EXERCISES 11.4

Determining Convergence or Divergence

Which of the series in Exercises 1–36 converge, and which diverge? Give reasons for your answers.

1. $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}}$ **2.** $\sum_{n=1}^{\infty} \frac{3}{n + \sqrt{n}}$ **3.** $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$

4. $\sum_{n=1}^{\infty} \frac{1 + \cos n}{n^2}$ **5.** $\sum_{n=1}^{\infty} \frac{2n}{3n - 1}$ **6.** $\sum_{n=1}^{\infty} \frac{n + 1}{n^2\sqrt{n}}$

7. $\sum_{n=1}^{\infty} \left(\frac{n}{3n + 1}\right)^n$ **8.** $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 2}}$ **9.** $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

10. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ **11.** $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ **12.** $\sum_{n=1}^{\infty} \frac{(\ln n)^3}{n^3}$

13. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$ **14.** $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$ **15.** $\sum_{n=1}^{\infty} \frac{1}{1 + \ln n}$

16. $\sum_{n=1}^{\infty} \frac{1}{(1 + \ln n)^2}$ **17.** $\sum_{n=2}^{\infty} \frac{\ln(n + 1)}{n + 1}$ **18.** $\sum_{n=1}^{\infty} \frac{1}{(1 + \ln^2 n)}$

19. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2 - 1}}$ **20.** $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$ **21.** $\sum_{n=1}^{\infty} \frac{1 - n}{n2^n}$

22. $\sum_{n=1}^{\infty} \frac{n + 2^n}{n^2 2^n}$ **23.** $\sum_{n=1}^{\infty} \frac{1}{3^{n-1} + 1}$ **24.** $\sum_{n=1}^{\infty} \frac{3^{n-1} + 1}{3^n}$

25. $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ **26.** $\sum_{n=1}^{\infty} \tan\frac{1}{n}$

27. $\sum_{n=1}^{\infty} \frac{10n + 1}{n(n + 1)(n + 2)}$ **28.** $\sum_{n=3}^{\infty} \frac{5n^3 - 3n}{n^2(n - 2)(n^2 + 5)}$

29. $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^{1.1}}$ **30.** $\sum_{n=1}^{\infty} \frac{\sec^{-1} n}{n^{1.3}}$ **31.** $\sum_{n=1}^{\infty} \frac{\coth n}{n^2}$

32. $\sum_{n=1}^{\infty} \frac{\tanh n}{n^2}$ **33.** $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ **34.** $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$

35. $\sum_{n=1}^{\infty} \frac{1}{1 + 2 + 3 + \cdots + n}$ **36.** $\sum_{n=1}^{\infty} \frac{1}{1 + 2^2 + 3^2 + \cdots + n^2}$

Theory and Examples

37. Prove **(a)** Part 2 and **(b)** Part 3 of the Limit Comparison Test.

38. If $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty}(a_n/n)$? Explain.

39. Suppose that $a_n > 0$ and $b_n > 0$ for $n \geq N$ (N an integer). If $\lim_{n\to\infty}(a_n/b_n) = \infty$ and $\sum a_n$ converges, can anything be said about $\sum b_n$? Give reasons for your answer.

40. Prove that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum {a_n}^2$ converges.

COMPUTER EXPLORATION

41. It is not yet known whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2 n}$$

converges or diverges. Use a CAS to explore the behavior of the series by performing the following steps.

a. Define the sequence of partial sums

$$s_k = \sum_{n=1}^{k} \frac{1}{n^3 \sin^2 n}.$$

What happens when you try to find the limit of s_k as $k \to \infty$? Does your CAS find a closed form answer for this limit?

b. Plot the first 100 points (k, s_k) for the sequence of partial sums. Do they appear to converge? What would you estimate the limit to be?

c. Next plot the first 200 points (k, s_k). Discuss the behavior in your own words.

d. Plot the first 400 points (k, s_k). What happens when $k = 355$? Calculate the number $355/113$. Explain from your calculation what happened at $k = 355$. For what values of k would you guess this behavior might occur again?

You will find an interesting discussion of this series in Chapter 72 of *Mazes for the Mind* by Clifford A. Pickover, St. Martin's Press, Inc., New York, 1992.