

EXERCISES 10.7

Areas Inside Polar Curves

Find the areas of the regions in Exercises 1–6.

1. Inside the oval limaçon $r = 4 + 2 \cos \theta$
2. Inside the cardioid $r = a(1 + \cos \theta)$, $a > 0$
3. Inside one leaf of the four-leaved rose $r = \cos 2\theta$
4. Inside the lemniscate $r^2 = 2a^2 \cos 2\theta$, $a > 0$
5. Inside one loop of the lemniscate $r^2 = 4 \sin 2\theta$
6. Inside the six-leaved rose $r^2 = 2 \sin 3\theta$

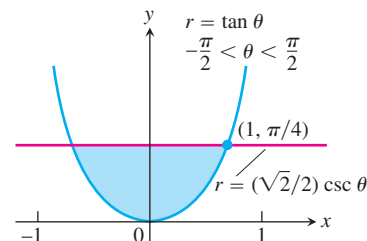
Areas Shared by Polar Regions

Find the areas of the regions in Exercises 7–16.

7. Shared by the circles $r = 2 \cos \theta$ and $r = 2 \sin \theta$
8. Shared by the circles $r = 1$ and $r = 2 \sin \theta$
9. Shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos \theta)$
10. Shared by the cardioids $r = 2(1 + \cos \theta)$ and $r = 2(1 - \cos \theta)$
11. Inside the lemniscate $r^2 = 6 \cos 2\theta$ and outside the circle $r = \sqrt{3}$
12. Inside the circle $r = 3a \cos \theta$ and outside the cardioid $r = a(1 + \cos \theta)$, $a > 0$
13. Inside the circle $r = -2 \cos \theta$ and outside the circle $r = 1$
14. a. Inside the outer loop of the limaçon $r = 2 \cos \theta + 1$
(See Figure 10.51.)

- b. Inside the outer loop and outside the inner loop of the limaçon $r = 2 \cos \theta + 1$

15. Inside the circle $r = 6$ above the line $r = 3 \csc \theta$
16. Inside the lemniscate $r^2 = 6 \cos 2\theta$ to the right of the line $r = (3/2) \sec \theta$
17. a. Find the area of the shaded region in the accompanying figure.



- b. It looks as if the graph of $r = \tan \theta$, $-\pi/2 < \theta < \pi/2$, could be asymptotic to the lines $x = 1$ and $x = -1$. Is it? Give reasons for your answer.
18. The area of the region that lies inside the cardioid curve $r = \cos \theta + 1$ and outside the circle $r = \cos \theta$ is not

$$\frac{1}{2} \int_0^{2\pi} [(\cos \theta + 1)^2 - \cos^2 \theta] d\theta = \pi.$$

Why not? What *is* the area? Give reasons for your answers.

Lengths of Polar Curves

Find the lengths of the curves in Exercises 19–27.

19. The spiral $r = \theta^2$, $0 \leq \theta \leq \sqrt{5}$
20. The spiral $r = e^\theta/\sqrt{2}$, $0 \leq \theta \leq \pi$
21. The cardioid $r = 1 + \cos \theta$
22. The curve $r = a \sin^2(\theta/2)$, $0 \leq \theta \leq \pi$, $a > 0$
23. The parabolic segment $r = 6/(1 + \cos \theta)$, $0 \leq \theta \leq \pi/2$
24. The parabolic segment $r = 2/(1 - \cos \theta)$, $\pi/2 \leq \theta \leq \pi$
25. The curve $r = \cos^3(\theta/3)$, $0 \leq \theta \leq \pi/4$
26. The curve $r = \sqrt{1 + \sin 2\theta}$, $0 \leq \theta \leq \pi\sqrt{2}$
27. The curve $r = \sqrt{1 + \cos 2\theta}$, $0 \leq \theta \leq \pi\sqrt{2}$
28. **Circumferences of circles** As usual, when faced with a new formula, it is a good idea to try it on familiar objects to be sure it gives results consistent with past experience. Use the length formula in Equation (3) to calculate the circumferences of the following circles ($a > 0$):
 - a. $r = a$
 - b. $r = a \cos \theta$
 - c. $r = a \sin \theta$

Surface Area

Find the areas of the surfaces generated by revolving the curves in Exercises 29–32 about the indicated axes.

29. $r = \sqrt{\cos 2\theta}$, $0 \leq \theta \leq \pi/4$, y -axis
30. $r = \sqrt{2}e^{\theta/2}$, $0 \leq \theta \leq \pi/2$, x -axis
31. $r^2 = \cos 2\theta$, x -axis
32. $r = 2a \cos \theta$, $a > 0$, y -axis

Theory and Examples

33. **The length of the curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$** Assuming that the necessary derivatives are continuous, show how the substitutions

$$x = f(\theta) \cos \theta, \quad y = f(\theta) \sin \theta$$

(Equations 2 in the text) transform

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

into

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

34. **Average value** If f is continuous, the average value of the polar coordinate r over the curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, with respect to θ is given by the formula

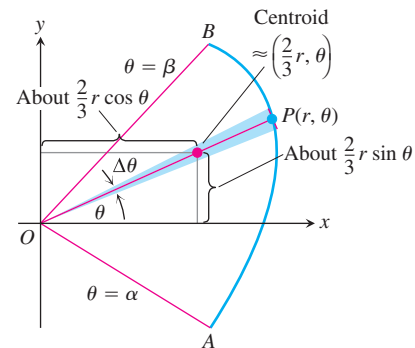
$$r_{\text{av}} = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(\theta) d\theta.$$

Use this formula to find the average value of r with respect to θ over the following curves ($a > 0$).

- a. The cardioid $r = a(1 - \cos \theta)$
 - b. The circle $r = a$
 - c. The circle $r = a \cos \theta$, $-\pi/2 \leq \theta \leq \pi/2$
35. **$r = f(\theta)$ vs. $r = 2f(\theta)$** Can anything be said about the relative lengths of the curves $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, and $r = 2f(\theta)$, $\alpha \leq \theta \leq \beta$? Give reasons for your answer.
 36. **$r = f(\theta)$ vs. $r = 2f(\theta)$** The curves $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, and $r = 2f(\theta)$, $\alpha \leq \theta \leq \beta$, are revolved about the x -axis to generate surfaces. Can anything be said about the relative areas of these surfaces? Give reasons for your answer.

Centroids of Fan-Shaped Regions

Since the centroid of a triangle is located on each median, two-thirds of the way from the vertex to the opposite base, the lever arm for the moment about the x -axis of the thin triangular region in the accompanying figure is about $(2/3)r \sin \theta$. Similarly, the lever arm for the moment of the triangular region about the y -axis is about $(2/3)r \cos \theta$. These approximations improve as $\Delta\theta \rightarrow 0$ and lead to the following formulas for the coordinates of the centroid of region AOB :



$$\bar{x} = \frac{\int \frac{2}{3} r \cos \theta \cdot \frac{1}{2} r^2 d\theta}{\int \frac{1}{2} r^2 d\theta} = \frac{\frac{2}{3} \int r^3 \cos \theta d\theta}{\int r^2 d\theta},$$

$$\bar{y} = \frac{\int \frac{2}{3} r \sin \theta \cdot \frac{1}{2} r^2 d\theta}{\int \frac{1}{2} r^2 d\theta} = \frac{\frac{2}{3} \int r^3 \sin \theta d\theta}{\int r^2 d\theta},$$

with limits $\theta = \alpha$ to $\theta = \beta$ on all integrals.

37. Find the centroid of the region enclosed by the cardioid $r = a(1 + \cos \theta)$.
38. Find the centroid of the semicircular region $0 \leq r \leq a$, $0 \leq \theta \leq \pi$.