10.7 Areas and Lengths in Polar Coordinates

This section shows how to calculate areas of plane regions, lengths of curves, and areas of surfaces of revolution in polar coordinates.

Area in the Plane

The region *OTS* in Figure 10.48 is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n nonoverlapping fan-shaped circular sectors based on a partition P of angle TOS. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\Delta\theta_k/2\pi$ times the area of a circle of radius r_k, or

$$A_k = \frac{1}{2} r_k^2 \, \Delta\theta_k = \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k .$$

The area of region *OTS* is approximately

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k .$$

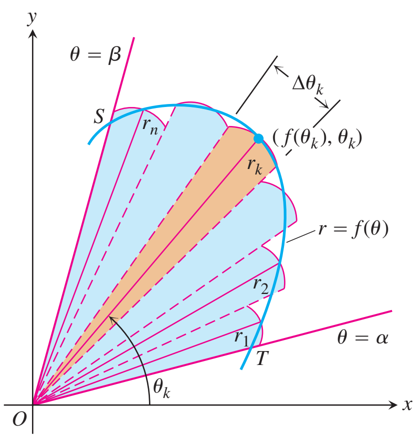

FIGURE 10.48 To derive a formula for the area of region *OTS*, we approximate the region with fan-shaped circular sectors.

If f is continuous, we expect the approximations to improve as the norm of the partition $\|P\| \to 0$, and we are led to the following formula for the region's area:

$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k$$
$$= \int_{\alpha}^{\beta} \frac{1}{2}\left(f(\theta)\right)^2 d\theta.$$

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta), \alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta.$$

This is the integral of the **area differential** (Figure 10.49)

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2}(f(\theta))^2 \, d\theta.$$

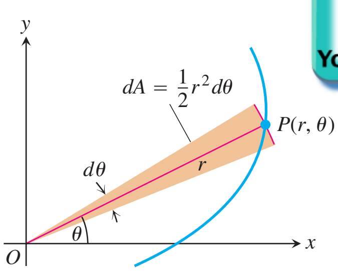

FIGURE 10.49 The area differential dA for the curve $n = f(\theta)$.

EXAMPLE 1 Finding Area

Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

Solution We graph the cardioid (Figure 10.50) and determine that the radius OP sweeps out the region exactly once as θ runs from 0 to 2π. The area is therefore

$$\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2}\cdot 4(1 + \cos\theta)^2 \, d\theta$$
$$= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta)\, d\theta$$
$$= \int_0^{2\pi} \left(2 + 4\cos\theta + 2\,\frac{1 + \cos 2\theta}{2}\right) d\theta$$
$$= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta)\, d\theta$$
$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi.$$

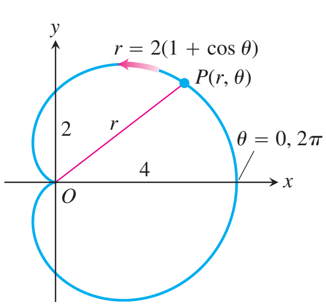

FIGURE 10.50 The cardioid in Example 1.

EXAMPLE 2 Finding Area

Find the area inside the smaller loop of the limaçon

$$r = 2\cos\theta + 1.$$

Solution After sketching the curve (Figure 10.51), we see that the smaller loop is traced out by the point (r, θ) as θ increases from $\theta = 2\pi/3$ to $\theta = 4\pi/3$. Since the curve is symmetric about the x-axis (the equation is unaltered when we replace θ by $-\theta$), we may calculate the area of the shaded half of the inner loop by integrating from $\theta = 2\pi/3$ to $\theta = \pi$. The area we seek will be twice the resulting integral:

$$A = 2\int_{2\pi/3}^{\pi} \frac{1}{2} r^2 \, d\theta = \int_{2\pi/3}^{\pi} r^2 \, d\theta.$$

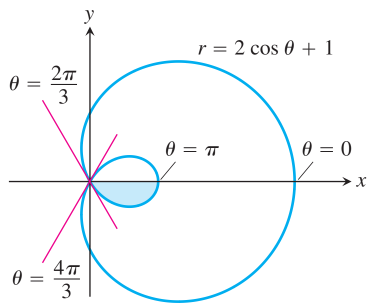

FIGURE 10.51 The limaçon in Example 2. Limaçon (pronounced LEE-ma-sahn) is an old French word for *snail*.

Since

$$\begin{aligned} r^2 &= (2\cos\theta + 1)^2 = 4\cos^2\theta + 4\cos\theta + 1 \\ &= 4\cdot\frac{1+\cos 2\theta}{2} + 4\cos\theta + 1 \\ &= 2 + 2\cos 2\theta + 4\cos\theta + 1 \\ &= 3 + 2\cos 2\theta + 4\cos\theta, \end{aligned}$$

we have

$$\begin{aligned} A &= \int_{2\pi/3}^{\pi} (3 + 2\cos 2\theta + 4\cos\theta)\, d\theta \\ &= \Big[3\theta + \sin 2\theta + 4\sin\theta\Big]_{2\pi/3}^{\pi} \\ &= (3\pi) - \left(2\pi - \frac{\sqrt{3}}{2} + 4\cdot\frac{\sqrt{3}}{2}\right) \\ &= \pi - \frac{3\sqrt{3}}{2}. \end{aligned}$$ ■

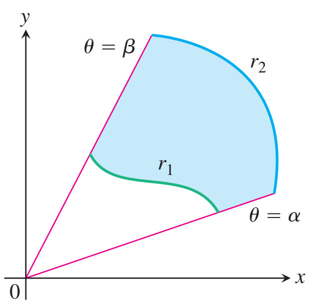

FIGURE 10.52 The area of the shaded region is calculated by subtracting the area of the region between r_1 and the origin from the area of the region between r_2 and the origin.

To find the area of a region like the one in Figure 10.52, which lies between two polar curves $r_1 = r_1(\theta)$ and $r_2 = r_2(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we subtract the integral of $(1/2)r_1^2\, d\theta$ from the integral of $(1/2)r_2^2\, d\theta$. This leads to the following formula.

Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\quad \alpha \le \theta \le \beta$

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2\, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2\, d\theta = \int_\alpha^\beta \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad (1)$$

EXAMPLE 3 Finding Area Between Polar Curves

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

Solution We sketch the region to determine its boundaries and find the limits of integration (Figure 10.53). The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$, and θ runs from $-\pi/2$ to $\pi/2$. The area, from Equation (1), is

$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \\ &= 2\int_0^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad \text{Symmetry} \\ &= \int_0^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\, d\theta \\ &= \int_0^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta = \int_0^{\pi/2}\left(2\cos\theta - \frac{1+\cos 2\theta}{2}\right) d\theta \\ &= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 2 - \frac{\pi}{4}. \end{aligned}$$ ■

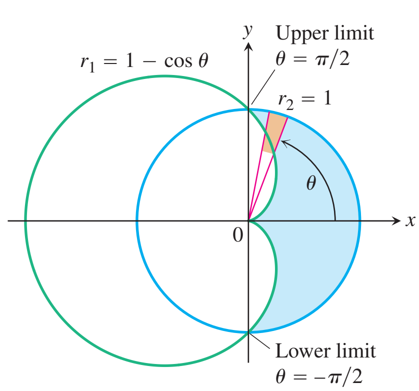

FIGURE 10.53 The region and limits of integration in Example 3.

Length of a Polar Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, by parametrizing the curve as

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta, \qquad \alpha \le \theta \le \beta. \tag{2}$$

The parametric length formula, Equation (1) from Section 6.3, then gives the length as

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

This equation becomes

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

when Equations (2) are substituted for x and y (Exercise 33).

Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{3}$$

You Try It

EXAMPLE 4 Finding the Length of a Cardioid

Find the length of the cardioid $r = 1 - \cos\theta$.

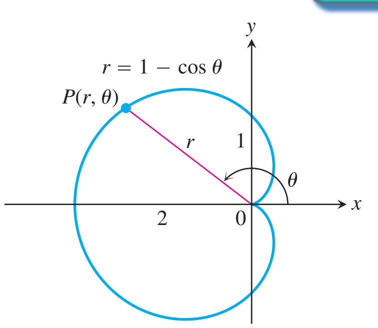

FIGURE 10.54 Calculating the length of a cardioid (Example 4).

Solution We sketch the cardioid to determine the limits of integration (Figure 10.54). The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from 0 to 2π, so these are the values we take for α and β.

With

$$r = 1 - \cos\theta, \qquad \frac{dr}{d\theta} = \sin\theta,$$

we have

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 - \cos\theta)^2 + (\sin\theta)^2$$
$$= 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 2 - 2\cos\theta$$

and

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$
$$= \int_{0}^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

$$= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$$

$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \qquad \sin\frac{\theta}{2} \ge 0 \quad \text{for} \quad 0 \le \theta \le 2\pi$$

$$= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = 8.$$

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Area of a Surface of Revolution

To derive polar coordinate formulas for the area of a surface of revolution, we parametrize the curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, with Equations (2) and apply the surface area equations for parametrized curves in Section 6.5.

Area of a Surface of Revolution of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the areas of the surfaces generated by revolving the curve about the x- and y-axes are given by the following formulas:

1. Revolution about the x-axis $(y \ge 0)$:

$$S = \int_\alpha^\beta 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{4}$$

2. Revolution about the y-axis $(x \ge 0)$:

$$S = \int_\alpha^\beta 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{5}$$

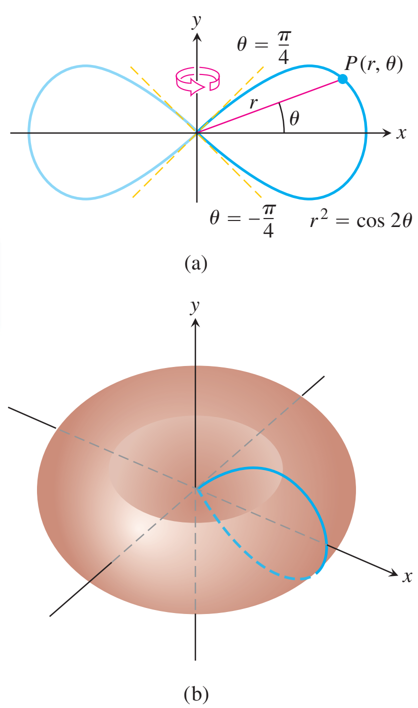

FIGURE 10.55 The right-hand half of a lemniscate (a) is revolved about the y-axis to generate a surface (b), whose area is calculated in Example 5.

EXAMPLE 5 Finding Surface Area

Find the area of the surface generated by revolving the right-hand loop of the lemniscate $r^2 = \cos 2\theta$ about the y-axis.

Solution We sketch the loop to determine the limits of integration (Figure 10.55a). The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from $-\pi/4$ to $\pi/4$, so these are the values we take for α and β.

We evaluate the area integrand in Equation (5) in stages. First,

$$2\pi r\cos\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2\pi\cos\theta\sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2}. \tag{6}$$

Next, $r^2 = \cos 2\theta$, so

$$2r\frac{dr}{d\theta} = -2\sin 2\theta$$

$$r\frac{dr}{d\theta} = -\sin 2\theta$$

$$\left(r\frac{dr}{d\theta}\right)^2 = \sin^2 2\theta.$$

Finally, $r^4 = (r^2)^2 = \cos^2 2\theta$, so the square root on the right-hand side of Equation (6) simplifies to

$$\sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2} = \sqrt{\cos^2 2\theta + \sin^2 2\theta} = 1.$$

All together, we have

$$\begin{aligned} S &= \int_\alpha^\beta 2\pi r\cos\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \text{Equation (5)} \\ &= \int_{-\pi/4}^{\pi/4} 2\pi\cos\theta \cdot (1)\, d\theta \\ &= 2\pi\Big[\sin\theta\Big]_{-\pi/4}^{\pi/4} \\ &= 2\pi\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] = 2\pi\sqrt{2}. \end{aligned}$$

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