EXERCISES 11.1

Finding Terms of a Sequence

Each of Exercises 1–6 gives a formula for the nth term a_n of a sequence $\{a_n\}$. Find the values of a_1, a_2, a_3, and a_4.

1. $a_n = \frac{1-n}{n^2}$

2. $a_n = \frac{1}{n!}$

3. $a_n = \frac{(-1)^{n+1}}{2n-1}$

4. $a_n = 2 + (-1)^n$

5. $a_n = \frac{2^n}{2^{n+1}}$

6. $a_n = \frac{2^n - 1}{2^n}$

Each of Exercises 7–12 gives the first term or two of a sequence along with a recursion formula for the remaining terms. Write out the first ten terms of the sequence.

7. $a_1 = 1, \quad a_{n+1} = a_n + (1/2^n)$

8. $a_1 = 1, \quad a_{n+1} = a_n/(n+1)$

9. $a_1 = 2, \quad a_{n+1} = (-1)^{n+1}a_n/2$

10. $a_1 = -2, \quad a_{n+1} = na_n/(n+1)$

11. $a_1 = a_2 = 1, \quad a_{n+2} = a_{n+1} + a_n$

12. $a_1 = 2, \quad a_2 = -1, \quad a_{n+2} = a_{n+1}/a_n$

Finding a Sequence's Formula

In Exercises 13–22, find a formula for the nth term of the sequence.

13. The sequence $1, -1, 1, -1, 1, \ldots$ — 1's with alternating signs

14. The sequence $-1, 1, -1, 1, -1, \ldots$ — 1's with alternating signs

15. The sequence $1, -4, 9, -16, 25, \ldots$ — Squares of the positive integers; with alternating signs

16. The sequence $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$ — Reciprocals of squares of the positive integers, with alternating signs

17. The sequence $0, 3, 8, 15, 24, \ldots$ — Squares of the positive integers diminished by 1

18. The sequence $-3, -2, -1, 0, 1, \ldots$ — Integers beginning with -3

19. The sequence $1, 5, 9, 13, 17, \ldots$ — Every other odd positive integer

20. The sequence $2, 6, 10, 14, 18, \ldots$ — Every other even positive integer

21. The sequence $1, 0, 1, 0, 1, \ldots$ — Alternating 1's and 0's

22. The sequence $0, 1, 1, 2, 2, 3, 3, 4, \ldots$ — Each positive integer repeated

Finding Limits

Which of the sequences $\{a_n\}$ in Exercises 23–84 converge, and which diverge? Find the limit of each convergent sequence.

23. $a_n = 2 + (0.1)^n$

24. $a_n = \frac{n + (-1)^n}{n}$

25. $a_n = \frac{1-2n}{1+2n}$

26. $a_n = \frac{2n+1}{1-3\sqrt{n}}$

27. $a_n = \frac{1-5n^4}{n^4+8n^3}$

28. $a_n = \frac{n+3}{n^2+5n+6}$

29. $a_n = \frac{n^2-2n+1}{n-1}$

30. $a_n = \frac{1-n^3}{70-4n^2}$

31. $a_n = 1 + (-1)^n$

32. $a_n = (-1)^n\left(1 - \frac{1}{n}\right)$

33. $a_n = \left(\frac{n+1}{2n}\right)\left(1 - \frac{1}{n}\right)$

34. $a_n = \left(2 - \frac{1}{2^n}\right)\left(3 + \frac{1}{2^n}\right)$

35. $a_n = \frac{(-1)^{n+1}}{2n-1}$

36. $a_n = \left(-\frac{1}{2}\right)^n$

37. $a_n = \sqrt{\frac{2n}{n+1}}$

38. $a_n = \frac{1}{(0.9)^n}$

39. $a_n = \sin\left(\frac{\pi}{2} + \frac{1}{n}\right)$

40. $a_n = n\pi \cos(n\pi)$

41. $a_n = \frac{\sin n}{n}$

42. $a_n = \frac{\sin^2 n}{2^n}$

43. $a_n = \frac{n}{2^n}$

44. $a_n = \frac{3^n}{n^3}$

45. $a_n = \frac{\ln(n+1)}{\sqrt{n}}$

46. $a_n = \frac{\ln n}{\ln 2n}$

47. $a_n = 8^{1/n}$

48. $a_n = (0.03)^{1/n}$

49. $a_n = \left(1 + \frac{7}{n}\right)^n$

50. $a_n = \left(1 - \frac{1}{n}\right)^n$

51. $a_n = \sqrt[n]{10n}$

52. $a_n = \sqrt[n]{n^2}$

53. $a_n = \left(\frac{3}{n}\right)^{1/n}$

54. $a_n = (n+4)^{1/(n+4)}$

55. $a_n = \frac{\ln n}{n^{1/n}}$

56. $a_n = \ln n - \ln(n+1)$

57. $a_n = \sqrt[n]{4^n n}$

58. $a_n = \sqrt[n]{3^{2n+1}}$

59. $a_n = \frac{n!}{n^n}$ (*Hint:* Compare with $1/n$.)

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Exercises

60. $a_n = \dfrac{(-4)^n}{n!}$

61. $a_n = \dfrac{n!}{10^{6n}}$

62. $a_n = \dfrac{n!}{2^n \cdot 3^n}$

63. $a_n = \left(\dfrac{1}{n}\right)^{1/(\ln n)}$

64. $a_n = \ln\left(1 + \dfrac{1}{n}\right)^n$

65. $a_n = \left(\dfrac{3n+1}{3n-1}\right)^n$

66. $a_n = \left(\dfrac{n}{n+1}\right)^n$

67. $a_n = \left(\dfrac{x^n}{2n+1}\right)^{1/n}, \quad x > 0$

68. $a_n = \left(1 - \dfrac{1}{n^2}\right)^n$

69. $a_n = \dfrac{3^n \cdot 6^n}{2^{-n} \cdot n!}$

70. $a_n = \dfrac{(10/11)^n}{(9/10)^n + (11/12)^n}$

71. $a_n = \tanh n$

72. $a_n = \sinh(\ln n)$

73. $a_n = \dfrac{n^2}{2n-1}\sin\dfrac{1}{n}$

74. $a_n = n\left(1 - \cos\dfrac{1}{n}\right)$

75. $a_n = \tan^{-1} n$

76. $a_n = \dfrac{1}{\sqrt{n}}\tan^{-1} n$

77. $a_n = \left(\dfrac{1}{3}\right)^n + \dfrac{1}{\sqrt{2^n}}$

78. $a_n = \sqrt[n]{n^2 + n}$

79. $a_n = \dfrac{(\ln n)^{200}}{n}$

80. $a_n = \dfrac{(\ln n)^5}{\sqrt{n}}$

81. $a_n = n - \sqrt{n^2 - n}$

82. $a_n = \dfrac{1}{\sqrt{n^2 - 1} - \sqrt{n^2 + n}}$

83. $a_n = \dfrac{1}{n}\displaystyle\int_1^n \dfrac{1}{x}\,dx$

84. $a_n = \displaystyle\int_1^n \dfrac{1}{x^p}\,dx, \quad p > 1$

Theory and Examples

85. The first term of a sequence is $x_1 = 1$. Each succeeding term is the sum of all those that come before it:

$$x_{n+1} = x_1 + x_2 + \cdots + x_n.$$

Write out enough early terms of the sequence to deduce a general formula for x_n that holds for $n \geq 2$.

86. A sequence of rational numbers is described as follows:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots, \frac{a}{b}, \frac{a+2b}{a+b}, \ldots.$$

Here the numerators form one sequence, the denominators form a second sequence, and their ratios form a third sequence. Let x_n and y_n be, respectively, the numerator and the denominator of the nth fraction $r_n = x_n/y_n$.

a. Verify that $x_1^2 - 2y_1^2 = -1$, $x_2^2 - 2y_2^2 = +1$ and, more generally, that if $a^2 - 2b^2 = -1$ or $+1$, then

$$(a+2b)^2 - 2(a+b)^2 = +1 \quad \text{or} \quad -1,$$

respectively.

b. The fractions $r_n = x_n/y_n$ approach a limit as n increases. What is that limit? (*Hint:* Use part (a) to show that $r_n^2 - 2 = \pm(1/y_n)^2$ and that y_n is not less than n.)

87. Newton's method The following sequences come from the recursion formula for Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Do the sequences converge? If so, to what value? In each case, begin by identifying the function f that generates the sequence.

a. $x_0 = 1, \quad x_{n+1} = x_n - \dfrac{x_n^2 - 2}{2x_n} = \dfrac{x_n}{2} + \dfrac{1}{x_n}$

b. $x_0 = 1, \quad x_{n+1} = x_n - \dfrac{\tan x_n - 1}{\sec^2 x_n}$

c. $x_0 = 1, \quad x_{n+1} = x_n - 1$

88. a. Suppose that $f(x)$ is differentiable for all x in $[0, 1]$ and that $f(0) = 0$. Define the sequence $\{a_n\}$ by the rule $a_n = nf(1/n)$. Show that $\lim_{n\to\infty} a_n = f'(0)$.

Use the result in part (a) to find the limits of the following sequences $\{a_n\}$.

b. $a_n = n\tan^{-1}\dfrac{1}{n}$

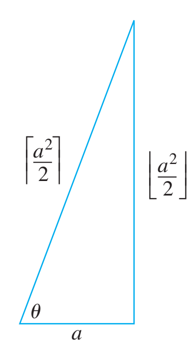
c. $a_n = n(e^{1/n} - 1)$

d. $a_n = n\ln\left(1 + \dfrac{2}{n}\right)$

89. Pythagorean triples A triple of positive integers a, b, and c is called a **Pythagorean triple** if $a^2 + b^2 = c^2$. Let a be an odd positive integer and let

$$b = \left\lfloor \frac{a^2}{2} \right\rfloor \quad \text{and} \quad c = \left\lceil \frac{a^2}{2} \right\rceil$$

be, respectively, the integer floor and ceiling for $a^2/2$.



a. Show that $a^2 + b^2 = c^2$. (*Hint:* Let $a = 2n + 1$ and express b and c in terms of n.)

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b. By direct calculation, or by appealing to the figure here, find

$$\lim_{a\to\infty} \frac{\left\lfloor \frac{a^2}{2} \right\rfloor}{\left\lceil \frac{a^2}{2} \right\rceil}.$$

90. The nth root of $n!$

a. Show that $\lim_{n\to\infty}(2n\pi)^{1/(2n)} = 1$ and hence, using Stirling's approximation (Chapter 8, Additional Exercise 50a), that

$$\sqrt[n]{n!} \approx \frac{n}{e} \quad \text{for large values of } n.$$

T **b.** Test the approximation in part (a) for $n = 40, 50, 60, \ldots$, as far as your calculator will allow.

91. a. Assuming that $\lim_{n\to\infty}(1/n^c) = 0$ if c is any positive constant, show that

$$\lim_{n\to\infty} \frac{\ln n}{n^c} = 0$$

if c is any positive constant.

b. Prove that $\lim_{n\to\infty}(1/n^c) = 0$ if c is any positive constant. (*Hint:* If $\epsilon = 0.001$ and $c = 0.04$, how large should N be to ensure that $|1/n^c - 0| < \epsilon$ if $n > N$?)

92. The zipper theorem Prove the "zipper theorem" for sequences: If $\{a_n\}$ and $\{b_n\}$ both converge to L, then the sequence

$$a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots$$

converges to L.

93. Prove that $\lim_{n\to\infty}\sqrt[n]{n} = 1$.

94. Prove that $\lim_{n\to\infty} x^{1/n} = 1, (x > 0)$.

95. Prove Theorem 2.

96. Prove Theorem 3.

In Exercises 97–100, determine if the sequence is nondecreasing and if it is bounded from above.

97. $a_n = \dfrac{3n+1}{n+1}$

98. $a_n = \dfrac{(2n+3)!}{(n+1)!}$

99. $a_n = \dfrac{2^n 3^n}{n!}$

100. $a_n = 2 - \dfrac{2}{n} - \dfrac{1}{2^n}$

Which of the sequences in Exercises 101–106 converge, and which diverge? Give reasons for your answers.

101. $a_n = 1 - \dfrac{1}{n}$

102. $a_n = n - \dfrac{1}{n}$

103. $a_n = \dfrac{2^n - 1}{2^n}$

104. $a_n = \dfrac{2^n - 1}{3^n}$

105. $a_n = ((-1)^n + 1)\left(\dfrac{n+1}{n}\right)$

106. The first term of a sequence is $x_1 = \cos(1)$. The next terms are $x_2 = x_1$ or $\cos(2)$, whichever is larger; and $x_3 = x_2$ or $\cos(3)$, whichever is larger (farther to the right). In general,

$$x_{n+1} = \max\{x_n, \cos(n+1)\}.$$

107. Nonincreasing sequences A sequence of numbers $\{a_n\}$ in which $a_n \geq a_{n+1}$ for every n is called a **nonincreasing sequence**. A sequence $\{a_n\}$ is **bounded from below** if there is a number M with $M \leq a_n$ for every n. Such a number M is called a **lower bound** for the sequence. Deduce from Theorem 6 that a nonincreasing sequence that is bounded from below converges and that a nonincreasing sequence that is not bounded from below diverges.

(*Continuation of Exercise 107.*) Using the conclusion of Exercise 107, determine which of the sequences in Exercises 108–112 converge and which diverge.

108. $a_n = \dfrac{n+1}{n}$

109. $a_n = \dfrac{1 + \sqrt{2n}}{\sqrt{n}}$

110. $a_n = \dfrac{1 - 4^n}{2^n}$

111. $a_n = \dfrac{4^{n+1} + 3^n}{4^n}$

112. $a_1 = 1, \quad a_{n+1} = 2a_n - 3$

113. The sequence $\{n/(n+1)\}$ has a least upper bound of 1 Show that if M is a number less than 1, then the terms of $\{n/(n+1)\}$ eventually exceed M. That is, if $M < 1$ there is an integer N such that $n/(n+1) > M$ whenever $n > N$. Since $n/(n+1) < 1$ for every n, this proves that 1 is a least upper bound for $\{n/(n+1)\}$.

114. Uniqueness of least upper bounds Show that if M_1 and M_2 are least upper bounds for the sequence $\{a_n\}$, then $M_1 = M_2$. That is, a sequence cannot have two different least upper bounds.

115. Is it true that a sequence $\{a_n\}$ of positive numbers must converge if it is bounded from above? Give reasons for your answer.

116. Prove that if $\{a_n\}$ is a convergent sequence, then to every positive number ϵ there corresponds an integer N such that for all m and n,

$$m > N \quad \text{and} \quad n > N \quad \Rightarrow \quad |a_m - a_n| < \epsilon.$$

117. Uniqueness of limits Prove that limits of sequences are unique. That is, show that if L_1 and L_2 are numbers such that $a_n \to L_1$ and $a_n \to L_2$, then $L_1 = L_2$.

118. Limits and subsequences If the terms of one sequence appear in another sequence in their given order, we call the first sequence a **subsequence** of the second. Prove that if two subsequences of a sequence $\{a_n\}$ have different limits $L_1 \neq L_2$, then $\{a_n\}$ diverges.

119. For a sequence $\{a_n\}$ the terms of even index are denoted by a_{2k} and the terms of odd index by a_{2k+1}. Prove that if $a_{2k} \to L$ and $a_{2k+1} \to L$, then $a_n \to L$.

120. Prove that a sequence $\{a_n\}$ converges to 0 if and only if the sequence of absolute values $\{|a_n|\}$ converges to 0.

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T Calculator Explorations of Limits

In Exercises 121–124, experiment with a calculator to find a value of N that will make the inequality hold for all $n > N$. Assuming that the inequality is the one from the formal definition of the limit of a sequence, what sequence is being considered in each case and what is its limit?

121. $|\sqrt[n]{0.5} - 1| < 10^{-3}$ **122.** $|\sqrt[n]{n} - 1| < 10^{-3}$

123. $(0.9)^n < 10^{-3}$ **124.** $2^n/n! < 10^{-7}$

125. Sequences generated by Newton's method Newton's method, applied to a differentiable function $f(x)$, begins with a starting value x_0 and constructs from it a sequence of numbers $\{x_n\}$ that under favorable circumstances converges to a zero of f. The recursion formula for the sequence is

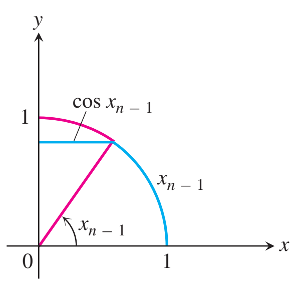
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

a. Show that the recursion formula for $f(x) = x^2 - a, a > 0$, can be written as $x_{n+1} = (x_n + a/x_n)/2$.

b. Starting with $x_0 = 1$ and $a = 3$, calculate successive terms of the sequence until the display begins to repeat. What number is being approximated? Explain.

126. (*Continuation of Exercise 125.*) Repeat part (b) of Exercise 125 with $a = 2$ in place of $a = 3$.

127. A recursive definition of $\pi/2$ If you start with $x_1 = 1$ and define the subsequent terms of $\{x_n\}$ by the rule $x_n = x_{n-1} + \cos x_{n-1}$, you generate a sequence that converges rapidly to $\pi/2$. **a.** Try it. **b.** Use the accompanying figure to explain why the convergence is so rapid.



128. According to a front-page article in the December 15, 1992, issue of the *Wall Street Journal*, Ford Motor Company used about $7\frac{1}{4}$ hours of labor to produce stampings for the average vehicle, down from an estimated 15 hours in 1980. The Japanese needed only about $3\frac{1}{2}$ hours.

Ford's improvement since 1980 represents an average decrease of 6% per year. If that rate continues, then n years from 1992 Ford will use about

$$S_n = 7.25(0.94)^n$$

hours of labor to produce stampings for the average vehicle. Assuming that the Japanese continue to spend $3\frac{1}{2}$ hours per vehicle, how many more years will it take Ford to catch up? Find out two ways:

a. Find the first term of the sequence $\{S_n\}$ that is less than or equal to 3.5.

T **b.** Graph $f(x) = 7.25(0.94)^x$ and use Trace to find where the graph crosses the line $y = 3.5$.

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the sequences in Exercises 129–140.

a. Calculate and then plot the first 25 terms of the sequence. Does the sequence appear to be bounded from above or below? Does it appear to converge or diverge? If it does converge, what is the limit L?

b. If the sequence converges, find an integer N such that $|a_n - L| \leq 0.01$ for $n \geq N$. How far in the sequence do you have to get for the terms to lie within 0.0001 of L?

129. $a_n = \sqrt[n]{n}$ **130.** $a_n = \left(1 + \frac{0.5}{n}\right)^n$

131. $a_1 = 1, \quad a_{n+1} = a_n + \frac{1}{5^n}$

132. $a_1 = 1, \quad a_{n+1} = a_n + (-2)^n$

133. $a_n = \sin n$ **134.** $a_n = n \sin \frac{1}{n}$

135. $a_n = \frac{\sin n}{n}$ **136.** $a_n = \frac{\ln n}{n}$

137. $a_n = (0.9999)^n$ **138.** $a_n = 123456^{1/n}$

139. $a_n = \frac{8^n}{n!}$ **140.** $a_n = \frac{n^{41}}{19^n}$

141. Compound interest, deposits, and withdrawals If you invest an amount of money A_0 at a fixed annual interest rate r compounded m times per year, and if the constant amount b is added to the account at the end of each compounding period (or taken from the account if $b < 0$), then the amount you have after $n + 1$ compounding periods is

$$A_{n+1} = \left(1 + \frac{r}{m}\right)A_n + b. \qquad (1)$$

a. If $A_0 = 1000, r = 0.02015, m = 12$, and $b = 50$, calculate and plot the first 100 points (n, A_n). How much money is in your account at the end of 5 years? Does $\{A_n\}$ converge? Is $\{A_n\}$ bounded?

b. Repeat part (a) with $A_0 = 5000, r = 0.0589, m = 12$, and $b = -50$.

c. If you invest 5000 dollars in a certificate of deposit (CD) that pays 4.5% annually, compounded quarterly, and you make no further investments in the CD, approximately how many years will it take before you have 20,000 dollars? What if the CD earns 6.25%?

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d. It can be shown that for any $k \geq 0$, the sequence defined recursively by Equation (1) satisfies the relation

$$A_k = \left(1 + \frac{r}{m}\right)^k \left(A_0 + \frac{mb}{r}\right) - \frac{mb}{r}. \quad (2)$$

For the values of the constants A_0, r, m, and b given in part (a), validate this assertion by comparing the values of the first 50 terms of both sequences. Then show by direct substitution that the terms in Equation (2) satisfy the recursion formula in Equation (1).

142. Logistic difference equation The recursive relation

$$a_{n+1} = ra_n(1 - a_n)$$

is called the *logistic difference equation*, and when the initial value a_0 is given the equation defines the *logistic sequence* $\{a_n\}$. Throughout this exercise we choose a_0 in the interval $0 < a_0 < 1$, say $a_0 = 0.3$.

a. Choose $r = 3/4$. Calculate and plot the points (n, a_n) for the first 100 terms in the sequence. Does it appear to converge? What do you guess is the limit? Does the limit seem to depend on your choice of a_0?

b. Choose several values of r in the interval $1 < r < 3$ and repeat the procedures in part (a). Be sure to choose some points near the endpoints of the interval. Describe the behavior of the sequences you observe in your plots.

c. Now examine the behavior of the sequence for values of r near the endpoints of the interval $3 < r < 3.45$. The transition value $r = 3$ is called a **bifurcation value** and the new behavior of the sequence in the interval is called an **attracting 2-cycle**. Explain why this reasonably describes the behavior.

d. Next explore the behavior for r values near the endpoints of each of the intervals $3.45 < r < 3.54$ and $3.54 < r < 3.55$. Plot the first 200 terms of the sequences. Describe in your own words the behavior observed in your plots for each interval. Among how many values does the sequence appear to oscillate for each interval? The values $r = 3.45$ and $r = 3.54$ (rounded to two decimal places) are also called bifurcation values because the behavior of the sequence changes as r crosses over those values.

e. The situation gets even more interesting. There is actually an increasing sequence of bifurcation values $3 < 3.45 < 3.54 < \cdots < c_n < c_{n+1} \cdots$ such that for $c_n < r < c_{n+1}$ the logistic sequence $\{a_n\}$ eventually oscillates steadily among 2^n values, called an **attracting 2^n-cycle**. Moreover, the bifurcation sequence $\{c_n\}$ is bounded above by 3.57 (so it converges). If you choose a value of $r < 3.57$ you will observe a 2^n-cycle of some sort. Choose $r = 3.5695$ and plot 300 points.

f. Let us see what happens when $r > 3.57$. Choose $r = 3.65$ and calculate and plot the first 300 terms of $\{a_n\}$. Observe how the terms wander around in an unpredictable, chaotic fashion. You cannot predict the value of a_{n+1} from previous values of the sequence.

g. For $r = 3.65$ choose two starting values of a_0 that are close together, say, $a_0 = 0.3$ and $a_0 = 0.301$. Calculate and plot the first 300 values of the sequences determined by each starting value. Compare the behaviors observed in your plots. How far out do you go before the corresponding terms of your two sequences appear to depart from each other? Repeat the exploration for $r = 3.75$. Can you see how the plots look different depending on your choice of a_0? We say that the logistic sequence is *sensitive to the initial condition* a_0.

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