10.8 Conic Sections in Polar Coordinates

Polar coordinates are important in astronomy and astronautical engineering because the ellipses, parabolas, and hyperbolas along which satellites, moons, planets, and comets approximately move can all be described with a single relatively simple coordinate equation. We develop that equation here.

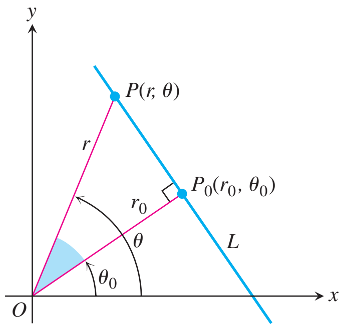

FIGURE 10.56 We can obtain a polar equation for line L by reading the relation $r_0 = r\cos(\theta - \theta_0)$ from the right triangle OP_0P.

Lines

Suppose the perpendicular from the origin to line L meets L at the point $P_0(r_0, \theta_0)$, with $r_0 \geq 0$ (Figure 10.56). Then, if $P(r, \theta)$ is any other point on L, the points P, P_0, and O are the vertices of a right triangle, from which we can read the relation

$$r_0 = r\cos(\theta - \theta_0).$$

The Standard Polar Equation for Lines

If the point $P_0(r_0, \theta_0)$ is the foot of the perpendicular from the origin to the line L, and $r_0 \geq 0$, then an equation for L is

$$r\cos(\theta - \theta_0) = r_0. \qquad (1)$$

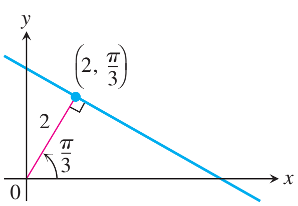

FIGURE 10.57 The standard polar equation of this line converts to the Cartesian equation $x + \sqrt{3}\,y = 4$ (Example 1).

EXAMPLE 1 Converting a Line's Polar Equation to Cartesian Form

Use the identity $\cos(A - B) = \cos A\cos B + \sin A\sin B$ to find a Cartesian equation for the line in Figure 10.57.

Solution

$$r\cos\left(\theta - \frac{\pi}{3}\right) = 2$$

$$r\left(\cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}\right) = 2$$

$$\frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta = 2$$

$$\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 2$$

$$x + \sqrt{3}\,y = 4$$

■

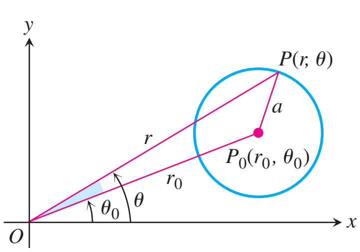

FIGURE 10.58 We can get a polar equation for this circle by applying the Law of Cosines to triangle OP_0P.

Circles

To find a polar equation for the circle of radius a centered at $P_0(r_0, \theta_0)$, we let $P(r, \theta)$ be a point on the circle and apply the Law of Cosines to triangle OP_0P (Figure 10.58). This gives

$$a^2 = r_0^2 + r^2 - 2r_0 r\cos(\theta - \theta_0).$$

If the circle passes through the origin, then $r_0 = a$ and this equation simplifies to

$$a^2 = a^2 + r^2 - 2ar\cos(\theta - \theta_0)$$
$$r^2 = 2ar\cos(\theta - \theta_0)$$
$$r = 2a\cos(\theta - \theta_0).$$

If the circle's center lies on the positive x-axis, $\theta_0 = 0$ and we get the further simplification

$$r = 2a\cos\theta$$

(see Figure 10.59a).

If the center lies on the positive y-axis, $\theta = \pi/2$, $\cos(\theta - \pi/2) = \sin\theta$, and the equation $r = 2a\cos(\theta - \theta_0)$ becomes

$$r = 2a\sin\theta$$

(see Figure 10.59b).

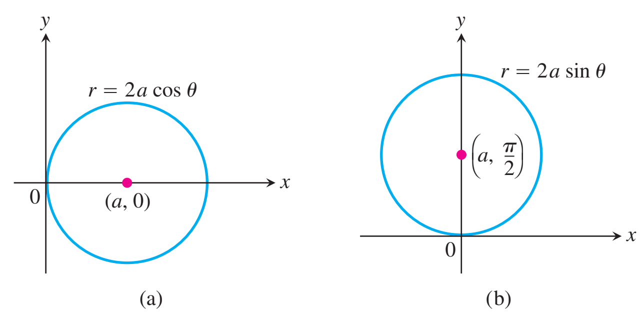

FIGURE 10.59 Polar equation of a circle of radius a through the origin with center on (a) the positive x-axis, and (b) the positive y-axis.

Equations for circles through the origin centered on the negative x- and y-axes can be obtained by replacing r with $-r$ in the above equations (Figure 10.60).

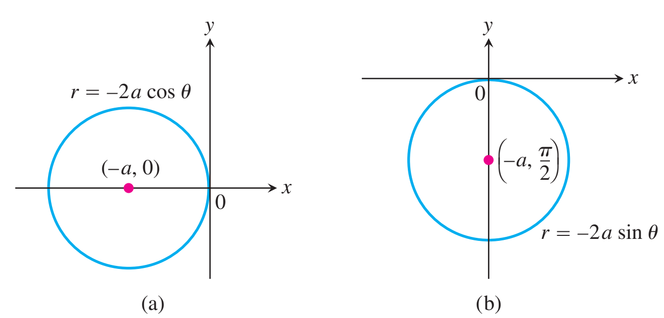

FIGURE 10.60 Polar equation of a circle of radius a through the origin with center on (a) the negative x-axis, and (b) the negative y-axis.

EXAMPLE 2 Circles Through the Origin

Radius	Center (polar coordinates)	Polar equation
3	$(3, 0)$	$r = 6 \cos \theta$
2	$(2, \pi/2)$	$r = 4 \sin \theta$
1/2	$(-1/2, 0)$	$r = -\cos \theta$
1	$(-1, \pi/2)$	$r = -2 \sin \theta$

■

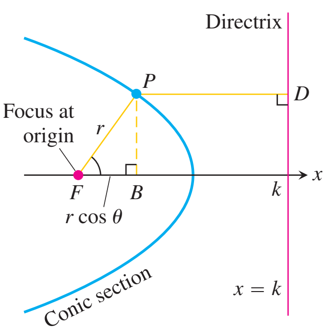

FIGURE 10.61 If a conic section is put in the position with its focus placed at the origin and a directrix perpendicular to the initial ray and right of the origin, we can find its polar equation from the conic's focus–directrix equation.

Ellipses, Parabolas, and Hyperbolas

To find polar equations for ellipses, parabolas, and hyperbolas, we place one focus at the origin and the corresponding directrix to the right of the origin along the vertical line $x = k$ (Figure 10.61). This makes

$$PF = r$$

and

$$PD = k - FB = k - r \cos \theta .$$

The conic's focus–directrix equation $PF = e \cdot PD$ then becomes

$$r = e(k - r \cos \theta),$$

which can be solved for r to obtain

Polar Equation for a Conic with Eccentricity e

$$r = \frac{ke}{1 + e \cos \theta}, \tag{2}$$

where $x = k > 0$ is the vertical directrix.

This equation represents an ellipse if $0 < e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$. That is, ellipses, parabolas, and hyperbolas all have the same basic equation expressed in terms of eccentricity and location of the directrix.

EXAMPLE 3 Polar Equations of Some Conics

$e = \frac{1}{2}$: ellipse $\quad r = \frac{k}{2 + \cos \theta}$

$e = 1$: parabola $\quad r = \frac{k}{1 + \cos \theta}$

$e = 2$: hyperbola $\quad r = \frac{2k}{1 + 2 \cos \theta}$

■

You may see variations of Equation (2) from time to time, depending on the location of the directrix. If the directrix is the line $x = -k$ to the left of the origin (the origin is still a focus), we replace Equation (2) by

$$r = \frac{ke}{1 - e\cos\theta}.$$

The denominator now has a $(-)$ instead of a $(+)$. If the directrix is either of the lines $y = k$ or $y = -k$, the equations have sines in them instead of cosines, as shown in Figure 10.62.

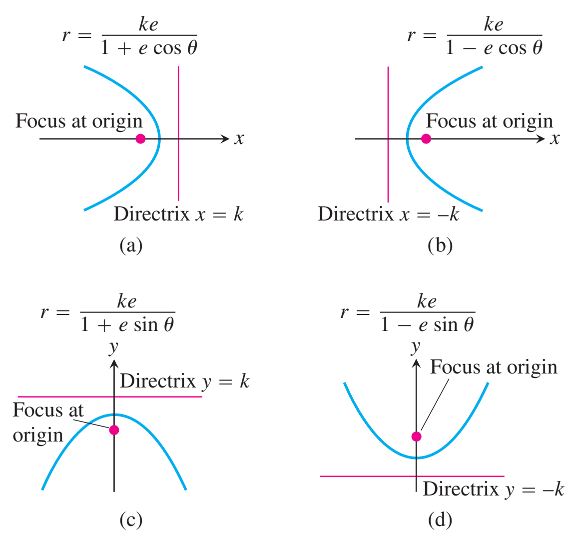

FIGURE 10.62 Equations for conic sections with eccentricity $e > 0$, but different locations of the directrix. The graphs here show a parabola, so $e = 1$.

EXAMPLE 4 Polar Equation of a Hyperbola

You Try It

Find an equation for the hyperbola with eccentricity $3/2$ and directrix $x = 2$.

Solution We use Equation (2) with $k = 2$ and $e = 3/2$:

$$r = \frac{2(3/2)}{1 + (3/2)\cos\theta} \quad \text{or} \quad r = \frac{6}{2 + 3\cos\theta}.$$

■

EXAMPLE 5 Finding a Directrix

Find the directrix of the parabola

$$r = \frac{25}{10 + 10\cos\theta}.$$

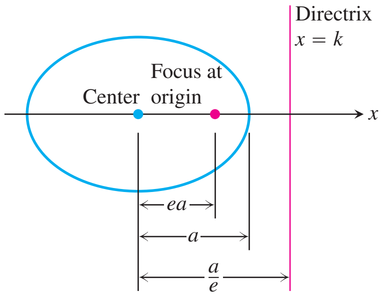

FIGURE 10.63 In an ellipse with semimajor axis a, the focus–directrix distance is $k = (a/e) - ea$, so $ke = a(1 - e^2)$.

Solution We divide the numerator and denominator by 10 to put the equation in standard form:

$$r = \frac{5/2}{1 + \cos\theta}.$$

This is the equation

$$r = \frac{ke}{1 + e\cos\theta}$$

with $k = 5/2$ and $e = 1$. The equation of the directrix is $x = 5/2$. ■

From the ellipse diagram in Figure 10.63, we see that k is related to the eccentricity e and the semimajor axis a by the equation

$$k = \frac{a}{e} - ea.$$

From this, we find that $ke = a(1 - e^2)$. Replacing ke in Equation (2) by $a(1 - e^2)$ gives the standard polar equation for an ellipse.

Polar Equation for the Ellipse with Eccentricity *e* and Semimajor Axis *a*

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \tag{3}$$

Notice that when $e = 0$, Equation (3) becomes $r = a$, which represents a circle.

Equation (3) is the starting point for calculating planetary orbits.

EXAMPLE 6 The Planet Pluto's Orbit

Find a polar equation for an ellipse with semimajor axis 39.44 AU (astronomical units) and eccentricity 0.25. This is the approximate size of Pluto's orbit around the sun.

Solution We use Equation (3) with $a = 39.44$ and $e = 0.25$ to find

$$r = \frac{39.44(1 - (0.25)^2)}{1 + 0.25\cos\theta} = \frac{147.9}{4 + \cos\theta}.$$

At its point of closest approach (perihelion) where $\theta = 0$, Pluto is

$$r = \frac{147.9}{4 + 1} = 29.58 \text{ AU}$$

from the sun. At its most distant point (aphelion) where $\theta = \pi$, Pluto is

$$r = \frac{147.9}{4 - 1} = 49.3 \text{ AU}$$

from the sun (Figure 10.64). ■

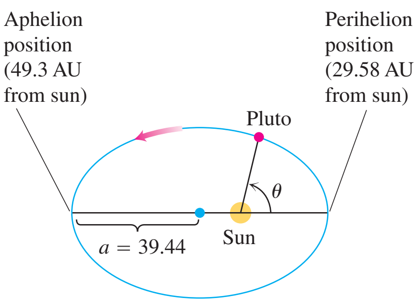

FIGURE 10.64 The orbit of Pluto (Example 6).