mathbf{A}} = \mathrm{tr}\mathbf{A},$$

$$\mathrm{II}_{\mathbf{A}} = \frac{1}{2}\left[(\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}\mathbf{A}^2\right],$$

$$\mathrm{III}_{\mathbf{A}} = \frac{1}{3}\left[\mathrm{II}_{\mathbf{A}}\mathrm{tr}\mathbf{A} - \mathrm{I}_{\mathbf{A}}\mathrm{tr}\mathbf{A}^2 + \mathrm{tr}\mathbf{A}^3\right].$$

Inserting the first and second expression into the third one we obtain

$$\begin{aligned} \mathrm{III}_{\mathbf{A}} &= \frac{1}{3}\left\{\frac{1}{2}\left[(\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}\mathbf{A}^2\right]\mathrm{tr}\mathbf{A} - \mathrm{tr}\mathbf{A}\mathrm{tr}\mathbf{A}^2 + \mathrm{tr}\mathbf{A}^3\right\} \\ &= \frac{1}{3}\left[\mathrm{tr}\mathbf{A}^3 - \frac{3}{2}\mathrm{tr}\mathbf{A}^2\mathrm{tr}\mathbf{A} + \frac{1}{2}(\mathrm{tr}\mathbf{A})^3\right]. \end{aligned}$$

4.3 Since $r_i = t_i$ for every eigenvalue λ_i we have exactly $n = \sum_{i=1}^{s} r_i$ eigenvectors, say $\boldsymbol{a}_i^{(k)}$ $(i = 1, 2, \ldots, s; k = 1, 2, \ldots, r_i)$. Let us assume, on the contrary, that they are linearly dependent so that

$$\sum_{i=1}^{s}\sum_{k=1}^{r_i} \alpha_i^{(k)} \boldsymbol{a}_i^{(k)} = 0,$$

where not all $\alpha_i^{(k)}$ are zero. Since linear combinations of the eigenvectors $\boldsymbol{a}_i = \sum_{k=1}^{r_i} \alpha_i^{(k)} \boldsymbol{a}_i^{(k)}$ associated with the same eigenvalue λ_i are again eigenvectors we arrive at

$$\sum_{i=1}^{s} \varepsilon_i \boldsymbol{a}_i = 0,$$

where ε_i are either one or zero (but not all). This relation establishes the linear dependence between eigenvectors corresponding to distinct eigenvalues, which contradicts the statement of Theorem 4.2. Applying then Theorem 1.3 for the space $\mathbb{C}^n$ instead of $\mathbb{V}$ we infer that the eigenvectors $\boldsymbol{a}_i^{(k)}$ form a basis of $\mathbb{C}^n$.

4.4 (4.40), (4.42):

$$\begin{aligned} \mathbf{P}_i\mathbf{P}_j &= \left(\sum_{k=1}^{r_i} \boldsymbol{a}_i^{(k)} \otimes \boldsymbol{b}_i^{(k)}\right)\left(\sum_{l=1}^{r_j} \boldsymbol{a}_j^{(l)} \otimes \boldsymbol{b}_j^{(l)}\right) = \sum_{k=1}^{r_i}\sum_{l=1}^{r_j} \delta_{ij}\delta^{kl} \boldsymbol{a}_i^{(k)} \otimes \boldsymbol{b}_j^{(l)} \\ &= \delta_{ij}\sum_{k=1}^{r_i} \boldsymbol{a}_i^{(k)} \otimes \boldsymbol{b}_j^{(k)} = \begin{cases} \mathbf{P}_i & \text{if } i = j, \\ \mathbf{0} & \text{if } i \neq j. \end{cases} \end{aligned}$$

4.5 By means of (4.40) and (4.42) we infer that $\mathbf{P}_i \boldsymbol{a}_j^{(l)} = \delta_{ij} \boldsymbol{a}_j^{(l)}$. Every vector $\boldsymbol{x}$ in $\mathbb{C}^n$ can be represented with respect the basis of this space $\boldsymbol{a}_i^{(k)}$ $(i = 1, 2, \ldots, s; k = 1, 2, \ldots, r_i)$ by $\boldsymbol{x} = \sum_{j=1}^{s}\sum_{k=1}^{r_j} x_j^{(k)} \boldsymbol{a}_j^{(k)}$. Hence,

$$\begin{aligned} \left(\sum_{i=1}^{s} \mathbf{P}_i\right)\boldsymbol{x} &= \sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{r_j} x_j^{(k)} \mathbf{P}_i \boldsymbol{a}_j^{(k)} \\ &= \sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=1}^{r_j} x_j^{(k)} \delta_{ij} \boldsymbol{a}_j^{(k)} = \sum_{j=1}^{s}\sum_{k=1}^{r_j} x_j^{(k)} \boldsymbol{a}_j^{(k)} = \boldsymbol{x}, \quad \forall \boldsymbol{x} \in \mathbb{C}^n, \end{aligned}$$

which immediately implies (4.46).

4.6 Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of $\mathbf{A} \in \mathbf{Lin}^n$. By the spectral mapping theorem (Theorem 4.1) we infer that $\exp(\lambda_i)$ $(i = 1, 2, \ldots, n)$ are eigenvalues of $\exp \mathbf{A}$. On use of (4.24) and (4.26) we can thus write: $\det[\exp(\mathbf{A})] = \prod_{i=1}^{n} \exp \lambda_i = \exp\left(\sum_{i=1}^{n} \lambda_i\right) = \exp(\mathrm{tr}\mathbf{A})$.

4.7 Let us consider the right hand side of (4.54) for example for $i = 1$. In this case we have

$$\prod_{\substack{j=1\\ j\neq 1}}^{3} \frac{\mathbf{A} - \lambda_j \mathbf{I}}{\lambda_1 - \lambda_j} = \frac{\mathbf{A} - \lambda_2 \mathbf{I}}{\lambda_1 - \lambda_2} \frac{\mathbf{A} - \lambda_3 \mathbf{I}}{\lambda_1 - \lambda_3}.$$

On use of (4.43), (4.44) and (4.46) we further obtain

$$\frac{\mathbf{A} - \lambda_2 \mathbf{I}}{\lambda_1 - \lambda_2} \frac{\mathbf{A} - \lambda_3 \mathbf{I}}{\lambda_1 - \lambda_3} = \frac{\sum_{i=1}^{3} (\lambda_i - \lambda_2) \mathbf{P}_i}{\lambda_1 - \lambda_2} \frac{\sum_{j=1}^{3} (\lambda_j - \lambda_3) \mathbf{P}_j}{\lambda_1 - \lambda_3}$$

$$= \frac{\sum_{i,j=1}^{3} (\lambda_i - \lambda_2)(\lambda_j - \lambda_3) \delta_{ij} \mathbf{P}_i}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} = \frac{\sum_{i=1}^{3} (\lambda_i - \lambda_2)(\lambda_i - \lambda_3) \mathbf{P}_i}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}$$

$$= \frac{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \mathbf{P}_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} = \mathbf{P}_1.$$

In a similar way, one verifies the Sylvester formula also for $i = 2$ and $i = 3$.

4.8 The characteristic equation of the tensor $\mathbf{A}$ takes the form:

$$\begin{vmatrix} -2-\lambda & 2 & 2 \\ 2 & 1-\lambda & 4 \\ 2 & 4 & 1-\lambda \end{vmatrix} = 0.$$

Writing out this determinant we get after some algebraic manipulations

$$\lambda^3 - 27\lambda - 54 = 0$$

Comparing this equation with (4.28) we see that

$$\mathrm{I}_{\mathbf{A}} = 0, \quad \mathrm{II}_{\mathbf{A}} = -27, \quad \mathrm{III}_{\mathbf{A}} = 54. \tag{S.16}$$

Inserting this result into the Cardano formula (4.31) and (4.32) we obtain

$$\vartheta = \arccos\left[\frac{2\mathrm{I}_{\mathbf{A}}^3 - 9\mathrm{I}_{\mathbf{A}}\mathrm{II}_{\mathbf{A}} + 27\mathrm{III}_{\mathbf{A}}}{2\left(\mathrm{I}_{\mathbf{A}}^2 - 3\mathrm{II}_{\mathbf{A}}\right)^{3/2}}\right]$$

$$= \arccos\left[\frac{27 \cdot 54}{2(3 \cdot 27)^{3/2}}\right] = \arccos(1) = 0,$$

$$\lambda_k = \frac{1}{3}\left\{\mathrm{I}_{\mathbf{A}} + 2\sqrt{\mathrm{I}_{\mathbf{A}}^2 - 3\mathrm{II}_{\mathbf{A}}}\cos\frac{1}{3}\left[\vartheta + 2\pi\left(k-1\right)\right]\right\}$$
$$= \frac{2}{3}\sqrt{3\cdot 27}\cos\left(\frac{2}{3}\pi\left(k-1\right)\right) = 6\cos\left(\frac{2}{3}\pi\left(k-1\right)\right), \quad k = 1,2,3.$$

Thus, we obtain two pairwise distinct eigenvalues ($s = 2$):

$$\lambda_1 = 6, \quad \lambda_2 = \lambda_3 = -3. \tag{S.17}$$

The Sylvester formula (4.54) further yields

$$\mathbf{P}_1 = \prod_{\substack{j=1\\ j\neq 1}}^{2} \frac{\mathbf{A} - \lambda_j\mathbf{I}}{\lambda_i - \lambda_j} = \frac{\mathbf{A} - \lambda_2\mathbf{I}}{\lambda_1 - \lambda_2} = \frac{\mathbf{A} + 3\mathbf{I}}{9} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}^j,$$

$$\mathbf{P}_2 = \prod_{\substack{j=1\\ j\neq 2}}^{2} \frac{\mathbf{A} - \lambda_j\mathbf{I}}{\lambda_i - \lambda_j} = \frac{\mathbf{A} - \lambda_1\mathbf{I}}{\lambda_2 - \lambda_1} = \frac{\mathbf{A} - 6\mathbf{I}}{-9} = \frac{1}{9}\begin{bmatrix} 8 & -2 & -2 \\ -2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}^j.$$

4.9 The spectral representation of $\mathbf{A}$ takes the form

$$\mathbf{A} = \sum_{i=1}^{s} \lambda_i \mathbf{P}_i = \lambda_1\mathbf{P}_1 + \lambda_2\mathbf{P}_2 = 6\mathbf{P}_1 - 3\mathbf{P}_2.$$

Thus,

$$\mathbf{A} = \sum_{i=1}^{s} \exp\left(\lambda_i\right)\mathbf{P}_i$$
$$= \exp\left(\lambda_1\right)\mathbf{P}_1 + \exp\left(\lambda_2\right)\mathbf{P}_2 = \exp\left(6\right)\mathbf{P}_1 + \exp\left(-3\right)\mathbf{P}_2$$
$$= \frac{\mathrm{e}^6}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j + \frac{\mathrm{e}^{-3}}{9}\begin{bmatrix} 8 & -2 & -2 \\ -2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$
$$= \frac{1}{9}\begin{bmatrix} \mathrm{e}^6 + 8\mathrm{e}^{-3} & 2\mathrm{e}^6 - 2\mathrm{e}^{-3} & 2\mathrm{e}^6 - 2\mathrm{e}^{-3} \\ 2\mathrm{e}^6 - 2\mathrm{e}^{-3} & 4\mathrm{e}^6 + 5\mathrm{e}^{-3} & 4\mathrm{e}^6 - 4\mathrm{e}^{-3} \\ 2\mathrm{e}^6 - 2\mathrm{e}^{-3} & 4\mathrm{e}^6 - 4\mathrm{e}^{-3} & 4\mathrm{e}^6 + 5\mathrm{e}^{-3} \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j.$$

4.10 Components of the eigenvectors $\boldsymbol{a} = a^i\boldsymbol{e}_i$ result from the equation system (4.16)

$$\left(\mathrm{A}^i_j - \delta^i_j\lambda\right)a^j = 0, \quad i = 1,2,3. \tag{S.18}$$

Setting $\lambda = 6$ we obtain only two independent equations

$$\begin{cases} -8a^1 + 2a^2 + 2a^3 = 0, \\ 2a^1 - 5a^2 + 4a^3 = 0. \end{cases}$$

Multiplying the first equation by two and subtracting from the second one we get $a^2 = 2a^1$ and consequently $a^3 = 2a^1$. Additionally we require that the eigenvectors have unit length so that

$$\left(a^1\right)^2 + \left(a^2\right)^2 + \left(a^3\right)^2 = 1, \tag{S.19}$$

which leads to

$$\boldsymbol{a}_1 = \frac{1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3.$$

Further, setting in the equation system (S.18) $\lambda = -3$ we obtain only one independent linear equation

$$a^1 + 2a^2 + 2a^3 = 0 \tag{S.20}$$

with respect to the components of the eigenvectors corresponding to this double eigenvalue. One of these eigenvectors can be obtained by setting for example $a^1 = 0$. In this case, $a^2 = -a^3$ and in view of (S.19)

$$\boldsymbol{a}_2^{(1)} = \frac{1}{\sqrt{2}}\boldsymbol{e}_2 - \frac{1}{\sqrt{2}}\boldsymbol{e}_3.$$

Requiring that the eigenvectors $\boldsymbol{a}_2^{(1)}$ and $\boldsymbol{a}_2^{(2)}$ corresponding to the double eigenvalue $\lambda = -3$ are orthogonal we get an additional condition $a^2 = a^3$ for the components of $\boldsymbol{a}_2^{(2)}$. Taking into account (S.19) and (S.20) this yields

$$\boldsymbol{a}_2^{(2)} = -\frac{4}{3\sqrt{2}}\boldsymbol{e}_1 + \frac{1}{3\sqrt{2}}\boldsymbol{e}_2 + \frac{1}{3\sqrt{2}}\boldsymbol{e}_3.$$

With the aid of the eigenvectors we can construct eigenprojections without the Sylvester formula by (4.42):

$$\begin{aligned}
\mathbf{P}_1 &= \boldsymbol{a}_1 \otimes \boldsymbol{a}_1 \\
&= \left(\frac{1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3\right) \otimes \left(\frac{1}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 + \frac{2}{3}\boldsymbol{e}_3\right) = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j,
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}_2 &= \boldsymbol{a}_2^{(1)} \otimes \boldsymbol{a}_2^{(1)} + \boldsymbol{a}_2^{(2)} \otimes \boldsymbol{a}_2^{(2)} = \left(\frac{1}{\sqrt{2}}\boldsymbol{e}_2 - \frac{1}{\sqrt{2}}\boldsymbol{e}_3\right) \otimes \left(\frac{1}{\sqrt{2}}\boldsymbol{e}_2 - \frac{1}{\sqrt{2}}\boldsymbol{e}_3\right) \\
&+ \left(-\frac{4}{3\sqrt{2}}\boldsymbol{e}_1 + \frac{1}{3\sqrt{2}}\boldsymbol{e}_2 + \frac{1}{3\sqrt{2}}\boldsymbol{e}_3\right) \otimes \left(-\frac{4}{3\sqrt{2}}\boldsymbol{e}_1 + \frac{1}{3\sqrt{2}}\boldsymbol{e}_2 + \frac{1}{3\sqrt{2}}\boldsymbol{e}_3\right) \\
&= \frac{1}{9}\begin{bmatrix} 8 & -2 & -2 \\ -2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j.
\end{aligned}$$

4.11 Since linearly independent vectors are non-zero it follows from (4.9) that $\boldsymbol{c}_i \cdot \overline{\boldsymbol{c}}_i \neq 0$ $(i = 1, 2, \ldots, m)$. Thus, the first vector can be given by

$$\boldsymbol{a}_1 = \frac{\boldsymbol{c}_1}{\sqrt{\boldsymbol{c}_1 \cdot \overline{\boldsymbol{c}}_1}},$$

such that $\boldsymbol{a}_1 \cdot \overline{\boldsymbol{a}}_1 = 1$. Next, we set

$$\boldsymbol{a}_2' = \boldsymbol{c}_2 - (\boldsymbol{c}_2 \cdot \overline{\boldsymbol{a}}_1)\,\boldsymbol{a}_1,$$

so that $\boldsymbol{a}_2' \cdot \overline{\boldsymbol{a}}_1 = 0$. Further, $\boldsymbol{a}_2' \neq \boldsymbol{0}$ because otherwise $\boldsymbol{c}_2 = (\boldsymbol{c}_2 \cdot \overline{\boldsymbol{a}}_1)\,\boldsymbol{a}_1 = (\boldsymbol{c}_2 \cdot \overline{\boldsymbol{a}}_1)\,(\boldsymbol{c}_1 \cdot \overline{\boldsymbol{c}}_1)^{-1/2}\,\boldsymbol{c}_1$ which implies a linear dependence between $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$. Thus, we can set $\boldsymbol{a}_2 = \boldsymbol{a}_2' / \sqrt{\boldsymbol{a}_2' \cdot \overline{\boldsymbol{a}}_2'}$. The third vector can be given by

$$\boldsymbol{a}_3 = \frac{\boldsymbol{a}_3'}{\sqrt{\boldsymbol{a}_3' \cdot \overline{\boldsymbol{a}}_3'}}, \quad \text{where} \quad \boldsymbol{a}_3' = \boldsymbol{c}_3 - (\boldsymbol{c}_3 \cdot \overline{\boldsymbol{a}}_2)\,\boldsymbol{a}_2 - (\boldsymbol{c}_3 \cdot \overline{\boldsymbol{a}}_1)\,\boldsymbol{a}_1,$$

so that $\boldsymbol{a}_3 \cdot \overline{\boldsymbol{a}}_1 = \boldsymbol{a}_3 \cdot \overline{\boldsymbol{a}}_2 = 0$. Repeating this procedure we finally obtain the set of vectors $\boldsymbol{a}_i$ satisfying the condition $\boldsymbol{a}_i \cdot \overline{\boldsymbol{a}}_j = \delta_{ij}, (i, j = 1, 2, \ldots, m)$. One can easily show that these vectors are linearly independent. Indeed, otherwise $\sum_{i=1}^{m} \alpha_i \boldsymbol{a}_i = \boldsymbol{0}$, where not all α_i are zero. Multiplying this vector equation scalarly by $\overline{\boldsymbol{a}}_j$ $(j = 1, 2, \ldots, m)$ yields, however, $\alpha_j = 0$ $(j = 1, 2, \ldots, m)$.

4.12 Comparing $(4.63)_1$ with $(4.71)_1$ we infer that the right eigenvectors $\boldsymbol{a}_i^{(k)}$ $(k = 1, 2, \ldots, t_i)$ associated with a complex eigenvalue λ_i are simultaneously the left eigenvalues associated with $\overline{\lambda}_i$. Since $\overline{\lambda}_i \neq \lambda_i$ Theorem 4.3 implies that $\boldsymbol{a}_i^{(k)} \cdot \boldsymbol{a}_i^{(l)} = 0$ $(k, l = 1, 2, \ldots, t_i)$.

4.13 Taking into account the identities $\mathrm{tr}\mathbf{W}^k = 0$, where $k = 1, 3, 5, \ldots$ (see Exercise 1.46) we obtain from (4.29)

$$\mathrm{I}_{\mathbf{W}} = \mathrm{tr}\mathbf{W} = 0,$$

$$\begin{aligned}
\mathrm{II}_{\mathbf{W}} &= \frac{1}{2}\left[(\mathrm{tr}\mathbf{W})^2 - \mathrm{tr}\mathbf{W}^2\right] \\
&= -\frac{1}{2}\mathrm{tr}\mathbf{W}^2 = -\frac{1}{2}\left(\mathbf{W} : \mathbf{W}^{\mathrm{T}}\right) = \frac{1}{2}\left(\mathbf{W} : \mathbf{W}\right) = \frac{1}{2}\left\|\mathbf{W}\right\|^2, \\
\mathrm{III}_{\mathbf{W}} &= \frac{1}{3}\left[\mathrm{tr}\mathbf{W}^3 - \frac{3}{2}\mathrm{tr}\mathbf{W}^2\mathrm{tr}\mathbf{W} + \frac{1}{2}(\mathrm{tr}\mathbf{W})^3\right] = 0,
\end{aligned}$$

or in another way

$$\mathrm{III}_{\mathbf{W}} = \det\mathbf{W} = \det\mathbf{W}^{\mathrm{T}} = -\det\mathbf{W}^{\mathrm{T}} = -\mathrm{III}_{\mathbf{W}} = 0.$$

4.14 Eigenvalues of the rotation tensor (Exercise 1.22)

$$\mathbf{R} = \mathrm{R}^i_{\cdot j}\boldsymbol{e}_i \otimes \boldsymbol{e}^j, \quad \text{where} \quad \left[\mathrm{R}^i_{\cdot j}\right] = \left[\mathrm{R}^{ij}\right] = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

result from the characteristic equation

$$\begin{vmatrix} \cos\alpha - \lambda & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{vmatrix} = 0.$$

Writing out this determinant we have

$$(1 - \lambda)\left(\lambda^2 - 2\lambda\cos\alpha + 1\right) = 0$$

and consequently

$$\lambda_1 = 1, \quad \lambda_{2/3} = \cos\alpha \pm \mathrm{i}\sin\alpha.$$

Components of the right eigenvectors $\boldsymbol{a} = a^i \boldsymbol{e}_i$ result from the equation system (4.16)

$$\left(\mathrm{R}^i_{\cdot j} - \delta^i_j \lambda\right) a^j = 0, \quad i = 1, 2, 3. \tag{S.21}$$

Setting first $\lambda = 1$ we obtain a homogeneous equation system

$$a^1 (\cos\alpha - 1) - a^2 \sin\alpha = 0,$$
$$a^1 \sin\alpha + a^2 (\cos\alpha - 1) = 0,$$

leading to $a^1 = a^2 = 0$. Thus, $\boldsymbol{a}_1 = a^3 \boldsymbol{e}_3$, where a^3 can be an arbitrary real number. The unit length condition requires further that

$$\boldsymbol{a}_1 = \boldsymbol{e}_3.$$

Next, inserting $\lambda = \cos\alpha \pm \mathrm{i}\sin\alpha$ into (S.21) yields

$$a^2 = \mp \mathrm{i} a^1, \quad a^3 = 0.$$

Thus, the right eigenvectors associated with the complex conjugate eigenvalues $\lambda_{2/3}$ are of the form $\boldsymbol{a}_{2/3} = a^1 (\boldsymbol{e}_1 \mp \mathrm{i}\boldsymbol{e}_2)$. Bearing in mind that any rotation tensor is orthogonal we infer that $\overline{\boldsymbol{a}}_{2/3} = \boldsymbol{a}_{3/2} = a^1 (\boldsymbol{e}_1 \pm \mathrm{i}\boldsymbol{e}_2)$ are the left eigenvectors associated with $\lambda_{2/3}$. Imposing the additional condition $\boldsymbol{a}_2 \cdot \overline{\boldsymbol{a}}_2 = \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 = 1$ (4.38) we finally obtain

$$\boldsymbol{a}_2 = \frac{\sqrt{2}}{2}(\boldsymbol{e}_1 - \mathrm{i}\boldsymbol{e}_2), \quad \boldsymbol{a}_3 = \frac{\sqrt{2}}{2}(\boldsymbol{e}_1 + \mathrm{i}\boldsymbol{e}_2).$$

The eigenprojections can further be expressed by (4.42) as

$$\mathbf{P}_1 = \boldsymbol{a}_1 \otimes \boldsymbol{a}_1 = \boldsymbol{e}_3 \otimes \boldsymbol{e}_3,$$

$$\mathbf{P}_2 = \boldsymbol{a}_2 \otimes \overline{\boldsymbol{a}}_2 = \frac{\sqrt{2}}{2}(\boldsymbol{e}_1 - \mathrm{i}\boldsymbol{e}_2) \otimes \frac{\sqrt{2}}{2}(\boldsymbol{e}_1 + \mathrm{i}\boldsymbol{e}_2)$$
$$= \frac{1}{2}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2) + \frac{1}{2}\mathrm{i}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_1),$$

$$\mathbf{P}_3 = \boldsymbol{a}_3 \otimes \overline{\boldsymbol{a}}_3 = \frac{\sqrt{2}}{2} (\boldsymbol{e}_1 + \mathrm{i}\boldsymbol{e}_2) \otimes \frac{\sqrt{2}}{2} (\boldsymbol{e}_1 - \mathrm{i}\boldsymbol{e}_2)$$
$$= \frac{1}{2} (\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2) - \frac{1}{2}\mathrm{i} (\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) .$$

4.15 First, we write

$$\left[\left(\mathbf{A}^2\right)^i_{\cdot j} \right] = \begin{bmatrix} -2 & 2 & 2 \\ 2 & 1 & 4 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 & 2 \\ 2 & 1 & 4 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 6 & 6 \\ 6 & 21 & 12 \\ 6 & 12 & 21 \end{bmatrix},$$

$$\left[\left(\mathbf{A}^3\right)^i_{\cdot j} \right] = \begin{bmatrix} 12 & 6 & 6 \\ 6 & 21 & 12 \\ 6 & 12 & 21 \end{bmatrix} \begin{bmatrix} -2 & 2 & 2 \\ 2 & 1 & 4 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 54 & 54 \\ 54 & 81 & 108 \\ 54 & 108 & 81 \end{bmatrix}.$$

Then,

$$p_{\mathbf{A}} (\mathbf{A}) = \mathbf{A}^3 - 27\mathbf{A} - 54\mathbf{I} = \begin{bmatrix} 0 & 54 & 54 \\ 54 & 81 & 108 \\ 54 & 108 & 81 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

$$-27 \begin{bmatrix} -2 & 2 & 2 \\ 2 & 1 & 4 \\ 2 & 4 & 1 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j - 54 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j.$$

4.16 The characteristic polynomial of $\mathbf{F}$ (4.23) can be represented by $p_{\mathbf{A}} (\lambda) = (1 - \lambda)^3$. Hence,

$$p_{\mathbf{F}} (\mathbf{F}) = (\mathbf{I} - \mathbf{F})^3 = \begin{bmatrix} 0 & -\gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^3 \boldsymbol{e}_i \otimes \boldsymbol{e}^j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}^j = \mathbf{0}.$$

Exercises of Chapter 5

5.1 By using $(1.101)_1$, (D.2) and (5.17) one can verify for example $(5.20)_1$ and $(5.21)_1$ within the following steps

$$\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) : \mathbf{X} = \mathbf{A}\mathbf{X} (\mathbf{B} + \mathbf{C}) = \mathbf{A}\mathbf{X}\mathbf{B} + \mathbf{A}\mathbf{X}\mathbf{C} = (\mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}) : \mathbf{X},$$

$$\mathbf{A} \odot (\mathbf{B} + \mathbf{C}) : \mathbf{X} = \mathbf{A} [(\mathbf{B} + \mathbf{C}) : \mathbf{X}] = \mathbf{A} (\mathbf{B} : \mathbf{X} + \mathbf{C} : \mathbf{X})$$
$$= \mathbf{A} (\mathbf{B} : \mathbf{X}) + \mathbf{A} (\mathbf{C} : \mathbf{X})$$
$$= (\mathbf{A} \odot \mathbf{B} + \mathbf{A} \odot \mathbf{C}) : \mathbf{X}, \quad \forall \mathbf{X} \in \mathbf{Lin}^n.$$

The proof of $(5.20)_2$ and $(5.21)_2$ is similar.

5.2 With the aid of (5.16), (5.17) and (1.138) we can write

$$(\mathbf{Y} : \mathbf{A} \otimes \mathbf{B}) : \mathbf{X} = \mathbf{Y} : (\mathbf{A} \otimes \mathbf{B} : \mathbf{X}) = \mathbf{Y} : \mathbf{A}\mathbf{X}\mathbf{B} = \mathbf{A}^{\mathrm{T}}\mathbf{Y}\mathbf{B}^{\mathrm{T}} : \mathbf{X},$$

$$(\mathbf{Y} : \mathbf{A} \odot \mathbf{B}) : \mathbf{X} = \mathbf{Y} : (\mathbf{A} \odot \mathbf{B} : \mathbf{X}) = \mathbf{Y} : [\mathbf{A}(\mathbf{B} : \mathbf{X})]$$
$$= (\mathbf{Y} : \mathbf{A})(\mathbf{B} : \mathbf{X}) = [(\mathbf{Y} : \mathbf{A})\mathbf{B}] : \mathbf{X}, \quad \forall \mathbf{X}, \mathbf{Y} \in \mathbf{L}\mathrm{in}^n.$$

5.3 Using the definition of the simple composition (5.40) and taking (5.17) into account we obtain

$$\mathbf{A}(\mathbf{B} \otimes \mathbf{C})\mathbf{D} : \mathbf{X} = \mathbf{A}(\mathbf{B} \otimes \mathbf{C} : \mathbf{X})\mathbf{D} = \mathbf{A}(\mathbf{BXC})\mathbf{D}$$
$$= (\mathbf{AB})\mathbf{X}(\mathbf{CD}) = (\mathbf{AB}) \otimes (\mathbf{CD}) : \mathbf{X},$$
$$\mathbf{A}(\mathbf{B} \odot \mathbf{C})\mathbf{D} : \mathbf{X} = \mathbf{A}(\mathbf{B} \odot \mathbf{C} : \mathbf{X})\mathbf{D} = \mathbf{A}[\mathbf{B}(\mathbf{C} : \mathbf{X})]\mathbf{D}$$
$$= \mathbf{ABD}(\mathbf{C} : \mathbf{X}) = (\mathbf{ABD}) \odot \mathbf{C} : \mathbf{X}, \quad \forall \mathbf{X} \in \mathbf{L}\mathrm{in}^n.$$

5.4 By means of (1.140), (5.17), (5.22) and (5.45) we can write

$$(\mathbf{A} \otimes \mathbf{B})^{\mathrm{T}} : \mathbf{X} = \mathbf{X} : (\mathbf{A} \otimes \mathbf{B}) = \mathbf{A}^{\mathrm{T}}\mathbf{X}\mathbf{B}^{\mathrm{T}} = \left(\mathbf{A}^{\mathrm{T}} \otimes \mathbf{B}^{\mathrm{T}}\right) : \mathbf{X},$$
$$(\mathbf{A} \odot \mathbf{B})^{\mathrm{T}} : \mathbf{X} = \mathbf{X} : (\mathbf{A} \odot \mathbf{B}) = (\mathbf{X} : \mathbf{A})\mathbf{B} = (\mathbf{B} \odot \mathbf{A}) : \mathbf{X},$$
$$(\mathbf{A} \odot \mathbf{B})^{\mathrm{t}} : \mathbf{X} = (\mathbf{A} \odot \mathbf{B}) : \mathbf{X}^{\mathrm{T}} = \mathbf{A}\left(\mathbf{B} : \mathbf{X}^{\mathrm{T}}\right)$$
$$= \mathbf{A}\left(\mathbf{B}^{\mathrm{T}} : \mathbf{X}\right) = \left(\mathbf{A} \odot \mathbf{B}^{\mathrm{T}}\right) : \mathbf{X}, \quad \forall \mathbf{X} \in \mathbf{L}\mathrm{in}^n.$$

Identities (5.51) and (5.52) follow immediately from (1.116) (5.23), (5.24) $(5.49)_1$ and (5.50) by setting $\mathbf{A} = \boldsymbol{a} \otimes \boldsymbol{b}$, $\mathbf{B} = \boldsymbol{c} \otimes \boldsymbol{d}$ or $\mathbf{A} = \boldsymbol{a} \otimes \boldsymbol{d}$, $\mathbf{B} = \boldsymbol{b} \otimes \boldsymbol{c}$, respectively.

5.5 Using (5.51) and (5.52) we obtain for the left and right hand sides different results:

$$(\boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c} \otimes \boldsymbol{d})^{\mathrm{tT}} = (\boldsymbol{a} \otimes \boldsymbol{c} \otimes \boldsymbol{b} \otimes \boldsymbol{d})^{\mathrm{T}} = \boldsymbol{c} \otimes \boldsymbol{a} \otimes \boldsymbol{d} \otimes \boldsymbol{b},$$
$$(\boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c} \otimes \boldsymbol{d})^{\mathrm{Tt}} = (\boldsymbol{b} \otimes \boldsymbol{a} \otimes \boldsymbol{d} \otimes \boldsymbol{c})^{\mathrm{t}} = \boldsymbol{b} \otimes \boldsymbol{d} \otimes \boldsymbol{a} \otimes \boldsymbol{c}.$$

5.6 (5.31), (5.32), (5.45):

$$(\mathcal{A} : \mathcal{B})^{\mathrm{T}} : \mathbf{X} = \mathbf{X} : (\mathcal{A} : \mathcal{B}) = (\mathbf{X} : \mathcal{A}) : \mathcal{B}$$
$$= \mathcal{B}^{\mathrm{T}} : (\mathbf{X} : \mathcal{A}) = \mathcal{B}^{\mathrm{T}} : \left(\mathcal{A}^{\mathrm{T}} : \mathbf{X}\right) = \left(\mathcal{B}^{\mathrm{T}} : \mathcal{A}^{\mathrm{T}}\right) : \mathbf{X},$$
$$(\mathcal{A} : \mathcal{B})^{\mathrm{t}} : \mathbf{X} = (\mathcal{A} : \mathcal{B}) : \mathbf{X}^{\mathrm{T}} = \mathcal{A} : \left(\mathcal{B} : \mathbf{X}^{\mathrm{T}}\right)$$
$$= \mathcal{A} : \left(\mathcal{B}^{\mathrm{t}} : \mathbf{X}\right) = \left(\mathcal{A} : \mathcal{B}^{\mathrm{t}}\right) : \mathbf{X}, \quad \forall \mathbf{X} \in \mathbf{L}\mathrm{in}^n.$$

5.7 In view of (1.115), (5.17) and (5.45) we write for an arbitrary tensor $\mathbf{X} \in \mathbf{L}\mathrm{in}^n$

$$(\mathbf{A} \otimes \mathbf{B})^{\mathrm{t}} : (\mathbf{C} \otimes \mathbf{D}) : \mathbf{X} = (\mathbf{A} \otimes \mathbf{B})^{\mathrm{t}} : (\mathbf{CXD}) = (\mathbf{A} \otimes \mathbf{B}) : (\mathbf{CXD})^{\mathrm{T}}$$
$$= (\mathbf{A} \otimes \mathbf{B}) : \left(\mathbf{D}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\right) = \mathbf{A}\mathbf{D}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{B}$$
$$= \left[\left(\mathbf{A}\mathbf{D}^{\mathrm{T}}\right) \otimes \left(\mathbf{C}^{\mathrm{T}}\mathbf{B}\right)\right] : \mathbf{X}^{\mathrm{T}} = \left[\left(\mathbf{A}\mathbf{D}^{\mathrm{T}}\right) \otimes \left(\mathbf{C}^{\mathrm{T}}\mathbf{B}\right)\right]^{\mathrm{t}} : \mathbf{X},$$

$$(\mathbf{A} \otimes \mathbf{B})^{\mathrm{t}} : (\mathbf{C} \odot \mathbf{D}) : \mathbf{X} = (\mathbf{A} \otimes \mathbf{B})^{\mathrm{t}} : [(\mathbf{D} : \mathbf{X})\, \mathbf{C}]$$

$$= (\mathbf{A} \otimes \mathbf{B}) : \left[(\mathbf{D} : \mathbf{X})\, \mathbf{C}^{\mathrm{T}}\right] = (\mathbf{D} : \mathbf{X})\, \mathbf{A}\mathbf{C}^{\mathrm{T}}\mathbf{B} = \left(\mathbf{A}\mathbf{C}^{\mathrm{T}}\mathbf{B}\right) \odot \mathbf{D} : \mathbf{X}.$$

5.8 By virtue of (5.51) and (5.52) we can write

$$\begin{aligned}
\boldsymbol{\mathcal{C}}^{\mathrm{T}} &= \left(\mathcal{C}^{ijkl}\boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}_l\right)^{\mathrm{T}} = \mathcal{C}^{ijkl}\boldsymbol{g}_j \otimes \boldsymbol{g}_i \otimes \boldsymbol{g}_l \otimes \boldsymbol{g}_k \\
&= \mathcal{C}^{jilk}\boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}_l, \\
\boldsymbol{\mathcal{C}}^{\mathrm{t}} &= \left(\mathcal{C}^{ijkl}\boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}_l\right)^{\mathrm{t}} = \mathcal{C}^{ijkl}\boldsymbol{g}_i \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_l \\
&= \mathcal{C}^{ikjl}\boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}_l.
\end{aligned}$$

According to (5.60) and (5.61) $\boldsymbol{\mathcal{C}}^{\mathrm{T}} = \boldsymbol{\mathcal{C}}^{\mathrm{t}} = \boldsymbol{\mathcal{C}}$. Taking also into account that the tensors $\boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k \otimes \boldsymbol{g}_l$ $(i, j, k, l = 1, 2, \ldots, n)$ are linearly independent we thus write

$$\mathcal{C}^{ijkl} = \mathcal{C}^{jilk} = \mathcal{C}^{ikjl}.$$

The remaining relations (5.70) are obtained in the same manner by applying the identities $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}}^{\mathrm{TtT}}$ and $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}}^{\mathrm{tTt}}$.

5.9 With the aid of (1.140), (5.16) and (5.79) we get

$$(\mathbf{Y} : \boldsymbol{\mathcal{T}}) : \mathbf{X} = \mathbf{Y} : (\boldsymbol{\mathcal{T}} : \mathbf{X}) = \mathbf{Y} : \mathbf{X}^{\mathrm{T}} = \mathbf{Y}^{\mathrm{T}} : \mathbf{X}, \quad \forall \mathbf{X}, \mathbf{Y} \in \mathbf{Lin}^n.$$

5.10 On use of (5.31), $(5.45)_2$ and (5.79) we obtain

$$(\boldsymbol{\mathcal{A}} : \boldsymbol{\mathcal{T}}) : \mathbf{X} = \boldsymbol{\mathcal{A}} : (\boldsymbol{\mathcal{T}} : \mathbf{X}) = \boldsymbol{\mathcal{A}} : \mathbf{X}^{\mathrm{T}} = \boldsymbol{\mathcal{A}}^{\mathrm{t}} : \mathbf{X}, \quad \forall \mathbf{X} \in \mathbf{Lin}^n.$$

The second identity (5.83) can be derived by means of (5.54), (5.78) and (5.81) as follows

$$\boldsymbol{\mathcal{A}}^{\mathrm{TtT}} = (\boldsymbol{\mathcal{I}} : \boldsymbol{\mathcal{A}})^{\mathrm{TtT}} = \left(\boldsymbol{\mathcal{A}}^{\mathrm{T}} : \boldsymbol{\mathcal{I}}\right)^{\mathrm{tT}} = \left(\boldsymbol{\mathcal{A}}^{\mathrm{T}} : \boldsymbol{\mathcal{I}}^{\mathrm{t}}\right)^{\mathrm{T}} = \left(\boldsymbol{\mathcal{A}}^{\mathrm{T}} : \boldsymbol{\mathcal{T}}\right)^{\mathrm{T}} = \boldsymbol{\mathcal{T}} : \boldsymbol{\mathcal{A}}.$$

The last identity (5.83) can finally be proved by

$$(\boldsymbol{\mathcal{T}} : \boldsymbol{\mathcal{T}}) : \mathbf{X} = \boldsymbol{\mathcal{T}} : (\boldsymbol{\mathcal{T}} : \mathbf{X}) = \boldsymbol{\mathcal{T}} : \mathbf{X}^{\mathrm{T}} = \mathbf{X} = \boldsymbol{\mathcal{I}} : \mathbf{X}, \quad \forall \mathbf{X} \in \mathbf{Lin}^n.$$

5.11 $\boldsymbol{\mathcal{C}}$ possesses the minor symmetry (5.61) by the very definition. In order to prove the major symmetry (5.60) we show that $\boldsymbol{\mathcal{C}} : \mathbf{X} = \mathbf{X} : \boldsymbol{\mathcal{C}}$, $\forall \mathbf{X} \in \mathbf{Lin}^n$. Indeed, in view of $(5.17)_1$, $(5.22)_1$ and (5.48)

$$\begin{aligned}
\boldsymbol{\mathcal{C}} : \mathbf{X} &= (\mathbf{M}_1 \otimes \mathbf{M}_2 + \mathbf{M}_2 \otimes \mathbf{M}_1)^{\mathrm{s}} : \mathbf{X} = (\mathbf{M}_1 \otimes \mathbf{M}_2 + \mathbf{M}_2 \otimes \mathbf{M}_1) : \mathrm{sym}\mathbf{X} \\
&= \mathbf{M}_1 (\mathrm{sym}\mathbf{X})\, \mathbf{M}_2 + \mathbf{M}_2 (\mathrm{sym}\mathbf{X})\, \mathbf{M}_1,
\end{aligned}$$

$$\begin{aligned}
\mathbf{X} : \boldsymbol{\mathcal{C}} &= \mathbf{X} : (\mathbf{M}_1 \otimes \mathbf{M}_2 + \mathbf{M}_2 \otimes \mathbf{M}_1)^{\mathrm{s}} \\
&= \mathrm{sym}\left[\mathbf{X} : (\mathbf{M}_1 \otimes \mathbf{M}_2 + \mathbf{M}_2 \otimes \mathbf{M}_1)\right] \\
&= \mathrm{sym}\left(\mathbf{M}_1\mathbf{X}\mathbf{M}_2 + \mathbf{M}_2\mathbf{X}\mathbf{M}_1\right) = \mathbf{M}_1 (\mathrm{sym}\mathbf{X})\, \mathbf{M}_2 + \mathbf{M}_2 (\mathrm{sym}\mathbf{X})\, \mathbf{M}_1.
\end{aligned}$$

5.12 (a) Let $\boldsymbol{e}_i$ $(i = 1, 2, 3)$ be an orthonormal basis in $\mathbb{E}^3$. By virtue of (5.77), (5.82) and (5.84) we can write

$$\mathfrak{I}^{\mathrm{s}} = \sum_{i,j=1}^{3} \frac{1}{2} \boldsymbol{e}_i \otimes (\boldsymbol{e}_i \otimes \boldsymbol{e}_j + \boldsymbol{e}_j \otimes \boldsymbol{e}_i) \otimes \boldsymbol{e}_j.$$

Using the notation

$$\mathbf{M}_i = \boldsymbol{e}_i \otimes \boldsymbol{e}_i, \quad i = 1, 2, 3, \quad \mathbf{M}_4 = \frac{\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1}{\sqrt{2}},$$

$$\mathbf{M}_5 = \frac{\boldsymbol{e}_1 \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_1}{\sqrt{2}}, \qquad \mathbf{M}_6 = \frac{\boldsymbol{e}_3 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_3}{\sqrt{2}} \tag{S.22}$$

and taking (5.23) into account one thus obtains the spectral decomposition of $\mathfrak{I}^{\mathrm{s}}$ as

$$\mathfrak{I}^{\mathrm{s}} = \sum_{p=1}^{6} \mathbf{M}_p \odot \mathbf{M}_p.$$

The only eigenvalue 1 is of multiplicity 6. Note that the corresponding eigentensors $\mathbf{M}_p$ $(p = 1, 2, \ldots, 6)$ form an orthonormal basis of $\mathbf{L}\mathrm{in}^3$.

(b) Using the orthonormal basis (S.22) we can write

$$\begin{aligned}\mathfrak{P}_{\mathrm{sph}} &= \frac{1}{3} (\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3) \odot (\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3) \\ &= \sum_{p,q=1}^{6} \mathcal{P}_{\mathrm{sph}}^{pq} \mathbf{M}_p \odot \mathbf{M}_q, \quad \text{where} \quad [\mathcal{P}_{\mathrm{sph}}^{pq}] = \frac{1}{3} \begin{bmatrix} 1&1&1&0&0&0 \\ 1&1&1&0&0&0 \\ 1&1&1&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}.\end{aligned}$$

Eigenvalues and eigenvectors of this matrix can be represented as

$$\Lambda_1 = 1, \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix},$$

$$\Lambda_2 = \Lambda_3 = \Lambda_4 = \Lambda_5 = \Lambda_6 = 0, \begin{Bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \begin{Bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Thus, the orthonormal eigentensors can be given by

$$\widehat{\mathbf{M}}_1 = \frac{1}{\sqrt{3}}\left(\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3\right) = \frac{1}{\sqrt{3}}\mathbf{I}, \quad \widehat{\mathbf{M}}_2 = -\frac{\sqrt{2}}{2}\mathbf{M}_1 + \frac{\sqrt{2}}{2}\mathbf{M}_2,$$

$$\widehat{\mathbf{M}}_3 = -\frac{\sqrt{6}}{6}\mathbf{M}_1 - \frac{\sqrt{6}}{6}\mathbf{M}_2 + \frac{\sqrt{6}}{3}\mathbf{M}_3, \quad \widehat{\mathbf{M}}_p = \mathbf{M}_p, \quad p = 4, 5, 6, \tag{S.23}$$

where the tensors $\mathbf{M}_q$, $(q = 1, 2, \ldots, 6)$ are defined by (S.22).

(c) For the super-symmetric counterpart of the deviatoric projection tensor $(5.87)_2$ $(n = 3)$ we can write

$$\boldsymbol{\mathcal{P}}^{\mathrm{s}}_{\mathrm{dev}} = \sum_{p,q=1}^{6} \mathcal{P}^{pq}_{\mathrm{dev}} \mathbf{M}_p \odot \mathbf{M}_q, \quad \text{where} \quad \left[\mathcal{P}^{pq}_{\mathrm{dev}}\right] = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, the eigenvalues of $\boldsymbol{\mathcal{P}}^{\mathrm{s}}_{\mathrm{dev}}$ are $\Lambda_1 = 0$, $\Lambda_q = 1$ $(q = 2, 3, \ldots, 6)$. The corresponding eigentensors are again given by (S.23).

(d) With respect to the orthonormal basis (S.22) the elasticity tensor (5.91) can be represented by

$$\boldsymbol{\mathcal{C}} = \sum_{p,q=1}^{6} \mathcal{C}^{pq} \mathbf{M}_p \odot \mathbf{M}_q,$$

where

$$\left[\mathcal{C}^{pq}\right] = \begin{bmatrix} 2G+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2G+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2G+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G & 0 \\ 0 & 0 & 0 & 0 & 0 & 2G \end{bmatrix}.$$

The eigentensors of $\boldsymbol{\mathcal{C}}$ are the same as of $\boldsymbol{\mathcal{P}}_{\mathrm{sph}}$ and $\boldsymbol{\mathcal{P}}^{\mathrm{s}}_{\mathrm{dev}}$ and are given by (S.23). The eigenvalues are as follows: $\Lambda_1 = 2G + 3\lambda$, $\Lambda_q = 2G$ $(q = 2, 3, \ldots, 6)$. They can be obtained as linear combinations of those ones of $\boldsymbol{\mathcal{P}}_{\mathrm{sph}}$ and $\boldsymbol{\mathcal{P}}^{\mathrm{s}}_{\mathrm{dev}}$.

Exercises of Chapter 6

6.1 (a) $f\left(\mathbf{QAQ}^{\mathrm{T}}\right) = \boldsymbol{a}\mathbf{QAQ}^{\mathrm{T}}\boldsymbol{b} \neq \boldsymbol{a}\mathbf{A}\boldsymbol{b}$.

(b) Since the components of $\mathbf{A}$ are related to an orthonormal basis we can write

$$f(\mathbf{A}) = \mathrm{A}^{11} + \mathrm{A}^{22} + \mathrm{A}^{33} = \mathrm{A}^{1}_{\cdot 1} + \mathrm{A}^{2}_{\cdot 2} + \mathrm{A}^{3}_{\cdot 3} = \mathrm{A}^{i}_{\cdot i} = \mathrm{tr}\mathbf{A}.$$

Trace of a tensor represents its isotropic function.

(c) For an isotropic tensor function the condition (6.1) $f(\mathbf{QAQ}^{\mathrm{T}}) = f(\mathbf{A})$ must hold on the whole definition domain of the arguments $\mathbf{A}$ and $\forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^3$. Let us consider a special case where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j, \quad \mathbf{Q} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j.$$

Thus,

$$\mathbf{A}' = \mathbf{QAQ}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

and consequently

$$f(\mathbf{A}) = \mathrm{A}^{11} + \mathrm{A}^{12} + \mathrm{A}^{13} = 1 \neq 0 = \mathrm{A}'^{11} + \mathrm{A}'^{12} + \mathrm{A}'^{13} = f\left(\mathbf{QAQ}^{\mathrm{T}}\right),$$

which means that the function $f(\mathbf{A})$ is not isotropic.

(d) $\det\mathbf{A}$ is the last principal invariant of $\mathbf{A}$ and represents thus its isotropic tensor function. Isotropy of the determinant can, however, be shown directly using the relation $\det(\mathbf{BC}) = \det\mathbf{B}\det\mathbf{C}$. Indeed,

$$\begin{aligned}\det\left(\mathbf{QAQ}^{\mathrm{T}}\right) &= \det\mathbf{Q}\,\det\mathbf{A}\,\det\mathbf{Q}^{\mathrm{T}} = \det\mathbf{Q}\,\det\mathbf{Q}^{\mathrm{T}}\det\mathbf{A} \\ &= \det\left(\mathbf{QQ}^{\mathrm{T}}\right)\det\mathbf{A} = \det\mathbf{I}\,\det\mathbf{A} = \det\mathbf{A}, \quad \forall \mathbf{Q} \in \mathbf{O}\mathrm{rth}^n.\end{aligned}$$

(e) Eigenvalues of a second-order tensor are uniquely defined by its principal invariants and represent thus its isotropic functions. This can also be shown in a direct way considering the eigenvalue problem for the tensor $\mathbf{QAQ}^{\mathrm{T}}$ as

$$\left(\mathbf{QAQ}^{\mathrm{T}}\right)\boldsymbol{a} = \lambda\boldsymbol{a}.$$

Mapping both sides of this vector equation by $\mathbf{Q}^{\mathrm{T}}$ yields

$$\left(\mathbf{Q}^{\mathrm{T}}\mathbf{QAQ}^{\mathrm{T}}\right)\boldsymbol{a} = \lambda\mathbf{Q}^{\mathrm{T}}\boldsymbol{a}.$$

Using the abbreviation $\boldsymbol{a}' = \mathbf{Q}^{\mathrm{T}}\boldsymbol{a}$ we finally obtain

$$\mathbf{A}\boldsymbol{a}' = \lambda\boldsymbol{a}'.$$

Thus, every eigenvalue of $\mathbf{QAQ}^{\mathrm{T}}$ is the eigenvalue of $\mathbf{A}$ and vice versa. In other words, the eigenvalues of these tensors are pairwise equal which immediately implies that they are characterized by the same value of $\lambda_{\max}$. The tensors obtained by the operation $\mathbf{QAQ}^{\mathrm{T}}$ from the original one $\mathbf{A}$ are called similar tensors.

6.2 Inserting

$$\mathbf{M} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right), \quad \mathbf{W} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathrm{T}}\right)$$

into (6.17) we obtain

$$\mathrm{tr}\mathbf{M} = \frac{1}{2}\left(\mathrm{tr}\mathbf{A} + \mathrm{tr}\mathbf{A}^{\mathrm{T}}\right) = \mathrm{tr}\mathbf{A},$$

$$\begin{aligned}
\mathrm{tr}\mathbf{M}^2 &= \frac{1}{4}\mathrm{tr}\left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right)^2 \\
&= \frac{1}{4}\left[\mathrm{tr}\mathbf{A}^2 + \mathrm{tr}\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right) + \mathrm{tr}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right) + \mathrm{tr}\left(\mathbf{A}^{\mathrm{T}}\right)^2\right] \\
&= \frac{1}{2}\left[\mathrm{tr}\mathbf{A}^2 + \mathrm{tr}\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right)\right],
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}\mathbf{M}^3 &= \frac{1}{8}\mathrm{tr}\left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right)^3 \\
&= \frac{1}{8}\Big\{\mathrm{tr}\mathbf{A}^3 + \mathrm{tr}\left(\mathbf{A}^2\mathbf{A}^{\mathrm{T}}\right) + \mathrm{tr}\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{A}\right) + \mathrm{tr}\left[\mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\right)^2\right] \\
&+ \mathrm{tr}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}^2\right) + \mathrm{tr}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{A}^{\mathrm{T}}\right) + \mathrm{tr}\left[\left(\mathbf{A}^{\mathrm{T}}\right)^2\mathbf{A}\right] + \mathrm{tr}\left(\mathbf{A}^{\mathrm{T}}\right)^3\Big\} \\
&= \frac{1}{4}\left[\mathrm{tr}\mathbf{A}^3 + 3\mathrm{tr}\left(\mathbf{A}^2\mathbf{A}^{\mathrm{T}}\right)\right],
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}\mathbf{W}^2 &= \frac{1}{4}\mathrm{tr}\left(\mathbf{A} - \mathbf{A}^{\mathrm{T}}\right)^2 \\
&= \frac{1}{4}\left[\mathrm{tr}\mathbf{A}^2 - \mathrm{tr}\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right) - \mathrm{tr}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right) + \mathrm{tr}\left(\mathbf{A}^{\mathrm{T}}\right)^2\right] \\
&= \frac{1}{2}\left[\mathrm{tr}\mathbf{A}^2 - \mathrm{tr}\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right)\right],
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}\left(\mathbf{M}\mathbf{W}^2\right) &= \frac{1}{8}\mathrm{tr}\left[\left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right)\left(\mathbf{A} - \mathbf{A}^{\mathrm{T}}\right)^2\right] \\
&= \frac{1}{8}\Big\{\mathrm{tr}\mathbf{A}^3 - \mathrm{tr}\left(\mathbf{A}^2\mathbf{A}^{\mathrm{T}}\right) - \mathrm{tr}\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{A}\right) + \mathrm{tr}\left[\mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\right)^2\right] \\
&+ \mathrm{tr}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}^2\right) - \mathrm{tr}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{A}^{\mathrm{T}}\right) - \mathrm{tr}\left[\left(\mathbf{A}^{\mathrm{T}}\right)^2\mathbf{A}\right] + \mathrm{tr}\left(\mathbf{A}^{\mathrm{T}}\right)^3\Big\}
\end{aligned}$$

$$= \frac{1}{4}\left[\mathrm{tr}\mathbf{A}^3 - \mathrm{tr}\left(\mathbf{A}^2\mathbf{A}^{\mathrm{T}}\right)\right],$$

$$\mathrm{tr}\left(\mathbf{M}^2\mathbf{W}^2\right) = \frac{1}{16}\mathrm{tr}\left[\left(\mathbf{A}+\mathbf{A}^{\mathrm{T}}\right)^2\left(\mathbf{A}-\mathbf{A}^{\mathrm{T}}\right)^2\right]$$

$$= \frac{1}{16}\mathrm{tr}\left\{\left[\mathbf{A}^2 + \mathbf{A}\mathbf{A}^{\mathrm{T}} + \mathbf{A}^{\mathrm{T}}\mathbf{A} + \left(\mathbf{A}^{\mathrm{T}}\right)^2\right]\left[\mathbf{A}^2 - \mathbf{A}\mathbf{A}^{\mathrm{T}} - \mathbf{A}^{\mathrm{T}}\mathbf{A} + \left(\mathbf{A}^{\mathrm{T}}\right)^2\right]\right\}$$

$$= \frac{1}{16}\mathrm{tr}\Big[\mathbf{A}^4 - \mathbf{A}^3\mathbf{A}^{\mathrm{T}} - \mathbf{A}^2\mathbf{A}^{\mathrm{T}}\mathbf{A} + \mathbf{A}^2\left(\mathbf{A}^{\mathrm{T}}\right)^2 + \mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{A}^2 - \mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{A}^{\mathrm{T}}$$

$$- \mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\right)^2\mathbf{A} + \mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\right)^3 + \mathbf{A}^{\mathrm{T}}\mathbf{A}^3 - \mathbf{A}^{\mathrm{T}}\mathbf{A}^2\mathbf{A}^{\mathrm{T}} - \mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{A}$$

$$+ \mathbf{A}^{\mathrm{T}}\mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\right)^2 + \left(\mathbf{A}^{\mathrm{T}}\right)^2\mathbf{A}^2 - \left(\mathbf{A}^{\mathrm{T}}\right)^2\mathbf{A}\mathbf{A}^{\mathrm{T}} - \left(\mathbf{A}^{\mathrm{T}}\right)^3\mathbf{A} + \left(\mathbf{A}^{\mathrm{T}}\right)^4\Big]$$

$$= \frac{1}{8}\left[\mathrm{tr}\mathbf{A}^4 - \mathrm{tr}\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right)^2\right],$$

$$\mathrm{tr}\left(\mathbf{M}^2\mathbf{W}^2\mathbf{M}\mathbf{W}\right)$$

$$= \frac{1}{64}\mathrm{tr}\Big[\mathbf{A}^4 - \mathbf{A}^3\mathbf{A}^{\mathrm{T}} - \mathbf{A}^2\mathbf{A}^{\mathrm{T}}\mathbf{A} + \mathbf{A}^2\left(\mathbf{A}^{\mathrm{T}}\right)^2 + \mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{A}^2 - \mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{A}^{\mathrm{T}}$$

$$- \mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\right)^2\mathbf{A} + \mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\right)^3 + \mathbf{A}^{\mathrm{T}}\mathbf{A}^3 - \mathbf{A}^{\mathrm{T}}\mathbf{A}^2\mathbf{A}^{\mathrm{T}} - \mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{A}$$

$$+ \mathbf{A}^{\mathrm{T}}\mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\right)^2 + \left(\mathbf{A}^{\mathrm{T}}\right)^2\mathbf{A}^2 - \left(\mathbf{A}^{\mathrm{T}}\right)^2\mathbf{A}\mathbf{A}^{\mathrm{T}} - \left(\mathbf{A}^{\mathrm{T}}\right)^3\mathbf{A} + \left(\mathbf{A}^{\mathrm{T}}\right)^4\Big]$$

$$\left[\mathbf{A}^2 - \mathbf{A}\mathbf{A}^{\mathrm{T}} + \mathbf{A}^{\mathrm{T}}\mathbf{A} - \left(\mathbf{A}^{\mathrm{T}}\right)^2\right]\Big\}$$

$$= \frac{1}{16}\mathrm{tr}\left[\left(\mathbf{A}^{\mathrm{T}}\right)^2\mathbf{A}^2\mathbf{A}^{\mathrm{T}}\mathbf{A} - \mathbf{A}^2\left(\mathbf{A}^{\mathrm{T}}\right)^2\mathbf{A}\mathbf{A}^{\mathrm{T}}\right].$$

Finally, $\mathrm{tr}\mathbf{A}^4$ should be expressed in terms of the principal traces $\mathrm{tr}\mathbf{A}^i$ ($i = 1,2,3$) presented in the functional basis (6.18). To this end, we apply the Cayley-Hamilton equation (4.91). Its composition with $\mathbf{A}$ yields

$$\mathbf{A}^4 - \mathrm{I}_{\mathbf{A}}\mathbf{A}^3 + \mathrm{II}_{\mathbf{A}}\mathbf{A}^2 - \mathrm{III}_{\mathbf{A}}\mathbf{A} = \mathbf{0},$$

so that

$$\mathrm{tr}\mathbf{A}^4 = \mathrm{I}_{\mathbf{A}}\mathrm{tr}\mathbf{A}^3 - \mathrm{II}_{\mathbf{A}}\mathrm{tr}\mathbf{A}^2 + \mathrm{III}_{\mathbf{A}}\mathrm{tr}\mathbf{A},$$

where $\mathrm{I}_{\mathbf{A}}$, $\mathrm{II}_{\mathbf{A}}$ and $\mathrm{III}_{\mathbf{A}}$ are given by (4.29). Thus, all the invariants of the functional basis (6.17) are expressed in a unique form in terms of (6.18).

6.3 By Theorem 6.1 ψ is an isotropic function of $\mathbf{C}$ and $\mathbf{L}_i$ ($i = 1,2,3$). Applying further (6.15) and taking into account the identities

$$\mathbf{L}_i\mathbf{L}_j = \mathbf{0}, \quad \mathbf{L}_i^k = \mathbf{L}_i, \quad \mathrm{tr}\mathbf{L}_i^k = 1, \ i \neq j, \ i,j = 1,2,3; \ k = 1,2,\ldots \qquad \text{(S.24)}$$

we obtain the following orthotropic invariants

$$\begin{aligned}
&\mathrm{tr}\mathbf{C}, \quad \mathrm{tr}\mathbf{C}^2, \quad \mathrm{tr}\mathbf{C}^3, \\
&\mathrm{tr}\left(\mathbf{CL}_1\right) = \mathrm{tr}\left(\mathbf{CL}_1^2\right), \quad \mathrm{tr}\left(\mathbf{CL}_2\right) = \mathrm{tr}\left(\mathbf{CL}_2^2\right), \quad \mathrm{tr}\left(\mathbf{CL}_3\right) = \mathrm{tr}\left(\mathbf{CL}_3^2\right), \\
&\mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_1\right) = \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_1^2\right), \ \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_2\right) = \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_2^2\right), \ \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_3\right) = \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_3^2\right), \\
&\mathrm{tr}\left(\mathbf{L}_i\mathbf{CL}_j\right) = \mathrm{tr}\left(\mathbf{CL}_j\mathbf{L}_i\right) = \mathrm{tr}\left(\mathbf{L}_j\mathbf{L}_i\mathbf{C}\right) = 0, \quad i \neq j = 1, 2, 3. \qquad \text{(S.25)}
\end{aligned}$$

Using the relation

$$\sum_{i=1}^{3} \mathbf{L}_i = \mathbf{I} \qquad \text{(S.26)}$$

one can further write

$$\begin{aligned}
\mathrm{tr}\left(\mathbf{C}\right) &= \mathbf{C} : \mathbf{I} = \mathbf{C} : \left(\mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3\right) \\
&= \mathbf{C} : \mathbf{L}_1 + \mathbf{C} : \mathbf{L}_2 + \mathbf{C} : \mathbf{L}_3 = \mathrm{tr}\left(\mathbf{CL}_1\right) + \mathrm{tr}\left(\mathbf{CL}_2\right) + \mathrm{tr}\left(\mathbf{CL}_3\right).
\end{aligned}$$

In the same manner we also obtain

$$\mathrm{tr}\left(\mathbf{C}^2\right) = \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_1\right) + \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_2\right) + \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_3\right).$$

Thus, the invariants $\mathrm{tr}\mathbf{C}$ and $\mathrm{tr}\mathbf{C}^2$ are redundant and can be excluded from the functional basis (S.25). Finally, the orthotropic strain energy function can be represented by

$$\begin{aligned}
\psi = \tilde{\psi}\,&[\mathrm{tr}\left(\mathbf{CL}_1\right), \mathrm{tr}\left(\mathbf{CL}_2\right), \mathrm{tr}\left(\mathbf{CL}_3\right), \\
&\mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_1\right), \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_2\right), \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_3\right), \mathrm{tr}\mathbf{C}^3]. \qquad \text{(S.27)}
\end{aligned}$$

Alternatively, a functional basis for the orthotropic material symmetry can be obtained in the component form. To this end, we represent the right Cauchy-Green tensor by $\mathbf{C} = \mathrm{C}^{ij}\boldsymbol{l}_i \otimes \boldsymbol{l}_j$. Then,

$$\begin{aligned}
&\mathrm{tr}\left(\mathbf{CL}_i\right) = \left(\mathrm{C}^{kl}\boldsymbol{l}_k \otimes \boldsymbol{l}_l\right) : \boldsymbol{l}_i \otimes \boldsymbol{l}_i = \mathrm{C}^{ii}, \quad i = 1, 2, 3, \\
&\mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_i\right) = \left(\mathrm{C}^{i1}\right)^2 + \left(\mathrm{C}^{i2}\right)^2 + \left(\mathrm{C}^{i3}\right)^2, \quad i = 1, 2, 3, \\
&\mathrm{tr}\left(\mathbf{C}^3\right) = \left(\mathrm{C}^{11}\right)^3 + \left(\mathrm{C}^{22}\right)^3 + \left(\mathrm{C}^{33}\right)^3 + 3\left(\mathrm{C}^{12}\right)^2\left(\mathrm{C}^{11} + \mathrm{C}^{22}\right) \\
&\qquad + 3\left(\mathrm{C}^{13}\right)^2\left(\mathrm{C}^{11} + \mathrm{C}^{33}\right) + 3\left(\mathrm{C}^{23}\right)^2\left(\mathrm{C}^{22} + \mathrm{C}^{33}\right) + 6\mathrm{C}^{12}\mathrm{C}^{13}\mathrm{C}^{23}.
\end{aligned}$$

Thus, the orthotropic strain energy function (S.27) can be given in another form as

$$\psi = \hat{\psi}\left[\mathrm{C}^{11}, \mathrm{C}^{22}, \mathrm{C}^{33}, \left(\mathrm{C}^{12}\right)^2, \left(\mathrm{C}^{13}\right)^2, \left(\mathrm{C}^{23}\right)^2, \mathrm{C}^{12}\mathrm{C}^{13}\mathrm{C}^{23}\right].$$

6.4 (6.52), (6.54), (6.93), (6.119), (6.123), (6.127) (6.131), (S.27):

$$\mathbf{S} = 6\frac{\partial\tilde{\psi}}{\partial \mathrm{tr}\mathbf{C}^3}\mathbf{C}^2 + 2\sum_{i=1}^{3}\frac{\partial\tilde{\psi}}{\partial \mathrm{tr}\left(\mathbf{C}\mathbf{L}_i\right)}\mathbf{L}_i + 2\sum_{i=1}^{3}\frac{\partial\tilde{\psi}}{\partial \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_i\right)}\left(\mathbf{C}\mathbf{L}_i + \mathbf{L}_i\mathbf{C}\right),$$

$$\mathfrak{C} = 36\frac{\partial^2\tilde{\psi}}{\partial \mathrm{tr}\mathbf{C}^3\partial \mathrm{tr}\mathbf{C}^3}\mathbf{C}^2 \odot \mathbf{C}^2 + 4\sum_{i,j=1}^{3}\frac{\partial^2\tilde{\psi}}{\partial \mathrm{tr}\left(\mathbf{C}\mathbf{L}_i\right)\partial \mathrm{tr}\left(\mathbf{C}\mathbf{L}_j\right)}\mathbf{L}_i \odot \mathbf{L}_j$$

$$+ 4\sum_{i,j=1}^{3}\frac{\partial^2\tilde{\psi}}{\partial \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_i\right)\partial \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_j\right)}\left(\mathbf{C}\mathbf{L}_i + \mathbf{L}_i\mathbf{C}\right) \odot \left(\mathbf{C}\mathbf{L}_j + \mathbf{L}_j\mathbf{C}\right)$$

$$+ 12\sum_{i=1}^{3}\frac{\partial^2\tilde{\psi}}{\partial \mathrm{tr}\left(\mathbf{C}\mathbf{L}_i\right)\partial \mathrm{tr}\mathbf{C}^3}\left(\mathbf{L}_i \odot \mathbf{C}^2 + \mathbf{C}^2 \odot \mathbf{L}_i\right)$$

$$+ 12\sum_{i=1}^{3}\frac{\partial^2\tilde{\psi}}{\partial \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_i\right)\partial \mathrm{tr}\mathbf{C}^3}\left[\mathbf{C}^2 \odot \left(\mathbf{C}\mathbf{L}_i + \mathbf{L}_i\mathbf{C}\right) + \left(\mathbf{C}\mathbf{L}_i + \mathbf{L}_i\mathbf{C}\right) \odot \mathbf{C}^2\right]$$

$$+ 4\sum_{i,j=1}^{3}\frac{\partial^2\tilde{\psi}}{\partial \mathrm{tr}\left(\mathbf{C}\mathbf{L}_i\right)\partial \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_j\right)}\left[\mathbf{L}_i \odot \left(\mathbf{C}\mathbf{L}_j + \mathbf{L}_j\mathbf{C}\right) + \left(\mathbf{C}\mathbf{L}_j + \mathbf{L}_j\mathbf{C}\right) \odot \mathbf{L}_i\right]$$

$$+ 12\frac{\partial\tilde{\psi}}{\partial \mathrm{tr}\mathbf{C}^3}\left(\mathbf{C} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{C}\right)^{\mathrm{s}} + 4\sum_{i=1}^{3}\frac{\partial\tilde{\psi}}{\partial \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_i\right)}\left(\mathbf{L}_i \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{L}_i\right)^{\mathrm{s}}.$$

6.5 Any orthotropic function $\mathbf{S}(\mathbf{C})$ is an isotropic function of $\mathbf{C}$ and $\mathbf{L}_i$ $(i = 1, 2, 3)$. The latter function can be represented by (6.100). Taking (S.24) and (S.26) into account we thus obtain

$$\mathbf{S} = \sum_{i=1}^{3}\left[\alpha_i\mathbf{L}_i + \beta_i\left(\mathbf{C}\mathbf{L}_i + \mathbf{C}\mathbf{L}_i\right) + \gamma_i\left(\mathbf{C}^2\mathbf{L}_i + \mathbf{L}_i\mathbf{C}^2\right)\right],$$

where α_i, β_i and γ_i $(i = 1, 2, 3)$ are some scalar-valued orthotropic functions of $\mathbf{C}$ (isotropic functions of $\mathbf{C}$ and $\mathbf{L}_i$ $(i = 1, 2, 3)$).

6.6 Applying (6.15) and taking into account the identities $\mathbf{L}_i^m = \mathbf{L}_i$, $\mathrm{tr}\mathbf{L}_i^m = 1$ $(i = 1, 2;\ m = 1, 2, \ldots)$ we obtain similarly to (S.25)

$$\mathrm{tr}\mathbf{C}, \quad \mathrm{tr}\mathbf{C}^2, \quad \mathrm{tr}\mathbf{C}^3,$$

$$\mathrm{tr}\left(\mathbf{C}\mathbf{L}_1\right) = \mathrm{tr}\left(\mathbf{C}\mathbf{L}_1^2\right), \quad \mathrm{tr}\left(\mathbf{C}\mathbf{L}_2\right) = \mathrm{tr}\left(\mathbf{C}\mathbf{L}_2^2\right),$$

$$\mathrm{tr}\left(\mathbf{L}_1\mathbf{L}_2\right) = \mathrm{tr}\left(\mathbf{L}_1\mathbf{L}_2^2\right) = \mathrm{tr}\left(\mathbf{L}_1^2\mathbf{L}_2\right)$$

$$= \left(\boldsymbol{l}_1 \otimes \boldsymbol{l}_1\right) : \left(\boldsymbol{l}_2 \otimes \boldsymbol{l}_2\right) = \left(\boldsymbol{l}_1 \cdot \boldsymbol{l}_2\right)^2 = \cos^2\phi,$$

$$\mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_1\right) = \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_1^2\right), \ \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_2\right) = \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_2^2\right), \ \mathrm{tr}\left(\mathbf{L}_1\mathbf{C}\mathbf{L}_2\right),$$

where ϕ denotes the angle between the fiber directions $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$. Thus, we can write

$$\psi = \tilde{\psi}\left[\mathrm{tr}\mathbf{C}, \mathrm{tr}\mathbf{C}^2, \mathrm{tr}\mathbf{C}^3, \mathrm{tr}\left(\mathbf{C}\mathbf{L}_1\right), \mathrm{tr}\left(\mathbf{C}\mathbf{L}_2\right),\right.$$
$$\left.\mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_1\right), \mathrm{tr}\left(\mathbf{C}^2\mathbf{L}_2\right), \mathrm{tr}\left(\mathbf{L}_1\mathbf{L}_2\right), \mathrm{tr}\left(\mathbf{L}_1\mathbf{C}\mathbf{L}_2\right)\right].$$

6.7 Using (6.57), (6.120), (6.122) and (6.127) we obtain

$$\mathbf{S} = 2\frac{\partial \psi}{\partial \mathbf{C}} + p\mathbf{C}^{-1} = 2c_1 \mathrm{I}_{\mathbf{C}},_{\mathbf{C}} + 2c_2 \mathrm{II}_{\mathbf{C}},_{\mathbf{C}} + p\mathbf{C}^{-1}$$
$$= 2c_1\mathbf{I} + 2c_2\left(\mathrm{I}_{\mathbf{C}}\mathbf{I} - \mathbf{C}\right) + p\mathbf{C}^{-1} = 2\left(c_1 + c_2\mathrm{I}_{\mathbf{C}}\right)\mathbf{I} - 2c_2\mathbf{C} + p\mathbf{C}^{-1},$$
$$\mathfrak{C} = 2\frac{\partial \mathbf{S}}{\partial \mathbf{C}} = 4c_2\left(\mathbf{I}\odot\mathbf{I} - \mathfrak{I}^{\mathrm{s}}\right) - 2p\left(\mathbf{C}^{-1}\otimes\mathbf{C}^{-1}\right)^{\mathrm{s}}.$$

6.8 Using the abbreviation $\Lambda_i = \lambda_i^2$ $(i = 1, 2, 3)$ for the eigenvalues of $\mathbf{C}$ we can write

$$\psi\left(\mathbf{C}\right) = \sum_{r=1}^{m} \frac{\mu_r}{\alpha_r}\left(\Lambda_1^{\alpha_r/2} + \Lambda_2^{\alpha_r/2} + \Lambda_3^{\alpha_r/2} - 3\right).$$

Assuming further that $\Lambda_1 \neq \Lambda_2 \neq \Lambda_3 \neq \Lambda_1$ and applying (6.69) we obtain

$$\mathbf{S} = 2\frac{\partial \psi}{\partial \mathbf{C}} = \sum_{r=1}^{m} \mu_r\left(\Lambda_1^{\alpha_r/2-1}\Lambda_1,_{\mathbf{C}} + \Lambda_2^{\alpha_r/2-1}\Lambda_2,_{\mathbf{C}} + \Lambda_3^{\alpha_r/2-1}\Lambda_3,_{\mathbf{C}}\right)$$
$$= \sum_{r=1}^{m} \mu_r\left(\Lambda_1^{\alpha_r/2-1}\mathbf{P}_1 + \Lambda_2^{\alpha_r/2-1}\mathbf{P}_2 + \Lambda_3^{\alpha_r/2-1}\mathbf{P}_3\right) = \sum_{r=1}^{m} \mu_r\mathbf{C}^{\alpha_r/2-1}.$$

Note that the latter expression is obtained by means of (7.2).

6.9 Using the identities

$$\mathbf{Q}^{\mathrm{T}}\mathbf{L}_i\mathbf{Q} = \mathbf{Q}\mathbf{L}_i\mathbf{Q}^{\mathrm{T}} = \mathbf{L}_i, \quad \forall \mathbf{Q} \in \mathfrak{g}_o$$

and taking (1.144) into account we can write

$$\mathrm{tr}\left(\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}\mathbf{L}_i\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}\mathbf{L}_j\right) = \mathrm{tr}\left(\mathbf{C}\mathbf{Q}^{\mathrm{T}}\mathbf{L}_i\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}\mathbf{L}_j\mathbf{Q}\right)$$
$$= \mathrm{tr}\left(\mathbf{C}\mathbf{L}_i\mathbf{C}\mathbf{L}_j\right), \quad \forall \mathbf{Q} \in \mathfrak{g}_o.$$

Further, one can show that

$$\mathrm{tr}\left(\mathbf{C}\mathbf{L}_i\mathbf{C}\mathbf{L}_i\right) = \mathrm{tr}^2\left(\mathbf{C}\mathbf{L}_i\right), \quad i = 1, 2, 3, \tag{S.28}$$

where we use the abbreviation $\mathrm{tr}^2\left(\bullet\right) = \left[\mathrm{tr}\left(\bullet\right)\right]^2$. Indeed, in view of the relation $\mathrm{tr}\left(\mathbf{C}\mathbf{L}_i\right) = \mathbf{C} : \left(\boldsymbol{l}_i \otimes \boldsymbol{l}_i\right) = \boldsymbol{l}_i\mathbf{C}\boldsymbol{l}_i$ we have

$$\operatorname{tr}(\mathbf{C}\mathbf{L}_i\mathbf{C}\mathbf{L}_i) = \operatorname{tr}(\mathbf{C}\boldsymbol{l}_i \otimes \boldsymbol{l}_i\mathbf{C}\boldsymbol{l}_i \otimes \boldsymbol{l}_i) = \boldsymbol{l}_i\mathbf{C}\boldsymbol{l}_i\operatorname{tr}(\mathbf{C}\boldsymbol{l}_i \otimes \boldsymbol{l}_i)$$
$$= \boldsymbol{l}_i\mathbf{C}\boldsymbol{l}_i\operatorname{tr}(\mathbf{C}\mathbf{L}_i) = \operatorname{tr}^2(\mathbf{C}\mathbf{L}_i), \quad i = 1, 2, 3.$$

Next, we obtain

$$\operatorname{tr}\left(\mathbf{C}^2\mathbf{L}_i\right) = \operatorname{tr}(\mathbf{C}\mathbf{I}\mathbf{C}\mathbf{L}_i) = \operatorname{tr}[\mathbf{C}(\mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3)\mathbf{C}\mathbf{L}_i] = \sum_{j=1}^{3}\operatorname{tr}(\mathbf{C}\mathbf{L}_j\mathbf{C}\mathbf{L}_i)$$

and consequently

$$\operatorname{tr}(\mathbf{C}\mathbf{L}_2\mathbf{C}\mathbf{L}_1) + \operatorname{tr}(\mathbf{C}\mathbf{L}_3\mathbf{C}\mathbf{L}_1) = \operatorname{tr}\left(\mathbf{C}^2\mathbf{L}_1\right) - \operatorname{tr}^2(\mathbf{C}\mathbf{L}_1),$$
$$\operatorname{tr}(\mathbf{C}\mathbf{L}_3\mathbf{C}\mathbf{L}_2) + \operatorname{tr}(\mathbf{C}\mathbf{L}_1\mathbf{C}\mathbf{L}_2) = \operatorname{tr}\left(\mathbf{C}^2\mathbf{L}_2\right) - \operatorname{tr}^2(\mathbf{C}\mathbf{L}_2),$$
$$\operatorname{tr}(\mathbf{C}\mathbf{L}_1\mathbf{C}\mathbf{L}_3) + \operatorname{tr}(\mathbf{C}\mathbf{L}_2\mathbf{C}\mathbf{L}_3) = \operatorname{tr}\left(\mathbf{C}^2\mathbf{L}_3\right) - \operatorname{tr}^2(\mathbf{C}\mathbf{L}_3).$$

The latter relations can be given briefly by

$$\operatorname{tr}(\mathbf{C}\mathbf{L}_j\mathbf{C}\mathbf{L}_i) + \operatorname{tr}(\mathbf{C}\mathbf{L}_k\mathbf{C}\mathbf{L}_i)$$
$$= \operatorname{tr}\left(\mathbf{C}^2\mathbf{L}_i\right) - \operatorname{tr}^2(\mathbf{C}\mathbf{L}_i), \quad i \neq j \neq k \neq i;\ i, j, k = 1, 2, 3.$$

Their linear combinations finally yield:

$$\operatorname{tr}(\mathbf{C}\mathbf{L}_i\mathbf{C}\mathbf{L}_j) = \frac{1}{2}\left[\operatorname{tr}\left(\mathbf{C}^2\mathbf{L}_i\right) + \operatorname{tr}\left(\mathbf{C}^2\mathbf{L}_j\right) - \operatorname{tr}\left(\mathbf{C}^2\mathbf{L}_k\right)\right]$$
$$- \frac{1}{2}\left[\operatorname{tr}^2(\mathbf{C}\mathbf{L}_i) + \operatorname{tr}^2(\mathbf{C}\mathbf{L}_j) - \operatorname{tr}^2(\mathbf{C}\mathbf{L}_k)\right],$$

where $i \neq j \neq k \neq i;\ i, j, k = 1, 2, 3$.

6.10 We begin with the directional derivative of $\operatorname{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_i\tilde{\mathbf{E}}\mathbf{L}_j\right)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{tr}\left[\left(\tilde{\mathbf{E}} + t\mathbf{X}\right)\mathbf{L}_i\left(\tilde{\mathbf{E}} + t\mathbf{X}\right)\mathbf{L}_j\right]\Big|_{t=0}$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\tilde{\mathbf{E}}\mathbf{L}_i\tilde{\mathbf{E}}\mathbf{L}_j + t\left(\mathbf{X}\mathbf{L}_i\tilde{\mathbf{E}}\mathbf{L}_j + \tilde{\mathbf{E}}\mathbf{L}_i\mathbf{X}\mathbf{L}_j\right) + t^2\mathbf{X}\mathbf{L}_i\mathbf{X}\mathbf{L}_j\right]\Big|_{t=0} : \mathbf{I}$$
$$= \left(\mathbf{X}\mathbf{L}_i\tilde{\mathbf{E}}\mathbf{L}_j + \tilde{\mathbf{E}}\mathbf{L}_i\mathbf{X}\mathbf{L}_j\right) : \mathbf{I} = \left(\mathbf{X}\mathbf{L}_i\tilde{\mathbf{E}}\mathbf{L}_j + \mathbf{L}_j\tilde{\mathbf{E}}\mathbf{L}_i\mathbf{X}\right) : \mathbf{I}$$
$$= \left(\mathbf{L}_i\tilde{\mathbf{E}}\mathbf{L}_j + \mathbf{L}_j\tilde{\mathbf{E}}\mathbf{L}_i\right) : \mathbf{X}^{\mathrm{T}} = \left(\mathbf{L}_i\tilde{\mathbf{E}}\mathbf{L}_j + \mathbf{L}_j\tilde{\mathbf{E}}\mathbf{L}_i\right)^{\mathrm{T}} : \mathbf{X}.$$

Hence,

$$\operatorname{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_i\tilde{\mathbf{E}}\mathbf{L}_j\right)_{,\tilde{\mathbf{E}}} = \mathbf{L}_i\tilde{\mathbf{E}}\mathbf{L}_j + \mathbf{L}_j\tilde{\mathbf{E}}\mathbf{L}_i.$$

For the second Piola-Kirchhoff stress tensor $\mathbf{S}$ we thus obtain

$$\mathbf{S} = \frac{\partial \psi}{\partial \tilde{\mathbf{E}}} = \frac{1}{2} \sum_{i,j=1}^{3} a_{ij} \mathbf{L}_i \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_j\right) + \frac{1}{2} \sum_{i,j=1}^{3} a_{ij} \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_i\right) \mathbf{L}_j$$

$$+ \sum_{\substack{i,j=1 \\ j \neq i}}^{3} G_{ij} \left(\mathbf{L}_i \tilde{\mathbf{E}} \mathbf{L}_j + \mathbf{L}_j \tilde{\mathbf{E}} \mathbf{L}_i\right)$$

$$= \sum_{i,j=1}^{3} a_{ij} \mathbf{L}_i \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_j\right) + 2 \sum_{\substack{i,j=1 \\ j \neq i}}^{3} G_{ij} \mathbf{L}_i \tilde{\mathbf{E}} \mathbf{L}_j.$$

By virtue of (5.42), (6.120), (6.123) and (6.127) the tangent moduli finally take the form

$$\mathfrak{C} = \frac{\partial \mathbf{S}}{\partial \tilde{\mathbf{E}}} = \sum_{i,j=1}^{3} a_{ij} \mathbf{L}_i \odot \mathbf{L}_j + 2 \sum_{\substack{i,j=1 \\ j \neq i}}^{3} G_{ij} \left(\mathbf{L}_i \otimes \mathbf{L}_j\right)^{\mathrm{s}}.$$

6.11 Setting (6.145) in (6.144) yields

$$\psi\left(\tilde{\mathbf{E}}\right) = \frac{1}{2} a_{11} \mathrm{tr}^2\left(\tilde{\mathbf{E}}\mathbf{L}_1\right) + \frac{1}{2} a_{22} \left[\mathrm{tr}^2\left(\tilde{\mathbf{E}}\mathbf{L}_2\right) + \mathrm{tr}^2\left(\tilde{\mathbf{E}}\mathbf{L}_3\right)\right]$$

$$+ a_{12} \left[\mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_1\right) \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_2\right) + \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_1\right) \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_3\right)\right]$$

$$+ a_{23} \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_2\right) \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_3\right) + \left(a_{22} - a_{23}\right) \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_2 \tilde{\mathbf{E}}\mathbf{L}_3\right)$$

$$+ 2G_{12} \left[\mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_1 \tilde{\mathbf{E}}\mathbf{L}_2\right) + \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_1 \tilde{\mathbf{E}}\mathbf{L}_3\right)\right].$$

Thus, we can write keeping in mind (S.28)

$$\psi\left(\tilde{\mathbf{E}}\right) = \frac{1}{2} a_{11} \mathrm{tr}^2\left(\tilde{\mathbf{E}}\mathbf{L}_1\right) + \frac{1}{2} a_{23} \left[\mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_2\right) + \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_3\right)\right]^2$$

$$+ a_{12} \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_1\right) \left[\mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_2\right) + \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_3\right)\right]$$

$$+ \frac{1}{2} \left(a_{22} - a_{23}\right) \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}_2 + \tilde{\mathbf{E}}\mathbf{L}_3\right)^2 + 2G_{12} \mathrm{tr}\left[\tilde{\mathbf{E}}\mathbf{L}_1 \tilde{\mathbf{E}}\left(\mathbf{L}_2 + \mathbf{L}_3\right)\right].$$

Using the abbreviation $\mathbf{L} = \mathbf{L}_1$ and taking (S.26) into account one thus obtains

$$\psi\left(\tilde{\mathbf{E}}\right) = \frac{1}{2} a_{11} \mathrm{tr}^2\left(\tilde{\mathbf{E}}\mathbf{L}\right) + \frac{1}{2} a_{23} \left[\mathrm{tr}\tilde{\mathbf{E}} - \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}\right)\right]^2$$

$$+ a_{12} \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}\right) \left[\mathrm{tr}\tilde{\mathbf{E}} - \mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}\right)\right] + 2G_{12} \left[\mathrm{tr}\left(\tilde{\mathbf{E}}^2\mathbf{L}\right) - \mathrm{tr}^2\left(\tilde{\mathbf{E}}\mathbf{L}\right)\right]$$

$$+ \frac{1}{2} \left(a_{22} - a_{23}\right) \left[\mathrm{tr}\tilde{\mathbf{E}}^2 - 2\mathrm{tr}\left(\tilde{\mathbf{E}}^2\mathbf{L}\right) + \mathrm{tr}^2\left(\tilde{\mathbf{E}}\mathbf{L}\right)\right].$$

Collecting the terms with the transversely isotropic invariants delivers

$$\psi\left(\tilde{\mathbf{E}}\right) = \frac{1}{2}a_{23}\mathrm{tr}^2\tilde{\mathbf{E}} + \frac{1}{2}\left(a_{22} - a_{23}\right)\mathrm{tr}\tilde{\mathbf{E}}^2 + \left(a_{23} - a_{22} + 2G_{12}\right)\mathrm{tr}\left(\tilde{\mathbf{E}}^2\mathbf{L}\right)$$
$$+\left(\frac{1}{2}a_{11} + \frac{1}{2}a_{22} - a_{12} - 2G_{12}\right)\mathrm{tr}^2\left(\tilde{\mathbf{E}}\mathbf{L}\right) + \left(a_{12} - a_{23}\right)\mathrm{tr}\tilde{\mathbf{E}}\mathrm{tr}\left(\tilde{\mathbf{E}}\mathbf{L}\right).$$

It is seen that the function $\psi(\tilde{\mathbf{E}})$ is transversely isotropic in the sense of the representation (6.29). Finally, considering (6.146) in the latter relation we obtain the isotropic strain energy function of the form (6.99) as

$$\psi\left(\tilde{\mathbf{E}}\right) = \frac{1}{2}\lambda\mathrm{tr}^2\tilde{\mathbf{E}} + G\mathrm{tr}\tilde{\mathbf{E}}^2.$$

6.12 The tensor-valued function (6.103) can be shown to be isotropic. Indeed,

$$\hat{g}\left(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{X}_j\mathbf{Q}^{\mathrm{T}}\right) = \mathbf{Q}''^{\mathrm{T}}g\left(\mathbf{Q}''\mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}\mathbf{Q}''^{\mathrm{T}}\right)\mathbf{Q}'', \quad \forall\mathbf{Q} \in \mathbf{O}\mathrm{rth}^n,$$

where $\mathbf{Q}''$ is defined by (6.39). Further, we can write taking (6.41) into account

$$\mathbf{Q}''^{\mathrm{T}}g\left(\mathbf{Q}''\mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}\mathbf{Q}''^{\mathrm{T}}\right)\mathbf{Q}'' = \mathbf{Q}''^{\mathrm{T}}g\left(\mathbf{Q}^*\mathbf{Q}'\mathbf{A}_i\mathbf{Q}'^{\mathrm{T}}\mathbf{Q}^{*\mathrm{T}}\right)\mathbf{Q}''$$
$$= \mathbf{Q}''^{\mathrm{T}}\mathbf{Q}^*g\left(\mathbf{Q}'\mathbf{A}_i\mathbf{Q}'^{\mathrm{T}}\right)\mathbf{Q}^{*\mathrm{T}}\mathbf{Q}'' = \mathbf{Q}\mathbf{Q}'^{\mathrm{T}}g\left(\mathbf{Q}'\mathbf{A}_i\mathbf{Q}'^{\mathrm{T}}\right)\mathbf{Q}'\mathbf{Q}^{\mathrm{T}}$$
$$= \mathbf{Q}\hat{g}\left(\mathbf{A}_i, \mathbf{X}_j\right)\mathbf{Q}^{\mathrm{T}},$$

which finally yields

$$\hat{g}\left(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{X}_j\mathbf{Q}^{\mathrm{T}}\right) = \mathbf{Q}\hat{g}\left(\mathbf{A}_i, \mathbf{X}_j\right)\mathbf{Q}^{\mathrm{T}}, \quad \forall\mathbf{Q} \in \mathbf{O}\mathrm{rth}^n.$$

Thus, the sufficiency is proved. The necessity is evident.

6.13 Consider the directional derivative of the identity $\mathbf{A}^{-k}\mathbf{A}^k = \mathbf{I}$. Taking into account (2.9) and using (6.116) we can write

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{A} + t\mathbf{X}\right)^{-k}\right|_{t=0}\mathbf{A}^k + \mathbf{A}^{-k}\left(\sum_{i=0}^{k-1}\mathbf{A}^i\mathbf{X}\mathbf{A}^{k-1-i}\right) = \mathbf{0}$$

and consequently

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{A} + t\mathbf{X}\right)^{-k}\right|_{t=0} = -\mathbf{A}^{-k}\left(\sum_{i=0}^{k-1}\mathbf{A}^i\mathbf{X}\mathbf{A}^{k-1-i}\right)\mathbf{A}^{-k}$$
$$= -\sum_{i=0}^{k-1}\mathbf{A}^{i-k}\mathbf{X}\mathbf{A}^{-1-i}.$$

Hence, in view of $(5.17)_1$

$$\mathbf{A}^{-k},_{\mathbf{A}} = -\sum_{j=1}^{k} \mathbf{A}^{j-k-1} \otimes \mathbf{A}^{-j}. \tag{S.29}$$

6.14 (2.4), (2.7), (5.16), $(5.17)_2$, (6.111):

$$\begin{aligned}(f\mathbf{G}),_{\mathbf{A}} : \mathbf{X} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\hat{f}(\mathbf{A}+t\mathbf{X})\, g(\mathbf{A}+t\mathbf{X})\right]\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\hat{f}(\mathbf{A}+t\mathbf{X})\Big|_{t=0} \mathbf{G} + f\frac{\mathrm{d}}{\mathrm{d}t}g(\mathbf{A}+t\mathbf{X})\Big|_{t=0} \\ &= (f,_{\mathbf{A}} : \mathbf{X})\, \mathbf{G} + f\,(\mathbf{G},_{\mathbf{A}} : \mathbf{X}) \\ &= (\mathbf{G} \odot f,_{\mathbf{A}} + f\mathbf{G},_{\mathbf{A}}) : \mathbf{X},\end{aligned}$$

$$\begin{aligned}(\mathbf{G} : \mathbf{H}),_{\mathbf{A}} : \mathbf{X} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[g(\mathbf{A}+t\mathbf{X}) : h(\mathbf{A}+t\mathbf{X})\right]\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}g(\mathbf{A}+t\mathbf{X})\Big|_{t=0} : \mathbf{H} + \mathbf{G} : \frac{\mathrm{d}}{\mathrm{d}t}h(\mathbf{A}+t\mathbf{X})\Big|_{t=0} \\ &= (\mathbf{G},_{\mathbf{A}} : \mathbf{X}) : \mathbf{H} + \mathbf{G} : (\mathbf{H},_{\mathbf{A}} : \mathbf{X}) \\ &= (\mathbf{H} : \mathbf{G},_{\mathbf{A}} + \mathbf{G} : \mathbf{H},_{\mathbf{A}}) : \mathbf{X}, \quad \forall \mathbf{X} \in \mathbf{Lin}^n,\end{aligned}$$

where $f = \hat{f}(\mathbf{A})$, $\mathbf{G} = g(\mathbf{A})$ and $\mathbf{H} = h(\mathbf{A})$.

6.15 In the case $n = 2$ (6.137) takes the form

$$\begin{aligned}\mathbf{0} &= \sum_{k=1}^{2} \mathbf{A}^{2-k} \sum_{i=1}^{k} (-1)^{k-i}\, \mathrm{I}_{\mathbf{A}}^{(k-i)} \left[\mathrm{tr}\left(\mathbf{A}^{i-1}\mathbf{B}\right)\mathbf{I} - \mathbf{B}\mathbf{A}^{i-1}\right] \\ &= \mathbf{A}\left[\mathrm{tr}(\mathbf{B})\,\mathbf{I} - \mathbf{B}\right] - \mathrm{I}_{\mathbf{A}}^{(1)}\left[\mathrm{tr}(\mathbf{B})\,\mathbf{I} - \mathbf{B}\right] + \mathrm{tr}(\mathbf{A}\mathbf{B})\,\mathbf{I} - \mathbf{B}\mathbf{A}\end{aligned}$$

and finally

$$\mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{A} - \mathrm{tr}(\mathbf{B})\,\mathbf{A} - \mathrm{tr}(\mathbf{A})\,\mathbf{B} + \left[\mathrm{tr}(\mathbf{A})\,\mathrm{tr}(\mathbf{B}) - \mathrm{tr}(\mathbf{A}\mathbf{B})\right]\mathbf{I} = \mathbf{0}. \tag{S.30}$$

Exercises of Chapter 7

7.1 By using (4.79) and (4.81) we can write

$$\mathbf{R}(\omega) = \mathbf{P}_1 + \mathrm{e}^{\mathrm{i}\omega}\mathbf{P}_2 + \mathrm{e}^{-\mathrm{i}\omega}\mathbf{P}_3.$$

Applying further (7.2) we get

$$\begin{aligned}\mathbf{R}^a(\omega) &= 1^a\mathbf{P}_1 + \left(\mathrm{e}^{\mathrm{i}\omega}\right)^a \mathbf{P}_2 + \left(\mathrm{e}^{-\mathrm{i}\omega}\right)^a \mathbf{P}_3 \\ &= \mathbf{P}_1 + \mathrm{e}^{\mathrm{i}a\omega}\mathbf{P}_2 + \mathrm{e}^{-\mathrm{i}a\omega}\mathbf{P}_3 = \mathbf{R}(a\omega).\end{aligned}$$

7.2 $(7.5)_1$, (S.14), (S.15):

$$\mathbf{U} = \sum_{i=1}^{s} \lambda_i \mathbf{P}_i = \sum_{i=1}^{s} \sqrt{\Lambda_i} \boldsymbol{a}_i \otimes \boldsymbol{a}_i = \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$$

$$+ \sqrt{\Lambda_1} \left(\frac{1}{\sqrt{1+\Lambda_1}} \boldsymbol{e}_1 + \sqrt{\frac{\Lambda_1}{1+\Lambda_1}} \boldsymbol{e}_2 \right) \otimes \left(\frac{1}{\sqrt{1+\Lambda_1}} \boldsymbol{e}_1 + \sqrt{\frac{\Lambda_1}{1+\Lambda_1}} \boldsymbol{e}_2 \right)$$

$$+ \sqrt{\Lambda_2} \left(\frac{1}{\sqrt{1+\Lambda_2}} \boldsymbol{e}_1 - \sqrt{\frac{\Lambda_2}{1+\Lambda_2}} \boldsymbol{e}_2 \right) \otimes \left(\frac{1}{\sqrt{1+\Lambda_2}} \boldsymbol{e}_1 - \sqrt{\frac{\Lambda_2}{1+\Lambda_2}} \boldsymbol{e}_2 \right)$$

$$= \frac{2}{\sqrt{\gamma^2+4}} \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \frac{\gamma}{\sqrt{\gamma^2+4}} \left(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 \right) + \frac{\gamma^2+2}{\sqrt{\gamma^2+4}} \boldsymbol{e}_2 \otimes \boldsymbol{e}_2$$

$$+ \boldsymbol{e}_3 \otimes \boldsymbol{e}_3.$$

7.3 The proof of the first relation (7.21) directly results from the definition of the analytic tensor function (7.15) and is obvious. In order to prove $(7.21)_2$ we first write

$$f(\mathbf{A}) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} f(\zeta) (\zeta \mathbf{I} - \mathbf{A})^{-1} \mathrm{d}\zeta, \; h(\mathbf{A}) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma'} h(\zeta') (\zeta' \mathbf{I} - \mathbf{A})^{-1} \mathrm{d}\zeta',$$

where the closed curve Γ' of the second integral lies outside Γ which, in turn, includes all eigenvalues of $\mathbf{A}$. Using the identity

$$(\zeta \mathbf{I} - \mathbf{A})^{-1} (\zeta' \mathbf{I} - \mathbf{A})^{-1} = (\zeta' - \zeta)^{-1} \left[(\zeta \mathbf{I} - \mathbf{A})^{-1} - (\zeta' \mathbf{I} - \mathbf{A})^{-1} \right]$$

valid both on Γ and Γ' we thus obtain

$$f(\mathbf{A}) h(\mathbf{A}) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma'} \oint_{\Gamma} f(\zeta) h(\zeta') (\zeta \mathbf{I} - \mathbf{A})^{-1} (\zeta' \mathbf{I} - \mathbf{A})^{-1} \mathrm{d}\zeta \mathrm{d}\zeta'$$

$$= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} f(\zeta) \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma'} \frac{h(\zeta')}{\zeta' - \zeta} \mathrm{d}\zeta' (\zeta \mathbf{I} - \mathbf{A})^{-1} \mathrm{d}\zeta$$

$$+ \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma'} h(\zeta') \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \frac{f(\zeta)}{\zeta - \zeta'} \mathrm{d}\zeta (\zeta' \mathbf{I} - \mathbf{A})^{-1} \mathrm{d}\zeta'.$$

Since the function $f(\zeta)(\zeta - \zeta')^{-1}$ is analytic in ζ inside Γ the Cauchy theorem (see, e.g. [5]) implies that

$$\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \frac{f(\zeta)}{\zeta - \zeta'} \mathrm{d}\zeta = 0.$$

Noticing further that

$$\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma'} \frac{h(\zeta')}{\zeta' - \zeta} \mathrm{d}\zeta' = h(\zeta)$$

we obtain

$$f(\mathbf{A})\,h(\mathbf{A}) = \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma} f(\zeta)\,\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma'} \frac{h(\zeta')}{\zeta'-\zeta}\mathrm{d}\zeta'\,(\zeta\mathbf{I}-\mathbf{A})^{-1}\,\mathrm{d}\zeta$$
$$= \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma} f(\zeta)\,h(\zeta)\,(\zeta\mathbf{I}-\mathbf{A})^{-1}\,\mathrm{d}\zeta$$
$$= \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma} g(\zeta)\,(\zeta\mathbf{I}-\mathbf{A})^{-1}\,\mathrm{d}\zeta = g(\mathbf{A}).$$

Finally, we focus on the third relation (7.21). It implies that the functions h and f are analytic on domains containing all the eigenvalues λ_i of $\mathbf{A}$ and $h(\lambda_i)$ $(i = 1, 2, \ldots, n)$ of $\mathbf{B} = h(\mathbf{A})$, respectively. Hence (cf. [24]),

$$f(h(\mathbf{A})) = f(\mathbf{B}) = \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma} f(\zeta)\,(\zeta\mathbf{I}-\mathbf{B})^{-1}\,\mathrm{d}\zeta, \tag{S.31}$$

where Γ encloses all the eigenvalues of $\mathbf{B}$. Further, we write

$$(\zeta\mathbf{I}-\mathbf{B})^{-1} = (\zeta\mathbf{I}-h(\mathbf{A}))^{-1} = \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma'} (\zeta-h(\zeta'))^{-1}\,(\zeta'\mathbf{I}-\mathbf{A})^{-1}\,\mathrm{d}\zeta', \tag{S.32}$$

where Γ' includes all the eigenvalues λ_i of $\mathbf{A}$ so that the image of Γ' under h lies within Γ. Thus, inserting (S.32) into (S.31) delivers

$$f(h(\mathbf{A})) = \frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma}\oint_{\Gamma'} f(\zeta)\,(\zeta-h(\zeta'))^{-1}\,(\zeta'\mathbf{I}-\mathbf{A})^{-1}\,\mathrm{d}\zeta'\mathrm{d}\zeta$$
$$= \frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma'}\oint_{\Gamma} f(\zeta)\,(\zeta-h(\zeta'))^{-1}\,\mathrm{d}\zeta\,(\zeta'\mathbf{I}-\mathbf{A})^{-1}\,\mathrm{d}\zeta'$$
$$= \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma'} f(h(\zeta'))\,(\zeta'\mathbf{I}-\mathbf{A})^{-1}\,\mathrm{d}\zeta'$$
$$= \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma'} g(\zeta')\,(\zeta'\mathbf{I}-\mathbf{A})^{-1}\,\mathrm{d}\zeta' = g(\mathbf{A}).$$

7.4 Inserting into the right hand side of (7.54) the spectral decomposition in terms of eigenprojections (7.1) and taking (4.46) into account we can write similarly to (7.17)

$$\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_i}(\zeta\mathbf{I}-\mathbf{A})^{-1}\,\mathrm{d}\zeta = \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_i}\left(\zeta\mathbf{I}-\sum_{j=1}^{s}\lambda_j\mathbf{P}_j\right)^{-1}\mathrm{d}\zeta$$
$$= \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_i}\left[\sum_{j=1}^{s}(\zeta-\lambda_j)\,\mathbf{P}_j\right]^{-1}\mathrm{d}\zeta = \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_i}\sum_{j=1}^{s}(\zeta-\lambda_j)^{-1}\,\mathbf{P}_j\mathrm{d}\zeta$$
$$= \sum_{j=1}^{s}\left[\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_i}(\zeta-\lambda_j)^{-1}\,\mathrm{d}\zeta\right]\mathbf{P}_j.$$

In the case $i \neq j$ the closed curve Γ_i does not include any pole so that

$$\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_i} (\zeta - \lambda_j)^{-1} \mathrm{d}\zeta = \delta_{ij}, \quad i, j = 1, 2, \ldots s.$$

This immediately leads to (7.54).

7.5 By means of (7.43) and (7.83) and using the result for the eigenvalues of $\mathbf{A}$ by (S.17), $\lambda_i = 6$, $\lambda = -3$ we write

$$\mathbf{P}_1 = \sum_{p=0}^{2} \rho_{1p} \mathbf{A}^p = -\frac{\lambda}{(\lambda_i - \lambda)} \mathbf{I} + \frac{1}{(\lambda_i - \lambda)} \mathbf{A} = \frac{1}{3}\mathbf{I} + \frac{1}{9}\mathbf{A},$$

$$\mathbf{P}_2 = \mathbf{I} - \mathbf{P}_1 = \frac{2}{3}\mathbf{I} - \frac{1}{9}\mathbf{A}.$$

Taking symmetry of $\mathbf{A}$ into account we further obtain by virtue of (7.56) and (7.84)

$$\begin{aligned}
\mathbf{P}_{1,\mathbf{A}} &= \sum_{p,q=0}^{2} \upsilon_{1pq} \left(\mathbf{A}^p \otimes \mathbf{A}^q\right)^{\mathrm{s}} \\
&= -\frac{2\lambda\lambda_i}{(\lambda_i - \lambda)^3} \mathfrak{J}^{\mathrm{s}} + \frac{\lambda_i + \lambda}{(\lambda_i - \lambda)^3} (\mathbf{I} \otimes \mathbf{A} + \mathbf{A} \otimes \mathbf{I})^{\mathrm{s}} - \frac{2}{(\lambda_i - \lambda)^3} (\mathbf{A} \otimes \mathbf{A})^{\mathrm{s}} \\
&= \frac{4}{81} \mathfrak{J}^{\mathrm{s}} + \frac{1}{243} (\mathbf{I} \otimes \mathbf{A} + \mathbf{A} \otimes \mathbf{I})^{\mathrm{s}} - \frac{2}{729} (\mathbf{A} \otimes \mathbf{A})^{\mathrm{s}}.
\end{aligned}$$

The eigenprojection $\mathbf{P}_2$ corresponds to the double eigenvalue $\lambda = -3$ and for this reason is not differentiable.

7.6 Since $\mathbf{A}$ is a symmetric tensor and it is diagonalizable. Thus, taking double coalescence of eigenvalues (S.17) into account we can apply the representations (7.77) and (7.78). Setting there $\lambda_a = 6$, $\lambda = -3$ delivers

$$\exp(\mathbf{A}) = \frac{\mathrm{e}^6 + 2\mathrm{e}^{-3}}{3} \mathbf{I} + \frac{\mathrm{e}^6 - \mathrm{e}^{-3}}{9} \mathbf{A},$$

$$\begin{aligned}
\exp(\mathbf{A})_{,\mathbf{A}} = {} & \frac{13\mathrm{e}^6 + 32\mathrm{e}^{-3}}{81} \mathfrak{J}^{\mathrm{s}} + \frac{10\mathrm{e}^6 - 19\mathrm{e}^{-3}}{243} (\mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A})^{\mathrm{s}} \\
& + \frac{7\mathrm{e}^6 + 11\mathrm{e}^{-3}}{729} (\mathbf{A} \otimes \mathbf{A})^{\mathrm{s}}.
\end{aligned}$$

Inserting

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & 2 \\ 2 & 1 & 4 \\ 2 & 4 & 1 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

into the expression for $\exp(\mathbf{A})$ we obtain

$$\exp(\mathbf{A}) = \frac{1}{9}\begin{bmatrix} e^6+8e^{-3} & 2e^6-2e^{-3} & 2e^6-2e^{-3} \\ 2e^6-2e^{-3} & 4e^6+5e^{-3} & 4e^6-4e^{-3} \\ 2e^6-2e^{-3} & 4e^6-4e^{-3} & 4e^6+5e^{-3} \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j,$$

which coincides with the result obtained in Exercise 4.9.

7.7 The computation of the coefficients series (7.89), (7.91) and (7.96), (7.97) with the precision parameter $\varepsilon = 1 \cdot 10^{-6}$ has required 23 iteration steps and has been carried out by using MAPLE-program. The results of the computation are summarized in Tables S.1 and S.2. On use of (7.90) and (7.92) we thus obtain

$$\exp(\mathbf{A}) = 44.96925\mathbf{I} + 29.89652\mathbf{A} + 4.974456\mathbf{A}^2,$$

$$\begin{aligned}\exp(\mathbf{A})_{,\mathbf{A}} = {} & 16.20582\mathfrak{J}^{\mathrm{s}} + 6.829754\,(\mathbf{I}\otimes\mathbf{A} + \mathbf{A}\otimes\mathbf{I})^{\mathrm{s}} + 1.967368\,(\mathbf{A}\otimes\mathbf{A})^{\mathrm{s}} \\ & +1.039719\left(\mathbf{I}\otimes\mathbf{A}^2 + \mathbf{A}^2\otimes\mathbf{I}\right)^{\mathrm{s}} + 0.266328\left(\mathbf{A}\otimes\mathbf{A}^2 + \mathbf{A}^2\otimes\mathbf{A}\right)^{\mathrm{s}} \\ & +0.034357\left(\mathbf{A}^2\otimes\mathbf{A}^2\right)^{\mathrm{s}}.\end{aligned}$$

Taking into account double coalescence of eigenvalues of $\mathbf{A}$ we can further write

$$\mathbf{A}^2 = (\lambda_a + \lambda)\,\mathbf{A} - \lambda_a\lambda\mathbf{I} = 3\mathbf{A} + 18\mathbf{I}.$$

Inserting this relation into the above representations for $\exp(\mathbf{A})$ and $\exp(\mathbf{A})_{,\mathbf{A}}$ finally yields

$$\exp(\mathbf{A}) = 134.50946\mathbf{I} + 44.81989\mathbf{A},$$

$$\exp(\mathbf{A})_{,\mathbf{A}} = 64.76737\mathfrak{J}^{\mathrm{s}} + 16.59809\,(\mathbf{I}\otimes\mathbf{A} + \mathbf{A}\otimes\mathbf{I})^{\mathrm{s}} + 3.87638\,(\mathbf{A}\otimes\mathbf{A})^{\mathrm{s}}.$$

Note that the relative error of this result in comparison to the closed-form solution used in Exercise 7.6 lies within 0.044%.

Exercises of Chapter 8

8.1 By (8.2) we first calculate the right and left Cauchy-Green tensors as

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}^j, \quad \mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}^j,$$

with the following eigenvalues $\Lambda_1 = 1$, $\Lambda_2 = 4$, $\Lambda_3 = 9$. Thus, $\lambda_1 = \sqrt{\Lambda_1} = 1$, $\lambda_2 = \sqrt{\Lambda_2} = 2$, $\lambda_3 = \sqrt{\Lambda_3} = 3$. By means of (8.11-8.12) we further obtain $\varphi_0 = \frac{3}{5}$, $\varphi_1 = \frac{5}{12}$, $\varphi_2 = -\frac{1}{60}$ and

Table S.1. Recurrent calculation of the coefficients $\omega_p^{(r)}$

r	$a_r\omega_0^{(r)}$	$a_r\omega_1^{(r)}$	$a_r\omega_2^{(r)}$
0	1	0	0
1	0	1	0
2	0	0	0.5
3	9.0	4.5	0
4	0	2.25	1.125
5	12.15	6.075	0.45
6	4.05	4.05	1.0125
...	...	...	...
23 $\left(\cdot 10^{-6}\right)$	3.394287	2.262832	0.377134
φ_p	44.96925	29.89652	4.974456

Table S.2. Recurrent calculation of the coefficients $\xi_{pq}^{(r)}$

r	$a_r\xi_{00}^{(r)}$	$a_r\xi_{01}^{(r)}$	$a_r\xi_{02}^{(r)}$	$a_r\xi_{11}^{(r)}$	$a_r\xi_{12}^{(r)}$	$a_r\xi_{22}^{(r)}$
1	1	0	0	0	0	0
2	0	0.5	0	0	0	0
3	0	0	0.166666	0.166666	0	0
4	4.5	1.125	0	0	0.041666	0
5	0	0.9	0.225	0.45	0	0.008333
6	4.05	1.0125	0.15	0.15	0.075	0
...	...	...	...			
23 $\left(\cdot 10^{-6}\right)$	2.284387	1.229329	0.197840	0.623937	0.099319	0.015781
η_{pq}	16.20582	6.829754	1.039719	1.967368	0.266328	0.034357

$$\mathbf{U} = \frac{3}{5}\mathbf{I} + \frac{5}{12}\mathbf{C} - \frac{1}{60}\mathbf{C}^2 = \frac{1}{5}\begin{bmatrix} 11 & -2 & 0 \\ -2 & 14 & 0 \\ 0 & 0 & 5 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}^j,$$

$$\mathbf{v} = \frac{3}{5}\mathbf{I} + \frac{5}{12}\mathbf{b} - \frac{1}{60}\mathbf{b}^2 = \frac{1}{5}\begin{bmatrix} 11 & 2 & 0 \\ 2 & 14 & 0 \\ 0 & 0 & 5 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}^j.$$

Eqs. (8.16-8.17) further yield $\varsigma_0 = \frac{37}{30}$, $\varsigma_1 = -\frac{1}{4}$, $\varsigma_2 = \frac{1}{60}$ and

$$\mathbf{R} = \mathbf{F}\left(\frac{37}{30}\mathbf{I} - \frac{1}{4}\mathbf{C} + \frac{1}{60}\mathbf{C}^2\right) = \frac{1}{5}\begin{bmatrix} 3 & 4 & 0 \\ -4 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}^j.$$

8.2 (4.44), (5.33), (5.47), (5.55), $(5.83)_1$:

$$\begin{aligned}\boldsymbol{\mathcal{P}}_{ij} : \boldsymbol{\mathcal{P}}_{kl} &= (\mathbf{P}_i \otimes \mathbf{P}_j + \mathbf{P}_j \otimes \mathbf{P}_i)^{\mathrm{s}} : (\mathbf{P}_k \otimes \mathbf{P}_l + \mathbf{P}_l \otimes \mathbf{P}_k)^{\mathrm{s}} \\ &= \left[(\mathbf{P}_i \otimes \mathbf{P}_j + \mathbf{P}_j \otimes \mathbf{P}_i)^{\mathrm{s}} : (\mathbf{P}_k \otimes \mathbf{P}_l + \mathbf{P}_l \otimes \mathbf{P}_k)\right]^{\mathrm{s}} \\ &= \frac{1}{2}\left\{\left[\mathbf{P}_i \otimes \mathbf{P}_j + \mathbf{P}_j \otimes \mathbf{P}_i + (\mathbf{P}_i \otimes \mathbf{P}_j)^{\mathrm{t}} + (\mathbf{P}_j \otimes \mathbf{P}_i)^{\mathrm{t}}\right]\right. \\ &\qquad \left. : (\mathbf{P}_k \otimes \mathbf{P}_l + \mathbf{P}_l \otimes \mathbf{P}_k)\right\}^{\mathrm{s}} \\ &= (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,(\mathbf{P}_i \otimes \mathbf{P}_j + \mathbf{P}_j \otimes \mathbf{P}_i)^{\mathrm{s}}, \quad i \neq j,\ k \neq l.\end{aligned}$$

In the case $i = j$ or $k = l$ the previous result should be divided by 2, whereas for $i = j$ and $k = l$ by 4, which immediately leads to (8.62).

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