EXERCISES 11.6

Determining Convergence or Divergence

Which of the alternating series in Exercises 1–10 converge, and which diverge? Give reasons for your answers.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$

2. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{3/2}}$

3. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$

4. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{10^n}{n^{10}}$

5. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{\ln n}$

6. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$

7. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{\ln n}{\ln n^2}$

8. $\sum_{n=1}^{\infty}(-1)^{n}\ln\left(1+\frac{1}{n}\right)$

9. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$

10. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3\sqrt{n+1}}{\sqrt{n}+1}$

Absolute Convergence

Which of the series in Exercises 11–44 converge absolutely, which converge, and which diverge? Give reasons for your answers.

11. $\sum_{n=1}^{\infty}(-1)^{n+1}(0.1)^n$

12. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.1)^n}{n}$

13. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\sqrt{n}}$

14. $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$

15. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^3+1}$

16. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{2^n}$

17. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n+3}$

18. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\sin n}{n^2}$

19. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$

20. $\sum_{n=2}^{\infty}(-1)^{n}\frac{1}{\ln(n^3)}$

21. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1+n}{n^2}$

22. $\sum_{n=1}^{\infty}\frac{(-2)^{n+1}}{n+5^n}$

23. $\sum_{n=1}^{\infty}(-1)^{n}n^2(2/3)^n$

24. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\sqrt[n]{10}\right)$

25. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\tan^{-1} n}{n^2+1}$

26. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$

27. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$

28. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n - \ln n}$

29. $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$

30. $\sum_{n=1}^{\infty} (-5)^{-n}$

31. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 + 2n + 1}$

32. $\sum_{n=2}^{\infty} (-1)^n \left(\frac{\ln n}{\ln n^2}\right)^n$

33. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}}$

34. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$

35. $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)^n}{(2n)^n}$

36. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n!)^2}{(2n)!}$

37. $\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{2^n n! n}$

38. $\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2 3^n}{(2n+1)!}$

39. $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$

40. $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n^2 + n} - n\right)$

41. $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n + \sqrt{n}} - \sqrt{n}\right)$

42. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n} + \sqrt{n+1}}$

43. $\sum_{n=1}^{\infty} (-1)^n \operatorname{sech} n$

44. $\sum_{n=1}^{\infty} (-1)^n \operatorname{csch} n$

Error Estimation

In Exercises 45–48, estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the entire series.

45. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ It can be shown that the sum is ln 2.

46. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{10^n}$

47. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.01)^n}{n}$ As you will see in Section 11.7, the sum is ln (1.01).

48. $\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n, \quad 0 < t < 1$

Approximate the sums in Exercises 49 and 50 with an error of magnitude less than 5×10^{-6}.

49. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}$ As you will see in Section 11.9, the sum is cos 1, the cosine of 1 radian.

50. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!}$ As you will see in Section 11.9, the sum is e^{-1}.

Theory and Examples

51. **a.** The series

$$\frac{1}{3} - \frac{1}{2} + \frac{1}{9} - \frac{1}{4} + \frac{1}{27} - \frac{1}{8} + \cdots + \frac{1}{3^n} - \frac{1}{2^n} + \cdots$$

does not meet one of the conditions of Theorem 14. Which one?

b. Find the sum of the series in part (a).

52. The limit L of an alternating series that satisfies the conditions of Theorem 14 lies between the values of any two consecutive partial sums. This suggests using the average

$$\frac{s_n + s_{n+1}}{2} = s_n + \frac{1}{2}(-1)^{n+2} a_{n+1}$$

to estimate L. Compute

$$s_{20} + \frac{1}{2} \cdot \frac{1}{21}$$

as an approximation to the sum of the alternating harmonic series. The exact sum is $\ln 2 = 0.6931\ldots$.

53. The sign of the remainder of an alternating series that satisfies the conditions of Theorem 14 Prove the assertion in Theorem 15 that whenever an alternating series satisfying the conditions of Theorem 14 is approximated with one of its partial sums, then the remainder (sum of the unused terms) has the same sign as the first unused term. (*Hint:* Group the remainder's terms in consecutive pairs.)

54. Show that the sum of the first $2n$ terms of the series

$$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \cdots$$

is the same as the sum of the first n terms of the series

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \cdots.$$

Do these series converge? What is the sum of the first $2n + 1$ terms of the first series? If the series converge, what is their sum?

55. Show that if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges.

56. Show that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then

$$\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n|.$$

57. Show that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge absolutely, then so does

a. $\sum_{n=1}^{\infty} (a_n + b_n)$

b. $\sum_{n=1}^{\infty} (a_n - b_n)$

c. $\sum_{n=1}^{\infty} k a_n$ (k any number)

58. Show by example that $\sum_{n=1}^{\infty} a_n b_n$ may diverge even if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge.

59. In Example 6, suppose the goal is to arrange the terms to get a new series that converges to $-1/2$. Start the new arrangement with the first negative term, which is $-1/2$. Whenever you have a sum that is less than or equal to $-1/2$, start introducing positive terms, taken in order, until the new total is greater than $-1/2$. Then add negative terms until the total is less than or equal to $-1/2$ again. Continue this process until your partial sums have been above the target at least three times and finish at or below it. If s_n is the sum of the first n terms of your new series, plot the points (n, s_n) to illustrate how the sums are behaving.

60. Outline of the proof of the Rearrangement Theorem (Theorem 17)

a. Let ϵ be a positive real number, let $L = \sum_{n=1}^{\infty} a_n$, and let $s_k = \sum_{n=1}^{k} a_n$. Show that for some index N_1 and for some index $N_2 \geq N_1$,

$$\sum_{n=N_1}^{\infty} |a_n| < \frac{\epsilon}{2} \quad \text{and} \quad |s_{N_2} - L| < \frac{\epsilon}{2}.$$

Since all the terms $a_1, a_2, \ldots, a_{N_2}$ appear somewhere in the sequence $\{b_n\}$, there is an index $N_3 \geq N_2$ such that if $n \geq N_3$, then $\left(\sum_{k=1}^{n} b_k\right) - s_{N_2}$ is at most a sum of terms a_m with $m \geq N_1$. Therefore, if $n \geq N_3$,

$$\left|\sum_{k=1}^{n} b_k - L\right| \leq \left|\sum_{k=1}^{n} b_k - s_{N_2}\right| + |s_{N_2} - L|$$

$$\leq \sum_{k=N_1}^{\infty} |a_k| + |s_{N_2} - L| < \epsilon.$$

b. The argument in part (a) shows that if $\sum_{n=1}^{\infty} a_n$ converges absolutely then $\sum_{n=1}^{\infty} b_n$ converges and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$. Now show that because $\sum_{n=1}^{\infty} |a_n|$ converges, $\sum_{n=1}^{\infty} |b_n|$ converges to $\sum_{n=1}^{\infty} |a_n|$.

61. Unzipping absolutely convergent series

a. Show that if $\sum_{n=1}^{\infty} |a_n|$ converges and

$$b_n = \begin{cases} a_n, & \text{if } a_n \geq 0 \\ 0, & \text{if } a_n < 0, \end{cases}$$

then $\sum_{n=1}^{\infty} b_n$ converges.

b. Use the results in part (a) to show likewise that if $\sum_{n=1}^{\infty} |a_n|$ converges and

$$c_n = \begin{cases} 0, & \text{if } a_n \geq 0 \\ a_n, & \text{if } a_n < 0, \end{cases}$$

then $\sum_{n=1}^{\infty} c_n$ converges.

In other words, if a series converges absolutely, its positive terms form a convergent series, and so do its negative terms. Furthermore,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n$$

because $b_n = (a_n + |a_n|)/2$ and $c_n = (a_n - |a_n|)/2$.

62. What is wrong here?:

Multiply both sides of the alternating harmonic series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots$$

by 2 to get

$$2S = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \frac{1}{6} + \cdots.$$

Collect terms with the same denominator, as the arrows indicate, to arrive at

$$2S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

The series on the right-hand side of this equation is the series we started with. Therefore, $2S = S$, and dividing by S gives $2 = 1$. (*Source:* "Riemann's Rearrangement Theorem" by Stewart Galanor, *Mathematics Teacher*, Vol. 80, No. 8, 1987, pp. 675–681.)

63. Draw a figure similar to Figure 11.9 to illustrate the convergence of the series in Theorem 14 when $N > 1$.