# EXERCISES 11.2

## Finding *n*th Partial Sums

In Exercises 1–6, find a formula for the $n$th partial sum of each series and use it to find the series' sum if the series converges.

Exercises

**1.** $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots$

**2.** $\frac{9}{100} + \frac{9}{100^2} + \frac{9}{100^3} + \cdots + \frac{9}{100^n} + \cdots$

**3.** $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + (-1)^{n-1}\frac{1}{2^{n-1}} + \cdots$

**4.** $1 - 2 + 4 - 8 + \cdots + (-1)^{n-1}2^{n-1} + \cdots$

**5.** $\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{(n+1)(n+2)} + \cdots$

**6.** $\frac{5}{1 \cdot 2} + \frac{5}{2 \cdot 3} + \frac{5}{3 \cdot 4} + \cdots + \frac{5}{n(n+1)} + \cdots$

## Series with Geometric Terms

In Exercises 7–14, write out the first few terms of each series to show how the series starts. Then find the sum of the series.

**7.** $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$

**8.** $\sum_{n=2}^{\infty} \frac{1}{4^n}$

Exercises

**9.** $\sum_{n=1}^{\infty} \frac{7}{4^n}$

**10.** $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$

**11.** $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$

**12.** $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$

**13.** $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$

**14.** $\sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{5^n}\right)$

## Telescoping Series

Use partial fractions to find the sum of each series in Exercises 15–22.

**15.** $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$

**16.** $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$

**17.** $\sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2n+1)^2}$

**18.** $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

**19.** $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

**20.** $\sum_{n=1}^{\infty} \left(\frac{1}{2^{1/n}} - \frac{1}{2^{1/(n+1)}}\right)$

**21.** $\sum_{n=1}^{\infty} \left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}\right)$

**22.** $\sum_{n=1}^{\infty} (\tan^{-1}(n) - \tan^{-1}(n+1))$

## Convergence or Divergence

Which series in Exercises 23–40 converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

**23.** $\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$

**24.** $\sum_{n=0}^{\infty} (\sqrt{2})^n$

**25.** $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{2^n}$

**26.** $\sum_{n=1}^{\infty} (-1)^{n+1} n$

**27.** $\sum_{n=0}^{\infty} \cos n\pi$

**28.** $\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n}$

**29.** $\sum_{n=0}^{\infty} e^{-2n}$

**30.** $\sum_{n=1}^{\infty} \ln \frac{1}{n}$

**31.** $\sum_{n=1}^{\infty} \frac{2}{10^n}$

**32.** $\sum_{n=0}^{\infty} \frac{1}{x^n}, \quad |x| > 1$

**33.** $\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n}$

**34.** $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$

**35.** $\sum_{n=0}^{\infty} \frac{n!}{1000^n}$

**36.** $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

**37.** $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

**38.** $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$

**39.** $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$

**40.** $\sum_{n=0}^{\infty} \frac{e^{n\pi}}{\pi^{ne}}$

## Geometric Series

In each of the geometric series in Exercises 41–44, write out the first few terms of the series to find $a$ and $r$, and find the sum of the series. Then express the inequality $|r| < 1$ in terms of $x$ and find the values of $x$ for which the inequality holds and the series converges.

**41.** $\sum_{n=0}^{\infty} (-1)^n x^n$

**42.** $\sum_{n=0}^{\infty} (-1)^n x^{2n}$

**43.** $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$

**44.** $\sum_{n=0}^{\infty} \frac{(-1)^n}{2}\left(\frac{1}{3 + \sin x}\right)^n$

In Exercises 45–50, find the values of $x$ for which the given geometric series converges. Also, find the sum of the series (as a function of $x$) for those values of $x$.

**45.** $\sum_{n=0}^{\infty} 2^n x^n$

**46.** $\sum_{n=0}^{\infty} (-1)^n x^{-2n}$

**47.** $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$

**48.** $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$

**49.** $\sum_{n=0}^{\infty} \sin^n x$

**50.** $\sum_{n=0}^{\infty} (\ln x)^n$

## Repeating Decimals

Express each of the numbers in Exercises 51–58 as the ratio of two integers.

**51.** $0.\overline{23} = 0.23\ 23\ 23\ldots$

**52.** $0.\overline{234} = 0.234\ 234\ 234\ldots$

**53.** $0.\overline{7} = 0.7777\ldots$

**54.** $0.\overline{d} = 0.dddd\ldots,$ where $d$ is a digit

**55.** $0.0\overline{6} = 0.06666\ldots$

**56.** $1.\overline{414} = 1.414\ 414\ 414\ldots$

**57.** $1.24\overline{123} = 1.24\ 123\ 123\ 123\ldots$

**58.** $3.\overline{142857} = 3.142857\ 142857\ldots$

## Theory and Examples

**59.** The series in Exercise 5 can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \quad \text{and} \quad \sum_{n=-1}^{\infty} \frac{1}{(n+3)(n+4)}.$$

Write it as a sum beginning with **(a)** $n = -2$, **(b)** $n = 0$, **(c)** $n = 5$.

**60.** The series in Exercise 6 can also be written as

$$\sum_{n=1}^{\infty} \frac{5}{n(n+1)} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{5}{(n+1)(n+2)}.$$

Write it as a sum beginning with **(a)** $n = -1$, **(b)** $n = 3$, **(c)** $n = 20$.

**61.** Make up an infinite series of nonzero terms whose sum is

**a.** 1 **b.** $-3$ **c.** 0.

**62.** (*Continuation of Exercise 61.*) Can you make an infinite series of nonzero terms that converges to any number you want? Explain.

**63.** Show by example that $\sum(a_n/b_n)$ may diverge even though $\sum a_n$ and $\sum b_n$ converge and no $b_n$ equals 0.

**64.** Find convergent geometric series $A = \sum a_n$ and $B = \sum b_n$ that illustrate the fact that $\sum a_n b_n$ may converge without being equal to $AB$.

**65.** Show by example that $\sum(a_n/b_n)$ may converge to something other than $A/B$ even when $A = \sum a_n$, $B = \sum b_n \neq 0$, and no $b_n$ equals 0.

**66.** If $\sum a_n$ converges and $a_n > 0$ for all $n$, can anything be said about $\sum(1/a_n)$? Give reasons for your answer.

**67.** What happens if you add a finite number of terms to a divergent series or delete a finite number of terms from a divergent series? Give reasons for your answer.

**68.** If $\sum a_n$ converges and $\sum b_n$ diverges, can anything be said about their term-by-term sum $\sum(a_n + b_n)$? Give reasons for your answer.

**69.** Make up a geometric series $\sum ar^{n-1}$ that converges to the number 5 if

**a.** $a = 2$ **b.** $a = 13/2$.

**70.** Find the value of $b$ for which

$$1 + e^b + e^{2b} + e^{3b} + \cdots = 9.$$

**71.** For what values of $r$ does the infinite series

$$1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + r^6 + \cdots$$

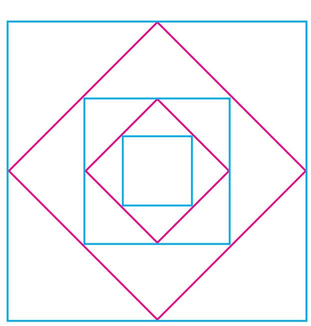
converge? Find the sum of the series when it converges.

**72.** Show that the error $(L - s_n)$ obtained by replacing a convergent geometric series with one of its partial sums $s_n$ is $ar^n/(1 - r)$.

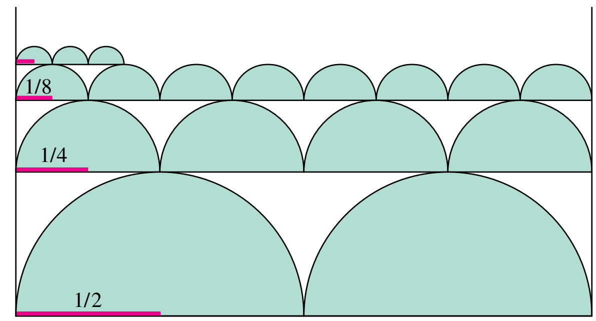
**73.** A ball is dropped from a height of 4 m. Each time it strikes the pavement after falling from a height of $h$ meters it rebounds to a height of $0.75h$ meters. Find the total distance the ball travels up and down.

**74.** (*Continuation of Exercise 73.*) Find the total number of seconds the ball in Exercise 73 is traveling. (*Hint:* The formula $s = 4.9t^2$ gives $t = \sqrt{s/4.9}$.)

**75.** The accompanying figure shows the first five of a sequence of squares. The outermost square has an area of 4 m$^2$. Each of the other squares is obtained by joining the midpoints of the sides of the squares before it. Find the sum of the areas of all the squares.



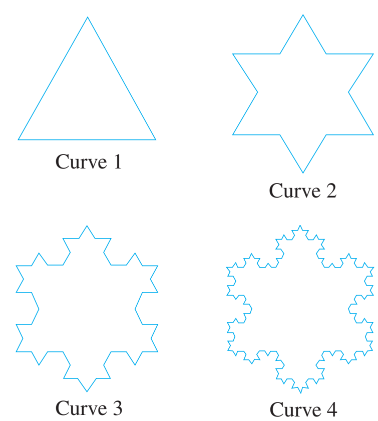
**76.** The accompanying figure shows the first three rows and part of the fourth row of a sequence of rows of semicircles. There are $2^n$ semicircles in the $n$th row, each of radius $1/2^n$. Find the sum of the areas of all the semicircles.




**77. Helga von Koch's snowflake curve** Helga von Koch's snowflake is a curve of infinite length that encloses a region of finite area. To see why this is so, suppose the curve is generated by starting with an equilateral triangle whose sides have length 1.

**a.** Find the length $L_n$ of the $n$th curve $C_n$ and show that $\lim_{n\to\infty} L_n = \infty$.

**b.** Find the area $A_n$ of the region enclosed by $C_n$ and calculate $\lim_{n\to\infty} A_n$.




**78.** The accompanying figure provides an informal proof that $\sum_{n=1}^{\infty}(1/n^2)$ is less than 2. Explain what is going on. (*Source:* "Convergence with Pictures" by P. J. Rippon, *American Mathematical Monthly*, Vol. 93, No. 6, 1986, pp. 476–478.)