Chapter 10 Additional and Advanced Exercises

Finding Conic Sections

1. Find an equation for the parabola with focus (4, 0) and directrix $x = 3$. Sketch the parabola together with its vertex, focus, and directrix.
2. Find the vertex, focus, and directrix of the parabola
$$x^2 - 6x - 12y + 9 = 0.$$
3. Find an equation for the curve traced by the point $P(x, y)$ if the distance from P to the vertex of the parabola $x^2 = 4y$ is twice the distance from P to the focus. Identify the curve.
4. A line segment of length $a + b$ runs from the x-axis to the y-axis. The point P on the segment lies a units from one end and b units from the other end. Show that P traces an ellipse as the ends of the segment slide along the axes.
5. The vertices of an ellipse of eccentricity 0.5 lie at the points $(0, \pm 2)$. Where do the foci lie?
6. Find an equation for the ellipse of eccentricity $2/3$ that has the line $x = 2$ as a directrix and the point (4, 0) as the corresponding focus.
7. One focus of a hyperbola lies at the point $(0, -7)$ and the corresponding directrix is the line $y = -1$. Find an equation for the hyperbola if its eccentricity is **(a)** 2, **(b)** 5.
8. Find an equation for the hyperbola with foci $(0, -2)$ and $(0, 2)$ that passes through the point (12, 7).
9. **a.** Show that the line
$$b^2xx_1 + a^2yy_1 - a^2b^2 = 0$$
is tangent to the ellipse $b^2x^2 + a^2y^2 - a^2b^2 = 0$ at the point (x_1, y_1) on the ellipse.

   b. Show that the line
$$b^2xx_1 - a^2yy_1 - a^2b^2 = 0$$
is tangent to the hyperbola $b^2x^2 - a^2y^2 - a^2b^2 = 0$ at the point (x_1, y_1) on the hyperbola.
10. Show that the tangent to the conic section
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$
at a point (x_1, y_1) on it has an equation that may be written in the form
$$Axx_1 + B\left(\frac{x_1y + xy_1}{2}\right) + Cyy_1 + D\left(\frac{x + x_1}{2}\right) + E\left(\frac{y + y_1}{2}\right) + F = 0.$$

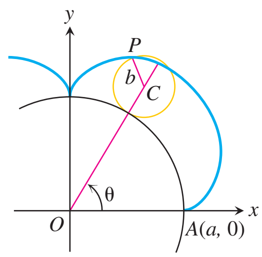
Equations and Inequalities

What points in the xy-plane satisfy the equations and inequalities in Exercises 11–18? Draw a figure for each exercise.

11. $(x^2 - y^2 - 1)(x^2 + y^2 - 25)(x^2 + 4y^2 - 4) = 0$
12. $(x + y)(x^2 + y^2 - 1) = 0$
13. $(x^2/9) + (y^2/16) \le 1$
14. $(x^2/9) - (y^2/16) \le 1$
15. $(9x^2 + 4y^2 - 36)(4x^2 + 9y^2 - 16) \le 0$
16. $(9x^2 + 4y^2 - 36)(4x^2 + 9y^2 - 16) > 0$
17. $x^4 - (y^2 - 9)^2 = 0$
18. $x^2 + xy + y^2 < 3$

Parametric Equations and Cycloids

19. Epicycloids When a circle rolls externally along the circumference of a second, fixed circle, any point P on the circumference of the rolling circle describes an *epicycloid*, as shown here. Let the fixed circle have its center at the origin O and have radius a.



Let the radius of the rolling circle be b and let the initial position of the tracing point P be $A(a, 0)$. Find parametric equations for the epicycloid, using as the parameter the angle θ from the positive x-axis to the line through the circles' centers.

20. a. Find the centroid of the region enclosed by the x-axis and the cycloid arch

$$x = a(t - \sin t), \quad y = a(1 - \cos t); \quad 0 \le t \le 2\pi.$$

b. Find the first moments about the coordinate axes of the curve

$$x = (2/3)t^{3/2}, \quad y = 2\sqrt{t}; \quad 0 \le t \le \sqrt{3}.$$

Polar Coordinates

21. a. Find an equation in polar coordinates for the curve

$$x = e^{2t}\cos t, \quad y = e^{2t}\sin t; \quad -\infty < t < \infty.$$

b. Find the length of the curve from $t = 0$ to $t = 2\pi$.

22. Find the length of the curve $r = 2\sin^3(\theta/3), 0 \le \theta \le 3\pi$, in the polar coordinate plane.

23. Find the area of the surface generated by revolving the first-quadrant portion of the cardioid $r = 1 + \cos\theta$ about the x-axis. (*Hint*: Use the identities $1 + \cos\theta = 2\cos^2(\theta/2)$ and $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$ to simplify the integral.)

24. Sketch the regions enclosed by the curves $r = 2a\cos^2(\theta/2)$ and $r = 2a\sin^2(\theta/2), a > 0$, in the polar coordinate plane and find the area of the portion of the plane they have in common.

Exercises 25–28 give the eccentricities of conic sections with one focus at the origin of the polar coordinate plane, along with the directrix for that focus. Find a polar equation for each conic section.

25. $e = 2, \quad r\cos\theta = 2$

26. $e = 1, \quad r\cos\theta = -4$

27. $e = 1/2, \quad r\sin\theta = 2$

28. $e = 1/3, \quad r\sin\theta = -6$

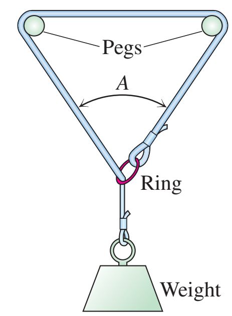
Theory and Examples

29. A rope with a ring in one end is looped over two pegs in a horizontal line. The free end, after being passed through the ring, has a weight suspended from it to make the rope hang taut. If the rope slips freely over the pegs and through the ring, the weight will descend as far as possible. Assume that the length of the rope is at least four times as great as the distance between the pegs and that the configuration of the rope is symmetric with respect to the line of the vertical part of the rope.

a. Find the angle A formed at the bottom of the loop in the accompanying figure.

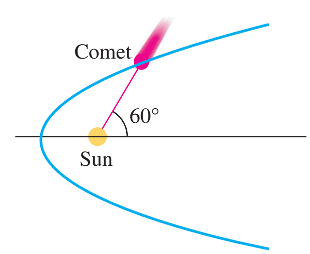
b. Show that for each fixed position of the ring on the rope, the possible locations of the ring in space lie on an ellipse with foci at the pegs.

c. Justify the original symmetry assumption by combining the result in part (b) with the assumption that the rope and weight will take a rest position of minimal potential energy.



30. Two radar stations lie 20 km apart along an east–west line. A low-flying plane traveling from west to east is known to have a speed of v_0 km/sec. At $t = 0$ a signal is sent from the station at $(-10, 0)$, bounces off the plane, and is received at $(10, 0)$ $30/c$ seconds later (c is the velocity of the signal). When $t = 10/v_0$, another signal is sent out from the station at $(-10, 0)$, reflects off the plane, and is once again received $30/c$ seconds later by the other station. Find the position of the plane when it reflects the second signal under the assumption that v_0 is much less than c.

31. A comet moves in a parabolic orbit with the sun at the focus. When the comet is 4×10^7 miles from the sun, the line from the comet to the sun makes a 60° angle with the orbit's axis, as shown here. How close will the comet come to the sun?



32. Find the points on the parabola $x = 2t, y = t^2, -\infty < t < \infty$, closest to the point (0, 3).

33. Find the eccentricity of the ellipse $x^2 + xy + y^2 = 1$ to the nearest hundredth.

34. Find the eccentricity of the hyperbola $xy = 1$.

35. Is the curve $\sqrt{x} + \sqrt{y} = 1$ part of a conic section? If so, what kind of conic section? If not, why not?

36. Show that the curve $2xy - \sqrt{2}\,y + 2 = 0$ is a hyperbola. Find the hyperbola's center, vertices, foci, axes, and asymptotes.

37. Find a polar coordinate equation for

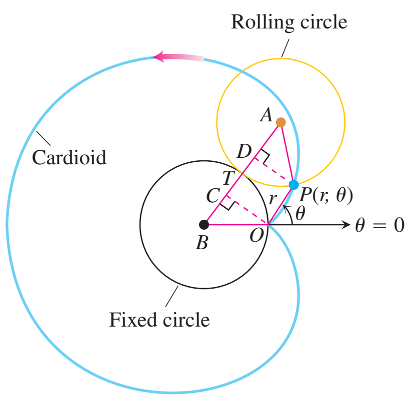
a. the parabola with focus at the origin and vertex at $(a, \pi/4)$;

b. the ellipse with foci at the origin and $(2, 0)$ and one vertex at $(4, 0)$;

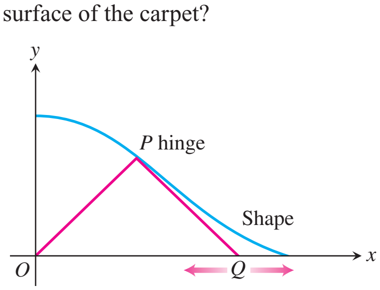
c. the hyperbola with one focus at the origin, center at $(2, \pi/2)$, and a vertex at $(1, \pi/2)$.

38. Any line through the origin will intersect the ellipse $r = 3/(2 + \cos\theta)$ in two points P_1 and P_2. Let d_1 be the distance between P_1 and the origin and let d_2 be the distance between P_2 and the origin. Compute $(1/d_1) + (1/d_2)$.

39. Generating a cardioid with circles Cardioids are special epicycloids (Exercise 18). Show that if you roll a circle of radius a about another circle of radius a in the polar coordinate plane, as in the accompanying figure, the original point of contact P will trace a cardioid. (*Hint:* Start by showing that angles OBC and PAD both have measure θ.)



40. A bifold closet door A bifold closet door consists of two 1-ft-wide panels, hinged at point P. The outside bottom corner of one panel rests on a pivot at O (see the accompanying figure). The outside bottom corner of the other panel, denoted by Q, slides along a straight track, shown in the figure as a portion of the x-axis. Assume that as Q moves back and forth, the bottom of the door rubs against a thick carpet. What shape will the door sweep out on the surface of the carpet?

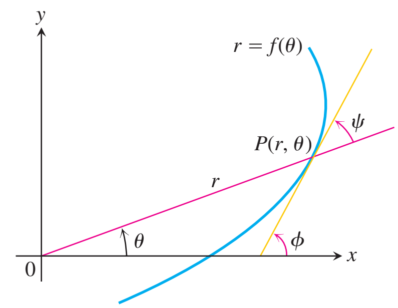


The Angle Between the Radius Vector and the Tangent Line to a Polar Coordinate Curve

In Cartesian coordinates, when we want to discuss the direction of a curve at a point, we use the angle ϕ measured counterclockwise from the positive x-axis to the tangent line. In polar coordinates, it is more convenient to calculate the angle ψ from the *radius vector* to the tangent line (see the accompanying figure). The angle ϕ can then be calculated from the relation

$$\phi = \theta + \psi, \tag{1}$$

which comes from applying the Exterior Angle Theorem to the triangle in the accompanying figure.



Suppose the equation of the curve is given in the form $r = f(\theta)$, where $f(\theta)$ is a differentiable function of θ. Then

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{2}$$

are differentiable functions of θ with

$$\begin{aligned} \frac{dx}{d\theta} &= -r\sin\theta + \cos\theta\,\frac{dr}{d\theta}, \\ \frac{dy}{d\theta} &= r\cos\theta + \sin\theta\,\frac{dr}{d\theta}. \end{aligned} \tag{3}$$

Since $\psi = \phi - \theta$ from (1),

$$\tan\psi = \tan(\phi - \theta) = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}.$$

Furthermore,

$$\tan\phi = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

because $\tan\phi$ is the slope of the curve at P. Also,

$$\tan\theta = \frac{y}{x}.$$

Hence

$$\tan\psi = \frac{\dfrac{dy/d\theta}{dx/d\theta} - \dfrac{y}{x}}{1 + \dfrac{y}{x}\dfrac{dy/d\theta}{dx/d\theta}} = \frac{x\dfrac{dy}{d\theta} - y\dfrac{dx}{d\theta}}{x\dfrac{dx}{d\theta} + y\dfrac{dy}{d\theta}}. \tag{4}$$

The numerator in the last expression in Equation (4) is found from Equations (2) and (3) to be

$$x\frac{dy}{d\theta} - y\frac{dx}{d\theta} = r^2.$$

Similarly, the denominator is

$$x\frac{dx}{d\theta} + y\frac{dy}{d\theta} = r\frac{dr}{d\theta}.$$

When we substitute these into Equation (4), we obtain

$$\tan\psi = \frac{r}{dr/d\theta}. \tag{5}$$

This is the equation we use for finding ψ as a function of θ.

41. Show, by reference to a figure, that the angle β between the tangents to two curves at a point of intersection may be found from the formula

$$\tan\beta = \frac{\tan\psi_2 - \tan\psi_1}{1 + \tan\psi_2\tan\psi_1}. \tag{6}$$

When will the two curves intersect at right angles?

42. Find the value of $\tan\psi$ for the curve $r = \sin^4(\theta/4)$.

43. Find the angle between the radius vector to the curve $r = 2a\sin 3\theta$ and its tangent when $\theta = \pi/6$.

T **44.** **a.** Graph the hyperbolic spiral $r\theta = 1$. What appears to happen to ψ as the spiral winds in around the origin?

b. Confirm your finding in part (a) analytically.

45. The circles $r = \sqrt{3}\cos\theta$ and $r = \sin\theta$ intersect at the point $(\sqrt{3}/2, \pi/3)$. Show that their tangents are perpendicular there.

46. Sketch the cardioid $r = a(1 + \cos\theta)$ and circle $r = 3a\cos\theta$ in one diagram and find the angle between their tangents at the point of intersection that lies in the first quadrant.

47. Find the points of intersection of the parabolas

$$r = \frac{1}{1 - \cos\theta} \quad \text{and} \quad r = \frac{3}{1 + \cos\theta}$$

and the angles between their tangents at these points.

48. Find points on the cardioid $r = a(1 + \cos\theta)$ where the tangent line is **(a)** horizontal, **(b)** vertical.

49. Show that parabolas $r = a/(1 + \cos\theta)$ and $r = b/(1 - \cos\theta)$ are orthogonal at each point of intersection $(ab \neq 0)$.

50. Find the angle at which the cardioid $r = a(1 - \cos\theta)$ crosses the ray $\theta = \pi/2$.

51. Find the angle between the line $r = 3\sec\theta$ and the cardioid $r = 4(1 + \cos\theta)$ at one of their intersections.

52. Find the slope of the tangent line to the curve $r = a\tan(\theta/2)$ at $\theta = \pi/2$.

53. Find the angle at which the parabolas $r = 1/(1 - \cos\theta)$ and $r = 1/(1 - \sin\theta)$ intersect in the first quadrant.

54. The equation $r^2 = 2\csc 2\theta$ represents a curve in polar coordinates.

a. Sketch the curve.

b. Find an equivalent Cartesian equation for the curve.

c. Find the angle at which the curve intersects the ray $\theta = \pi/4$.

55. Suppose that the angle ψ from the radius vector to the tangent line of the curve $r = f(\theta)$ has the constant value α.

a. Show that the area bounded by the curve and two rays $\theta = \theta_1$, $\theta = \theta_2$, is proportional to $r_2^2 - r_1^2$, where (r_1, θ_1) and (r_2, θ_2) are polar coordinates of the ends of the arc of the curve between these rays. Find the factor of proportionality.

b. Show that the length of the arc of the curve in part (a) is proportional to $r_2 - r_1$, and find the proportionality constant.

56. Let P be a point on the hyperbola $r^2\sin 2\theta = 2a^2$. Show that the triangle formed by OP, the tangent at P, and the initial line is isosceles.