

## 1.7

## Graphing with Calculators and Computers

A graphing calculator or a computer with graphing software enables us to graph very complicated functions with high precision. Many of these functions could not otherwise be easily graphed. However, care must be taken when using such devices for graphing purposes and we address those issues in this section. In Chapter 4 we will see how calculus helps us to be certain we are viewing accurately all the important features of a function's graph.

## Graphing Windows

When using a graphing calculator or computer as a graphing tool, a portion of the graph is displayed in a rectangular **display** or **viewing window**. Often the default window gives an incomplete or misleading picture of the graph. We use the term *square window* when the units or scales on both axes are the same. This term does not mean that the display window itself is square in shape (usually it is rectangular), but means instead that the  $x$ -unit is the same as the  $y$ -unit.

When a graph is displayed in the default window, the  $x$ -unit may differ from the  $y$ -unit of scaling in order to fit the graph in the display. The viewing window in the display is set by specifying the minimum and maximum values of the independent and dependent variables. That is, an interval  $a \leq x \leq b$  is specified as well as a range  $c \leq y \leq d$ . The machine selects a certain number of equally spaced values of  $x$  between  $a$  and  $b$ . Starting with a first value for  $x$ , if it lies within the domain of the function  $f$  being graphed, and if  $f(x)$  lies inside the range  $[c, d]$ , then the point  $(x, f(x))$  is plotted. If  $x$  lies outside the domain of  $f$ , or  $f(x)$  lies outside the specified range  $[c, d]$ , the machine just moves on to the next  $x$ -value since it cannot plot  $(x, f(x))$  in that case. The machine plots a large number of points  $(x, f(x))$  in this way and approximates the curve representing the graph by drawing a short line segment between each plotted point and its next neighboring point, as we might do by hand. Usually, adjacent points are so close together that the graphical representation has the appearance of a smooth curve. Things can go wrong with this procedure and we illustrate the most common problems through the following examples.

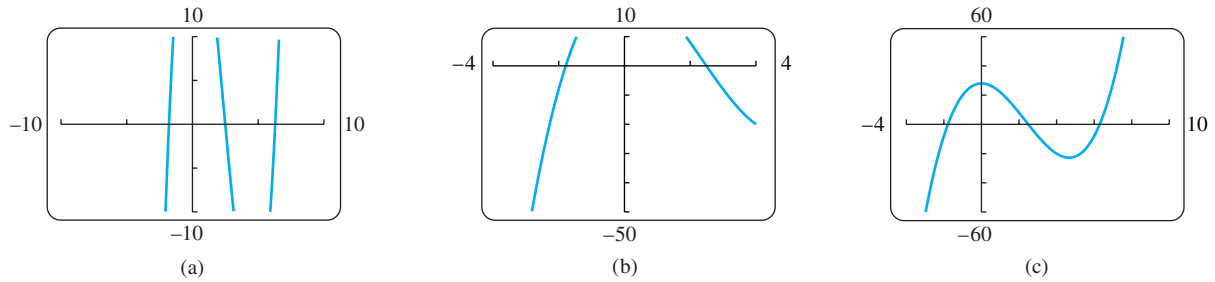
**EXAMPLE 1** Choosing a Viewing Window

Graph the function  $f(x) = x^3 - 7x^2 + 28$  in each of the following display or viewing windows:

- (a)  $[-10, 10]$  by  $[-10, 10]$       (b)  $[-4, 4]$  by  $[-50, 10]$       (c)  $[-4, 10]$  by  $[-60, 60]$

**Solution**

- (a) We select  $a = -10$ ,  $b = 10$ ,  $c = -10$ , and  $d = 10$  to specify the interval of  $x$ -values and the range of  $y$ -values for the window. The resulting graph is shown in Figure 1.78a. It appears that the window is cutting off the bottom part of the graph and that the interval of  $x$ -values is too large. Let's try the next window.
- (b) Now we see more features of the graph (Figure 1.78b), but the top is missing and we need to view more to the right of  $x = 4$  as well. The next window should help.
- (c) Figure 1.78c shows the graph in this new viewing window. Observe that we get a more complete picture of the graph in this window and it is a reasonable graph of a third-degree polynomial. Choosing a good viewing window is a trial-and-error process which may require some troubleshooting as well.



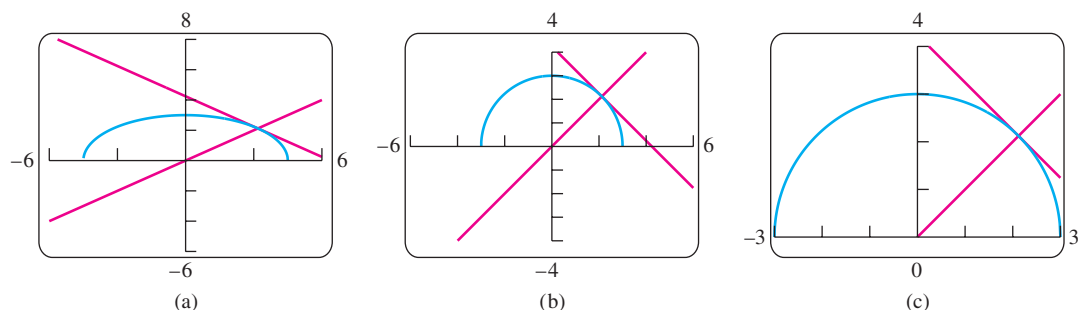
**FIGURE 1.78** The graph of  $f(x) = x^3 - 7x^2 + 28$  in different viewing windows (Example 1). ■

### EXAMPLE 2 Square Windows

When a graph is displayed, the  $x$ -unit may differ from the  $y$ -unit, as in the graphs shown in Figures 1.78b and 1.78c. The result is distortion in the picture, which may be misleading. The display window can be made square by compressing or stretching the units on one axis to match the scale on the other, giving the true graph. Many systems have built-in functions to make the window “square.” If yours does not, you will have to do some calculations and set the window size manually to get a square window, or bring to your viewing some foreknowledge of the true picture.

Figure 1.79a shows the graphs of the perpendicular lines  $y = x$  and  $y = -x + 3\sqrt{2}$ , together with the semicircle  $y = \sqrt{9 - x^2}$ , in a nonsquare  $[-6, 6]$  by  $[-6, 8]$  display window. Notice the distortion. The lines do not appear to be perpendicular, and the semicircle appears to be elliptical in shape.

Figure 1.79b shows the graphs of the same functions in a square window in which the  $x$ -units are scaled to be the same as the  $y$ -units. Notice that the  $[-6, 6]$  by  $[-4, 4]$  viewing window has the same  $x$ -axis in both figures, but the scaling on the  $x$ -axis has been compressed in Figure 1.79b to make the window square. Figure 1.79c gives an enlarged view with a square  $[-3, 3]$  by  $[0, 4]$  window.



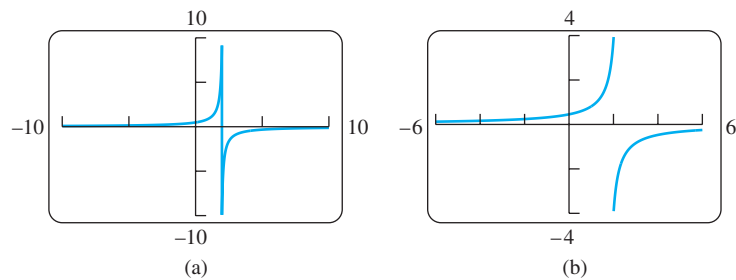
**FIGURE 1.79** Graphs of the perpendicular lines  $y = x$  and  $y = -x + 3\sqrt{2}$ , and the semicircle  $y = \sqrt{9 - x^2}$ , in (a) a nonsquare window, and (b) and (c) square windows (Example 2). ■

If the denominator of a rational function is zero at some  $x$ -value within the viewing window, a calculator or graphing computer software may produce a steep near-vertical line segment from the top to the bottom of the window. Here is an example.

**EXAMPLE 3** Graph of a Rational Function

Graph the function  $y = \frac{1}{2-x}$ .

**Solution** Figure 1.80a shows the graph in the  $[-10, 10]$  by  $[-10, 10]$  default square window with our computer graphing software. Notice the near-vertical line segment at  $x = 2$ . It is not truly a part of the graph and  $x = 2$  does not belong to the domain of the function. By trial and error we can eliminate the line by changing the viewing window to the smaller  $[-6, 6]$  by  $[-4, 4]$  view, revealing a better graph (Figure 1.80b).



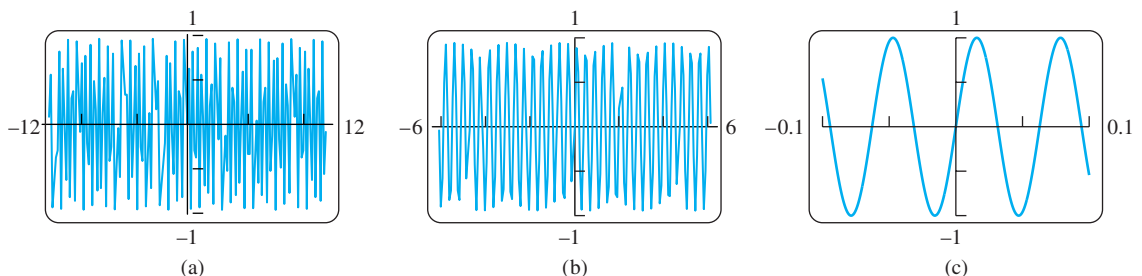
**FIGURE 1.80** Graphs of the function  $y = \frac{1}{2-x}$  (Example 3).

Sometimes the graph of a trigonometric function oscillates very rapidly. When a calculator or computer software plots the points of the graph and connects them, many of the maximum and minimum points are actually missed. The resulting graph is then very misleading.

**EXAMPLE 4** Graph of a Rapidly Oscillating Function

Graph the function  $f(x) = \sin 100x$ .

**Solution** Figure 1.81a shows the graph of  $f$  in the viewing window  $[-12, 12]$  by  $[-1, 1]$ . We see that the graph looks very strange because the sine curve should oscillate periodically between  $-1$  and  $1$ . This behavior is not exhibited in Figure 1.81a. We might experiment with a smaller viewing window, say  $[-6, 6]$  by  $[-1, 1]$ , but the graph is not better (Figure 1.81b). The difficulty is that the period of the trigonometric function  $y = \sin 100x$  is very small ( $2\pi/100 \approx 0.063$ ). If we choose the much smaller viewing window  $[-0.1, 0.1]$  by  $[-1, 1]$  we get the graph shown in Figure 1.81c. This graph reveals the expected oscillations of a sine curve.

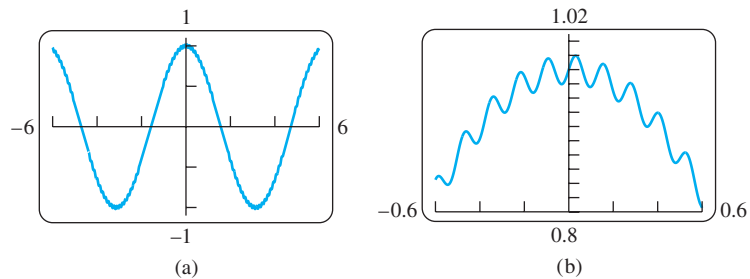


**FIGURE 1.81** Graphs of the function  $y = \sin 100x$  in three viewing windows. Because the period is  $2\pi/100 \approx 0.063$ , the smaller window in (c) best displays the true aspects of this rapidly oscillating function (Example 4).

**EXAMPLE 5** Another Rapidly Oscillating Function

Graph the function  $y = \cos x + \frac{1}{50} \sin 50x$ .

**Solution** In the viewing window  $[-6, 6]$  by  $[-1, 1]$  the graph appears much like the cosine function with some small sharp wiggles on it (Figure 1.82a). We get a better look when we significantly reduce the window to  $[-0.6, 0.6]$  by  $[0.8, 1.02]$ , obtaining the graph in Figure 1.82b. We now see the small but rapid oscillations of the second term,  $\frac{1}{50} \sin 50x$ , added to the comparatively larger values of the cosine curve.



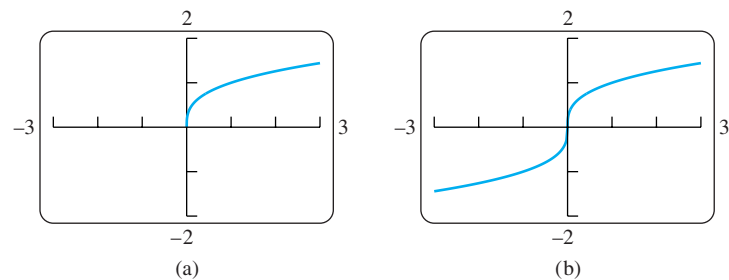
**FIGURE 1.82** In (b) we see a close-up view of the function

$y = \cos x + \frac{1}{50} \sin 50x$  graphed in (a). The term  $\cos x$  clearly dominates the second term,  $\frac{1}{50} \sin 50x$ , which produces the rapid oscillations along the cosine curve (Example 5). ■

**EXAMPLE 6** Graphing an Odd Fractional Power

Graph the function  $y = x^{1/3}$ .

**Solution** Many graphing devices display the graph shown in Figure 1.83a. When we compare it with the graph of  $y = x^{1/3} = \sqrt[3]{x}$  in Figure 1.38, we see that the left branch for  $x < 0$  is missing. The reason the graphs differ is that many calculators and computer soft-



**FIGURE 1.83** The graph of  $y = x^{1/3}$  is missing the left branch in (a). In (b) we graph the function  $f(x) = \frac{x}{|x|} \cdot |x|^{1/3}$  obtaining both branches. (See Example 6.)

ware programs calculate  $x^{1/3}$  as  $e^{(1/3)\ln x}$ . (The exponential and logarithmic functions are studied in Chapter 7.) Since the logarithmic function is not defined for negative values of  $x$ , the computing device can only produce the right branch where  $x > 0$ .

To obtain the full picture showing both branches, we can graph the function

$$f(x) = \frac{x}{|x|} \cdot |x|^{1/3}.$$

This function equals  $x^{1/3}$  except at  $x = 0$  (where  $f$  is undefined, although  $0^{1/3} = 0$ ). The graph of  $f$  is shown in Figure 1.83b. ■

### Empirical Modeling: Capturing the Trend of Collected Data

In Example 3 of Section 1.4, we verified the reasonableness of Kepler's hypothesis that the period of a planet's orbit is proportional to its mean distance from the sun raised to the  $3/2$  power. If we cannot hypothesize a relationship between a dependent variable and an independent variable, we might collect data points and try to find a curve that "fits" the data and captures the trend of the scatterplot. The process of finding a curve to fit data is called **regression analysis** and the curve is called a **regression curve**. A computer or graphing calculator finds the regression curve by finding the particular curve which minimizes the sum of the squares of the vertical distances between the data points and the curve. This method of **least squares** is discussed in the Section 14.7 exercises.

There are many useful types of regression curves, such as straight lines, power, polynomial, exponential, logarithmic, and sinusoidal curves. Many computers or graphing calculators have a regression analysis feature to fit a variety of regression curve types. The next example illustrates using a graphing calculator's linear regression feature to fit data from Table 1.5 with a linear equation.

#### EXAMPLE 7 Fitting a Regression Line

Starting with the data in Table 1.5, build a model for the price of a postage stamp as a function of time. After verifying that the model is "reasonable," use it to predict the price in 2010.

**Solution** We are building a model for the price of a stamp since 1968. There were two increases in 1981, one of three cents followed by another of two cents. To make 1981 comparable with the other listed years, we lump them together as a single five-cent increase, giving the data in Table 1.6. Figure 1.84a gives the scatterplot for Table 1.6.

**TABLE 1.5** Price of a U.S. postage stamp

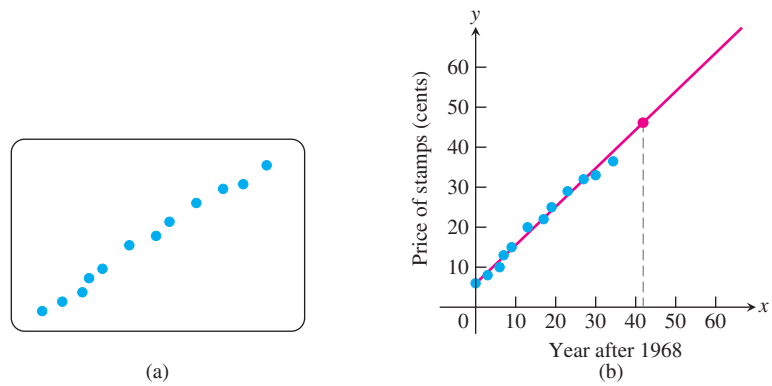
Year $x$	Cost $y$
1968	0.06
1971	0.08
1974	0.10
1975	0.13
1977	0.15
1981	0.18
1981	0.20
1985	0.22
1987	0.25
1991	0.29
1995	0.32
1998	0.33
2002	0.37

**TABLE 1.6** Price of a U.S. postage stamp since 1968

$x$	0	3	6	7	9	13	17	19	23	27	30	34
$y$	6	8	10	13	15	20	22	25	29	32	33	37

Since the scatterplot is fairly linear, we investigate a linear model. Upon entering the data into a graphing calculator (or computer software) and selecting the linear regression option, we find the regression line to be

$$y = 0.94x + 6.10.$$



**FIGURE 1.84** (a) Scatterplot of  $(x, y)$  data in Table 1.6. (b) Using the regression line to estimate the price of a stamp in 2010. (Example 7).

Figure 1.84b shows the line and scatterplot together. The fit is remarkably good, so the model seems reasonable.

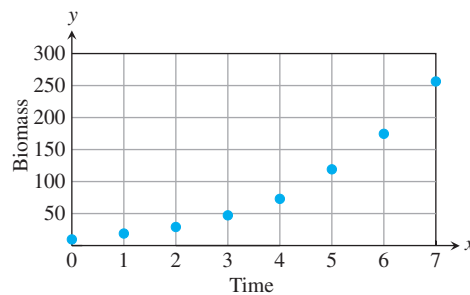
Evaluating the regression line, we conclude that in 2010 ( $x = 42$ ), the price of a stamp will be

$$y = 0.94(42) + 6.10 \approx 46 \text{ cents.}$$

The prediction is shown as the red point on the regression line in Figure 1.84b. ■

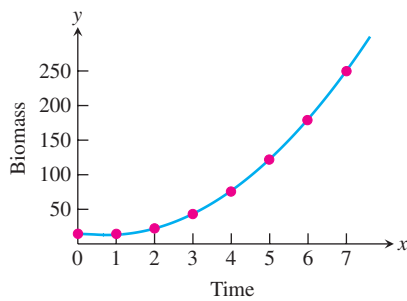
### EXAMPLE 8 Finding a Curve to Predict Population Levels

We may want to predict the future size of a population, such as the number of trout or catfish living in a fish farm. Figure 1.85 shows a scatterplot of the data collected by R. Pearl for a collection of yeast cells (measured as **biomass**) growing over time (measured in hours) in a nutrient.



**FIGURE 1.85** Biomass of a yeast culture versus elapsed time (Example 8).  
(Data from R. Pearl, “The Growth of Population,” *Quart. Rev. Biol.*, Vol. 2 (1927), pp. 532–548.)

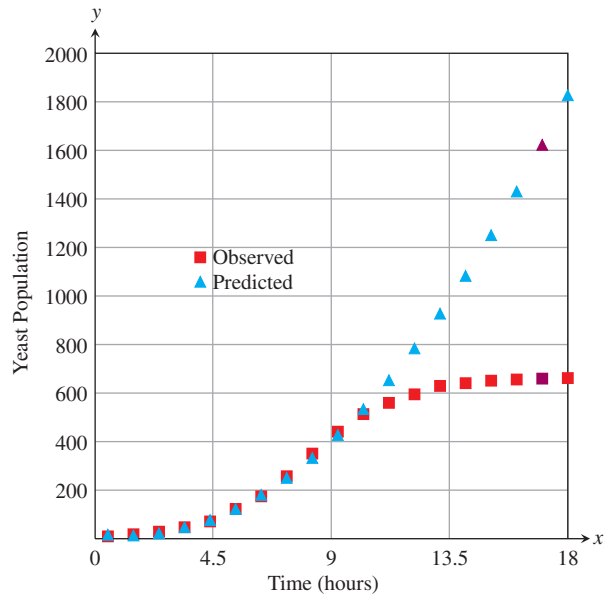
The plot of points appears to be reasonably smooth with an upward curving trend. We might attempt to capture this trend by fitting a polynomial (for example, a quadratic  $y = ax^2 + bx + c$ ), a power curve ( $y = ax^b$ ), or an exponential curve ( $y = ae^{bx}$ ). Figure 1.86 shows the result of using a calculator to fit a quadratic model.



**FIGURE 1.86** Fitting a quadratic to Pearl's data gives the equation  $y = 6.10x^2 - 9.28x + 16.43$  and the prediction  $y(17) = 1622.65$  (Example 8).

The quadratic model  $y = 6.10x^2 - 9.28x + 16.43$  appears to fit the collected data reasonably well (Figure 1.86). Using this model, we predict the population after 17 hours as  $y(17) = 1622.65$ . Let us examine more of Pearl's data to see if our quadratic model continues to be a good one.

In Figure 1.87, we display all of Pearl's data. Now you see that the prediction of  $y(17) = 1622.65$  grossly overestimates the observed population of 659.6. Why did the quadratic model fail to predict a more accurate value?



**FIGURE 1.87** The rest of Pearl's data (Example 8).

The problem lies in the danger of predicting beyond the range of data used to build the empirical model. (The range of data creating our model was  $0 \leq x \leq 7$ .) Such *extrapolation* is especially dangerous when the model selected is not supported by some underlying rationale suggesting the form of the model. In our yeast example, why would we expect a quadratic function as underlying population growth? Why not an exponential function? In the face of this, how then do we predict future values? Often, calculus can help, and in Chapter 9 we use it to model population growth. ■

### Regression Analysis

Regression analysis has four steps:

1. Plot the data (scatterplot).
2. Find a regression equation. For a line, it has the form  $y = mx + b$ , and for a quadratic, the form  $y = ax^2 + bx + c$ .
3. Superimpose the graph of the regression equation on the scatterplot to see the fit.
4. If the fit is satisfactory, use the regression equation to predict  $y$ -values for values of  $x$  not in the table.